تمرین اول احتمال

نیما بهرنگ ۹۶۱۰۰۱۱۴

۶ اردیبهشت ۱۳۹۸

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پرسش ۱

طبق قانون احتمال کل عمل می کنیم

1. $E[X] = \Sigma x.P(x)$
   X=k: both be alive after k steps which means that both guns shouldn't fire after k step.
   fact1: $\forall 0 \geq x \leq 1 \Sigma_{k=1}^{\infty} ax^k = \dfrac{ax}{1-x}$
   $x = (1-p)(1-q)$
   $P(k) = x^k$
   $E[X] = \Sigma_{k=1}^{\infty} k.P(k) = \Sigma_{k=1}^{\infty} kx^k$
   $=> E[X] = \Sigma_{k=1}^{\infty} x^k + \Sigma_{k=2}^{\infty} x^k + ... = \Sigma_{k=1}^{\infty} \dfrac{x^k}{1-x}$
   $= \dfrac{x}{1-x^2}, x = (1-p)(1-q)$

2. X=k: both killed after k steps which means that both guns should fire after exactly k step.
   $x = (1-p)(1-q)$
   $P(k) = x^{k-1}pq$
   $E[X] = \Sigma_{k=1}^{\infty} k.P(k) = \Sigma_{k=1}^{\infty} kx^{k-1}pq$
   $=> E[X] = \Sigma_{k=1}^{\infty} pqx^{k-1} + \Sigma_{k=2}^{\infty} pqx^{k-1} + ... = \Sigma_{k=1}^{\infty} \dfrac{pqx^{k-1}}{1-x}$
   $= \dfrac{pq}{1-x^2}, x = (1-p)(1-q)$

پرسش ۲

Probability of a color appear in our event: 1 - it doesn't appear = $1 - \frac{\binom{60}{20}}{\binom{70}{20}}$
introduce Y_i a new indicator that show if color i exists in our outcome.
$X = \Sigma Y_i$ and all of them are equal.We have seven color so the expected value of number of colors according to the linearity of expectation is $7 \times (1 - \frac{\binom{60}{20}}{\binom{70}{20}})$

پرسش ۳

Memory less: $P_x(X > m+n|X > m) = P_x(X > n)$

Conditional: $P(a|b) = \dfrac{P(ab)}{P(b)}$

If its geometric($P_x(X = k) = (1-p)^{k-1}p$) $\Rightarrow$

$$P_x(X > m+n|X > n) = \frac{P_x(X > m+n)}{P_x(X > n)} =$$

$$\frac{(1-p)^{n+m}}{(1-p)^n} = (1-p)^m = P_x(X > m)$$

If we have a Memory-less so $P_x(X = k) = P_x(X > k) - P_x(X > k+1) =$

$$= P_x(X > k) - P_x(X > k+1|X > 1) \times P_x(X > 1) =$$

$$= P_x(X > k) - P_x(X > k) \times P_x(X > 1) = (1 - P_x(X > 1)) \times P_x(X > k)$$

now we calculate the ratio of sequence: $\dfrac{P_x(X = k+1)}{P_x(X = k)} = \dfrac{P_x(X > k)}{P_x(X > k+1)} =$

$$\frac{P_x(X > k)}{P_x(X > k+1|X > 1) \times P_x(X > 1)} =$$

$$= \frac{P_x(X > k)}{P_x(X > k) \times P_x(X > 1)} = Constant = \frac{1}{P_x(X > 1)}$$

So its exactly the definition of a geometric RV.

پرسش ۴

$$fact1 : bydefinition : \Sigma_k \frac{\binom{D}{k}\binom{(N)-(D)}{(n)-(x)}}{\binom{(N)}{(n)}} = 1$$

$$fact2 : \binom{m}{n} = \frac{m.(m-1)}{n.(n-1)}\binom{m-2}{n-2}$$

we know that expected of a hypergeometric RV is $\frac{nD}{N}$

$$Var(X) = E[(X-E[X])^2] = E[X^2] - E[X]^2 = E[X(X-1)+X] - E[X]^2 = E[X(X-1)] + E[X] - E[X]^2$$

$$E[X(X-1)] = \Sigma x(x-1)P(X=x) = \Sigma x(x-1)\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} =$$

$$\Sigma_x D.(D-1)\frac{\binom{D-2}{x-2}\binom{(N-2)-(D-2)}{(n-2)-(x-2)}}{\frac{N(N-1)}{n(n-1)} \times \binom{(N-2)}{(n-2)}} = \frac{D(D-1).n(n-1)}{N(N-1)}$$

$$Var(X) = \frac{D(D-1).n(n-1)}{N(N-1)} + \frac{nD}{N} - (\frac{nD}{N})^2 = n\frac{D}{N}(1-\frac{D}{N})(1-\frac{n-1}{N-1})$$