Constrained optimization

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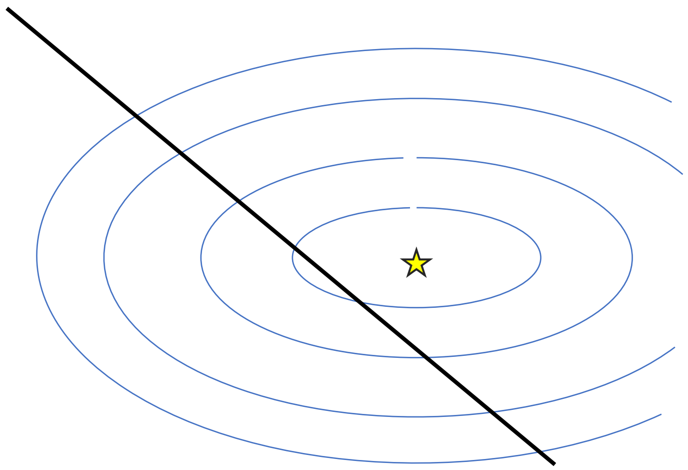
Optimization with constraint

- Linearly-Constrained Quadratic Program (LCQP)

$$\min_x ||Ax - b||^2$$
$$s.t \;\; Cx = d$$



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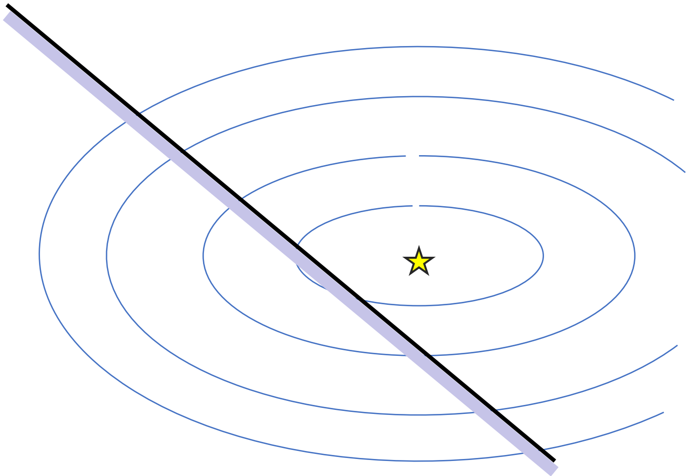

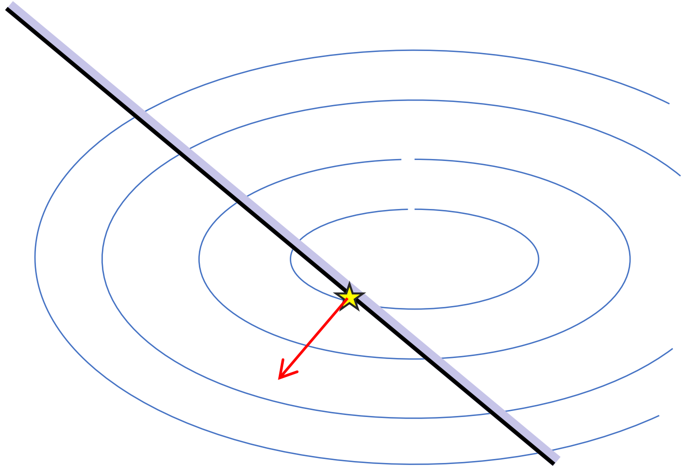

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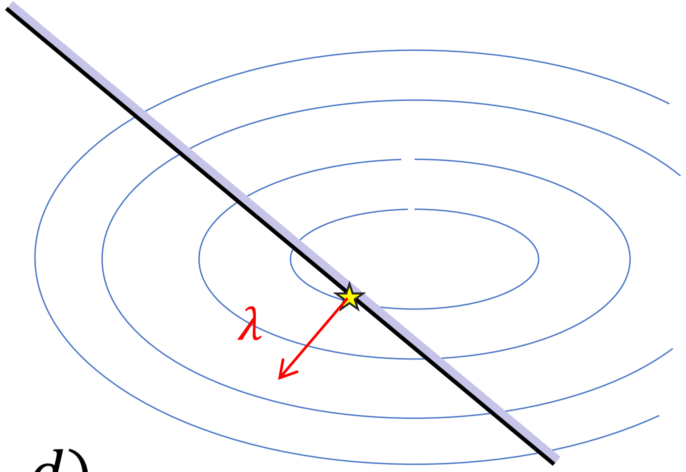


λ is called a Lagrange multiplier

$$\mathcal{L}(x, \lambda) = ||Ax - b||^2 + \lambda^T(Cx - d)$$

Duality in optimization

- The dual variable is an auxiliary quantity
  - Needed to assert the optimality condition

$$\min_x \max_{\lambda} \mathcal{L}(x, \lambda)$$

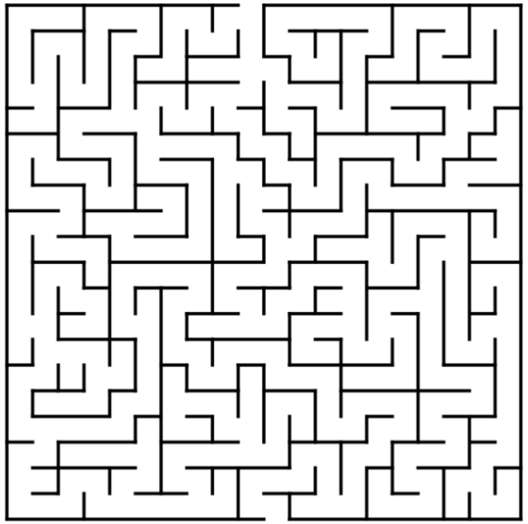
$$\begin{aligned} &\min_x c^T x \\ &s.t. \quad Ax \geq b \end{aligned} \qquad \text{(LP)}$$

$$\begin{aligned} &\max_{\lambda} b^T \lambda \\ &s.t. \quad A^T \lambda = c, \lambda \geq 0 \end{aligned} \qquad \text{(LP*)}$$

Primal problem

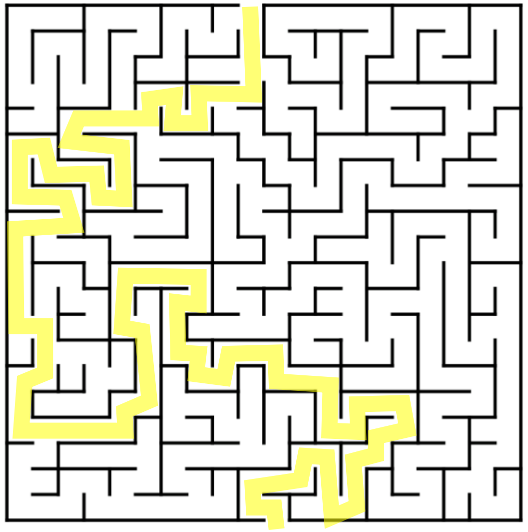
Dual problem

Duality in optimization



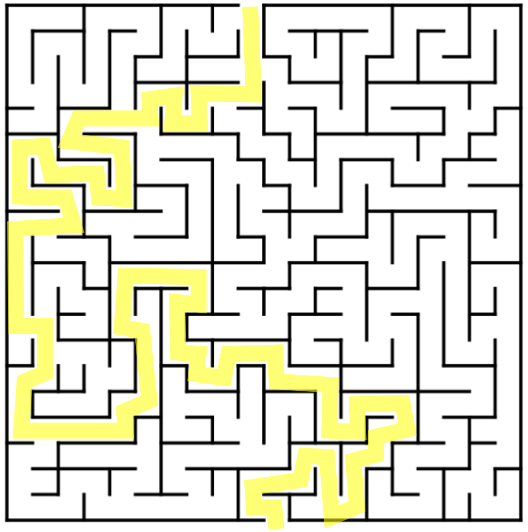
IS IT FEASIBLE ?

Duality in optimization

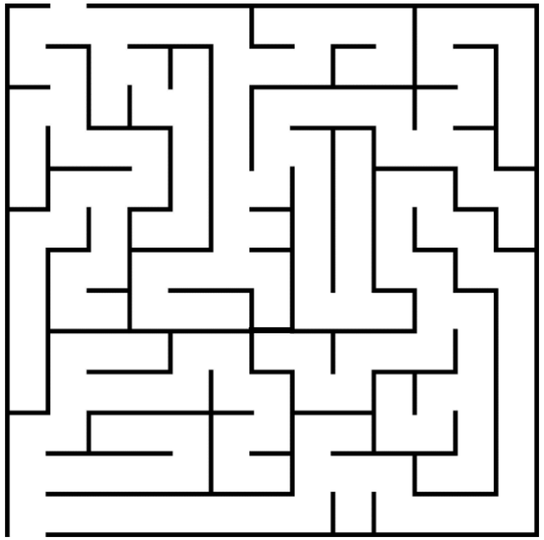


Feasible = I can demonstrate the existence of a path

Duality in optimization

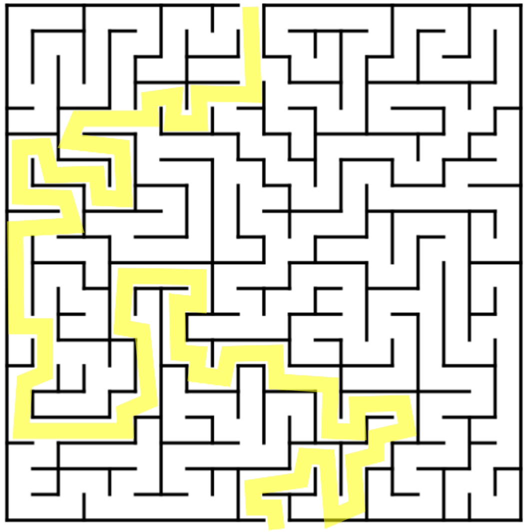


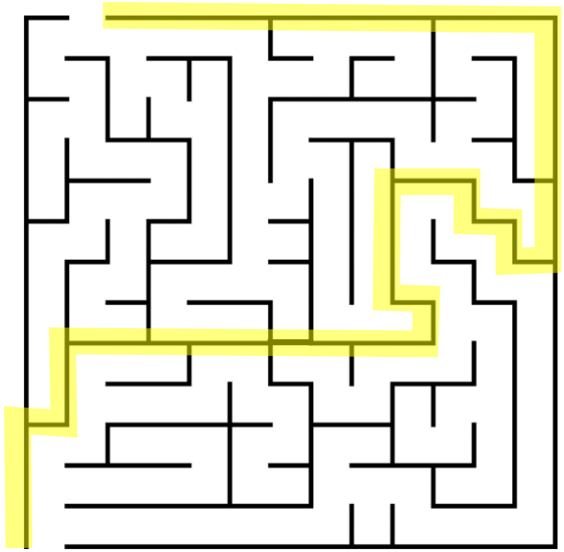
Feasible = I can demonstrate the existence of a path



IS THIS ONE FEASIBLE ?

Duality in optimization





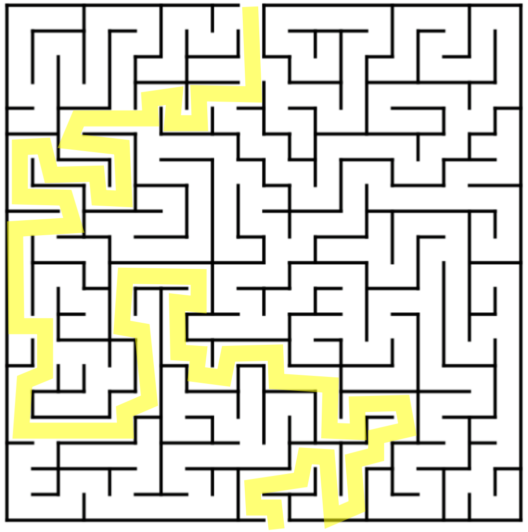
Feasible = I can demonstrate the existence of a path

white

Unfeasible = I can demonstrate the existence of a ~~wall~~

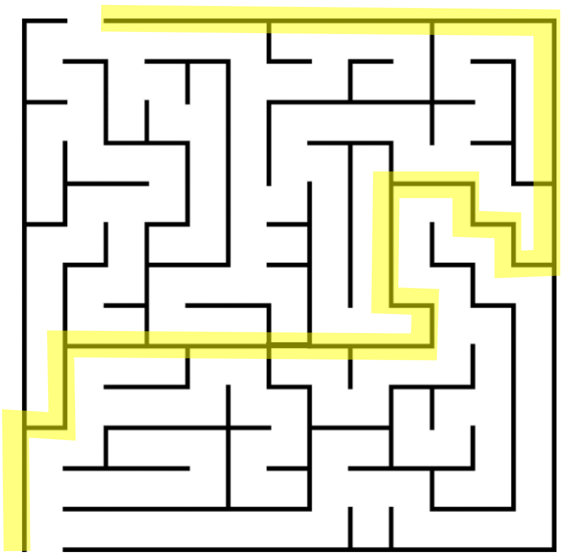
black path

Duality in optimization



Feasible = I can demonstrate the existence of a path

Primal problem



Unfeasible = I can demonstrate the existence of a wall

Dual problem

KKT conditions

$$\min_x c(x)$$
$$s.t. \quad g(x) \geq 0$$

- x, λ are solutions if they respects:

$$\mathcal{L}(x, \lambda) = c(x) - \lambda^T g(x)$$ Lagrangian

$$\nabla_x \mathcal{L} = \nabla c - \lambda^T \nabla g = 0$$ Gradient normal to constraints

$$\nabla_\lambda \mathcal{L} = g(x) \geq 0$$ Constraint satisfied

$$\lambda^T g(x) = 0$$ Complementarity

Dynamics for simulation

- Complementarity problem

$$\ddot{q} = M^{-1}(\tau - b + J^T f)$$

$$J\ddot{q} + \dot{J}\dot{q} \geq 0 \quad \perp \quad f \geq 0$$

no penetration *no pulling*

one or the other

- Equivalent to a principled QP

$$\min_{\ddot{q}} \left\| \ddot{q} - \ddot{q}_{free} \right\|_M \ \text{ s.t. } \quad J\ddot{q} + \dot{J}\dot{q} \geq 0$$