Screening Two Types

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1. Bundling multiple heterogeneous products
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Here: screening two types

- Impose only quasilinearity

Results

A general characterization of optimal mechanisms

Two applications

1. Bundling
2. Vertical and horizontal differentiation

Model

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Two types $\{t_1, t_2\}$, probabilities $1 - q, q$

A set of "alternatives" A

Value $v(t, a)$, $v(t, 0) = 0$

- Payoff $v(t, a) - p$

Cost $c(a)$ normalized to zero normalization

Goal: profit-maximizing IC&IR mechanisms

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To guarantee existence: $\{(v(t_1,a), v(t_2,a))\}_{a\in A}$ closed and bounded.

Application 1: Bundling

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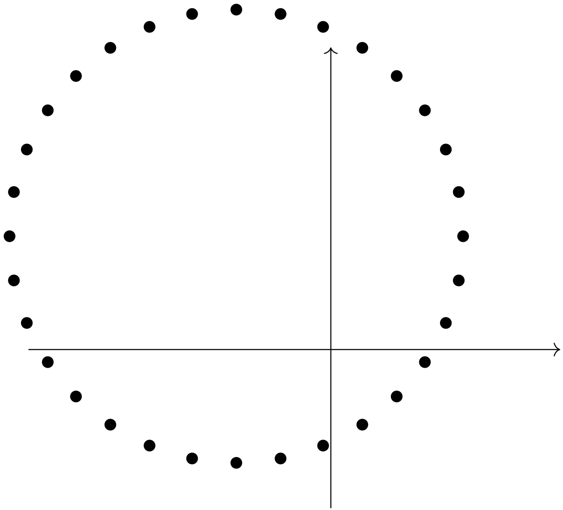

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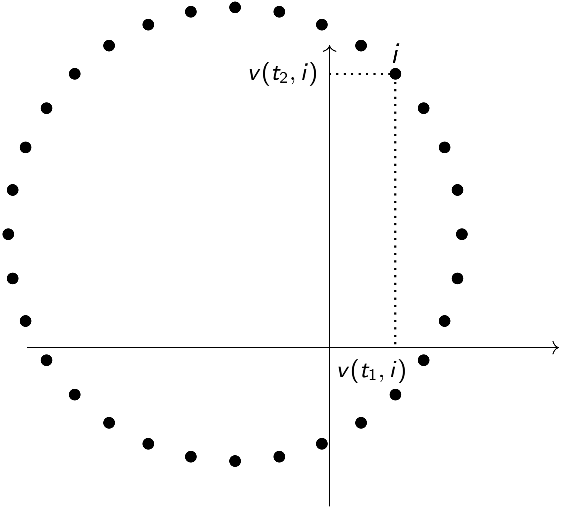
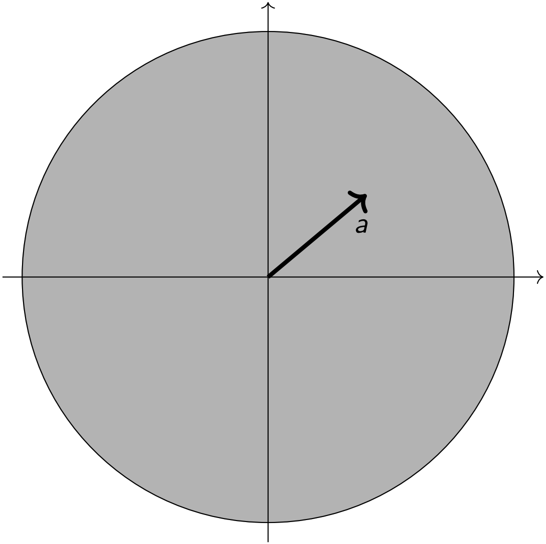


Application 2: Vertical and Horizontal differentiation

$A =$ all points within the circle.



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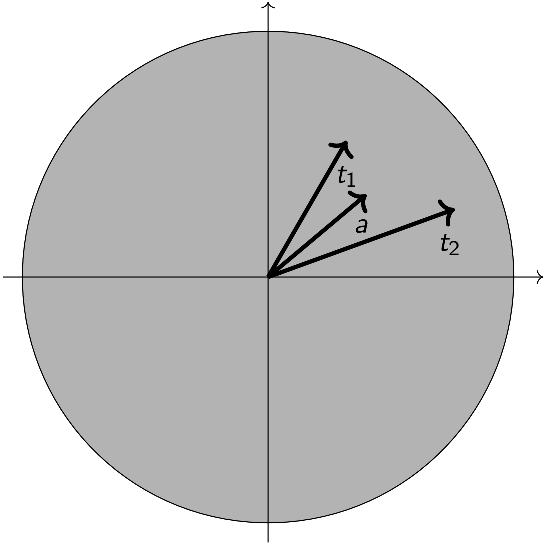

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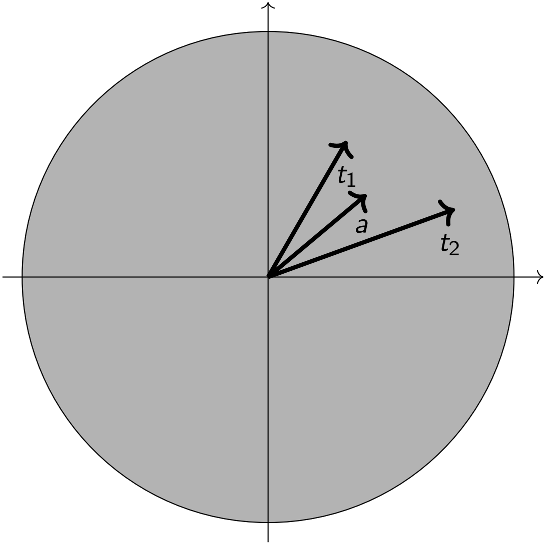

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more

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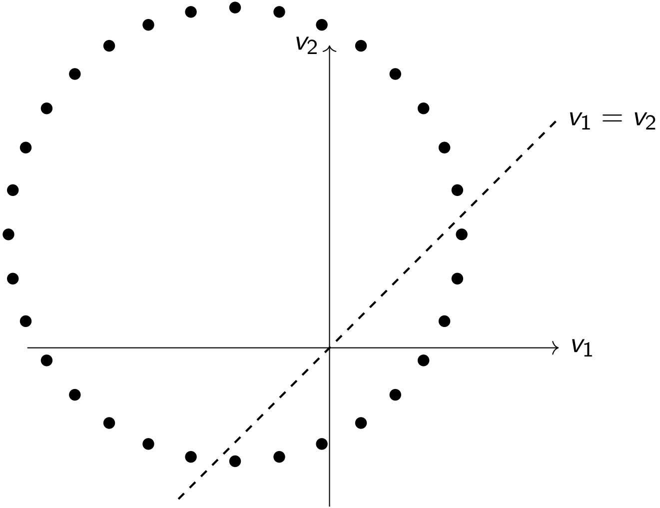

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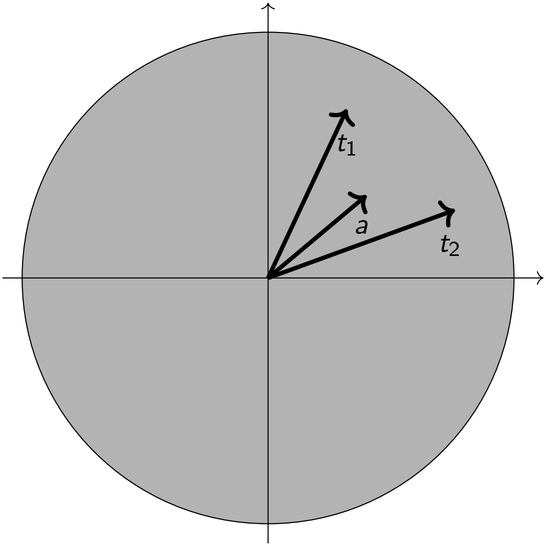

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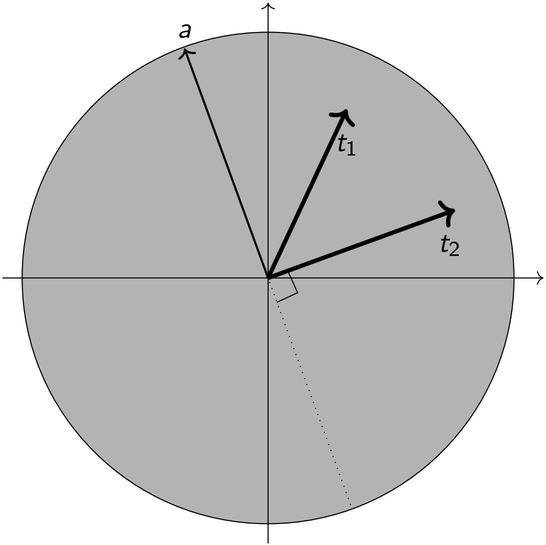

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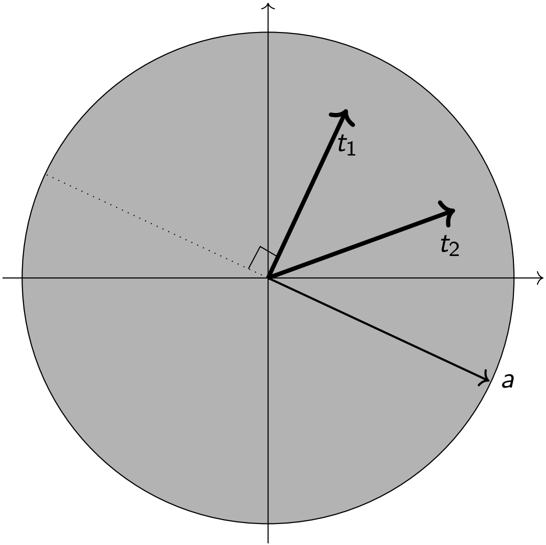

Back to General Model

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Consider the first-best mechanism:

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First-best is feasible (is IC) if

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Proposition

If (1) $\Rightarrow$ First-best mechanism is feasible and therefore optimal.
If not (1) $\Rightarrow$ see next slide.

Result continued

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Proposition (continued)

Suppose $v(t_2, \bar{a}_1) > v(t_1, \bar{a}_1)$.

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1. t_2 *is "the high type":*
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1. t_2 *is "the high type":*
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    b. *Its IC binds (pins down payment given t_1's allocation-payment)*
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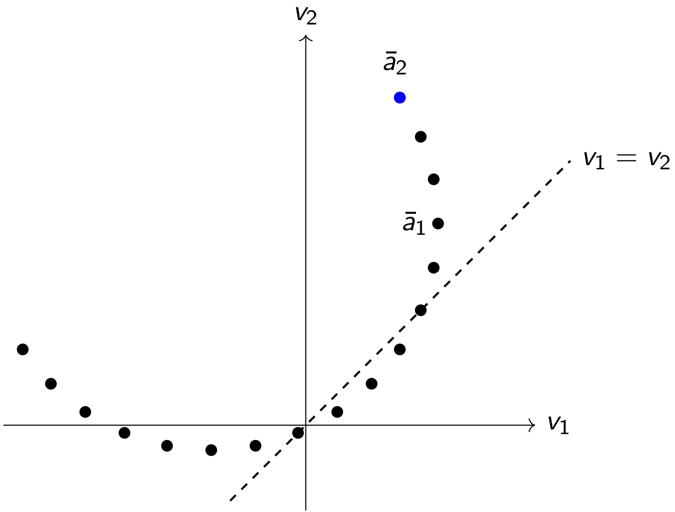
   - c. *Its IR binds (pins down payment given t_1's allocation)*
   - d. *Allocation $\in \arg\max v(t_1, a) - qv(t_2, a)$, s.t. $v(t_2, a) \geq v(t_1, a)$*

Allocation of t_1 when $v(t_2, \bar{a}_1) > v(t_1, \bar{a}_1)$



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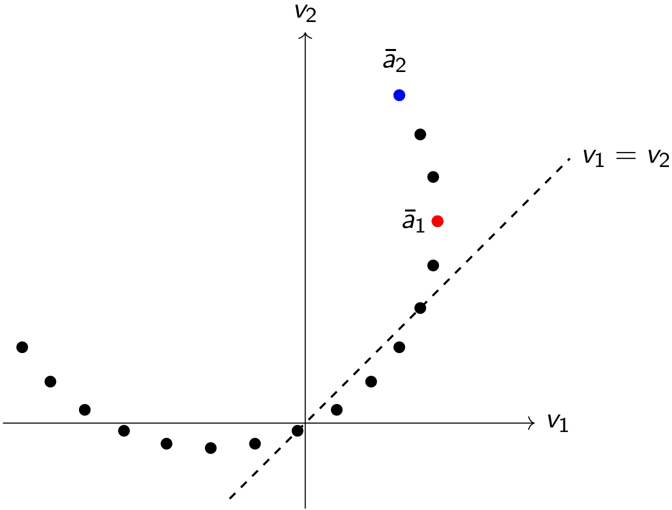

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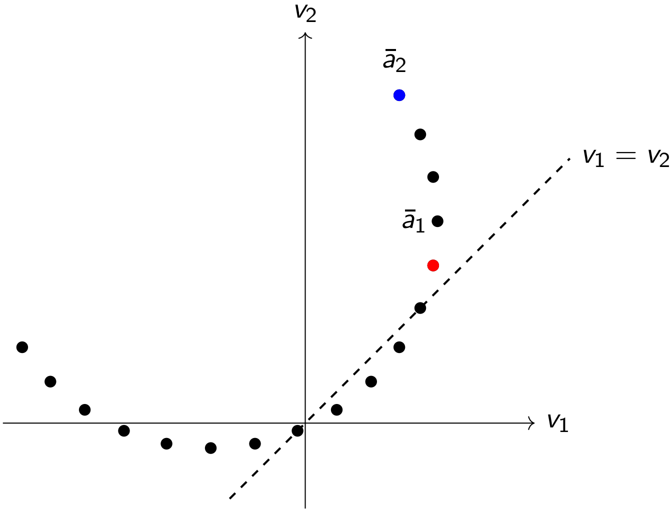

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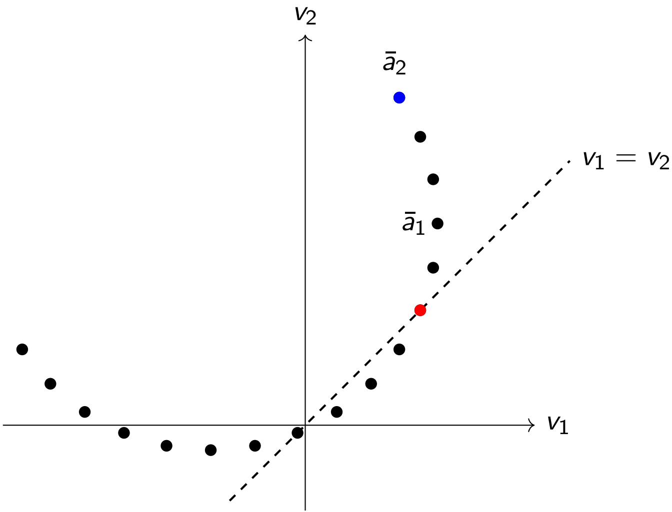

Proof

Suppose FSE impossible: $v(t_2, \bar{a}_1) > v(t_1, \bar{a}_1)$

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2. t_2's allocation is efficient

3. IR_1 binds

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   - Sub-claim: t_1's allocation is efficient.
     - Suppose not: t_1's allocation a_1 is inefficient
     - Make it "more efficient" by randomizing: a_1 w.p. $1 - \epsilon$, $\bar{a}_1$ w.p. ϵ
     - Charge t_1 more to keep her utility the same, improve revenue
   - Then $u(t_2) >_{IC2} v(t_2, \bar{a}_1) - p(t_1) \geq_{IR1} v(t_2, \bar{a}_1) - v(t_1, \bar{a}_1) > 0$. Can't be!
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4. Solve problem subject to IR2: $\max_a v(t_1, a) - qv(t_2, a)$ s.t. $v(t_2, a) \geq v(t_1, a)$

Next

Two applications:

1. Vertical + horizontal differentiation
2. Bundling

Vertical + Horizontal differentiation

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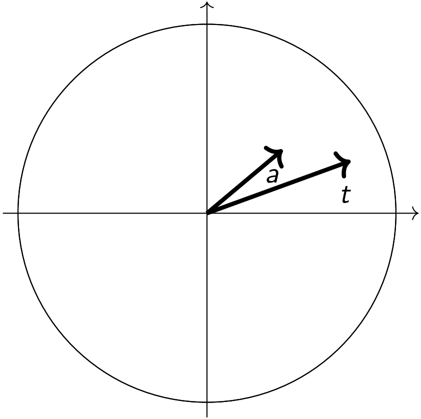

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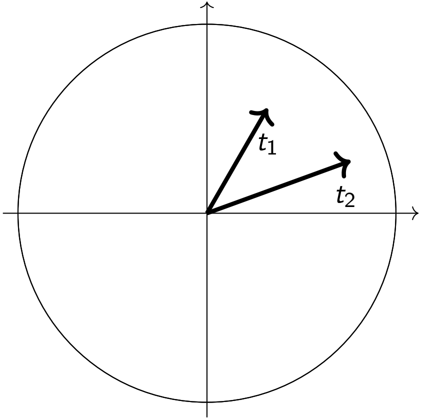

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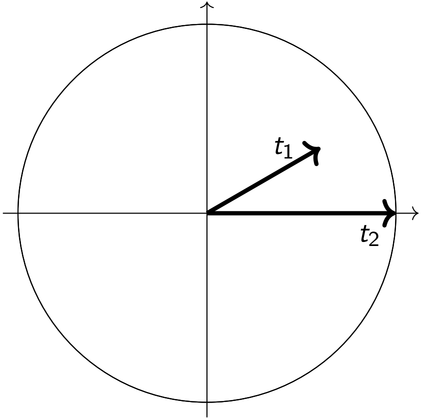

Vertical + Horizontal differentiation

$c(a) = c \cdot s(a)$, $v(t, a) = a \cdot t$. Suppose $c < 1$.

t_2's efficient alternative: unit horizontal vector.

1. $|t_1| < c$: outside option $\Rightarrow$ FSE possible.
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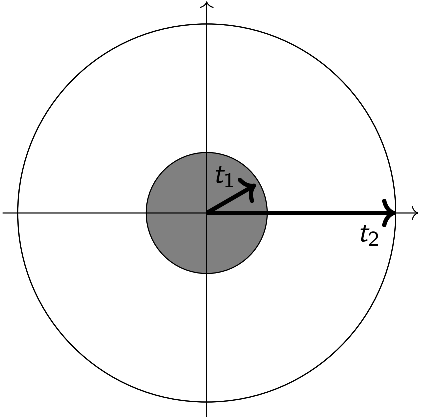

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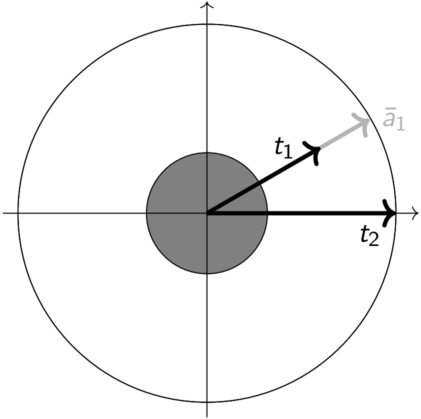

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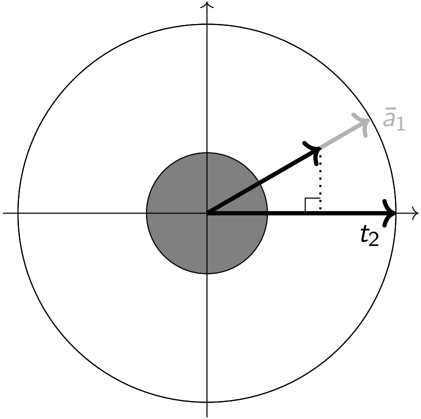

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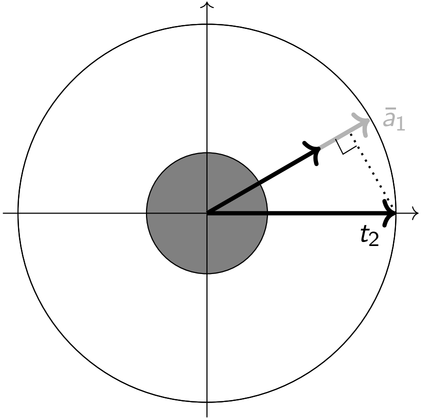

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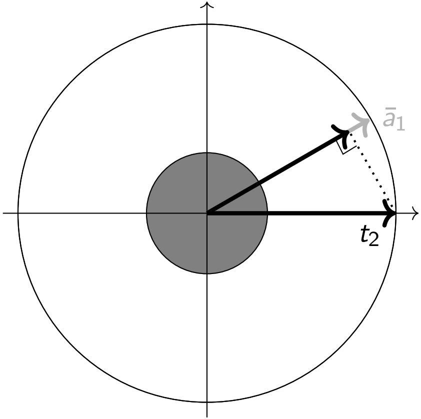

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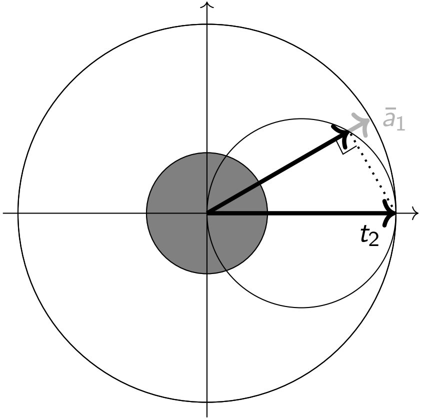
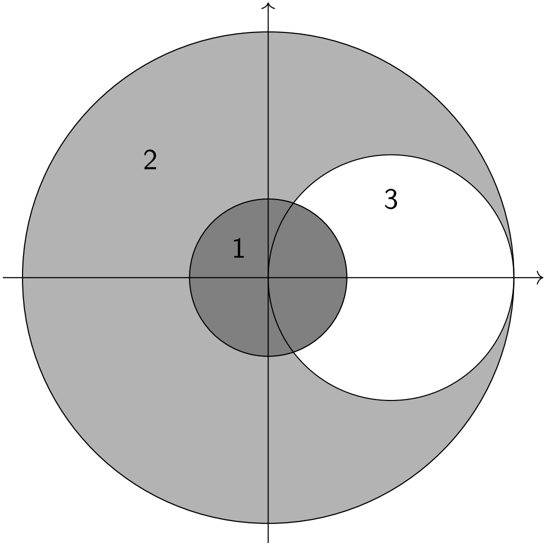


Vertical + Horizontal differentiation result

1 & 2: FSE. 3?



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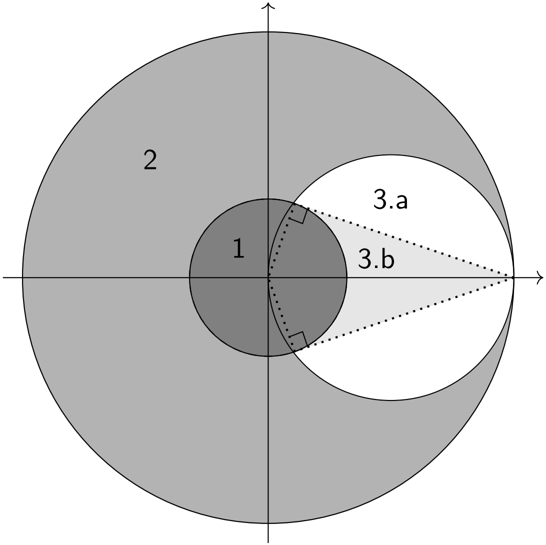

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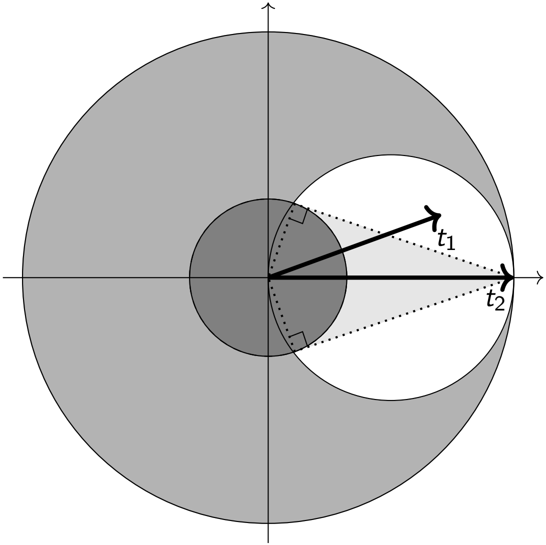

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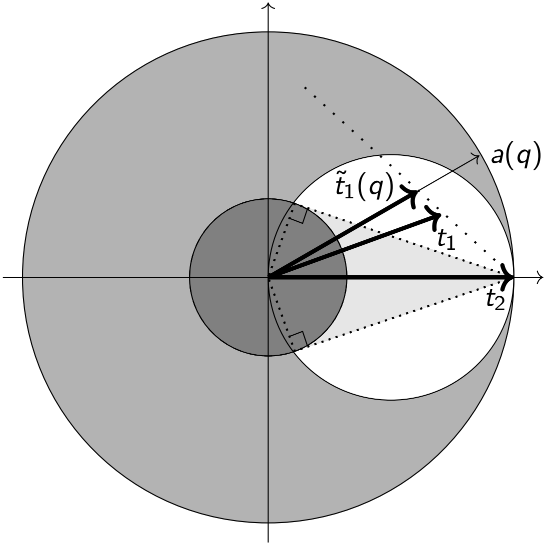

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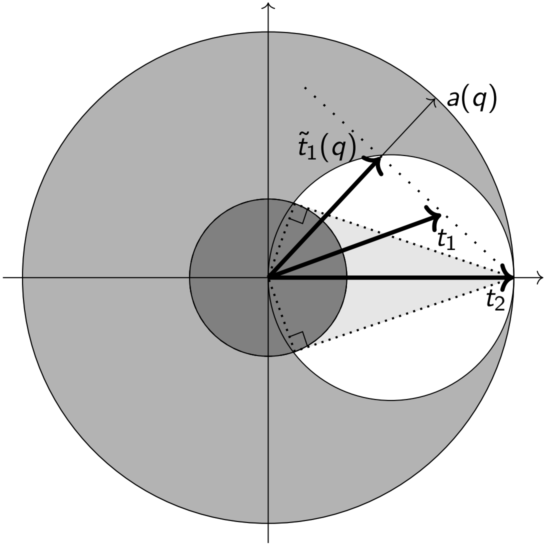

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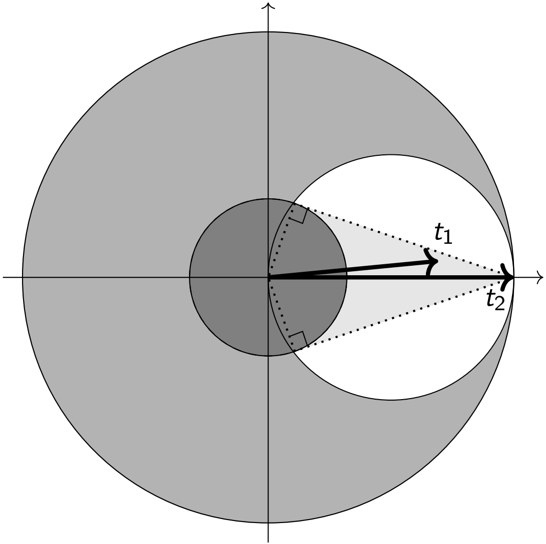

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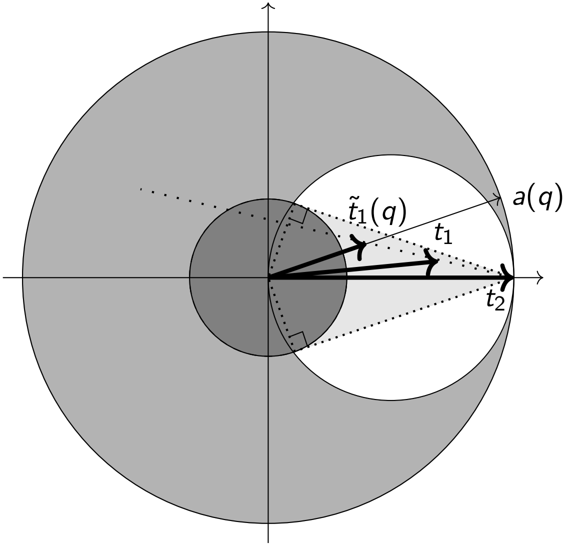

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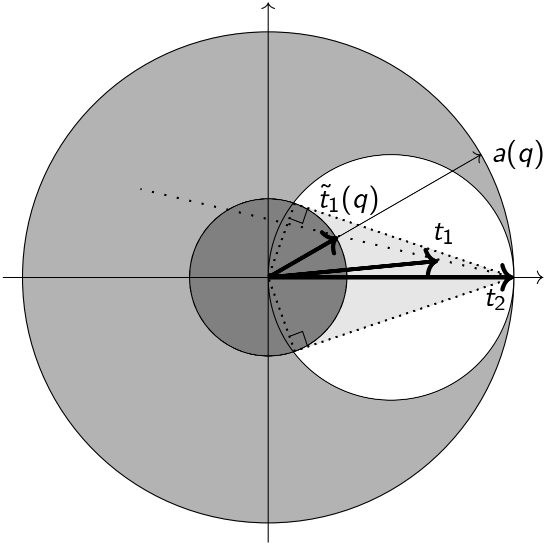

Bundling result

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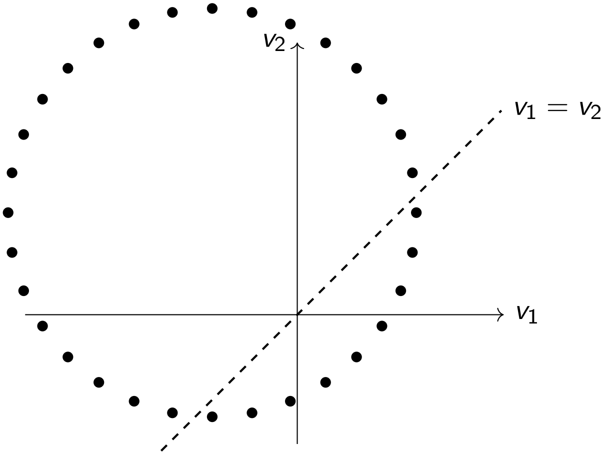

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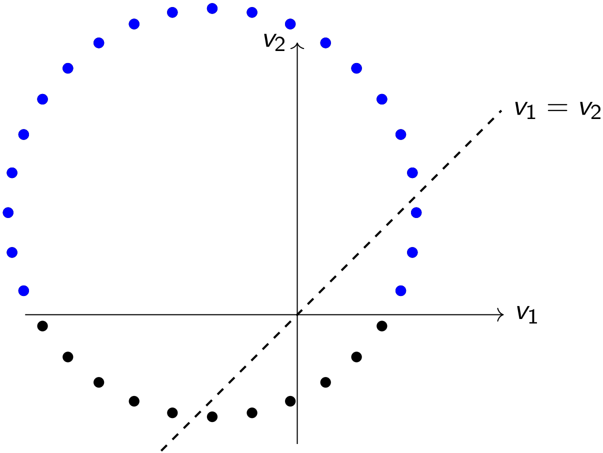

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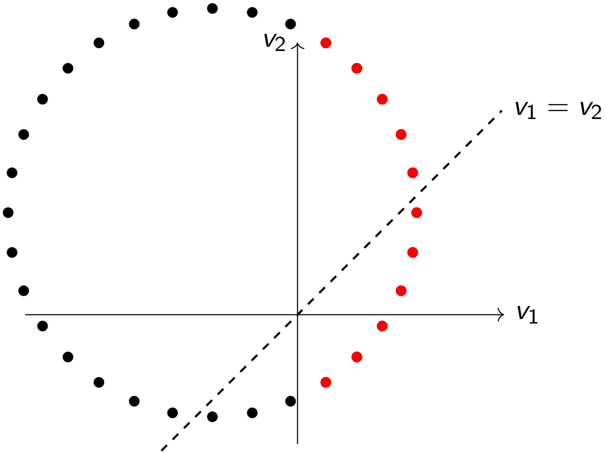

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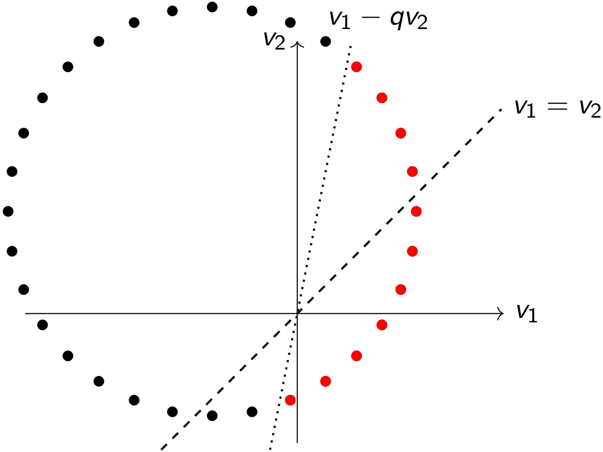

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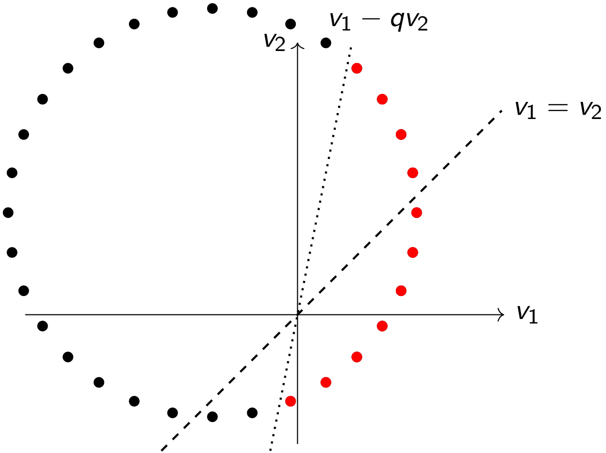
$$\text{As } q \uparrow,\ u(t_2) = v(t_2, \color{red}{S}) - v(t_1, \color{red}{S}) \text{ decreases}$$



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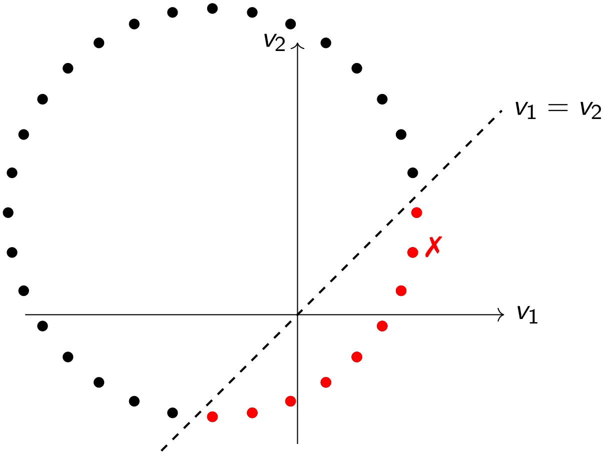

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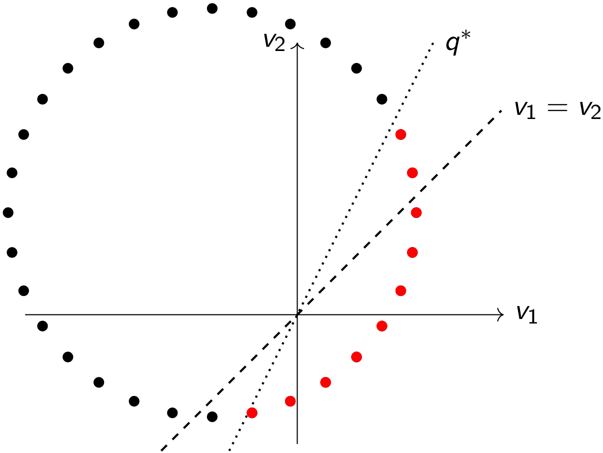

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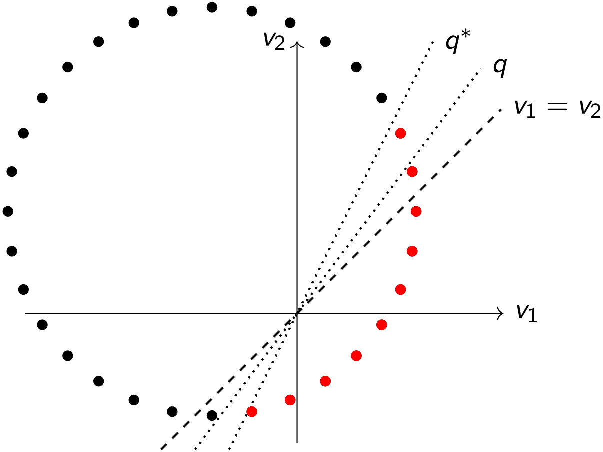

Related Literature

Classic models only consider vertically differentiated products

- Mussa and Rosen (1978), Maskin and Riley (1984)

Imperfect competition of single-product firms

- Horizontal differentiation: Hotelling (1929), Salop (1979)
- Horizontal + vertical differentiation: Villas-Boas (1999), Armstrong and Vickers (2001), and Rochet and Stole (2002)

Multi-product bundling: optimal mechanisms are complex and difficult to characterize

- Even with two products with additive and independently drawn values (Daskalakis et al., 2014, Thirumulanathan et al., 2019)

Applications that don't satisfy single-crossing, study two types

- Selling information: Bergemann, Bonatti, Smolin (2018)
- Screening with self-control: Galperti (2015)

More in the paper

1. A more general model that doesn't require randomization here
2. Use the result to characterize when randomization helps here

Conclusion

A general characterization of optimal mechanisms with two types

- A simple comparison specifies which type is high and which is low

Two applications

1. Bundling
   - Products might be added to distort allocation
2. Vertical and horizontal differentiation
   - Allocation is distorted away from the low type

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Thanks!

Normalizing costs and value of outside option to zero

$$\begin{aligned}\max_{a_1,a_2\in\Delta(A)\ p_1,p_2\in\mathbb{R}} \quad & (1-q)(p_1-c(a_1))+q(p_2-c(a_2)) \\ \text{subject to} \quad & v_1(a_1)-p_1\geq v_1(0), \\ & v_2(a_2)-p_2\geq v_2(0), \\ & v_1(a_1)-p_1\geq v_1(a_2)-p_2, \\ & v_2(a_2)-p_2\geq v_2(a_1)-p_1.\end{aligned}$$

Define $\tilde{v}_t(a)=v_t(a)-c(a)-v_t(0)$, $r_t=p_t-c(a_t)$.

$$\begin{aligned}\max_{a_1,a_2\in\Delta(A)\ r_1,r_2\in\mathbb{R}} \quad & (1-q)r_1+qr_2 \\ \text{subject to} \quad & \tilde{v}_1(a_1)-r_1\geq 0, \\ & \tilde{v}_2(a_2)-r_2\geq 0, \\ & \tilde{v}_1(a_1)-r_1\geq \tilde{v}_1(a_2)-r_2, \\ & \tilde{v}_2(a_2)-r_2\geq \tilde{v}_2(a_1)-r_1.\end{aligned}$$

back

“Weak” Increasing Differences

$\cancel{A \subset \mathbb{R}_+}$ $A \subset \mathbb{R}, 0 \in A$

- $\forall a > a', v(t_2, a) - v(t_2, a') > v(t_1, a) - v(t_1, a')$

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- $\forall a > a', v(t_2, a) - v(t_2, a') > v(t_1, a) - v(t_1, a')$
    - ~~when~~ $\cancel{a' = 0 : v(t_2, a) > v(t_1, a), \forall a \neq 0.}$

Then

1. ~~IR2 is implied by IR1 and IC2 and can be relaxed~~
2. ...

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"Standard" consequences:

1. Only local constraints bind, each type's IR might bind.
2. With two types: IC1, IC2, IR1, IR2 might all bind.

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Generalized Model

Two types $\{t_1, t_2\}$, probabilities $1 - q, q$

A set of "alternatives" A

Value $v(t, a)$, $v(t, 0) = 0$

- Payoff $v(t, a) - p$

Cost $c(a)$ normalized to zero

Mechanisms: $(x, p) : \{t_1, t_2\} \to A \times \mathbb{R}$

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Result

$$0 \geq v(t_1, \bar{a}_2) - v(t_2, \bar{a}_2); 0 \geq v(t_2, \bar{a}_1) - v(t_1, \bar{a}_1) \tag{1}$$

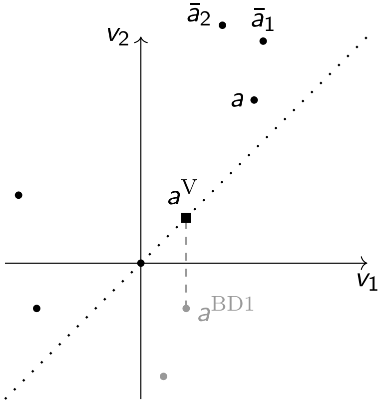
Proposition

If (1) $\Rightarrow$ FSE is feasible.

Otherwise suppose (WLOG) $v(t_2, \bar{a}_1) > v(t_1, \bar{a}_1)$. Then for all distributions

1. *t_2 is "the high type":*
   - a. *Its allocation is efficient: it gets $\bar{a}_2$*
   - b. ~~*Its IC binds*~~
2. *t_1 is "the low type":*
   - c. *Its IR binds (pins down payment given t_1's allocation)*
   - d. *Allocation: see next slide. It determines whether IC2 or IR2 binds.*

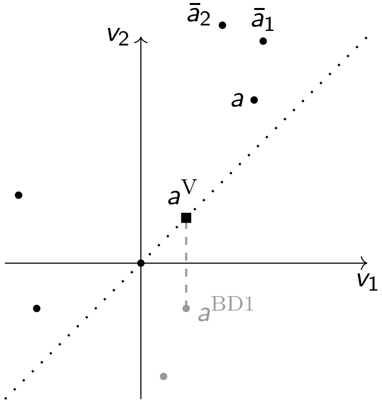
Allocation of t_1 when $v(t_2, \bar{a}_1) > v(t_1, \bar{a}_1)$



Allocation of t_1 when $v(t_2, \bar{a}_1) > v(t_1, \bar{a}_1)$

Maximize $v_1 - qv_2$ over $\{a | v_2 \geq v_1\} + a^{\mathrm{V}}$.

- Maximizer is a^{V}: Give t_1 alternative a^{BD1}. IR2 binds.
- Maximizer is not a^{V}: Give t_1 the maximizer. IC2 binds.



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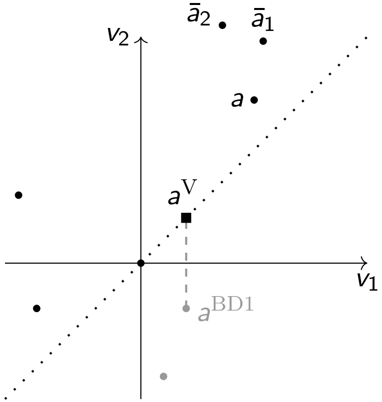

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General Proof: Suppose FSE impossible: $v(t_2, \bar{a}_1) > v(t_1, \bar{a}_1)$

Previously

1. IC2 binds
2. t_2's allocation is efficient
3. IR_1 binds
4. Solve problem subject to IR2: $\max_a v(t_1, a) - qv(t_2, a)$ s.t. $v(t_2, a) \geq v(t_1, a)$

Now:

1. ~~IC2 binds~~
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   - Either IC2 binds or IR2 binds.
   - If IC2 binds: same as before.
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     - make t_2's allocation efficient and keep her utility at 0
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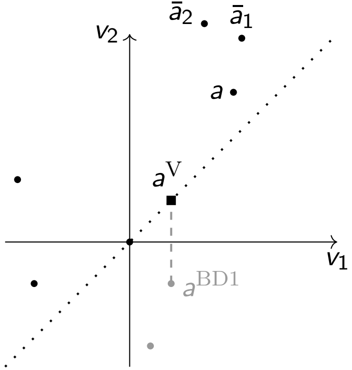

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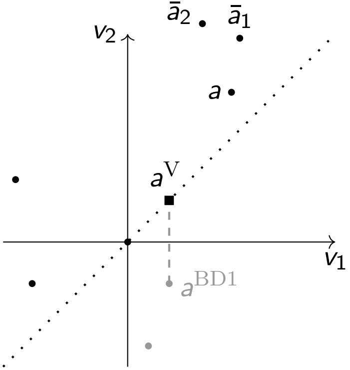

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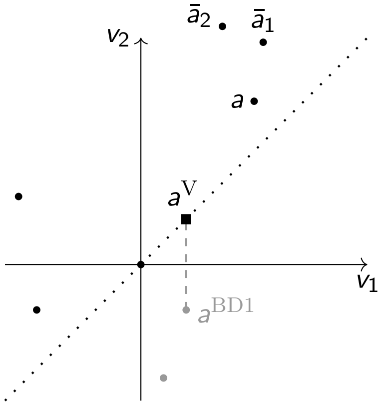
When does randomization help?

Proposition

Randomization helps (for some q) if and only if a^{BD1} is on the diagonal.



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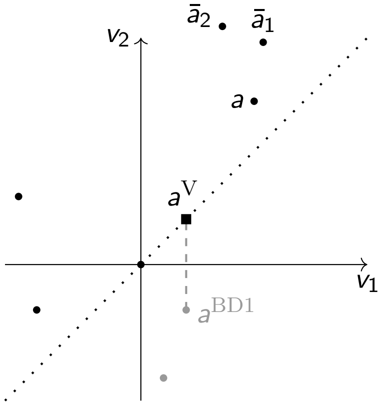

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