

A Field-theoretic networks model

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1 Fundamentals

Our goal is to construct networks out of locally interacting fields inside some space. These fields could represent agents moving or communicating in real space-time or inside some parameter space which characterise their personality and interests.

For technical simplicity, we begin with networks put on familiar manifolds, endowed with some [pseudo]-Riemannian metric¹.

1.1 The construction

We will assume that nodes (agents) are sources, J_i which are concentrated around points x_i . They emanate a field ϕ ². The corresponding term in the action is assumed to be the minimal coupling term $\int J_i \phi$. This field ϕ could represent the extent to which an agent moves from its source J_i in real space or in parameter space how much an agent is willing to explore or tolerate values of parameters (e.g. political views) other than its own mean.

1.1.1 The Adjacency Matrix

The most natural choice for the Adjacency matrix will then be something based on the correlation between the fields emanated from two different sources:

$$A_{ij} \sim \langle \phi(x_i, t_i) \phi(x_j, t_j) \rangle$$

We will comment on the choice of times t_i, t_j later. For now, what matters is that the normalized two-point function could be used as an ensemble-averaged probability of having a link.

1.1.2 Action and Partition Function

We will first assume a real-time field theory (not a finite-temperature Gibbs ensemble.) We start with a mainly quadratic action with a perturbation which allows fields of various agents

¹Later we will explore the possibility of making the “base space” a network itself, or removing the metric and working with a topological theory.

²For now we will assume a single ϕ . Subsequently we may explore having multiple ϕ components as well as large N $O(N)$ models.

to interact. Since we have assumed that all agents emanate the same ϕ , we may start with an action of the following form:

$$S = \int d^{(n+1)}x \left(\phi^\dagger D \phi + \sum_i \phi^\dagger J_i + h.c. \right) \quad (1)$$

Where D is a linear operator such as $\partial_t^2 - \vec{\partial}^2 + m^2$ if we consider a relativistic *phi* of mass m , or $\partial_t + D\vec{\partial}^2 + \mu$ in the case of diffusion, with diffusion constant D and loss μ , or any other type of linear operator. We will mostly focus on the case of diffusion.

The partition function of a quadratic theory may be evaluated using Gaussian integration, though caution needs to be exercised regarding possible zero modes of the action. these may be taken care of by either calculating the zero mode contributions separately or by regularizing the action by adding an appropriate $i\varepsilon$. For simplicity, suppose we have a real field $\phi = \phi^\dagger$. Schematically, the partition function becomes ($J \equiv \sum_i J_i$):

$$\begin{aligned} Z &= \int [d\phi] \exp \left[i \int (\phi D \phi + J \phi) \right] \\ &= \det(D/\pi)^{-1/2} \exp \left[i \int J D^{-1} J / 4 \right] \end{aligned} \quad (2)$$

Let us also recall that the 2-point function in quadratic theories is the Green's function. In particular, for the time ordered 2-point function:

$$\begin{aligned} \langle T[\phi(x_i)\phi(x_j)] \rangle &= \theta(t_i - t_j) \frac{\delta}{\delta J_i} \frac{\delta}{\delta J_j} \ln Z + (i \leftrightarrow j) \\ &= \frac{1}{4} \theta(t_i - t_j) D^{-1}(x_i - x_j) + (i \leftrightarrow j) \\ &= G_F(x_i - x_j; t_i - t_j) \end{aligned} \quad (3)$$

With G_F being the Feynman propagator, or the time-ordered Green's function.

2 Network of Random Walkers

Let us now work out a specific example. Consider the case where the nodes are sources for agents that do a random walk in an n -D Euclidean space. The spread of the density of these random walkers can be represented by a heat (diffusion) equation. The action is given by:

$$S = \int d^{(n+1)}x \left(\phi(\partial_t + D\vec{\partial}^2 + \mu)\phi + J\phi \right) \quad (4)$$

Here ϕ is related to the density of the random walkers. We are assuming that the sources $J = \sum_i J_i$ are static and need to specify how they are distributed over space.

We will first assume the sources are localized as delta functions:

$$J_i(x) = J_0 \delta(x - x_i)$$

and that they are uniformly spread in space. The ensemble average network is made of the time-ordered Green's functions between the sources:

$$A_{ij} \sim \langle T[\phi(x_i)\phi(x_j)] \rangle = \theta(t_i - t_j) \frac{\exp\left[-\frac{(x_i - x_j)^2}{4D(t_i - t_j)}\right]}{(4\pi D(t_i - t_j))^{n/2}} + (i \leftrightarrow j)$$

But there is also a time involved in the diffusion process. We will look at two scenarios: 1) one agent diffuses from x_i and gets to the source at x_j ; 2) two agents start from x_i and x_j and they meet at a random point in space.

First look at the case where all fields from all sources diffuse for a fixed amount of time t and then we calculate the probability of the field starting from x_i reaching x_j within time t .

$$\begin{aligned} A_{ij}^1(t) &= \langle \phi(x_i, t)\phi(x_j, 0) \rangle + \langle \phi(x_j, t)\phi(x_i, 0) \rangle \\ &= 2\theta(t) \frac{\exp\left[-\frac{(x_i - x_j)^2}{4Dt}\right]}{(4\pi Dt)^{n/2}} \end{aligned} \quad (5)$$

In the other case two agents start at the same time from x_i and x_j , both travelling for a time t and then meet somewhere in space. This time we have two propagators meeting at a random point y , which we need to integrate over:

$$\begin{aligned} A_{ij}^2(t) &= \int d^n y \langle \phi(x_i, 0)\phi(y, t) \rangle \langle \phi(x_j, 0)\phi(y, t) \rangle \\ &= \frac{\theta(t)}{(4\pi Dt)^n} \int d^n y \exp\left[-\frac{(x_i - y)^2}{4Dt} - \frac{(x_j - y)^2}{4Dt}\right] \\ &= \frac{\theta(t)}{(4\pi Dt)^n} \exp\left[-\frac{(x_i - x_j)^2}{8Dt}\right] \int d^n y \exp\left[-\frac{\left(\frac{x_i + x_j}{2} - y\right)^2}{2Dt}\right] \\ &= \frac{\theta(t)}{(4\pi D(2t))^{n/2}} \exp\left[-\frac{(x_i - x_j)^2}{4D(2t)}\right] \end{aligned} \quad (6)$$

This is exactly the the same expression as the previous case where only one agent travelled and eventually found the source of another agent, the only difference is that one agent needs to travel twice the time that two agents needed to cross each other at a random point, which is intuitively correct:

$$A_{ij}^2(t) = A_{ij}^1(2t)$$

2.1 Degree Distribution

The ensemble average of the degree k_i of each node i is easily found by integrating over all other nodes. This immediately makes it clear that in this symmetric setting with uniform distribution of nodes, all nodes will more or less have the same degree, plus possible Poisson-like fluctuations, which exist in any finite sample from a uniform distribution. Let us assume that we have N nodes in a space of volume L^n , and define the node density as $\lambda \equiv N/L^n$. The average degree of i is:

$$\langle k_i(t) \rangle = \lambda \int d^n x_j A_{ij}^1(t) = \lambda$$

And since it does not depend on x_i or t the distribution can only depend on local fluctuations which make for a Poisson distribution. Recall that in a Poisson distribution the mean and standard deviation are equal and thus we expect:

$$\langle k_i \rangle = \sigma(k_i) = \lambda$$

ACTUALLY CALCULATE!!!

3 Breaking the Spatial Symmetry

One evident way of avoiding the highly symmetrical case above which basically generated an analog of and Erdős-Renyí network is to prepare a configuration in which the degree k_i would depend on x_i . This means that we need to break the spatial symmetry, either by imposing a different distribution of nodes in space, or by introducing a space-dependent interaction term.

3.1 Nima 06/25/14

3.2 Space-dependent Interaction Term

We want to have add a space-dependent potential to the action (4) to break the spatial symmetry and see if we can get a different degree distribution. We want the interaction to be of a type that would connect two fields ϕ emanated from different sources. So, naturally, it should contain ϕ^2 . Take the following action now (we set $D = 1$, because it can be absorbed as a length scale):

$$S = \int d^{(n+1)}x \left(\phi(\partial_t + \vec{\partial}^2 + \mu)\phi + J\phi + \lambda(x)\phi^2(x) \right) \quad (7)$$

If we calculate the partition function for this action and try to evaluate the effect of this potential we see that it effectively introduces a space-dependent mass term. The Green's functions will be modified to the following in the Fourier basis:

$$G(x, t_x; y, t_y) \sim \int \frac{d\omega d^n k}{(2\pi)^{n+1}} \frac{e^{i\omega(t_x - t_y) + ik \cdot (x - y)}}{i\omega - k^2 + \mu + \lambda(x)} + c.c. \quad (8)$$

But since there is this dependence on x the form of the retarded and advances Green's functions will differ in whether $\lambda(x)$ comes from the starting or the end point:

$$\begin{aligned} G_R(x, t_x; y, t_y) &= \theta(t_y - t_x) \int \frac{d\omega d^n k}{(2\pi)^{n+1}} \frac{e^{i\omega(t_x - t_y) + ik \cdot (x - y)}}{i\omega - k^2 + \mu + \lambda(y) + i\varepsilon} \\ G_A(x, t_x; y, t_y) &= \theta(t_x - t_y) \int \frac{d\omega d^n k}{(2\pi)^{n+1}} \frac{e^{i\omega(t_x - t_y) + ik \cdot (x - y)}}{i\omega - k^2 + \mu + \lambda(x) - i\varepsilon} \end{aligned} \quad (9)$$

(Check PLUS/MINUS SIGNS!!) If $\lambda(x)$ was small, it would basically be a small perturbation and the Green's function could be written as the expansion:

$$G_R(x, t_x; y, t_y) = \theta(t_y - t_x) \int \frac{d\omega d^n k e^{i\omega(t_x - t_y) + ik \cdot (x - y)}}{(2\pi)^{n+1}} \frac{1}{i\omega - k^2 + \mu} \sum_{j=1}^{\infty} \frac{(-1)^j \lambda(y)}{i\omega - k^2 + \mu} \quad (10)$$

If we only keep the first order perturbation for when $\lambda(y) \ll \mu$ we have:

$$G_R(x, t_x; y, t_y) \approx \theta(t_y - t_x) \int \frac{d\omega d^n k e^{i\omega(t_x - t_y) + ik \cdot (x - y)}}{(2\pi)^{n+1}} \times \frac{1}{i\omega - k^2 + \mu} \left(1 - \frac{\lambda(y)}{i\omega - k^2 + \mu} \right) \quad (11)$$

This can be rewritten in terms of two propagators by inserting a Dirac delta function:

$$\begin{aligned} \frac{\lambda(y)}{i\omega - k^2 + \mu} &= \lambda(y) \int \frac{d\omega' d^{n-1} k'}{(2\pi)^n} \int \frac{dt_z d^n z e^{i(\omega' - \omega)t_z + i(k' - k) \cdot z}}{(2\pi)^{n+1}} \frac{1}{i\omega' - k'^2 + \mu} \\ &= \int \frac{dt_z d^n z e^{-i\omega t_z - ik \cdot z}}{(2\pi)^{n+1}} \lambda(y) G_R^0(z, t_z) / \theta(t_z) \end{aligned} \quad (12)$$

Where G_R^0 is the unperturbed Green's function which only depends on the difference between coordinates. If we combine the exponents of this one and use the $\delta^{n-1}(k - k')\delta(\omega - \omega')$ that's implicitly in the integral to replace $\exp[i\omega t_y + ik \cdot y] \rightarrow \exp[i\omega' t_y + ik' \cdot y]$ we are able to write the perturbed Green's function as the product of two unperturbed ones, if $t_y > t_z > t_x$ (which makes yields $\theta(t_y - t_x) = \theta(t_y - t_z)\theta(t_z - t_x)$), setting $\Delta x = x - y, \Delta t = t_x - t_y$:

$$\begin{aligned} G_R(x, t_x; y, t_y) &= G_R^0(\Delta x, \Delta t) - \int \frac{dt_z d^n z}{(2\pi)^{n+1}} \lambda(y) G_R^0(\Delta x - z, \Delta t - t_z) G_R^0(z, t_z) \\ &= G_R^0(\Delta x, \Delta t) - \int \frac{dt_z d^n z}{(2\pi)^{n+1}} \lambda(y) G_R^0(x - z, t_x - t_z) G_R^0(z - y, t_z - t_y) \\ &= G_R^0(\Delta x, \Delta t) (1 - \lambda(y)) \end{aligned} \quad (13)$$

(Take care of the dimensionality!! λ is dimensionful, but the integration and G_R^0 should absorb its dimensions. However, this is lost once $G_R^0 G_R^0 \rightarrow G_R^0$. Some dimensionful factors must either emerge from this conversion, or everything should be made dimensionless.) Which is because as we showed before the product of two consecutive unperturbed propagators G_R^0 is just a single propagation. A very similar thing happens for the advanced Green's function G_A .

3.2.1 The Perturbed Adjacency Matrix

Again, consider the two scenarios: A_{ij}^1 for direct walk from $i \rightarrow j$ or $j \rightarrow i$; A_{ij}^2 for two random walks starting at the same time and meeting somewhere. In the first case we just have the time-ordered Green's function:

$$\begin{aligned} A_{ij}^1 &= G_T(x_i, t_i; x_j, t_j) = G_A + G_R \\ &= \theta(t_y - t_x) \int \frac{d\omega d^n k e^{i\omega(t_x - t_y) + ik \cdot (x - y)}}{(2\pi)^{n+1}} \\ &\quad \times \left[\frac{\theta(t_y - t_x)}{i\omega - k^2 + \mu + \lambda(x)} + \frac{\theta(t_x - t_y)}{i\omega - k^2 + \mu + \lambda(y)} \right] \end{aligned} \quad (14)$$

3.2.2 Simultaneous Random-walkers

For the other one, A_{ij}^2 we have two retarded Green's functions only, meeting at some random point in space. If we just take the full Green's function and expand in terms of $\lambda(x)$ all such terms necessarily have one of the endpoints in them. For when $\lambda(x) \ll \mu$ we can keep up to first order and we have:

$$\begin{aligned} A_{ij}^2 &= \int \frac{dt_z d^n z}{(2\pi)^{n+1}} G_R(x-z, -t) G_R(y-z, -t) \\ &\approx \int \frac{dt_z d^n z}{(2\pi)^{n+1}} G_R^0(x-z, -t) G_R^0(y-z, -t) (1 - \lambda(z))^2 \\ &\approx \int \frac{dt_z d^n z}{(2\pi)^{n+1}} G_R^0(x-z, -t) G_R^0(y-z, -t) (1 - 2\lambda(z)) \end{aligned} \quad (15)$$

As a simple example, consider the case where $\lambda(x) = \lambda \delta^{n-1}(x)$:

$$A_{ij}^2 \approx G^0(x-y, 2t) - 2\lambda G_R^0(x, -t) G_R^0(y, -t) \quad (16)$$

3.3 Nima 06/26/14

3.4 Adjacency Matrix from the Path Integral

Firstly, let us note that when $\mu = 0$, there is a priori no scale to define a perturbative expansion for $\lambda(x)$ because ω, k may have any values and expansion in $\lambda(x)$ becomes invalid when $i\omega + k^2 < \lambda(x)$. In this case A_{ij}^2 may not be written in terms of G_R^0 and the full G_R needs to be calculated. This is because even a small mass $\lambda(x)$ will avoid an IR divergence.

3.4.1 Possible Resolutions

The problem above shows that a perturbative expansion in term of $\lambda(x)$ is not valid when $\mu = 0$. This is basically saying that when there is a single field ϕ all orders in $\lambda(x)$ play a role in the result of diffusion from one point to another. There is no way to separate the interactions with $\lambda(x)$ from regular diffusion in space and it has the same effect as a space-dependent mass (which is a loss term in diffusion.)

The other issue is, the $\lambda(x)$ term is a perturbation and should be smaller in order than the zeroth order, direct propagation. This means that in this model if the $\lambda(x)$ term takes over, it is already too big to be considered a perturbation. So we need to modify the model if we want $\lambda(x)$ to play a dominant role.

3.5 Nima 06/27/14

3.5.1 Large N and Off-diagonal Interactions

Now consider the case where each source J_i is emitting its own unique “agent field” ϕ_i . Take the unperturbed action to be:

$$S = \int d^{(n+1)}x \sum_i \left(\phi_i (\partial_t + \vec{\partial}^2 + \mu) \phi_i + J_i \phi_i + \sum_j \lambda_{ij}(x) \phi_i(x) \phi_j(x) \right) \quad (17)$$

This way different agents can only talk to each other through the interaction term $\lambda_{ij}(x)$. The Green's functions $G_{ij} \sim \langle \phi_i \phi_j \rangle$ are modified to:

$$G_{ij}(x, t_x; y, t_y) \sim \int \frac{d\omega d^n k e^{i\omega(t_x - t_y) + ik \cdot (x - y)}}{(2\pi)^{n+1}} ((i\omega - k^2 + \mu)I + \lambda(x))_{ij}^{-1} + c.c. \quad (18)$$

Where I is the identity matrix. Thus, when $i \neq j$ it will be only the $\lambda_{ij}(x)$ terms which make up the adjacency matrix. For example, in the case of two simultaneous random-walkers the above expression gets an expansion of the form:

$$\begin{aligned} A_{ij}^{(2)} &= \sum_k \int \frac{d^{n+1}z}{(2\pi)^{n+1}} G_{Rik}(x_i - z, -t) G_{Rjk}(x_j - z, -t) \\ &\approx \int \frac{dt_0 d^n z_0}{(2\pi)^{n+1}} G_R^0(x - z_0, t_0 - t) \\ &\times \sum_{i=1}^{\infty} \prod_{k=0}^i \int \frac{d^{n+1}z_k}{(2\pi)^{n+1}} \lambda_{l_k l_{k+1}}(z_k) G_R^0(z_{k-1} - z_k, t_{k-1} - t_k) \\ &\times G_R^0(y - z_{\infty}, t_{\infty} - t) \end{aligned} \quad (19)$$

The problem again will be that if the loss term μ vanishes, there won't be any scale to compare λ against to see whether a perturbative expansion is valid or not.

3.5.2 IR Cutoff

One way to resolve the problem with perturbation on λ is to have an IR cutoff. For instance, if the random walkers only walk for a certain time, say each random walk is a day long, we will have a natural cutoff for t , which in turn means cutoff for ω . If we denote this "day" as t_{\max} , the natural IR cutoff will be:

$$i\omega_{IR} \sim \frac{1}{t_{\max}} \quad (20)$$

This way, if $\lambda \ll i\omega_{IR}$ we can use perturbation theory for λ .

Suppose we have such a weak interaction and that the loss (mass term) $\mu = 0$. In this case we may keep only up to the first order terms.

$$A_{ij}^{(2)} \approx \delta_{ij} G^0(x_i - x_j, 2t) - \int \frac{d^n z}{(2\pi)^n} \lambda_{ij}(z) G_R^0(x_i - z, -t) G_R^0(x_j - z, -t) \quad (21)$$

As we see, for $i \neq j$, only the λ term contributes to lowest order and the zeroth order is absent. Thus in this model, even though $\lambda < i\omega_{IR}$ may be small, the λ term is what forms the network because it is the lowest order term.

As a simple example, consider the case where $\lambda(x) = \lambda \delta^{n-1}(x)$:

$$A_{ij}^2 \approx G^0(x - y, 2t) - 2 \int \frac{d^n z}{(2\pi)^n} \lambda G_{Rjk}^0(x_i - z, -t) G_{Rjk}^0(x_j - z, -t) \quad (22)$$

4 Nima 06/16/2014

Each of us who makes an edit can add a new section like this with the date.