Problem Set 8

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Problem 1:
Solution:

Here, we consider linearly polarized light with frequency ω incident on a 1 electron atom with a wave function that resembles a 3D harmonic oscillator with frequency ω_0. We can determine the differential cross section for this problem. Given that the electron is ejected with momentum $\hbar k_f$. We start with the expression,

$$\frac{d\sigma}{d\Omega} = \frac{2\pi\hbar c}{\omega A_0^2}\frac{dW}{d\Omega}$$

with W as the transition rate obtained from Fermi's golden rule. From this, we obtain

$$= \frac{4\pi^2\alpha\hbar}{m_e\omega}|\langle k_f|e^{i\frac{\omega}{c}\hat{n}\cdot\mathbf{r}}\hat{\epsilon}\cdot\mathbf{p}|i\rangle|^2\frac{m_e k_f L^3}{\hbar^2(2\pi)^3}$$
$$= \frac{L^3e^2}{2\pi m_e c\omega}k_f(\mathbf{k}_f\cdot\hat{\epsilon})^2|\langle\mathbf{k}_f|e^{i\frac{\omega}{c}\hat{n}\cdot\mathbf{r}}|\psi_0\rangle|^2$$

Requiring now the matrix element $\langle\mathbf{k}_f|e^{i\frac{\omega}{c}\hat{n}\cdot\mathbf{r}}|\psi_0\rangle$. We have that ψ_0 is the ground state wavefunction for the 3D Harmonic oscillator. It has the form, $\psi_0(r) = (\frac{m\omega}{\pi\hbar})^{3/4}e^{-\frac{m\omega r^2}{2\hbar}}$. Hence,

$$\langle\mathbf{k}_f|e^{i\frac{\omega}{c}\hat{n}\cdot\mathbf{r}}|\psi_0\rangle = \frac{1}{L^{3/2}}(\frac{m\omega}{\pi\hbar})^{3/4}\int d^3r e^{i\frac{\omega}{c}\hat{n}\cdot\mathbf{r}}e^{-i\mathbf{k}_f\cdot\mathbf{r}}e^{-\frac{m\omega r^2}{2\hbar}}$$

Here, we will assume that $k_f - \frac{\omega}{c}\hat{n}||\hat{z}$ and that $\mathbf{q} = \mathbf{k}_f - \frac{\omega}{c}\hat{n}$.

$$\frac{1}{L^{3/2}}(\frac{m\omega}{\pi\hbar})^{3/4}\int_0^L dz\int_0^L dy\int_0^L dx e^{-iqz}e^{-\frac{m\omega_0}{2\hbar^2}(x^2+y^2+z^2)}$$

$$= \frac{1}{L^{3/2}}(\frac{m\omega}{\pi\hbar})^{3/4}e^{-\frac{\hbar q^2}{2m\omega_0}q^2} = \frac{1}{L^{3/2}}(\frac{m\omega}{\pi\hbar})^{3/4}e^{-\frac{\hbar}{2m\omega_0}(k_f^2+\frac{\omega^2}{c^2}-2\frac{\omega}{c}\mathbf{k}_f\cdot\hat{n})}$$

Now that we have the matrix element, we can substitute for our expression of the differential absorption cross-section.

$$\frac{d\sigma}{d\Omega} = \frac{L^3e^2}{2\pi m_e c\omega}k_f(\mathbf{k}_f\cdot\hat{\epsilon})^2\frac{1}{L^3}(\frac{m\omega}{\pi\hbar})^{3/2}e^{-\frac{\hbar}{m\omega_0}(k_f^2+\frac{\omega^2}{c^2}-2\frac{\omega}{c}k_f\cos\theta)}$$

$$= \frac{L^3e^2}{2\pi m_e c\omega}k_f^3(\sin^2\theta\cos^2\phi)\frac{1}{L^3}(\frac{m\omega}{\pi\hbar})^{3/2}e^{-\frac{\hbar}{m\omega_0}(k_f^2+\frac{\omega^2}{c^2}-2\frac{\omega}{c}k_f\cos\theta)}$$

Giving us the final expression,

$$\frac{d\sigma}{d\Omega} = \frac{4\alpha\hbar^2k_f^3}{m^2\omega\omega_0}\sqrt{\frac{\pi\hbar}{m\omega_0}}\exp[-\frac{\hbar}{m\omega_0}(k_f^2+(\frac{\omega}{c})^2)]\sin^2\theta\cos^2\phi\exp[(\frac{2\hbar k_f\omega}{m\omega_0 c}\cos\theta)]$$

Problem 2:
Solution:

We now attempt to find the probability $|\phi(\mathbf{p})|d^3p$ for an electron in the ground state of the Hydrogen atom. We have that the ground state of the Hydrogen atom has the wave function,

$$\psi_0(r) = \frac{1}{\sqrt{\pi}a_0^{3/2}}e^{-2r/a_0}$$

Hence, we must find the Fourier transform of ψ_0 to get $\phi_0(\mathbf{p})$. This is given by (taking $\mathbf{p}||\hat{z}$),

$$\phi_0(\mathbf{p}) = \frac{1}{\sqrt{2\pi\hbar}}\int d^3\mathbf{r}\psi_0(r)e^{-\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar}} = \frac{1}{\sqrt{2\pi\hbar}\sqrt{\pi}a_0^{3/2}}\int d^3\mathbf{r}e^{-\frac{ipr\cos\theta_p\cos\theta}{\hbar}}e^{-2r/a_0} =$$

$$\frac{1}{\pi\sqrt{2\hbar}a_0^{3/2}}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta\int_0^{\infty}dr r^2e^{-\frac{ipr\cos\theta\cos\theta_p}{\hbar}}e^{-2r/a_0} =$$

$$\frac{1}{2^{1/2}\hbar^{3/2}\pi a_0^3}\int_{-1}^{1} d(\cos\theta)\int_0^\infty drr^2 e^{-r/a_0}e^{-\frac{ipr\cos\theta}{\hbar}} =$$

$$\frac{1}{2^{1/2}\hbar^{3/2}\pi a_0^3}\int_0^\infty drr^2 e^{-r/a_0}\frac{\hbar}{pr}\sin(\frac{pr}{\hbar}) =$$

$$\frac{2^{1/2}}{\hbar^{3/2}\pi a_0^3}\int_{-1}^{1}\frac{d\mu}{(a_0^{-1}+\frac{ip}{\hbar\mu})^3} =$$

$$\frac{2^{1/2}}{\hbar^{3/2}\pi a_0^3}\frac{2a_0^3\hbar^4}{(\hbar^2+a_0^2p^2)^2}$$

Giving us, $\phi_0(\mathbf{p}) = \frac{2^{3/2}\hbar^{5/2}}{\pi(1+\frac{a_0^2p^2}{\hbar^2})^2}$. Hence,

$$|\phi(\mathbf{p})|^2 d^3p = \frac{2^3\hbar^5}{\pi^2(1+\frac{a_0^2p^2}{\hbar^2})^4}d^3p$$

Problem 3:
Solution:

Consider now two non-identical spin 1 particles that hav e no angular momentum with $s = 0, 1, 2$. From the total spin, we can tell that the particles are bosons and hence our symmetric under exchange. Now if we take these to be identical, we can seek restrictions on the quantum number of the particles. This now implies that $s_1 = s_2 = 1$, hence giving the possible s values $0 \leq s \leq 2$. Now looking at the CG-coefficients we have that

$$\langle s_1m_1s_2m_1|sm_s\rangle = (-1)^{s_1+s_2-s}\langle s_2m_2s_1m_1|sm_s\rangle = (-1)^{2-s}\langle 1m_21m_1|sm_s\rangle =$$

$$(-1)^s\langle m_2m_1|sm_s\rangle$$

Hence, we write out the state

$$|sm_s\rangle = \sum_{m_1,m_2}(-1)^s\langle m_2m_1|sm_s\rangle|m_1m_2\rangle = (-1)^s\sum_{m_1,m_2}\langle m_2m_1|sm_s\rangle|m_1m_2\rangle$$

Now let's perform an exchange operation between the two particles, which will result in

$$|sm_s\rangle = (-1)^s\sum_{m_1,m_2}\langle m_2m_1|sm_s\rangle|m_2m_1\rangle$$

$$= (-1)^s|sm_s\rangle$$

For this to be a boson, the outcome of that exchange must be symmetric. Hence, s must be odd. Since the only allowable values for s are $s = 0, 1, 2$ we must have that $s \neq 1$.

Problem 4:
Solution:

Here we have now three spin 0 particles at the corners of an equilateral triangle, such that the z axis goes through the center of the triangle. We can determine the restrictions on the magnetic quantum numbers corresponding to J_z (m_j). Since the three particles can rotate freely about the z axis and the system had three-fold symmetry, we have that the states of the system are also eigenstates of three-fold rotation operator given by,

$$\hat{D} = \exp(-i\frac{2\pi}{3\hbar}n\hat{J}_z)$$

which has eigenvalues $d_j = \exp(-i\frac{2\pi}{3}nm_j)$ with m_j as the eigenvalue of J_z with n as an integer. For this system to be bosonic, the particles must be symmetric under exchange. We can exchange by swapping particles between different points of the triangle which has the effect of rotating by $2\pi/3$. Hence, for this to be symmetric under exchange we must have that this rotation does not result in a sign change. Hence, we require that $\exp(-i\frac{2\pi}{3}nm_j) = 1$ which puts a constraint on the eigenvalues of J_z.