Problem Set 5

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Problem 1:
Solution:

(a)
If we look at the components of Δ_j for each side of the superconductor, we see that

$$\Delta_j = \sqrt{\frac{n_j}{2}} e^{i\phi_j}$$

with ϕ_j as the superconducting phase, which we already know to be constant for the entire superconductor bulk. We can also write, $n_j = \frac{j_j}{ev_j} \propto \frac{j_j}{\nabla \phi_j}$. We know that j_j is constant as it is the superconducting persistent current and that ϕ_j is also constant. Hence, n_j must be constant over the whole region.

(b)
We can now look at the Hamiltonian H,

$$\mathcal{H} = \begin{pmatrix} eV & K \\ K & -eV \end{pmatrix}$$

when $K = 0$, there is no tunnel current and we have.

$$\mathcal{H} = \begin{pmatrix} eV & 0 \\ 0 & -eV \end{pmatrix}$$

with eigenvalues $\pm eV$. Hence, the eigenstates must be

$$|\psi\rangle_+ = \begin{pmatrix} |\Delta_1|e^{i\phi_1} \\ 0 \end{pmatrix}$$

$$|\psi\rangle_- = \begin{pmatrix} 0 \\ |\Delta_2|e^{i\phi_2} \end{pmatrix}$$

In this case, we have $\Delta_1 = \sqrt{\frac{n_1}{2}}e^{i\phi}$ and $\Delta_2 = \sqrt{\frac{n_2}{2}}e^{i\phi}$, with $\phi_1 = \phi_2 = \phi$. Here, we have that $n_i = |\Delta_i|^2 e^{-i\phi_i}$.

(c)
If there's no tunneling ($K = 0$), the energy required (energy gained) for the Cooper pair to make it from side 1 to side 2 would be twice the electron energy (2 eV). If $\mu_1 - \mu_2 = eV$ is at least 2 eV, then Cooper pair can be transported across the junction. Hence, this argument is consistent with taht of the voltage drop across the junction since the energy gained by the Cooper pair will correspond to the potential applied across the junction.

(d)
The off diagonal component of $\mathcal{H}$ describes the process where Cooper pairs tunnel across the barrier. The current response with this applied bias is related to the phase difference across the junction. The current response is simply,

$$I = I_0 \sin(\phi_1 - \phi_2)$$

which shows that the current will just vary sinusoidally with the phase difference across the junction.
(e)
Starting with the Schrodinger equation, we can come up with a set of coupled differential equations for Δ_1 and Δ_2. We start with,

$$i\hbar\frac{\partial\psi_1}{\partial t} = eV\psi_1 + K\psi_2$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = K\psi_1 - eV\psi_2$$

which comes out to,

$$i\hbar\frac{\partial}{\partial t}(|\Delta_1|e^{i\phi_1}) = eV|\Delta_1|e^{i\phi_1} + K|\Delta_2|e^{i\phi_2}$$

$$= i\hbar\frac{\partial}{\partial t}(\sqrt{\frac{n_{s,1}}{2}}e^{i\phi_1}) = eV\sqrt{\frac{n_{s,1}}{2}}e^{i\phi_1} + K\sqrt{\frac{n_{s,2}}{2}}e^{i\phi_2}$$

$$i\hbar\frac{\partial}{\partial t}(|\Delta_2|e^{i\phi_2}) = K|\Delta_1|e^{i\phi_1} - eV|\Delta_2|e^{i\phi_2}$$

$$= i\hbar\frac{\partial}{\partial t}(\sqrt{\frac{n_{s,2}}{2}}e^{i\phi_1}) = K\sqrt{\frac{n_{s,1}}{2}}e^{i\phi_1} - eV\sqrt{\frac{n_{s,2}}{2}}e^{i\phi_2}$$

Expanding the left side for each case gives,

$$\frac{\partial}{\partial t}(\sqrt{\frac{n_{s,j}}{2}}e^{i\phi_j}) = e^{i\phi_j}\frac{\partial}{\partial t}\sqrt{\frac{n_{s,j}}{2}} + \sqrt{\frac{n_{s,j}}{2}}\frac{\partial}{\partial t}e^{i\phi_j} = e^{i\phi_j}\frac{\dot{n}_{s,j}}{4}\sqrt{\frac{2}{n_{s,j}}} + i\sqrt{\frac{n_{s,j}}{2}}e^{i\phi_j}\dot{\phi}_j$$

Hence, we have the coupled equations

$$i\hbar e^{i\phi_1}\frac{\dot{n}_{s,1}}{4}\sqrt{\frac{2}{n_{s,1}}} - \hbar\sqrt{\frac{n_{s,1}}{2}}e^{i\phi_1}\dot{\phi}_1 = eV\sqrt{\frac{n_{s,1}}{2}}e^{i\phi_1} + K\sqrt{\frac{n_{s,2}}{2}}e^{i\phi_2}$$

$$i\hbar e^{i\phi_2}\frac{\dot{n}_{s,2}}{4}\sqrt{\frac{2}{n_{s,2}}} - \hbar\sqrt{\frac{n_{s,2}}{2}}e^{i\phi_2}\dot{\phi}_2 = K\sqrt{\frac{n_{s,1}}{2}}e^{i\phi_1} - eV\sqrt{\frac{n_{s,2}}{2}}e^{i\phi_2}$$

The first and second equations simplify to,

$$\hbar\frac{\dot{n}_{s,1}}{4} - \hbar\frac{n_{s,1}}{2i}\dot{\phi}_1 = eV\frac{n_{s,1}}{2i} + \frac{K}{2i}\sqrt{n_{s,1}n_{s,2}}e^{i\theta}$$

$$\hbar\frac{\dot{n}_{s,2}}{4} - \frac{\hbar}{2i}n_{s,2}\dot{\phi}_2 = \frac{K}{2i}\sqrt{n_{s,1}n_{s,2}}e^{-i\theta} - \frac{eV}{2i}n_{s,2}$$

Subtracting the two above equations gives,

$$\hbar(\frac{\dot{n}_{s,1}}{4} - \frac{\dot{n}_{s,2}}{4}) - (\hbar\frac{n_{s,1}}{2i} - \frac{\hbar}{2i}n_{s,2}\dot{\phi}_2) = (eV\frac{n_{s,1}}{2i} + \frac{eV}{2i}n_{s,2}) + K\sqrt{n_{s,1}n_{s,2}}\sin\theta$$

This simplifies to 4 equations,

$$\dot{n}_{s1} = \frac{2K}{\hbar}\sqrt{n_{s1}n_{s2}}\sin\theta$$

$$\dot{n}_{s2} = -\frac{2K}{\hbar}\sqrt{n_{s1}n_{s2}}\sin\theta$$

$$\dot{\phi}_1 = -\frac{K}{\hbar}\sqrt{\frac{n_{s2}}{n_{s1}}}\cos\theta - \frac{eV}{\hbar}$$

$$\dot{\phi}_2 = -\frac{K}{\hbar}\sqrt{\frac{n_{s1}}{n_{s2}}}\cos\theta + \frac{eV}{\hbar}$$

(f)
If we take $n_{s1} \approx n_{s2}$, we have that

$$\dot{n}_s = \frac{2K}{\hbar}n_s\sin\theta$$

Hence,

$$I_J = e\dot{n}_s = \frac{2eK}{\hbar}n_s\sin\theta = I_0\sin\theta$$

We also have,

$$\begin{aligned}\dot{\theta} &= -\frac{K}{\hbar}\left(\sqrt{\frac{n_{s1}}{n_{s2}}} - \sqrt{\frac{n_{s2}}{n_{s1}}}\right) + 2\frac{eV}{\hbar}\\ &\approx 2\frac{eV}{\hbar}\end{aligned}$$

Hence,

$$V = \frac{\hbar\dot{\theta}}{2e}$$

Problem 2:
Solution:

(a) The charge on the island of the JJ has the form,

$$Q = C_g(\phi - V_g) + C_J\phi$$

Hence, the energy of this island is given by (to charge it from 0 to $-2eN$)

$$U = \int_0^{-2eN}\phi dQ = \int_0^{-2eN}\phi dQ = \int_{\phi_1}^{\phi_2}\phi d[C_g(\phi - V_g) + C_J\phi]$$

$$= (C_g + C_J)\int_{\phi_1}^{\phi_2}\phi d\phi = \frac{C_g + C_J}{2}\phi^2|_{\phi_1}^{\phi_2} = \frac{C_g + C_J}{2}[\phi_2^2 - \phi_1^2]$$

with $\phi_1 = \frac{V_g C_g}{C_g + C_J}$ and $\phi_2 = \frac{V_g C_g - 2eN}{C_g + C_J}$. Let $C_\Sigma = C_g + C_J$. Hence,

$$\begin{aligned} U &= \frac{C_\Sigma}{2}[\frac{(V_g C_g - 2eN)^2}{C_\Sigma^2} - \frac{(V_g C_g)^2}{C_\Sigma^2}] \\ &= \frac{1}{2C_\Sigma}[(V_g C_g - 2eN)^2 - (V_g C_g)^2] \\ &= \frac{1}{2C_\Sigma}[(2eN)^2 - 2eNC_g V_g] \end{aligned}$$

If $N_g = C_g V_g / 2e$ and $E_c = (2e)^2/2C_\Sigma$, then this expression simplifies to,

$$\begin{aligned} &= E_c N^2 - \frac{(2e)^2}{C_\Sigma} N N_g = E_c N^2 - 2E_c N_g N = E_c(N^2 - N_g N) \\ &= E_c(N - N_g)^2 - N_g^2 E_c \end{aligned}$$

Hence, $U = E_c(N - N_g)^2 - N_g^2 E_c$.

(b) We now have the Hamiltonian,

$$\mathcal{H} = E_c(N - N_g)^2 - E_J \cos\theta$$

where we have the conjugate charge states,

$$|\theta\rangle = \sum_{N=-\infty}^{\infty} e^{-iN\theta} |N\rangle$$

such that $\langle N|\theta\rangle = e^{-iN\theta}$. We can evaluate $\langle\theta'|\theta\rangle$. Hence,

$$\langle\theta'|\theta\rangle = \sum_{N=-\infty}^{\infty} \langle N|N\rangle e^{iN(\theta'-\theta)} = \sum_{N=-\infty}^{\infty} e^{iN(\theta'-\theta)}$$

which is simply the Fourier transform of $e^{iN(\theta'-\theta)}$. We know this to be, $2\pi\delta(\theta - \theta')$. Hence,

$$\langle\theta'|\theta\rangle = 2\pi\delta(\theta - \theta')$$

(c)
Likewise, we can find the inverse relation for $|N\rangle$. Hence, we can find the inverse Fourier transform that's given by

$$|N\rangle = \frac{1}{\sqrt{2\pi}} \sum_\theta e^{iN\theta} |\theta\rangle$$

$$= \frac{1}{2\pi}\int_\theta d\theta e^{iN\theta}|\theta\rangle$$

Hence,

$$|N\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{iN\theta}|\theta\rangle$$

(d) Provided the operator expression for $\hat{N}$,

$$\hat{N} = \int_0^{2\pi}\frac{d\theta}{2\pi}|\theta\rangle\frac{\partial}{i\partial\theta}\langle\theta|$$

we can first expand out,

$$\frac{\partial}{\partial\theta}\langle\theta| = \frac{\partial}{\partial\theta}\sum_N e^{iN\theta}\langle N| = i\sum_N Ne^{iN\theta}\langle N|$$

Hence,

$$|\theta\rangle\frac{\partial}{i\partial\theta}\langle\theta| = \sum_N e^{-iN\theta}|N\rangle i\sum_N Ne^{iN\theta}\langle N| = \sum_N N|N\rangle\langle N|$$

Hence,

$$\hat{N} = \frac{1}{2\pi}\int_0^{2\pi} d\theta\sum_N N|N\rangle\langle N|$$

Giving,

$$\hat{N} = \sum_N N|N\rangle\langle N|$$

We see that $\hat{N}$ and $\hat{\theta}$ are Fourier pairs. We also know that $\hat{x}$ and $\hat{p}$ are Fourier pairs. Hence, we can say that $\hat{\theta} = i\frac{\partial}{\partial N}$. Hence, we can act the commutator on $|N\rangle$.

$$[\hat{\theta},\hat{N}]|N\rangle = [i\frac{\partial}{\partial N},\hat{N}]|N\rangle = i(\frac{\partial}{\partial N}(N|N\rangle) - N\frac{\partial}{\partial N}|N\rangle)$$

$$= i(N\frac{\partial|N\rangle}{\partial N} + |N\rangle - N\frac{\partial|N\rangle}{\partial N})$$

$$= i|N\rangle$$

Hence, $[\hat{\theta}, \hat{N}] = i$.

(e) Here, we have

$$e^{\pm i\hat{\theta}}|\theta\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{\pm i\theta}|\theta\rangle\langle\theta|\theta\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{\pm i\theta}|\theta\rangle = e^{\pm i\theta}|\theta\rangle$$

(f) We have the operator $e^{i\hat{\theta}}|N\rangle = |N+1\rangle$.

$$e^{i\hat{\theta}}|N\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{\pm i\theta}|\theta\rangle\langle\theta| \int_0^{2\pi} \frac{d\theta}{2\pi} e^{iN\theta}|\theta\rangle$$

$$= \int_0^{2\pi} \frac{d\theta}{2\pi} e^{\pm i\theta} e^{iN\theta}|\theta\rangle = |N \pm 1\rangle$$

Hence, we have that

$$e^{\pm i\hat{\theta}} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{\pm i\theta} \sum_N e^{iN\theta} e^{-iN\theta}|N\rangle\langle N| = \int_0^{2\pi} \frac{d\theta}{2\pi} \sum_N e^{\pm i\theta}|N\rangle\langle N|$$

$$= \sum_N |N \pm 1\rangle\langle N|$$

(g) Now, we can finally express our Hamiltonian in the number basis. We started with our expression as,

$$\mathcal{H} = E_c(N - N_g)^2 - E_J \cos\theta$$

where, we can replace with a complete set of states (as operator)

$$\mathcal{H} = E_c \sum_N (N - N_g)^2 |N\rangle\langle N| - \frac{E_J}{2} \sum_N e^{i\theta}|N\rangle\langle N| + e^{-i\theta}|N\rangle\langle N|$$

$$= E_c \sum_N (N - N_g)^2 |N\rangle\langle N| - \frac{E_J}{2} \sum_N |N+1\rangle\langle N| + |N-1\rangle\langle N|$$

$$= E_c \sum_N (N - N_g)^2 |N\rangle\langle N| - \frac{E_J}{2} \sum_N |N+1\rangle\langle N| + |N\rangle\langle N+1|$$

Problem 3:

Solution:
Now, provided the Hamiltonian that is diagonal in the charge basis

$$\mathcal{H} = E_C \sum_N (N - N_g)^2 |N\rangle\langle N|$$

(a) We can plot the eigenstates for $N = -2, -1, .0, 1, 2$. This is shown in Figure 1. We take $E_c = 1$ here.
(b) We can show that the the intersection of the states at $|N\rangle$ and $|N+1\rangle$ occurs at $N_g = N_g +$. The corresponding eigenvalues for each state is given by,

$$E_N = E_c(N - N_g)^2$$

$$E_{N+1} = E_c(N + 1 - N_g)^2$$

Now solving the expression,

$$(N - N_g)^2 = (N + 1 - N_g)^2$$

gives us that the eigenvalues intersect at $N_g = N + 1/2$. Plugging this in to E_N gives us $E_{cross} = E_c/4$. If we look at the next highest energy level (E_{N+2}), at this point we have that $E_{N+2} = \frac{3}{2}E_c$. We can see that this is well separated from $E_c/4$. Hence, the lowest lying energy levels are decoupled and therefore addressable which is ideal behavior for a qubit.

(c) We can go back to our Hamiltonian expression, in the limit where $E_J << E_C$ for $N = 0$. The expression comes out to,

$$\mathcal{H} = E_c N_g^2 |0\rangle\langle 0| - \frac{E_J}{2}(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

For $N_g = 1/2 + \Delta_g$, we have

$$\mathcal{H} = E_c(1/2 + \Delta_g)^2 |0\rangle\langle 0| - \frac{E_J}{2}(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

$$= E_c(\Delta_g^2 + \frac{1}{4})|0\rangle\langle 0| + E_c\Delta_g|0\rangle\langle 0| - \frac{E_J}{2}(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

$$= E_c(\Delta_g^2 + \frac{1}{4})I + E_c\Delta_g\sigma_z - \frac{E_J}{2}\sigma_x$$

(d)
For $\Delta_g = 0$, we have

$$\mathcal{H} = \frac{E_c}{4}I - \frac{E_J}{2}\sigma_x = \begin{pmatrix} E_c/4 & -E_J/2 \\ -E_J/2 & E_c/4 \end{pmatrix}$$

The eigenvalues of this are $E_{\pm} = \frac{1}{4}(E_c \pm 2E_J)$. The corresponding eigenvectors are,

$$|\Psi\rangle_+ = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle)$$

$$|\Psi\rangle_- = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

(e) Here, we have the eigenstates of the Hamiltonian $|\Psi\rangle_+$ and $|\Psi\rangle_-$. We can apply the Hadammard gate on one of these states to show the effect of rotation.

$$H|\Psi\rangle_- = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Hence, we can see that the Hadammard has the effect of rotating the state. We can now look at the effect of applying a Hadammard operation to the system's Hamiltonian.

$$\mathcal{H}_{qubit} = H\mathcal{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} E_c/4 & -E_J/2 \\ -E_J/2 & E_c/4 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} E_c/4 - E_j/2 & E_c/4 - E_j/2 \\ -E_c/4 - E_j/2 & E_c/4 + E_j/2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(-\frac{E_j}{2}\sigma_z + \frac{E_c}{4}I + (E_c/4 + E_j/2)\sigma_x)$$

$$= -E_z(\sigma_z + X_{cont}\sigma_x)$$

Bonus Problem:
Here, we consider the general spin Hamiltonian

$$\mathcal{H} = E_c\sum_N (N - N_g)^2|N\rangle\langle N| - \frac{E_J}{2}\sum_N |N\rangle\langle N+1| + |N+1\rangle\langle N|$$

We can consider the first 2 lowest-lying states and solve for the eigenvalues. The Hamiltonian can be written as,

$$\mathcal{H} = E_c N_g^2|0\rangle\langle 0| + E_c(1 - N_g)^2|1\rangle\langle 1| - \frac{E_J}{2}[|0\rangle\langle 1| + |1\rangle\langle 0|]$$

$$= N_g^2|0\rangle\langle 0| + (1-N_g)^2|1\rangle\langle 1| - \frac{E_J}{2E_c}[|0\rangle\langle 1| + |1\rangle\langle 0|]$$

$$= \begin{pmatrix} N_g^2 & -\frac{E_J}{2E_c} \\ -\frac{E_J}{2E_c} & (1-N_g)^2 \end{pmatrix} = \begin{pmatrix} N_g^2 & -\gamma \\ -\gamma & (1-N_g)^2 \end{pmatrix}$$

Now, diagonalizing the Hamiltonian will give us eigenenergies

$$E_{\pm} = \frac{1}{2}(1 - 2N_g + 2N_g^2 \pm \sqrt{1 + 4\gamma^2 - 4N_g + 4N_g^2})$$

Here we want the qubit to have $E_J/2 >> E_c\Delta_g$. Hence, we want that $\Gamma >> 1$. So we can take $\Gamma = 1.5$. This is plotted in Figure 2. We can show that the eigenenergies are insensitive to changes in Δ_g, by expanding in order of N_g. If we substitute $N_g = \frac{1}{4} + \Delta_g$. Expanding the lowest state energy to leading order, we get (for $\gamma = 1.5$).

$$E_- \approx -1.2$$

Hence, $\frac{\partial E_-}{\partial \Delta_g} = 0$. This suggests that the