Problem Set 4

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Problem 1:
Solution:

(a)
We can evaluate $\sigma_x^A = \sigma_x^A \otimes I_B$. We have,

$$\sigma_x^A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Evaluating $\sigma_x \sigma_z = \sigma_x^A \otimes \sigma_z^B$ gives us,

$$\sigma_x^A \otimes \sigma_z^B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

(b)
From this result of (a), we can determine the unitary evolution operator $U = e^{-i\mathcal{H}t}$ for $\mathcal{H} = \omega\sigma_x\sigma_z$. Hence, we have

$$U = \exp(-i\mathcal{H}t) = \exp(-i\omega t\sigma_x\sigma_z) = \exp(-i\omega t\sigma_x^A \otimes \sigma_z^B)$$

$$= \exp(-i\omega t \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix})$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix} \\ \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

This result is not the same as $U_{prod} = \exp(-i\omega t\sigma_x) \otimes \exp(-i\omega t\sigma_z)$. If we expand out first $e^{-i\omega t\sigma_x}$, we get

$$\exp(-i\omega t\sigma_x) = \exp(-\frac{i\omega t}{4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix})$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-\frac{i\omega t}{4}} & 0 \\ 0 & e^{\frac{i\omega t}{4}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2 \begin{pmatrix} \cos\frac{\omega t}{4} & i\sin\frac{\omega t}{4} \\ i\sin\frac{\omega t}{4} & \cos\frac{\omega t}{4} \end{pmatrix}$$

For $\exp(-i\omega t\sigma_z)$ we have,

$$\exp(-i\omega t\sigma_z) = \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix}$$

Hence, $U_{prod} = \exp(-i\omega t\sigma_x) \otimes \exp(-i\omega t\sigma_z)$.

$$U_{prod} = 2 \begin{pmatrix} \cos\frac{\omega t}{4} & i\sin\frac{\omega t}{4} \\ i\sin\frac{\omega t}{4} & \cos\frac{\omega t}{4} \end{pmatrix} \otimes \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix}$$

(c)
Considering the unitary operator,

$$U(\theta,\phi) = \begin{pmatrix} \cos\theta/2 & -ie^{-i\phi}\sin\theta/2 \\ -ie^{i\phi}\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

which act on the qubit basis $\{|0\rangle, |1\rangle\}$. Let $U_1 = U(\theta_1, \phi_1)$ and $U_2 = U(\theta_2, \phi_2)$.

(i) If we take $[U_1 \otimes U_2]|00\rangle$, we get

$$[U_1 \otimes U_2]|00\rangle = [U_1 \otimes U_2]|0\rangle_1|0\rangle_2 = (U_1|0\rangle_1) \otimes (U_2|0\rangle_2) = |\psi\rangle_1 \otimes |\psi\rangle_2$$

Hence, the state are separable.

(ii)
Here we have $[U_1 \otimes U_2](|00\rangle + |01\rangle)/\sqrt{2}$.

$$\frac{1}{\sqrt{2}}([U_1 \otimes U_2]|0\rangle_1|0\rangle_2 + [U_1 \otimes U_2]|0\rangle_1|1\rangle_2)$$

$$= \frac{1}{\sqrt{2}}(|\psi\rangle_1 \otimes |\psi\rangle_2 + |\psi\rangle_1 \otimes |\phi\rangle_2)$$

$$= \frac{|\psi\rangle_1}{\sqrt{2}} \otimes (|\psi\rangle_2 + |\phi\rangle_2)$$

$$= \frac{1}{\sqrt{2}}|\psi\rangle_1 \otimes |\chi\rangle_2$$

Hence, the states are separable, not entangled.

(iii)
Here we have, $[U_1 \otimes U_2](|00\rangle - |11\rangle)/\sqrt{2}$. Expanding gives us,

$$\frac{1}{\sqrt{2}}([U_1 \otimes U_2]|0\rangle_1|0\rangle_2 - [U_1 \otimes U_2]|1\rangle_1|1\rangle_2)$$

$$= \frac{1}{\sqrt{2}}(|\psi\rangle_1\psi\rangle_2 - |\phi\rangle_1|\phi\rangle_2)$$

Hence, states are entangled, not separable.

(d) If we have a two-qubit Hamiltonian

$$\mathcal{H} = a\sigma_x.\sigma_x + b\sigma_y\sigma_y$$

(i)

The eigenstates of the Hamiltonian are not entangled.

(ii) Given the initial state $|0\rangle_A \otimes |0\rangle_B$, we can find the evolved composite state vector $|\psi(t)\rangle = \mathcal{U}|0\rangle_A \otimes |0\rangle_B$. We can now write out the form of the unitary,

$$\mathcal{U} = \exp(-ita\sigma_x^A \otimes \sigma_x^B) + \exp(-itb\sigma_y^A \otimes \sigma_y^B)$$

In matrix form, this gives us

$$\mathcal{U} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} \cos at & -i\sin at \\ -i\sin at & \cos at \end{pmatrix} \\ \frac{1}{2}\begin{pmatrix} \cos at & -i\sin at \\ -i\sin at & \cos at \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

$+$

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} e^{-ibt} + ie^{ibt} & e^{-ibt} - ie^{ibt} \\ ie^{-ibt} + e^{ibt} & ie^{-ibt} - e^{ibt} \end{pmatrix} \\ \frac{1}{2}\begin{pmatrix} e^{-ibt} + ie^{ibt} & e^{-ibt} - ie^{ibt} \\ ie^{-ibt} + e^{ibt} & ie^{-ibt} - e^{ibt} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} e^{-ibt} + ie^{ibt} + \cos at & e^{-ibt} - ie^{ibt} - i\sin \\ ie^{-ibt} + e^{ibt} - i\sin at & ie^{-ibt} - e^{ibt} + \cos \end{pmatrix} \\ \frac{1}{2}\begin{pmatrix} e^{-ibt} + ie^{ibt} + \cos at & e^{-ibt} - ie^{ibt} - i\sin at \\ ie^{-ibt} + e^{ibt} - i\sin at & ie^{-ibt} - e^{ibt} + \cos at \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

If we can now apply the unitary to our initial state $|0\rangle_A \otimes |0\rangle_B$ written out in matrix form as,

$$|0\rangle_A \otimes |0\rangle_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence,

$$\mathcal{U}|0\rangle_A \otimes |0\rangle_B = \begin{pmatrix} e^{-ibt} + ie^{ibt} + \cos at & 0 \\ ie^{-ibt} + e^{ibt} - i\sin at & 0 \end{pmatrix}$$

Hence, our evolved composite state is

$$|\psi(t)\rangle = \begin{pmatrix} e^{-ibt} + ie^{ibt} + \cos at & 0 \\ ie^{-ibt} + e^{ibt} - i\sin at & 0 \end{pmatrix}$$

Hence, the evolved state becomes entangled.

(iii) Provided our evolved state, we can compute the corresponding density matrix. The full density matrix is just the outer product of this matrix. $\rho = |\psi\rangle\langle\psi|$. This computation was performed in Mathematica. From this, the reduced density matrix is simply $\rho^A_{red} = Tr_B\rho$.

$\rho^A_{red} = (e^{-ibt} + ie^{ibt} + \cos at)^2 + (ie^{-ibt} + ie^{ibt} - i\cos at)(e^{-ibt} + ie^{ibt} + \cos at)$
The purity is defined as $Tr\rho^2_A$. From this, we get

$$Tr\rho^2_A = (e^{-ibt} + e^{ibt} + \cos at)^4$$

(iv) Finally, we can obtain the concurrence of the state. This is defined as,

$$C = 2|a\delta - \beta\gamma|$$

From this, we obtain

$$C = 2((e^{-ibt} + ie^{ibt} + \cos at)(0) - (0)(ie^{-ibt} + e^{ibt} - i\sin at) = 0$$

Problem 2:
Solution:

(a) We can express the commutator $[A_1 \otimes A_2, B_1 \otimes B_2]$ as,

$$[A_1 \otimes A_2, B_1 \otimes B_2] = 2(A_1 \otimes A_2)(B_1 \otimes B_2) - \{A_1 \otimes A_2, B_1 \otimes B_2\}$$

$$= 2A_1B_1 \otimes A_2B_2 - (A_1B_1 \otimes A_2B_2 + B_1A_1 \otimes B_2A_2)$$

$$= A_1B_1 \otimes A_2B_2 + A_1B_1 \otimes A_2B_2 - A_1B_1 \otimes A_2B_2 - B_1A_1 \otimes B_2A_2$$

$$= \frac{1}{2}[A_1B_1 \otimes A_2B_2 + A_1B_1 \otimes A_2B_2 - B_1A_1 \otimes A_2B_2 - B_1A_1 \otimes B_2A_2$$

$$+A_1B_1 \otimes A_2B_2 - A_1B_1 \otimes B_2A_2 + B_1A_1 \otimes A_2B_2 - B_1A_1 \otimes B_2A_2]$$

$$= \frac{1}{2}[(A_1B_1 - B_1A_1) \otimes (A_2B_2 + B_2A_2) + (A_1B_1 + B_1A_1) \otimes (A_2B_2 - B_2A_2)]$$

$$= \frac{1}{2}([A_1, B_1] \otimes \{A_2, B_2\} + \{A_1, B_1\} \otimes [A_2, B_2])$$

(b) Provided that we have the system Hamiltionain,

$$\mathcal{H} = \mathcal{H}_S + \mathcal{H}_D + \mathcal{H}_I$$

where we have $\mathcal{H}_S = \hat{h}_S \otimes \mathbf{1}_D$, $\mathcal{H}_D = \mathbf{1}_S \otimes \hat{h}_D$, and $\mathcal{H}_I = \hat{A}_S \otimes \hat{A}_D$. The observable we care about is $\hat{O}_S \otimes \mathbf{1}_D$. For this measurement to be quantum non-demolitian, we must have that $[\hat{O}_S \otimes \mathbf{1}_D, \mathcal{H}] = 0$. Hence, we can check to see if each term of $\mathcal{H}$ commutes with the observable. This gives us,

$$[\hat{O}_S \otimes \mathbf{1}_D, \hat{h}_S \otimes \mathbf{1}_D] = (\hat{O}_S \otimes \mathbf{1}_D)(\hat{h}_S \otimes \mathbf{1}_D) - (\hat{h}_S \otimes \mathbf{1}_D)(\hat{O}_S \otimes \mathbf{1}_D)$$

$$= \hat{O}_S\hat{h}_S \otimes \mathbf{1}_D\mathbf{1}_D - \hat{h}_S\hat{O}_S \otimes \mathbf{1}_D\mathbf{1}_D$$
$$= [\hat{O}_S, \hat{h}_S] \otimes \mathbf{1}_D$$

This leads us to make our first QND constraint to be that $[\hat{h}_S, \hat{O}_S] = 0$. We can also look at the interaction Hamiltonian $\mathcal{H}_I$.

$$[\mathcal{H}_I, \hat{O}_S\otimes\mathbf{1}_D] = [\hat{A}_S\otimes\hat{A}_D, \hat{O}_S\otimes\mathbf{1}_D] = (\hat{A}_S\otimes\hat{A}_D)(\hat{O}_S\otimes\mathbf{1}_D) - (\hat{O}_S\otimes\mathbf{1}_D)(\hat{A}_S\otimes\hat{A}_D)$$

$$= \hat{A}_S\hat{O}_S \otimes \hat{A}_D\mathbf{1}_D - \hat{O}_S\hat{A}_S \otimes \mathbf{1}_D\hat{A}_D$$
$$= \hat{A}_S\hat{O}_S \otimes \hat{A}_D - \hat{O}_S\hat{A}_S \otimes \hat{A}_D$$
$$= [\hat{A}_S, \hat{O}_S] \otimes \hat{A}_D$$

The above would be 0 if $[\hat{A}_S, \hat{O}_S] = 0$. Hence, we can make our other QND constraint to be for $[\hat{A}_S, \hat{O}_S] = 0$.

(c) Here we consider the Hamiltonian of a spin system to be,

$$\mathcal{H} = \omega_e S_z + \omega_n I_z + A_{||} S_z I_z$$

with S_z corresponding to the electron spin and I_z corresponding to the nuclear spin. If the nuclear spin is the system and the electron spin is the detector, we can verify that the nuclear spin projector is a QND observable. Hence, we must satisfy that $[\mathcal{H}, I_z] = 0$. If we do this term-by-term, we get

$$[\omega_e S_z, I_z] = \omega_e[S_z, I_z] = \omega_e(S_z I_z - I_z S_z) = \omega_e(S_z I_z - S_z I_z) = 0$$

Obviously, $[I_z, I_z] = 0$. Now for the final term,

$$A_{||}[I_z, S_z I_z] = A_{||}(I_z S_z I_z - S_z I_z I_z) = A_{||}(S_z I_z I_z - S_z I_z I_z) = 0$$

Hence, $\langle I_z \rangle$ is QND observable.

(d) We now need to correlate $\langle I_z \rangle$ with an observable of the system. We can do this by performing frequency selective pulses to rotate the electron spin, conditioned on the nuclear spin projection $\langle I_z \rangle$. To examine this, we can first diagonalize the Hamiltonian. In matrix form, we have

$$\begin{aligned}
\mathcal{H} &= \frac{\omega_e}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{\omega_n}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{A_{||}e^2}{4}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\
&= \frac{\omega_e}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{\omega_n}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{A_{||}e^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\
&= \begin{pmatrix}\frac{\omega_e}{2} + \frac{\omega_n}{2} + \frac{A_{||}e^2}{4} & 0\\ 0 & \frac{A_{||}e^2}{4} - \frac{\omega_n}{2} - \frac{\omega_e}{2}\end{pmatrix}
\end{aligned}$$

Hence, the eigenvalues of the system are

$$E_+ = \frac{\omega_e}{2} + \frac{\omega_n}{2} + \frac{A_{||}e^2}{4}$$

$$E_- = \frac{A_{||}e^2}{4} - \frac{\omega_n}{2} - \frac{\omega_e}{2}$$

with corresponding eigenvectors $|\psi\rangle_+ = |1\rangle_e \otimes |1\rangle_n$ and $|\psi\rangle_+ = |0\rangle_e \otimes |0\rangle_n$.

(e) Figure 1 shows the energy splitting diagram for this system.

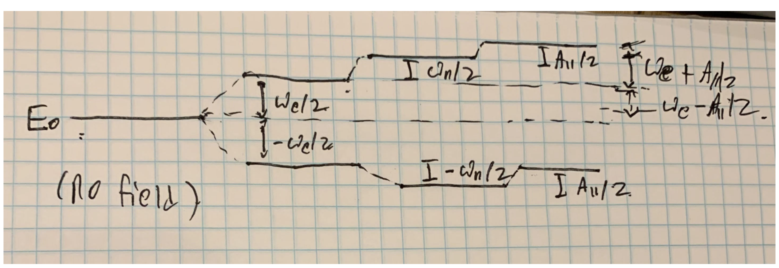

Figure 1: Energy splitting diagram.

(f) If we consider only the spin contributions, we have the Hamiltonian $\mathcal{H}_s = \omega_e S_z + A_{II} S_z I_z$. In diagonal form, we have

$$\mathcal{H}_s = \begin{pmatrix}\frac{\omega_e}{2} + \frac{A_{||}e^2}{4} & 0\\ 0 & \frac{A_{||}e^2}{4} - \frac{\omega_e}{2}\end{pmatrix}$$

Hence, the corresponding resonant frequencies for the above diagnonalzied Hamiltonian are $\omega_\pm = |\frac{A_{||}}{2} \pm \omega_e| = \omega_e \pm A_{||}2$ (doing this assumes that the electron resonant frequency is much larger than the nuclear's). These transitions are sketched in Figure 1.

(g) We can now imagine applying a microwave field that's resonant with the electron spin transition. This will rotate the spin about the x axis in the Bloch sphere at a rate Ω_0. We will get a a drive at a frequency of $\Omega = \sqrt{\Omega_0^2 + A_{||}^2}$ about the axis $\hat{\mathbf{n}} = (\Omega_0/\Omega, 0, A_{||}/\Omega)$. If we take the electron's state to be at $|0\rangle$ initially, we find the minimum overlap of the final state due to our off-resonant pulse. We can take the evolved state to be $|\psi(t)\rangle = \mathcal{U}_R(t)|0\rangle$. Here, we take $\mathcal{U}_R$ to be,

$$\mathcal{U}_R = \begin{pmatrix} \cos(\Omega t/2) + \frac{iA_{||}}{\Omega}\sin(\frac{\Omega t}{2}) & -\frac{i\Omega_0}{\Omega}e^{-i\phi}\sin(\frac{\Omega t}{2}) \\ -\frac{i\Omega_0}{\Omega}e^{i\phi}\sin(\frac{\Omega t}{2}) & \cos(\frac{\Omega t}{2}) - \frac{iA_{||}}{\Omega}\sin(\frac{\Omega t}{2}) \end{pmatrix}$$

Hence,

$$|\Psi(t)\rangle = \mathcal{U}_R|0\rangle = \begin{pmatrix} \cos(\Omega t/2) + \frac{iA_{||}}{\Omega}\sin(\frac{\Omega t}{2}) \\ -\frac{i\Omega_0}{\Omega}e^{i\phi}\sin(\frac{\Omega t}{2}) \end{pmatrix}$$

This gives us,

$$\langle\psi(t)|0\rangle = \cos(\Omega t/2) - \frac{iA_{||}}{\Omega}\sin(\frac{\Omega t}{2})$$

Hence,

$$|\langle\psi(t)|0\rangle|^2 = \cos^2(\Omega t/2) + (\frac{A_{||}}{\Omega})^2\sin^2(\frac{\Omega t}{2})$$

After plugging into mathematica, we get that the minimum of this function is

$$(1 + (\frac{\Omega_0}{A_{||}})^2)^{-1}$$

(h) Now we drive the nuclear projection $1\rangle_n$. The corresponding Hamiltonian is

$$\mathcal{H}_p = \Omega_0 S_x \otimes |1_n\rangle\langle 1_n|$$

We can now see that I_z is still a QND observable in the presence of $\mathcal{H}_p$. Let's first expand $\mathcal{H}_p$.

$$\mathcal{H}_p = \frac{\Omega_0}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{\Omega_0}{2}\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

Since the resulting tensor only has diagonal elements, we can see that it also commutes with σ_z.

(i) The evolution operator will have the form $\mathcal{U}_p = \exp(-i\mathcal{H}_p\tau)$. This will simply be,

$$\mathcal{U}_p = \exp(\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & -i\tau\frac{\Omega_0}{2} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & -i\tau\frac{\Omega_0}{2} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix})$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\tau\frac{\Omega_0}{2}} \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\tau\frac{\Omega_0}{2}} \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

(j) If the system is initialized in the state $|0\rangle_e \otimes (\alpha|0\rangle_n + \beta|1\rangle_n)$, we can find the state after applying a π pulse to the system (assuming that the π pulse is along the z-axis), $|\psi\rangle = |1\rangle_e \otimes (\alpha|1\rangle_n + \beta|0\rangle_n)$. Hence, we can see that the state is entangled.

(k) Since the nuclear and electron spins are correlated, we would expect the re-initialization of the electron spin to effect the nuclear spin state. Hence, it's possible that the nuclear spin state will flip after re-initializing the electron spin.

(l) Following re-initialization of the electron spin, we have the following state

$$|\psi\rangle = |0\rangle_e \otimes (\alpha|1\rangle_n + \beta|0\rangle_n) = \alpha|0\rangle_e \otimes |1\rangle_n + \beta|0\rangle_e \otimes |0\rangle_n$$

The corresponding density matrix is given by $|\psi\rangle\langle\psi|$.

$$|\psi\rangle\langle\psi| = \alpha|0\rangle_e \otimes |1\rangle_n\langle 1|_n \otimes \langle 0|_e + \beta|0\rangle_e \otimes |0\rangle_n\langle 0|_n \otimes \langle 0|_e$$

(m) After a second π pulse, we have the state

$$|\psi\rangle = |1\rangle_e \otimes (\alpha|0\rangle_n + \beta|1\rangle_n)$$

Clearly, this state is still entangled.

(n) Under further applications of π pulses, we get

$$|1\rangle_e \otimes (\alpha|0\rangle_n + \beta|1\rangle_n) \rightarrow |0\rangle_e \otimes (\alpha|1\rangle_n + \beta|0\rangle_n) \rightarrow |1\rangle_e \otimes (\alpha|0\rangle_n + \beta|1\rangle_n)$$

Hence, the system will always remain entangled.

(o) The signal to noise ratio is given by,

$$SNR = \frac{N}{\sqrt{N}} = \sqrt{N}$$

Here $N = r\tau_m$ with r as the spin-contrast count rate and τ_m as the measurement time. Hence, we have $SNR = \sqrt{r\tau_m}$. Our count rate is $r = 0.1r_0 = 0.1(10^5)$ phot/s $= 10^4$ phot/s. If we want an $SNR = 1$, this will require $\tau_m = 10^{-4}$ s $= 100$ μs. We will need this time plus the time to initialize the qubit. This now places a more stringent requirement on the coherence time of the nuclear spin, requiring a longer coherence time.

Bonus

Now provided the Hamiltonian,

$$\mathcal{H}_{flip-flop} = A_\perp(S_-I_+ + S_+I_-)/2$$

(a) We can show that $\langle I_z\rangle$ is not a QND observable with respect to $\mathcal{H}_{flip-flop}$. For this, we want to show that $[I_z, \mathcal{H}_{flip-flop}] \neq 0$. If we expand out, we get

$$[I_z, A_\perp(S_-I_+ + S_+I_-)/2] =$$

$$\frac{A_\perp}{2}(I_z(S_-I_+ + S_+I_-) - (S_-I_+ + S_+I_-)I_z) =$$

$$\frac{A_\perp}{16}(\sigma_z^n((\sigma_x^e - i\sigma_y^e)(\sigma_x^n + i\sigma_y^n) + (\sigma_x^e + i\sigma_y^e)(\sigma_x^n - i\sigma_y^n)) - ((\sigma_x^e - i\sigma_y^e)(\sigma_x^n + i\sigma_y^n) + (\sigma_x^e + i\sigma_y^e)(\sigma_x^n - i\sigma_y^n))\sigma_z^n)$$

From this, we can already spot out terms corresponding to $\sigma_z^n\sigma_x^n - \sigma_x^n\sigma_z^n = [\sigma_z^n, \sigma_x^n]$ and $\sigma_z^n\sigma_y^n - \sigma_y^n\sigma_z^n = [\sigma_z^n, \sigma_y^n]$. We know that $[\sigma_z^n, \sigma_x^n] \neq 0$ and $[\sigma_z^n, \sigma_y^n] \neq 0$. Hence, $[I_z, \mathcal{H}_{flip-flop}] \neq 0$. This means that $\langle I_z\rangle$ is not a QND observable.