Problem Set 4

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Problem 1
Solution:
Consider now a robotic arm with the EOM,

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

with q as a joint angle, M as inertia, C as centrifugal force, g as the gravitational force, and τ as the applied torque.

(a) Taking $g = 0$ and a controller input,

$$\tau = K(r - q) + K_d\dot{q}$$

we can first check asymptotic tracking, first defining a Lyapunov function of the system to be

$$V(q) = V(e) = \frac{1}{2}(\dot{q}^2 M(q) + K(q-r)^2) = \frac{1}{2}(\dot{e}^2 M(q) + Ke^2) = \frac{1}{2}(x_2^2 M(x_1) + Kx_1^2)$$

with $e = q - r$ as the error of the system, defined as the difference in the coordinate of the robot and the reference. Hence, we can write this as a system defining $x_1 = q - r = e$ and $x_2 = \dot{x}_1$.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{M(x_1)}(\tau - C(x_1, x_2)x_2) = \frac{1}{M(x_1)}(-Kx_1 + K_d x_2 - g(x_1) - C(x_1, x_2)x_2)$$

We want that $\dot{V} < 0$. Evaluating $\dot{V}$ gives,

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} f_1(x_1, x_2) + \frac{\partial V}{\partial x_1} f_2(x_1, x_2)$$

We have,

$$\frac{\partial V}{\partial t} = \frac{1}{2}(2x_2 \dot{x}_2 M(x_1) + x_2^2 \dot{M}(x_1) + 2Kx_1 x_2)$$

$$\frac{\partial V}{\partial x_1} = \frac{1}{2}(x_2^2 \frac{\partial M(x_1)}{\partial x_1} + 2Kx_1)$$

$$\frac{\partial V}{\partial x_2} = x_2 M(x_1)$$

Hence,

$$\begin{aligned}
\dot{V} = (x_2 \dot{x}_2 M(x_1) + \frac{1}{2}x_2^2 \dot{M}(x_1) + Kx_1 x_2) + (\frac{1}{2}x_2^2 \frac{\partial M(x_1)}{\partial x_1} + Kx_1)x_2 \\
+ x_2(-Kx_1 + K_d x_2 - g(x_1) - C(x_1, x_2)x_2) \\
= x_2^2(\frac{1}{2}\dot{M} - C) + x_2 \dot{x}_2 M + K_d x_2^2 - g x_2 + \frac{1}{2}x_2^2 \frac{\partial M(x_1)}{\partial x_1}
\end{aligned}$$

Here we take $g = 0$, hence the expression becomes

$$\dot{V} = x_2^2(\frac{1}{2}\dot{M} - C) + x_2 \dot{x}_2 M + K_d x_2^2 + \frac{1}{2}x_2^2 \frac{\partial M(x_1)}{\partial x_1}$$

We have $\dot{M} - 2C$ as skew-symmetric, so this term will be < 0 here. The $K_d x_2^2 < 0$ here since $K_d < 0$ and we expect both $M < 0$ and $\frac{\partial M(x_1)}{\partial x_1}$. Hence, we get that $\dot{V} < 0$. We can also easily see that $V(0) = 0$ and $V > 0$ here. Hence, we satisfy all the conditions for the system to be asymptotically stable. Hence, asymptotic tracking can be achieved here.

(b) Now we have that $g \neq 0$. In this case, the expression for $\dot{V}$ is given below.

$$\dot{V} = x_2^2(\frac{1}{2}\dot{M} - C) + x_2 \dot{x}_2 M + K_d x_2^2 - g x_2 + \frac{1}{2}x_2^2 \frac{\partial M(x_1)}{\partial x_1}$$

Since $g < 0$ (due to gravitational field), we expect the corresponding term above should be > 0. Hence, if this is sufficiently large, there will be a point when $\dot{V} > 0$. This suggests that we must modify the Lyapunov function such that the corresponding term cancels out. Hence, we seek a term added to V such that a term $+gx_2$ emerges in $\dot{V}$ to cancel out the term $-gx_2$. We can take it such that $\dot{g}(x_1) = 0$. If we take the modified Lyapunov function,

$$V(x_1, x_2) = \frac{1}{2}(x_2^2 M(x_1) + Kx_1^2) + gx_2$$

We now get the derivative,

$$\frac{\partial V}{\partial x_2} = x_2 M(x_1) + g$$

Hence, this gives $\dot{V}$,

$$\dot{V} = x_2^2(\frac{1}{2}\dot{M} - C) + x_2\dot{x}_2 M + K_d x_2^2 + \frac{1}{2}x_2^2\frac{\partial M(x_1)}{\partial x_1}$$

which cancelled out the $-gx_2$ term. Hence, we now again have that $\dot{V} < 0$.

(c) We now seeks a control law such the relationship between the control input v and q is linear. We can write the dynamical equation,

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau(q, \dot{q}, u)$$

where we seek a control law $\tau(q, \dot{q}, u)$ with control input such that $\ddot{q} = u$ so that the dynamical response is linear in the control input u and is independent of the nonlinear contributions in the dynamics. Hence, we can pick a τ such that all non-linear contributions on the left hand side of the above equation vanish. A natural choice would be the following,

$$\tau(q, \dot{q}, u) = C(q, \dot{q})\dot{q} + g(q) + \frac{u}{M(q)}$$

plugging this in gives us,

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) - \tau(q, \dot{q}, u)$$

$$= M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) - (C(q, \dot{q})\dot{q} + g(q) + \frac{u}{M(q)})$$

$$= M(q)\ddot{q} - \frac{u}{M(q)} = 0$$

Which is equivalent to $\ddot{q} = u$. Hence, our choice for τ achieves our desired result.

(d) The robustness of the system is quantified by the output's sensitivity to variations on the control input. In the first case (part (a)), the controller is considered to be robust as it has a strong dependence on the dynamics and no dependence on nonlinear parameters, making it insensitive to perturbations on the input. In part (b)/(c), the controller has a strong dependence on $g(q)$- the gravitational force. Here, the gravitational force has a strong tendency to fluctuate, resulting in an output that is sensitive to these fluctuations. Hence, this controller is less robust as small deviations from the actual value of $g(q)$ would result in accumulated error to the dynamics.

Problem 2
Solution:
Consider the system,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_1 - 2x_2 - 4x_1^3$$

(a) If we take the Lyapunov function, $V(x) = 0.5(x_1^2 + x_2^2)$, we can first check asymptotic stability at the origin by evaluating $\dot{V}$.

$$\dot{V}(x_1, x_2) = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$$

$$= \frac{1}{2}((2x_1)(x_2) + (2x_2)(-2x_1 - 2x_2 - 4x_1^3))$$

$$= -x_1x_2 - 2x_2^2 - 4x_1^3x_2$$

This expression for $\dot{V}(x_1, x_2)$ is plotted in Figure 1. Here, we can clearly see that there are regions where $\dot{V} > 0$ as is evident from the expression above. Hence, this Lyapunov function does not demonstrate global asymptotic stability. However in the regions of the plot where $\dot{V} < 0$, we can claim that the system is asymptotically stable.

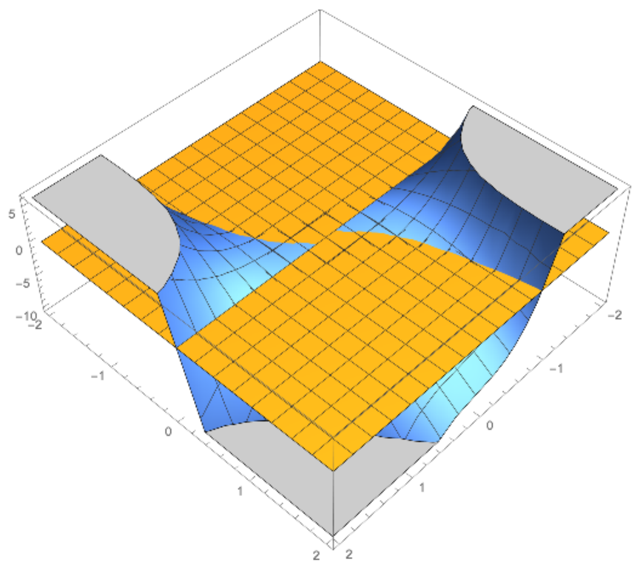

Figure 1: $\dot{V}(x_1, x_2)$ for Problem 2. $\dot{V}$ in blue and the $f = 0$ plane in orange.

(b) For a globally asymptotic behavior, we seek a Lyapunov function that $\dot{V} < 0$ (except at the origin), $V(0) = 0$, $V(x) > 0$, and that $V \to \infty$ as $||x|| \to \infty$. Considering our derived expression for $\dot{V}$, we obtained

$$\dot{V}(x_1, x_2) = -x_1x_2 - 2x_2^2 - 4x_1^3x_2$$

Hence, it is clear that we would have an asymptotically stable system if $\dot{V}(x_1, x_2) = -2x_2^2$ which requires a Lyapunov function to cancel out every other term above. Hence, we can solve for V such that

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = -2x_2^2$$

Since there are two variables in this PDE ($\frac{\partial V}{\partial x_1}$ and $\frac{\partial V}{\partial x_2}$), we need two equations. To solve this, we can add the constraint that $\frac{\partial V}{\partial x_2} = x_2$ which is the same as before and satisfies the requirement of global asymptotic stability. This results in,

$$\dot{V} = \frac{\partial V}{\partial x_1}x_2 + x_2(-2x_1 - 2x_2 - 4x_1^3) = -2x_2^2$$

Solving for $\frac{\partial V}{\partial x_1}$ gives,

$$\frac{\partial V}{\partial x_1} = 2x_1 + 4x_1^3$$

Integrating gives,

$$V(x_1, x_2) = V(x_1 = 0, x_2) + x_1^2 + x_1^4 = x_2^2 + x_1^2 + x_1^4$$

Resulting in our new candidate Lyapunov function. We can verify that this is valid by evaluating $\dot{V}$.

$$\dot{V} = (2x_1 + 4x_1^3)x_2 + (2x_2)(-2x_1 - 2x_2 - 4x_1^3) = -2x_2^2 < 0$$

Since this gives us $\dot{V} < 0$ for $x_2 \neq 0$, we have that the system is asymptotically stable. Since $V \to \infty$ as $x_1, x_2 \to \infty$, the function choice is radially unbounded. This function choice also satisfies that $V(0) = 0$ and $V(x) > 0$. Hence,

$$V(x_1, x_2) = x_1^2 + x_2^2 + x_1^4$$

would satisfies our criteria for this system to be globally asymptotically stable.

Problem 3

Solution:

Take the system,

$$\dot{x}_1 = x_1(\sin^2(\frac{\pi x_2}{2} - 1))$$
$$\dot{x}_2 = -x_2 + u$$

(a) If $u = 0$, then our system becomes

$$\dot{x}_1 = x_1(\sin^2(\frac{\pi x_2}{2} - 1))$$
$$\dot{x}_2 = -x_2$$

we can take a candidate Lyapunov function to be $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$. Hence, we can evaluate $\dot{V}$,

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$$

$$= x_1^2(\sin^2(\frac{\pi x_2}{2}) - 1) - x_2^2$$

Since the term $\sin^2(\frac{\pi x_2}{2})$ is upper bounded by 1, the term $(\sin^2(\frac{\pi x_2}{2}) - 1)$ is upper bounded by 0. Hence, the entire expression for $\dot{V}$ satisfies that $\dot{V} < 0$.

Given that $V(0) = 0$, $V(x) > 0$, and that $V(x)$ is radially unbounded, the system is globally asymptotically stable.

(b) We can now demonstrated that for bounded input $u(t)$, the state $x(t)$ is bounded. For this, we must show that $\dot{V} \leq 0$. Here,

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \frac{\partial V}{\partial t} \\ &= x_1^2(\sin^2(\frac{\pi x_2}{2}) - 1) - x_2^2 + u(x_2 + 2\dot{u})\end{aligned}$$

We saw from before that $x_1^2(\sin^2(\frac{\pi x_2}{2}) - 1) - x_2^2 \leq 0$. Since u is bounded, $\dot{u} < 0$. For $|x_2| > 1$, $x_2^2 > ux_2$. Hence, $\dot{V} \leq 0$ for a bounded control input u. We can conclude that for bounded u, we will always have bounded x.

(c) Given $u(t) = 1$, $x_1(0) = a$, and $x_2(0) = 1$, we can show that $x_1(t) = a$ and $x_2(t) = 1$. Here, we can go back to Lyapunov's direct method. We can show that $\dot{V} \leq 0$ to prove that a solution starting in $x_1 = a, x_2 = 1$ will stay there. We have from before,

$$\dot{V} = x_1^2(\sin^2(\frac{\pi x_2}{2}) - 1) - x_2^2 + u(x_2 + 2\dot{u}) = x_1^2(\sin^2(\frac{\pi x_2}{2}) - 1) - x_2^2 + (x_2 + 0) = x_1^2(\sin^2(\frac{\pi x_2}{2}) - 1) - x_2^2 + x_2$$

At $x_1 = a, x_2 = 1$,

$$\dot{V} = a^2(\sin^2(\frac{\pi}{2}) - 1) - (1)^2 + (1) = 0$$

which shows that for this choice of initial solutions, $\dot{V}$ is invariant. Hence, we expect to stay at this solution for any t. $x_1 = a$ and $x_2 = 1$ for all t.

(d) We can now show input-to-state stability. Take the candidate Lyapunov function to be,

$$V = \frac{1}{2}(x_1^2 + x_2^4)$$

when we have no control input, $u = 0$ we get that

$$\dot{V} = x_1^2(\sin^2(\frac{\pi x_2}{2}) - 1) - x_2^4 < 0$$

Now take $u \neq 0$, we get

$$\dot{V} = x_1^2 \sin^2(\frac{\pi x_2}{2}) - (x_1^2 + x_2^4) + x_2^3 u$$

$$\leq x_1^2 \sin^2(\frac{\pi x_2}{2} - (x_1^2 + x_2^4) + |x_2|^3 |u|$$

We will take the term $-(x_1^2 + x_2^4)$ to dominate $|x_2|^3|u|$, allowing us to write

$$\dot{V} =\leq (1-\theta)(x_1^2 \sin^2(\frac{\pi x_2}{2}) - (x_1^2 + x_2^4)) + \theta(x_1^2 \sin^2(\frac{\pi x_2}{2}) - (x_1^2 + x_2^4)) + |x_2|^3|u|$$

for $0 < \theta < 1$. We expect,

$$\theta(x_1^2 \sin^2(\frac{\pi x_2}{2}) - (x_1^2 + x_2^4)) + |x_2|^3 |u| \leq 0$$

for $|x_2| \leq |u|/\theta$ and $|x_1| \geq (|u|/\theta)^2$.
This implies that,

$$\max\{|x_1|, |x_2|\} \geq \max\{\frac{|u|}{\theta}, (\frac{|u|}{\theta})^2\}$$

We define the class $\mathcal{K}$ function ρ as ,

$$\rho(r) = \max\{\frac{r}{\theta}, (\frac{r}{\theta})^2\}$$

satisfying the inequality,

$$\dot{V} \leq \theta(x_1^2 \sin^2(\frac{\pi x_2}{2}) - (x_1^2 + x_2^4)) + |x_2|^3 |u| \ \forall ||x||_\infty \geq \rho(|u|)$$

This inequality satisfies the condition that we look for in input-to-state stability. Moreover, $V(x)$ is positive definite and radially unbounded. Hence, this system is input-to-state stable.

Problem 4

Solution:

Here, we consider the system

$$\dot{x}_1 = x_1(\sin^2(\pi x_2/2) - 1)$$

$$\dot{x}_2 = -x_2$$

SOS programming can be implemented to construct a Lyapunov function to prove asymptotic stability. We want to solve for the coefficients of the 2x2 matrix Q below that satisfy our constraints,

$$p(x) = Z(x)^T Q Z(x)$$

with $Z(x) = [x_1^2 x_2^2]$. Here we enforce that $Q \geq 0$ so that it is positive semi-definite. Our two constraints are shown below,

$$V(x) - (x_1^2 + x_2^2) \geq 0$$

$$\frac{\partial V}{\partial x} f(x) \geq 0$$

This is implemented in Matlab and the code used for the program is shown below,

```
clear; echo on;
syms x1 x2;
vars = [x1; x2];

% Constructing the vector field dx/dt = f
f = [x1*(sin(pi*x2/2)^2 - 1);
     -x2];

% =============================================
% First, initialize the sum of squares program
prog = sosprogram(vars);

% =============================================
% The Lyapunov function V(x):
[prog,V] = sospolyvar(prog,[x1^2; x2^2],'wscoeff');

% =============================================
% Next, define SOSP constraints

% Constraint 1 : V(x) - (x1^2 + x2^2) >= 0
prog = sosineq(prog,V-(x1^2+x2^2));

% Constraint 2: -dV/dx*f >= 0
```

```
expr = -(diff(V,x1)*f(1)+diff(V,x2)*f(2))*(x3^2+1);
prog = sosineq(prog,expr);

% =============================================
% And call solver
solver_opt.solver = 'sedumi';
prog = sossolve(prog,solver_opt);

% =============================================
% Finally, get solution
SOLV = sosgetsol(prog,V)
echo off;
```