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Digital Communications

Homework #5

1. In class we said that for differential encoding and decoding of M-ary PSK, we can obtain the proper decision statistics using

$$R_1 = R_1^i R_1^{i-1} + R_2^i R_2^{i-1}$$
$$R_2 = R_2^i R_1^{i-1} - R_1^i R_2^{i-1}$$

Show that this eliminates the unknown phase. (Hint: use the definitions for R_1^i and R_1^{i-1})

We had

$$R_1^i = \frac{A_c T_b}{2} \cos\left[\frac{2\pi}{M} m_i(t) + \theta_o\right]$$
$$R_2^i = \frac{A_c T_b}{2} \sin\left[\frac{2\pi}{M} m_i(t) + \theta_o\right]$$

We had the identity that

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{2}$$

and also

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$$

So for R_1

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \frac{\cancel{\cos(\alpha+\beta)} + \cos(\alpha-\beta)}{2} + \frac{\cos(\alpha-\beta) - \cancel{\cos(\alpha+\beta)}}{2}$$
$$= \frac{\cancel{2}\cos(\alpha-\beta)}{\cancel{2}} = \cos(\alpha-\beta)$$

Then we have

$$\boxed{R_1 = \cos\left(\frac{2\pi}{M}\left[m_i(t) - m_{i-1}(t)\right] + \cancel{\theta_0 - \theta_0}\right) = \cos\left(\frac{2\pi}{M}\left[m_i(t) - m_{i-1}(t)\right]\right)}$$

Similarly, we have the identity that

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}$$

And another identity that

$$\sin(\alpha) - \sin(-\alpha) = 2\sin(\alpha)$$

So for R_2

$$\sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{\cancel{\sin(\alpha+\beta)} + \sin(\alpha-\beta)}{2} - \frac{\cancel{\sin(\alpha+\beta)} + \sin(-\alpha+\beta)}{2}$$

$$= \frac{\cancel{2}\sin(\alpha-\beta)}{\cancel{2}} = \sin(\alpha-\beta)$$

Then we have

$$\boxed{R_2 = \sin\left(\frac{2\pi}{M}\left[m_i(t) - m_{i-1}(t)\right] + \cancel{\theta_0 - \theta_0}\right) = \sin\left(\frac{2\pi}{M}\left[m_i(t) - m_{i-1}(t)\right]\right)}$$

2. The noise equivalent bandwidth of a filter is defined as the width of a fictitious rectangular filter transfer function such that the power in the rectangular band is equal to the power associated with the actual filter over positive frequencies. In other words if we define

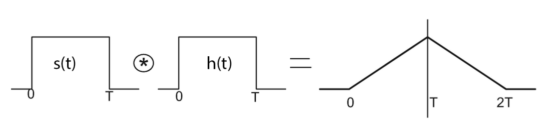
$$P_{pos} = \int_0^\infty |H(f)|^2\,df$$

Then we wish to find B_{eq} such that

$$B_{eq} = \frac{1}{\left|H(f_0)\right|^2}\int_0^\infty |H(f)|^2\,df$$

If square pulses with unit energy ($E = \int_{-\infty}^{\infty} |p(t)|^2\,dt = 1$) are used to transmit PCM data, find the noise equivalent bandwidth of the matched filter. Assume that the symbol duration is T_s.

The noise-equivalent bandwidth B_{eq} is the bandwidth of an ideal lowpass filter that has the same maximum power gain as $H(f)$ and passes the same amount of white noise power to its output as does the actual system frequency response $H(f)$.



The matched filter is defined as $h(t) = s(T-t)$

That is for $s(t) = \Pi(\frac{t}{T})$ we have $h(t) = \Pi(\frac{T-t}{T}) = \Pi(\frac{t}{T})$

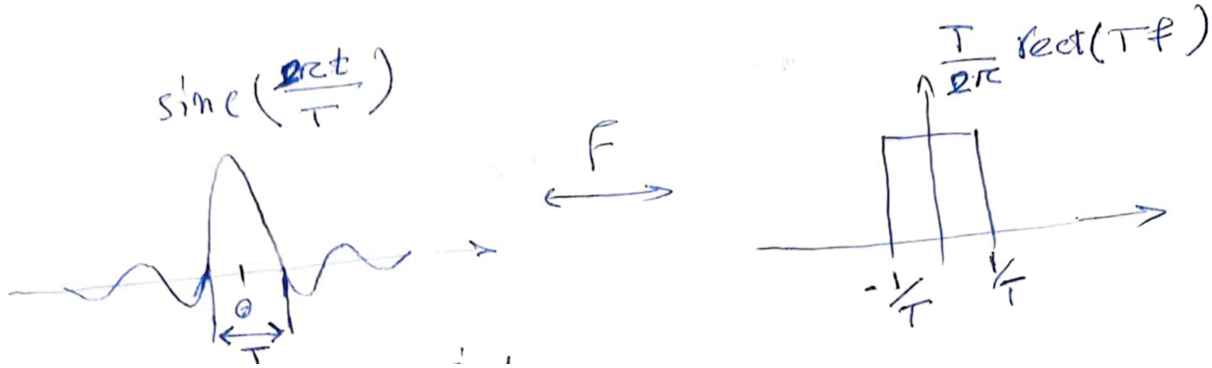
$\mathcal{F}[\Pi(\frac{t}{T})] = T\,\mathrm{sinc}(fT)$

① $\int_0^\infty |H(f)|^2 df = \int_0^\infty |T\text{sinc}(fT)|^2 df = \frac{\pi}{2}$

② $\left|H(f_0)\right|^2 = \left|T\text{sinc}(f_0 T)\right|^2$ is maximized at $f_0 = 0$ where it is evaluated to 1

① & ② ⇒. $$B_{eq} = \frac{1}{\left|H(f_0)\right|^2}\int_0^\infty |H(f)|^2 df = \frac{\pi}{2}$$

3. Repeat problem 2 for sinc pulses. Note that T_s is the time between pulses not the pulse duration. What does this say about a matched filter?

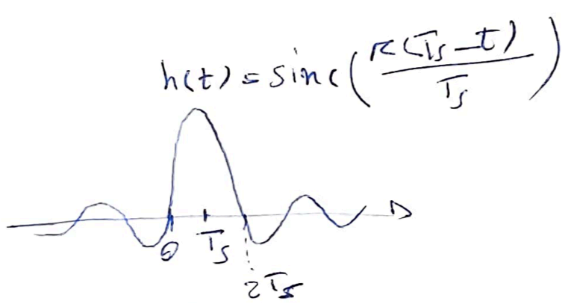


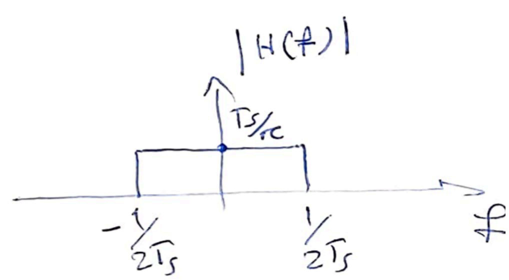
$\mathscr{F}[f(t-t_0)] = F(\omega)e^{j\omega t_0}$

$\mathscr{F}[f(-t)] = F(-\omega)$

$h(t) = s(T-t)$

$H(f) = e^{j2\pi fT}s(-f)$





$H(f) = e^{j2\pi fT}[\frac{T_s}{\pi}\Pi(-2T_s f)]$

$$
\begin{aligned}
B_{eq} &= \frac{1}{\left|H\left(f_0\right)\right|^2} \int_0^{\infty} |H(f)|^2 df \\
&= \frac{(\frac{T_s}{\pi})^2 \frac{2}{2T_s}}{(\frac{T_s}{\pi})^2} = \frac{1}{T_s}
\end{aligned}
$$

4. Use the results in problems 2 and 3 to determine the signal-to-noise ratio at the output of the matched filter.

$$
\begin{aligned}
\text{SNR} &= \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_n(f)} df = \frac{2}{N_o} \int_{-\infty}^{\infty} |X(f)|^2 df = \frac{2}{N_o} \int_{-\infty}^{\infty} |x(t)|^2 dt \\
&= \frac{2E_s}{N_o}
\end{aligned}
$$

And this is true regardless of the shape of the filter as long as it **matches** the shape of our pulse (assuming AWGN channel).

5. A constant voltage signal of V volts is corrupted by noise signal. In other words, a sample of the received signal voltage is modeled as $R = V + N$ where N is a random variable with a Gaussian pdf. Given that N has a mean of 0 volts and a standard deviation of $V/10$ volts. What is the probability that the random variable R is less than zero? (Note that the Q-function is available in Matlab.)

From what is described in the question we can see that R is a Gaussian random variable with mean V and variance $V^2/100$

$R \sim \mathcal{N}(\mu = V, \sigma = V/10)$

The probability density function of a Gaussian Distribution is

$$
p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
$$

That is for R we have

$$
p_R(r) = \frac{1}{V/10\sqrt{2\pi}} e^{-\frac{(r-V)^2}{2V^2/100}}
$$

Then we want

$$P(R < 0) = \int_{-\infty}^{0} \frac{1}{V/10\sqrt{2\pi}} e^{-\frac{(r-V)^2}{2V^2/100}}\, dr$$

Writing this in terms of the Q-function

$$P(R < r) = 1 - P(R > r) = 1 - Q\left(\frac{r-\mu}{\sigma}\right)$$

$$P(R < 0) = 1 - P(R > 0) = 1 - Q\left(\frac{0-V}{\frac{V}{10}}\right) = 1 - Q(-10)$$

Which evaluates to almost zero as the Matlab one-liner `1-qfunc(-10)` suggests.