Problem Set 1

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Problem 1
Solution:

(a)
Here, we consider the set of equations describing the nonlinear dynamics of a single-link manipulator with flexible joints.

$$\begin{aligned} I\ddot{q}_1 + Mgl\sin q_1 + k(q_1 - q_2) &= 0 \\ J\ddot{q}_2 - k(q_1 - q_2) &= u \end{aligned} \tag{1}$$

with J as the moment of inertia of the actuator, I as the moment of inertia of the link, q_1 and q_2 as the angular positions of the links, k as the spring constant, u as applied torque, l as the length of the link, and M as the mass of the link. To derive these expressions, we can start with the angular equivalent to Newton's second law, where we write out the expression for torque with respect to angular acceleration for angles q_1 and q_2.

$$\begin{aligned} I\ddot{q}_1 &= -Mgl\sin q_1 - k(q_1 - q_2) \\ J\ddot{q}_2 &= u + k(q_1 - q_2) \end{aligned} \tag{2}$$

The first expression in Eqt. (2) (corresponding to $\ddot{q}_1$) describes the motion of the link. Due to gravity, the link will experience a force going downwards perpendicular to the lever arm of $-Mg$, while the lever arm has length l.

Hence, the torque contribution here will be $\tau = F_g l \sin\theta = -Mgl \sin q_1$. To the left of the link, there is a restoring torque $k(q_1 - q_2)$. Hence, the net torque is simply, $-Mgl \sin q_1 - k(q_1 - q_2)$ which is equated to $I\ddot{q}_1$.

We can perform the same analysis for the actuator side of the system. Here, the actuator provides an applied torque u, while the restoring torque from the spring is $k(q_1 - q_2)$. The net torque is simply $u + k(q_1 - q_2)$. From the angular version of Newton's second law, this is equated to $J\ddot{q}_2$. The two torque expressions for coordinates q_1 and q_2 are written explicitly in Eqt. (2) and these can be re-expressed as in Eqt. (1).

(b)
We can now write out the state equations of this system. We let $x_4 = \dot{q}_1$. $x_3 = \dot{q}_2$, $x_2 = q_1$, $x_1 = q_2$. Hence, we arrive at Eqt. (3).

$$\begin{aligned} I\dot{x}_4 + Mgl \sin x_2 + k(x_2 - x_1) &= 0 \\ J\dot{x}_3 - k(x_2 - x_1) &= u \end{aligned} \tag{3}$$

Rearranging terms, we arrive at the following coupled 4 equations (Eqt. (4)).

$$\begin{aligned} \dot{x}_4 &= \frac{1}{I}(Mgl \sin x_2 - k(x_2 - x_1)) \\ \dot{x}_3 &= \frac{1}{J}(u + k(x_2 - x_1)) \\ \dot{x}_2 &= x_4 \\ \dot{x}_1 &= x_3 \end{aligned} \tag{4}$$

Eqt. (4) gives the set of nonlinear state equations for this dynamical system.

Problem 2
Solution:
(a)

Here we consider a rotating bar with friction due to viscosity, as given in Eqt. (5).

$$ml^2\ddot{\theta} = -mgl \sin\theta - kl^2\dot{\theta} + C \tag{5}$$

with θ as the angle between the bar and vertical axis, C as applied torque, k as the friction constant, and m as the mass of the bar. Here we assign the state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$. This gives us the state equations shown in Eqt. (6).

$$\begin{aligned} \dot{x}_2 = \frac{1}{ml^2}(-mgl\sin x_1 - kl^2 x_2 + C) \\ \dot{x}_1 = x_2 \end{aligned} \tag{6}$$

(b)
Next, we seek the equilibria of the system by solving for the condition $\dot{x}_1 = \dot{x}_2 = 0$. Clearly, we must have $x_2 = 0$. We also must satisfy that $\sin x_1 = \frac{C}{mgl}$. Hence, our equilibrium point is $(x_1^0, x_2^0) = (0, \arcsin(\frac{C}{mgl}))$.

(c)
We can now linearize this system about the equilibrium point. This allows us to use the form,

$$\dot{x} = \frac{\partial f(p)}{\partial x}(x - p)$$

where p is the equilibrium point and $\frac{\partial f(p)}{\partial x}$ is the Jacobian centered around p. Given that $f_1 = \dot{x}_1(x_1, x_2)$ and $f_2 = \dot{x}_2(x_1, x_2)$. We write the Jacobian as,

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -k/m \end{pmatrix}$$

Evaluating this at our equilibrium point, we get Eqt. (7).

$$\frac{\partial f(p)}{\partial x} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -k/m \end{pmatrix} \tag{7}$$

We get the eigenvalues, $\lambda_{\pm} = \frac{-kl \pm \sqrt{(kl)^2 - 4glm^2}}{2lm}$. Since both eigenvalues here are negative, this point should be stable.

Problem 3
Solution:

(a) We now have the system in Eqt. (8),

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + x_1^3/6 - x_2
\end{aligned}
\tag{8}
$$

At equilibrium, we find the points $(0, \pm\sqrt{6})$. The Jacobian of the system is given below,

$$
\frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ -1 + \frac{1}{3}x_1^2 & -1 \end{pmatrix}
$$

For each of the two equilibria, we get eigenvalues (same pair for each), $\lambda_\pm = 1, -2$. This suggests that each equilibrium point is a saddle point. The corresponding phase portrait for this is plotted in Figure 1(a).

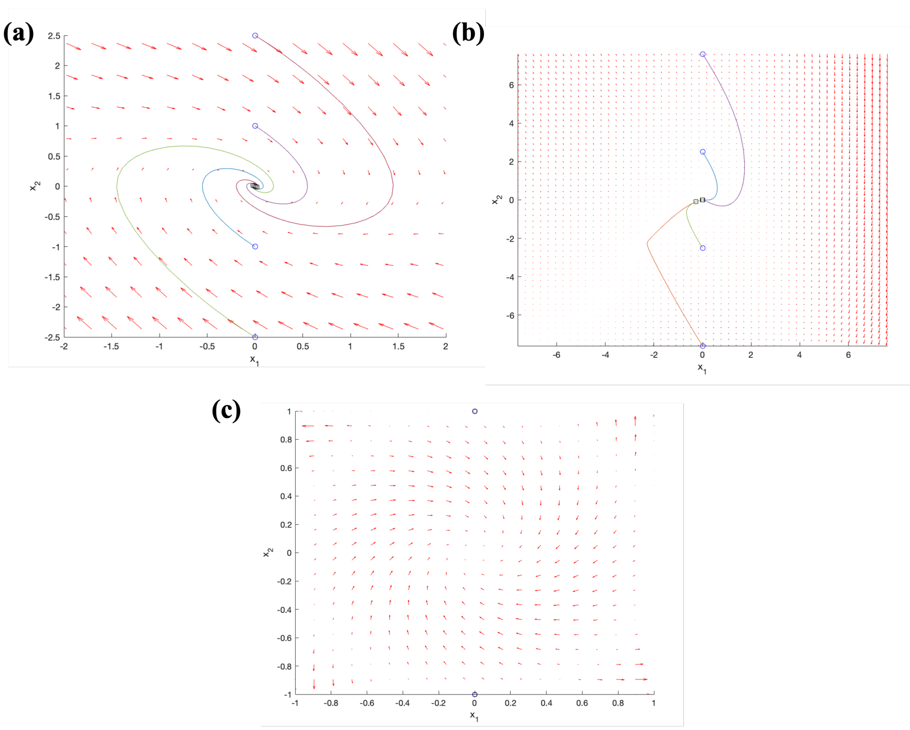

Figure 1: Problem 1

(b) Given the system in Eqt. (9),

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3
\end{aligned}
\tag{9}
$$

Under equilibrium conditions, we have $x_1 = x_2$ and $0.1x_1 - 2x_1 - x_1^2 - 0.1x_1^3 = 0$. This gives us roots, $(0, 0), (-0.7449, -0.7449), (-2.5505, -2.550))$. Our Jacobian is given below,

$$\frac{\partial f}{\partial x} = \begin{pmatrix} -1 & 1 \\ 0.1x_1 - 0.3x_1^2 & -2 \end{pmatrix}$$

Hence, for each fixed point we get eigenvalues $(\lambda_+, \lambda_-) = \{(-1, -2), (-1.59512, -1.405), (-1.5+1.3j, -1.5-1.3j)\}$. Since all these eigenvalues have negative real components, we conclude that all of the equilibria are stable. The phase portrait of the system is plotted in Figure 1(b).

(c) Considering the system in Eqt. (10),

$$\begin{aligned} \dot{x}_1 &= (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= (x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{aligned} \tag{10}$$

At equilibrium, we have solutions $(-0.997, 0.06757)$ and $(-0.33598, -0.941867)$. The Jacobian here is given by,

$$\frac{\partial f}{\partial x} = \begin{pmatrix} -1 + x_1^2 + 2x_1(x_1 - x_2) + x_2^2 & 1 - x_1^2 + 2(x_1 - x_2)x_2 - x_2^2 \\ 0.1x_1 - 0.3x_1^2 & -1 + x_1^2 + x_2^2 + 2x_2(x_1 + x_2) \end{pmatrix}$$

The eigenvalues corresponding to each fixed point are: $(\lambda_+, \lambda_-) = \{(2.14, -0.15), (2.43, -0.432)\}$. Hence, each fixed point here is a saddle point. The phase portrait of the system is plotted in Figure 1(c).

Problem 4
Solution:
(a)
Considering the pendulum, whose dynamics is described by the equation below.

$$\ddot{\theta} = -g \sin\theta - \dot{\theta}$$

From this, we can write the system below ($x_1 = \theta$).

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -g \sin x_1 - x_2 \end{aligned} \tag{11}$$

For equilibrium, we need that $x_2 = 0$ and $\sin x_1 = 0$. Hence, $x_1 = 0, n\pi$ with n as an integer. Solving this numerically, we can plot the trajectories x_1, x_2 as a function of time for different initial conditions. This will be used to analyze the stability behavior for different points. We consider the points $(x_1^0, x_2^0) = (0,0), (0.5, 0.5), (3.14, 0), (6.28, 0)$. Note that the points $(0,0), (3.14, 0), (6.28, 0)$ are chosen to be close to the equilibria $(0,0), (\pi, 0), (2\pi, 0)$. The trajectories as a function of time are plotted in Figure 2.
As shown in Figure 2(a), starting with x_1 we see that at the $(0,0)$ equilibrium, the solution x_1 stays constant over time. Hence, the solution is asymptotically stable here. Likewise, x_2 remains at its fixed point in Figure 2(b) (at 0). This suggests that the system is asymptotically stable about its fixed point (0,0) [this is expected, since we are at the fixed point and by definition, it should not change here]. Now moving far away from any equilibrium point, moving to $(0.5, 0.5)$ we see that initially x_1 is far from the equilibrium, but then after the system evolves for $t = 5$, it quickly converges and oscillates around $x_1 = 0$. As $t \to \infty$, the oscillations damp. Likewise for x_2 at this equilibrium, we see the same convergence behavior as x_1. Hence, this the system is also asymptotically stable about $(0.5, 0.5)$, which is not an equilibrium point. Finally, we look at points $(3.14, 0), (6.28, 0)$ close to equilibria as plotted in Figure 2(a)/(b). Here, we find that the solutions start close to the equilibria and remain close as time evolves. This provides more conclusive evidence that the system is asymptotically stable about its equilibria. Hence, for all of these we attribute the equilibria stability to the fact that the system tends to oscillate about the points, with amplitudes damping over time.

(b)
Plotted in Figure 2(c) is the phase portrait of our damped pendulum. The red quivers in the plot depict the vector field for coordinates (x_1, x_2)- the angle and angular velocities of the pendulum. The colored curves represent contours of dynamical system in this phase space. Not that the contours resemble spirals moving in the clockwise direction, as indicated by the orientation of the vector field. This clockwise rotation about the (0,0) equilibrium suggests that the system is stable about this point. This verifies the behavior of the time-dependent trajectories seen in Figures 2(a)/(b).

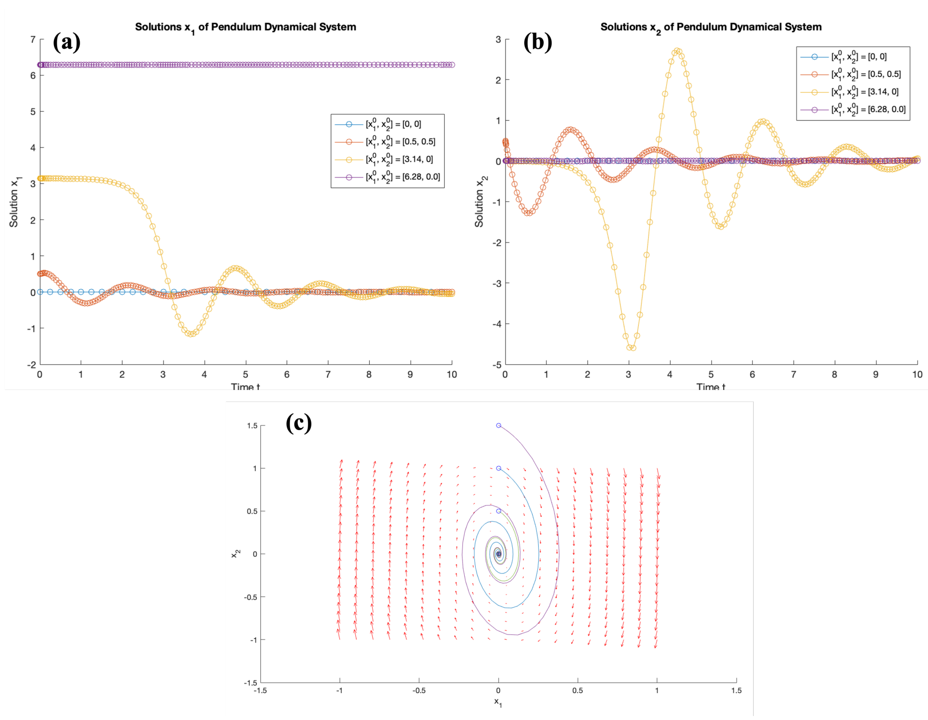

Figure 2: Dynamical behavior of a damped pendulum. (a) Solutions x_1 representing angular coordinate of damped pendulum versus time, for various initial conditions. (b)Solutions x_2 representing angular velocity coordinate of damped pendulum versus time, for various initial conditions. (c) Phase portrait of $x_1 - x_2$ of pendulum system.