Problem Set 5

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Problem 1:
Solution:

Here we consider the condition,

$$\int_{x_1}^{x_2} dx\sqrt{2m(E-V(x))} = \hbar\pi(n+\frac{1}{2})$$

using the potential $V(x) = \frac{1}{2}\omega x^2$.

$$\int_{x_1}^{x_2} dx\sqrt{2m(E-\frac{1}{2}m\omega x^2)} =$$

$$\frac{1}{2}\sqrt{m(2E-m\omega^2x^2)}[x+\frac{2E}{\omega\sqrt{2E-m\omega^2x^2}}\arctan(\frac{\sqrt{m}\omega x}{\sqrt{2E-m\omega^2x^2}})]|_{x=x_1}^{x=x_2} = \pi\frac{E}{\omega}$$

For the Harmonic oscillator, we have turning points at $x_1 = -\sqrt{\frac{2E}{m\omega^2}}$ and $x_2 = \sqrt{\frac{2E}{m\omega^2}}$. Hence, we get

$$\pi\hbar(n+\frac{1}{2}) = \pi\frac{E_n}{\omega}$$

This gives us the result,

$$E_n = \hbar\omega(n+\frac{1}{2})$$

Problem 2:
Solution:

Here, we can imagine a 3d particle that's bound to a spherically symmetrical potential. The wavefunctions have the form, $R_l(r) = u_l(r)/r$. We can start with the Schrodinger equatiton,

$$(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}r + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r))R_l(r) = ER_l(r)$$

We only care about the s states here, so we take $l = 0$. The equation will reduce to,

$$(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + V(r))u(r) = Eu(r)$$

Now, near the classical turning point, we can approximate the potential as a linear potential

$$V(r) = E - \alpha(r - a)$$

with α as a constant. Hence,

$$(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + E - \alpha(r - a))u(r) = Eu(r)$$

$$(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} - \alpha(r - a))u(r) = 0$$

we can let $r_0 = (\frac{\hbar^2}{2m\alpha})^{1/3}$ and $x = \frac{r-a}{r_0}$. Hence, the equation reduces to

$$(\frac{\partial^2}{\partial x^2} + x)u(x) = 0$$

Allowing us to solve this equation using the boundary condition that $u(x = -\frac{a}{r_0}) = 0$. We are applying this boundary condition since we cannot allow the wave function $R(r)$ to explode at $r = 0$. Hence, $u = 0$ at this point to avoid divergence. The solution to this equation is given by,

$$u(x) = c_1 Ai((-1)^3 x) + c_2 Bi((-1)^3 x)$$

with Ai and Bi as Airy functions and c_1 and c_2 as coefficients determined from the boundary conditions of the problem. However, we want this in analytical form. Hence, we can transform the solution u to a Bessel function, $u(x) = x^{1/2} f(\frac{2}{3}x^{3/2}) = x^{1/2} f(z)$ with f. This allows us to transform the expression to the $n = 1/3$ Bessel equation,

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{z}\frac{\partial f}{\partial z} + (1 - \frac{(1/3)^2}{z^2})f = 0$$

This has the solution,

$$f(z) = c_1 J_0(z)$$

with,

$$J_0(z) = \sqrt{\frac{2}{\pi z}} \cos(z - \frac{\pi}{4})$$

Hence, the solution will be

$$u(x) = x^{1/2} \sqrt{\frac{3}{\pi x^{3/2}}} \cos(\frac{2}{3}x^{3/2} - \frac{\pi}{4})$$

$$= \sqrt{\frac{3}{\pi}} x^{-3/2} \cos(\frac{2}{3}x^{3/2} - \frac{\pi}{4})$$

Expanding out gives,

$$u_{\pm}(x) = \frac{2}{\sqrt{\kappa(x)}}(\exp(\frac{i}{\hbar}\int^x \kappa(x')dx' - \frac{\pi}{4}) + \exp(\frac{-i}{\hbar}\int^x \kappa(x')dx' - \frac{\pi}{4}))$$

Allowing us to substitute in $x = \frac{r-a}{r_0}$.

$$u_I(r) = \sqrt{\frac{3}{\pi}}\sqrt{\frac{r_0^3}{(r-a)^3}}(\exp(\frac{i}{\hbar}\int_r^a \kappa(r')dr' - \frac{\pi}{4}) + \exp(-\frac{i}{\hbar}\int_r^a \kappa(r')dr' - \frac{\pi}{4}))$$

which is our solution for the side that's left of the turning point. To the right of the turning point, we have

$$u_{II} \sim \frac{1}{\sqrt{\kappa(r)}} \exp(-\int_a^r dr' \kappa(r'))$$

We have that $u(r = 0) = 0$. To meet our condition that $u(0) = 0$, we must have that

$$\frac{2\pi}{h}\int_0^a \kappa(r')dr' - \frac{\pi}{4} = n + \frac{\pi}{2}$$

with n as an integer. This expression simplifies to,

$$2\int_0^a \kappa(r')dr' = 2\int_0^a \sqrt{2m(E-V(r))}dr' = (n+\frac{3}{4})h$$

which is the quantization condition for bound s states. Now for a potential that goes as r^{-p}, we have

$$2\lim_{r\to\infty}\int_0^a \sqrt{2m(E-r^{-p})}dr = 2\int_0^a \lim_{r\to\infty}\sqrt{2m(E-r^{-p})}dr = 2\int_0^a \sqrt{2mE}dr = 2\sqrt{2mE} = (n+\frac{3}{4})h$$

so we can see that there are an infinite number of states here. Hence, the quantization condition is satisfied here. Now, if $p > 2$ we get that near $r = 0$, the potential will blow up rapidly. Hence, the integral will diverge at 0. Hence, the quantization condition will not be satisfied.

Problem 3:
Solution:

Here, we can write the Hamiltonian of the system as

$$\mathcal{H} = \frac{p^2}{2m} - \lambda\delta(x)$$

At $t = 0$, the potential suddenly changes to μ. We can now determine the probability of transition from the old ground state to the new ground state. We know that the initial ground state has the form,

$$\psi(x,t=-\epsilon) = \begin{cases} \sqrt{\frac{-m\lambda}{\hbar^2}}e^{\frac{-m\lambda}{\hbar^2}x} & x<0 \\ \sqrt{\frac{-m\lambda}{\hbar^2}}e^{\frac{m\lambda}{\hbar^2}x} & x>0 \end{cases}$$

the new ground state is given by,

$$\psi(x,t=+\epsilon) = \begin{cases} \sqrt{\frac{-m\mu}{\hbar^2}}e^{\frac{-m\mu}{\hbar^2}x} & x<0 \\ \sqrt{\frac{-m\mu}{\hbar^2}}e^{\frac{m\mu}{\hbar^2}x} & x>0 \end{cases}$$

Hence, the probability of transition from the old to new state is given by $\langle\psi(x,t=-\epsilon)|\psi(x,t=+\epsilon)\rangle$. We can evaluate this for the left and right hand side. For $x < 0$, we have

$$\langle\psi_0(x,t=-\epsilon)|\psi_0(x,t=+\epsilon)\rangle = \sqrt{\frac{m\lambda}{\hbar^2}}\sqrt{\frac{m\mu}{\hbar^2}}\int_{-\infty}^0 e^{\frac{-m(\lambda+\mu)}{\hbar^2}x}dx = -\frac{\sqrt{\mu\lambda}}{\lambda+\mu}$$

Then for $x > 0$,

$$\langle \psi_0(x, t = -\epsilon) | \psi_0(x, t = +\epsilon) \rangle = -\frac{\sqrt{\mu\lambda}}{\lambda + \mu}$$

Hence, for combining both sides we have the probability amplitude: $\langle \psi_0(x, t = -\epsilon) | \psi_0(x, t = +\epsilon) \rangle = -\frac{2\sqrt{\mu\lambda}}{\lambda+\mu}$. Giving us probability,

$$|\langle \psi_0(x, t = -\epsilon) | \psi_0(x, t = +\epsilon) \rangle|^2 = \frac{4\mu\lambda}{(\lambda + \mu)^2}$$

For $\mu = \lambda/2$, we have

$$|\langle \psi_0(x, t = -\epsilon) | \psi_0(x, t = +\epsilon) \rangle|^2 = \frac{2\lambda^2}{(3/2\lambda)^2} = \frac{8}{9}$$

Let $\gamma = \mu/\lambda$. Hence, our total probability expression becomes

$$|\langle \psi_0(x, t = -\epsilon) | \psi_0(x, t = +\epsilon) \rangle|^2 = \frac{4\gamma}{(1 + \gamma)^2} \sim \frac{1}{\gamma}$$

Hence, for $\gamma = \mu/\lambda >> 1$ the transition probability will be very small. Conversely, for $\gamma = \mu/\lambda << 1$ the transition probability will be very large.

Problem 4:
Solution:

Considering now a spin 1/2 particle with magnetic moment μ in a magnetic field,

$$\mathbf{B}(t) = B_0(\cos\theta(t), \sin\theta(t), 0)$$

such that,

$$\theta(t) = \frac{\pi}{2} \frac{1}{1 + e^{t/t_0}}$$

The Hamiltonian of the system is given by, $\mathcal{H} = -\mu \cdot \mathbf{B}$. This comes out to (general case),

$$\mathcal{H} = -\frac{\mu}{2}(\cos\theta(t)\sigma_x + \sin\theta(t)\sigma_y)$$

Early on or when $t \to -\infty$, we have that the dominant term of the Hamiltonian corresponds to σ_y (with $\theta = \pi/2$ in this limit). For later times or when $t \to +\infty$, the dominant part of the Hamiltonian corresponds to σ_y. Hence,

$$\mathcal{H}(t = -\infty) = -\frac{1}{2}\mu B_0 \sigma_y$$

$$\mathcal{H}(t = +\infty) = -\frac{1}{2}\mu B_0 \sigma_x$$

Now for each case, we can look at the limit of large t_0 and small t_0 corresponding to the adiabatic and sudden approximations that will be used.

(i) We can now find the probability as a function of time for the particle to be in its ground state for $t >> t_0$. Here, the particle is in state $|y; -\rangle$ but the ground state is $|x; -\rangle$. For large t_0 (adiabatic), Hence, in the adiabatic limit we expect the particle to stay in the ground state with probability $P = 1$. In the sudden limit, we can the probability $|\langle -; y|x; -\rangle|^2$
We find,

$$\langle -; y|x; -\rangle = \frac{1}{2}(1, -1)^T(1, -i) = \frac{1+i}{2}$$

Hence, $\langle -; y|x; -\rangle|^2 = \frac{1}{2}$.

(ii) We can now compute the expected value $\langle \mathbf{S}(t)\rangle$ for$t >> t_0$. We start with the wave function of the system given by,

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}E_+ t}|x; +\rangle\langle +; x|y; +\rangle + e^{-\frac{i}{\hbar}E_- t}|x; -\rangle\langle -; x|y; +\rangle$$

$$= \frac{1}{\sqrt{2}}(e^{i\pi/4}e^{i\pi/2\omega_B t}|x; +\rangle + e^{-i\pi/4}e^{-i\pi/2\omega_B t}|x; -\rangle)$$

with $\omega_B = \frac{\mu B_0}{\hbar}$.

$$= \frac{1}{\sqrt{2}}(ie^{\frac{i}{2}\omega_B t}|x; +\rangle + e^{-\frac{i}{2}\omega_B t}|x; -\rangle)$$

We're interested in computing $\langle\psi|\sigma_x|\psi\rangle, \langle\psi|\sigma_y|\psi\rangle, \langle\psi|\sigma_z|\psi\rangle$. Hence,

$$\langle\psi|\sigma_z|\psi\rangle = \frac{\hbar}{4}(-ie^{-i/2\omega_B t}\langle x; +| + e^{i/2\omega_B t}\langle x; -|)(ie^{i/2\omega_B t}|x; -\rangle + e^{-i/2\omega_B t}|x; +\rangle)\frac{\hbar i}{4}(e^{it/\omega_b} - e^{it/\omega_b}) =$$

$$= -\frac{\hbar}{2}\sin\omega_B t$$

For $\langle\psi|\sigma_x|\psi\rangle$, we get

$$\frac{1}{2}([ie^{-\frac{i}{2}\omega_B t}\langle x;+| + e^{\frac{i}{2}\omega_B t}\langle x;-|]\sigma_x[ie^{\frac{i}{2}\omega_B t}|x;+\rangle + e^{-\frac{i}{2}\omega_B t}|x;-\rangle])$$

$$= \frac{1}{2}([ie^{-\frac{i}{2}\omega_B t}\langle x;+| + e^{\frac{i}{2}\omega_B t}\langle x;-|][ie^{\frac{i}{2}\omega_B t}|x;+\rangle - e^{-\frac{i}{2}\omega_B t}|x;-\rangle]) = \frac{1}{2}$$

Then for $\langle\psi|\sigma_y|\psi\rangle = 0$. Hence,

$$\langle \mathbf{S}\rangle = \hbar(1/2, 0, -\frac{1}{2}\sin\omega_B t)$$

This is the result for the sudden limit. In the adiabatic limit, we simply have $\langle \mathbf{S}\rangle = (0, \frac{\hbar}{2}, 0)$.

(ii) For t_0 in the sudden limit, we should compare the timescale to $t_0 << \frac{\hbar}{\Delta E}$ which corresponds to the time it takes for the wave function to evolve from the y to x and ΔE is the corresponding energy scale. Like wise, in the adiabatic limit we must have $t_0 >> \frac{\hbar}{\Delta E}$.

Problem 5:
Solution:

(a)
Here, we take the Hamiltonian $H(\mathbf{R}) = (X\sigma_x + Y\sigma_y + Z\sigma_z)$. This can be expanded as ,

$$H(\mathbf{R}) = \begin{pmatrix} Z & X - iY \\ X + iY & -Z \end{pmatrix} = -R\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -d\cos\theta \end{pmatrix}$$

This Hamiltonian has eigenvalues $\pm\sqrt{x^2+y^2+z^2} = \pm|\mathbf{R}|$. Hence, the ground state energy of the system is $-|\mathbf{R}|$. Its corresponding eigenvector is,

$$|0\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2e^{i\phi} \end{pmatrix} e^{i\alpha(R,\theta,\phi)}$$

with α as our phase.

(b) Now we can compute Berry's potential given by,

$$\mathbf{A}_0(\mathbf{R}) = i\langle\Psi_0(\mathbf{R})|\nabla_{\mathbf{R}}\Psi_0(\mathbf{R})\rangle$$

Hence, we need $\nabla_{\mathbf{R}}\Psi_0(\mathbf{R})$. Here,

$$\frac{\partial\Psi_0}{\partial R}\hat{r} = i\begin{pmatrix}\cos\theta/2\\ \sin\theta/2e^{i\phi}\end{pmatrix}e^{i\alpha(R,\theta,\phi)}\frac{\partial\alpha}{\partial R}$$

$$\frac{\partial\Psi_0}{\partial\theta}\hat{\theta} = \frac{1}{R}\frac{\partial\Psi_0}{\partial\theta}$$

$$= \frac{1}{R}\begin{pmatrix}(\cos\theta/2)\frac{\partial\alpha}{\partial\theta} - \frac{1}{2}\sin\theta/2)\\ e^{i\phi}(\frac{1}{2}(\cos(\theta/2) + i\frac{\partial}{\partial\theta}))\end{pmatrix}e^{i\alpha}$$

$$\frac{\partial\Psi_0}{\partial\phi}\hat{\phi} = \frac{1}{R\sin\theta}\frac{\partial\Psi_0}{\partial\phi} = \frac{i}{R\sin\theta}(\begin{pmatrix}\cos\theta/2\frac{\partial\alpha}{\partial\phi}\\ i\sin(\theta/2)e^{i\phi}(1+\frac{\partial\alpha}{\partial\phi})\end{pmatrix}e^{i\alpha(R,\theta,\phi)})$$

We can now obtain the inner product $\langle\Psi_0|\nabla_{\mathbf{R}}\Psi_0\rangle$. Computing this for each component gives,

$$\langle\Psi_0|\nabla_R\Psi_0\rangle = i(\sin^2\theta/2 + \cos^2\theta/2)\frac{\partial\alpha}{\partial R} = i\frac{\partial\alpha}{\partial R}$$

For the θ component, we have

$$\langle\Psi_0|\nabla_\theta\Psi_0\rangle = \cos^2(\theta/2)\frac{\partial\alpha}{\partial\theta} - \frac{1}{2}\cos(\theta/2)\sin(\theta/2) + \frac{1}{2}\cos(\theta/2)\sin(\theta/2) + i\sin^2(\theta/2)\frac{\partial\alpha}{\partial\theta}$$

$$= \frac{1}{R}\frac{\partial\alpha}{\partial\theta}(\cos^2(\theta/2) + i\sin^2(\theta/2))$$

Lastly, we can compute this for the ϕ component. We get,

$$\langle\Psi_0|\nabla_\phi\Psi_0\rangle = \frac{i}{R\sin\theta}(\cos^2\theta\frac{\partial\alpha}{\partial\phi} + i\sin^2\theta/2(1+\frac{\partial\alpha}{\partial\phi}))$$

Hence, these three components define $\mathbf{A} = i(\langle\Psi_0|\nabla_R\Psi_0\rangle, \langle\Psi_0|\nabla_\theta\Psi_0\rangle, \langle\Psi_0|\nabla_\phi\Psi_0\rangle)$ in spherical coordinates. Now to obtain the Berry curvature, we can compute the curl of $\mathbf{A}$. This is defined as,

$$\mathbf{V} = \nabla\times\mathbf{A} = \frac{1}{R\sin\theta}(\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi})\hat{\mathbf{R}} + \frac{1}{r}(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial}{\partial R}(RA_\phi))\hat{\theta} + \frac{1}{R}(\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial\theta})\hat{\phi}$$

With,

$$A_R = -\frac{\partial\alpha}{\partial R}$$

$$A_\theta = \frac{1}{R}\frac{\partial\alpha}{\partial\theta}(\cos^2(\theta/2) + i\sin^2(\theta/2))$$
$$A_\phi = -\frac{1}{R\sin\theta}(\cos^2(\theta/2)\frac{\partial\alpha}{\partial\phi} + \sin^2\theta/2(1 + \frac{\partial\alpha}{\partial\phi}))$$

Letting Mathematica do the heavy lifting to compute the above expression, we arrive at

$$\mathbf{V} = -\frac{\hat{R}}{2R^2}$$

(c) We can now find a point where $\mathbf{A}_0$ is singular which occurs in the vector potential of a magnetic monopole. We can find a guage α where $|\Psi_0$ is continuous everywhere except at $X = Y = 0, Z > 0$ (south pole). We can also do this for the north pole. For the south pole, we want $\alpha = 0$. The corresponding components of $\mathbf{A}_0$ here will be (corresponding to $\theta = \pi$),

$$A_R = 0$$

$$A_\theta = 0$$
$$A_\phi = -\frac{1}{R}$$

(d) We can now compute the Berry phase for a given Guage. Let's take $\alpha = 0$.

$$\gamma = \oint_C \mathbf{A}\cdot d\mathbf{R}$$

such that we integrate over ϕ.

$$\gamma = \int_0^{2\pi}(-\frac{1}{R\sin\theta}(\sin^2\theta/2))d\phi$$

$$= -\frac{2\pi\sin^2\theta/2}{R\sin\theta}$$

This relates to the solid angle subtended by C, as it relates to the phase difference over the closed loop C for any chosen point R, θ. This expression is only true for the guage that we chose. Hence, this quantitty is not guage invariant.