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Digital Communications

Homework #2

1. Consider a Gaussian Random Variable with mean 0 and variance 1.
   a) Using the Matlab function randn, generate 10 instances of the random variable drawn from that distribution. What are the max and min values? What are the mean and standard deviation of those 10 values? Do these values match the expected values? Why or why not?
   b) Repeat for 100 and 10000 instances.
   c) Use the Matlab function hist create and plot a histogram. Plot the theoretical distribution for a Gaussian random variable superimposed on the histogram.

MATLAB/Octave code

```
clc;
clear;
total_draws = {10, 100, 10000};
x_gauss = -3:.1:3;
y_gauss = normpdf(x_gauss, 0, 1);
figure;
for i = 1:length(total_draws)
    x = randn([total_draws{i}, 1]);
    max_x = max(x);
    min_x = min(x);
    mean_x = mean(x);
    std_x = std(x);
    fprintf('Max of %d draws: %f\n', total_draws{i}, max_x);
    fprintf('Min of %d draws: %f\n', total_draws{i}, min_x);
    fprintf('Mean of %d draws: %f\n', total_draws{i}, mean_x);
    fprintf('Std of %d draws: %f\n', total_draws{i}, std_x);
    fprintf('-------------\n');
```

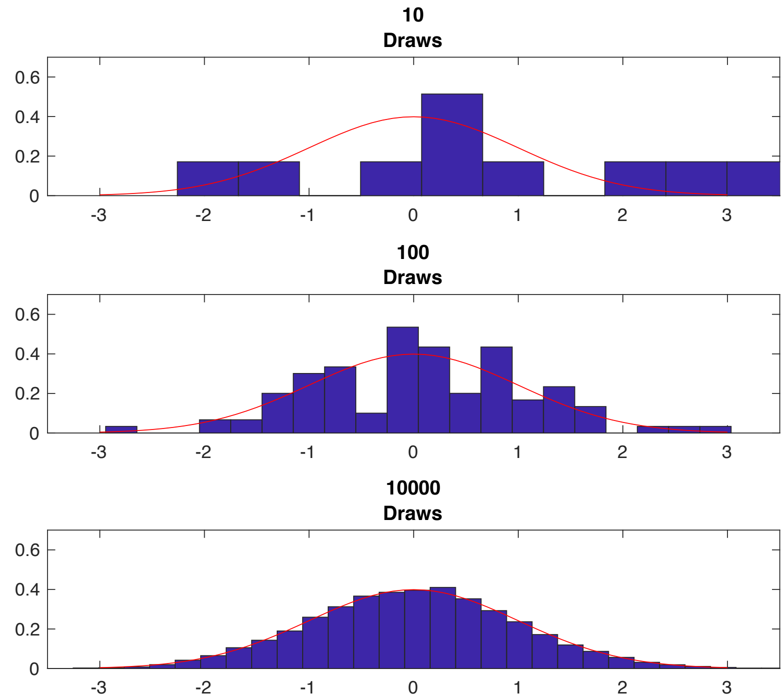
```
    % Since I wanted to use hist() instead of histogram()
    [heights, locations] = hist(x, 10*i);
    width = locations(2)-locations(1);
    heights = heights / (total_draws{i}*width);
    subplot(length(total_draws), 1, i);
    bar(locations, heights, 'hist')
    hold on

    plot(x_gauss, y_gauss, 'r');
    title([total_draws{i} "Draws"]);
    xlim([-3.5 3.5]);
    ylim([0 .7]);
end
```

Output

```
Max of 10 draws: 3.578397
Min of 10 draws: -2.258847
Mean of 10 draws: 0.624282
Std of 10 draws: 1.769885
-------------
Max of 100 draws: 3.034923
Min of 100 draws: -2.944284
Mean of 100 draws: 0.080239
Std of 100 draws: 1.047152
-------------
Max of 10000 draws: 3.569868
Min of 10000 draws: -3.742182
Mean of 10000 draws: -0.001427
Std of 10000 draws: 0.989485
```

Plot



Explanation

The sample mean $\overline{x}$ and sample variance s^2 can be regarded as estimators of the true population parameters, population mean μ and population variance σ^2. This is an implication of the **law of large numbers** indicating that the variability of the estimator decreases with more samples.

2. Random Variables Part II

a. Using Matlab create 10,000 instances of the random variable $Y = \sqrt{X_1^2 + X_2^2}$ where X_1 and X_2 are Gaussian random variables with mean 0 and variance $= 1/2$. Plot the histogram.

b. What type of random variable is Y? Superimpose a plot of the probability density function.

Explanation

The polar random variable $Y = \sqrt{X_1^2 + X_2^2}$ follows a Rayleigh distribution $f(y;\sigma) = \frac{y}{\sigma^2} e^{-y^2/\left(2\sigma^2\right)}$

MATLAB/Octave code

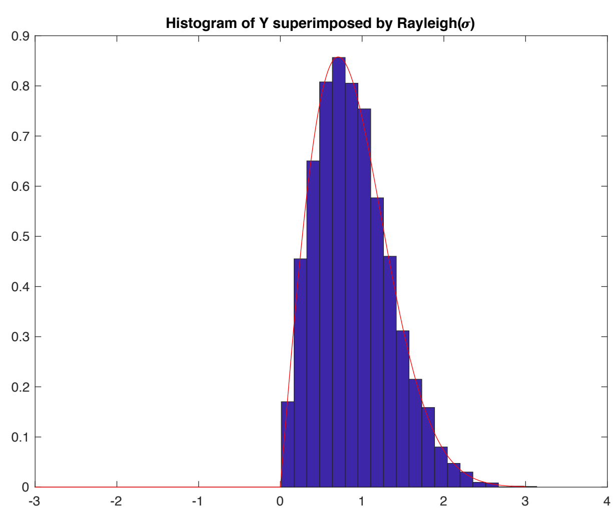
```
clc;
clear;
N = 10000;
x1 = 1.0/sqrt(2)*randn([N, 1]);
x2 = 1.0/sqrt(2)*randn([N, 1]);

y = sqrt(x1.^2 + x2.^2);

[heights, locations] = hist(y, 20);
width = locations(2)-locations(1);
heights = heights / (10000*width);
bar(locations, heights, 'hist')
title('Histogram of Y superimposed by Rayleigh(\sigma)');
hold on

x_rayl = -3:.01:3;
y_rayl = raylpdf(x_rayl, 1.0/sqrt(2));
plot(x_rayl, y_rayl, 'r');
```

Plot



3. Random Processes

a. Using Matlab create 10,000 sample functions of the random process $X(t) = 2\cos(2\pi t + \theta)$ over the time period $0 < t < 10$ where θ is a uniform random variable distributed on $(0,2\pi)$.

MATLAB/Octave code

```
t = linspace(0+eps, 10-eps, 10000);
theta = 2*pi*rand(1, 10000);
x = 2 * cos(2*pi*t + theta);
```

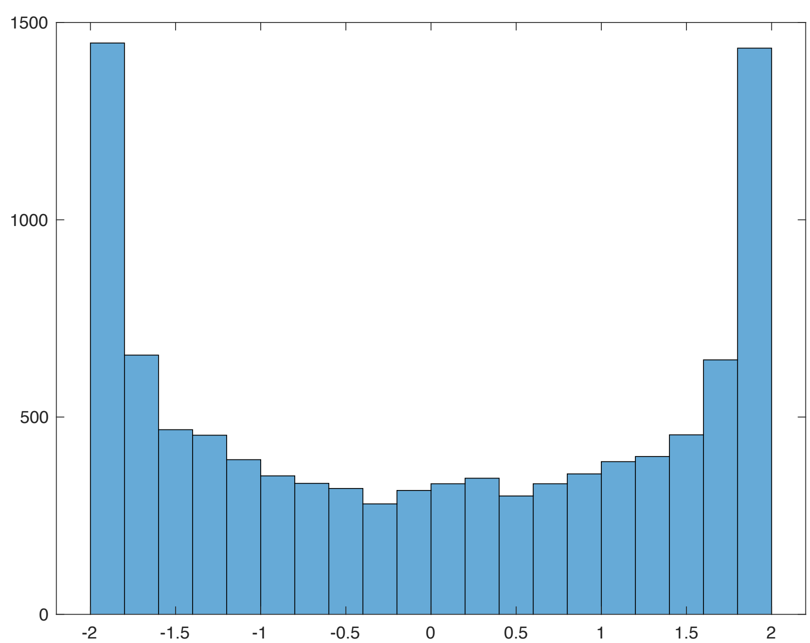
b. Determine and plot the histogram of $X(t = 1)$.

MATLAB/Octave code

```
drawX=@(t) 2 * cos(2*pi*t + 2*pi*rand);

draws = 1:10000;
for i=1:10000
    draws(i) = drawX(1);
end
histogram(draws)
```

Plot



4. A random variable x has a PDF

$$f(x) = \begin{cases} \frac{3}{32}\left(-x^2 + 8x - 12\right) & 2 < x < 6 \\ 0 & \text{else} \end{cases}$$

a. Demonstrate that it is a valid PDF.

First we need to show that $f(x)$ is larger than or equal to zero from $-\infty$ to $+\infty$.
The function $f(x) = \frac{3}{32}\left(-x^2 + 8x - 12\right)$ is a concave function with its maximum at x=4. It also has two roots, namely 2 and 6, which are the boundaries of the piecewise function. So we can confirm that it is not negative anywhere.

Then using integration we have to make sure that the area under the curve is equal to one.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) &= \int_{2}^{6} \frac{3}{32}\left(-x^2 + 8x - 12\right) dx \\ &= \frac{3}{32}\left(-\int_{2}^{6} x^2 dx + \int_{2}^{6} 8x\, dx - \int_{2}^{6} 12 dx\right) \\ &= \frac{3}{32}\left(-\frac{208}{3} + 128 - 48\right) = 1 \end{aligned}$$

b. Find the mean.

$$E(X) = \int_{-\infty}^{\infty} x\ f(x) = \int_{2}^{6} x \frac{3}{32}\left(-x^2 + 8x - 12\right) dx\ = 4$$

c. Find the second moment and the variance.

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2\ f(x) = \int_{2}^{6} x^2 \frac{3}{32}\left(-x^2 + 8x - 12\right) dx \\ &= \frac{3}{32}\left(-\frac{7744}{5} + 2560 - 832\right) = \frac{84}{5} \end{aligned}$$

$$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2\ f(x) = \int_{2}^{6} (x-4)^2 \frac{3}{32}\left(-x^2 + 8x - 12\right) dx\ = \frac{4}{5}$$

We can confirm this answer:

$$Var(X) = E(X^2) - E^2(X) = \left(\frac{84}{5}\right) - 16 = \frac{4}{5} = 0.8$$

Bonus Questions:

1. Suppose that the occurrence of an event B increases the probability that an accused person is guilty; that is, if A is the event that the defendant is guilty then $P(A \mid B) \geq P(A)$. The prosecutor finds that B did not occur. What can you say about the defendant's conditional probability of being guilty?

Hint: You need to compare $P\left(A \mid B^C\right)$ and $P(A)$.

From the Law of Total Probability we have:

$P(A) = P(A \cap B) + P\left(A \cap B^c\right)$

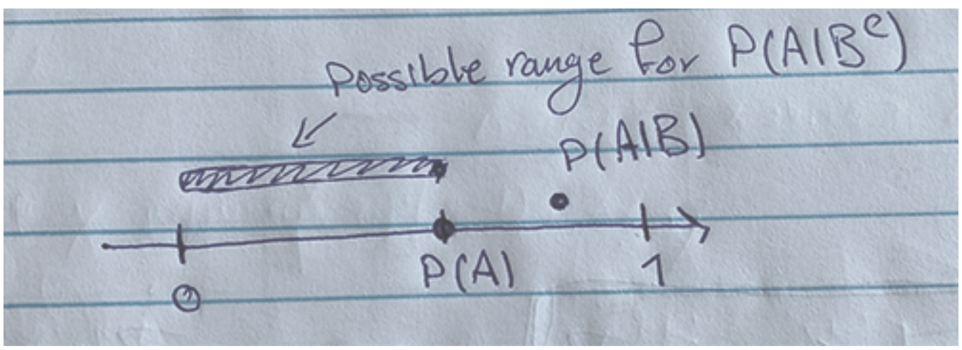
That is

$P(A) = P(A|B)P(B) + P\left(A \cap B^c\right)P(B^c)$

Which is a convex combination of two values:

$P(A) = \alpha P(A|B) + (1-\alpha)P\left(A|B^c\right)$

Showing on the real line, we have



That is, given that $P(B)$ and $P(B^c)$ are undetermined, we have $P(A|B^c) \in [0,\, P(A)]$

If $P(B) = 0$ then it must be $P(A|B^c) = P(A)$ for the equation to hold.

2. 10 people attend a conference wearing 10 different hats. Before they enter the conference room, each person places his/her hat on the table. After the conference, each person randomly picks a hat on the table. Let X be the number of the persons who happen to pick their own hats. What is the expected value of X, i.e., $E[X]$?

Let X_i be an indicator random variable showing if attendee i has received his/her own hat. Then we will have

$$X = \sum_{i=1}^{n} X_i$$

For n=10 people, there are 10 hats and each attendee picks a hat uniformly at random with probability of getting the right hat equal to $\frac{1}{n}$. That is $E[X_i] = \frac{1}{n}$

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^{n} X_i\right] \\ &= \sum_{i=1}^{n} E\left[X_i\right] \\ &= \sum_{i=1}^{n} \frac{1}{n} \\ &= 1 \end{aligned}$$