A Field-theoretic networks model

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1 Fundamentals

Our goal is to construct networks out of locally interacting fields inside some space. These fields could represent agents moving or communicating in real space-time or inside some parameter space which characterizes their personality and interests.

For technical simplicity, we begin with networks put on familiar manifolds, endowed with some [pseudo]-Riemannian metric[1].

1.1 The construction

We will assume that nodes (agents) are sources, J_i which are concentrated around points x_i. They emanate a field ϕ[2]. The corresponding term in the action is assumed to be the minimal coupling term $\int J_i\phi$. This field ϕ could represent the extent to which an agent moves from its source J_i in real space or in parameter space how much an agent is willing to explore or tolerate values of parameters (e.g. political views) other than its own mean.

1.1.1 The Adjacency Matrix

The most natural choice for the Adjacency matrix will then be something based on the correlation between the fields emanated from two different sources:

$$A_{ij} \sim \langle \phi(x_i, t_i)\phi(x_j, t_j)\rangle$$

We will comment on the choice of times t_i, t_j later. For now, what matters is that the normalized two-point function could be used as an ensemble-averaged probability of having a link.

1.1.2 Action and Partition Function

We will first assume a real-time field theory (not a finite-temperature Gibbs ensemble.) We start with a mainly quadratic action with a perturbation which allows fields of various agents to interact. Since we have assumed that all agents emanate the same ϕ, we may start with an action of the following form:

$$S = \int d^{(n+1)}x \left(\phi^\dagger D\phi + \sum_i \phi^\dagger J_i + h.c.\right) \tag{1}$$

Where D is a linear operator such as $\partial_t^2 - \vec{\partial}^2 + m^2$ if we consider a relativistic *phi* of mass m, or $\partial_t + D\vec{\partial}^2 + \mu$ in the case of diffusion, with diffusion constant D and loss μ, or any other type of linear operator. We will mostly focus on the case of diffusion.

The partition function of a quadratic theory may be evaluated using Gaussian integration, though caution needs to be exercised regarding possible zero modes of the action. these may be taken care of by either calculating the zero mode contributions separately or by regularizing the action by adding an appropriate $i\varepsilon$. For simplicity, suppose we have a real field $\phi = \phi^\dagger$. Schematically, the partition function becomes ($J \equiv \sum_i J_i$):

$$Z = \int [d\phi] \exp\left[i\int (\phi D\phi + J\phi)\right]$$

[1]Later we will explore the possibility of making the "base space" a network itself, or removing the metric and working with a topological theory.

[2]For now we will assume a single ϕ. Subsequently we may explore having multiple ϕ components as well as large N $O(N)$ models.

$$= \det(D/\pi)^{-1/2} \exp\left[i \int JD^{-1}J/4\right] \tag{2}$$

Let us also recall that the 2-point function in quadratic theories is the Green's function. In particular, for the time ordered 2-point function:

$$\begin{aligned}
\langle T[\phi(x_i)\phi(x_j)]\rangle &= \theta(t_i - t_j)\frac{\delta}{\delta J_i}\frac{\delta}{\delta J_j}\ln Z + (i \leftrightarrow j) \\
&= \frac{1}{4}\theta(t_i - t_j)D^{-1}(x_i - x_j) + (i \leftrightarrow j) \\
&= G_F(x_i - x_j; t_i - t_j)
\end{aligned} \tag{3}$$

With G_F being the Feynman propagator, or the time-ordered Green's function.

2 Network of Random Walkers

Let us now work out a specific example. Consider the case where the nodes are sources for agents that do a random walk in an n-D Euclidean space. The spread of the density of these random walkers can be represented by a heat (diffusion) equation. The action is given by:

$$S = \int d^{(n+1)}x \left(\phi(\partial_t + D\vec{\partial}^2 + \mu)\phi + J\phi\right) \tag{4}$$

Here ϕ is related to the density of the random walkers. We are assuming that the sources $J = \sum_i J_i$ are static and need to specify how they are distributed over space.

We will first assume the sources are localized as delta functions:

$$J_i(x) = J_0\delta(x - x_i)$$

and that they are uniformly spread in space. The ensemble average network is made of the time-ordered Green's functions between the sources:

$$A_{ij} \sim \langle T[\phi(x_i)\phi(x_j)]\rangle = \theta(t_i - t_j)\frac{\exp\left[-\frac{(x_i-x_j)^2}{4D(t_i-t_j)}\right]}{(4\pi D(t_i - t_j))^{n/2}} + (i \leftrightarrow j)$$

But there is also a time involved in the diffusion process. We will look at two scenarios: 1) one agent diffuses from x_i and gets to the source at x_j; 2) two agents start from x_i and x_j and they meet at a random point in space.

First look at the case where all fields from all sources diffuse for a fixed amount of time t and then we calculate the probability of the field starting from x_i reaching x_j within time t.

$$\begin{aligned}
A^1_{ij}(t) &= \langle\phi(x_i,t)\phi(x_j,0)\rangle + \langle\phi(x_j,t)\phi(x_i,0)\rangle \\
&= 2\theta(t)\frac{\exp\left[-\frac{(x_i-x_j)^2}{4Dt}\right]}{(4\pi Dt)^{n/2}}
\end{aligned} \tag{5}$$

In the other case two agents start at the same time from x_i and x_j, both travelling for a time t and then meet somewhere in space. This time we have two propagators meeting at a random point y, which we need to integrate over:

$$
\begin{aligned}
A^2_{ij}(t) &= \int d^n y \, \langle \phi(x_i, 0)\phi(y, t)\rangle \, \langle \phi(x_j, 0)\phi(y, t)\rangle \\
&= \frac{\theta(t)}{(4\pi D t)^n} \int d^n y \exp\left[-\frac{(x_i - y)^2}{4Dt} - \frac{(x_j - y)^2}{4Dt}\right] \\
&= \frac{\theta(t)}{(4\pi D t)^n} \exp\left[-\frac{(x_i - x_j)^2}{8Dt}\right] \int d^n y \exp\left[-\frac{\left(\frac{x_i + x_j}{2} - y\right)^2}{2Dt}\right] \\
&= \frac{\theta(t)}{(4\pi D(2t))^{n/2}} \exp\left[-\frac{(x_i - x_j)^2}{4D(2t)}\right]
\end{aligned} \tag{6}
$$

This is exactly the the same expression as the previous case where only one agent travelled and eventually found the source of another agent, the only difference is that one agent needs to travel twice the time that two agents needed to cross each other at a random point, which is intuitively correct:

$$A^2_{ij}(t) = A^1_{ij}(2t)$$

2.1 Degree Distribuition

The ensemble average of the degree k_i of each node i is easily found by integrating over all other nodes. This immediately makes it clear that in this symmetric setting with uniform distribution of nodes, all nodes will more or less have the same degree, plus possible Poisson-like fluctuations, which exist in any finite sample from a uniform distribution. Let us assume that we have N nodes in a space of volume L^n, and define the node density as $\lambda \equiv N/L^n$. The average degree of i is:

$$\langle k_i(t)\rangle = \lambda \int d^n x_j A^1_{ij}(t) = \lambda$$

And since it does not depend on x_i or t the distribution can only depend on local fluctuations which make for a Poisson distribution. Recall that in a Poisson distribution the mean and standard deviation are equal and thus we expect:

$$\langle k_i\rangle = \sigma(k_i) = \lambda$$

ACTUALLY CALCULATE!!!

3 Breaking the Spatial Symmetry

One evident way of avoiding the highly symmetrical case above which basically generated an analog of and Erdös-Renyí network is to prepare a configuration in which the degree k_i would depend on x_i. This means that we need to break the spatial symmetry, either by imposing a different distribution of nodes in space, or by introducing a space-dependent interaction term.

3.1 Nima 06/25/14

3.2 Space-dependent Interaction Term

We want to have add a space-dependent potential to the action (4) to break the spatial symmetry and see if we can get a different degree distribution. We want the interaction to be of a type that would connect two fields ϕ emanated from different sources. So, naturally, it should contain ϕ^2. Take the following action now(we set $D = 1$, because it can be absorbed as a length scale):

$$S = \int d^{(n+1)}x \left(\phi(\partial_t + \vec{\partial}^2 + \mu)\phi + J\phi + \lambda(x)\phi^2(x) \right) \tag{7}$$

If we calculate the partition function for this action and try to evaluate the effect of this potential we see that it effectively introduces a space-dependent mass term. The Green's functions will be modified to the following in the Fourier basis:

$$G(x, t_x; y, t_y) \sim \int \frac{d\omega d^n k}{(2\pi)^{n+1}} \frac{e^{i\omega(t_x - t_y) + ik\cdot(x-y)}}{i\omega - k^2 + \mu + \lambda(x)} + c.c. \tag{8}$$

But since there is this dependece on x the form of the retarded and advances Green's functions will differ in whether $\lambda(x)$ comes from the starting or the end point:

$$\begin{aligned} G_R(x, t_x; y, t_y) &= \theta(t_y - t_x) \int \frac{d\omega d^n k}{(2\pi)^{n+1}} \frac{e^{i\omega(t_x - t_y) + ik\cdot(x-y)}}{i\omega - k^2 + \mu + \lambda(y) + i\varepsilon} \\ G_A(x, t_x; y, t_y) &= \theta(t_x - t_y) \int \frac{d\omega d^n k}{(2\pi)^{n+1}} \frac{e^{i\omega(t_x - t_y) + ik\cdot(x-y)}}{i\omega - k^2 + \mu + \lambda(x) - i\varepsilon} \end{aligned} \tag{9}$$

(Check PLUS/MINUS SIGNS!!) If $\lambda(x)$ was small, it would basically be a small perturbation and the Green's fnction could be written as the expansion:

$$G_R(x, t_x; y, t_y) = \theta(t_y - t_x) \int \frac{d\omega d^n k e^{i\omega(t_x - t_y) + ik\cdot(x-y)}}{(2\pi)^{n+1}} \frac{1}{i\omega - k^2 + \mu} \sum_{j=1}^{\infty} \frac{(-1)^j \lambda(y)}{i\omega - k^2 + \mu} \tag{10}$$

If we only keep the first order perturbation for when $\lambda(y) \ll \mu$ we have:

$$\begin{aligned} G_R(x, t_x; y, t_y) \approx& \theta(t_y - t_x) \int \frac{d\omega d^n k e^{i\omega(t_x - t_y) + ik\cdot(x-y)}}{(2\pi)^{n+1}} \\ & \times \frac{1}{i\omega - k^2 + \mu} \left(1 - \frac{\lambda(y)}{i\omega - k^2 + \mu} \right) \end{aligned} \tag{11}$$

This can be rewritten in terms of two propagators by inserting a Dirac delta function:

$$\begin{aligned} \frac{\lambda(y)}{i\omega - k^2 + \mu} &= \lambda(y) \int \frac{d\omega' d^{n-1} k'}{(2\pi)^n} \int \frac{dt_z d^n z e^{i(\omega' - \omega)t_z + i(k' - k)\cdot z}}{(2\pi)^{n+1}} \frac{1}{i\omega' - k'^2 + \mu} \\ &= \int \frac{dt_z d^n z e^{-i\omega t_z - ik\cdot z}}{(2\pi)^{n+1}} \lambda(y) G_R^0(z, t_z)/\theta(t_z) \end{aligned} \tag{12}$$

Where G_R^0 is the unperturbed Green's function which only depends on the difference between coordinates. If we combine the exponents of this one and use the $\delta^{n-1}(k-k')\delta(\omega-\omega')$ that's implicitly in the integral to replace $\exp[i\omega t_y + ik\cdot y] \to \exp[i\omega' t_y + ik'\cdot y]$ we are able to write the perturbed Green's function as the product of two unperturbed ones, if $t_y > t_z > t_x$ (which makes yields $\theta(t_y - t_x) = \theta(t_y - t_z)\theta(t_z - t_x)$), setting $\Delta x = x - y, \Delta t = t_x - t_y$:

$$
\begin{aligned}
G_R(x,t_x;y,t_y) &= G_R^0(\Delta x,\Delta t) - \int \frac{dt_z d^n z}{(2\pi)^{n+1}}\lambda(y) G_R^0(\Delta x - z, \Delta t - t_z) G_R^0(z,t_z) \\
&= G_R^0(\Delta x,\Delta t) - \int \frac{dt_z d^n z}{(2\pi)^{n+1}}\lambda(y) G_R^0(x - z, t_x - t_z) G_R^0(z - y, t_z - t_y) \\
&= G_R^0(\Delta x,\Delta t)(1-\lambda(y))
\end{aligned} \tag{13}
$$

(Take care of the dimensionalty!! λ is dimensionful, but the integration and G_R^0 should absorb its dimensions. However, this is lost once $G_R^0 G_R^0 \to G_R^0$. Some dimensionful factors must either emerge from this conversion, or everything should be made dimensionless.)Which is because as we showed before the product of two consecutive unperturbed propagators G_R^0 is just a single propagation. A very similar thing happens for the advanced Green's function G_A.

3.2.1 The Perturbed Adjacency Matrix

Again, consider the two scenarios: A_{ij}^1 for direct walk from $i \to j$ or $j \to i$; A_{ij}^2 for two random walks starting at the same time and meeting somewhere. In the fisrt case we just have the time-ordered Green's function:

$$
\begin{aligned}
A_{ij}^1 =& G_T(x_i,t_i;x_j,t_j) = G_A + G_R \\
=& \theta(t_y - t_x)\int \frac{d\omega d^n k e^{i\omega(t_x - t_y) + ik\cdot(x-y)}}{(2\pi)^{n+1}} \\
&\times \left[\frac{\theta(t_y - t_x)}{i\omega - k^2 + \mu + \lambda(x)} + \frac{\theta(t_x - t_y)}{i\omega - k^2 + \mu + \lambda(y)}\right]
\end{aligned} \tag{14}
$$

3.2.2 Simultaneous Random-walkers

For the other one, A_{ij}^2 we have two retarded Green's functions only, meeting at some random point in space. If we just take the full Green's function and expand in terms of $\lambda(x)$ all such terms necessarily have one of the endpoints in them. For when $\lambda(x) \ll \mu$ we can keep up to first order and we have:

$$
\begin{aligned}
A_{ij}^2 &= \int \frac{dt_z d^n z}{(2\pi)^{n+1}} G_R(x-z,-t) G_R(y-z,-t) \\
&\approx \int \frac{dt_z d^n z}{(2\pi)^{n+1}} G_R^0(x-z,-t) G_R^0(y-z,-t)(1-\lambda(z))^2 \\
&\approx \int \frac{dt_z d^n z}{(2\pi)^{n+1}} G_R^0(x-z,-t) G_R^0(y-z,-t)(1-2\lambda(z))
\end{aligned} \tag{15}
$$

As a simple example, consider the case where $\lambda(x) = \lambda\delta^{n-1}(x)$:

$$
A_{ij}^2 \approx G^0(x-y,2t) - 2\lambda G_R^0(x,-t) G_R^0(y,-t) \tag{16}
$$

3.3 Nima 06/26/14

3.4 Adjacency Matrix from the Path Integral

Firstly, let us note that when $\mu = 0$, there is a priori no scale to define a perturbative expansion for $\lambda(x)$ because ω, k may ave any values and expansion in $\lambda(x)$ becomes invalid when $i\omega + k^2 < \lambda(x)$. In this case A^2_{ij} my not be written in terms of G^0_R and the full G_R needs to be calculated. This is because even a small mass $\lambda(x)$ will avoid an IR divergence.

3.4.1 Possible Resolutions

The problem above shows that a perturbative expansion in term of $\lambda(x)$ is not valid when $\mu = 0$. This is basically saying that when there is a single field ϕ all orders in $\lambda(x)$ play a role in the result of diffusion from one point to another. There is no way to separate the interactions with $\lambda(x)$ from regular diffusion in space and it has the same effect as a space-dependent mass (which is a loss term in diffusion.)

The other issue is, the $\lambda(x)$ term is a perturbation and should be smaller in order than the zeroth order, direct propagation. This means that in this model if the $\lambda(x)$ term takes over, it is already too big to be considered a perturbation. So we need to modify the model if we want $\lambda(x)$ to play a dominant role.

3.5 Nima 06/27/14

3.5.1 Large N and Off-diagonal Interactions

Now consider the case where each source J_i is emitting its own unique "agent field" ϕ_i. Take the unperturbed action to be:

$$S = \int d^{(n+1)}x \sum_i \left(\phi_i(\partial_t + \vec{\partial}^2 + \mu)\phi_i + J_i\phi_i + \sum_j \lambda_{ij}(x)\phi_i(x)\phi_j(x) \right) \tag{17}$$

This way different agents can only talk to each other through the interaction term $\lambda_{ij}(x)$. The Green's functions $G_{ij} \sim \langle \phi_i \phi_j \rangle$ are modified to:

$$G_{ij}(x, t_x; y, t_y) \sim \int \frac{d\omega d^n k e^{i\omega(t_x - t_y) + ik\cdot(x-y)}}{(2\pi)^{n+1}} \left((i\omega - k^2 + \mu)I + \lambda(x)\right)^{-1}_{ij} + c.c. \tag{18}$$

Where I is the identity matrix. Thus, when $i \neq j$ it will be only the $\lambda_{ij}(x)$ terms which make up the adjacency matrix. For example, in the case of two simultaneous random-walkers the above expression gets an expansion of the form:

$$
\begin{aligned}
A^{(2)}_{ij} &= \sum_k \int \frac{d^{n+1}z}{(2\pi)^{n+1}} G_{Rik}(x_i - z, -t) G_{Rjk}(x_j - z, -t) \\
&\approx \int \frac{dt_0 d^n z_0}{(2\pi)^{n+1}} G^0_R(x - z_0, t_0 - t) \\
&\times \sum_{i=1}^{\infty} \prod_{k=0}^{i} \int \frac{d^{n+1}z_k}{(2\pi)^{n+1}} \lambda_{l_k l_{k+1}}(z_k) G^0_R(z_{k-1} - z_k, t_{k-1} - t_k) \\
&\times G^0_R(y - z_\infty, t_\infty - t)
\end{aligned} \tag{19}
$$

The problem again will be that if the loss term μ vanishes, there won't be any scale to compare λ against to see whether a perturbative expansion is valid or not.

3.5.2 IR Cutoff

One way to resolve the problem with perturbation on λ is to have an IR cutoff. For instance, if the random walkers only walk for a certain time, say each random walk is a day long, we will have a natural cutoff for t, which in turn means cutoff for ω. If we denote this "day" as $t_{\max}$, the natural IR cutoff will be:

$$
i\omega_{IR} \sim \frac{1}{t_{\max}} \tag{20}
$$

This way, if $\lambda << i\omega_{IR}$ we can use perturbation theory for λ.

Suppose we have such a weak interaction and that the loss (mass term) $\mu = 0$. In this case we may keep only up to the first order terms.

$$
\begin{aligned}
A^{(2)}_{ij}(t) &\approx \delta_{ij} G^0(x_i - x_j, 2t) - \int \frac{d^n z}{(2\pi)^n} \lambda_{ij}(z) G^0_R(x_i - z, -t) G^0_R(x_j - z, -t) \\
&= \delta_{ij} A^1_{ij}(2t) - \int \frac{d^n z}{(2\pi)^n} \lambda_{ij}(z) A^1_{iz}(t) A^1_{jz}(t)
\end{aligned} \tag{21}
$$

As we see, for $i \neq j$, only the λ term contributes to lowest order and the zeroth order is absent. Thus in this model, even though $\lambda < i\omega_{IR}$ may be small, the λ term is what forms the network because it is the lowest order term.

4 Concrete Examples

From now on

$$
A_{ij} = A^{(2)}_{ij}(t_{\max})
$$

As a simple example, consider the case where $\lambda_{ij}(x) = \lambda (2\pi)^n \delta^n(x)$. We will call this the case of a single "hub" at $x = 0$. For $i \neq j$ this results in an adjacency matrix of the form:

$$
A^{(2)}_{ij}(t) \approx -\lambda G^0_R(x_i, -t) G^0_R(x_j, -t) \tag{22}
$$

Although we derived this formula starting from diffusion, it actually holds for any general quadratic action for the nonperturbed part (it will only result in a different G^0.)

4.1 Degree Distribution

From this, it is actually very easy to calculate the degree and its distribution. The distribution of the nodes in space is given by the source current:

$$J(x) = \sum_i J_i(x) = \sum_{i=0}^{N} J_0 (2\pi)^n \delta^n(x - x_i)$$

The degree of a node is simply given by:

$$\begin{aligned} k_i = \sum_j A_{ij} &= J_0^{-1} \int \frac{d^n z}{(2\pi)^n} (J^T(z) A)_i \\ &= J_0^{-1} G^0(x_i, -t_{\max}) \int \frac{d^n z}{(2\pi)^n} J(z) G^0(z, -t_{\max}) \end{aligned} \tag{23}$$

4.1.1 Uniform Spatial Distribution

Eq. (23) is the general form of the degree for a weakly coupled hub to lowest order. If the distribution of the nodes is uniform over space such that the re are N nodes in a volume V with

$$V \gg \left\langle x^2 \right\rangle^{n/2}$$

where $\langle x^2 \rangle$ denotes the average distance that the fields ϕ_i may travel in $t_{\max}$, then the degree reduces to:

$$\begin{aligned} k_i &= \frac{N}{V} G^0(x_i, -t_{\max}) \int_V d^n z G^0(z, -t_{\max}) \\ &= c_0 \frac{N}{V} G^0(x_i, -t_{\max}) \end{aligned} \tag{24}$$

Which clearly is only finite if the Green's function has afinite extent (the necessary condition for $\langle x^2 \rangle < \infty$ as well. This is the first important result in the model, namely that the degree distribution will only depend on the propagation from a point to the hubs. As long as $c_0 < \infty$ the distribution will not depend on it.

There is another important point in eq. (24), namely that the degree strongly depends on the the geographical location of the hubs and the node in question. We will work out concrete examples below.

4.1.2 Random Walkers

Restricting to the specific case of the diffusion equation, the adjacency matrix elements become ($t = t_{\max}$:

$$\begin{aligned} G^0(x, t) &= (2\pi t)^{-n/2} \exp\left[-\frac{x^2}{4t} \right] \\ A^{(2)}_{ij}(t) &\approx -\lambda (2\pi t)^{-n} \exp\left[-\frac{x_i^2 + x_j^2}{4t} \right] \end{aligned} \tag{25}$$

And the degrees are given by:

$$k_i = c_0 \frac{N}{V} (2\pi t)^{-n/2} \exp\left[-\frac{x_i^2}{4t}\right]$$

where we used:

$$c_0 = \int d^n z G^0(z, -t) = 1$$

From this, it is actually very easy to calculate the degree distribution. Since the degree only depends on position, and more precisely, on the magnitude x_i^2, all nodes with the same x_i^2 will have more or less the same degree. The only determining factor will then be the density of the pints in that particular radius. the distribution of the degrees can then be visualized as how many points δN are between r and $r + \delta r$. These will be the points which have degree $k(r)$. It follows that:

$$\int P(k)dk = 2N = 2\int dN = 2\int \frac{dN}{dk} dk$$

$$P(k) = 2\frac{dN}{dk} = 2\frac{dN/dr}{dk/dr} \tag{26}$$

In the case of diffusion this yields:

$$\begin{aligned} \frac{dN}{dk} &= \frac{N}{V} \Omega_{n-1} r_k^{n-1} \\ P(k) &= 4t(2\pi t)^{n/2} \frac{\Omega_{n-1} r_k^{n-1}}{r_k \exp\left[-\frac{r_k^2}{4t}\right]} \\ &= 4t \frac{\Omega_{n-1}(4t \log k)^{\frac{n-2}{2}}}{k} \end{aligned} \tag{27}$$

In the special case of two spatial dimensions $n = 2$ this reduces to a power law $p(k) \propto k^\gamma$ with an exponent $\gamma = -1$.

4.1.3 Adjusting the Node Distribution and the Power Law

As we saw the degree distribution depends on the node distribution in space as well as the degrees. One thing we notice immediately in (26) is that if

$$\frac{dN}{dr} \propto \exp[-\alpha r^2] = k^{\alpha/t}$$

We will have the degree distribution

$$P(k) \propto k^{-1+\alpha/t}$$

This may look exciting, as it allows for adjusting the degree distribution, but it is actually rather unphysical for constructing degrees $\gamma < -1$ as it requires the density of the nodes to grow as a quadratic exponential with their distance from the hub. Thus, if we wish to adjust the degree distribution, it is probably not feasible to do it through the node distribution. Our other options are changing the hub distribution $\lambda(x)$ and the metric of the space. We will examine these options below.

4.1.4 Other Power Law Degree Distributions from Hub Distribution

Here we wish examine the possibility of getting different degree distributions which are power laws with an exponent other than -1 by using different distributions for the hubs. For a general hub distribution $\lambda(x)$, The adjacency matrix is given by (21). Again, let's first assume the case where:

$$\lambda_{ij}(x) = \lambda(x)$$

so that it couples universally to all the fields ϕ_i. The degrees get slightly modified:

$$\begin{aligned}
k_i &= J_0^{-1} \int \frac{d^n y}{(2\pi)^n} \lambda(y) G^0(x_i - y, -t) \int \frac{d^n z}{(2\pi)^n} J(z) G^0(z - y, -t) \\
&= c_0 \frac{N}{V} \int \frac{d^n y}{(2\pi)^n} \lambda(y) G^0(x_i - y, -t) \\
c_0 &= \int \frac{d^n z}{(2\pi)^n} J(z) G^0(z, -t)
\end{aligned} \tag{28}$$

Now, finding the degree distribution will in general be more complicated than the procedure we took in (26) because of the integration involved. However, if $\lambda(x)$ enjoys any spatial symmetries, it simplifies again. Take for example a spherically symmetric distribution. Then $\lambda(x) = \lambda(|x|)$. This way from symmetry we will have $k_i = k(|x_i|)$. This yields a degree distribution of the form:

$$P(k) = c_0^{-1} \Omega_{n-1} r_k^{n-1} \left(\int \frac{d^n y}{(2\pi)^n} \lambda(y) G^0(\vec{r}_k - y, -t) \right)^{-1} \tag{29}$$

4.1.5 Solving for Hub Distribution

Now suppose we are observing a certain degree distribution $P(k)$ and we wonder what distribution of the hubs $\lambda(x)$ yields such a $P(k)$. To solve for $\lambda(x)$ we first recall that the Greens function is the inverse of the differential operators which define the equation of motion:

$$\begin{aligned}
\Delta &\equiv \partial_t + \partial^2 \\
\Delta G(x,t) &= (2\pi)^{n+1} \delta(t) \delta(x) \\
if \quad \Delta\phi(x,t) &= f(x,t) \quad \Rightarrow \phi(x,t) = \int \frac{dt_z dz^n}{(2\pi)^{n+1}} G(x - z, t - t_z) f(z, t_z)
\end{aligned} \tag{30}$$

Therefore, if we introduce time integral into (29) we may use this relation to solve for $\lambda(x)$. But before doing that, notice that $k = k(r)$ and in (29) we must write k in terms of r_k in order to be able to use the Green's function equation. This can be done using (26):

$$\int_0^k P(k') dk' = \frac{\Omega_{n-1}}{n} r_k^n$$

After solving this equation and obtaining $P(k(r))$ we may proceed as follows:

$$\int \frac{dt_y d^n y}{(2\pi)^n} \lambda(y) \delta(t_y) G^0(\vec{r} - y, -t - t_y) = \frac{\Omega_{n-1} r^{n-1}}{c_0 P(k(r))} \tag{31}$$

And therefore:

$$\lambda(x)\delta(t) = \Delta\left(\frac{\Omega_{n-1} r^{n-1}}{c_0 P(k(r))}\right) \tag{32}$$

4.1.6 Example of Deriving Power Laws

Imagine a case where the degree distribution is some power law of arbitrary exponent β:

$$P(k;t) = c(t)k^{\beta}$$

Where we included the time dependence, because without it the equations may not be solvable and because we saw previously in (27) that $P(k)$ will explicitly depend on time. We wish to find a hub distribution $\lambda(x)$ which results in such a distribution. First we need to find $P(k(r))$. If $\beta \neq -1$:

$$\begin{aligned}
\int_{k_0}^{k} P(k')dk' = (\beta+1)^{-1}\left(k^{\beta+1} - k_0^{\beta+1}\right) &= \frac{\Omega_{n-1}}{n} r^n \\
k &= c_1\left(r^n - r_0^n\right)^{\frac{1}{\beta+1}} \\
c_1 &= \left(\frac{\Omega_{n-1}(\beta+1)}{n}\right)^{\frac{1}{\beta+1}}
\end{aligned} \tag{33}$$

Plugging into (32) yields:

$$\lambda(x)\delta(t) = \Delta\left(\frac{\Omega_{n-1} r^{n-1}}{c_2\left(r^n - r_0^n\right)^{\frac{\beta}{\beta+1}}}\right) \tag{34}$$

For the case of diffusion (random walk) we have:

$$\left(\partial_t + \partial^2\right) f(r) = \left(\partial_t + \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\right) + \frac{L^2}{r^2}\right) f(r)$$

Where L^2 is the "angular momentum" part coming from the spherical symmetry. Since we assumed no dependence on the angles in $P(k)$, $L = 0$ and we have

$$\begin{aligned}
\lambda(r)\delta(t) &= \Omega_{n-1}\left(\partial_t + \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\right)\right)\left(\frac{r^{n-1}}{c_2(t)\left(r^n - r_0^n\right)^{\frac{\beta}{\beta+1}}}\right) \\
&= \frac{\Omega_{n-1}}{c_2(t)}\left(-\partial_t \ln(c_2) + \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\right)\right)\left(\frac{r^{n-1}}{\left(r^n - r_0^n\right)^{\frac{\beta}{\beta+1}}}\right)
\end{aligned} \tag{35}$$

As we see, there is no way to get a time-independent $\lambda(x)$ from this. On the other hand, having a time-dependent $\lambda(x)$ was not in contradiction with (29), though we would prefer at least a slowly varying hub distribution $\lambda(x)$. For instance, suppose having $c_2(t)^{-1} \sim \delta(t)$. Of course we cant calculate $\partial_t \ln c_2$ from this, but we can regularize this by assuming that:

$$c_2^{-1}(t) = (\pi\tau)^{-1/2} \exp\left[-\frac{t^2}{\tau^2}\right] \approx \delta(t), \quad \tau \ll t$$

This leads to:

$$\lambda(r) = \Omega_{n-1}\left(-\frac{t}{\tau^2} + \frac{n-1}{r}\partial_r + \partial_r^2\right)\left(\frac{r^{n-1}}{(r^n - r_0^n)^{\frac{\beta}{\beta+1}}}\right) \tag{36}$$

In the special case of $\beta = -1$ we get:

$$k = k_0 \exp\left[\frac{\Omega_{n-1}}{nc(t)}r^n\right]$$

and

$$\begin{aligned}\lambda(r;t)\delta(t) &= \left(\partial_t + \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\right)\right)\left(\frac{r^{n-1}}{\frac{c_0}{c(t)}r^{n-1}k(r)}\right)\\ &= c_0^{-1}c(t)k_0^{-1}\left(\partial_t \ln c(t) + \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\right)\right)\exp\left[-\frac{\Omega_{n-1}}{nc(t)}r^n\right]\\ &= c_0^{-1}c(t)\left(\partial_t \ln c(t) + 2(n-1)r^{n-1}\right)k(r)\end{aligned} \tag{37}$$

In (2+1)D, i.e. $n = 2$, the result states that the hubs will need to have a Gaussian times at most linear terms in r to yield $P(k) \propto k^{-1}$. Indeed if we use the same Gaussian regularization as before for $c(t) \sim \delta(t)$ and use the fact that $\tau \ll t$ we find:

$$\begin{aligned}\lambda(r;t) &= c_0^{-1}\left(2\frac{t}{\tau^2} + 2r\right)\exp\left[-\frac{\pi}{c(t)}r^2\right]\\ &\approx c_3 t\delta(r)\end{aligned} \tag{38}$$

5 Introducing Metric

6 Nima 06/16/2014

Each of us who makes an edit can add a new section like this with the date.

7 00

Assume we want to find the probability that the degree of a node is equal to $z = \sum_i f(r_{0i})$ This means that we have to find the probability that that the number density of nodes in every slice times the value of $f(r)$ for that slice sums up to this value. If the number densities are more or less independent of each other, this probability will be the product of the probabilities for each section:

$$P(z) = \sum_{\{\sum_i y_i = z\}} P(y_1)P(y_2)\cdots P(y_n) \tag{39}$$

This is essentially the convolution of these multiple probabilities, which is compatible with the characteristic function being the product of the characteristic functions. However, this convolution involves breaking down the path in space into many infinitesimal pieces and then in each of those infinitesimal sections we are allowing the value of y_i to fluctuate freely (though always positive) in such a way that the sum of all y's remains constant and equal to the desired z. This is reminiscent of a path integral. The path is broken down into many little pieces and the sum of the path traversed in y remains constant. If we do not restrict ourselves to finding the probability of one value of z and instead freely sum over all possible configurations of y, while still keeping the number of points constant we find:

$$\int [dy] P(\int y dx) \tag{40}$$

it will be like assuming that the number of points

8 Basics of the model

The network emerges from local interactions. If it becomes a scale-free network, the interactions become long-range. The exponent in the distribution of degrees may be related to the 2 point correlation functions.

I want something that:

1. lets me label fields to distinguish between different nodes, like ϕ_i,
2. allows for correlations between different fields, $\sim \langle \phi_i \phi_j \rangle \neq 0$ for $i \neq j$,

For the first item, a trivial thing to do would be to work with an $O(N)$ model in which each node would be represented by it's own field ϕ_i. But we can also have many particle states infield theories with a single field species. These are convenient because then for item 2 we would not need a theory in which one particle can transform into another. For item 2 we need an interacting field theory which allows propagation of different field species into each other.

One way to achieve this is to have one field ϕ and a number of currents J_i that it couples to. The field ϕ will act as a mediator between the nodes J_i. We will ignore the dynamics of J_i and assume that their spatial position is fixed.

8.1 Nodes as field sources

We will assume that ϕ is dynamical and that is what generates the network for J_i. Each J_i is localized at a certain space-time point, though it may have dynamics and flow:

$$J_i(x) = \alpha \delta^d (x - x_i)$$

We assume that J_i are "sources" of ϕ and couple to it as:

$$S_i = \int \phi J_i \sqrt{|g|} d^d x$$

Where g represents the metric on the space-time. We will assume some general dynamical action for ϕ of the following form:

$$S_\phi \equiv \int \sqrt{|g|} d^d x \left[\phi \left((i\partial_t)^w - \left(\nabla^2 \right)^z + m^2 \right) \phi + \lambda \phi^4 \right] \tag{41}$$

Where ∇ denotes the spatial gradient. We will fix the powers w, z in the operator $\Delta \equiv (i\partial_t)^w - \vec{\partial}^{2z} + m^2$ in the quadratic part of the action later below. This theory theory has Galilean symmetry (though for special values $a = 2z$ it also becomes Lorentz invariant) and we do not need any symmetry beyond that for what follows.

8.2 N and the "Quantum" regime

The full action is $S = S_\phi + \sum_i S_i$. We can Naively use this action and calculate a real time partition function:

$$Z \equiv \int D\phi \exp[iS]$$

But note that this is not a theory of actual particles and therefore $\hbar$ doesn't exist here. In order to have a "semi-classical", however, we would need to have some parameter replacing $\hbar$. In addition, doing any calculation in this theory would be hopeless unless we could relate it to some weakly interacting theory, e.g. if λ would be small. But a priori there is no reason for any of this to be true.
[3]

[3]OR: This is where the large number of nodes N comes in. To each node, we associated a source current J_i. The sum of all these "node sources" is like the total source on this space:

$$J \equiv \sum_i J_i$$

But since there are $N \to \infty$ sources which generate non-zero ϕ wherever they are, we end up with:

$$\phi J \sim N \phi J_i \to \infty$$

unless $\phi \to 0$, which clearly is not what we want because then there would be no interaction between individual J_i. Therefore, we should pull out an N from the definition of ϕ to find the part of interaction that does not grow with N and which rather shows the individual ϕJ_i interaction. ....

We now define an "average" source as:

$$J \equiv \frac{1}{N} \sum_i J_i$$

Which has the property that:

$$\int J d^d x \sqrt{|g|} = \alpha$$

So this division by N also makes sure that J as a field does not diverge on the whole space as $N \to \infty$. Then by reabsorbing the N into the definition of ϕ through:

$$\phi \to N\phi$$

We get:

$$S = \frac{1}{N^2} \int \sqrt{|g|} d^d x \left[\phi \Delta \phi + \frac{\lambda}{N^2} \phi^4 + N^2 J \phi \right] \tag{42}$$

Now we can take N^2 to be the $\hbar$ in the model. This means that for large N^2 a semi-classical analysis is not meaningful because we are deep in the quantum regime and all the loop effects should be considered. In addition we now have λ/N^2 as the coupling in the quartic term, which means that interactions of this ϕ are weak and we can use the perturbative description. The source term still has no net power of N.

8.3 n point functions

Recall that $\ln Z[J]$ is the generator of "connected" diagrams, while $Z[J]$ alone generates all diagrams (connected or disconnected.) For very weak couplings, i.e. large N, we effectively have a quadratic theory:

$$Z|_{\lambda \to 0} \sim C_0 \exp \left[-\frac{N^2}{4} J \Delta^{-1} J \right] \tag{43}$$

With $C_0 \propto \det(\Delta)^{-1/2}$ coming from integrating out ϕ. Obviously at $\lambda = 0$ the theory is free and thus the connected part of the 4 point function $\langle \phi\phi\phi\phi \rangle_C = 0$. This is evident from:

$$\langle \phi\phi\phi\phi \rangle_C \sim \frac{\delta^4}{\delta J^4} \ln Z[J] \bigg|_{\lambda \to 0} = 0$$

Never the less a contraction of four ϕ fields is possible in this theory, with the ϕ's pairwise contracted. This is calculated from Z itself rather than $\ln Z$ because it's a disconnected diagram.

$$\langle \phi\phi\phi\phi \rangle \sim \frac{1}{Z[J]} \frac{\delta^4}{\delta J^4} Z[J] \bigg|_{J \to 0, \lambda \to 0} \sim \Delta^{-1} \Delta^{-1}$$

8.4 Adjacency matrix

Consider the matrix of correlations of the field ϕ at the points of insertion of the node source currents J_i:

$$\begin{aligned}\frac{\delta}{\delta J_i(x)}\frac{\delta}{\delta J_j(y)}\ln Z|_{\lambda\to 0} &= \langle\phi(x_i)\phi(x_j)\rangle \\ &= -\frac{\delta^d(x-x_i)\delta^d(y-x_j)}{N^2}\frac{N^2}{4}\Delta^{-1}(x_i;x_j) \qquad (44)\end{aligned}$$

We wish to define the network using this sort of correlation matrix (which is the scattering matrix.) We define the adjacency matrix to be:

$$A_{ij} \equiv \frac{\delta}{\delta J_i}\frac{\delta}{\delta J_j}\ln Z = -\frac{\delta^d(x-x_i)\delta^d(y-x_j)}{4}\langle\phi(x_i)\phi(x_j)\rangle \equiv \langle\phi_i\phi_j\rangle \qquad (45)$$

8.4.1 Degree matrix

Naively, we can define a degree matrix as follows:

$$D_{ij} = \delta_{ij}\sum_k A_{ik} = N\delta_{ij}\frac{\delta}{\delta J_i}\frac{\delta}{\delta J}\ln Z|_{\lambda\to 0}$$

The bad thing about this matrix is that its construction seems so arbitrary from the point of view of field theory. It does not have good parallel in field theory.

For an unweighted undirected network this definition is obviously giving the correct number. But for more general networks, the quantity obtained above may not even be the thing we were interested in. For example in a neural network we have a weighted graph where many of the wights may even be negative. There it really depends on what we are looking for when we say "degree." There are a number of other quantities which reduce to the above quantity for unweighted undirected graphs, but which take very different forms in other graphs. For example, since for an unweighted undirected network $A_{ij} = 0, 1$ we have:

$$D_{ii} = \sum_j A_{ij} = \sum_j A_{ij}A_{ij} = (AA^T)_{ii} = (A^TA)_{ii}$$

These two forms, however, have very different meanings in an unweighted directed network: one is the in-degree, the other the out-degree.

$$D^{\text{in}}_{ii} = (AA^T)_{ii}, \qquad D^{\text{out}}_{ii} = (A^TA)_{ii}$$

For weighted graphs, this measures the sum of the squares of the wights and so is a positive semi-definite matrix even when A has negative components.

Note that ϕ_i just represents the configuration of the N fields. It does *not* comprise a complete basis necessarily and the above is thus not like an identity operator insertion.

We will adopt this formulation for network variables and try to find parallels of familiar network measures in terms of concepts in field theory.

8.5 Paths

In an undirected unweighted network paths of a l can be found by multiplying the adjacency matrix A, l times into itself because A is the transformation which moves us along the edges. Moreover we have:

$$(A^l)_{ij} : \# \text{ of paths of length } l \text{ from } i \to j$$

In a directed graph $A \neq A^T$. Multiplying by A takes us in the direction of the edges and A^T moves us opposite to the edge directions. Thus:

$$(A^l)_{ij} : \# \text{ of directed paths of length } l \text{ from } i \to j$$

and $\{(A^T)^l\}_{ij}$ is the number of backward paths going from $j \to i$.

Other network measures, such as clustering, can also be calculated in this manner. The clustering coefficient, for instance, counts the number of triangles, which are loops of length 3. To find a triangle all one needs to calculate is $(A^3)_{ii}$ where the two indices are the same because the starting- and end-point are the same. Thus, the total number of triangles is found by taking the trace of this:

$$\# \text{ triangles} : \mathrm{Tr}[A^3] = \mathrm{Tr}\left[\langle \Phi\Phi \rangle^3\right]$$

9 Generating Function

$$\begin{aligned}
G(q) &= \sum_k P(k) q^k \\
\langle k \rangle &= \sum_k k P(k) = q \frac{d}{dq} G(q) \Big|_{q \to 0} \\
\langle f(k) \rangle &= \sum_k P(k) f(k) = f\left(\frac{d}{d \ln q}\right) G(q) \Big|_{q \to 0}
\end{aligned} \tag{46}$$

First of all, for weighted graphs, k comes from a continuous spectrum of numbers and thus the sum in k needs to be promoted to an integral.

$$\sum_k \to \int dk$$

Second, since we need $d/d \ln q$ for all our calculations, it's better if we change variables to:

$$q = e^w$$

This way, denoting $\partial_w = \frac{d}{dw}$ we get:

$$\begin{aligned}
G(w) &\equiv \int dk P(k) e^{kw} \\
\langle k \rangle &= \int k dk P(k) = \partial_w G(w)|_{w \to -\infty} \\
\langle f(k) \rangle &= \int dk P(k) f(k) = f(\partial_w) G(w)|_{w \to -\infty}
\end{aligned} \tag{47}$$

These are true if $G(w)$ is normalized so that:

$$\langle\rangle = G(-\infty) = 1$$

So, in order to account for normalization we must use:

$$\langle f(k)\rangle = \int dk P(k) f(k) = \frac{1}{G(w)} f\left(\partial_w\right) G(w)\Big|_{w\to-\infty}$$

Note that this is in general different from using $\ln G$ in calculating $\langle f\rangle$ because the former includes disconnected diagrams. Additionally, these relations tell us an important fact about the generating function: It's the Laplace (or Fourier) transform of the degree distribution. This means that the degree distribution itself is found through an inverse Laplace transformation of the generating function:

$$P(k) = \frac{i}{2\pi}\int_{-i\infty}^{i\infty} dw G(w) e^{-kw} \tag{48}$$

Which is just an inverse Fourier transform. This method is different from what people usually do with generating functionals in unweighted networks where k is just a positive integer or zero. using this definition, $\langle k\rangle$ becomes:

$$\begin{aligned}\langle k\rangle &= \int_0^\infty dk k P(k) = \frac{i}{2\pi}\int_0^\infty dk k \int_{-i\infty}^{i\infty} dw G(w) e^{-kw} \\ &= \frac{-i}{2\pi}\partial_{w_0}\int_{-i\infty}^{i\infty} dw G(w)\int_0^\infty dk e^{-k(w-w_0)}\Big|_{w_0\to 0} \\ &= \frac{-i}{2\pi}\partial_{w_0}\int_{-i\infty}^{i\infty}\frac{dw G(w)}{w-w_0}\Big|_{w_0\to 0} = \partial_{w_0} G(w_0)\Big|_{w_0\to 0}\end{aligned} \tag{49}$$

Which suggests that in this representation the $w_0 \to_0$ is the correct choice.

9.1 Multi-dimensional case

In the above generalization of the generating function to continuous weights we had only one variable w in the generating function. But since we are trying to relate our network to a geometric model which may be living in some space with some number of dimensions, we wish to work backwards and see if we can generalize the integrals above to higher dimensional versions. Let's start by assuming that the dw integral is the leftover radial part of a higher dimensional integral over $d^d w$. We will first eulideanize the integral[4] so that the dot product can be written as $\vec{A}\cdot\vec{B} = |A||B|\cos\theta$. We then assume that both w and k are magnitudes of n dimensional vectors. Working backwards, this yields:

$$\int k dk P(k) \equiv \int d^d k \pi(k), \quad \Rightarrow P(k) = k^{d-2}\int d\Sigma^{d-1}\pi(\vec{k})$$

[4] Any two vectors A, B both of which are in a Euclidean subspace constitute a $2D$ plane (unless they are parallel.) The plane is simply all linear combinations $aA + bB$. If we are in a signature p, q, the local symmetries of the space can be found from the Euclidean transformations of $p+q$ dimensions by inserting p appropriate i's in the transformations and again two vectors can be put on the same plane.

Now, we also need to transform the w integral:

$$\int_{-i\infty}^{i\infty} dw G(w) e^{-kw} = i \int_0^\infty dw G(iw) \left(e^{ikw} - e^{-ikw}\right)$$
$$= -k \int w dw d\cos\theta G(iw) e^{i\vec{k}\cdot\vec{w}}$$
$$\equiv -k \int d^d w \gamma(w) e^{i\vec{k}\cdot\vec{w}}$$
$$\Rightarrow G(iw) = \int d\Sigma^{d-2} w^{d-2} \gamma(\vec{w}) \tag{50}$$

So, in terms of π and γ, $\langle k \rangle$ becomes:

$$\langle k \rangle = \int d^d k \pi(\vec{k})$$
$$= -\int k^2 dk \int d^d w \gamma(\vec{w}) e^{ikw\cos\theta}$$
$$= \int d^d k \int d^d w \frac{k^{3-d}}{\Sigma^{d-1}} \gamma(\vec{w}) e^{ikw\cos\theta} \tag{51}$$

Note that these are essentially identities, as long as such transformations exist. We made no special assumptions in writing these. From the second line we see that the degree distribution is:

$$P(k) = k \int d^d w \gamma(\vec{w}) e^{ikw\cos\theta} = k^{d+1} \int d^d y \gamma\left(\frac{\vec{y}}{k}\right) e^{iy\cos\theta}$$

This is telling us that if γ is a homogeneous function, we will have a power law distribution $P(k)$:

$$\text{if}: \gamma(\lambda x) = \lambda^\eta \gamma(x), \quad \Rightarrow P(k) \propto k^{d+1-\eta}$$

The reason why we complicate the formulas this much is because in the geometric model all the integrals will likely be d dimensional and we want to be able to read off what $P(k)$ is. For instance, if we write $\langle k \rangle$ with two integral over $d^d k$ and $d^d w$ as in (51) we have:

$$\langle k \rangle = \int d^d k \int d^d w \Delta(w,k) e^{ikw\cos\theta} \tag{52}$$
$$\Delta(w,k) = \frac{k^{3-d}}{\Sigma^{d-1}} \gamma(\vec{w})$$
$$P(k) = k^{d-2} \Sigma^{d-1} \int d^d w \Delta(w,k) e^{ikw\cos\theta} \tag{53}$$

Note that in principle Δ can be a function of both w and k.

9.2 Degree Distribution

For brevity, we define $\phi_i \equiv \phi(x_i)$. The adjacency matrix is defined through

$$A_{ij} \equiv \langle \phi_i \phi_j \rangle$$

For the degree, as we argued, there may be a number of definitions in general. Let's restrict ourselves to the following two (assume the network is undirected for now):

$$D_i^1 \equiv \sum_j A_{ij}, \qquad D_i^2 \equiv (AA^T)_{ii}$$

Let's try to find a function of ϕ which would be a generating function for the degree. Let's work with the average degree equation $\langle k \rangle = \partial_w G(0)$. To get average degree D^1 from the partition function one would do:

$$\langle k \rangle = \frac{1}{N}\sum_i D_i^1 = \frac{1}{N}\sum_{i,j} \frac{\delta}{\delta J_i}\frac{\delta}{\delta J_j} \ln Z\Bigg|_{J\to 0} \tag{54}$$

For higher powers of k though, we only need the adjacency matrix, which consists of 2pt functions $\langle \phi_i \phi_j \rangle$ and not higher n-point functions. For example:

$$\langle k^n \rangle = \frac{1}{N}\sum_i \left(D_i^1\right)^n = \frac{1}{N}\sum_i \left\langle \phi_i \sum_j \phi_j \right\rangle^n$$

The distribution must have a few properties. First, it must be normalized. Second, it must produce the correct average degree. Average degree may also be calculated in the following way:

$$\langle k \rangle = \frac{1}{N}\sum_{i,j} \langle \phi_i \phi_j \rangle \equiv \frac{1}{N}\sum_{i,j} S(x_i, x_j)$$

Where $S(x_i, x_j)$ is the Green's function for ϕ for propagating from x_i to x_j. Assuming there are no preferred points in space, we can assume that $S(x_i, x_j) = S(x_i - x_j) \equiv S_{ij}$. If we assume a free particle kinetic action for ϕ, $S(x) = S(|x|)$, though it may have extra spin and polarization indices. Thus, the weight of nodes can only depend on $|x|$ (where the metric may have an arbitrary signature and is not necessarily Euclidean.) Taking $NS(x)/V$ at a fixed x, where V is the volume of the space, and integrating the angular parts is like counting the number of all nodes that may have connections of weight $S(x)$. This is *almost* the generating function. To be more precise, $S(x)$ tells us the distribution of "weights" for the nodes. To find the degree, one sums over weights times the multiplicity of the appearance of each weight. This is an alternative way of describing the average degree. This duality resembles duality between a function and its Fourier transform. We are essentially expressing the average degree in two different bases:

$$\langle k \rangle = \int k dk P(k) = \int d^d x S(|x|) = \Sigma^{d-1} \int x^{d-1} dx S(x)$$

10 Example model

Suppose we have a free scalar ϕ that's coupled to node currents J_i. Assume the world is a circle S^1 and the action is:

$$S = \int dt \left(|\partial_t \phi|^2 + m^2 |\phi|^2 + \sum_i J_i \phi \right)$$

Now assume that the nodes J_i are distributed at random on the S^2. We want to construct a network from $\langle\phi_i\phi_j\rangle$. This is given by $(t_{ij} \equiv t_i - t_j)$:

$$A_{ij} = \langle\phi_i\phi_j\rangle = C\int \frac{d\omega}{2\pi}\frac{e^{i\omega(t_i-t_j)}}{\omega^2+m^2} = C\frac{e^{-m|t_{ij}|}}{2m}$$

Where we closed the contour from above. C is the normalization factor found by dividing out the partition function at $J=0$:

$$C^{-1} = Z[0] = \langle\rangle \propto m^{-2}$$

And the final normalized correlations A_{ij} become flatter and decrease in level when m decreases and becomes completely flat at $m=0$. Because the space S^1 is finite and periodic the normalization of A_{ij} gets a factor related to the period L:

$$C^{-1} = \int_{-L/2}^{L/2} dt\frac{e^{-m|t|}}{m} = \frac{2}{m^2}\left(1-e^{-mL/2}\right)$$

Now let us build a binary network out of A_{ij} by setting correlations below a certain level to be 0 and above it to be 1. This is nothing but choosing a certain distance from each node and connecting it to all the nodes within that distance. This way we would expect all nodes to have similar degrees, with only variations because of the random nature of the distribution of points.

Figure 1: Left: Degree distributions for various thresholds (distances) in the 1D theory close to zero (only nodes very close to each node are connected.) The overlaid curves are Poisson distributions which seem to fit the distributions well. Right distributions for various thresholds covering the entire space (circle S^1.) The distribution for threshold near 0 is the mirror of threshold near $0.5T$.

We wish to understand what the distribution of degrees will be as a function of m and L. First look at the limit $L\to\infty$. Denote the density of the nodes by $n = N/L$. The probability of finding a node per unit length is then $p = n/N = 1/L$. Thus, the distribution of the number of nodes up to a fixed distance l from each node follows a Poisson distribution as long as $l \ll L$. This distribution can be derived from the exact binomial expression in the standard fashion. The probability of going a distance l away from a node and find k nodes within that distance (which is equivalent to having degree k for that node) is found from the appropriate term in the binomial distribution. We are going $l \gg 1$ length units and in each step there is probability p to find a node and $(1-p)$ not to find one:

$$1 = (1-p+p)^l = \sum_{k=0}^{l}\binom{l}{k}(1-p)^{l-k}p^k \approx e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}, \quad \lambda \equiv \frac{lp}{e} = \frac{l}{Le}$$

Because it's a Poisson distribution both the mean and the variance are λ.

This was obviously an Erdös-Rényi (ER) network. Let us try to understand what field theory function would represent the degree distribution in this network. In terms of the correlation matrix A_{ij} connecting all nodes within distance l of each other is like setting to 1 the correlations which are above a threshold a and 0 below it. At $L \to \infty$ the threshold a is:

$$a = A_{ij} = \frac{m}{4} e^{-ml}, \qquad l = cm^{-1}$$

$l \propto m^{-1}$ is of course the correlation length. The degree distribution does not look like any familiar transformation of the correlation function A_{ij}.

10.1 Continuous weight network

What about the continuous network? Instead of picking different distances to connect nodes, we are connecting all nodes with some weight that falls as the correlation function A_{ij}, or in this case exponentially with their distance. We have N points on a length T distributed at random. The sum of the weights that each node gets is like a Monte Carlo sample probing the area under A_{ij}. Thus on average each node is getting:

$$\begin{aligned} \langle k \rangle &\approx \frac{N}{T} \mathrm{Re} \left[\int_0^T dt \int_0^\infty \frac{dp}{2\pi} \frac{e^{-pt}}{p^2 - m^2 + i\varepsilon} \right] \\ &= \frac{N}{T} \int_{-T}^{T} dt \frac{e^{-m|t|}}{2m} = \frac{N}{Tm^2} \left(1 - e^{-mT} \right) \end{aligned} \tag{55}$$

In the regime $m \gg 1/T$ the average degree will go like m^{-2}. However, we should also remember that because by construction this is a Monte Carlo sampling, it will have the "curse of $\sqrt{n}$" as a limit to its precision, where $n = N^2$ is the number of samples in our case. More precisely, we expect the result of a probability p to be measured around:

$$p \pm 2\sqrt{\frac{p(1-p)}{n}}$$

In our case we can put $p = 1/(Tm)^2$, so that $\langle k \rangle = NTp$. Since $m \ll T$ we have:

$$\sqrt{\frac{p(1-p)}{n}} \approx \sqrt{\frac{p}{n}} = \frac{1}{NTm}$$

And the precision expected for $\langle k \rangle$ will then be:

$$\langle k \rangle \approx NT \left\{ p \pm 2\sqrt{\frac{p}{n}} \right\} = \left(\frac{N}{T} \pm 2m \right) \frac{1}{m^2}$$

The result of our numerical simulations yield:

$$\langle k \rangle \approx \left(\frac{N}{T} + \frac{m}{2} \right) \frac{1}{m^2}$$

Which is well within the expected precision.

10.2 Navid's method with uniform distribution

The idea is to use the characteristic function ϕ to find the distribution. For a uniform distribution ϕ is:

$$\phi(a,b) = \frac{e^{itb} - e^{ita}}{it(b-a)}$$

over a small interval $[r, r+dr]$ we get:

$$\phi = \frac{e^{itr}}{it}$$

The contribution to the degree of a node at $r = 0$ from nodes on the interval $[r, r+dr]$ is given by their number $n(r)dr$ times their weight $G(r)$:

$$d(r)dr = G(r)n(r)dr$$

We want to find the total degree and its distribution:

$$d = \int d(r)dr$$

over each interval the random variables are independent of other intervals and so the characteristic function over the whole space is the product of the individual characteristic functions.

10.3 Degree distribution

The degree d_a of node a is:

$$d_a = \int dt' G(t' - t_a) D(t' - t_a)$$

Where $D(t)$ is the density of nodes in the interval $[t, t+dt)$. The origin is arbitrary. How can the probability density depend on t' then? Once we pick a fixed node a and start counting the number of nodes that may fall within a given distance from that node we have broken the symmetry and now the distribution of nodes does depend on the distance from a. Thus it will depend on $t = t' - t_a$. Since we had a uniform distribution of nodes, the probability distribution of the nodes on an interval is going to be a Poisson distribution. The probability of finding n nodes in the interval $[0, t]$ is:

$$C_n(t) = \frac{1}{n!}\left(\frac{Nt}{L}\right)^n e^{-Nt/L} \equiv \frac{\tau^n}{n!}e^{-\tau}$$

Where $\tau \equiv Nt/L$. The probability of finding n nodes in $[t, t+dt)$ is therefore:

$$\rho_n(t) = \frac{dC_n(t)}{dt} = \frac{N}{L}\frac{\tau^{n-1}}{(n-1)!}e^{-\tau}\left(1 - \frac{\tau}{n}\right)$$

The density of nodes is then given by the number of nodes times the probability of finding that many nodes in the interval:

$$
\begin{aligned}
\frac{L}{N}D(t) &= \frac{L}{N}\sum_{n=0}^{N} n\rho_n(t) = e^{-\tau}\sum_{n=1}^{N} n\left(\frac{\tau^{n-1}}{(n-1)!} - \frac{\tau^n}{n!}\right) \\
&= e^{-\tau}\sum_{n=1}^{N}\left((n-1+1)\frac{\tau^{n-1}}{(n-1)!} - \frac{\tau^n}{(n-1)!}\right) \\
&= e^{-\tau}\sum_{n=0}^{N-1}\left(\frac{(n+1)\tau^n}{n!} - \frac{\tau^{n+1}}{n!}\right) \\
&= e^{-\tau}\left[\tau\sum_{n=0}^{N-2}\frac{\tau^n}{n!} + (1-\tau)\sum_{n=0}^{N-1}\frac{\tau^n}{n!}\right] \\
&\approx 1 + O(\tau^{-N}) - \frac{\tau^N e^{-\tau}}{(N-1)!} \approx 1 \quad (\text{if } N \gg \tau)
\end{aligned} \tag{56}
$$

The $O(\tau^{-N})$ terms come from the fact that our sums go only up to $N-1$ while to get a normalized Poisson distribution it has to go to ∞ to cancel $e^{-\tau}$. But in any case Thus:

$$
d_a = \int_0^L d\tau G(\tau)\frac{\tau^N e^{-\tau}}{N!} = \frac{\partial_\lambda^N}{N!}\int_0^L d\tau G(\tau)e^{-\lambda\tau}\bigg|_{\lambda\to 1}
$$

Which claims that the degree of node a (and thus any other node) is on average given through a derivative of the Laplace transform of the Green's function. The Laplace transform

The initial distribution of the nodes on the circle was assumed to be uniform with density $\lambda = N/L$. For a given node a at t_a, the probability of finding another node b at t_b in the interval $t_b \in [t', t'+dt_b]$ is λdt. The weighting of the degrees comes from the Green's function $d_{ab} = G(t_b - t_a)$ and so the total degree from the nodes in this interval is number degree from a to b is given by:

$$
P_b d_{ab} = \frac{N}{L}G(t_{ab})dt_b
$$

Now to find all pairs of nodes that give this degree, we need to integrate over the position of a, but in order to look for the ones that have exactly $k_{ab} = NG(t_{ab})/L$ as the weight we want to look for the ones where $t_{ab} = \tau$ is fixed, thus having a $\delta(t_{ab} - \tau)$ in them. The density of the degree $NG(\tau)/L$ is then:

$$
\begin{aligned}
\rho(\tau) &= \frac{1}{N}\frac{N}{L}\int k_{ab}\delta(t_{ab}-\tau)dt_a dt_b \\
&= \frac{1}{L}\int k_{ab}\delta(t_{ab}-\tau)dt_{ab}dt \\
&= \int k_{ab}\delta(t_{ab}-\tau)dt_{ab}
\end{aligned} \tag{57}
$$

We may also write the delta function in an appropriate basis (assuming $L \to \infty$ we convert the Fourier series with frequencies $k_n = 2n\pi/L$ to a Fourier transform):

$$
\begin{aligned}
\rho(\tau) &= \frac{N}{L}\int G(t)\delta(t-\tau)dt \\
&= \frac{N}{L}\int \frac{dk}{2\pi}\int G(t)e^{ik(t-\tau)}dt \\
&\equiv \int \phi_\tau(k)dk \\
\phi(k) &= \int \frac{N^2}{L}G(t)e^{ik(t-\tau)}\frac{dt}{2\pi}
\end{aligned} \tag{58}
$$

10.4 Moments of the degree distribution

The simulations show that similar to the Poisson distribution the distribution becomes wider as the mean grows larger. So the natural question to ask is: Is the distribution Poisson? Checking the second moment, however, reveals that it's not. The variance is orders of magnitude smaller than the mean.

10.5 Massive vs massless

In d dimensions the Green's function of the Euclidean theory with action $\int d^d x(|\partial\phi|^2 + m^2|\phi|^2)$ in $d \geq 2$ and $m \neq 0$ is given by:

$$
\begin{aligned}
G(x;0) &= \int \frac{d^d p}{(2\pi)^d}\frac{e^{ip\cdot x}}{p^2+m^2} \\
&= \int \frac{d\Omega_{d-2}p^{d-1}dpd\cos\theta}{(2\pi)^d}\frac{e^{ipx\cos\theta}}{p^2+m^2} \\
&= \frac{\Omega_{d-2}}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{p^{d-1}dpe^{ipx}}{px(p^2+m^2)} \\
&\sim \frac{\Omega_{d-2}}{2(2\pi)^{d-1}}m^{d-3}\frac{e^{-mx}}{x}
\end{aligned} \tag{59}
$$

Which has the form of a familiar Yukawa potential in any dimension. The average degree obtained from this correlation function is then:

$$
\begin{aligned}
\langle k\rangle &= \frac{N}{L^d}\int d^d x G(x;0) \\
&= \frac{N}{L^d}\frac{\Omega_{d-2}\Omega_{d-1}}{2(2\pi)^{d-1}}m^{d-3}\int x^{d-2}dxe^{-mx} \\
&= \frac{N}{L^d}\frac{\Omega_{d-2}\Omega_{d-1}}{2(2\pi)^{d-1}}\Gamma(d-2)m^{-2}
\end{aligned} \tag{60}
$$

Which again depends on m^{-2}.

The massless case, however, has a drastically different behavior. When $m = 0$ we no longer have the two distinct poles $\pm im$. Instead the Laplace transform Green's function reduces to:

$$G(x;0) = \frac{\Omega_{d-2}}{(2\pi)^d} 2 \int_0^\infty \frac{p^{d-3} dp e^{-px}}{px}$$
$$= \frac{\Omega_{d-2}}{(2\pi)^d} \frac{2\Gamma(d-3)}{|x|^{d-2}} \tag{61}$$

Therefore The network which is literally A_{ij} then has the following weighted degree distributions. As we see in figure 2 the distribution, though still similar to a Poisson distribution, does not match Poisson as perfectly as before.

Figure 2: Left: Degree distributions for continuous network in the 1D theory close to zero (only nodes very close to each node are connected.) Right: more detailed plot of the same region with more nodes. The distribution doesn't fit Poisson distributions as nicely as in the discrete model. Especially, the peaks are a lot sharper.

Since the distribution of the points

We can use a number of tricks to find $P(k)$ which make the relation to Laplace and Fourier transform explicit. First we will use $1 = x \int_0^\infty dk e^{-kx}$:

$$\langle k \rangle = \int_0^\infty dk \int x d^d x S(x) e^{-kx}$$

But our degrees in this generalized network may also be negative and we must do the changes such that k can run from $-\infty$ to ∞. And so:

$$P(k) = \frac{1}{k} \int x d^d x S(x) e^{-kx} \tag{62}$$

This can be expressed in one higher dimension as an integral over $\cos\theta$ where θ is the angle between the two vectors $\vec{k}$ and $\vec{x}$ (NOT QUITE!!!):

$$\langle k \rangle = \int_0^\infty k dk \int x d^{d+1} x S(x) e^{-kx\cos\theta}$$

We have assumed homogeneity.

take $S(x)$ and integrate it over the Since according to [44] this has two delta functions, we can define this using two integrals:

$$\langle k \rangle = \int \int d^d x d^d y \frac{\delta^2 \ln Z}{\delta J(x) \delta J(y)}$$

Now in order to define a generating function, we need to express this in terms of some "suitable" integral transformation. Fact is that only the equation for $\langle k \rangle$ is not sufficient for defining $G(k)$ uniquely. Clearly we would need information about higher moments as well.

We will now choose a generating function which defines “a” distribution which gives the correct form for $\langle k\rangle$. What it yields for higher moments $\langle k^n\rangle$ is very familiar from the field theory perspective, but whether or not these moments would make sense from the networks' perspective needs yet to be examined.

Thus if we find a description of $\langle k\rangle$ which looks like $\int kdkP(k)$ we can read off the distribution function. We know that the Laplace (or Fourier) transform of the any function would yield a kind of distribution for the moments (or spectrum in case of Fourier.) Thus the distribution function $P(k)$ should be related to the Fourier transform of $\langle\phi_i\phi_j\rangle$. With some abuse of notation, the Fourier transform of the 2-pt function is:

$$\langle k\rangle = \frac{1}{N}\sum_{i,j}\langle\phi_i\phi_j\rangle = \frac{1}{N}\sum_{i,j}\int\frac{d^dk}{(2\pi)^d}\left\langle\phi(\vec{k})\phi(-\vec{k})\right\rangle e^{i\vec{k}\cdot(x_i-x_j)} \tag{63}$$

Now we can use the random nature of x_i to conclude that[5]:

$$\sum_{i,j} e^{i\vec{k}\cdot(\vec{x}_i-\vec{x}_j)} \approx N\sum_j e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_j)} \approx N\int d^dx e^{i\vec{k}\cdot\vec{x}} \tag{64}$$

Thus:

$$\begin{aligned}
\langle k\rangle &\approx \int\frac{d^dk}{(2\pi)^d}\left\langle\phi(\vec{k})\phi(-\vec{k})\right\rangle\int d^dx e^{i\vec{k}\cdot\vec{x}} \\
&= \int\frac{d^dk}{(2\pi)^d}\left\langle\phi(\vec{k})\phi(-\vec{k})\right\rangle\delta(|k|) \\
&= \int\frac{d^dk}{(2\pi)^d}\mathrm{Im}\left[\left\langle\phi(\vec{k})\phi(-\vec{k})\right\rangle\right]
\end{aligned} \tag{65}$$

So, in this representation, if the moment frequencies $\vec{k}$ could be related to the degree k_i of the nodes, the degree distribution will be related to the “density of states”, which is the imaginary part of the We wish to make this expression similar to $\langle k\rangle = \partial_w G(w)|_{w\to 0}$ in order to read off the distribution function $P(k)$. We start by $x\to x-x_0$ and to use x_0 as a means of generating powers of k. Then we take the $d\cos\theta$ integral with θ being the angle between $\vec{k}$ and $(\vec{x}-\vec{x}_0)$.

$$\begin{aligned}
&= \int x^{d-1}dxd\Omega^{d-2}d\cos\theta e^{ikx\cos\theta} \\
&= \int x^{d-2}dxd\Omega^{d-2}\frac{e^{ikx}-e^{-ikx}}{ik} \\
&\approx N\int d^dx(2\pi)^d\delta(kx) = \frac{N(2\pi)^d}{k^d}
\end{aligned} \tag{66}$$

Thus identifying $w\sim(x_i-x_j)$ we find that:

$$P(k)\sim\frac{\mathrm{Vol}_{d-1}k^{d-2}}{(2\pi)^d}\langle\phi(k)\phi(-k)\rangle \tag{67}$$

[5]This is done in Euclidean signature, but the generalization to other signatures is straightforward.

Where Vol_{d-1} is the volume of the homogeneous $d-1$ dimensional subspace[6]. Thus, up to k^{d-2}, the distribution function becomes the 2pt function in momentum space. Therefore, for example, a degree cutoff becomes a momentum (or energy scale) cutoff in this description. Also, if the degree distribution becomes a power law distribution, the exponent will be related to the anomalous dimension of the propagator because of this.

10.6 Example and Motivation: Granger causality and time series correlation network

A type of network which may resemble field theory is the network formed by evaluating correlations between time series data for stocks. If we calculate an equal time correlation of these stock data, the network is:

$$A_{ij} = \langle \phi_i \phi_j \rangle = \frac{1}{T} \int_0^T dt \phi_i(t) \phi_j(t)$$

But sometimes there is a time lag between when one stock changes until its effects reach another stock. Therefore, to find causal relations, it is useful to calculate correlations with a time difference (Granger causality.)

$$A_{ij}(\tau) \sim \langle \phi_i(0) \phi_j(\tau) \rangle = \frac{1}{T} \int_0^T dt \phi_i(t) \phi_j(t+\tau)$$

We can imagine that the nodes of this network are spread over a space and put the indices i, j on the coordinates:

$$A_{ij}(\tau) \sim \langle \phi_i(0) \phi_j(\tau) \rangle = \frac{1}{T} \int_0^T dt \phi(x_i; t) \phi(x_j; t+\tau)$$

Because the effects may take some time to "propagate" or "diffuse" from one point to another. This is like having a field ϕ which is space and time dependent, while the kind of adjacency matrix we are interested in does not involve integration over space. Or, one might say, that ϕ does not really "propagate" in the space. This makes sense, because obviously the value of the correlation $A_{ij}(\tau)$ not only depends on τ but also the nodes themselves, and hence x_i, x_j. If there existed some geometric representation of the network in some space, then the value of A_{ij} might have been inferable from the points x_i, x_j.

The clues from above will serve as guidelines for our calculations here. Specifically, we want:

1) A geometric network model where the position and metric distance of nodes is related to the connectivity or links between the nodes.

Thus we will look for network models which make use of geometry and metric of the space in forming the network.

[6] This depends on geometry. In a Euclidean space this would be some spherical shell, but in hyperbolic spaces it is usually infinite.

11 Generators of bulk symmetry and their boundary counterparts

11.1 Homogeneous Euclidean space-time

11.2 Symmetries of the space vs symmetries of action

11.3 Time translation and scaling

12 Solutions at the boundary

12.1 Weight of the fields and their correlations

12.2 Interactions

12.3 Renormalization group flow and the anomalous dimension

13 Classical and "Quantum" Limits

13.1 Classical causal Lorentz model

13.2 Classical diffusion model

13.3 Quantum regime

14 Relating a network to a field theory

Consider an undirected weighted network with N nodes. The connections of the network can be represented by the "Adjacency matrix", A, which in this case is a symmetric matrix (if we allow negative weights for the adjacency matrix, it will not be positive semi-definite as is usually the case in networks.) But in order to derive some other properties for the network, such as various centrality measures for the nodes, we sometimes need to attribute numbers to the nodes themselves. For these calculations, node i may get assigned a number ϕ_i. The adjacency matrix then propagates these numbers along the edges of the network based on the weights of the edges.

14.1 Field correlation, scattering matrix and adjacency matrix

In some networks, such as time-series correlation network in stock markets, the adjacency matrix is constructed out of correlations in some functions attributed to the nodes. Say we denote the time-series associated with historical stock data of node i by $\phi_i(t)$. The adjacency matrix then is constructed through:

$$A_{ij} = \int dt \phi_i(t)\phi_j(t) \equiv \langle \phi_i \phi_j \rangle$$

This is a two point correlation function in a one dimensional (t being its sole dimension) field theory. In this sense the adjacency matrix is actually the "scattering matrix":

$$A : \text{Adjacency matrix} \sim \text{Scattering matrix}$$

14.2 Causality as a network

One case in which this formulation can give rise to undirected unweighted networks is when the fields ϕ are like "classical point like particles" with delta function distributions. If there would be some sort of a kinetic term which would generate correlations $\langle \phi_i \phi_j \rangle$...

For example if we wanted to reconstruct the causal set network, in which points which fall into one point's future light cone are connected to that point we could use the familiar relativistic dynamics of massive fields. $\langle \phi_i \phi_j \rangle$ is the propagator and if the fields are massive, it will have a nonzero value only if the fields fall into each other's light cone in the classical regime. Let's briefly recall how causality worked for scalar Klein-Gordon fields in Lorentzian signature. Let the metric be of signature $(-, +, ...)$ so that:

$$x^2 = -x_0^2 + x_i^2$$

The free part of the action of real fields is:

$$S = \int d^d x \phi(\hbar^2 \partial^2 - m^2)\phi$$

The propagator can be calculated in momentum space

$$\begin{aligned} x^2 < 0 \rightarrow \langle \phi_i(x)\phi_j(0) \rangle &= \int \frac{d^d p}{(2\pi\hbar)^d} \frac{e^{ip\cdot x/\hbar}}{p^2 + m^2} \\ &= \int \frac{d^{d-1} p}{(2\pi\hbar)^{d-1}} \frac{e^{ip\cdot x/\hbar}}{2\sqrt{p_i^2 + m^2}} \\ &= \frac{Vol_{d-2}}{(2\pi\hbar)^{d-1}} \int_m^\infty dE \left(E^2 - m^2\right)^{(d-1)/2} e^{-iEt/\hbar} \\ &\sim e^{-imt/\hbar} \end{aligned} \tag{68}$$

Which follows from the fact that since x is time-like we can transform it such that it becomes the temporal direction. This amplitude is non-vanishing and just states that it is possible to propagate from one point to any other point inside the future[7] light cone. But when the separation x is space-like we can change the basis such that $x' = (0, r(...))$. This time we

[7] or past, depending on whether we use retarded or advanced Green's functions through the small imaginary part $\pm i\varepsilon$ which we omitted here.

get:

$$
\begin{aligned}
x^2 < 0 \to \langle \phi_i(x)\phi_j(0)\rangle &= \delta_{ij}\int \frac{d^{d-1}p}{(2\pi\hbar)^{d-1}}\frac{e^{ip\cdot x/\hbar}}{2\sqrt{p_i^2+m^2}} \\
&= \int_0^\infty \frac{p^{d-2}dpd\Omega_{d-2}}{(2\pi\hbar)^{d-1}}\frac{e^{-ipr\cos\theta/\hbar}}{2\sqrt{p^2+m^2}} \\
&= i\int_{-}\infty^\infty \frac{p^{d-3}dpd\Omega_{d-3}}{(2\pi\hbar)^{d-1}}\frac{e^{-ipr/\hbar}}{2r\sqrt{p^2+m^2}} \\
&\sim e^{-mr/\hbar}
\end{aligned}
\tag{69}
$$

Where in the last line we perform a contour integral[8]. As we can see this function is decaying exponentially outside of the light cone and in the classical limit $\hbar \to 0$ this goes to zero. Therefore the classical limit of this theory may only have interactions which happen entirely inside the light-cone.

[8] $\sqrt{p^2+m^2}$ has imaginary branch cuts and if we close the contour from above we it will wrap around the top branch and we can perform the integral by changing $p \to ip$