Problem Set 6

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Problem 1:
Solution:

(a) We now have that $|\psi'\rangle = N\hat{a}|\psi\rangle$ with N as a normalization factor. Hence, we need to satisfy the condition that $\langle\psi'|\psi'\rangle = 1$.

$$\langle\psi'|\psi'\rangle = N^2\langle\psi|\hat{a}^\dagger\hat{a}|\psi\rangle = N^2\langle\hat{n}\rangle$$

with $\langle\hat{n}\rangle$ as the expectation value of the number operator. This gives us the normalization constant,

$$N = \frac{1}{\sqrt{\langle\hat{n}\rangle}}$$

Hence, we have the following normalized state

$$|\psi'\rangle = \frac{1}{\sqrt{\langle\hat{n}\rangle}}\hat{a}|\psi\rangle$$

(b) We can now compare expressions for the $\langle n'\rangle$ and $\langle n\rangle$. For $\langle n\rangle$, we have

$$\langle n\rangle = \langle\psi|a^\dagger a|\psi\rangle =$$

Likewise, for $\langle n'\rangle$ we have

$$\langle n'\rangle = \langle\psi'|a^\dagger a|\psi'\rangle = \frac{1}{\langle n\rangle}\langle\psi|a^\dagger a^\dagger aa|\psi\rangle = \frac{1}{\langle n\rangle}\langle\psi|a^\dagger[1 - aa^\dagger]a|\psi\rangle$$

$$= \frac{1}{\langle n\rangle}(\langle\psi|a^\dagger a|\psi\rangle - \langle\psi|(a^\dagger a)^2|\psi\rangle) = \frac{1}{\langle n\rangle}(\langle\psi|n|\psi\rangle - \langle\psi|n^2|\psi\rangle) = \langle n\rangle - 1$$

hence, we have that $\langle n'\rangle = \langle n\rangle - 1$.

(c) Now we consider a superposition of Fock states,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |10\rangle)$$

For the average photon number, we can evaluate $\langle\psi|a^\dagger a|\psi\rangle$. We get,

$$\langle\psi|a^\dagger a|\psi\rangle = \frac{1}{2}[\langle 0| + \langle 10|]a^\dagger a[|0\rangle + |10\rangle]) = [10\langle 10|10\rangle] = 10$$

Hence, $\langle\hat{n}\rangle = 5$ for this superposition of Fock states.

(d) If one photon is absorbed by the cavity, we have that $|\psi'\rangle = \frac{1}{\sqrt{\langle n\rangle}}\hat{a}|\psi\rangle$. This gives us,

$$|\psi'\rangle = \frac{1}{\sqrt{5}\sqrt{2}}\hat{a}[|0\rangle + |10\rangle] = \frac{\sqrt{10}}{\sqrt{5}\sqrt{2}}|9\rangle = \frac{\sqrt{5}}{\sqrt{5}}|9\rangle = |9\rangle$$

Hence, the average number of photons is

$$\langle n\rangle = \langle 9|a^\dagger a|9\rangle = 9$$

Hence, the average photon number is 9. This is consistent with what was found in part (b).

Problem 2:
Solution:

We define eigenstates of the exponential operator as,

$$|\phi\rangle = \sum_n^\infty e^{in\phi}|n\rangle$$

The associated phase distribution is given by,

$$P(\phi) = \frac{1}{2\pi}|\langle\phi|\psi\rangle|^2 = \frac{1}{2\pi}|\sum_{n=0}^\infty e^{-in\phi}C_n|^2$$

Given the state $|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle\langle 0| + |1\rangle\langle 1|]$, we can find the phase distribution

$$P(\phi) = \frac{1}{2\pi}[\langle\phi|0\rangle\langle 0|\phi\rangle + \langle\phi|1\rangle\langle 1|\phi\rangle]$$

We can evaluate the matrix element,

$$\langle m|\phi\rangle = \sum_n e^{in\phi}\langle m|n\rangle = \sum_n e^{in\phi}\delta_{mn} = e^{im\phi}$$

Hence,

$$P(\phi) = \frac{1}{4\pi}(1+1) = \frac{1}{2\pi}$$

For the superposition state Ψ, we have the phase distribution ,

$$P(\phi) = \frac{1}{4\pi}|1 + e^{-i\phi}e^{i\theta}|^2 = \frac{1}{4\pi}4\cos^2(\frac{\phi-\theta}{2}) = \frac{1}{\pi}\cos^2(\frac{\phi-\theta}{2})$$

Problem 3:
Solution:

Here we consider the Jaynes-Cummings Hamiltonian,

$$\mathcal{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega a^\dagger a + \hbar\lambda(\sigma_+ a + \sigma_- a^\dagger)$$

(a) First, we can expand in the basis $\{|g,n\rangle, |e,n\rangle\}$ which gives us the following.

$$\mathcal{H} = \frac{1}{2}\hbar\omega_0(|en-1\rangle\langle en-1| - |gn-1\rangle\langle gn-1|) + \hbar\omega a^\dagger a(|gn-1\rangle\langle gn-1 + |en\rangle\langle en)$$

$$+\hbar\lambda(a|en-1\rangle\langle gn| + |gn\rangle\langle en-1|a^\dagger)$$

Evaluating the matrix elements for each gives us the following for the nth manifold of the Hamiltonian.

$$\mathcal{H}_n = \begin{pmatrix} n\hbar\omega + \frac{1}{2}\hbar\omega_0 & \hbar\lambda\sqrt{n} \\ \hbar\lambda\sqrt{n} & \hbar n\omega - \frac{1}{2}\hbar\omega_0 \end{pmatrix}$$

(b) Now isolating the manifold corresponding to N excitations, we can write $\mathcal{H}$ in terms of the cavity frequency ω and the detuning $\Delta = \omega - \omega_0$ and solve for the eigenvalues $E_{\pm,N}$ and corresponding states $|\pm, N\rangle$. We can write this Hamiltonian as,

$$\mathcal{H}_N = \begin{pmatrix} \hbar N\omega + \frac{1}{2}\hbar\omega_0 & \hbar\lambda\sqrt{N} \\ \hbar\lambda\sqrt{N} & \hbar N\omega - \frac{1}{2}\hbar\omega_0 \end{pmatrix}$$

Making the substitution $\Delta = \omega - \omega_0$ gives us,

$$\mathcal{H}_N = \begin{pmatrix} N\hbar\omega - \frac{1}{2}\hbar\Delta & \hbar\lambda\sqrt{N} \\ \hbar\lambda\sqrt{N} & N\hbar\omega + \frac{1}{2}\hbar\Delta \end{pmatrix}$$

$$= \hbar\omega NI - \frac{1}{2}\hbar\Delta\sigma_z + 2\hbar\lambda\sqrt{N}\sigma_x$$

Let $A = -\frac{1}{2}\hbar\Delta$ and $B = 2\hbar\lambda\sqrt{N}$. Writing this in explicit matrix form,

$$\mathcal{H}_N = \begin{pmatrix} N\hbar\omega + A & B \\ B & N\hbar\omega - A \end{pmatrix}$$

For the Nth manifold, we have eigenvalues

$$E_\pm = \frac{1}{2}(\hbar\omega N \pm \sqrt{4A^2 + 4B^2 + 4\hbar N\omega A + (\hbar N\omega)^2})$$

The corresponding eigenstates are then (using $\theta = \arctan(B/A)$).

$$|N, +\rangle = \cos(\frac{\theta}{2})|e, N-1\rangle + \sin(\frac{\theta}{2})|g, N\rangle$$

$$|N, -\rangle = \cos(\frac{\theta}{2})|g, N\rangle - \sin(\frac{\theta}{2})|e, N-1\rangle$$

(c) When $\Delta = 0$, we have $A = 0$. Hence,

$$E_\pm = \frac{1}{2}(\hbar\omega N \pm \sqrt{4B^2 + (\hbar N\omega)^2})$$

Hence, the energy splitting is ΔE

$$\Delta E = 2\sqrt{4B^2 + (\hbar N\omega)^2} = 2\sqrt{8\hbar^2\lambda^2 N + (\hbar N\omega)^2} \propto \sqrt{N}$$

Hence, we see that the energy splitting scales as $\sqrt{N}$.

(d) Now if we consider the dispersive regime (when $\Delta >> \lambda$), the energies and eigenstates can be determined by performing a Taylor expansion to first order in $\mu = \lambda/\Delta$. We have,

$$E_{\pm} = \frac{1}{2}(\hbar\omega N \pm \sqrt{\hbar^2\Delta^2 + 4\hbar^2\lambda^2 N - 2\hbar^2 N\omega\Delta + (\hbar N\omega)^2})$$

$$= \frac{\hbar}{2}(\frac{\omega N}{\sqrt{\Delta}} \pm \sqrt{\Delta + 4\lambda N\mu - 2N\omega + \frac{(N\omega)^2}{\Delta}})$$

Now, expanding in μ gives us to first order the following

$$E_{\pm} \approx \frac{\hbar}{2}(\omega N \pm (\sqrt{\frac{(\Delta - N\omega)^2}{\Delta}} + \frac{2\Delta\lambda N\sqrt{\frac{(\Delta-N\omega)^2}{\Delta}}\mu}{(\Delta - N\omega)^2})$$

Correspondingly, the eigenstates can be approximated as a Taylor series expansion to first order. Here,

$$\theta = \arctan\frac{B}{A} = \arctan -\frac{4\sqrt{N}\lambda}{\Delta} = \arctan(-4\sqrt{N}\mu)$$

Giving us,

$$\theta \approx -4\sqrt{N}\mu$$

Hence, our eigenstates in the dispersive limit are

$$|N,+\rangle = \cos(-2\sqrt{N}\mu)|e, N-1\rangle + \sin(-2\sqrt{N}\mu)|g, N\rangle$$

$$|N,-\rangle = \cos(-2\sqrt{N}\mu)|g, N\rangle - \sin(-2\sqrt{N}\mu)|e, N-1\rangle$$

(e) In the dispersive limit, we have

$$\Delta E = \hbar(\sqrt{\frac{(\Delta - N\omega)^2}{\Delta}} + \frac{2\Delta\lambda N\sqrt{\frac{(\Delta-N\omega)^2}{\Delta}}\mu}{(\Delta - N\omega)^2})$$

and for $\Delta = 0$ we have,

$$\Delta E = 2\sqrt{8\hbar^2\lambda^2 N + (\hbar N\omega)^2}$$

From this, we can see that in the dispersive limit, the energy difference is linear in N but goes as $\sqrt{N}$ for the case of $\Delta = 0$.

(f) In the dispersive limit, the detuning is much larger than the Rabbi frequency. In this region, dissipation is dominant. Hence, this regime is useful in quantum information as it is a way to quantify the degree of dissipation/decoherence of a quantum device. When the detuning is large, the frequency of the field is far away from the frequency of the atom.

Problem 4:
Solution:

(a) Here the transition matrix for each beam splitter is given by,

$$T_i = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix}$$

for a 50:50 beam-splitter, we have $r = r' = it = it'. = i/\sqrt{2}$. Hence,

$$T_1 = T_2 = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Hence, for the entire system we get

$$\begin{aligned} T &= \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &= i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = iI \end{aligned}$$

The intermediate matrix flips the two states $|0\rangle$ and $|1\rangle$ due to the beam mirrors. Hence, the output for the input photon state $|0\rangle$ is $i|0\rangle$. Physically, this would represent a $\pi/2$ phase shift of the input photon state.

(b) If the room has a bomb, the probabilities of detecting a photon at D_0 or D_1 can be determined for a single photon at the input in $|0\rangle$. After the first beam-splitter, the photon will accumulate a probability of 0.5 for passing through the upper arm of the MZI. The probability to end up at one of the detectors will be then be 0.25 after the second beam splitter.

(c) We can now design a setup to determine if a bomb has blown up or not. Starting on the detector side of the setup, we have probabilities P_{D1} and P_{D2} for landing at the first or second detector. Including the bomb, the transition matrix describing this setup is given by

$$T_{bomb} = T_{BS}T_{bomb}T_{flip}T_{BS} = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$$

where T_{BS} is transition matrix for each 50:50 beam splitter, T_{bomb} is the transition matrix for the bomb (destroys state corresponding to the bottom arm and retains state in upper arm), and T_{flip} flips the states corresponding to each arm corresponding to the mirror operation. This is the transition matrix corresponding to the case when the bomb is present. When the bomb isn't present, we have the following

$$T_{nobomb} = T_{BS}T_{flip}T_{BS} = i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence if we apply T_{bomb} and T_{nobomb} to the input state, we should expect two different outcomes. We can perform the actual measurement and compare it to both of these possible outcomes. If it matches what we get from applying T_{bomb}, then the bomb is present. Otherwise, the bomb is not present. Now iterating the procedure, we should expect the total probability to correspond to $T_{bomb}T_{nobomb}^{n-1}$for n iterations, if the bomb has not already blown up. If the bomb has blown up, it should correspond to T_{nobomb}^{n}.

(d) Now, if the beam splitters are not 50:50 we should have the following transition matrix (instead transmission probability of p for BS1 and $1-p$ for BS 2.

$$T = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} t' & r \\ r' & t \end{pmatrix}$$

$$T = \begin{pmatrix} \frac{p}{\sqrt{2}} & i\frac{1-p}{\sqrt{2}} \\ i\frac{1-p}{\sqrt{2}} & \frac{p}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1-p}{\sqrt{2}} & i\frac{p}{\sqrt{2}} \\ i\frac{p}{\sqrt{2}} & \frac{1-p}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} i(1-p)+ip^2 & 0 \\ 0 & i(1-p)^2+ip^2 \end{pmatrix} = (i(1-p)^2+ip^2)I$$

Hence, applying this to the $|0\rangle$ state will result in the state $(i(1-p)^2+ip^2)|0\rangle$. The probability that the photon from the port initially in $|0\rangle$ to end up at one of the detectors is $p(1-p)$. To determine whether or not the bomb is present, we can apply the same technique in part (c), but this time replace the transition matrices corresponding to two 50:50 beam splitters with two separate transition matrices corresponding to probabilities $(1-p)$ and p. The probability of detecting a bomb without it blowing up is $p(1-p)$.