# Problem Set 10

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**Problem 1:**
*Solution:*

Here we consider a 1-dimensional scattering problem where a particle scatters with a reflection and transmission coefficient $r$ and $t$. These coefficients satisfy the condition that $|t|^2 + |r|^2 = 1$. Hence, we can can derive a 1D version of the optical theorem.
(a) Now given $f(\theta = \pi) = r$ and $f(\theta = 0) = t - 1$. These are given by,

$$f(\theta = 0) = \langle \theta = 0|T|\theta = 0\rangle = 1 - \langle \theta = 0|S|\theta = 0\rangle 1 - t$$

$$f(\theta = \pi) = \langle \theta = \pi|T|\theta = 0\rangle = -r$$

Hence,

$$|f(\theta = \pi)|^2 = |r|^2$$

$$|f(\theta = 0)|^2 = (1 - t)(1 - t^\star) = 1 - t^\star - t - t^\star + |t|^2$$

Now if we add these probabilities $|f(\theta = \pi)|^2 + |f(\theta = 0)|^2$ we get

$$|f(\theta = 0)|^2 + |f(\theta = \pi)|^2 = 2 - t - t^\star = 2Re[f(\theta = 0)]$$

This gives us the outcomes for $1 - t$ and $1 - t^\star$,

$$1 - t = Re[f(\theta = 0)] + iIm[f(\theta = 0)]$$

$$1 - t^\star = Re[f(\theta = 0)] - iIm[f(\theta = 0)]$$

which we obtained from the above expression. These above expressions relate $|f(\theta)|^2$ to $Im(f(\theta = 0)$ and $Re(f(\theta = 0)$ giving a relationship for the 1D optical theorem.

(b) We can now verify the optical theorem for a spherically symmetric potential in terms of $\delta_l$. We start with our expression from the partial waves expansion,

$$f(\theta) = \frac{1}{k}\sum_l (2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta)$$

Now evaluating this at $\theta = 0$,

$$f(\theta = 0) = \frac{1}{k}\sum_l (2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos(\theta = 0))$$

$$= \frac{1}{k}\sum_l (2l+1)e^{i\delta_l}\sin\delta_l$$

We are only interested in the imaginary part of this scattering amplitude,

$$Imf(\theta = 0) = \frac{1}{k}\sum_l (2l+1)\sin^2\delta_l$$

This will be the left hand side of the expression. Evaluating the right hand side requires finding the total cross section,

$$\int d\Omega |f(\theta)|^2 = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta |f(\theta)|^2$$

$$= \frac{2\pi}{k^2}\int_{-1}^{1} d(\cos\theta)\sum_{l,l'}(2l+1)(2l'+1)e^{i(\delta_l - \delta_{l'})}P_l(\cos\theta)P_l'(\cos\theta)\sin\delta_l\sin\delta_{l'}$$

Recall the orthonormality condition that $\int_{-1}^{1} P_l(x)P_{l'}(x) = \frac{2\delta_{l,l'}}{2l+1}$. Reducing to,

$$\frac{2\pi}{k^2}\sum_{l,l'}(2l+1)(2l'+1)e^{i(\delta_l - \delta_{l'})}\frac{2\delta_{l,l'}}{2l+1}\sin\delta_l\sin\delta_{l'} = \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2\delta_l$$

This gives us,

$$\frac{4\pi}{k}Imf(\theta = 0) = \int d\Omega |f(\Omega)|^2$$

**Problem 2:**
*Solution:*

Here we compute the lifetime of a 2p state of Hydrogen which undergoes spontaneous emission of a photon. For instance, we can consider the transition from $2p_z$ to the $1s$ state. This requires employing Fermi's golden rule, given by

$$\frac{1}{\tau} = \frac{2\pi}{\hbar} \sum_{\epsilon,\lambda} |\langle \phi_{2p_z} n_{\lambda,\epsilon} + 1 = 1 | V | \phi_{2p_z}, n_{\lambda,\epsilon} = 0 \rangle|^2 \delta(E_{2p} - E_{1s} - \hbar\omega_\epsilon)$$

Here, $V$ is simply a perturbative potential corresponding to the coupling of the quantized electromagnetic field to the atom. The coupling will have the form $V = -\frac{e}{mc}\mathbf{A}\cdot\mathbf{P}$. $\mathbf{A}$ is the gauge of the field and $\mathbf{P}$ is the momentum operator for the atom. Here $\mathbf{A}$ will take the following form,

$$\mathbf{A}(\mathbf{r}) = \sum_{\lambda,\mathbf{k}} \sqrt{\frac{2\pi\hbar c^2}{\Omega\omega_k}} \hat{\epsilon}_{k,\lambda}(a_{k,\lambda} + a^\dagger_{-k,\lambda}) e^{i\mathbf{k}\cdot\mathbf{r}}$$

which results in the following coupling potential ($\Omega$ here is the volume).

$$V = -\frac{e}{m} \sum_{k,\lambda} \sqrt{\frac{2\pi\hbar}{\Omega\omega_k}} \hat{\epsilon}_{k,\lambda} \cdot \mathbf{p} e^{i\mathbf{k}\cdot\mathbf{r}} (a_{k,\lambda} + a^\dagger_{-k,\lambda})$$

Employing the dipole approximation and the fact that we are only considering stimulated emission, this would give us,

$$V = -\frac{e}{m} \sum_{k,\lambda} \sqrt{\frac{2\pi\hbar}{\Omega\omega_k}} \hat{\epsilon}_{k,\lambda} \cdot \mathbf{p} a^\dagger_{-k,\lambda}$$

We are now ready to compute the corresponding matrix element.

$$\langle \phi_{1s} n_{\lambda,\epsilon} + 1 = 1 | V | \phi_{2p_z}, n_{\lambda,\epsilon} = 0 \rangle = -\frac{e}{m} \sum_{k,\lambda} \sqrt{\frac{2\pi\hbar}{\Omega\omega_k}} \langle \phi_{1s} | \hat{\epsilon}_{k,\lambda} \cdot \mathbf{p} | \phi_{2p_z} \rangle \langle n_{\lambda,\epsilon} + 1 = 1 | a^\dagger_{-k,\lambda} | n_{\lambda,\epsilon} = 0 \rangle$$

$$= -\frac{e}{m} \sum_{k,\lambda} \sqrt{\frac{2\pi\hbar}{\Omega\omega_k}} \langle \phi_{1s} | \hat{\epsilon}_{k,\lambda} \cdot \mathbf{p} | \phi_{2p_z} \rangle$$

Let's expand the matrix element involving $\hat{\epsilon}_{k,\lambda} \cdot \mathbf{p}$. We know from the Heisenberg equation of motion that, $\mathbf{p} = \frac{m}{i\hbar}[\mathbf{r}, \mathcal{H}_0]$. Hence,

$$\hat{\epsilon}_{k,\lambda} \cdot \mathbf{p} = \frac{m}{i\hbar}[(\hat{\epsilon}_{k,\lambda} \cdot \mathbf{r})\mathcal{H}_0 - \mathcal{H}_0(\hat{\epsilon}_{k,\lambda} \cdot \mathbf{r})]$$

Hence, the matrix element we evaluate is

$$\langle \phi_{1s} | \hat{\epsilon}_{k,\lambda} \cdot \mathbf{p} | \phi_{2p_z} \rangle = \frac{m}{i\hbar}(E_{2p} - E_{1s}) \langle \phi_{1s} | \hat{\epsilon}_{k,\lambda} \cdot \mathbf{r} | \phi_{2p_z} \rangle$$

$$= \frac{m}{i\hbar}(E_{2p} - E_{1s})(\hat{\epsilon}_{q,\lambda} \cdot \hat{z}) \langle \phi_{1s} | z | \phi_{2p_z} \rangle$$

Now evaluating the matrix element,

$$\langle \phi_{1s} | z | \phi_{2p_z} \rangle = \frac{1}{\sqrt{\pi} a_0^{3/2}} \frac{1}{4\sqrt{2\pi} a_0} \int_0^\infty dr r^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta e^{-r/a_0} (r e^{-r/2a_0}) \cos\theta (r\cos\theta)$$

$$= \frac{2\pi}{4\sqrt{2}\pi a_0^{5/2}} \int_0^\infty dr r^4 e^{-3r/a_0} \int_0^\pi d\theta \cos^2\theta \sin\theta = \frac{2\pi}{4\sqrt{2}\pi a_0^{5/2}} (\frac{8a_0^5}{81})(\frac{2}{3}) = \frac{\sqrt{2}128}{243} a_0$$

To evaluate the total rate, we can work in the continuous limit by converting the discrete sum over $\mathbf{q}$ states to an integral

$$\frac{1}{\tau} = \sum_\lambda \int_{\Omega_q} d^3\mathbf{q} [\frac{e^2}{2\pi\hbar^2 cq} (E_{2p} - E_{1s})^2 (\hat{\epsilon} \cdot \hat{z})^2 | \langle \phi_{1s} | z | \phi_{2p_z} \rangle |^2 \delta(E_{2ps} - E_{1s} - \hbar qc)]$$

$$= \sum_\lambda \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta (\hat{\epsilon}_{k,\lambda} \cdot \hat{z})^2 [\frac{e^2 (E_{2p} - E_{1s})^3}{2\pi\hbar^4 c^3} | \langle \phi_{1s} | z | \phi_{2p} \rangle |^2]$$

$$= 2\pi \int_0^\pi d\theta \sin^3\theta [\frac{e^2 (E_{2p} - E_{1s})^3}{2\pi\hbar^4 c^3} | \langle \phi_{1s} | z | \phi_{2p} \rangle |^2$$

$$= \frac{8\pi}{3} [\frac{e^2 (E_{2p} - E_{1s})^3}{2\pi\hbar^4 c^3} | \langle \phi_{1s} | z | \phi_{2p} \rangle |^2 = \frac{4}{3} [\frac{e^2 (E_{2p} - E_{1s})^3}{\hbar^4 c^3} \frac{2(128)^2}{243^2}] = \frac{8(128)^2 (E_{2p} - E_{1s})^3}{3(243)^2 \hbar^4 c^3} = (\frac{2}{3})^8 \frac{me^{10}}{\hbar^6 c^3}$$

Here, $E_{1s} = -\frac{m_e e^4}{2(4\pi\epsilon_0)\hbar^2}$ and $E_{2p} = -\frac{m_e e^4}{8(4\pi\epsilon_0)\hbar^2}$. This gives us an expression now for the transition time $\tau$. Evaluating this gives us the numerical result that $\tau \sim 1.6$ ns.

**Problem 3:**
*Solution:*

Now consider a gas of atoms in equilibrium where the population of atoms $n$ is given by $n = e^{-E_n/kT}$. Given that the energy of photons we consider is $\hbar\omega = E_n - E_m$, we can determine a relation for the population of photons. We know that during emission, we have a transition from the Fock state $|n_q\rangle$ to $|n_q + 1\rangle$. Hence, the transition rate will go as

$$|\langle n_q + 1|a + a^\dagger|n_q\rangle|^2 = n_q + 1$$

Giving an emission rate,

$$\frac{1}{\tau_{emiss}} = (n_q + 1)A$$

Then for absorption, we have the opposite process given by

$$|\langle n_q - 1|a + a^\dagger|n_q\rangle|^2 = n_q$$

Hence, our rates are

$$\frac{1}{\tau_{emiss}} = (n_q + 1)A$$

$$\frac{1}{\tau_{abs}} = n_q A$$

Giving us a relationship for the photon numbers in states $n$ and $m$ (should be equal rates),

$$\frac{N_n}{\tau_{emiss}} = \frac{N_m}{\tau_{abs}}$$

Where $N_n = e^{-E_n/kT}$ and $N_m = e^{-E_m/kT}$.