تمرین سوم احتمال

نیما بهرنگ ۹۶۱۰۰۱۱۴

۳ خرداد ۱۳۹۸

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پرسش ۱

طبق قانون احتمال کل عمل می کنیم

1. $p_X = \Sigma_y p_{X,Y}(x,y)$

$$p_X = \begin{cases} 0.4 & 1 \\ 0.1 & 2 \\ 0 & 3 \\ 0.1 & 4 \\ 0.4 & 5 \end{cases}$$

$$p_Y = \begin{cases} 0.1 & 1 \\ 0.3 & 2 \\ 0.2 & 3 \\ 0.4 & 4 \end{cases}$$

2. They are independent as $\forall x, y : p_{X,Y}(x,y) = p_X(x) \times p_Y(y)$

3. $E[W] = \Sigma w.P_W(w)$
   $E[X|Y=y] = \Sigma x.p_{X|Y}(x|y) = \Sigma x.\dfrac{p_{X,Y}(x,y)}{p_Y(y)}$

$$E[X|Y=y] = \begin{cases} & 1 \\ & 2 \\ 0 & 3 \\ 0.1 & 4 \\ 0.4 & 5 \end{cases}$$

پرسش ۲

Probability of a color appear in our event: 1 - it doesn't appear = $1 - \frac{\binom{60}{20}}{\binom{70}{20}}$
introduce Y_i a new indicator that show if color i exists in our outcome.
$X = \Sigma Y_i$ and all of them are equal.We have seven color so the expected value of number of colors according to the linearity of expectation is $7 \times (1 - \frac{\binom{60}{20}}{\binom{70}{20}})$

پرسش ۳

Memory less: $P_x(X > m+n|X > m) = P_x(X > n)$

Conditional: $P(a|b) = \dfrac{P(ab)}{P(b)}$

If its geometric($P_x(X = k) = (1-p)^{k-1}p$) $\Rightarrow$

$$P_x(X > m+n|X > n) = \frac{P_x(X > m+n)}{P_x(X > n)} =$$

$$\frac{(1-p)^{n+m}}{(1-p)^n} = (1-p)^m = P_x(X > m)$$

If we have a Memory-less so $P_x(X = k) = P_x(X > k) - P_x(X > k+1) =$

$$= P_x(X > k) - P_x(X > k+1|X > 1) \times P_x(X > 1) =$$

$$= P_x(X > k) - P_x(X > k) \times P_x(X > 1) = (1 - P_x(X > 1)) \times P_x(X > k)$$

now we calculate the ratio of sequence: $\dfrac{P_x(X = k+1)}{P_x(X = k)} = \dfrac{P_x(X > k)}{P_x(X > k+1)} =$

$$\frac{P_x(X > k)}{P_x(X > k+1|X > 1) \times P_x(X > 1)} =$$

$$= \frac{P_x(X > k)}{P_x(X > k) \times P_x(X > 1)} = Constant = \frac{1}{P_x(X > 1)}$$

So its exactly the definition of a geometric RV.

پرسش ۴

$$fact1 : bydefinition : \Sigma_k \frac{\binom{D}{k}\binom{(N)-(D)}{(n)-(x)}}{\binom{(N)}{(n)}} = 1$$

$$fact2 : \binom{m}{n} = \frac{m.(m-1)}{n.(n-1)}\binom{m-2}{n-2}$$

we know that expected of a hypergeometric RV is $\frac{nD}{N}$

$$Var(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2 = E[X(X-1) + X] - E[X]^2 = E[X(X-1)] + E[X] - E[X]^2$$

$$E[X(X-1)] = \Sigma x(x-1)P(X = x) = \Sigma x(x-1)\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} =$$

$$\Sigma_x D.(D-1)\frac{\binom{D-2}{x-2}\binom{(N-2)-(D-2)}{(n-2)-(x-2)}}{\frac{N(N-1)}{n(n-1)} \times \binom{(N-2)}{(n-2)}} = \frac{D(D-1).n(n-1)}{N(N-1)}$$

$$Var(X) = \frac{D(D-1).n(n-1)}{N(N-1)} + \frac{nD}{N} - (\frac{nD}{N})^2 = n\frac{D}{N}(1 - \frac{D}{N})(1 - \frac{n-1}{N-1})$$

پرسش ۵

we can make a new indicator like Y_i which show card

پرسش ۶

پرسش ۷

برای اینکه آقای خسته به جایگاه برود لازم است که هر ۹ تا ماشین جلوی او رفته باشند و بعد از تعدادی ماشین، او انتخاب شود.

$$\forall k >= 10 : P_x(k) = ((\binom{k-1}{9})\frac{1}{3}^{10}) \times (\frac{1}{3}^{k-10} \times 2^{k-10})$$

$$E[X] = \Sigma_{k=10}^{\infty} k.P_x(k)$$

پرسش۸

$$E[X] = \Sigma_{k=1}^{\infty}\frac{i}{2^i} = \Sigma_{k=1}^{\infty}\frac{1}{2^i} + \Sigma_{k=2}^{\infty}\frac{1}{2^i} + \Sigma_{k=3}^{\infty}\frac{1}{2^i} + ...$$
$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + ... = 2$$

پرسش ۹

1.

پرسش ۱۰

1. correct
   $A \searrow B => P(A|B) \leq P(A) => \frac{P(AB)}{P(B)} \leq P(A) => \frac{P(AB)}{P(A)} \leq P(B) => P(B|A) \leq P(B) => B \searrow A$

2. incorrect
   $A = C = 1, 2, 3, B = 1, 4$
   $P(A) = \frac{3}{4} \geq P(A|B) = \frac{1}{2}$
   $P(B) = \frac{2}{4} \geq P(B|C) = \frac{1}{3}$
   $P(C) = \frac{3}{4} \leq P(C|A) = \frac{1}{1}$

3. incorrect
   $A = 1, 2, 3, C = 3, 4, 5, B = 3, 6, 7, 8, 9$
   $P(A) = \frac{3}{9} \geq P(A|B) = \frac{1}{5}$
   $P(C) = \frac{3}{9} \geq P(C|B) = \frac{1}{5}$
   $P(AC) = \frac{1}{9} \leq P(C|A) = \frac{1}{5}$

پرسش ۱۱

k: initial money

bankruptcy: when our money get 0

A_k : probability of getting bankrupt when we have k money

$$P(A_k) = P(A_k|win)P(win) + P(A_k|loose)P(loose)$$

$$P(A_k) = P(A_{k+1})p + P(A_{k-1})(1-p)$$

we can solve it with discrete math method as an second degree equation

$$x = p.x^2 + (1-p) => x = \frac{1 \pm \sqrt{1-4p(1-p)}}{2p}$$

$x = \frac{1 \pm |2p-1|}{2p} : p \leq \frac{1}{2} => x = \frac{1-(1-2p)}{2p} = 1$: always bankrupt

$$p > \frac{1}{2} => x = \frac{1-(2p-1)}{2p} = \frac{1-p}{p} => x_k = (\frac{1-p}{p})^k$$

پرسش۱۲

۱. با استقرا حدس خود را ثابت می کنیم

$\frac{g}{r+b+g}$

for one ball it's obvious.

for more than one ball we use induction on total number of balls.

A_r : first ball be red A_g : first ball be green A_b : first ball be blue G : last balls be green $P(G) = P(G|A_r)P(A_r) + P(G|A_g)P(A_g) + P(G|A_b)P(A_b)$

we can calculate $P(G|A_k)$ with the induction we use as the total number of balls reduced by one.

$P(G) = \frac{g}{g+r+b-1} \times \frac{r}{r+g+b} + \frac{g-1}{g+r+b-1} \times \frac{g}{r+g+b} + \frac{g}{g+r+b-1} \times \frac{b}{r+g+b} = \frac{g}{r+g+b}$

so its always same.

۲. همانند مثال قبل از استقرا استفاده می کنیم

$\frac{r}{r+b+g}$

for one ball it's obvious.

for more than one ball we use induction on total number of balls.

A_r : first ball be red A_g : first ball be green A_b : first ball be blue R : red balls finished first $P(R) = P(R|A_r)P(A_r) + P(R|A_g)P(A_g) + P(R|A_b)P(A_b)$

we can calculate $P(R|A_k)$ with the induction we use as the total number of balls reduced by one.

$P(R) = \frac{r-1}{g+r+b-1} \times \frac{r}{r+g+b} + \frac{r}{g+r+b-1} \times \frac{g}{r+g+b} + \frac{r}{g+r+b-1} \times \frac{b}{r+g+b} = \frac{r}{r+g+b}$

so its always same.