Problem Set 1

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January 26, 2021

Problem 1:
Solution:

(a)
Here, we write the Pauli spin operators in matrix and outer product representations ($\sigma = (\sigma_x, \sigma_y, \sigma_z)$).

$$\sigma_x = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = i(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = |\downarrow\rangle\langle\downarrow| - |\uparrow\rangle\langle\uparrow| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(b)
We can now evaluate the expected value of the spin operator for different states.
(i)

$$\langle\uparrow|\sigma|\uparrow\rangle = \langle\uparrow|(\sigma_x, \sigma_y, \sigma_z)|\uparrow\rangle$$

$$\langle\uparrow|\sigma_x|\uparrow\rangle = \frac{1}{2}\langle\uparrow|\sigma_+ + \sigma_-|\uparrow\rangle = \frac{1}{2}[\langle\uparrow|\sigma_+|\uparrow\rangle + \langle\uparrow|\sigma_-|\uparrow\rangle] = 0$$

$$\langle\uparrow|\sigma_y|\uparrow\rangle = \frac{1}{2i}\langle\uparrow|\sigma_+ - \sigma_-|\uparrow\rangle = 0$$
$$\langle\uparrow|\sigma_z|\uparrow\rangle = 1$$

Giving, $\langle\sigma\rangle = (0,0,1)$.
(ii)

$$\langle\downarrow|\sigma|\downarrow\rangle = \langle\downarrow|(\sigma_x,\sigma_y,\sigma_z)|\downarrow\rangle$$
$$\langle\downarrow|\sigma_x|\downarrow\rangle = \frac{1}{2}\langle\downarrow|\sigma_+ + \sigma_-|\downarrow\rangle = 0$$
$$\langle\downarrow|\sigma_y|\downarrow\rangle = \frac{1}{2i}\langle\downarrow|\sigma_+ - \sigma_-|\downarrow\rangle = 0$$
$$\langle\downarrow|\sigma_z|\downarrow\rangle = -1$$

Giving, $\langle\sigma\rangle = (0,0,-1)$.
(iii)

$$\frac{1}{2}(\langle\uparrow| - \langle\downarrow|)\sigma(|\uparrow\rangle - |\downarrow\rangle) = \frac{1}{2}(\langle\uparrow| - \langle\downarrow|)(\sigma_x,\sigma_y,\sigma_z)(|\uparrow\rangle - |\downarrow\rangle)$$
$$\frac{1}{2}(\langle\uparrow| - \langle\downarrow|)\sigma_x(|\uparrow\rangle - |\downarrow\rangle) = \frac{1}{4}(\langle\uparrow| - \langle\downarrow|)(\sigma_+ + \sigma_-)(|\uparrow\rangle - |\downarrow\rangle) = \frac{1}{4}[\langle\uparrow|\uparrow\rangle + \langle\downarrow|\downarrow\rangle] = \frac{1}{2}$$

$$\frac{1}{2}(\langle\uparrow| - \langle\downarrow|)\sigma_y(|\uparrow\rangle - |\downarrow\rangle) = \frac{1}{2}\langle\downarrow|\sigma_+ + \sigma_-|\downarrow\rangle = \frac{1}{4i}(-\langle\uparrow|\uparrow\rangle - \langle\downarrow|\downarrow\rangle) = -\frac{1}{2i}$$
$$\frac{1}{2}(\langle\uparrow| - \langle\downarrow|)\sigma_z(|\uparrow\rangle - |\downarrow\rangle) = \frac{1}{2}(\langle\uparrow| - \langle\downarrow|)(|\uparrow\rangle + |\downarrow\rangle) = 0$$
Giving, $\langle\sigma\rangle = \frac{1}{2}(1,i,0)$.

(iv)
$$\frac{1}{2}(\langle\uparrow| - i\langle\downarrow|)\sigma(|\uparrow\rangle + i|\downarrow\rangle) = \frac{1}{2}(\langle\uparrow| - i\langle\downarrow|)(\sigma_x,\sigma_y,\sigma_z)(|\uparrow\rangle + i|\downarrow\rangle)$$
$$\frac{1}{2}(\langle\uparrow| - i\langle\downarrow|)\sigma_x(|\uparrow\rangle + i|\downarrow\rangle) = \frac{1}{4}(\langle\uparrow| - i\langle\downarrow|)[\sigma_+ + \sigma_-](|\uparrow\rangle + i|\downarrow\rangle) = \frac{1-i}{4}$$

$$\frac{1}{2}(\langle\uparrow| - i\langle\downarrow|)\sigma_y(|\uparrow\rangle + i|\downarrow\rangle) = \frac{1}{4i}(\langle\uparrow| - i\langle\downarrow|)\sigma_+ - \sigma_-(|\uparrow\rangle + i|\downarrow\rangle) = 0$$

$$\frac{1}{2}(\langle\uparrow| - i\langle\downarrow|)\sigma_z(|\uparrow\rangle + i|\downarrow\rangle) = \frac{1}{2}(\langle\uparrow| - i\langle\downarrow|)(|\uparrow\rangle - i|\downarrow\rangle) = 0$$

Giving, $\langle\sigma\rangle = \frac{1-i}{4}(1,0,0)$.

(c)
Now provided that we have a magnetic field $\mathbf{B} = (B_x, 0, 0)$, we can write out the Hamiltonian for the system in matrix form, from the Zeeman interaction

$$\hat{H}_{Ze} = -\hat{\mu}\cdot\mathbf{B} = \frac{1}{2}g\mu_b\sigma\cdot\mathbf{B} = \frac{1}{2}g\mu_b B_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

We can now diagonalize $\hat{H}_{Ze}$ to obtain its eigenvalues and corresponding eigenvectors. From this, we can see that σ_x has eigenvalues $\lambda_\pm = \pm 1$ with corresponding eigenvectors $\mathbf{v}_\pm = (1,1), (1-,1)$. Hence, the energy levels and corresponding states of the Hamiltonian are,

$$\epsilon_\pm = \pm\frac{1}{2}g\mu_b B_x$$

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}(|\downarrow\rangle + |\uparrow\rangle)$$

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$$

(d)
We can now look at the time-dependence of each state by solving the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\mathcal{U}(t,t_0) = \hat{H}_{Ze}\mathcal{U}(t,t_0)$$

with $\mathcal{U}(t,t_0)$ as the time-evolution unitary operator. Here, we get the solution

$$\mathcal{U}(t,t_0=0) = \exp(\frac{-i\hat{H}_{Ze}t}{\hbar}) = \exp(\frac{-ig\mu_b B_x\hat{S}_x t}{2\hbar})$$

We can now operate $\mathcal{U}(t,0)$ on our initial eigenstates, giving

$$|\Psi_+(t)\rangle = \mathcal{U}(t,0)|\Psi_+\rangle = \mathcal{U}(t,0)[\frac{1}{\sqrt{2}}(|\downarrow\rangle + |\uparrow\rangle)] = \frac{1}{\sqrt{2}}[\exp(\frac{-i\epsilon_- t}{\hbar})|\downarrow\rangle + \exp(\frac{-i\epsilon_+ t}{\hbar})|\uparrow\rangle]$$

$$= \frac{1}{\sqrt{2}}[\exp(\frac{+ig\mu_b B_x t}{2\hbar})|\downarrow\rangle + \exp(\frac{-ig\mu_b B_x t}{2\hbar})|\uparrow\rangle]$$

$$|\Psi_-(t)\rangle = \mathcal{U}(t,0)|\Psi_-\rangle = \mathcal{U}(t,0)[\frac{1}{\sqrt{2}}(|\downarrow\rangle + |\uparrow\rangle)]$$

$$= \mathcal{U}(t,0)|\Psi_-\rangle = \mathcal{U}(t,0)[\frac{1}{\sqrt{2}}(-|\downarrow\rangle + |\uparrow\rangle)]$$

$$= \frac{1}{\sqrt{2}}[-\exp(\frac{ig\mu_b B_x t}{2\hbar})|\downarrow\rangle + \exp(\frac{-ig\mu_b B_x t}{2\hbar})|\uparrow\rangle]$$

We can now compute the expected value $\langle\sigma(t)\rangle$. Let $\omega = \frac{g\mu_b B_x}{2\hbar}$. Performing this element wise, we have

$$\langle\psi_+|\sigma_x|\psi_+\rangle$$

$$= \frac{1}{4}[\langle\uparrow|e^{i\omega t} + \langle\downarrow|e^{-i\omega t}]|(\sigma_+ + \sigma_-)|[e^{-i\omega t}|\uparrow\rangle + e^{i\omega t}|\downarrow\rangle]$$

$$= \frac{1}{4}[\langle\uparrow|e^{i\omega t} + \langle\downarrow|e^{-i\omega t}]|\sigma_-|e^{i\omega t}|\uparrow\rangle = \frac{1}{4}\langle\downarrow|\downarrow\rangle = \frac{1}{4}$$

$$\langle\psi_+|\sigma_y|\psi_+\rangle = \frac{1}{4i}[\langle\uparrow|e^{i\omega t} + \langle\downarrow|e^{-i\omega t}]|(\sigma_+ - \sigma_-)|[e^{-i\omega t}|\uparrow\rangle + e^{i\omega t}|\downarrow\rangle] = -\frac{1}{4}$$

$$\langle\psi_+|\sigma_z|\psi_+\rangle = \frac{1}{2}[\langle\uparrow|e^{i\omega t} + \langle\downarrow|e^{-i\omega t}]|(\sigma_z)|[e^{-i\omega t}|\uparrow\rangle + e^{i\omega t}|\downarrow\rangle]$$

$$= \frac{1}{2}[\langle\uparrow|e^{i\omega t} + \langle\downarrow|e^{-i\omega t}]|[e^{-i\omega t}|\uparrow\rangle - e^{i\omega t}|\downarrow\rangle]$$

$$= e^{i\omega t}e^{-i\omega t} + 0 + 0 - e^{i\omega t}e^{-i\omega t} = 0$$

Hence,

$$\langle\sigma\rangle = \frac{1}{4}(1, -1, 0)$$

Problem 2:
Solution:

(a)
We can show that for Hermitian matrices,

$$e^{iHt} = R^\dagger e^{iDt} R$$

for D as a diagonal matrix and R as a matrix with columns comprising the eigenvectors of H ($H = R^\dagger DR$). Hence, we can write

$$e^{iHt} = \sum_{k=0}^{\infty} \frac{1}{k!}(iHt)^k = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} H^k = \sum_{k=0}^{\infty} \frac{(it)^k}{k!}(R^\dagger DR)^k = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} R^\dagger D^k R = R^\dagger(\sum_{k=0}^{\infty} \frac{1}{k!}(iDt)^k)R$$

$$= R^\dagger e^{iDt} R$$

(b)
Now if $H^2 = I$, this becomes

$$e^{iHt} = R^\dagger(\sum_{k=0}^{\infty} \frac{1}{k!}(itI)^k)R = R^\dagger e^{itI} R = e^{it} R^\dagger IR = e^{it} R^\dagger R = e^{it} I$$