Problem Set 2

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Problem 1
Solution:

(a)
Here we consider the predator-prey model in Eqt. (1).

$$\begin{aligned} \dot{x}(t) &= x(t)(a + bx(t) + cy(t)) \\ \dot{y}(t) &= y(t)(d + ex(t) + fy(t)) \end{aligned} \tag{1}$$

Constraining that $b \leq 0$ and $f \leq 0$, we can vary c and e to obtain the following models: (i) predator-prey, (ii) prey-predator, (iii) competitive, and (iv) symbiotic. For each, we must have that

$$\begin{aligned} \text{(i) } & c > 0, e < 0 \\ \text{(ii) } & c < 0, e > 0 \\ \text{(iii) } & c < 0, e < 0 \\ \text{(iv) } & c > 0, e > 0 \end{aligned} \tag{2}$$

(b) Now, take that $x(0) \geq 0$ and $y(0) \geq 0$. We can show that the solutions to the IVP can not be such that $x(t) < 0$ or $y(t) < 0$ for $t \geq 0$. We can first substitute for the case where $x, y < 0$. Hence, we get

$$\begin{aligned} \dot{x}(t) &= -x(t)(a - bx(t) - cy(t)) = -ax + bx^2 + cxy \\ \dot{y}(t) &= -y(t)(d - ex(t) - fy(t)) = -dy + fy^2 + exy \end{aligned} \tag{3}$$

Form this, we see that both solutions will always increase over time. Hence, if we plot the vector field and contours under this condition, we expect the field vectors to point in the outward direction from the equilibrium point (or contour). This is because the time derivatives of both x and y will be positive. However, substituting $x < 0$ and $y < 0$ at $t \geq 0$ would violate this condition. Hence, we cannot have $x, y < 0$ for $t \geq 0$ and maintain the condition that $x(0), y(0) \geq 0$.

(c) Given this system, we find it's equillibria for $a = 3$, $b = f = -1$, $d = 2$ by solving for x, y when $\dot{x}, \dot{y} = 0$. We consider cases: $(c, e) = (-2, -1), (-2, 1), (2, -1), (2, 1)$. Solutions for each case is done in Mathematica. These are found to be,

$$
\begin{gathered}
[c, e] = [-2, -1] : (x, y) = \{(0, 2), (1, 1), (3, 0), (0, 0)\} \\
[c, e] = [-2, 1] : (x, y) = \{(1/3, 5/3), (0, 2), (3, 0), (0, 0)\} \\
[c, e] = [2, -1] : (x, y) = \{(0, 2), (7/3, -1/3), (3, 0), (0, 0)\} \\
[c, e] = [2, 1] : (x, y) = \{(-7, -5), (0, 2), (3, 0), (0, 0)\}
\end{gathered}
$$

We can now find the stability of each equilibrium by looking at the system's Jacobian:

$$
\frac{\partial f}{\partial \mathbf{x}} = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{pmatrix} = \begin{pmatrix} a + 2bx + cy & cx \\ ey & d + ex + 2by \end{pmatrix}
$$

Hence, for each condition and equilibrium point, we can find the eigenvalues of the Jacobian to understand the stability. The diagonalization was performed in Mathematica and the results are shown below. For each, there is also a comment on the stability of the point as determined from the sign of the eigenvalues.

$[c, e] = [-2, -1]$:

$$
\begin{gathered}
(0, 2), \lambda = -1, -2, \text{ stable node} \\
(1, 1), \lambda = -1 \pm \sqrt{2}, \text{ saddle} \\
(3, 0), \lambda = -3, -1, \text{ stable node} \\
(0, 0), \lambda = 3, 2, \text{ unstable node}
\end{gathered}
$$

$[c, e] = [-2, 1]$:

$$(1/3, 5/3), \lambda = \frac{1}{3}(-3 \pm \sqrt{10}i), \text{ stable focus}$$
$$(0, 2), \lambda = -1, -2, \text{ stable node}$$
$$(3, 0), \lambda = 5, -3, \text{ saddle point}$$
$$(0, 0), \lambda = 3, 2, \text{ unstable node}$$

$[c, e] = [2, -1]$:

$$(0, 2), \lambda = 7, -2, \text{ stable node}$$
$$(7/3, -1/3), \lambda = \frac{1}{3}(-3 \pm \sqrt{30}i), \text{ saddle point}$$
$$(3, 0), \lambda = -1, -3, \text{ stable node}$$
$$(0, 0), \lambda = 3, 2, \text{ unstable node}$$

$[c, e] = [2, 1]$:

$$(-7, -5), \lambda = 6 \pm \sqrt{71}, \text{ saddle point}$$
$$(0, 2), \lambda = 7, -2, \text{ saddle point}$$
$$(3, 0), \lambda = 5, -3, \text{ saddle point}$$
$$(0, 0), \lambda = 3, 2, \text{ unstable node}$$

From the values of c, e, we would expect $(-2, -1)$ to be competitive, $(-2, 1)$ to be prey-predator, $(2, -1)$ to be predator-prey, and (2,1) to be symbiotic. However, looking at equllibrium points for each model, we find multiple types of equillibria. Most of the equillibria for the competitive model are stable, which would suggest that both species would die off at these points. This is indeed in agreement with expectations of being a competitive model. For the prey-predator, we saddle, stable and unstable points. The unstable points here suggest that both populations grow, the stable points suggest that both populations decrease, and saddle points suggest that both populations remain constant over time.We again see similar behavior for the remaining model cases. We can now obtain phase portraits for each case. These are plotted in Figure 1.

(d) We can now consider the case for $a = e = 1, b = f = 0, c = d = -1$. Hence, the system becomes

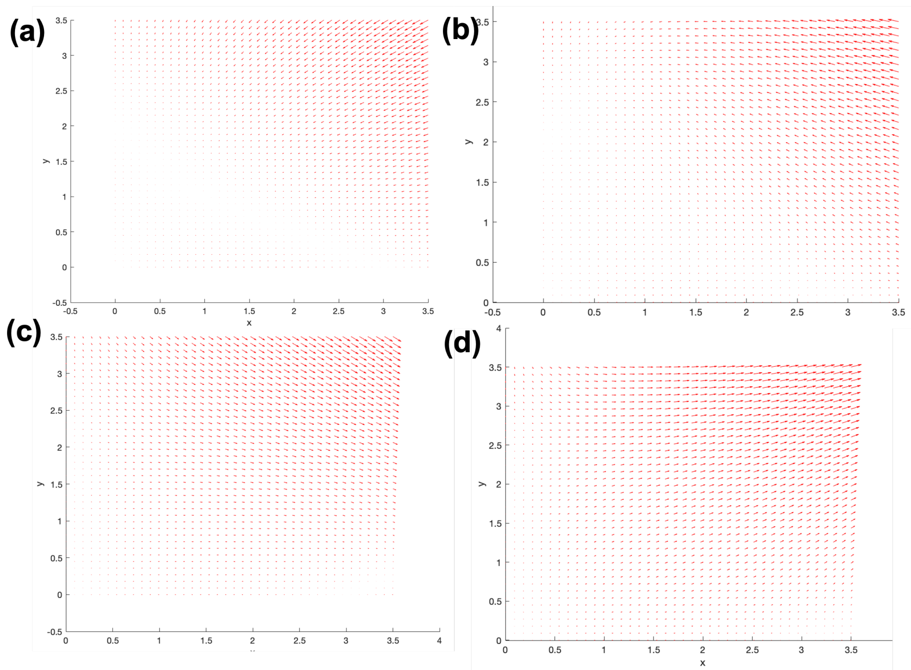

Figure 1: Problem 1(b): (a) [c,e] = (-2,-1) , (b) [c,e] = (-2,1) , (c) [c,e] = (2,-1) , (d) [c,e] = (2,1)

$$\begin{aligned} \dot{x}(t) &= x(t)(1 - y(t)) \\ \dot{y}(t) &= y(t)(-1 + x(t)) \end{aligned} \tag{4}$$

Hence, the equillibrium point satisfying this system is $(x, y) = (0, 0)$. Using the attached code, the eigenvalues of the Jacobian are $\lambda = \pm 1$. Hence, this is a saddle point. Therefore, both populations should remain constant about this point. The corresponding phase portrait for this is plotted in Figure 2.
(e) Now we take $a = e = 1$, $b = f = 0$, and $c = d = -1$. We can demonstrate the existence of a periodic orbit here. From part (c), we saw that the eigenvalues for this equillibrium are ± 1. Since one of these eigenvalues is negative, our closed and bounded set cannot contain this point (the origin). Instead, we can take our closed and bounded set to be the entire region excluding the origin. We also know that every trajectory starting in this set will end in the set. From Poincare-Bendixson's criterion, if we take this as our set, then it must contain a periodic orbit.

Problem 2

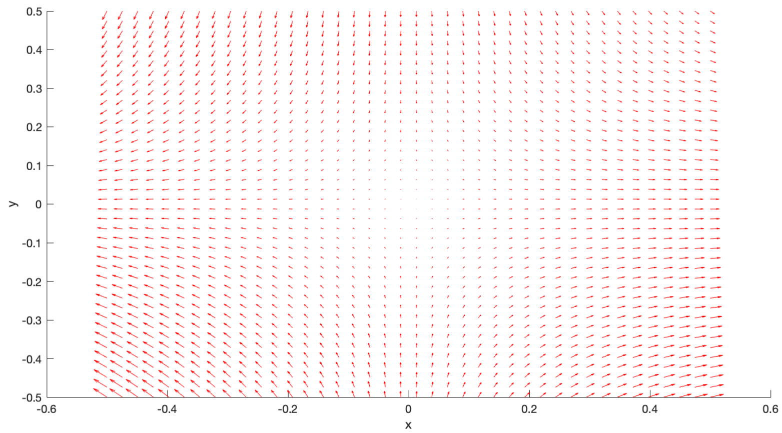

Figure 2: Problem 1(c)

Solution:
Consider the system below,

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1(1 - x_1^2 - x_2^2)\\ \dot{x}_2 &= -x_1 + x_2(1 - x_1^2 - x_2^2)\end{aligned} \tag{5}$$

This system has a Jacobian shown below,

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{pmatrix} 1 - 3x_1^3 - x_2^2 & 1 - 2x_1x_2 \\ -1 - 2x_2x_1 & 1 - x_1^2 - 3x_2^2 \end{pmatrix} \tag{6}$$

At the fixed point $(0, 0)$ it becomes.

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{7}$$

This has eigenvalues $1 \pm j$. By the Poincare-Bendixon criterion, we need that the eigenvalues have only positive real parts for a system to be a periodic orbit within a closed system. We indeed find that the both eigenvalues have real positive components. This the only equilibrium point in the region where this holds true.This shows that every trajectory within out closed subset M stays within M.

Problem 3

Solution:

For each function $f(x)$, we check if its continuously differentiable, globally Lipschitz, locally Lipschitz or continuous.

(a) For $f(x) = x^2 + |x|$, we can look at the function's derivative w.r.t x,

$$f'(x) = \begin{cases} 2x - 1, x < 0 \\ 2x + 1, x > 0 \\ \text{undef.}\ , x = 0 \end{cases}$$

This function is not continuously differentiable at $x = 0$, as can be seen from above. The function is not locally Lipschitz at $x = 0$ since it's undefined there and not Lipschitz anywhere else since it's not bounded. Hence, it's not globally Lipschitz either. The function is piecewise continuous at $x = 0$ and continuous everywhere else.

(b) For the function $f(x) = x + sign(x)$, we can write it's derivative

$$f'(x) = \begin{cases} 1, x > 0 \\ 1, x < 0 \\ 1, x = 0 \end{cases}$$

This is a continuously differentiable function. Since it's derivative is bounded everywhere by 1, it is also globally Lipschitz (not just locally Lipschitz). However, the function is not continuous at $x = 0$ since it can take on the values $+1$ and -1 at that point.

(c) For the function $\sin(x) sign(x)$, we can write it's derivative as

$$f(x) = \begin{cases} \text{undef.}\ , x = 0 \\ \cos x, x > 0 \\ -\cos x, x < 0 \end{cases}$$

Hence, the function is not continuously differentiable as seen in the derivative above. Since its norm everywhere is bounded by 1, it is also Lipschitz continuous. The function is piecewise continuous everywhere (not continuous).

(d) For the function $f(x) = \tan x$, we have its derivative $f'(x) = \sec^2 x$. Hence, the function is continuously differentiable for $-n\pi/2 < x < n\pi/2$ with n as an odd integer. The function is locally Lipschitz continuous within this region ($-n\pi/2 < x < n\pi/2$ with n as an odd integer), but not globally Lipschitz since its derivative shoots off to infinity at $x = n\pi/2$. From this, we can also say that the function is not continuous at $x = n\pi/2$.

(e) Now given the function,

$$f(x) = \begin{pmatrix} -x_1 + a|x_2| \\ -(a+b)x_1 + bx_1^2 - x_1x_2 \end{pmatrix}$$

we have its Jacobian,

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{pmatrix} -1 & a(1(x_2 > 0) - 1(x_2 < 0) + undef(x_2 = 0)) \\ -(a+b) + 2bx_1 - x_2 & -x_1 \end{pmatrix}$$

Hence, the function is continuously differentiable everywhere except $x_2 = 0$, continuous everywhere, and is locally Lipschitz for everywhere except $x_2 = 0$.

Problem 4
Solution:

Now given the state equation below,

$$\begin{aligned} \dot{x}_1 &= -x_1 + \frac{2x_2}{1+x_2^2} \\ \dot{x}_2 &= -x_2 + \frac{2x_1}{1+x_1^2} \end{aligned} \tag{8}$$

We can show that the system has a unique solution for all times. We can demonstrate uniqueness if our system is piecewise continuous in t and Lipschitz in x. Clearly, the solutions are piecewise continuous in t. We can demonstrate the Lipschitz criterium by evaluating the Jacobian to show that its norm is bounded. The Jacobian is written as,

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{pmatrix} -1 & 2\frac{1+x_2^2}{(1+2x_2^2)^2} \\ 2\frac{1+x_1^2}{(1+2x_1^2)^2} & -1 \end{pmatrix}$$

Using the l_2-norm, we can write $||\frac{\partial f}{\partial \mathbf{x}}||_2$ as,

$$||\frac{\partial f}{\partial \mathbf{x}}||_2 = (2(\frac{1+x_1^2}{(1+2x_1^2)^2})^2 + (\frac{1+x_2^2}{(1+2x_2^2)^2})^2 + 2)^{1/2}$$

The maximum of this function is at $(x_1, x_2) = (0, 0)$. Evaluating this at $(x_1, x_2) = (0, 0)$, we find that $||\frac{\partial f}{\partial \mathbf{x}}||_2 = \sqrt{10}$. Hence, $||\frac{\partial f}{\partial \mathbf{x}}||_2$ is bounded by $\sqrt{10}$, allowing us to assign $\sqrt{10}$ as the Lipschitz constant for f. Hence, this system is Lipchitz continuous for $t \geq 0$ for all points. Hence, the system does have a unique solution.