Problem Set 3

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Problem 1:
Solution:

(a) Here we have two chips with states either in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ or $\{|X\rangle, |-X\rangle\}$. We can determine the density matrix for each case. For the first case, we have

$$\rho^{(1)} = \sum_j w_j |\phi_j\rangle\langle\phi_j| = \frac{1}{2}[w_1^1|\uparrow\rangle\langle\uparrow| + w_2^1|\downarrow\rangle\langle\downarrow|] = \frac{1}{2}[w_1^1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + w_2^1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}]$$

$$= \frac{1}{2}\begin{pmatrix} w_1^1 & 0 \\ 0 & w_2^1 \end{pmatrix}$$

If there are equal numbers of spins in either state, then $w_1^1 = w_2^1 = 1/2$. Hence,

$$\rho^{(1)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Likewise, we can write out the density matrix $\rho^{(2)}$ for the set $\{|X\rangle, |-X\rangle\}$.

$$\rho^{(2)} = \sum_j w_j |\phi_j\rangle\langle\phi_j| = \frac{1}{2}[w_1^2|X\rangle\langle X| + w_2^2|-X\rangle\langle -X|]$$

$$= \frac{1}{2}[w_1^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + w_2^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}]$$

$$= \frac{1}{2}\begin{pmatrix} w_1^2 + w_2^2 & w_1^2 - w_2^2 \\ w_1^2 - w_2^2 & w_1^2 + w_2^2 \end{pmatrix}$$

If $w_1^2 = w_2^2 = 1/2$ as is the case, the expression reduces to

$$\rho^{(2)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) Provided the expressions for the density matrices $\rho^{(1)}$ and $\rho^{(2)}$ in part (a), we see that they are the same. If we were to be idle, we would compute the expected value for some observable A and find that this observable would be the same in either case because the density matrices are the same. However, the density matrices are different, corresponding to different states. So, we can imagine doing a measurement with $\rho^{(1)}$ and using that to measure $\rho^{(2)}$. In this case, we would find that the measurement of $\rho^{(2)}$ would be different. There would also be error associated with this. Hence, if we were to use this measurement approach, we would be able to distinguish the two chips.

Problem 2:
Solution:

(a) Provided a state represented by the density matrix ρ' that evolves according to the unitary U, we get

$$\rho' = U\rho U^\dagger$$

If we take the trace of both sides we get,

$$\mathrm{Tr}\rho' = \mathrm{Tr}U\rho U^\dagger = \mathrm{Tr}UU^\dagger\rho = \mathrm{Tr}I\rho = \mathrm{Tr}\rho = 1$$

Hence, ρ' is also pure.

(b) Given that the unitary evolution has the form $U_t = \exp(-iHt/\hbar)$, we can write the time evolution of $\rho(t)$ as (with ρ_0 as the initial state), $\rho(t) = U_t\rho_0 U_t^\dagger$. Then the time derivative $\dot{\rho}(t)$ is given by,

$$\dot{\rho}(t) = \frac{d}{dt}U_t\rho_0 U_t^\dagger = \dot{U}_t\rho_0 U_t^\dagger + U_t\rho_0\dot{U}_t^\dagger$$

$$= \frac{-i}{\hbar} H U_t \rho_0 U_t^\dagger + \frac{i}{\hbar} U_t \rho_0 U_t^\dagger H = \frac{-i}{\hbar}(H\rho(t) - \rho(t)H) = \frac{-i}{\hbar}[H, \rho(t)]$$

Hence, we have derived the quantum Liouville equation:

$$\dot{\rho}(t) = \frac{-i}{\hbar}[H, \rho(t)]$$

(c) We can now apply the quantum Liouville equation to describe the determine the time-dependent state for a spin-1/2 particle undergoing Rabi oscillations with the rotating wave approximation. Here we can use the Hamiltonian H,

$$H = \begin{pmatrix} -\delta & \Omega_0 e^{-i\phi} \\ \Omega_0 e^{i\phi} & \delta \end{pmatrix}$$

with $\delta = \omega - \omega_0$ and $\Omega_0 = g\mu_B B_1/\hbar$. This allows us to write the coupled equations as,

$$\begin{pmatrix} \dot{\rho}_{00} & \dot{\rho}_{01} \\ \dot{\rho}_{10} & \dot{\rho}_{11} \end{pmatrix} = -\frac{i}{\hbar}[\begin{pmatrix} -\delta & \Omega_0 e^{-i\phi} \\ \Omega_0 e^{i\phi} & \delta \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} - \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \begin{pmatrix} -\delta & \Omega_0 e^{-i\phi} \\ \Omega_0 e^{i\phi} & \delta \end{pmatrix}]$$

$$= -\frac{i}{\hbar} \begin{pmatrix} \Omega_0(-\rho_{01} e^{i\phi} + \rho_{10} e^{-i\phi}) & -e^{-i\phi}\Omega_0\rho_{00} - 2\delta\rho_{01} + e^{-i\phi}\Omega_0\rho_{11} \\ e^{i\phi}\Omega\rho_{00} + 2\delta\rho_{10} - e^{i\phi}\Omega\rho_{11} & e^{i\phi}\Omega_0\rho_{01} - e^{-i\phi}\Omega_0\rho_{10} \end{pmatrix}$$

(d) We can now find the time dependent density, which evolves according to the Hamiltonian above. This would take the form of $\rho_t = U_t \rho_0 U_t^\dagger$ where ρ_0 is the initial state (in the up state) and U_t is the unitary corresponding to the Hamiltonian that evolves the system. Hence,

$$\rho_t = \frac{1}{2} \begin{pmatrix} \cos(\Omega t/2) + i\frac{\delta}{\Omega}\sin(\Omega t/2) & -i\frac{\Omega_0}{\Omega} e^{-i\phi}\sin(\frac{\Omega}{2}t) \\ -i\frac{\Omega_0}{\Omega} e^{i\phi}\sin(\frac{\Omega}{2}t) & \cos(\Omega t/2) - i\frac{\delta}{\Omega}\sin(\frac{\Omega}{2}t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times$$

$$\begin{pmatrix} \cos(\Omega t/2) - i\frac{\delta}{\Omega}\sin(\Omega t/2) & i\frac{\Omega_0}{\Omega} e^{-i\phi}\sin(\frac{\Omega}{2}t) \\ i\frac{\Omega_0}{\Omega} e^{i\phi}\sin(\frac{\Omega}{2}t) & \cos(\Omega t/2) + i\frac{\delta}{\Omega}\sin(\frac{\Omega}{2}t) \end{pmatrix}$$

$$= \left(\begin{matrix} (\cos(\Omega t/2) - i\frac{\delta}{\Omega}\sin(\Omega/2))(\cos(\Omega t/2) + i\frac{\delta}{\Omega}\sin(\Omega/2)) & \frac{ie^{-i\phi}\Omega_0}{\Omega}(\cos(\Omega t/2) + i\frac{\delta}{\Omega}\sin(\Omega/2))\sin(\Omega t/2 \\ -\frac{ie^{i\phi}\Omega_0}{\Omega}(\cos(\Omega t/2) - i\frac{\delta}{\Omega}\sin(\Omega/2))\sin(\Omega t/2) & \frac{\Omega_0^2}{\Omega^2}\sin^2(\Omega t/2) \end{matrix} \right.$$

Note: sorry...not enough room to fit the whole matrix....

Problem 3:
Solution:

We can define the fidelity as,

$$F(|\phi\rangle\langle\phi|, \rho) = \sqrt{\langle\phi|\rho|\phi\rangle} = \sqrt{\mathrm{Tr}(|\phi\rangle\langle\phi|\rho)}$$

If we have the density matrix,

$$\rho = (1-p)|\psi\rangle\langle\psi| + p\sigma_z|\psi\rangle\langle\psi|\sigma_z$$

We can first determine the expected value $\langle\psi|\rho|\psi\rangle$. Hence,

$$\langle\psi|\rho|\psi\rangle = (1-p)\langle\psi|\psi\rangle\langle\psi|\psi\rangle + p\langle\psi|\sigma_z|\psi\rangle\langle\psi|\sigma_z|\psi\rangle$$
$$= (1-p) + p|\langle\psi|\sigma_z|\psi\rangle|^2$$

This results in the fidelity,

$$F(|\psi\rangle\langle\psi|, \rho) = \sqrt{(1-p) + p|\langle\psi|\sigma_z|\psi\rangle|^2}$$

(b) The states $|\psi\rangle$ that are immune to decoherence are the pure states. In this case, these are the eigenstates of σ_z. The ones that are vulnerable are the mixed states. One example of this are the eigenstates of σ_x.

(c) Here we can look at the average of the square fidelity between initial and final states, considering states on the Bloch sphere. We can start with our expression derived for F^2,

$$F^2 = 1 - p + p|\langle\psi|\sigma_z|\psi\rangle|^2$$

If we now include the angular dependence of the states , $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$. We can crank out $\langle\psi|\sigma_z|\psi\rangle$ first.

$$\langle\psi|\sigma_z|\psi\rangle = [\langle 1|e^{-i\phi}\sin\frac{\theta}{2} + \langle 0|\cos\frac{\theta}{2}]|\sigma_z[\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle]$$

$$= \sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2} = \cos\theta$$

Giving us,

$$F^2(\theta,\phi) = 1 - p + p\cos^2\theta$$

Since we're interested in the average fidelity over the Bloch sphere, we want to compute

$$\langle F^2\rangle_{\theta,\phi} = \int_{\partial S} F^2(\theta,\phi)d\phi d\theta$$

$$= \int_\phi\int_\theta F^2(\theta,\phi)|\mathbf{r}_\theta \times \mathbf{r}_\phi|d\theta d\phi$$

Here $|\mathbf{r}_\theta \times \mathbf{r}_\phi| = 4\sin\phi$. Hence,

$$\langle F^2\rangle_{\theta,\phi} = \int_0^\pi d\phi[4\sin\phi]\int_0^{2\pi} d\theta[1 - p + p\cos^2\theta]$$

$$= 8\int_0^{2\pi} d\theta[1 - p + p\cos^2\theta] = 8\pi(2 - p)$$

If we normalize by the surface area, we would divide by 4π. Hence, $\langle F^2\rangle_{\theta,\phi} = 4 - 2p$.

Problem 4:
Solution:

Consider the following master equation for this problem,

$$\dot{\rho} = \sum_j R_j\rho R_j^\dagger - -\frac{1}{2}R_j^\dagger R_j\rho - \frac{1}{2}R_j^2 R_j$$

(a) If we have the density matrix,

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$$

For each case of relaxation operators, we can determine the corresponding expression for $\dot{\rho}$.

(i) $R = \sqrt{\Gamma}\sigma_z$. Here we have that,

$$\dot{\rho} = \Gamma\sigma_z\rho\sigma_z - \frac{\Gamma}{2}\sigma_z\sigma_z\rho - \frac{\Gamma}{2}\rho\sigma_z\sigma_z$$

$$= \Gamma[\sigma_z\rho\sigma_z - \frac{1}{2}(\sigma_z\sigma_z\rho + \rho\sigma_z\sigma_z)]$$

$$= \Gamma[\sigma_z\rho\sigma_z - \frac{1}{2}(I\rho + \rho I)]$$

$$= \Gamma(\sigma_z\rho\sigma_z - \rho)$$

(ii) $R = \sqrt{\Gamma}\sigma_-$. Here, we have that

$$\dot{\rho} = \Gamma[(\sigma_x - i\sigma_y)\rho(\sigma_x - i\sigma_y)^\dagger - \frac{1}{2}((\sigma_x - i\sigma_y)^\dagger(\sigma_x - i\sigma_y)\rho + \rho((\sigma_x - i\sigma_y)^\dagger(\sigma_x - i\sigma_y))]$$

$$= \Gamma[(\sigma_x - i\sigma_y)\rho(\sigma_x + i\sigma_y^\dagger) - \frac{1}{2}((\sigma_x + i\sigma_y^\dagger)(\sigma_x - i\sigma_y)\rho + \rho((\sigma_x + i\sigma_y^\dagger)(\sigma_x - i\sigma_y))]$$

$$= \Gamma[(\sigma_x - i\sigma_y)\rho(\sigma_x + i\sigma_y) - ((I + \sigma_z)\rho + \rho(I + \sigma_z))]$$

(iii) $R_1 = \sqrt{\Gamma}\sigma_+$ and $R_2 = \sqrt{\Gamma}\sigma_-$.

$$\dot{\rho} = \Gamma[(\sigma_x - i\sigma_y)\rho(\sigma_x + i\sigma_y) - ((I + \sigma_z)\rho + \rho(I + \sigma_z))] + \Gamma[(\sigma_x - i\sigma_y)\rho(\sigma_x + i\sigma_y) - ((I + \sigma_z)\rho + \rho(I + \sigma_z))]$$

$$= 2\Gamma[(\sigma_x - i\sigma_y)\rho(\sigma_x + i\sigma_y) - ((I + \sigma_z)\rho + \rho(I + \sigma_z))]$$

(iv) $R = \sqrt{\Gamma}\sigma_x$

$$\dot{\rho} = \Gamma\sigma_x\rho\sigma_x - \frac{\Gamma}{2}\sigma_x\sigma_x\rho - \frac{\Gamma}{2}\rho\sigma_x\sigma_x$$

$$= \Gamma[\sigma_x\rho\sigma_x - \frac{1}{2}(\sigma_x\sigma_x\rho + \rho\sigma_x\sigma_x)]$$

$$= \Gamma[\sigma_x\rho\sigma_x - \frac{1}{2}(I\rho + \rho I)] = \Gamma(\sigma_x\rho\sigma_x - \rho)$$

(b) For each case (i)-(iv), we can represent these as coupled differential equations of the form $\dot{x} = Gx$ (with x as a flattened form of ρ). For each case, the matrix simplifications were performed in Mathematica.
(i)

$$\dot{\rho} = \Gamma \begin{pmatrix} \rho_{11} - \rho_{00} & -(\rho_{01} + \rho_{10}) \\ -(\rho_{01} + \rho_{10}) & \rho_{11} - \rho_{00} \end{pmatrix}$$

$$\begin{pmatrix} \dot{\rho}_{00} \\ \dot{\rho}_{01} \\ \dot{\rho}_{11} \\ \dot{\rho}_{10} \end{pmatrix} = \Gamma \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{11} \\ \rho_{10} \end{pmatrix}$$

(ii)

$$\dot{\rho} = \Gamma \begin{pmatrix} \rho_{11} - 2\rho_{00} & 2ip_{01} + p_{10} \\ p_{01} - 2ip_{10} & \rho_{00} - 2\rho_{11} \end{pmatrix}$$

$$\begin{pmatrix} \dot{\rho}_{00} \\ \dot{\rho}_{01} \\ \dot{\rho}_{11} \\ \dot{\rho}_{10} \end{pmatrix} = \Gamma \begin{pmatrix} -2 & 0 & 1 & 0 \\ 0 & 2i & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2i \end{pmatrix} \begin{pmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{11} \\ \rho_{10} \end{pmatrix}$$

(iii)

$$\dot{\rho} = 2\Gamma \begin{pmatrix} \rho_{11} - 2\rho_{00} & 2ip_{01} + p_{10} \\ p_{01} - 2ip_{10} & \rho_{00} - 2\rho_{11} \end{pmatrix}$$

$$\begin{pmatrix} \dot{\rho}_{00} \\ \dot{\rho}_{01} \\ \dot{\rho}_{11} \\ \dot{\rho}_{10} \end{pmatrix} = 2\Gamma \begin{pmatrix} -2 & 0 & 1 & 0 \\ 0 & 2i & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2i \end{pmatrix} \begin{pmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{11} \\ \rho_{10} \end{pmatrix}$$

(iv)

$$\dot{\rho} = \Gamma \begin{pmatrix} \rho_{11} - \rho_{00} & \rho_{10} - \rho_{01} \\ \rho_{01} - \rho_{10} & \rho_{00} - \rho_{11} \end{pmatrix}$$

$$\begin{pmatrix} \dot{\rho}_{00} \\ \dot{\rho}_{01} \\ \dot{\rho}_{11} \\ \dot{\rho}_{10} \end{pmatrix} = \Gamma \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{11} \\ \rho_{10} \end{pmatrix}$$

(c) Refer to attached code.

(d) We can now compare the decoherence properties for each of the cases considered. In case (i), we see that the x component decoheres more quickly than in the other two directions, beyond 2 ms. The state is initialized in the up state, so we see the z component gradually die off over the time period. This is sensible since the relaxation operator used here involves σ_z. In case (ii), the state starts in the up state and then drops off very quickly, at a rate faster than in the other two directions. This is sensible because we are using the negative ladder operator to force the state from the up to down state. The decoherence process in case (iii) is similar to case (ii), but with a faster rate at which is drops off. The spin trajectory in these two cases is also quite similar, as the state dephases in the equitorial direction. Comparing case (iii) to case (iv), we see that the trajectories are almost identical. However, the z component drops off more quickly in case (iii), than in case (iv). I believe that case (iii) is more representative of an enviromental decoherence process since it involves σ_+ and σ_- operators. This case involves more than 1 decoherence process, which is most likely in a physical qubit.