Problem Set 5

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Problem 1
Solution:
Consider now a static system defined by $y = h(u)$ for input and output $u, y \in \mathbb{R}^p$ and $h(u)$ as globally Lipschitz continuous.

(a) We can look at the finite-gain $\mathcal{L}_p$ stability with $p \in \{1, 2, \infty\}$ when $h(0) = 0$. This system is finite gain stable if there exists non-negative constants γ and β such that,

$$||(Hu)_\tau||_{\mathcal{L}_p} \leq \gamma ||u_\tau||_{\mathcal{L}_p} + \beta$$

where γ is the gain of the system. Here H is the mapping from $\mathcal{L}_p \to \mathcal{L}_q$ defined by $h(u)$. Since the system is globally Lipschitz, we have that $||h'(u)||_{\mathcal{L}_p} \leq L$ or $\frac{||h(u)||_{\mathcal{L}_p}}{||u||_{\mathcal{L}_p}} \leq L$.
with L as the Lipschitz constant. Hence, expanding the inequality gives us

$$||h(u)_\tau||_{\mathcal{L}_p} \leq L||u_\tau||_{\mathcal{L}_p} = \gamma||u_\tau||_{\mathcal{L}_p} + \beta$$

which follows the form we sought out for finite-gain $\mathcal{L}_p$-stability. Here our constants would be $\gamma = L \geq 0$ and $\beta = 0$ (since $h(0) = 0$). We can now look at each case for $p = 1, 2, \infty$. For $p = 1, 2$ we have,

$$(\int_0^\infty ||h^T(u(\tau))h(u(\tau))||^p d\tau)^{1/p} \leq L(\int_0^\infty ||(u^T(\tau)u(\tau)||^p d\tau)^{1/p}$$

Here, we have

$$\frac{(\int_0^\infty ||h^T(u(\tau))h(u(\tau))||^p d\tau)^{1/p}}{(\int_0^\infty ||(u^T(\tau)u(\tau)||^p d\tau)^{1/p}} \leq L = \gamma$$

which satisfies our condition for finite-gain $\mathcal{L}_p$ stability. Then we have the case that $p = \infty$,

$$\sup_{\tau \geq 0} ||h(u((\tau))|| \leq L \sup_{\tau \geq 0} ||u((\tau)||$$

Giving,

$$\frac{\sup_{\tau \geq 0} ||h(u((\tau))||}{\sup_{\tau \geq 0} ||u((\tau)||} \leq \sup_{\tau \geq 0} \frac{||h(u(\tau)||}{||u(\tau)||} \leq \sup_{\tau \geq 0} \frac{L||u(\tau)||}{||u(\tau)||} \leq L$$

Where again we get $\gamma = L$ and $\beta = 0$, demonstrating finite-gain $\mathcal{L}_p$ stability for $p = \infty$.

(b) Now we consider the finite-gain $\mathcal{L}_p$-stability for $p \in \{1, 2\}$ when $h(0) \neq 0$. For $p \in \{1, 2\}$, we have

$$(\int_0^\infty ||h^T(u(\tau))h(u(\tau))||^p d\tau)^{1/p} \leq L(\int_0^\infty ||(u^T(\tau)u(\tau)||^p d\tau)^{1/p} + h(0)$$

$$= \gamma ||u_\tau||_{\mathcal{L}_p} + \beta$$

Here, $\gamma = L$ again and $\beta = h(0)$ which is finite-gain $\mathcal{L}_p$-stable only for $\beta = h(0) \geq 0$.

Problem 2

Solution:

Now consider the system,

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{Q} + g(q) = u$$

for $q \in \mathbb{R}^m$ as the joint angles of a robot and $u \in \mathbb{R}^m$ as a control input. M here is a positive definite inertia matrix, $C(q, \dot{q})\dot{q}$ accounts for centrifugal/-coriolis forces, $D\dot{q}$ accounts for damping with D as a positive semidefinite matrix and g accounts for gravity. We also have that $\dot{M}(q) - 2C(q, \dot{q}) = 0$ is a skew-symmetric matrix.

(a) We have the energy function,

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$$

with $P(q)$ as the potential energy from gravity. We can demonstrate that the map from u to $\dot{q}$ is passive. Here we must satisfy:

$$u^T\dot{q} \geq \dot{V} = \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial \dot{q}}\ddot{q}$$

We can assign $x_1 = q$ and $x_2 = \dot{q}$, allowing us to rewrite the system as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{M(x_1)}(u - g(x_1) - Dx_2 - C(x_1,x_2)x_2)$$

Expanding $\dot{V}$ gives us (with $V = \frac{1}{2}x_2^T M(x_1)x_2 + P(x_1)$),

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$$

$$= (\frac{1}{2}x_2^T x_2\frac{\partial M}{\partial x_1} + \frac{\partial P}{\partial x_1})\dot{x}_1 + (x_2^T M(x_1))\frac{1}{M(x_1)}(u - g(x_1) - Dx_2 - C(x_1,x_2)x_2)$$

$$= \frac{1}{2}x_2^T x_2\dot{M} + g(x_1)\dot{x}_1 - x_2^T x_2 C(x_1,x_2) + x_2^T u - x_2^T g(x_1) - x_2^T x_2 D$$

$$= x_2^T x_2(\dot{M} - 2C(x_1,x_2)) + \frac{1}{2}x_2 g(x_1) - x_2^T x_2 D + x_2^T u$$

Here, we can see that $x_2^T x_2(\dot{M} - 2C(x_1,x_2)), \frac{1}{2}x_2 g(x_1), -x_2^T x_2 D \leq 0$. Hence, we can write that

$$x_2^T x_2(\dot{M} - 2C(x_1,x_2)) + \frac{1}{2}x_2 g(x_1) - x_2^T x_2 D \leq x_2^T u$$

or $\dot{V} \leq \dot{q}^T u$ which satisfies our condition for output strict passivity.

(b) Now given $u = -K_d x_2 + v$ with K_d as a diagonal matrix whose map v to $\dot{q}$ is strictly passive. Here we must satisfy,

$$v^T x_2 \geq \dot{V} + x_2^T\rho(x_2)$$

for $x_2^T \rho(x_2) > 0$.
We have our expression for $\dot{V}$,

$$\dot{V} = x_2^T x_2(\dot{M} - 2C(x_1, x_2)) + \frac{1}{2}x_2 g(x_1) - x_2^T x_2 D + x_2^T(-K_d x_2 + v)$$

$$= x_2^T x_2(\dot{M} - 2C(x_1, x_2)) + \frac{1}{2}x_2 g(x_1) - x_2^T x_2 D - x_2^T x_2 K_d + x_2^T v$$

Giving,

$$x_2^T x_2(\dot{M} - 2C(x_1, x_2)) + \frac{1}{2}x_2 g(x_1) - x_2^T x_2 D - x_2^T(K_d x_2) \leq x_2^T v$$

with $\dot{V} = x_2^T x_2(\dot{M} - 2C(x_1, x_2)) + \frac{1}{2}x_2 g(x_1) - x_2^T x_2 D$ and $x_2^T \rho(x_2) = -x_2^T(K_d x_2)$. This satisfies our condition that $v^T x_2 \geq \dot{V} + x_2^T \rho(x_2)$ for $x_2^T \rho(x_2) > 0$.

(c) Consider now $u = -K_d \dot{q}$, we can show that the origin is asymptotically stable. We have that $V(0) = 0$ and$V(x_1, x_2) \geq 0$ Here,

$$\dot{V} = x_2^T x_2(\dot{M} - 2C(x_1, x_2)) + \frac{1}{2}x_2 g(x_1) - x_2^T x_2 D + x_2^T(-K_d x_2)$$

$$= x_2^T x_2(\dot{M} - 2C(x_1, x_2)) + \frac{1}{2}x_2 g(x_1) - x_2^T x_2 D + x_2^T x_2 - K_d x_2 \leq 0$$

where we can see that since $\dot{M} - 2C(x_1, x_2)$ is skew-symmetric, the first term is negative, $\frac{1}{2}x_2 g(x_1)$ is negative due to gravity, the third term is negative since $D > 0$, and the last term is negative since $K_d < 0$ for all x_1, x_2 $(q, \dot{q})$. Hence, $\dot{V} \leq 0$ is asymptotically stable. For this to be globally asymptotically stable, we must also stipulate that V is radially unbounded. Here, $\dot{V} = 0.5 x_2^T M(x_1) x_2 + P(x_1)$. Hence, if use this Lyapunov function, we must have that $M \to \infty$ and $P \to \infty$ as $x_1 \to \infty$.

Problem 3
Solution:
(a) Considering the system,

$$\dot{y} = -z^2$$
$$\dot{z} = -z + y^2 + yz$$

with Jacobian,

$$A = \begin{pmatrix} 0 & -2z \\ 2y+z & y-1 \end{pmatrix}$$

evaluated at the equillibrium point gives eigenvalues -1, 0. Hence Lyapunov's indirect method is not applicable here. We can apply the central manifold theorem, where we must satisfy

$$\mathcal{N}(h(y)) = \frac{dh}{dy}[A_1 y + g_1(y, h(y))] - A_2 h(y) - g_2(y, h(y)) = 0$$

with $h(0) = 0$ and $dh(0)/dy = 0$. Here $A_1 = 0$ and $A_2 = -1$. $g_1 = -z^2$ and $g_2 = y^2 + yz$. Hence,

$$h'(y)(-h^2(y)) - h(y) - (y^2 + yh(y)) = 0$$

with $h(y) = \phi(y) + \mathcal{O}(||y||^p)$. Starting with $\phi(y) = 0$ giving $h(y) = \mathcal{O}(||y||)$. Here we get, $\dot{y} = \mathcal{O}(||y||^2)$ which is inconclusive. To second order, we can take $\phi(y) = h_2 y^2$, giving $h(y) = h_2 y^2 + \mathcal{O}(||y||^3)$. We can solve for h_2 by substituting $h(y)$ into the central manifold equation.

$$2h_2 y(-(h_2 y^2)^2) - h_2 y^2 - (y^2 + y h_2 y^2)) = 0$$

Giving,

$$-y^2 = h_2 y^2 - h_2 y^3 - 2h_2^3 y^5$$

Giving, $h_2 = -1$. Hence, $h(y) = -y^2 + \mathcal{O}(||y||^2)$. Substituting this in gives,

$$\dot{y} = -y^4 + \mathcal{O}(||y||^5)$$

From this we can see that that the system is asymptotically stable since $\dot{y} < 0$ for $y \neq 0$.

(b) We now have the system,

$$\dot{y}_1 = -y_2 + y_1 z$$

$$\dot{y}_2 = y_1 + y_2 z$$

$$\dot{z} = -z - (y_1^2 + y_2^2) + z^2$$

where we take $\phi(y) = p_1 y_1^2 + p_2 y_2^2$. Hence, $h(y) = p_1 y_1^2 + p_2 y_2^2 + \mathcal{O}(|y_1|^2 + |y_2|^2)$. From this, we can write expression for $\mathcal{N}(h(y))$,

$$\mathcal{N}(h(y)) = \frac{dh}{dy}^T [A_1 y + g_1(y, h(y))] - A_2 h(y) - g_2(y, h(y)) = 0$$

where $\frac{dh}{dy} = \begin{pmatrix} \frac{\partial h}{\partial y_1} \\ \frac{\partial h}{\partial y_2} \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and $g_1 = \begin{pmatrix} g_1^1 \\ g_1^2 \end{pmatrix}$. Here $g_1 \in \mathbb{R}^{2\times 1}$, $g_2 \in \mathbb{R}$, $A_1 \in \mathbb{R}^{2\times 2}$, and $A_2 \in \mathbb{R}$. Hence, we can write the system as

$$\dot{y} = A_1 y + g_1(y, z)$$

$$\dot{z} = A_2 z + g_2(y, z)$$

where

$$g_1(y, z) = 0$$
$$g_2(y, z) = -(y_1^2 + y_2^2) + z^2$$
$$A_1 = \begin{pmatrix} z & -1 \\ 1 & z \end{pmatrix}$$
$$A_2 = -1$$

Hence expanding $\mathcal{N}$ gives us,

$$\mathcal{N} = \begin{pmatrix} \frac{\partial h}{\partial y_1} \\ \frac{\partial h}{\partial y_2} \end{pmatrix}^T \begin{pmatrix} z & -1 \\ 1 & z \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \frac{\partial h}{\partial y_1} \\ \frac{\partial h}{\partial y_2} \end{pmatrix}^T \begin{pmatrix} 0 \\ 0 \end{pmatrix} + h(y) + (y_1^2 + y_2^2) - z^2$$

$$= \begin{pmatrix} \frac{\partial h}{\partial y_1} \\ \frac{\partial h}{\partial y_2} \end{pmatrix}^T \begin{pmatrix} y_1 z - y_2 \\ y_1 + y_2 z \end{pmatrix} + h(y) + y_1^2 + y_2^2 - z^2$$

$$= \frac{\partial h}{\partial y_1}(y_1 h(y) - y_2) + \frac{\partial h}{\partial y_2}(y_1 + y_2 h(y)) + h(y) + y_1^2 + y_2^2 - h(y)^2 = 0$$

If we assume $h(y) = p_1 y_1^2 + p_2 y_2^2 + \mathcal{O}(|y_1|^2 + |y_2|^2)$, we get

$$\mathcal{N} = \frac{\partial h}{\partial y_1}(y_1 h(y) - y_2) + \frac{\partial h}{\partial y_2}(y_1 + y_2 h(y)) + h(y) + y_1^2 + y_2^2 - h(y)^2$$

$$= 2p_1 y_1 (y_1(p_1 y_1^2 + p_2 y_2^2) - y_2) + 2p_2 y_2 (y_1 + y_2(p_1 y_1^2 + p_2 y_2^2)) + (p_1 y_1^2 + p_2 y_2^2) + y_1^2 + y_2^2 - (p_1 y_1^2 + p_2 y_2^2)^2 = 0$$

Allowing us to match up terms y_1^2 and y_2^2 to find p_1 and p_2. Matching them up gives us $p_1y_1^2 = -y_1^2$ and $p_2y_2^2 = -y_2^2$, hence $p_1 = p_2 = -1$. This gives us, $h(y) = -(y_1^2 + y_2^2) + \mathcal{O}(||y||^3)$. The system becomes,

$$\dot{y}_1 = -y_2 - y_1(y_1^2 + y_2^2) + \mathcal{O}(||y||^4)$$

$$\dot{y}_2 = y_1 - y_2(y_1^2 + y_2^2) + \mathcal{O}(||y||^4)$$

Hence, we can see that the above linearize system in unstable at the origin. We conclude that the system is unstable.

Problem 4

Solution:

(a) If we have no saturation and no dead-band, then the controller $F(s) = K$ giving the transfer function ($G(s) = (s + 0.5)^{-2}$) ,

$$H(s) = \frac{KG(s)}{1 + KG(s)} = \frac{K}{(1 + K(s + 0.5)^{-2})(s + 0.5)^2} = \frac{K}{(s + 0.5)^2 + K}$$

We can guarantee stability when $KG(s) < 1$ or,

$$\frac{K}{(s + 0.5)^2} < 1$$

Hence, this feedback loop is stable when $K < (s + 0.5)^2$ (for $K > 0$).

(b) When we have saturation and deadband, we can find the K that yields finite-gain $\mathcal{L}_2$ stability. We have finite-gain $\mathcal{L}_2$ stability if,

$$\gamma = \sup_{e \in \mathcal{L}_2} \frac{||S(e(s))||_{\mathcal{L}_2}}{||e(s)||_{\mathcal{L}_2}} < \infty$$

If we restrict ourselves to values of e that involve K, we can compute $||S(e)||^2_{\mathcal{L}_2}$ such that $S(e(s)) = KG(s)e(s)$. Hence, working in frequency space gives us

$$||S(e)||^2_{\mathcal{L}_2} = \frac{K^2}{2\pi} \int_{-\infty}^{\infty} G(j\omega)G^\star(j\omega)e(j\omega)e^\star(j\omega)d\omega$$

$$\leq \frac{K^2}{2\pi} \sup_{\omega} \lambda_{max}(|G(j\omega)|^2) \int_{-\infty}^{\infty} e(j\omega)e^\star(j\omega)d\omega$$

$$= K^2 \sup_\omega \sigma^2_{max}(G(j\omega))||e||^2_{\mathcal{L}_2}$$

with $\sigma^2_{max}(G(j\omega)$ as the maximum singular value of $G(j\omega)$. Hence, we want

$$\gamma = \sup_{e\in\mathcal{L}_2} \frac{K^2 \sup_\omega \sigma^2_{max}(G(j\omega))||e||^2_{\mathcal{L}_2}}{||e(s)||^2_{\mathcal{L}_2}} =$$

$$= \sup_{e\in\mathcal{L}_2} K^2 \sup_\omega \sigma^2_{max}(G(j\omega))$$

The above now suggests that as long as K is finite, we can have finite-gain $\mathcal{L}_2$ stability.

Problem 5

Solution:

Considering our definition of finite-gain $\mathcal{L}_p$ stability that,

$$\gamma_p = \sup_u \frac{||y||_{\mathcal{L}_p}}{||u||_{\mathcal{L}_p}} < \infty$$

(a) We satisfy that $||y(t(||_{\mathcal{L}_p} \leq C||u(t)||_{\mathcal{L}_p}$, where the following hold

$$\sup_t |y(t)| \leq C \sup_t ||u(t)|$$

$$\int_0^\infty y^2 dt \leq C^2 \int_0^\infty u^2 dt$$

which suggests that that this is finite gain $\mathcal{L}_\infty$ stable and finite gain $\mathcal{L}_2$ stable.

(b) Again, we satisfy that $||y(t(||_{\mathcal{L}_p} \leq C||u(t)||_{\mathcal{L}_p}$, where the following hold

$$\sup_t |y(t)| \leq C \sup_t ||u(t)|$$

$$\int_0^\infty y^2 dt \leq C^2 \int_0^\infty u^2 dt$$

which suggests that that this is finite gain $\mathcal{L}_\infty$ stable and finite gain $\mathcal{L}_2$ stable.

(c) We satisfy that $||y(t(||_{\mathcal{L}_p} \leq C||u(t)||_{\mathcal{L}_p}$, where the following hold

$$\sup_t |y(t)| \leq C \sup_t ||u(t)|$$

which suggests that that this is finite gain $\mathcal{L}_\infty$ stable. If we now use an input $u(t) = e^t$ which will give an inifinite output for

$$\int_0^\infty y^2 dt \geq C^2 \int_0^\infty e^{2t} dt$$

demonstrating that it is not $\mathcal{L}_2$. So this is $\mathcal{L}_\infty$ but not $\mathcal{L}_2$ stable. Finally, we have

(d) Finally, we have

$$\sup_t |y(t)| \leq C \sup_t ||u(t)|$$

$$\int_0^\infty y^2 dt \leq C^2 \int_0^\infty u^2 dt$$

which again suggests that that this is finite gain $\mathcal{L}_\infty$ stable and finite gain $\mathcal{L}_2$ stable.