Problem Set 7

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Problem 1
Solution:
Consider here the system,

$$\dot{x}_1 = x_1 + x_2$$
$$\dot{x}_2 = 3x_1^2 x_2 + x_1 + u$$
$$y = -x_1^3 + x_2$$

(a) We can verify that the system is input-output linearizable by checking if a system degree exists. Taking the Lie derivative gives us,

$$L_f h(x) = \frac{\partial h}{\partial x} f(x) = \frac{\partial h}{\partial x_1}\dot{x}_1 + \frac{\partial h}{\partial x_2}\dot{x}_2$$

$$= -3x_1^2(x_1 + x_2) + 3x_1^2 x_2 + x_1 + u$$

where we see that the controller u pops up. Hence, the relative degree is $\rho = 1$. This implies that the system is input-output linearizable.
(b) We can transform the system into normal form. We seek a from,

$$\dot{\eta} = \frac{\partial \phi(x)}{\partial x} f(x)|_{x=T^{-1}([\xi,\eta])}$$

$$\dot{\xi} = A_c \xi + B_c \gamma(x)(u - \alpha(x))$$

$$y = C_c \xi$$

where $\gamma(x) = L_g L_f^{\rho-1} h(x)$ and $\alpha(x) = -\frac{L_f^\rho h(x)}{L_g L_f^{\rho-1} h(x)}$. We can choose the controller to be $u = \alpha(x) + \frac{1}{\gamma(x)} v$. Let's first find the transformation $T(x)$. Clearly, $\eta = h(x) = -x_1^3 + x_2$ here. For $\phi(x)$, we must satisfy that

$$\frac{\partial \phi(x)}{\partial x_2} = 0$$

This allows for x_1 to be a reasonable choice (which also satisfies that $\phi(0) = 0$). This gives the transformation $T(x) = [\eta, \xi] = [x_1, -x_1^3 + x_2]$. This give the transformation, $x_1 = \eta$ and $x_2 = \xi + \eta^3$. First obtaining $\dot{\eta}$,

$$\dot{\eta} = \frac{\partial \phi(x)}{\partial x_1} f(x)|_{x=T^{-1}[\eta,\xi]} = [\frac{\partial \phi}{\partial x_1} f_1(x) + \frac{\partial \phi}{\partial x_2} f_2(x)]_{x=T^{-1}[\eta,\xi]} = \frac{\partial \phi}{\partial x_1} f_1(x)|_{x=T^{-1}[\eta,\xi]} = f_1(x)|_{x=T^{-1}[\eta,\xi]}$$

$$= \eta + \xi + \eta^3$$

Hence, this gives us the following expression in normal form

$$\dot{\eta} = \eta + \xi + \eta^3$$

$$\dot{\xi} = \xi + v$$

$$y = \xi$$

This transformation is valid in $\mathbb{R}^2$.

(c) The zero dynamics here occur at $Z^\star = \{x \in D | -x_1^3 + x_2 = 0\}$. This gives that, $x_2 = x_1^3$. Hence, $\dot{\eta} = \eta + \eta^3$. This system is not minimum phase.

Problem 2

Solution:

Consider the system,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - 2x_2 + u$$

with control law $u = -sign(x_1 + x_2)$. We can show that this corresponds to a sliding mode controller. Suppose that we want to bring the system to the

manifold $s = a_1 x_1 + x_2 = 0$. We have $x_2 = -a_1 x_2$ where $a_1 > 0$. Hence, we can take the manifold to be $s = x_1 + x_2$. For a sliding gain controller, we must satisfy that $u = -\beta(x) sign(s)$ where $\beta(x) \geq \phi(x) + \beta_0$. Where $\phi(x)$ satisfies $|\frac{a_1 x_2 + h(x)}{g(x)}| \leq \phi(x)$. Here, $|x_2 - x_1 - 2x_2| = |x_1 + x_2| \leq \phi(x) \leq \phi(x) + \beta_0$. Given this form, we can identify that $\beta(x) = 1$ in the controller satisfies our condition and our choice for manifold is $s = x_1 + x_2$. This follows the form $u(s) = -g(x) sign(s)$ which need for a sliding mode controller, hence this is a sliding mode controller.
We can verify that this controller works by writing $\dot{V}$. Take $V = \frac{1}{2} s^T s$. Hence,

$$\dot{V} = s\dot{s} = s(a_1 \dot{x}_1 + \dot{x}_2) = s(a_1 x_2 + -x_1 - 2x_2 + u)$$
$$= s(a_1 x_2 + -x_1 - 2x_2 - sign(x_1 + x_2)) = s(-x_1 - x_2 - sign(x_1 + x_2))$$

which we can see satisfies that $\dot{V} < 0$ for $s \neq 0$.

Problem 3
Solution:
Consider now the system,

$$\dot{x}_1 = x_2 + \sin(x_1)$$
$$\dot{x}_2 = \theta_1 x_1^2 + (1 + \theta_2) u$$

with $1 + \theta_2 > 0$.
(a) We can now design a sliding mode controller for the system. We can take our manifold to be $s = a_1 x_1 + x_2$ with $a_1 > 0$. We can assume that Lyapunov function, $V = \frac{1}{2} s^T s$ and require that $\dot{V} < 0$ for $s \neq 0$. This gives us,

$$\dot{V}(s) = s\dot{s} = s(a_1 \dot{x}_1 + \dot{x}_2) = s(a_1(x_2 + \sin(x_1)) + (\theta_1 x_1^2 + (1 + \theta_2) u))$$

Our controller should take the form $u = f(x) - \beta(x) sign(s)$ where $f(x)$ is designed to cancel out the nonlinear terms in the system and $\beta(x)$ is the prefactor to our sliding mode controller. Hence, we get a controller of the form ($\beta(x) = 1$),

$$u = \frac{1}{1 + \theta_2}(-a_1(x_2 + \sin(x_1) - \theta_1 x_1^2)) - sign(s)$$

. Hence substituting this into our expression for $\dot{V}$ gives,

$$\dot{V} = s(a_1 \dot{x}_1 + \dot{x}_2) = s(a_1(x_2 + \sin(x_1)) + (\theta_1 x_1^2 + (1 + \theta_2)(\frac{1}{1 + \theta_2}(-a_1(x_2 + \sin(x_1) - \theta_1 x_1^2)) - sign(s))))$$

$$= -s[sign(s))] = -|s| < 0 \text{ for} s \neq 0$$

which satisfies our stability criteria. Hence, our controller is

$$u = \frac{1}{1+\theta_2}(-a_1(x_2 + \sin(x_1) - \theta_1 x_1^2)) - sign(s)$$

(b) Now take $\theta_1, \theta_2 \in [0, 2]$ within the region $\{x \in \mathbb{R}^2 | x_1 \in [-1, 1], x_2 \in [-1, 1]\}$. Provided these constraints, we can obtain a sliding mode controller. Here we can take θ_1, θ_2. Here, we'd be able to use the same controller as before but instead add another constant term for the case that $x_1 = x_2 = 0$. This gives (added -1 at the end),

$$u = \frac{1}{1+\theta_2}(-a_1(x_2 + \sin(x_1) - \theta_1 x_1^2)) - sign(s) - 1$$