تمرین اول احتمال

نیما بهرنگ ۹۶۱۰۰۱۱۴

۶ تیر ۱۳۹۸

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پرسش ۱

1. *$Cov(X,Y) = E[XY] - E[X]E[Y]$
   *$E[E[Y|X]] = E[Y]$
   *$E[E[XY|X]] = E[XY]$
   so
   $Cov(X, E[Y|X]) = E[XE[Y|X]] - E[X]E[E[Y|X]] =$
   $= E[E[XY|X]] - E[X]E[Y] = E[E[XY|X]] - E[X]E[Y] =$
   $= E[XY] - E[X]E[Y] = Cov(X,Y)$

2. * $Cov(X+Z,Y) = Cov(X,Y) + Cov(Z,Y)$
   proof:
   E is linear
   $Cov(X+Z,Y) = E[(X+Y-E[X]-E[Y])(Z-E[Z])] =$
   $= E[(X-E[X])(Z-E[Z]) + (Y-E[Y])(Z-E[Z])] =$
   $= E[(X-E[X])(Z-E[Z])] + E[(Y-E[Y])(Z-E[Z])] = Cov(X,Z) + Cov(Y,Z)$
   * $Cov(X,X) = Var(X)$
   *$Cov(X,Y) = Cov(Y,X)$
   so
   lets think that both X,Y have distribution like Z
   $Cov(X+Y, X-Y) = Cov(X,X) + Cov(Y,X) - Cov(Y,Y) - Cov(X,Y) = Var(X) - Var(Y) = Var(Z) - Var(Z) = 0$

پرسش ۲

1. * $\int x^k dx = \dfrac{x^{k+1}}{k+1} => \int_0^1 x^k dx = \dfrac{1}{k+1}$
   $\int_0^1 G(s)ds = \int_0^1 E(s^X)ds = \int \int_0^1 f(x)s^x dsdx =$
   $\int f(x)\dfrac{1}{x+1}dx = E[\dfrac{1}{1+x}]$

2. * $M_X(t) = E[e^{tX}] = e^{t\mu+\dfrac{(t\sigma)^2}{2}}$
   so
   $Y = log(X) => E[X] = E[e^Y] = M_1(Y) = e^{\mu+\dfrac{\sigma^2}{2}}$
   $var(X) = E[X^2] - E[X]^2$
   $E[X^2] = E[e^{2Y}] = M_2(Y) = e^{2\mu+\dfrac{4\sigma^2}{2}}$
   $var(X) = e^{2\mu+\dfrac{4\sigma^2}{2}} - (e^{\mu+\dfrac{\sigma^2}{2}})^2$

پرسش ۳

$$*e^x = \Sigma\frac{x^n}{n!}$$

$$*\ M_X^{(k)}(0) = E[X^k]$$

$*M_X$ for normal standard is: $e^{\dfrac{t^2}{2}}$

we use its moment generative function

$$e^{\dfrac{t^2}{2}} = \Sigma_{n=0}\frac{t^{2n}}{2^n n!}$$

so we derive it k times:

$$e^{\dfrac{t^2}{2}^{(k)}} = \Sigma_{n\geq\frac{k}{2}}\frac{2n\times(2n-1)...\times(2n-k+1)t^{2n-k}}{2^n n!}$$

as we can see if k is 2j+1 and t=0 then all the elements are equal to zero but if its 2j then the first one is t^{2n-2j} and for t=0,n=$\frac{k}{2}$=j its equal to one so its coefficient is the sum of whole sigma which is equal to:

$$\frac{2j\times(2j-1)...\times 1}{2^{2j}(2j)!} = \frac{(2j)!}{2^{2j}(2j)!}$$

پرسش ۴

X_i if toss number i is head

$$M_n = \frac{X_1 + ... + X_n}{n}$$

$$Var(M_n) = \frac{\sigma_X^2}{n}$$

$$P|M_n - f| > 0.1 \leq 0.1$$

by Chebyshev we have: $P(|M_n - f| > \epsilon) \leq \frac{\sigma_M^2}{\epsilon^2} =>$

$$P(|M_n - f| > 0.1) \leq \frac{\sigma_M^2}{0.01} \leq 0.1 =>$$

$$\sigma_M^2 \leq 0.001 => \frac{p.(1-p)}{n} \leq 0.001 => n \geq 250$$

پرسش ۵

1. as it says: $P(limX_n = c) = 1$
   if we want to say that it also converge in probability, we have to show that: $\forall\epsilon > 0 : limP(|X_n - c| > \epsilon) = 0$
   if we show $limP(|X_n - c| = 0) = 1$ so we have prove the above theorem.
   by the definition: $P(limX_n = c) = 1 => limP(X_n = c) = 1 => limP(X_n - c = 0) = 1 => limP(|X_n - c| = 0) = 1$

2. we use central limit theorem and try to use standard normal RV to get the answer
   as $X_i$are independent and identical with $\mu = 0, \sigma^2 = \frac{1}{12}$
   $S_n = \Sigma X_i => P(S_n \leq 5) = \Phi(\frac{5 - n\mu}{\sigma\sqrt{n}}), n = 100 = \Phi(\sqrt{3}) = 0.9584$
   so the answer is $1 - \Phi(1.732) \approx= 0.0416$
   we have this formula in book and as the result of similarity of sum of n number of random variable to normal distribution

پرسش ۶

1.

2. as the CDF of exponential is $1 - e^{-\lambda x}$ for positive x
$P\{X <= k\} = 1 - e^{-k\lambda}$
$W = min(X, Y) => F_W(w) = 1 - P(X > w, Y > w) =$
$1 - P\{X > w\}P\{Y > w\} = 1 - (1 - F_X(w))(1 - F_Y(w)) = 1 - e^{-2w\lambda} =>$
$f_W(w) = F' = 2\lambda e^{-2w\lambda}$
so its just like a exponential with parameter 2λ
$E[W] = \frac{1}{2\lambda}$
now for second one $P\{X <= k\} = 1 - e^{-k\lambda}$
$W = min(2X, Y) => F_W(w) = 1 - P(X > \frac{w}{2}, Y > w) =$
$1 - P\{X > w\}P\{Y > w\} = 1 - (1 - F_X(w))(1 - F_Y(w)) = 1 - e^{-\frac{3}{2}w\lambda} =>$
$f_W(w) = F' = \frac{3}{2}\lambda e^{-\frac{3}{2}w\lambda}$
so its just like a exponential with parameter $\frac{3}{2}\lambda$
$E[W] = \frac{2}{3\lambda}$

پرسش ۷

1. $E[|X - Y|] = \int\int_{x>y} x - ydxdy + \int\int_{y>x} y - xdxdy = 2\int\int_{x>y} x - ydxdy$
$2\int_0^\alpha(\int_y^\alpha xdx - y\int_y^\alpha dx)dy = 2\int_0^\alpha(\frac{\alpha^2}{2} - \frac{y^2}{2} - y\alpha + y^2)dy =$
$2(\frac{\alpha^3}{2} - \frac{\alpha^3}{6} - \frac{\alpha^3}{2} + \frac{\alpha^3}{3}) = \frac{\alpha^3}{3}$

2. $max(a, b) - min(a, b) = |a - b|$
directly from last part we get $\frac{1}{3}$
$max(a, b) + min(a, b) = a + b$
so its the PMF of sum of two uniform and we use the convolution on them:
$f_{X+Y}(Z) = \int f_X(z - y)f_Y(y)dy$
as we solve it before, there is two case: 0<z<1, 1<z<2
its only important when $f_Y(y) = 1$ other place its 0, so: $0 \leq z < 1$:
$\int_0^1 f_X(z - y)dy = \int_0^z dy = z$
$1 \leq z \leq 2 : \int_0^1 f_X(z - y)dy = \int_{z-1}^1 dy = 2 - z$
$f_{X+Y}(z) = z : 0 \leq z < 1, f_{X+Y}(z) = 2 - z : 1 \leq z \leq 2$

پرسش۸

for any two X_i we know the probability of $X_i > X_j$ as they are iid, it means it must be $\frac{1}{2}$

$$P(X_i > X_{i-1}, X_i > X_{i-2}, ...X_i > X_1) = P(X_i > X_{i-1}) \times ... \times P(X_i > X_1) = \frac{1}{2^i}$$

so Y_i : person i do record

$$E[\Sigma Y_i] = \Sigma E[Y_i] = 1 + \frac{1}{2} + ... = 2$$

$$E[Y_i^2] = 1^2 \times P(record) + 0^2 \times (1 - P(record)) = E[Y_i]$$

Y_i are independent so

$$var(\Sigma Y_i) = \Sigma var(Y_i) = \Sigma E[Y^2] - E[Y]^2 = \Sigma_{i=1} \frac{1}{2^{i-1}} - \frac{1}{2^i} = 1 - \frac{1}{2^n}$$

پرسش ۹

پرسش ۱۰

1. * $E[Cos(\theta)] = 0$ as it is odd between 0 and 2π so $f_\theta(a) = \frac{1}{2\pi}$
   $Cov(Sin(\theta), Cos(\theta)) = E[Sin(\theta)Cos(\theta)] - E[Sin(\theta)]E[Cos(\theta)] = E[Sin(\theta)Cos(\theta)] =$
   $\int_0^{2\pi} cos(x)sin(x)\frac{1}{2\pi}dx =$
   $\frac{1}{2\pi}\int \frac{1}{2}sin(2x) = \frac{-1}{8\pi}cos(2x)from(0, 2\pi) = 0 => uncorrelated$

2. * $var(X+Y) = var(X) + var(Y) - 2cov(X, Y)$
   $\rho(X+Y, X-Y) = \frac{cov(X+Y, X-Y)}{\sigma_{X+Y}\sigma_{X-Y}} = \frac{var(X) - var(Y)}{\sigma_{X+Y}\sigma_{X-Y}} =$
   $\frac{var(X) - var(Y)}{\sqrt{var(X+Y)var(X-Y)}} = \frac{var(X) - var(Y)}{\sqrt{var(X)^2 + var(Y)^2 + 2var(x)var(y)}} =$
   $\frac{var(X) - var(Y)}{\sqrt{(var(X) + var(y))^2}} = \frac{var(X) - var(Y)}{var(X) + var(Y)}$

پرسش ۱۱

1. $f_X(x) = \int f(x,y)dy = \int \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-z}dy$
   $z = \frac{1}{2(1-\rho^2)} \times [(\frac{x-\mu_x}{\sigma_x}^2) - (\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}) + (\frac{y-\mu_y}{\sigma_y}^2)]$
   we can prove that it integral is equal to:
   $\frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}$
   which is exactly the normal distribution with mean μ_x and variance σ_x^2
   same for y we have:
   $f_Y(y) = \int \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-z}dx = \frac{1}{\sqrt{2\pi}\sigma_y}e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}$
   prove:

2.

پرسش۱۲