Problem Set 2

Nima Leclerc (nleclerc@seas.upenn.edu)

ESE 523 (Quantum Engineering)
School of Engineering and Applied Science
University of Pennsylvania

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Problem 1:
Solution:

If we take the Schrodinger equation to be,

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = \mathcal{H}|\psi\rangle$$

such that $|\psi\rangle = \mathcal{U}|\psi_R\rangle$, we can now express the equation in a rotating frame basis. Hence,

$$i\hbar\frac{\partial}{\partial t}\mathcal{U}|\psi_R\rangle = \mathcal{H}\mathcal{U}|\psi_R\rangle$$

with $\mathcal{U} = \exp(-\frac{i\mathcal{A}t}{\hbar})$ we have,

$$i\hbar\frac{\partial}{\partial t}[\exp(-\frac{i\mathcal{A}t}{\hbar})|\psi_R\rangle] = \mathcal{H}\exp(-\frac{i\mathcal{A}t}{\hbar})|\psi_R\rangle$$

Hence,

$$i\hbar[\frac{-i\mathcal{A}}{\hbar}\exp(-\frac{i\mathcal{A}t}{\hbar})|\psi_R\rangle + \exp(-\frac{i\mathcal{A}t}{\hbar})\frac{\partial|\psi_R\rangle}{\partial t}] = \mathcal{H}\exp(-\frac{i\mathcal{A}t}{\hbar})|\psi_R\rangle$$

$$\mathcal{U}[\mathcal{A}|\psi_R\rangle + i\hbar\frac{\partial|\psi_R\rangle}{\partial t}] = \mathcal{H}\mathcal{U}|\psi_R\rangle$$

$$\mathcal{A}|\psi_R\rangle + i\hbar\frac{\partial|\psi_R\rangle}{\partial t} = \mathcal{U}^\dagger\mathcal{H}\mathcal{U}|\psi_R\rangle$$

Allowing to write the Schrodinger equation in our new basis,

$$i\hbar\frac{\partial|\psi_R\rangle}{\partial t} = [\mathcal{U}^\dagger\mathcal{H}\mathcal{U} - \mathcal{A}]|\psi_R\rangle$$

Hence, $\mathcal{H}_R = \mathcal{U}^\dagger\mathcal{H}\mathcal{U} - \mathcal{A}$.

Problem 2:
Solution:

(a)
Consider the magnetic field applied to a single spin system $\mathbf{B}$ with both AC and DC components:

$$\mathbf{B}_{DC} = B_0\hat{z}$$

$$\mathbf{B}_{AC} = B_1\cos(\omega t + \phi)\hat{x}$$

we can now construct a Zeeman Hamiltonian describing the dynamics of the system,

$$\hat{H} = -\boldsymbol{\mu}\cdot\boldsymbol{B} = -\frac{\mu_b g_s}{\hbar}(\sigma_x, \sigma_y, \sigma_z)\cdot(B_1\cos(\omega t+\phi), 0, B_0)$$

$$= -\frac{\mu_b g_s}{\hbar}[B_1\cos(\omega t+\phi)\sigma_x + B_0\sigma_z]$$

$$= -\frac{\mu_b g_s}{\hbar}[B_1\cos(\omega t+\phi)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + B_0\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}] = \frac{\mu_b g_s}{\hbar}\begin{pmatrix}B_0 & B_1\cos(\omega t+\phi)\\ B_1\cos(\omega t+\phi) & -B_0\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}\omega_0 & 2\cos(\omega t+\phi)\\ 2\cos(\omega t+\phi) & -\omega_0\end{pmatrix}$$

(b)
In this case, we can take the unitary operating on the initial state $|+\rangle_x$,

$$U(t)|+\rangle_x = \exp(-\frac{iHt}{\hbar})|+\rangle_x = \frac{1}{\sqrt{2}}(I\cos(\frac{\Omega t}{2}) + \frac{\delta}{\Omega}\sin(\frac{\Omega t}{2}))\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(I\cos(\frac{\Omega t}{2}) + \frac{\delta}{\Omega}\sin(\frac{\Omega t}{2}))\begin{pmatrix}1\\ -1\end{pmatrix} = U_0|-\rangle_x$$

From this, we can see that if the prefactor U_0 is positive, then this would result in counterclockwise rotation.

(c)

We can transform our Hamiltonian to a rotating frame of reference, so

$$\mathcal{H}_R = \mathcal{U}\mathcal{H}\mathcal{U}^\dagger - \mathcal{A}$$

such that $\mathcal{U} = \exp(-\frac{it\mathcal{H}}{\hbar})$. We have that,

$$\mathcal{H} = \frac{\hbar}{2}\begin{pmatrix} \omega & 2\Omega_0\cos(\omega t + \phi) \\ 2\Omega_0\cos(\omega t + \phi) & -\omega \end{pmatrix}$$

$$\mathcal{A} = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & 0 \\ 0 & -\omega_0 \end{pmatrix}$$

Giving us,

$$\mathcal{H}_R = \frac{\hbar}{2}\exp(\frac{it}{2}\begin{pmatrix} \omega_0 & 0 \\ 0 & -\omega_0 \end{pmatrix})\begin{pmatrix} \omega & 2\Omega_0\cos(\omega t + \phi) \\ 2\Omega_0\cos(\omega t + \phi) & -\omega \end{pmatrix}\exp(\frac{-it}{2}\begin{pmatrix} \omega_0 & 0 \\ 0 & -\omega_0 \end{pmatrix})$$
$$-\frac{\hbar}{2}\begin{pmatrix} \omega_0 & 0 \\ 0 & -\omega_0 \end{pmatrix}$$

Plugging the above expression into Mathematica simplifies to,

$$\begin{aligned}\mathcal{H}_R &= \frac{\hbar}{2}\begin{pmatrix} \omega_0 - \omega & 2\Omega_0\cos(\omega t + \phi)e^{i\omega t} \\ 2\Omega_0\cos(\omega t + \phi)e^{-i\omega t} & \omega - \omega_0 \end{pmatrix} \\ &= \frac{\hbar}{2}\begin{pmatrix} -\delta & 2\Omega_0\cos(\omega t + \phi)e^{i\omega t} \\ 2\Omega_0\cos(\omega t + \phi)e^{-i\omega t} & \delta \end{pmatrix}\end{aligned}$$

(d)

Here we have that,

$$\begin{aligned}\mathcal{H}_R &= \frac{\hbar}{2}\begin{pmatrix} -\delta & 2\Omega_0\cos(\omega t + \phi)e^{i\omega t} \\ 2\Omega_0\cos(\omega t + \phi)e^{-i\omega t} & \delta \end{pmatrix} \\ &= \frac{\hbar}{2}\begin{pmatrix} -\delta & \Omega_0(e^{i(\omega t+\phi)} + e^{-i(\omega t+\phi)})e^{i\omega t} \\ \Omega_0(e^{i(\omega t+\phi)} + e^{-i(\omega t+\phi)})e^{-i\omega t} & \delta \end{pmatrix} \\ &= \frac{\hbar}{2}\begin{pmatrix} -\delta & \Omega_0(e^{i(2\omega t+\phi)} + e^{-i\phi}) \\ \Omega_0(e^{-i(2\omega t+\phi)} + e^{i\phi}) & \delta \end{pmatrix}\end{aligned}$$

In the rotating wave approximation, we can drop out terms in which the frequency of the oscillations vary as twice of the AC drive (2ω). Hence,

$$\mathcal{H}_R \approx \frac{\hbar}{2}\begin{pmatrix} -\delta & \Omega_0(e^{i\phi}+e^{-i\phi}) \\ \Omega_0(e^{i\phi}+e^{-i\phi}) & \delta \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} -\delta & 2\Omega_0\cos\phi \\ 2\Omega_0\cos\phi & \delta \end{pmatrix}$$

From this, we can observe that $\mathcal{H}_R = \mathbf{\Omega}\cdot\mathbf{S}$. Such that,

$$\mathbf{\Omega} = (\Omega_x, \Omega_y, \Omega_z) = (2\Omega_0\cos\phi, 0, -\delta)$$

(e)
Here, $U_R = \exp(-it\frac{\mathcal{H}_R}{\hbar})$. If we expand out, we get

$$U_R = \exp(-it\frac{\Omega_x\sigma_x + \Omega_z\sigma_z}{\hbar}) = \exp(-\frac{it}{\hbar}\Omega_x\sigma_x)\exp(-\frac{it}{\hbar}\Omega_x\sigma_z)$$

$$= \begin{pmatrix} \cos 2\Omega_0 t\cos\phi/\hbar & -i\sin 2\Omega_0 t\cos\phi/\hbar \\ -i\sin 2\Omega_0 t\cos\phi/\hbar & \cos 2\Omega_0 t\cos\phi/\hbar \end{pmatrix}\begin{pmatrix} \exp(-i\delta t/\hbar) & 0 \\ 0 & \exp(i\delta t/\hbar) \end{pmatrix}$$

$$= \begin{pmatrix} \exp(-i\delta t/\hbar)\cos 2\Omega_0 t\cos\phi/\hbar & -i\exp(i\delta t/\hbar)\sin 2\Omega_0 t\cos\phi/\hbar \\ -i\exp(-i\delta t/\hbar)\sin 2\Omega_0 t\cos\phi/\hbar & \exp(i\delta t/\hbar)\cos 2\Omega_0 t\cos\phi/\hbar \end{pmatrix}$$

Now if $\Omega = \sqrt{\Omega_0^2+\delta^2}$. We have that,

$$U_R = \begin{pmatrix} \cos(\Omega t/2) + i\delta/\Omega & -i\frac{\Omega_0}{\Omega}e^{-i\phi}\sin(\frac{\Omega t}{2}) & \\ -i\frac{\Omega_0}{\Omega}e^{i\phi}\sin(\frac{\Omega t}{2}) & & \cos(\frac{\Omega t}{2}) - \frac{-i\delta}{\Omega}\sin(\Omega t/2) \end{pmatrix}$$

Problem 3:
Solution:

For each pulse sequence operated on the states $|\uparrow\rangle$, $|X\rangle$, and $|Y\rangle$.

(i) $(\frac{\pi}{2})_X - (\frac{\pi}{2})_X$
For each state, we get

$$R_X(\frac{\pi}{2})R_X(\frac{\pi}{2})|\uparrow\rangle = \begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 \\ -i\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 \\ -i\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow\rangle$$

$$R_X(\frac{\pi}{2})R_X(\frac{\pi}{2})|X\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 \\ -i\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 \\ -i\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = |X\rangle$$

$$R_X(\frac{\pi}{2})R_X(\frac{\pi}{2})|Y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 \\ -i\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 \\ -i\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} i \\ -1 \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = |-Y\rangle$$

(ii) $(\frac{\pi}{2})_Z - (\frac{\pi}{2})_Y$

$$R_Z(\frac{\pi}{2})R_Y(\frac{\pi}{2})|\uparrow\rangle = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = |Y\rangle$$

$$R_Y(\frac{\pi}{2})R_Z(\frac{\pi}{2})|X\rangle = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow\rangle$$

$$R_Y(\frac{\pi}{2})R_Z(\frac{\pi}{2})|Y\rangle = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} i \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = |-X\rangle$$

(iii) $(\frac{\pi}{2})_X - (\pi)_Y - (\frac{\pi}{2})_X$

For each state, we have

$$R_X(\frac{\pi}{2})R_Y(\pi)R_X(\frac{\pi}{2})|\uparrow\rangle = \begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 \\ -i\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 \\ -i\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\uparrow\rangle$$

$$R_X(\frac{\pi}{2})R_Y(\pi)R_X(\frac{\pi}{2})|X\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 \\ -i\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 \\ -i\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \end{pmatrix} = |-X\rangle$$

$$R_X(\frac{\pi}{2})R_Y(\pi)R_X(frac\pi 2)|Y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 \\ -i\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 \\ -i\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} i \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -i \\ 1 \end{pmatrix} = |-Y\rangle$$

(iv)
This must have the pulse sequence $(\pi/2)_Y - (\pi/2)_Y$.
(v)
This must have the pulse sequence $(\pi/2)_X - (\pi/2)_Z - (\pi/2)_X$.