Problem Set 3

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Problem 1
Solution:

(a)
Consider now the system,

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= x_1^6 - x_2^3 \end{aligned} \tag{1}$$

We have the set,

$$\mathcal{S} := \{(x_1, x_2) \in \mathbb{R}^2 | 0 \leq x_1 \leq 1, x_2 \geq x_1^3, x_1^2 \geq x_2\}$$

We can determine if the set $\mathcal{S}$ is forward invariant. For $\mathcal{S}$ to be forward invariant, we must have that if a point starts in $(x_1, x_2) = (0, 0)$ starts in $\mathcal{S}$ at $t = 0$, it must stay in $\mathcal{S}$ as the system evolves for $t > 0$. Hence, we can look at the boundaries of $\mathcal{S}$ evaluate the signs of $\dot{x}_1, \dot{x}_2$ to ensure that both satisfy $\dot{x}_1 \geq 0, \dot{x}_2 \geq 0$. Hence, lets start by picking a point at the boundary $x_2 \geq x_1^3$ with $0 \leq x_1 \leq 1$. For this boundary, we take $(x_1, x_2) = (0.5, 0.125)$. Evaluating $\dot{x}_1$ and $\dot{x}_2$, we get $(\dot{x}_1, \dot{x}_2) = (0, 0.01367)$. Here, $\dot{x}_1, \dot{x}_2 \geq 0$. Hence, the system satisfies the forward invariant criterium at this boundary. We can now look at the other boundary, $x_1^2 \geq x_2$ with $0 \leq x_1 \leq 1$. Here, we can again take the point $x_1 = 0.5$. Hence, we have $(x_1, x_2) = (0.5, \sqrt{0.5})$.

Here, $(\dot{x}_1, \dot{x}_2) = (0.582, -0.337)$. Hence, the trajectory points inwards at this point along the second boundary. Since both trajectories point inwards for both boundaries, we can conclude that the set is forward is forward invariant for this system. Refer to Figure 1 for the phase portrait of the system to gain validate this statement.

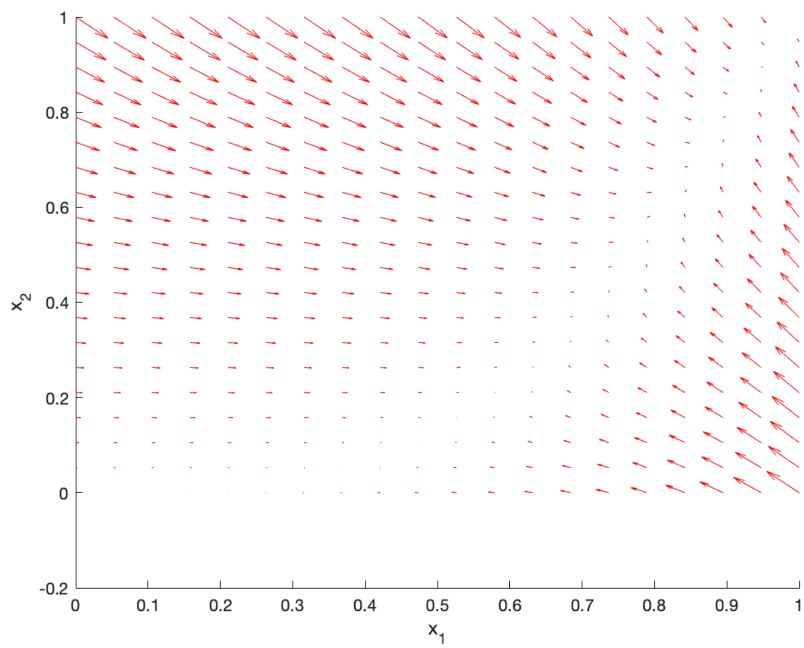

Figure 1: Problem 1

(b) We can now find the system's equilibria by solving,

$$\cdot x_1 = -x_1^3 + x_2 = 0$$

$$\cdot x_2 = x_1^6 - x_2^3 = 0$$

Giving points, $(x_1, x_2) = \{(0,0), (1,1), (-(-1)^{1/3}, 1), , (-(-1)^{2/3}, 1)\}$.
To determine the stability of each point, must come up with the system's Jacobian and evaluate the eigenvalues at each point.

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$$\frac{\partial f}{\partial \mathbf{x}} = \begin{pmatrix} -3x_1^2 & 1 \\ 6x_1^5 & -3x_2^2 \end{pmatrix}$$

Hence, we get

$$\lambda_{\pm} = \frac{1}{2}(-x_1^2 - 3x_2^2 \pm \sqrt{x_1^4 + 24x_1^5 - 6x_1^2x_2^2 + 9x_2^4})$$

For each point, $(x_1, x_2) = \{(0,0), (1,1), (-(-1)^{1/3}, 1), , (-(-1)^{2/3}, 1)\}$ we get eigenvalues $(\lambda_+, \lambda_-) = \{(0,0), (0.645, -4.65), (-2+i\sqrt{5}, -2-i\sqrt{5}), (0.645, -4.65)\}$. This makes these points (in the order they were written): inconclusive, saddle point, stable focus, and saddle point.

(c) The stability of the origin is unknown since the eigenvalues are both 0. However, since the set $\mathcal{S}$ is forward invariant, and the origin lies within this set, then by the invariance principle we can conclude that this origin is asymptotically stable. Qualitatively, this is because the trajectories will get pushed inwards towards the origin as time evolves.

Problem 2
Solution:
We consider now the system below,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2 - 2x_2^2)$$

(a)We can find the system's equillibria and stability properties first. Solving,

$$x_2 = 0$$

$$-x_2 + x_2(1 - x_1^2 - 2x_2^2) = 0$$

Giving us an equillibrium of $(x_1, x_2) = (0, 0)$. We consider the Jacobian evaluated at the equill. point,

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -1 - 2x_1x_2 & 1 - x_1^2 - 6x_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

This has eigenvalues, $\lambda_{\pm} = 0.5 \pm 0.8666i$. Hence, the point is unstable since the real parts of each eigenvalue are positive.

(b) Now, we have the set $\mathcal{S} = \{(x_1, x_2) \in \mathbb{R}^2 | 0.5 \leq x_1^2 + x_2^2 \leq 1\}$. We can take a point on the boundary of this region and check if the trajectory will evolve inwards for $t > 0$. Hence, we can take the points $(x_1, x_2) = (0, 1)$ and $(x_1, x_2) = (0, \sqrt{0.5})$. These correspond to the inward and outward boundaries of the region. Evaluating, $(\dot{x}_1, \dot{x}_2)$ for each, we get $(\dot{x}_1, \dot{x}_2) = (1, -1)$ at (0,1) and $(\dot{x}_1, \dot{x}_2) = (0.707, 0)$ at $(0, \sqrt{0.5})$. Each of these time derivatives have signs indicating that the trajectories will point inwards for $t > 0$. Hence, the trajectories will stay in the set $\mathcal{S}$, making this forward invariant.

(c) From part (a), we claimed that using the indirect method that the system is unstable since the matrix was positive definite. However, applying the invariance principle we find that the system at the equill. point is forward invariant and hence asymptocially stable when we're within the forward invariant set $\mathcal{S}$. Hence, this is not a globally asymptotically stable set but is asymptocially stable when confined in the set $\mathcal{S}$.

Problem 3

Solution:

We consider now the system,

$$\dot{x}_1 = x_2 - (2x_1^2 + x_2^2)x_1$$
$$\dot{x}_2 = -x_1 - 2(2x_1^2 + x_2^2)x_2$$

We can demonstrate global asymptotic stability. For this, we must come up with a Lyapunov function $V(x)$ such that $V(0) = 0$ and $V(x) > 0$, $V(x) \to \infty$ as $||x|| \to \infty$, such that $\dot{V}(x) < 0$ at $p = 0$. Hence, we start with a candidate Lyapunov function $V(x)$ below.

$$V(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= p_{11}x_1^2 + p_{12}x_1x_2 + p_{21}x_1x_2 + p_{22}x_2^2$$

The above expression satisfies the condition that $V(0) = 0$. We must also satisfy $V(x) > 0$ and that V is radially unbounded. We can see that if $p_{11} > 0$ and $p_{22} > 0$ and $p_{11}, p_{22} > p_{12}, p_{21}$, then $V(x)$ is radially unbounded ($\to \infty$ as

$||x|| \to \infty$). We can satisfy the conditions we seek by setting $p_{12} = p_{21} = 0$. Giving us,

$$V(x) = p_{11}x_1^2 + p_{22}x_2^2$$

which satisfies that $V(0) = 0$, $V(x) > 0$, and is radially unbounded. Finally, we can check the condition for $\dot{V} < 0$.

$$\begin{aligned}
\dot{V}(x) &= 2p_{11}x_1\dot{x}_1 + p_{12}(\dot{x}_1x_2 + \dot{x}_2x_1) + p_{21}(\dot{x}_1x_2 + \dot{x}_2x_1) + 2p_{22}x_2\dot{x}_2 \\
&= 2(p_{11}x_1\dot{x}_1 + p_{22}x_2\dot{x}_2) + (p_{12} + p_{21})(\dot{x}_1x_2 + \dot{x}_2x_1) \\
&= 2(p_{11}x_1(x_2 - (2x_1^2 + x_2^2)x_1) + p_{22}x_2(-x_1 - 2(2x_1^2 + x_2^2)x_2)) \\
&\quad +(p_{12} + p_{21})(x_2(x_2 - (2x_1^2 + x_2^2)x_1) + x_1(-x_1 - 2(2x_1^2 + x_2^2)x_2))
\end{aligned}$$

$$= 2p_{11}x_1(-2x_1^3+x_2-x_1x_2^2)-2p_{22}x_2(x_1+4x_1^2x_2+2x_2^3)-(p_{12}+p_{21})(x_1^2+6x_1^3x_2-x_2^2+3x_1x_2^3)$$

with $p_{12} = p_{21} = 0$.

$$\begin{aligned}
\dot{V}(x) &= 2p_{11}x_1(-2x_1^3 + x_2 - x_1x_2^2) - 2p_{22}x_2(x_1 + 4x_1^2x_2 + 2x_2^3) \\
&= -4p_{11}x_1^4 + 2p_{11}x_1x_2 - 2p_{11}x_1^2x_2^2 - 2p_{22}x_1x_2 - 8p_{22}x_1^2x_2^2 - 4p_{22}x_2^4 \\
&= 2(p_{11} - p_{22})x_1x_2 - 4p_{11}x_1^2x_2^2 - 4p_{11}x_1^4 - 4p_{22}x_2^4 - 8p_{22}x_1^2x_2^2
\end{aligned}$$

Hence, we can see that if $p_{11} > 0$ and $p_{22} > 0$ and $p_{22} > p_{11}$, we can guarantee that $\dot{V}(x) < 0$. This is the last criteria that we needed to satisfy for global asymptotic stability.

Problem 4

Solution:

We now have the system,

$$\dot{x}_1 = x_1^2 + x_1x_2 + u$$

$$\dot{x}_2 = x_1x_2^2 + x_1$$

we can find a control law $u(x)$ for the origin to become globally asymptotically stable. We need that $V(0) = 0$, $V \to \infty$ as $||x|| \to \infty$, and $V(x) > 0$. Constructing a Lyapunov function,

$$V(x_1, x_2) = p_{11}x_1^2 + p_{12}x_1x_2 + p_{21}x_1x_2 + p_{22}x_2^2$$

Looking at the time derivative $\dot{V}(x)$, we have

$$\dot{V} = 2(p_{11}x_1\dot{x}_1 + p_{22}x_2\dot{x}_2) + (p_{12} + p_{21})(\dot{x}_1x_2 + \dot{x}_2x_1)$$

$$= 2(p_{11}x_1(x_1^2+x_1x_2+u)+p_{22}x_2(x_1x_2^2+x_1))+(p_{12}+p_{21})((x_1^2+x_1x_2+u)x_2+(x_1x_2^2+x_1)x_1)$$

This simplifies to,

$$\dot{V} = (p_{12}+p_{21})(x_2u+x_1x_2^2+x_1^2(1+x_2+x_2^2))+2(p_{22}x_1x_2(1+x_2^2)+p_{11}x_1(u+x_1(x_1+x_2)))$$

We can again make the simplifying assumption that $p_{12} = p_{21} = 0$ giving us,

$$\dot{V} = 2(p_{22}x_1x_2(1+x_2^2)+p_{11}x_1(u+x_1(x_1+x_2))) = 2(p_{22}x_1x_2+p_{22}x_1x_2^3+p_{11}x_1u+p_{11}x_1^3+p_{11}x_1^2x_2)$$

We take that $p_{11}, p_{22} > 0$ to satisfy that $V(x) > 0$ and that its radially unbounded. We can look at several cases: (1) $x_1 > 0$, $x_2 > 0$, (2) $x_1 < 0$, $x_2 < 0$, (3) $x_1 > 0$, $x_2 < 0$, (4) $x_1 < 0$, $x_2 > 0$ and design $u(x)$ to work for all 4 cases. For case (1), we need

$$u < -\frac{p_{22}x_1x_2 + p_{22}x_1x_2^3 + p_{11}x_1^3 + p_{11}x_1^2x_2}{p_{11}x_2}$$

Hence, one choice would be

$$u(x_1, x_2) = -b\frac{p_{22}x_1x_2 + p_{22}x_1x_2^3 + p_{11}x_1^3 + p_{11}x_1^2x_2}{p_{11}x_2}$$

with $b > 1$.
For case (2) , we have $x_1 < 0$, $x_2 < 0$. Giving us the condition,

$$u < -\frac{p_{22}x_1x_2 + p_{22}x_1x_2^3 + p_{11}x_1^3 + p_{11}x_1^2x_2}{p_{11}x_2}$$

Hence, given that $p_{22}x_1x_2+p_{11}x_1^3+p_{11}x_1^2x_2 > p_{22}x_1x_2^3$ we can have the control law,

$$u(x) = -c\frac{p_{22}x_1x_2 + p_{22}x_1x_2^3 + p_{11}x_1^3 + p_{11}x_1^2x_2}{p_{11}x_2}$$

with $c > 0$.
For cases (3) and (4), we get the same result.

Problem 5
Solution:
Consider now the system of equations,

$$\begin{aligned} J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + u_1 \\ J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + u_2 \\ J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + u_3 \end{aligned} \tag{2}$$

(a) Considering first the case $u_1 = u_2 = u_3 = 0$, we get

$$\begin{aligned} J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 \\ J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 \\ J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 \end{aligned} \tag{3}$$

where we write the Jacobian as,

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{pmatrix} 0 & \frac{(J_2-J_3)\omega_3}{J_1} & \frac{(J_2-J_3)\omega_2}{J_1} \\ \frac{(J_3-J_1)\omega_3}{J_2} & 0 & \frac{(J_3-J_1)\omega_1}{J_2} \\ \frac{(J_1-J_2)\omega_2}{J_3} & \frac{(J_1-J_2)\omega_1}{J_3} & 0 \end{pmatrix}$$

Evaluated at the origin, this becomes

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Giving us eigenvalues of all zeros, hence the system is stable under this condition.

(b) Now we consider the case where the control law is $u_i = -k_i\omega_i$ with $k_i > 0$. Here, our system becomes

$$\begin{aligned} \dot{\omega}_1 &= \frac{1}{J_1}((J_2 - J_3)\omega_2\omega_3 - k_1\omega_1) \\ \dot{\omega}_2 &= \frac{1}{J_2}((J_3 - J_1)\omega_3\omega_1 - k_2\omega_2) \\ \dot{\omega}_3 &= \frac{1}{J_3}((J_1 - J_2)\omega_1\omega_2 - k_3\omega_3) \end{aligned}$$

Hence, we can write a Lyapunov function for the system as shown below,

$$V(\omega_1,\omega_2,\omega_3) = \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}^T \begin{pmatrix}p_{11} & p_{12} & p_{13}\\ p_{21} & p_{22} & p_{23}\\ p_{31} & p_{32} & p_{33}\end{pmatrix} \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}$$

We can only look at diagonal components from above, hence the expression simplifies to

$$V(\omega_1,\omega_2,\omega_3) = p_{11}\omega_1^2 + p_{22}\omega_2^2 + p_{33}\omega_3^2$$

Expanding out $\dot{V}$, we get

$$\dot{V}(\omega_1,\omega_2,\omega_3) = 2(p_{11}\omega_1\dot{\omega}_1 + p_{22}\omega_2\dot{\omega}_2 + p_{33}\omega_3\dot{\omega}_3)$$

Hence,

$$\dot{V}(\omega_1,\omega_2,\omega_3) = 2(p_{11}\omega_1(\frac{1}{J_1}((J_2 - J_3)\omega_2\omega_3 - k_1\omega_1))$$

$$+p_{22}\omega_{22}(\frac{1}{J_2}((J_3 - J_1)\omega_3\omega_1 - k_2\omega_2)) + p_{33}\omega_{33}(\frac{1}{J_3}((J_1 - J_2)\omega_1\omega_2 - k_3\omega_3))$$

To satisfy the conditions that $V > 0$ and that the function is radially unbounded, we must stipulate that $p_{11}, p_{22}, p_{33} > 0$. Hence, for now we can take these to be 1. Resulting in,

$$\dot{V}(\omega_1,\omega_2,\omega_3) = 2(\omega_1(\frac{1}{J_1}((J_2 - J_3)\omega_2\omega_3 - k_1\omega_1))$$

$$+\omega_2(\frac{1}{J_2}((J_3 - J_1)\omega_3\omega_1 - k_2\omega_2)) + \omega_3(\frac{1}{J_3}((J_1 - J_2)\omega_1\omega_2 - k_3\omega_3))$$

$$= 2((\frac{J_2 - J_3}{J_1} + \frac{J_3 - J_1}{J_2} + \frac{J_1 - J_2}{J_3})\omega_1\omega_2\omega_3 - (k_1\omega_1^2 + k_2\omega_2^2 + k_3\omega_3^2))$$

Hence, from this expression we have that $k_i > 0$ making the second set of terms negative and these terms are greater than the first set of terms. This makes $\dot{V} < 0$. Given that we've satisfied $V(0) = 0$, $V(\omega) > 0$, $\dot{V} < 0$, and that the function is radially unbounded, we have that the function is globally asymptotically stable.