Problem Set 7

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Problem 1:
Solution:

To derive the 2D density of states for a free particle, we can first define the area of a single state box in k space

$$\Omega_{box} = \frac{\pi^2}{L^2}$$

and the area of a circle is $\Omega_{circ} = \pi k^2$. Hence, the number of states will be given by (including spin degeneracy of 2)

$$N(k) = \frac{2}{4}\frac{\Omega_{circ}}{\Omega_{box}} = \frac{k^2 L^2}{2\pi}$$

We are interested in, $\rho(E) = \frac{\partial N}{\partial E}$. Here,

$$\frac{\partial N}{\partial E} = \frac{\partial}{\partial E}[\frac{L^2}{2\pi}(\sqrt{\frac{2mE}{\hbar^2}})^2] = \frac{\partial}{\partial E}[\frac{mL^2 E}{\pi\hbar^2}] = \frac{mL^2}{\pi\hbar^2}$$

Giving us,

$$\rho(E)dEd\Omega = \frac{mL^2}{\pi\hbar^2}dEd\phi$$

Problem 2:
Solution:

Here we have an STM tip, incident on a sample. System 1 is the tip and system 2 is the sample, each described with the following Hamiltonians

$$\mathcal{H}_1 = \frac{|\mathbf{p}_1|^2}{2m} + eV$$

$$\mathcal{H}_2 = \frac{|\mathbf{p}_2|^2}{2m} + V_2(\mathbf{x}_2)$$

with V as the voltage of the tip and V_2 as the potential of the sample. The coupling at a point between the 2 systems is described by,

$$U = u(|\mathbf{x}_1 = 0\rangle\langle\mathbf{x}_2 = 0| + |\mathbf{x}_2 = 0\rangle\langle\mathbf{x}_1 = 0|)$$

(a) If the electron is initially in the state $|\mathbf{p}_1\rangle$, with energy $\epsilon(\mathbf{p}_1) = |\mathbf{p}_1|^2/2m + eV$.

We can now find the tunneling rate from system 1 to system 2. We can do this using Fermi's golden rule

$$\frac{1}{\tau_{1\to 2,n}} = \frac{2\pi}{\hbar}|\langle n|U|\mathbf{p}_1\rangle|^2\delta(E_2 - E_1)$$

where $|n\rangle$ is some arbitrary state in system 2 that electrons from system 1 will tunnel into. Hence, we can move forward to compute the matrix element $\langle n|U|\mathbf{p}_1\rangle$.

$$\langle n|U|\mathbf{p}_1\rangle = u[\langle n|\mathbf{x}_1 = 0\rangle\langle\mathbf{x}_2 = 0|\mathbf{p}_1\rangle + \langle n|\mathbf{x}_2 = 0\rangle\langle\mathbf{x}_1 = 0|\mathbf{p}_1\rangle]$$

$$= u\langle n|\mathbf{x}_2 = 0\rangle\langle\mathbf{x}_1 = 0|\mathbf{p}_1\rangle = u\psi^\star_{2,n}(\mathbf{x}_2 = 0)\psi_{1,\mathbf{p}_1}(\mathbf{x}_1 = 0)$$

with $\psi_{1,\mathbf{p}_1}(\mathbf{x}_1 = 0) = \frac{1}{L}e^{i\mathbf{p}_1\cdot\mathbf{x}_1}$. Hence,

$$|\langle n|U|\mathbf{p}_1\rangle|^2 = \frac{1}{L^2}e^{i\mathbf{p}_1\cdot\mathbf{x}_1}e^{-i\mathbf{p}_1\cdot\mathbf{x}_1}|u|^2\psi^\star_{2,n}(\mathbf{x}_2 = 0)\psi_{2,n}(\mathbf{x}_2 = 0) = \frac{|u|^2}{L^2}|\psi_{2,n}|^2$$

Hence, to get our final scattering rate, we must sum over all the final states in system 2, n. This will reduce to,

$$\frac{1}{\tau_{1\to 2}} = \frac{2\pi}{\hbar}\frac{|u|^2}{L^2}\sum_n |\psi_{2,n}|^2\delta(E_1 - E_{2,n})$$

$$= \frac{2\pi}{\hbar}\frac{|u|^2}{L^2}\sum_n |\psi_{2,n}|^2\delta(eV + \frac{p_1^2}{2m} - E_{2,n})$$

(b) We can now sum over all the scattering rates for electrons that are allowed to tunnel from system 1 to system 2, within the energy range $[E_F, E_F + eV]$. Hence, we can solve for the tunneling current given by

$$I = \sum_{E_F \leq E_{\mathbf{p_1}} \leq E_F + eV} \frac{e}{\tau(\mathbf{p}_1)}$$

which will translate into an integral over energy, for each initial momentum state $\mathbf{p}_1$.

$$I = e\sum_{\mathbf{p}_1} \int_{E_F}^{E_F+eV} dE\delta(E - E_{\mathbf{p}_1})\frac{1}{\tau(\mathbf{p_1})}$$

then substituting in our expression for the scattering rate, we have

$$I = \frac{2\pi e}{\hbar}\frac{|u|^2}{L^2}\sum_n\sum_{\mathbf{p}_1}\int_{E_F}^{E_F+eV} dE\delta(E - E_{\mathbf{p}_1})|\psi_{2,n}|^2\delta(eV + \frac{p_1^2}{2m} - E_{2,n})$$

$$\approx eV\frac{2\pi e}{\hbar}\frac{|u|^2}{L^2}\sum_{\mathbf{p}_1}\delta(E_F - E(\mathbf{p}_1))\rho_2(\mathbf{p}_1, 0) = \frac{2\pi e^2|u|^2V}{\hbar L^2}\sum_{\mathbf{p}_1}\delta(E_F - E(\mathbf{p}_1))\rho_2(E_F, 0)$$

with ρ as density of states. hence, our conductance is given by

$$G = \frac{2\pi e^2}{\hbar}|u|^2\rho_1(E_F, \mathbf{x}_1 = 0)\rho_2(E_F, \mathbf{x}_2 = 0)$$

with $\rho_1 = \sum_{\mathbf{p}_1}\delta(E - E_{\mathbf{p}_1})$
(c) We can now evaluate $\rho_1(\mathbf{x})$. We have that,

$$\rho_1(E, \mathbf{x}) = \sum_{\mathbf{p}_1}|\psi_{1,\mathbf{p}_1}|^2\delta(E - E_{1,n}) = \frac{1}{(\frac{2\pi\hbar}{L})^2L^2}\int d^2\mathbf{p}_1\delta(E - E_{\mathbf{p}_1})$$

$$= \frac{1}{(2\pi\hbar)^2}\int d^2\mathbf{p}_1\delta(E - \frac{p_1^2}{2m} - eV) = \frac{1}{(2\pi\hbar)^2}\int d^2\mathbf{p}_1\delta(E - \frac{p_1^2}{2m})$$

$$= \frac{2\pi}{(2\pi\hbar)^2}\int_0^\infty dp_1 p_1\delta(E - \frac{p_1^2}{2m}) = \frac{2\pi}{(2\pi\hbar)^2}\int_0^\infty dp_1 p_1\delta(-(p_1^2 - 2mE))$$

$$= \frac{2\pi}{(2\pi\hbar)^2}\int_0^\infty dp_1 p_1\delta(E - \frac{p_1^2}{2m}) = \frac{2\pi}{(2\pi\hbar)^2}\int_0^\infty dp_1 p_1\frac{1}{2\sqrt{2mE}}[\delta(-p_1 - \sqrt{2mE}) + \delta(-p_1 + \sqrt{2mE})]$$

$$
\begin{aligned}
&= \frac{2\pi}{2(2\pi\hbar)^2\sqrt{2mE}}[\sqrt{2mE}(\theta(\sqrt{2mE}) - \theta(-\sqrt{2mE}))] \\
&= \frac{1}{4\pi\hbar^2}[\theta(\sqrt{2mE}) - \theta(-\sqrt{2mE})]
\end{aligned}
$$