A Field-theoretic networks model

October 3, 2014

Contents

1 Fundamentals

Our goal is to construct networks out of locally interacting fields inside some space. These fields could represent agents moving or communicating in real space-time or inside some parameter space which characterizes their personality and interests.

For technical simplicity, we begin with networks put on familiar manifolds, endowed with some [pseudo]-Riemannian metric[1].

1.1 The construction

We will assume that nodes (agents) are sources, J_i which are concentrated around points x_i. They emanate a field ϕ[2]. The corresponding term in the action is assumed to be the minimal coupling term $\int J_i\phi$. This field ϕ could represent the extent to which an agent moves from its source J_i in real space or in parameter space how much an agent is willing to explore or tolerate values of parameters (e.g. political views) other than its own mean.

1.1.1 The Adjacency Matrix

The most natural choice for the Adjacency matrix will then be something based on the correlation between the fields emanated from two different sources:

$$A_{ij} \sim \langle \phi(x_i, t_i)\phi(x_j, t_j)\rangle$$

We will comment on the choice of times t_i, t_j later. For now, what matters is that the normalized two-point function could be used as an ensemble-averaged probability of having a link.

1.1.2 Action and Partition Function

We will first assume a real-time field theory (not a finite-temperature Gibbs ensemble.) We start with a mainly quadratic action with a perturbation which allows fields of various agents to interact. Since we have assumed that all agents emanate the same ϕ, we may start with an action of the following form:

$$S = \int d^{(n+1)}x \left(\phi^\dagger D\phi + \sum_i \phi^\dagger J_i + h.c.\right) \tag{1}$$

Where D is a linear operator such as $\partial_t^2 - \vec{\partial}^2 + m^2$ if we consider a relativistic *phi* of mass m, or $\partial_t + D\vec{\partial}^2 + \mu$ in the case of diffusion, with diffusion constant D and loss μ, or any other type of linear operator. We will mostly focus on the case of diffusion.

The partition function of a quadratic theory may be evaluated using Gaussian integration, though caution needs to be exercised regarding possible zero modes of the action. these

[1] Later we will explore the possibility of making the "base space" a network itself, or removing the metric and working with a topological theory.

[2] For now we will assume a single ϕ. Subsequently we may explore having multiple ϕ components as well as large N $O(N)$ models.

may be taken care of by either calculating the zero mode contributions separately or by regularizing the action by adding an appropriate $i\varepsilon$. For simplicity, suppose we have a real field $\phi = \phi^\dagger$. Schematically, the partition function becomes ($J \equiv \sum_i J_i$):

$$\begin{aligned} Z &= \int [d\phi] \exp\left[i \int (\phi D \phi + J\phi)\right] \\ &= \det(D/\pi)^{-1/2} \exp\left[i \int J D^{-1} J/4\right] \end{aligned} \tag{2}$$

Let us also recall that the 2-point function in quadratic theories is the Green's function. In particular, for the time ordered 2-point function:

$$\begin{aligned} \langle T[\phi(x_i)\phi(x_j)]\rangle &= \theta(t_i - t_j) \frac{\delta}{\delta J_i} \frac{\delta}{\delta J_j} \ln Z + (i \leftrightarrow j) \\ &= \frac{1}{4} \theta(t_i - t_j) D^{-1}(x_i - x_j) + (i \leftrightarrow j) \\ &= G_F(x_i - x_j; t_i - t_j) \end{aligned} \tag{3}$$

With G_F being the Feynman propagator, or the time-ordered Green's function.

2 Network of Random Walkers

Let us now work out a specific example. Consider the case where the nodes are sources for agents that do a random walk in an n-D Euclidean space. The spread of the density of these random walkers can be represented by a heat (diffusion) equation. The action is given by:

$$S = \int d^{(n+1)}x \left(\phi(\partial_t + D\vec{\partial}^2 + \mu)\phi + J\phi\right) \tag{4}$$

Here ϕ is related to the density of the random walkers. We are assuming that the sources $J = \sum_i J_i$ are static and need to specify how they are distributed over space.

We will first assume the sources are localized as delta functions:

$$J_i(x) = J_0 \delta(x - x_i)$$

and that they are uniformly spread in space. The ensemble average network is made of the time-ordered Green's functions between the sources:

$$A_{ij} \sim \langle T[\phi(x_i)\phi(x_j)]\rangle = \theta(t_i - t_j) \frac{\exp\left[-\frac{(x_i - x_j)^2}{4D(t_i - t_j)}\right]}{(4\pi D(t_i - t_j))^{n/2}} + (i \leftrightarrow j)$$

But there is also a time involved in the diffusion process. We will look at two scenarios: 1) one agent diffuses from x_i and gets to the source at x_j; 2) two agents start from x_i and x_j and they meet at a random point in space.

First look at the case where all fields from all sources diffuse for a fixed amount of time t and then we calculate the probability of the field starting from x_i reaching x_j within time t.

$$\begin{aligned} A^1_{ij}(t) &= \langle \phi(x_i,t)\phi(x_j,0)\rangle + \langle \phi(x_j,t)\phi(x_i,0)\rangle \\ &= 2\theta(t)\frac{\exp\left[-\frac{(x_i-x_j)^2}{4Dt}\right]}{(4\pi Dt)^{n/2}} \end{aligned} \tag{5}$$

In the other case two agents start at the same time from x_i and x_j, both travelling for a time t and then meet somewhere in space. This time we have two propagators meeting at a random point y, which we need to integrate over:

$$\begin{aligned} A^2_{ij}(t) &= \int d^n y \, \langle \phi(x_i,0)\phi(y,t)\rangle \, \langle \phi(x_j,0)\phi(y,t)\rangle \\ &= \frac{\theta(t)}{(4\pi Dt)^n}\int d^n y \exp\left[-\frac{(x_i-y)^2}{4Dt} - \frac{(x_j-y)^2}{4Dt}\right] \\ &= \frac{\theta(t)}{(4\pi Dt)^n}\exp\left[-\frac{(x_i-x_j)^2}{8Dt}\right]\int d^n y \exp\left[-\frac{\left(\frac{x_i+x_j}{2}-y\right)^2}{2Dt}\right] \\ &= \frac{\theta(t)}{(4\pi D(2t))^{n/2}}\exp\left[-\frac{(x_i-x_j)^2}{4D(2t)}\right] \end{aligned} \tag{6}$$

This is exactly the the same expression as the previous case where only one agent travelled and eventually found the source of another agent, the only difference is that one agent needs to travel twice the time that two agents needed to cross each other at a random point, which is intuitively correct:

$$A^2_{ij}(t) = A^1_{ij}(2t)$$

2.1 Degree Distribuition

The ensemble average of the degree k_i of each node i is easily found by integrating over all other nodes. This immediately makes it clear that in this symmetric setting with uniform distribution of nodes, all nodes will more or less have the same degree, plus possible Poisson-like fluctuations, which exist in any finite sample from a uniform distribution. Let us assume that we have N nodes in a space of volume L^n, and define the node density as $\lambda \equiv N/L^n$. The average degree of i is:

$$\langle k_i(t)\rangle = \lambda \int d^n x_j A^1_{ij}(t) = \lambda$$

And since it does not depend on x_i or t the distribution can only depend on local fluctuations which make for a Poisson distribution. Recall that in a Poisson distribution the mean and standard deviation are equal and thus we expect:

$$\langle k_i\rangle = \sigma(k_i) = \lambda$$

ACTUALLY CALCULATE!!!

3 Breaking the Spatial Symmetry

One evident way of avoiding the highly symmetrical case above which basically generated an analog of and Erdös-Renyí network is to prepare a configuration in which the degree k_i would depend on x_i. This means that we need to break the spatial symmetry, either by imposing a different distribution of nodes in space, or by introducing a space-dependent interaction term.

3.1 Nima 06/25/14

3.2 Space-dependent Interaction Term

We want to have add a space-dependent potential to the action (4) to break the spatial symmetry and see if we can get a different degree distribution. We want the interaction to be of a type that would connect two fields ϕ emanated from different sources. So, naturally, it should contain ϕ^2. Take the following action now(we set $D = 1$, because it can be absorbed as a length scale):

$$S = \int d^{(n+1)}x \left(\phi(\partial_t + \vec{\partial}^2 + \mu)\phi + J\phi + \lambda(x)\phi^2(x)\right) \tag{7}$$

If we calculate the partition function for this action and try to evaluate the effect of this potential we see that it effectively introduces a space-dependent mass term. The Green's functions will be modified to the following in the Fourier basis:

$$G(x, t_x; y, t_y) \sim \int \frac{d\omega d^n k}{(2\pi)^{n+1}} \frac{e^{i\omega(t_x - t_y) + ik\cdot(x-y)}}{i\omega - k^2 + \mu + \lambda(x)} + c.c. \tag{8}$$

But since there is this dependece on x the form of the retarded and advances Green's functions will differ in whether $\lambda(x)$ comes from the starting or the end point:

$$\begin{aligned} G_R(x, t_x; y, t_y) &= \theta(t_y - t_x) \int \frac{d\omega d^n k}{(2\pi)^{n+1}} \frac{e^{i\omega(t_x - t_y) + ik\cdot(x-y)}}{i\omega - k^2 + \mu + \lambda(y) + i\varepsilon} \\ G_A(x, t_x; y, t_y) &= \theta(t_x - t_y) \int \frac{d\omega d^n k}{(2\pi)^{n+1}} \frac{e^{i\omega(t_x - t_y) + ik\cdot(x-y)}}{i\omega - k^2 + \mu + \lambda(x) - i\varepsilon} \end{aligned} \tag{9}$$

(Check PLUS/MINUS SIGNS!!) If $\lambda(x)$ was small, it would basically be a small perturbation and the Green's fnction could be written as the expansion:

$$G_R(x, t_x; y, t_y) = \theta(t_y - t_x) \int \frac{d\omega d^n k e^{i\omega(t_x - t_y) + ik\cdot(x-y)}}{(2\pi)^{n+1}} \frac{1}{i\omega - k^2 + \mu} \sum_{j=1}^{\infty} \frac{(-1)^j \lambda(y)}{i\omega - k^2 + \mu} \tag{10}$$

If we only keep the first order perturbation for when $\lambda(y) \ll \mu$ we have:

$$\begin{aligned} G_R(x, t_x; y, t_y) \approx& \theta(t_y - t_x) \int \frac{d\omega d^n k e^{i\omega(t_x - t_y) + ik\cdot(x-y)}}{(2\pi)^{n+1}} \\ & \times \frac{1}{i\omega - k^2 + \mu} \left(1 - \frac{\lambda(y)}{i\omega - k^2 + \mu}\right) \end{aligned} \tag{11}$$

This can be rewritten in terms of two propagators by inserting a Dirac delta function:

$$\begin{aligned}\frac{\lambda(y)}{i\omega - k^2 + \mu} &= \lambda(y)\int \frac{d\omega' d^{n-1}k'}{(2\pi)^n}\int \frac{dt_z d^n z e^{i(\omega'-\omega)t_z + i(k'-k)\cdot z}}{(2\pi)^{n+1}}\frac{1}{i\omega' - k'^2 + \mu} \\ &= \int \frac{dt_z d^n z e^{-i\omega t_z - ik\cdot z}}{(2\pi)^{n+1}}\lambda(y) G_R^0(z, t_z)/\theta(t_z) \end{aligned} \tag{12}$$

Where G_R^0 is the unperturbed Green's function which only depends on the difference between coordinates. If we combine the exponents of this one and use the $\delta^{n-1}(k-k')\delta(\omega-\omega')$ that's implicitly in the integral to replace $\exp[i\omega t_y + ik\cdot y] \to \exp[i\omega' t_y + ik'\cdot y]$ we are able to write the perturbed Green's function as the product of two unperturbed ones, if $t_y > t_z > t_x$ (which makes yields $\theta(t_y - t_x) = \theta(t_y - t_z)\theta(t_z - t_x)$), setting $\Delta x = x - y, \Delta t = t_x - t_y$:

$$\begin{aligned} G_R(x, t_x; y, t_y) &= G_R^0(\Delta x, \Delta t) - \int \frac{dt_z d^n z}{(2\pi)^{n+1}}\lambda(y) G_R^0(\Delta x - z, \Delta t - t_z) G_R^0(z, t_z) \\ &= G_R^0(\Delta x, \Delta t) - \int \frac{dt_z d^n z}{(2\pi)^{n+1}}\lambda(y) G_R^0(x - z, t_x - t_z) G_R^0(z - y, t_z - t_y) \\ &= G_R^0(\Delta x, \Delta t)(1 - \lambda(y)) \end{aligned} \tag{13}$$

(Take care of the dimensionalty!! λ is dimensionful, but the integration and G_R^0 should absorb its dimensions. However, this is lost once $G_R^0 G_R^0 \to G_R^0$. Some dimensionful factors must either emerge from this conversion, or everything should be made dimensionless.)Which is because as we showed before the product of two consecutive unperturbed propagators G_R^0 is just a single propagation. A very similar thing happens for the advanced Green's function G_A.

3.2.1 The Perturbed Adjacency Matrix

Again, consider the two scenarios: A_{ij}^1 for direct walk from $i \to j$ or $j \to i$; A_{ij}^2 for two random walks starting at the same time and meeting somewhere. In the fisrt case we just have the time-ordered Green's function:

$$\begin{aligned} A_{ij}^1 =& G_T(x_i, t_i; x_j, t_j) = G_A + G_R \\ =& \theta(t_y - t_x)\int \frac{d\omega d^n k e^{i\omega(t_x - t_y) + ik\cdot(x-y)}}{(2\pi)^{n+1}} \\ &\times \left[\frac{\theta(t_y - t_x)}{i\omega - k^2 + \mu + \lambda(x)} + \frac{\theta(t_x - t_y)}{i\omega - k^2 + \mu + \lambda(y)}\right] \end{aligned} \tag{14}$$

3.2.2 Simultaneous Random-walkers

For the other one, A_{ij}^2 we have two retarded Green's functions only, meeting at some random point in space. If we just take the full Green's function and expand in terms of $\lambda(x)$ all such terms necessarily have one of the endpoints in them. For when $\lambda(x) \ll \mu$ we can keep up to first order and we have:

$$\begin{aligned} A_{ij}^2 &= \int \frac{dt_z d^n z}{(2\pi)^{n+1}} G_R(x-z,-t) G_R(y-z,-t) \\ &\approx \int \frac{dt_z d^n z}{(2\pi)^{n+1}} G_R^0(x-z,-t) G_R^0(y-z,-t)(1-\lambda(z))^2 \\ &\approx \int \frac{dt_z d^n z}{(2\pi)^{n+1}} G_R^0(x-z,-t) G_R^0(y-z,-t)(1-2\lambda(z)) \end{aligned} \tag{15}$$

As a simple example, consider the case where $\lambda(x) = \lambda\delta^{n-1}(x)$:

$$A_{ij}^2 \approx G^0(x-y,2t) - 2\lambda G_R^0(x,-t) G_R^0(y,-t) \tag{16}$$

3.3 Nima 06/26/14

3.4 Adjacency Matrix from the Path Integral

Firstly, let us note that when $\mu = 0$, there is a priori no scale to define a perturbative expansion for $\lambda(x)$ because ω, k may ave any values and expansion in $\lambda(x)$ becomes invalid when $i\omega + k^2 < \lambda(x)$. In this case A_{ij}^2 my not be written in terms of G_R^0 and the full G_R needs to be calculated. This is because even a small mass $\lambda(x)$ will avoid an IR divergence.

3.4.1 Possible Resolutions

The problem above shows that a perturbative expansion in term of $\lambda(x)$ is not valid when $\mu = 0$. This is basically saying that when there is a single field ϕ all orders in $\lambda(x)$ play a role in the result of diffusion from one point to another. There is no way to separate the interactions with $\lambda(x)$ from regular diffusion in space and it has the same effect as a space-dependent mass (which is a loss term in diffusion.)

The other issue is, the $\lambda(x)$ term is a perturbation and should be smaller in order than the zeroth order, direct propagation. This means that in this model if the $\lambda(x)$ term takes over, it is already too big to be considered a perturbation. So we need to modify the model if we want $\lambda(x)$ to play a dominant role.

3.5 Nima 06/27/14

3.5.1 Large N and Off-diagonal Interactions

Now consider the case where each source J_i is emitting its own unique "agent field" ϕ_i. Take the unperturbed action to be:

$$S = \int d^{(n+1)}x \sum_i \left(\phi_i(\partial_t + \vec{\partial}^2 + \mu)\phi_i + J_i\phi_i + \sum_j \lambda_{ij}(x)\phi_i(x)\phi_j(x) \right) \tag{17}$$

This way different agents can only talk to each other through the interaction term $\lambda_{ij}(x)$. The Green's functions $G_{ij} \sim \langle \phi_i \phi_j \rangle$ are modified to:

$$G_{ij}(x,t_x;y,t_y) \sim \int \frac{d\omega d^n k e^{i\omega(t_x-t_y)+ik\cdot(x-y)}}{(2\pi)^{n+1}} \left((i\omega - k^2 + \mu)I + \lambda(x)\right)_{ij}^{-1} + c.c. \tag{18}$$

Where I is the identity matrix. Thus, when $i \neq j$ it will be only the $\lambda_{ij}(x)$ terms which make up the adjacency matrix. For example, in the case of two simultaneous random-walkers the above expression gets an expansion of the form:

$$\begin{aligned}
A_{ij}^{(2)} &= \sum_k \int \frac{d^{n+1}z}{(2\pi)^{n+1}} G_{Rik}(x_i - z, -t) G_{Rjk}(x_j - z, -t) \\
&\approx \int \frac{dt_0 d^n z_0}{(2\pi)^{n+1}} G_R^0(x - z_0, t_0 - t) \\
&\times \sum_{i=1}^{\infty} \prod_{k=0}^{i} \int \frac{d^{n+1}z_k}{(2\pi)^{n+1}} \lambda_{l_k l_{k+1}}(z_k) G_R^0(z_{k-1} - z_k, t_{k-1} - t_k) \\
&\times G_R^0(y - z_\infty, t_\infty - t)
\end{aligned} \tag{19}$$

The problem again will be that if the loss term μ vanishes, there won't be any scale to compare λ against to see whether a perturbative expansion is valid or not.

3.5.2 IR Cutoff

One way to resolve the problem with perturbation on λ is to have an IR cutoff. For instance, if the random walkers only walk for a certain time, say each random walk is a day long, we will have a natural cutoff for t, which in turn means cutoff for ω. If we denote this "day" as $t_{\max}$, the natural IR cutoff will be:

$$i\omega_{IR} \sim \frac{1}{t_{\max}} \tag{20}$$

This way, if $\lambda << i\omega_{IR}$ we can use perturbation theory for λ.

Suppose we have such a weak interaction and that the loss (mass term) $\mu = 0$. In this case we may keep only up to the first order terms.

$$\begin{aligned}
A_{ij}^{(2)}(t) &\approx \delta_{ij} G^0(x_i - x_j, 2t) - \int \frac{d^n z}{(2\pi)^n} \lambda_{ij}(z) G_R^0(x_i - z, -t) G_R^0(x_j - z, -t) \\
&= \delta_{ij} A_{ij}^1(2t) - \int \frac{d^n z}{(2\pi)^n} \lambda_{ij}(z) A_{iz}^1(t) A_{jz}^1(t)
\end{aligned} \tag{21}$$

As we see, for $i \neq j$, only the λ term contributes to lowest order and the zeroth order is absent. Thus in this model, even though $\lambda < i\omega_{IR}$ may be small, the λ term is what forms the network because it is the lowest order term.

4 Concrete Examples

From now on

$$A_{ij} = A_{ij}^{(2)}(t_{\max})$$

As a simple example, consider the case where $\lambda_{ij}(x) = \lambda(2\pi)^n \delta^n(x)$. We will call this the case of a single "hub" at $x = 0$. For $i \neq j$ this results in an adjacency matrix of the form:

$$A_{ij}^{(2)}(t) \approx -\lambda G_R^0(x_i, -t) G_R^0(x_j, -t) \tag{22}$$

Although we derived this formula starting from diffusion, it actually holds for any general quadratic action for the nonperturbed part (it will only result in a different G^0.)

4.1 Degree Distribution

From this, it is actually very easy to calculate the degree and its distribution. The distribution of the nodes in space is given by the source current:

$$J(x) = \sum_i J_i(x) = \sum_{i=0}^{N} J_0(2\pi)^n \delta^n(x - x_i)$$

The degree of a node is simply given by:

$$\begin{aligned} k_i = \sum_j A_{ij} &= J_0^{-1} \int \frac{d^n z}{(2\pi)^n} (J^T(z)A)_i \\ &= J_0^{-1} G^0(x_i, -t_{\max}) \int \frac{d^n z}{(2\pi)^n} J(z) G^0(z, -t_{\max}) \end{aligned} \tag{23}$$

4.1.1 Uniform Spatial Distribution

Eq. (23) is the general form of the degree for a weakly coupled hub to lowest order. If the distribution of the nodes is uniform over space such that the re are N nodes in a volume V with

$$V \gg \langle x^2 \rangle^{n/2}$$

where $\langle x^2 \rangle$ denotes the average distance that the fields ϕ_i may travel in $t_{\max}$, then the degree reduces to:

$$\begin{aligned} k_i &= \frac{N}{V} G^0(x_i, -t_{\max}) \int_V d^n z G^0(z, -t_{\max}) \\ &= c_0 \frac{N}{V} G^0(x_i, -t_{\max}) \end{aligned} \tag{24}$$

Which clearly is only finite if the Green's function has afinite extent (the necessary condition for $\langle x^2 \rangle < \infty$ as well. This is the first important result in the model, namely that the degree distribution will only depend on the propagation from a point to the hubs. As long as $c_0 < \infty$ the distribution will not depend on it.

There is another important point in eq. (24), namely that the degree strongly depends on the the geographical location of the hubs and the node in question. We will work out concrete examples below.

4.1.2 Random Walkers

Restricting to the specific case of the diffusion equation, the adjacency matrix elements become ($t = t_{\max}$:

$$\begin{aligned} G^0(x,t) &= (2\pi t)^{-n/2} \exp\left[-\frac{x^2}{4t}\right] \\ A_{ij}^{(2)}(t) &\approx -\lambda(2\pi t)^{-n} \exp\left[-\frac{x_i^2 + x_j^2}{4t}\right] \end{aligned} \tag{25}$$

And the degrees are given by:

$$k_i = c_0 \frac{N}{V} (2\pi t)^{-n/2} \exp\left[-\frac{x_i^2}{4t}\right]$$

where we used:

$$c_0 = \int d^n z G^0(z, -t) = 1$$

From this, it is actually very easy to calculate the degree distribution. Since the degree only depends on position, and more precisely, on the magnitude x_i^2, all nodes with the same x_i^2 will have more or less the same degree. The only determining factor will then be the density of the pints in that particular radius. the distribution of the degrees can then be visualized as how many points δN are between r and $r + \delta r$. These will be the points which have degree $k(r)$. It follows that:

$$\int P(k) dk = N = \int dN = \int \frac{dN}{dk} dk$$

$$P(k) = \frac{dN}{dk} = \frac{dN/dr}{dk/dr} \tag{26}$$

In the case of diffusion this yields:

$$\begin{aligned}
\frac{dN}{dk} &= \frac{N}{V} \Omega_{n-1} r_k^{n-1} \\
P(k) &= 4t(2\pi t)^{n/2} \frac{\Omega_{n-1} r_k^{n-1}}{r_k \exp\left[-\frac{r_k^2}{4t}\right]} \\
&= 4t \frac{\Omega_{n-1}(4t \log k)^{\frac{n-2}{2}}}{k} \qquad (27)
\end{aligned}$$

In the special case of two spatial dimensions $n = 2$ this reduces to a power law $p(k) \propto k^\gamma$ with an exponent $\gamma = -1$.

4.1.3 Adjusting the Node Distribution and the Power Law

As we saw the degree distribution depends on the node distribution in space as well as the degrees. One thing we notice immediately in (26) is that if

$$\frac{dN}{dr} \propto \exp[-\alpha r^2] = k^{\alpha/t}$$

We will have the degree distribution

$$P(k) \propto k^{-1+\alpha/t}$$

This may look exciting, as it allows for adjusting the degree distribution, but it is actually rather unphysical for constructing degrees $\gamma < -1$ as it requires the density of the nodes to grow as a quadratic exponential with their distance from the hub. Thus, if we wish to adjust the degree distribution, it is probably not feasible to do it through the node distribution. Our other options are changing the hub distribution $\lambda(x)$ and the metric of the space. We will examine these options below.

4.1.4 Other Power Law Degree Distributions from Hub Distribution

Here we wish examine the possibility of getting different degree distributions which are power laws with an exponent other than -1 by using different distributions for the hubs. For a general hub distribution $\lambda(x)$, The adjacency matrix is given by (21). Again, let's first assume the case where:

$$\lambda_{ij}(x) = \lambda(x)$$

so that it couples universally to all the fields ϕ_i. The degrees get slightly modified:

$$\begin{aligned} k_i &= J_0^{-1} \int \frac{d^n y}{(2\pi)^n} \lambda(y) G^0(x_i - y, -t) \int \frac{d^n z}{(2\pi)^n} J(z) G^0(z - y, -t) \\ &= c_0 \frac{N}{V} \int \frac{d^n y}{(2\pi)^n} \lambda(y) G^0(x_i - y, -t) \\ c_0 &= \int \frac{d^n z}{(2\pi)^n} J(z) G^0(z, -t) \end{aligned} \tag{28}$$

Now, finding the degree distribution will in general be more complicated than the procedure we took in (26) because of the integration involved. However, if $\lambda(x)$ enjoys any spatial symmetries, it simplifies again. Take for example a spherically symmetric distribution. Then $\lambda(x) = \lambda(|x|)$. This way from symmetry we will have $k_i = k(|x_i|)$. This yields a degree distribution of the form:

$$P(k) = c_0^{-1} \Omega_{n-1} r_k^{n-1} \left(\frac{d}{dr_k} \int \frac{d^n y}{(2\pi)^n} \lambda(y) G^0(\vec{r}_k - y, -t) \right)^{-1} \tag{29}$$

4.1.5 Solving for Hub Distribution

Now suppose we are observing a certain degree distribution $P(k)$ and we wonder what distribution of the hubs $\lambda(x)$ yields such a $P(k)$. To solve for $\lambda(x)$ we first recall that the Greens function is the inverse of the differential operators which define the equation of motion:

$$\begin{aligned} \Delta &\equiv \partial_t + \partial^2 \\ \Delta G(x,t) &= (2\pi)^{n+1} \delta(t) \delta(x) \\ if \quad \Delta\phi(x,t) = f(x,t) \quad &\Rightarrow \phi(x,t) = \int \frac{dt_z dz^n}{(2\pi)^{n+1}} G(x - z, t - t_z) f(z, t_z) \end{aligned} \tag{30}$$

Therefore, if we introduce time integral into (29) we may use this relation to solve for $\lambda(x)$. But before doing that, notice that $k = k(r)$ and in (29) we must write k in terms of r_k in order to be able to use the Green's function equation. This can be done using (26):

$$\int_0^k P(k')dk' = \frac{\Omega_{n-1}}{n} r_k^n$$

After solving this equation and obtaining $P(k(r))$ we may proceed as follows:

$$\int \frac{dt_y d^n y}{(2\pi)^n} \lambda(y) \delta(t_y) G^0(\vec{r} - y, -t - t_y) = \frac{\Omega_{n-1} r^{n-1}}{c_0 P(k(r))} \tag{31}$$

And therefore:

$$\lambda(x)\delta(t) = \Delta\left(\frac{\Omega_{n-1}r^{n-1}}{c_0 P(k(r))}\right) \tag{32}$$

4.1.6 Example of Deriving Power Laws

Imagine a case where the degree distribution is some power law of arbitrary exponent β:

$$P(k;t) = c(t)k^{\beta}$$

Where we included the time dependence, because without it the equations may not be solvable and because we saw previously in (27) that $P(k)$ will explicitly depend on time. We wish to find a hub distribution $\lambda(x)$ which results in such a distribution. First we need to find $P(k(r))$. If $\beta \neq -1$:

$$\begin{aligned}
\int_{k_0}^{k} P(k')dk' = (\beta+1)^{-1}\left(k^{\beta+1} - k_0^{\beta+1}\right) &= \frac{\Omega_{n-1}}{n}r^n \\
k &= c_1\left(r^n - r_0^n\right)^{\frac{1}{\beta+1}} \\
c_1 &= \left(\frac{\Omega_{n-1}(\beta+1)}{n}\right)^{\frac{1}{\beta+1}}
\end{aligned} \tag{33}$$

Plugging into (32) yields:

$$\lambda(x)\delta(t) = \Delta\left(\frac{\Omega_{n-1}r^{n-1}}{c_2\left(r^n - r_0^n\right)^{\frac{\beta}{\beta+1}}}\right) \tag{34}$$

For the case of diffusion (random walk) we have:

$$\left(\partial_t + \partial^2\right) f(r) = \left(\partial_t + \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\right) + \frac{L^2}{r^2}\right) f(r)$$

Where L^2 is the "angular momentum" part coming from the spherical symmetry. Since we assumed no dependence on the angles in $P(k)$, $L = 0$ and we have

$$\begin{aligned}
\lambda(r)\delta(t) &= \Omega_{n-1}\left(\partial_t + \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\right)\right)\left(\frac{r^{n-1}}{c_2(t)\left(r^n - r_0^n\right)^{\frac{\beta}{\beta+1}}}\right) \\
&= \frac{\Omega_{n-1}}{c_2(t)}\left(-\partial_t \ln(c_2) + \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\right)\right)\left(\frac{r^{n-1}}{\left(r^n - r_0^n\right)^{\frac{\beta}{\beta+1}}}\right)
\end{aligned} \tag{35}$$

As we see, there is no way to get a time-independent $\lambda(x)$ from this. On the other hand, having a time-dependent $\lambda(x)$ was not in contradiction with (29), though we would prefer at least a slowly varying hub distribution $\lambda(x)$. For instance, suppose having $c_2(t)^{-1} \sim \delta(t)$. Of course we cant calculate $\partial_t \ln c_2$ from this, but we can regularize this by assuming that:

$$c_2^{-1}(t) = (\pi\tau)^{-1/2}\exp\left[-\frac{t^2}{\tau^2}\right] \approx \delta(t), \quad \tau \ll t$$

This leads to:

$$\lambda(r) = \Omega_{n-1}\left(-\frac{t}{\tau^2} + \frac{n-1}{r}\partial_r + \partial_r^2\right)\left(\frac{r^{n-1}}{(r^n - r_0^n)^{\frac{\beta}{\beta+1}}}\right) \tag{36}$$

In the special case of $\beta = -1$ we get:

$$k = k_0 \exp\left[\frac{\Omega_{n-1}}{nc(t)}r^n\right]$$

and

$$\begin{aligned}\lambda(r;t)\delta(t) &= \left(\partial_t + \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\right)\right)\left(\frac{r^{n-1}}{\frac{c_0}{c(t)}r^{n-1}k(r)}\right)\\ &= c_0^{-1}c(t)k_0^{-1}\left(\partial_t \ln c(t) + \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\right)\right)\exp\left[-\frac{\Omega_{n-1}}{nc(t)}r^n\right]\\ &= c_0^{-1}c(t)\left(\partial_t \ln c(t) + 2(n-1)r^{n-1}\right)k(r)\end{aligned} \tag{37}$$

In (2+1)D, i.e. $n = 2$, the result states that the hubs will need to have a Gaussian times at most linear terms in r to yield $P(k) \propto k^{-1}$. Indeed if we use the same Gaussian regularization as before for $c(t) \sim \delta(t)$ and use the fact that $\tau \ll t$ we find:

$$\begin{aligned}\lambda(r;t) &= c_0^{-1}\left(2\frac{t}{\tau^2} + 2r\right)\exp\left[-\frac{\pi}{c(t)}r^2\right]\\ &\approx c_3 t\delta(r)\end{aligned} \tag{38}$$

5 Introducing Metric

6 Nima 06/16/2014

Each of us who makes an edit can add a new section like this with the date.