Digital Communications

Nima Mohammadi

Homework #1

1 Convolution of a sinc function and an impulse

1a Determine the convolution of sinc(t) with $\delta(t-2)$

We have that

$$x(t) * \delta(t \pm a) = x(t \pm a)$$

Then

$$\text{sinc}(t) * \delta(t-2) = \text{sinc}(t-2)$$

In Frequency domain

$$x(t) * y(t) \xleftrightarrow{F} X(f)Y(f)$$

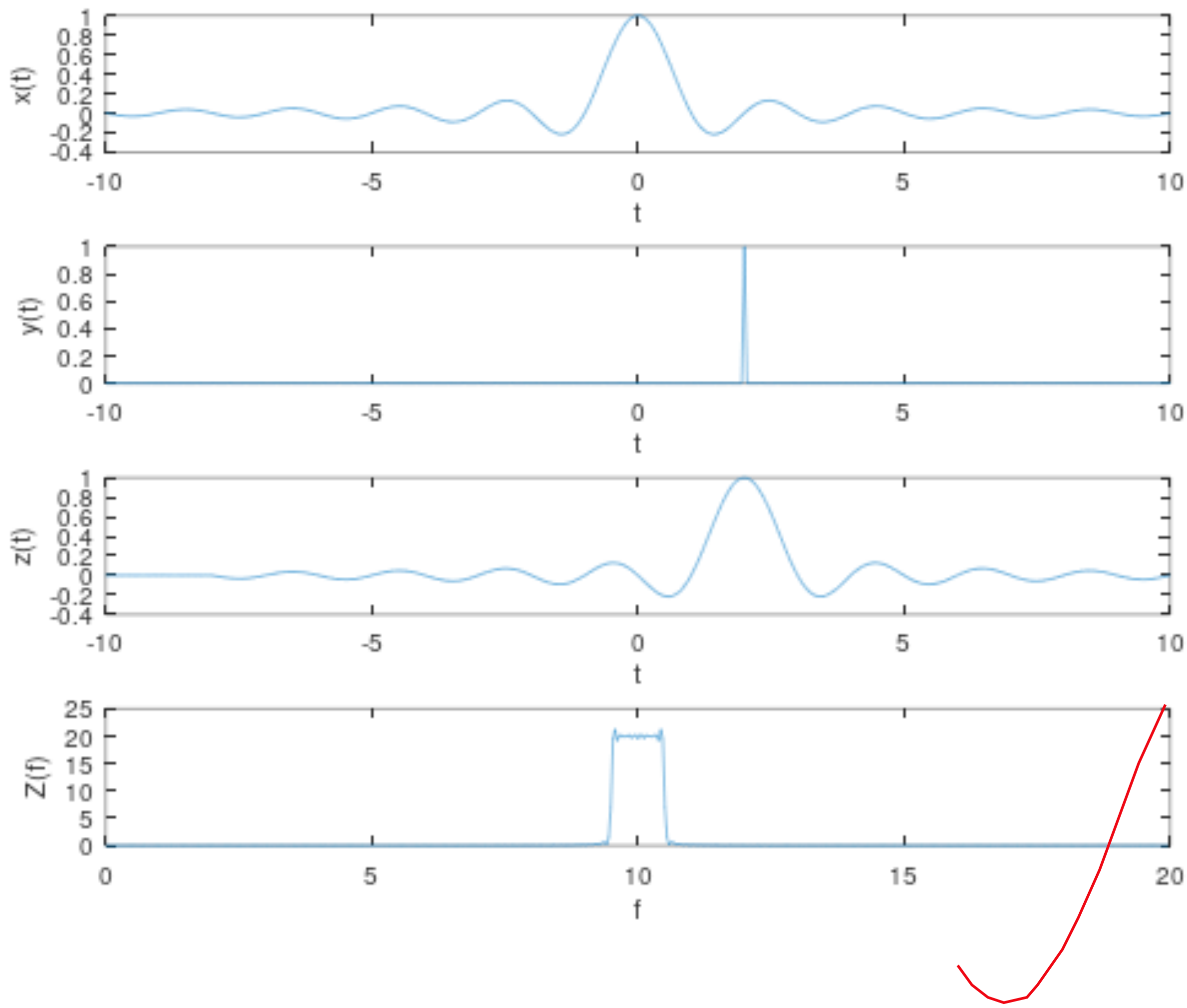
$$\begin{cases} \text{sinc}(t) \xleftrightarrow{F} \Pi(f) \\ \delta(t-2) \xleftrightarrow{F} e^{-j2\omega} \end{cases}$$

$$\Rightarrow \text{sinc}(t) * \delta(t-2) \xleftrightarrow{F} \Pi(f) \cdot e^{-j2\omega}$$

1b - The MATLAB code implementation

MATLAB/Octave code

```
dt = 0.05;
t = -10:dt:10;
x = sinc(t);
y = zeros(1,length(t));
ind = find(t==2);
y(ind) = 1;
z = conv(x, y, 'same');

figure;
subplot(4,1,1);
plot(t, x)
xlabel('t')
ylabel('x(t)')

subplot(4,1,2);
plot(t, y)
xlabel('t')
ylabel('y(t)')

subplot(4,1,3);
plot(t, z)
xlabel('t')
ylabel('z(t)')

N = 512;
Zf = (fft(z,N));
fr=(0:N-1)/N/dt;

subplot(4,1,4);
plot(fr, fftshift(abs(Zf)));
xlabel('f')
ylabel('Z(f)')
```

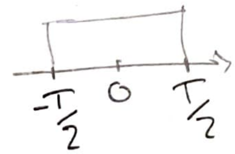


x(t)
y(t)
z(t)
Z(f)
t
f

② Convolve two square pulses (in time domain) of width T, $\Pi\left(\frac{t}{T}\right)$.
Plot the resulting time waveform and its spectrum

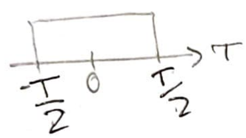
$\Pi(t) \longrightarrow$

$x_1(t) = x_2(t) = \Pi\left(\frac{t}{T}\right) \rightarrow$



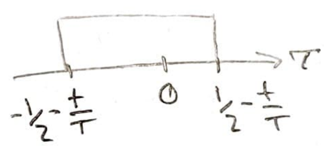
$$x_1(t) * x_2(t) \equiv \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$x_1(\tau) = \Pi\left(\frac{\tau}{T}\right) \rightarrow$



$x_2(t-\tau) = \Pi\left(\frac{t-\tau}{T}\right) = \Pi\left(\frac{t}{T} - \frac{\tau}{T}\right)$

① Tim-shift $\quad \Pi\left(\tau + \frac{t}{T}\right) \longrightarrow$

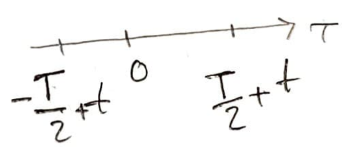


② Time-reversal

$\Pi\left(-\tau + \frac{t}{T}\right) \rightarrow$

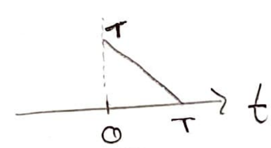
③ Time-scaling

$\Pi\left(-\frac{\tau}{T} + \frac{t}{T}\right) \rightarrow$

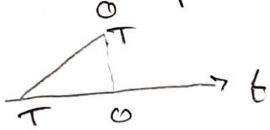


Convolution

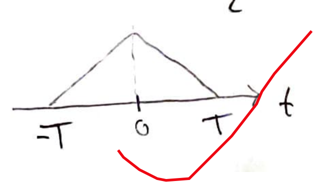
① $0 < t < T \;\rightarrow\; \int_{-\frac{T}{2}+t}^{\frac{T}{2}} 1.1\, d\tau = \frac{T}{2} - \left(-\frac{T}{2} + t\right) = T - t$



② $-T < t < 0 \;\rightarrow\; \int_{-\frac{T}{2}}^{\frac{T}{2}+t} 1.1\, d\tau = \frac{T}{2} + t - \left(-\frac{T}{2}\right) = T + t$

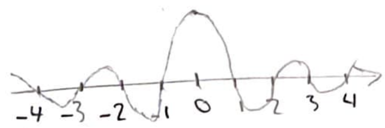


$x_1(t) * x_2(t) \rightarrow$

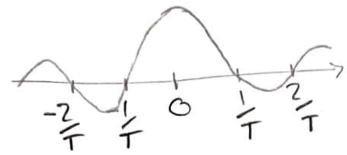


Spectrum:

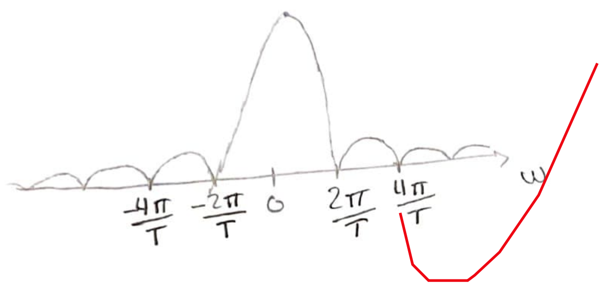
$$\mathrm{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$$



$$\Pi\left(\frac{t}{T}\right) \xrightarrow{f} T\,\mathrm{sinc}(fT)$$



$$\Rightarrow x_1(t) * x_2(t) \xrightarrow{f} X_1(f) \cdot X_2(f)$$



③ Let $x(t) = \text{sinc}(2Wt)$ and $y(t) = t$

If $z(t) = x(t)y(t)$, what is $Z(f)$?

$$\text{sinc}(2wt) = \frac{\sin(\pi 2wt)}{\pi(2wt)}$$

$$\Rightarrow \overbrace{x(t)\,y(t)}^{z(t)} = \frac{\sin(\pi 2wt)}{\pi(2w\cancel{t})} \cdot \cancel{t} = \frac{1}{\pi 2w} \sin(\pi 2wt)$$

$$\sin(2\pi f_c t) \longleftrightarrow -.5j(\delta(f-f_c) - \delta(f+f_c))$$ ← From Page 46 of textbook

$$\Rightarrow Z(f) = \frac{1}{2\pi w}(-.5)j(\delta(f-w) - \delta(f+w))$$ ← Via linearity property

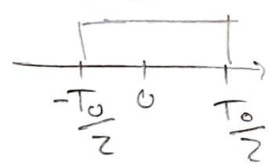
④ Determine whether each of the following signals is an enery signal or a power signal, and evaluate the normalized energy or power.

$$T_0 = \frac{1}{f_c}$$

Energy signal: $0 < E < \infty$ (finite total energy) and
$P = 0$ (zero average power)

Power signal $0 < P < \infty$ (finite average power) and
$E = \infty$ (infinite total energy)

$\boxed{4a}$ $w(t) = \Pi\left(\frac{t}{T_0}\right)$ ←



$$E = \int_{-\infty}^{\infty} |w(t)|^2 \, dt$$

$$= \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |1|^2 \, dt = t \Big|_{-\frac{T_0}{2}}^{\frac{T_0}{2}} = T_0$$

This is an energy signal of T_0 Joules. Averge power is equal to zero.

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |w(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} 1 \, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left[t\right]_{-\frac{T_0}{2}}^{\frac{T_0}{2}} = \lim_{T\to\infty} \frac{1}{2T} \left[\frac{T_0}{2} - \left(-\frac{T_0}{2}\right)\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} (T_0) = 0 \quad \Rightarrow 0 < E < \infty \text{ and } P = 0$$

$\boxed{4b}$ $$w(t) = \Pi\left(\frac{t}{T_0}\right) \cos(2\pi f_0 t) = \begin{cases} \cos(2\pi f_0 t) & -\frac{T_0}{2} < t < \frac{T_0}{2} \\ 0 & \text{O.W.} \end{cases}$$

$$E = \int_{-\infty}^{\infty} |w(t)|^2 dt = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos^2(2\pi f_0 t) \, dt$$

$$= \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{1 + \cos(4\pi f_0 t)}{2} dt = \left[\frac{t}{2} + \frac{1}{4\pi f_0} \sin(4\pi f_0 t)\right]_{-\frac{T_0}{2}}^{\frac{T_0}{2}}$$

$$= \left[\frac{T_0}{4} + \frac{1}{4\pi f_0} \sin(2\pi f_0 T_0)\right] - \left[-\frac{T_0}{4} + \frac{1}{4\pi f_0} \sin(-2\pi f_0 T_0)\right]$$

$$= \frac{T_0}{2} + \underbrace{\frac{1}{2\pi f_0} \sin(2\pi f_0 T_0)}_{=0} \quad \Rightarrow 0 < E < \infty \quad \Rightarrow \text{Energy signal}$$

$$\Rightarrow P = 0$$

$\boxed{4c}$ $w(t) = \cos^2(\underbrace{2\pi f_0}_{\omega_0} t)$

$$E = \lim_{T\to\infty} \int_{-T}^{T} \cos^4(\omega_0 t)\,dt = \lim_{T\to\infty} \int_{-T}^{T} \cos^3(\omega_0 t)\cdot\cos(\omega_0 t)\,dt$$

$$= \lim_{T\to\infty} \int_{-T}^{T} \frac{1}{8}\left(4\cos(2\omega_0 t) + \cos(4\omega_0 t) + 3\right) dt$$

$$= \lim_{T\to\infty} \frac{1}{8}\left[\frac{4\sin(2\omega_0 t)}{2\omega_0} + \frac{\sin(4\omega_0 t)}{4\omega_0} + 3t\right]_{-T}^{T} = \infty$$

$E = \infty$ Joules $\Rightarrow$ Not an energy signal

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \cos^4(\omega_0 t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left(\frac{1}{8}\left(\frac{8\sin(2\omega_0 T)}{2\omega_0} + \frac{2\sin(4\omega_0 T)}{4\omega_0} + 6T\right)\right)$$

$$= \lim \frac{\sin(2\omega_0 T)}{4T\omega_0} + \frac{\sin(4\omega_0 T)}{32T\omega_0} + \frac{6T}{16T} = 0 + 0 + \frac{6}{16}$$

$\Rightarrow P = \frac{3}{8}$ Watts and $E = \infty$ $\Rightarrow$ Power signal

(5) Using the convolution property, find the spectrum for

$$w(t) = \underbrace{\sin(2\pi f_1 t)}_{x_1(t)} \underbrace{\cos(2\pi f_2 t)}_{x_2(t)}$$

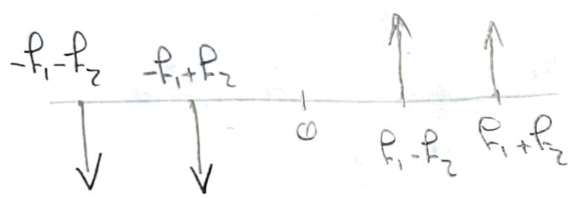
$$x_1(t) \xleftrightarrow{f} X_1(f) = -\frac{j}{2}\left(\delta(f-f_1) - \delta(f+f_1)\right)$$

$$x_2(t) \xleftrightarrow{f} X_2(f) = \frac{j}{2}\left(\delta(f-f_2) + \delta(f+f_2)\right)$$

← From "Fourier Transform for Periodic Signals" section of the textbook

From the Convolution property:

$$w(t) \xleftrightarrow{f} W(f) = -\frac{j}{4}\left[\delta(f-f_1-f_2) + \delta(f-f_1+f_2) - \delta(f+f_1-f_2) - \delta(f+f_1+f_2)\right]$$



⑥ The signal $x(t) = \cos(10t)$ is input to a linear system that has an impulse response $h(t) = \begin{cases} e^{-\frac{t}{T_0}} & t>0 \\ 0 & t<0 \end{cases}$

6a What is the output time signal?

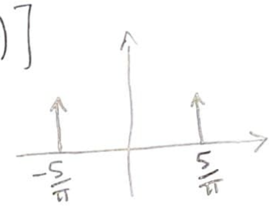
$$\mathfrak{F}\left[\cos(2\pi At + B)\right] = \int_{-\infty}^{\infty} \frac{e^{j(2\pi At+B)} + e^{-j(2\pi At+B)}}{2} e^{-j2\pi ft}\, dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{j2\pi(A-f)t+B}\, dt + \int_{-\infty}^{\infty} e^{-j2\pi(A+f)t+B}\, dt\right]$$

$$= \frac{1}{2} e^{B}\left[\delta(f-A) + \delta(f+A)\right]$$

$$\Rightarrow \mathfrak{F}[x(t)] = \mathfrak{F}[\cos(10t)] = \mathfrak{F}\left[\cos\left(2\pi \frac{5}{T_0} t\right)\right]$$

$$= \frac{1}{2}\left[\delta\left(f - \frac{5}{T_0}\right) + \delta\left(f + \frac{5}{T_0}\right)\right]$$



$$\Rightarrow \mathfrak{F}\left[e^{-\frac{t}{T_0}}\right] = \int_{0}^{\infty} e^{-\frac{t}{T_0}} e^{-j2\pi ft}\, dt = \frac{1}{\frac{1}{T_0} + (2\pi f)^2}$$

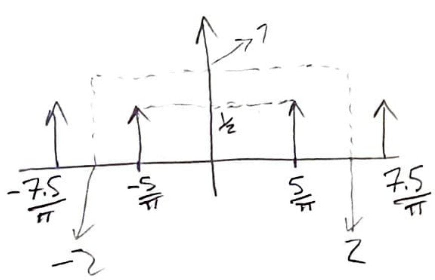
$$\Rightarrow X(f)\cdot H(f) = \frac{1}{2}\, \frac{1}{\frac{1}{T_0}(2\pi f)^2} \times \left[\delta\left(f - \frac{5}{\pi}\right) + \delta\left(f + \frac{5}{\pi}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{\frac{1}{T_0} + 100}\right] \times \left[\delta\left(f - \frac{5}{\pi}\right) + \delta\left(f + \frac{5}{\pi}\right)\right]$$

$$\frac{1}{2}\left[\mathfrak{F}^{-1}\left(\frac{\delta\left(f - \frac{5}{\pi}\right)}{\frac{1}{T_0}+100}\right) + \mathfrak{F}^{-1}\left(\frac{\delta\left(f + \frac{5}{\pi}\right)}{\frac{1}{T_0}+100}\right)\right] = \frac{\cos(10t)}{\frac{1}{T_0} + 100}$$ ✗ -1

6b Interference signal $z(t) = \cos\left(15t + \frac{\pi}{8}\right)$

$$\mathfrak{F}\left[\cos\left(15t + \frac{\pi}{8}\right)\right] = \frac{e^{\frac{\pi}{8}}}{2}\left[\delta\left(f - \frac{15}{2\pi}\right) + \delta\left(f + \frac{15}{2\pi}\right)\right]$$



$$\Rightarrow H(f) = 4\pi\left(\frac{1}{4} f\right) = 4\pi\left(\frac{\omega}{8\pi}\right)$$

$$\Rightarrow h(t) = 4\,\mathrm{sinc}(4\pi t)$$ ✓