Problem Set 9

Nima Leclerc (nleclerc@seas.upenn.edu)

PHYS 532 (Quantum Mechanics II)
School of Engineering and Applied Science
University of Pennsylvania

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Problem 1:
Solution:

Considering now 2 identical spin-1/2 fermions in an infinite potential well, we can write out the ground state wave functions for both the para and ortho states.

(a) The spatial wave function of the system is given by $\phi_1(x) = \sqrt{\frac{2}{L}} \sin(\frac{\pi x}{L})$ which has energy $E_1 = \frac{\hbar^2}{4mL^2}$. For the triplet state, we have 3 states:

$$\psi_{trip}(x_1, x_2, m_1, m_2) = \phi_0(x_1, x_2)\chi_1(m_1, m_2)$$

$$= (\phi_1(x_1)\phi_2(x_1) - \phi_1(x_2)\phi_2(x_2))\chi_\uparrow(m_1)\chi_\uparrow(m_2)$$

Hence, explicitly the three states are:

$$\psi^1_{trip} = \frac{2}{L}(\sin\frac{\pi x_1}{L}\sin\frac{2\pi x_1}{L} - \sin\frac{\pi x_2}{L}\sin\frac{2\pi x_2}{L})\chi_\uparrow(m_1)\chi_\uparrow(m_2)$$

$$\psi^2_{trip} = \frac{2}{L}(\sin\frac{\pi x_1}{L}\sin\frac{2\pi x_1}{L} - \sin\frac{\pi x_2}{L}\sin\frac{2\pi x_2}{L})\chi_\uparrow(m_1)\chi_\downarrow(m_1)\chi_\downarrow(m_2)$$

$$\psi^3_{trip} = \frac{2}{\sqrt{2}L}(\sin\frac{\pi x_1}{L}\sin\frac{2\pi x_2}{L})(\chi_\uparrow(m_1)\chi_\uparrow(m_2) - \chi_\downarrow(m_1)\chi_\downarrow(m_2))$$

with energy $E = \frac{2\hbar^2}{mL^2}$.

(b) For the singlet state, we have

$$\psi_{sing}(x_1, x_2, m_1, m_2) = \frac{1}{\sqrt{2}}\phi_0(x_1)\phi_0(x_2)[\chi_\uparrow(m_1)\chi_\downarrow(m_2) - \chi_\downarrow(m_1)\chi_\uparrow(m_2)]$$

$$= \frac{1}{\sqrt{2}}\frac{2}{L}\sin(\frac{\pi x_1}{L})\sin(\frac{2\pi x_2}{L})[\chi_\uparrow(m_1)\chi_\downarrow(m_2) - \chi_\downarrow(m_1)\chi_\uparrow(m_2)]$$

with energy $E = \frac{2\hbar^2}{mL^2}$

(c) We now include the interaction between the two particles given by $V = -\lambda\delta(x_1 - x_2)$. If we use the treatment of first order perturbation theory, we have a breaking in the degeneracy of the singlet and triplet states. This will correspond to the energy splitting,

$$\Delta E = -\lambda\langle\Psi|\delta(x_1 - x_2)|\Psi\rangle = \frac{1}{2}\int dx_1 dx_2 \delta(x_1 - x_2)|\phi_1(x_1)\phi_2(x_2) \pm \phi_1(x_2)\phi_2(x_1)|^2$$

From this, we can see that the integral goes to 0 if the wave function is symmetric, but is non-zero if it is anti-symmetric. Hence, this results in an energy splitting (breaks singlet-triplet degeneracy) if it's anti-symmetric.

Problem 2:
Solution:

Here we have the potential,

$$V(r) = \begin{cases} V_0, r \leq R \\ 0, r > R \end{cases}$$

(a) Now we determine the scattering amplitude $f(\theta)$ in the first Born approximation. In this approximation, we have that

$$f^{(1)}(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty rV(r)\sin qr dr$$

with $q = 2k\sin\theta/2$. Evaluating this integral gives us (for $r \leq R$),

$$f^{(1)}(\theta) = -\frac{2mV_0}{\hbar^2 q}\int_0^R r\sin qr dr = -\frac{2mV_0 R^3}{\hbar^2(qR)^2}[\frac{\sin qR}{qR} - \cos qR]$$

From this scattering amplitude, we can also evaluate the differential scattering cross-section $d\sigma/d\Omega$.

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{4m^2V_0^2R^6}{\hbar^4(qR)^4}(\frac{\sin qR}{qR} - \cos qR)^2$$

(b) Recall that we obtain the Born approximation from the expansion,

$$T = V + V\frac{1}{E - H_0 + i\epsilon}V + \cdots$$

Hence for this to work in our case (to take only first order), we must have that

$$\frac{|V\frac{1}{E-H_0+i\epsilon}V|}{|V|} = |\frac{V_0}{E - \frac{p^2}{2m} + i\epsilon}| << 1$$

(c) In the limit that $kR << 1$, we can find the differential cross section.

$$\frac{d\sigma}{d\Omega} \approx \frac{4m^2V_0^2R^6}{\hbar^4(qR)^4}(\frac{qR}{qR} - (1 - \frac{(qR)^2}{2}))^2$$

$$= \frac{4m^2V_0^2R^6}{\hbar^4(qR)^4}(\frac{(qR)^2}{2})^2 = \frac{m^2V_0^2R^6}{\hbar^4}$$

which we can see is independent of angle. We can now evaluate the total cross-section by integrating $\frac{d\sigma}{d\Omega}$,

$$\sigma = \int \frac{d\sigma}{d\Omega}d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \frac{m^2V_0^2R^6}{\hbar^4} = 4\pi\frac{m^2V_0^2R^6}{\hbar^4}$$

Problem 3:
Solution:

Now consider a hard sphere with $V(r) = 0$ for $r > a$ and $V(r) = \infty$ for $r < a$.

(a) We can derive an expression for the s wave phase shift. From the below expression, we have

$$j_l(ka)\cos\delta_l - n_l(ka)\sin\delta_l = 0$$

and the relation $\tan\delta_l = \frac{j_l(ka)}{n_l(ka)}$ follows from it. Now, if we set $l = 0$ for an s-wave we have

$$\tan\delta_0 = \frac{\sin ka/ka}{-\cos ka/ka} = -\tan ka$$

Giving us, $\delta_0 = -ka$.

(b) Now we can find the total cross section in the limit that $k \to 0$. This would correspond to small ka. We have the differential cross section,

$$\frac{d\sigma}{d\Omega} = \frac{\sin^2\delta_0}{k^2} = \frac{\sin^2(-ka)}{k^2} \approx \frac{(ka)^2}{k^2} = a^2$$

Hence, evaluating the total cross section gives us

$$\sigma = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta a^2 = 4\pi a^2$$

Hence, we can see that the total cross section is a factor of 4 larger than the geometric cross section.