Problem Set 1

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Problem 1:
Solution:

(a)
We are interested in determining if the Runge-Lenz vector is a constant of motion for a classical particle in a coulomb potential. We have that

$$\mathbf{R} = \frac{1}{m}\mathbf{p} \times \mathbf{L} - \frac{e^2\mathbf{r}}{r}$$

The Hamiltonian of the system is simply,

$$H = \frac{p^2}{2m} - \frac{e}{r} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} - \frac{e^2}{r}$$

We can show that,

$$\frac{\partial \mathbf{R}}{\partial t} = \{\mathbf{R}, H\} = 0$$

If we write $\mathbf{R}$ first in indicinal notation,

$$R_i = \frac{\epsilon_{ijk}}{m} p_j L_k - \frac{e^2}{r} r_i = \frac{\epsilon_{ijk}}{m} p_j (\epsilon_{kmn} r_m p_n) - \frac{e^2}{r} r_i$$

$$= \frac{\epsilon_{ijk}\epsilon_{kmn}}{m} p_j p_n r_m - \frac{e^2}{r} r_i$$

We want, $\frac{\partial R_i}{\partial t} = 0$.

$$\frac{\partial R_i}{\partial t} = \frac{\epsilon_{ijk}\epsilon_{kmn}}{m}[\frac{\partial p_j}{\partial t}(p_n r_m) + p_j(r_m\frac{\partial p_n}{\partial t} + p_n\frac{\partial r_m}{\partial t})] - \frac{e^2}{r}r_i$$
$$= \frac{\epsilon_{ijk}\epsilon_{kmn}}{m}[p_n r_m\frac{\partial p_j}{\partial t} + p_j r_m\frac{\partial p_n}{\partial t} + p_j p_n\frac{\partial r_m}{\partial t}] - \frac{e^2}{r}r_i$$
$$= \frac{\epsilon_{ijk}\epsilon_{kmn}}{m}[\frac{\partial p_j}{\partial t}(p_n r_m) + p_j(r_m\frac{\partial p_n}{\partial t} + p_n\frac{\partial r_m}{\partial t})] - \frac{e^2}{r}r_i$$
$$= \frac{\epsilon_{ijk}\epsilon_{kmn}}{m}[-p_n r_m\frac{\partial V}{\partial r_j} - p_j r_m\frac{\partial V}{\partial r_n} + p_j p_n\frac{p_m}{m}] - \frac{e^2}{mr}(p_i - \frac{r_i p}{r})$$
$$= \frac{\epsilon_{ijk}\epsilon_{kmn}}{m}[p_n r_m\frac{e^2 r_j}{r^3} + p_j r_m\frac{e^2 r_n}{r^3} + \frac{1}{m}p_j p_n p_m] - \frac{e^2}{mr}(p_i - \frac{r_i p}{r})$$
$$= 0$$

Hence, the quantity $\mathbf{R}$ is conserved.

Problem 2:

Solution:

(a)

Here we consider the 2D Harmonic oscillator with Hamiltonian,

$$\mathcal{H} = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2)$$

which has energy eigenvalues $\epsilon_{n_x,n_y} = \hbar\omega(n_x + n_y + 1)$. If $N = n_x + n_y$ corresponding to an energy level ϵ_N, then from enumeration we would find that each energy ϵ_N would be associated with $N+1$ degenerate states.

(b)

We can write the Hamiltonian expressed in (a) in terms of raising and lowering operators $a_x^\dagger, a_x, a_y^\dagger, a_y$. This becomes

$$\mathcal{H} = \hbar\omega(a_x^\dagger a_x + a_y^\dagger a_y + 1)$$

which we can write as,

$$\mathcal{H} = \hbar\omega(\begin{pmatrix} a_x^\dagger & a_y^\dagger \end{pmatrix} I_2 \begin{pmatrix} a_x \\ a_y \end{pmatrix} + 1)$$

with I_2 as the identity. We can let $\hat{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$. Applying the transformation, $\hat{a}' = M\hat{a} = U^\dagger \hat{a} U$. Hence,

$$\mathcal{H} = \hbar\omega(\hat{a}^\dagger I_2 \hat{a} + 1) = \hbar\omega(\hat{a}'^\dagger (M^{-1})^\dagger I_2 M^{-1}\hat{a}' + 1) = \hbar\omega(\hat{a}'^\dagger (M^{-1})^\dagger M^{-1}\hat{a}' + 1)$$

Hence, we can see that for this transformation on $\hat{a}$ to be invariant, we must have that $(M^{-1})^\dagger M^{-1} = I_2$, implying that $M^\dagger M = I_2$. Hence, M must be a 2x2 unitary matrix. This tells us that the dynamical symmetry group of the transformation is $U(2)$.

(c) In part (b), we identified that the symmetry group of the transformation M is $U(2)$. If h^c with $c = 1, 2, 3, 4$ are the infinitesimal generators of M such that $M = 1 - i\epsilon h^c$, then for these generators to commute with the Hamiltonian, they must be generators for $U(2)$. Hence, the generators for M must be,

$$h^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$h^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$h^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$h^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since M is unitary implying that h^c is Hermitian, we can write the Hamiltonian as

$$\mathcal{H} = \hbar\omega(\hat{a}'^\dagger M M^\dagger a' + 1) = \hbar\omega(\hat{a}'^\dagger (1 - i\epsilon h^c)(1 + i\epsilon h^c) a' + 1)$$

$$= \hbar\omega(U\hat{a}'^\dagger U^\dagger U \hat{a}' U^\dagger + 1)$$

$$= \hbar\omega((1 - i\epsilon G^c)\hat{a}'^\dagger (1 + i\epsilon G^c)(1 - i\epsilon G^c)\hat{a}'(1 + i\epsilon G^c) + 1)$$

$$= \hbar\omega((1 - i\epsilon G^c)\hat{a}'^\dagger \hat{a}'(1 + i\epsilon G^c) + 1)$$

From this, we can identify

$$M\hat{a} = U^\dagger \hat{a} U$$

$$= (1 - i\epsilon h^c)\hat{a} = (1 + i\epsilon G^c)\hat{a}(1 - i\epsilon G^c)$$

After simplification, we get that (dropping out (ϵ^2) terms).

$$h^c \hat{a} \approx [G^c, \hat{a}]$$

Hence,

$$\begin{aligned}\hat{a}^\dagger h^c \hat{a} &= \hat{a}^\dagger([G^c, \hat{a}]) \\ &= G^c\end{aligned}$$

$$G^c = \sum_{ij} h^c_{ij} a^\dagger_i a_j$$

Our conserved quantity is U. Here, we can write out

$$\frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2) = \hbar\omega(\hat{a}'^\dagger (M^{-1})^\dagger M^{-1}\hat{a}' + 1)$$

$$= \hbar\omega(U\hat{a}'^\dagger U^\dagger U \hat{a}' U^\dagger + 1)$$

$$= \hbar\omega((1 - i\epsilon G^c)\hat{a}'^\dagger \hat{a}'(1 + i\epsilon G^c) + 1)$$

which allows us to relate the generators to x, y, p_x, p_y.
(d) In part (a) we found that for an energy level corresponding to $N = n_x + n_y$, there would be $N + 1$ degeneracies. However if we set $N = 2k$, we can now express the degeneracies in terms of k. The generators used here follow the Lie algebra (using $K = \frac{\hbar}{2}G$). ,

$$[K_i, K_j] = i\hbar\epsilon_{ijk}K_k$$

such that $K^2 = K_1^1 + K_2^2 + K_3^2$. From this follows that, the operator K^2 will have eigenvalues $\hbar k(k + 1)$. We know that operators with this Lie algebra should have $2k + 1$ degeneracies. Hence, there should be $N + 1 = 2k + 1$ degeneracies.

(e) For a three-dimensional harmonic oscillator, we should have the dynamical symmetry group $U(3)$.