Problem Set 6

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Problem 1
Solution:

(a) Stabilization via linearization aims to find a control law u such that the origin is uniformly asymptotically stable for the system $\dot{x} = f(x, t, u(x, t))$ when the system is linearized over the state coordinates x and control coordinate u, such that u is a linear controller $u = -Kx$. We normally would use this as a first-pass approach for stabilization, since it is easy to implement. This technique tends to work well when there are no strong linearities and relies on the assumption that the linearization parameters A and B are controllable or stabilizable and when the controller is linear (such that $\dot{x} = Ax + Bu$). This is a local approach and it guarantees asymptotic stability about a point, but not global asymptotic stability.

(b) Integral control is convenient when one is not confident of the parameters used in a control system. In these cases, small perturbations in the control system parameters can lead to dramatic shifts in the system's equillibria. Hence, using integral control which does not depend on these parameters become advantageous. Here, we formulate the problem as a tracking and rejection problem where we define the error as the difference between the system's output and some desired trajectory. We then seek a control law such that the error $e \to 0$ as $t \to \infty$ (approach steady state) when a disturbance w is present. This approach effectivly integrates away the accumulated error in the system. However, this assumes that the constant K_2 used for the integrator is invertible. The main advantage in using integral control is that the approach is robust to disturbances that may arise from noise or other sources in the system.

(c) In full-state linearization, we design a controller that is linear in some control v, which has the advantage that the output of the system $y(t)$ need not be known, because of this it's widely applied to problems where the system output is difficult to obtain. Here, the state equation is completly linearized. However, this has the drawback that it's not as robust as integral control since it requires the system parameters to be exact and the dynamics must have the form $\dot{x} = Ax + B\gamma(x)(u - \alpha(x))$ where (A, B) are controllable and $\gamma(x)$ is non-singular.

(d) In input-output linearization, we can generate a system by taking multiple Lie derivatives of the initial system until it appears that the initial system becomes linear in the control variable u. Here, the input-output mapping is linearized but it's fine to only partially linearize the state equation. For this, we can utilize the diffeomorphic transformation $z = T(x)$ to obtain a linear relationship. The relative degree in this system is essentially the order of the derivative needed for the control u to appear.

(e) Verisig is an approach used to verify the safety of feedback systems using a neural network as the controller. In contrast to previous approaches, the controller is now parameterized by the weights of a DNN which are learned from the system dynamics. The drawbacks of this approach are that it requires a large amount of data to train the DNN. However, the system is robust since it does not rely on system parameters to be known precisely a-priori.

Problem 2

Solution:

Consider now the system,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\hat{l}}\sin x_1 - \frac{\hat{k}}{\hat{m}}x_2 + \frac{1}{\hat{m}\hat{l}^2}T \tag{1}$$

where $\hat{m} = \hat{l} = \hat{k} = 1$ and $g = 10$ are the model parameters, $x(0) = (\pi, 1)$ as the initial condition. However, the actual parameters are $m = 1.3$, $l = 1.2$, and $k = 1.5$. Take a control objective to stabilize about the equilibrium $x_d = (\delta, 0)$ with $\delta = 1.5$.

(a) First we consider a feedback controller based on linearization. Here, we have that $\dot{x} = Ax + Bu$ with $A = \frac{\partial f}{\partial x}|_{x=0,u=0}$ and $B = \frac{\partial f}{\partial u}|_{x=0,u=0}$. We attempt to stabilize about $(\delta, 0)$. From this, we can solve for the controller T such that steady state is achieved $f = 0$. Solving now,

$$\dot{x}_1 = 0$$
$$\dot{x}_2 = -\frac{g}{\hat{l}}\sin\delta + \frac{1}{\hat{m}\hat{l}^2}T_d$$

results in $T_d = mgl\sin\delta$. Let's transform coordinates now such that $\bar{x} = x - x_d$ and $\bar{T} = T - T_d$. The transformed system becomes,

$$\dot{\bar{x}}_1 = \bar{x}_2$$
$$\dot{\bar{x}}_2 = -g/\hat{l}(\sin(\bar{x}_1 + \delta) - \sin\delta) - \hat{k}/\hat{m}\bar{x}_2 + \frac{1}{m\hat{l}^2}\bar{T}$$

allowing us to linearize $\dot{\bar{x}} = A\bar{x} + B\bar{T}$ with,

$$A = \begin{pmatrix} 0 & 1 \\ -g/\hat{l}\cos(\delta) & -\hat{k}/m \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ g/\hat{m}\hat{l}^2 \end{pmatrix}$$

resulting in the controller (with $\bar{T} = -K\bar{x}$),

$$T = \hat{m}\hat{l}g\sin\delta - K\begin{pmatrix} x_1 - \delta \\ x_2 \end{pmatrix}$$

Here, we will take $K = [k_1 k_2]$ with $k_1 > -g/\hat{l}\cos\delta/(g\hat{l}^2)$ and $k_2 > -\hat{k}/\hat{m}$. Since this controller now depends on the model parameters $\hat{m}, \hat{l}, , \hat{K},$, we cannot obtain the exact control objective since they differ from the parameters in the physical system. Using these model parameters, we can still plot the trajectories and control input. Here we pick $k_1 = 0.05$, $k_2 = 0.01$. The trajectories and torque are plotted in Figure 1(a)/(b). Note that the trajectories plotted are far off from our desired stabilization point.

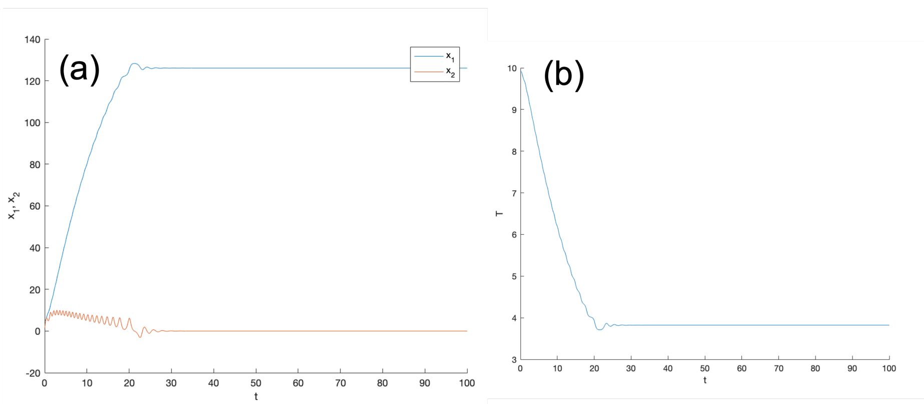

Figure 1: Problem 2(a).

(b) We can now implement integral control for the system. We found previously that the estimate $T_d = \hat{m}\hat{l}g\sin\delta$. Using this, the equilibrium would shift significantly. If we now use integral control instead, we can define the error as $\dot{\sigma} = x_1 - \delta$. We can pick the controller,

$$T(x,\sigma) = -k_1x_1 - k_2x_2 - k_3\sigma = -\bar{K}\begin{pmatrix} x \\ \sigma \end{pmatrix}$$

with $\sigma_d = \frac{1}{k_3}(T_d + k_1x_{1d} + k_2x_{2d})$ Hence, we should linearize about d_d, T_d, σ_d. Then,

$$\bar{A} = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, \bar{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$$

here,

$$A = \begin{pmatrix} 0 & 1 \\ -g/\hat{l}\cos\delta & -\hat{k}/\hat{m} \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1/\hat{m}\hat{l}^2 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

To satisfy the condition that the system is Hurwitz, we must have that $\hat{k}/\hat{m} + k_2/\hat{m}\hat{l}^2 > 0$, $(\hat{k}/\hat{m} + k_2/\hat{m}\hat{l}^2)(g/l\cos\delta + k_1/\hat{m}\hat{l}^2) > 0$, and $k_3/\hat{m}\hat{l}^2 > 0$. Hence, using our settings we can pick $k_1 = 1.1$, $k_2 = 1.1$, $k_3 = 1.1$ for the control law,

$$u = -k_1(x_1 - \delta) - k_2x_2 - k_3\sigma$$
$$\dot{\sigma} = \theta - \delta$$

Hence, using our choice of k_1, k_2, k_3 above we can solve the system below.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -g/\hat{l}\sin(x_1 + \delta) - (\hat{k}/m)x_2 + g/\hat{m}\hat{l}^2(-k_1x_1 - k_2x_2 - k_3\sigma)$$
$$\dot{\sigma} = x_1 - \delta$$

Figures 2(a)/(b) shows the outputs x_1/x_2 and the controller T. Note that both x_1 and x_2 approach our desired equillibria of $x_1 = 1.5$ and $x_2 = 0$ we seek for stabilization as $t \to \infty$.

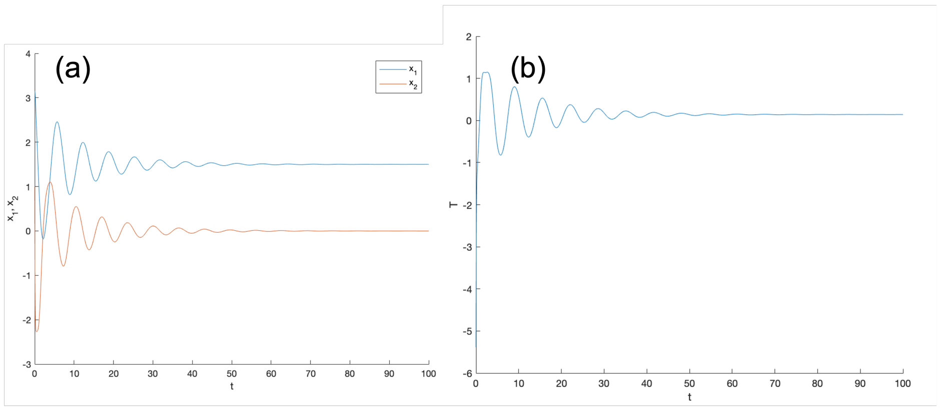

Figure 2: Problem 2(b).

(c) Given the system,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\hat{l}}\sin x_1 - \frac{\hat{k}}{\hat{m}}x_2 + \frac{1}{\hat{m}\hat{l}^2}T$$

Designing via full-state feedback linearization, we have a T to cancel out the system's nonlinearities and stabilize the linear components with coefficients k_1, k_2.

$$T = \hat{m}\hat{l}^2(-\frac{g}{\hat{l}}\sin x_1 - k_1x_1 - k_2x_2)$$

From this, we can see that the controller has robustness issues since it depends on the system parameters. However, we can still proceed. Substituting into the system gives us,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{\hat{k}}{\hat{m}}x_2 - \hat{m}\hat{l}^2(k_1x_1 + k_2x_2)$$

The corresponding matrix that linearizes the system is given by,

$$A = \begin{pmatrix} 0 & 1 \\ -\hat{m}\hat{l}^2k_1 & -\hat{m}\hat{l}^2k_2 - \hat{k}/\hat{m} \end{pmatrix}$$

This system has eigenvalues, $\lambda_{\pm} = \frac{1}{2m}(-k - k_2 \hat{l}^2 \hat{m}^2 \pm \sqrt{-4k_1 \hat{l}^2 \hat{m}^2 + (k + k_2 \hat{l}^2 \hat{m}^2)^2})$. If $\Re\{\lambda_{\pm}\} < 0$, then we have a properly designed controller. Hence, we must choose k_1, k_2 to satisfy this condition. Hence, we get that $k_1 > 0$ and $k_2 > \hat{k}/\hat{l}^2 \hat{m}^2$. This gives our final control law, where we can now pick $k_1 = 0.1$ and $k_2 = 1.1$. Hence, the control input becomes

$$T = (-10 \sin x_1 - 0.1 x_1 - 1.1 x_2)$$

We can now obtain the corresponding trajectories and control input. This is plotted in Figure 3. Clearly, the trajectory plotted here is far away from equillibrium.

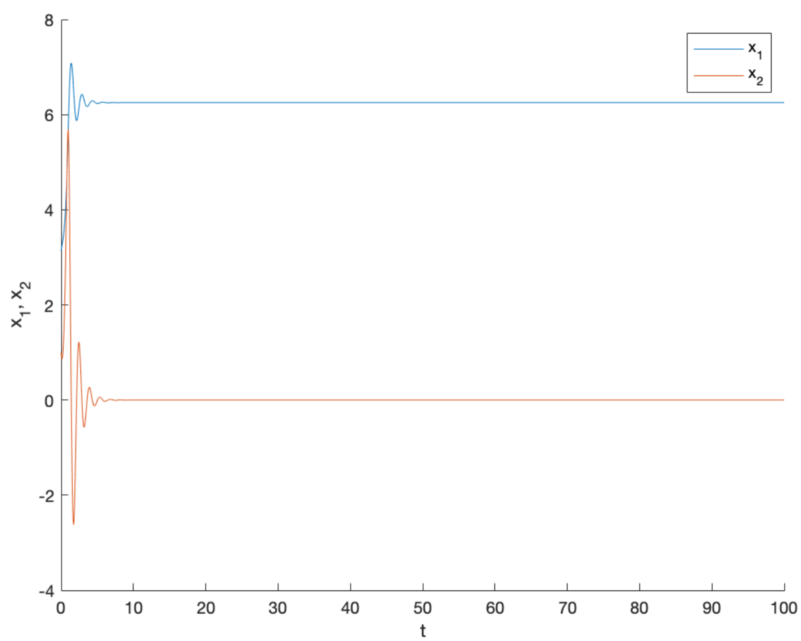

Figure 3: Problem 2(c).

(d) Here, the approaches in (a) and (c) rely too heavily on the system parameters- making the trajectories sensitive to small changes in input parameters. However, these two approaches came with the advantage of ease of implementation. However, the approach in (b) is robust to these variations on system parameters. I would ultimatly choose (c) since it does not depend on system paramaters and easily stabilizes at the desired point.

Problem 3

Solution:

(a) Consider the system below,

$$\dot{x}_1 = x_1 + x_2$$

$$\dot{x}_2 = x_1 x_2^2 - x_1 + x_3$$

$$\dot{x}_3 = u$$

we now want to linearize the system such that $\dot{\bar{x}} = A\bar{x} + B\bar{u}$, with $\bar{x} = x - x_d$ and h $\bar{u} = u - u_d$, with x_d and u_d as the equillibria of the system. We will also assume that u is a linear controller where $u = -Kx$. We can take $x_d = (x_{1d}, x_{2d}, x_{3d}) = (0, 0, 0)$. Hence, $\dot{\bar{x}} = A\bar{x} + B\bar{u} = \dot{x} = Ax + Bu$. Again, $A = \frac{\partial f}{\partial x}|_{x=0.u=0}$ and $B = \frac{\partial f}{\partial u}|_{x=0.u=0}$. We obtain,

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

we can pick our controller to be,

$$u = -k_1 x_1 - k_2 x_2 - k_3 x_3$$

The system should satisfy the condition that $A - BK$ is Hurwitz. Here,

$$A - BK = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} k_1 & k_2 & k_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -k_1 & -k_2 & -k_3 \end{pmatrix}$$

with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. However, the expressions for these eigenvalues are too large to write down here. Our controller would be,

$$u = -k_1 x_1 - k_2 x_2 - k_3 x_3$$

for k_1, k_2, k_3 chosen such that $\Re\{\lambda_1, \lambda_2, \lambda_3\} < 0$.

(b) Here we have the system,

$$\dot{x}_1 = x_1 + x_2$$

$$\dot{x}_2 = 3x_1^2 x_2 + 1 + u$$

If we take the equillibrium point $(x_{1,d}, x_{2,d}) = (0, 0)$, then substituting into the above relation gives $u_d = -1$. Here, we have (with A, B evaluated at $(x_1, x_2, u) = (0, 0, -1)$.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we can design a linear controller $\bar{u} = -K\bar{x} = -k_1 x_1 - k_2 x_2$ which stabilizes the system. The controller would become,

$$u = u_d + \bar{u} = -1 - k_1 x_1 - k_2 x_2$$

We should impose that $A - BK$ is Hurwitz. This matrix becomes,

$$A - B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 - k_1 & -k_2 \end{pmatrix}$$

which have eigenvalues $\lambda_\pm = \frac{1}{2}(1 - k_2 \pm \sqrt{5 - 4k_1 + 2k_2 + k_2^2})$. Hence our controller becomes,

$$u = 1 - k_1 x_1 - k_2 x_2$$

for k_1 and k_2 chosen such that $\Re\{(1 - k_2 \pm \sqrt{5 - 4k_1 + 2k_2 + k_2^2})\} < 0$.

Problem 4

Solution:

Consider now the system,

$$\dot{x}_1 = 8x_2 + x_3$$

$$\dot{x}_2 = -x_2 + x_3$$
$$\dot{x}_3 = x_1^4 - x_1^2 - x_3 + u$$

we can design our controller to cancel out the nonlinearities and establish stability of the system. Hence, we can choose $u = -x_1^4 + x_1^2 - k_1x_1 - k_2x_2 - k_3x_3$ where k_1, k_2, k_3 are coefficients to be determined. We essentially want the linear feedback control input to stabilize the system. The system then becomes,

$$\dot{x}_1 = 8x_2 + x_3$$
$$\dot{x}_2 = -x_2 + x_3$$
$$\dot{x}_3 = -k_1x_1 - k_2x_2 - k_3x_3$$

This linearized form has a matrix A below,

$$A = \begin{pmatrix} 0 & 8 & 1 \\ 0 & -1 & 1 \\ -k_1 & -k_2 & -k_3 \end{pmatrix}$$

This matrix will have eigenvalues $\lambda_1, \lambda_2, \lambda_3$ which are too complex to write out explicitly here. However, k_1, k_2, k_3 can determined such that $\Re\{\lambda_1, \lambda_2, \lambda_3\} < 0$ again using our controller,

$$u = -x_1^4 + x_1^2 - k_1x_1 - k_2x_2 - k_3x_3$$

Problem 5
Solution:
Consider the system,

$$\dot{x}_1 = -x_1 + x_2 - x_3$$
$$\dot{x}_2 = -x_1x_3 - x_2 + u$$
$$\dot{x}_3 = -x_1 + u$$
$$y = x_3$$

We can see if the system has a relative degree by computing $\dot{y}$. Here,

$$\dot{y} = \dot{x}_3 = -x_1 + u$$

Here, u appears. Hence the system has relative degree 1 in $\mathbb{R}^3$. Hence, it is input-output linearizable since we can use the control law,

$$u = \frac{1}{L_g L_f^{\rho-1} h(x)}(-L_f^{\rho} h(x) + v)$$

(b) We can now convert the state equation into normal form. We can take $\xi = y$ and $\eta = [\eta_1 \ \eta_2]^T = [x_1 \ x_2]^T$ since the relative order of the system is $\rho = 1$ and the overall dimension is $n = 3$. To express in normal form, we must have that

$$\dot{\eta} = \frac{\partial \phi}{\partial x} f(x)|_{x=T^{-1}(z)}$$

$$\dot{\xi} = A_c \xi + B_c \gamma(x)[u - \alpha(x)]$$

$$y = C_c \xi$$

Hence, we have the transformation

$$z = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ h(x) \end{pmatrix} = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ x_3 \end{pmatrix}$$

Here $\phi_1(x)$ and $\phi_2(x)$ must satisfy that $\phi_1(0) = \phi_2(0) = 0$ and,

$$\frac{\partial \phi_1}{\partial x} g(x) = \frac{\partial \phi_2}{\partial x} g(x)$$

Hence,

$$\frac{\partial \phi_1}{\partial x_1}(0) + \frac{\partial \phi_1}{\partial x_2} + \frac{\partial \phi_1}{\partial x_3} = 0$$

Hence, we must satisfy $\frac{\partial \phi_1}{\partial x_1} + \frac{\partial \phi_1}{\partial x_2} = 0$

The solution to this differential equation (with $\phi_1(0) = 0$) is $\phi_1 = (x_3 - x_2) = \xi - \eta_2$. We can also take $\phi_1 = \phi_2$. Again, $\eta_1 = x_1$, $\eta_2 = x_2$, and $\xi = x_3$. Hence,

$$\dot{\eta}_1 = -\phi_1 + \phi_1 - \xi = -\xi$$

$$\dot{\eta}_2 = -\phi_1 \xi - \phi_2 + u = -\phi_1(\xi - 1) + u$$

$$\dot{\xi} = -\eta_1 + u$$

$$y = \xi$$

Hence, our normal system is

$$\dot{\eta}_1 = -\xi$$

$$\dot{\eta}_2 = -(\xi - \eta_2)(\xi - 1) + u$$

$$\dot{\xi} = -\eta_1 + u$$
$$y = \xi$$

where our domain for this transformation to be valid is in $\mathbb{R}^3$.

(c) We can characterize the zero dynamics by $Z^\star = \{x \in \mathbb{R}^3 | x_3 = 0\}$ and take $u = 0$. Hence, we get

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = x_2$$
$$\dot{x}_3 = -x_1$$

This system has eigenvalues $\lambda = \pm 1, 0$. Hence, the system is not minimum phase given that it's not asymptotically stable.