Problem Set #1

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Please:

- Type your answers and submit them in a pdf.
- Attach your codes.

1 The Span of Control Model of Lucas (80 pts)

In this exercise we will solve the well-known Lucas (1978), which is often used for macroeconomic models of entrepreneurship and income distribution. Then, we will use the model to think about the distributional consequences of demographics changes (e.g. immigration) that changes the skill distribution of the economy.

Consider a simple, static economy populated by a continuum of agents. We normalize the population to 1. Each individual realizes an (entrepreneurial) ability level $z \geq 0$, which is drawn from a common distribution $g(z)$. Thus, the cross-section distribution of abilities in characterized by such a distribution, and $\int g(z)\,dz = 1$. For this exercise, we will assume that the density $g(z)$, is exponential, i.e.: $g(z) = \lambda e^{-\lambda z}$, where $\lambda > 0$. Notice that then the c.d.f. is given by $G(z) = \Pr(Z \leq z) = 1 - e^{-\lambda z}$. Finally, each individual is endowed with one unit of time that can be used either as a worker or to provide their ability z as an entrepreneur, i.e. as a team leader.

Output in the economy is produced by teams. Each team is composed of one leader and a group of workers n. The output of the team given by $y = z * 2\sqrt[2]{n}$, that is, it is determined by the ability z of the team leader and the workers under his/her command. Notice that output is subject to decreasing returns to scale.

Preferences are quite simple: Individuals value consumption linearly, $u(c) = c$. Moreover, agents will consume their income. Markets are quite simple. There is a competitive labor market, where workers earn wages $w > 0$ for their unit of labor. Entrepreneurs hire labor n, pay w and keep the difference between output and the cost of labor:

$$\pi^*(z) = \max_n \left\{ z * 2\sqrt[2]{n} - wn \right\}.$$

1. For any $w > 0$:

   (a) Individual Rationality: Solve for the profits $\pi^*(z)$, optimal hiring $n^*(z)$ for each z. Solve for the threshold $\bar{z}$ for which an individual would be indifferent between being a worker or being an entrepreneur.

(b) Aggregates: Write down the formulae and provide a graph displaying the aggregate supply of labor and the aggregate demand for labor. Hint: The latter is an integral that can be solved by repeated (twice) integrating by parts. Alternatively, it can be expressed using the Incomplete Gamma function, for which Matlab has an automated function.

2. Equilibrium: Write a Matlab (Julia, Python, Excel) code to solve for the equilibrium $\bar{w}$. Please solve and interpret the equilibrium values the threshold $\bar{z}$ and the total amount of workers for economies with $\lambda = 1$, $\lambda = 2$ and $\lambda = 3$.

3. Explain why larger values of λ would mean that the population is less skilled. Also, relative to a baseline of $\lambda = 1$, please describe the set of individuals' z for which they would be better in an economy with $\lambda = 2$ and those who would be worse off. Include in your description how these comparisons vary across occupations.

4. How would your answers to **1.a.** and **1.b.** above change if the production function of each individual firm were given by a more general $y = z * (1/\nu) n^\nu$, where $\nu \in (0, 1)$. Solve for 2 above with $\nu = 0.85$.

2 HARA Preferences and Aggregation (20 pts)

In this exercise we consider a generalization of the preferences in class and consider the so-called hyperbolic absolute risk aversion (HARA) preferences. Consider a two-period ($t = 0, 1$) economy, populated by $i = 0, 1, 2, ...I$ types of individuals.

$$U = \frac{1-\theta}{\theta}\left(\frac{c_0}{1-\theta} + b\right)^\theta + \beta\frac{1-\theta}{\theta}\left(\frac{c_1}{1-\theta} + b\right)^\theta$$

where, given the parameter $\theta \neq 1$, the consumption levels c_0 and c_1 must satisfy $\frac{c}{1-\theta} + b \geq 0$.[1]

There are fractions $\pi_i > 0$, with $\sum_{i=1}^{I} \pi_i = 1$, of each of these types. Each individual in type i has income y_0^i in period $t = 0$ and income y_1^i in period $t = 1$. There is no uncertainty (or risk) so the only asset needed is a one-period bond with return R. All agents, regardless of their type i, have frictionless access to trading in this bond (borrowing or lending.) Thus, the problem of the agent is given by maximizing their utility U subject to the present-value condition

$$c_0 + \frac{c_1}{R} = y_0 + \frac{y_1}{R}.$$

Notice that the Inada conditions do not necessarily hold. Hence solving the household problem must also check non-negativity constraints, $c_0^i, c_1^i \geq 0$, for all types i.

1. Show that if the non-negative constraints $c_0^i, c_1^i \geq 0$, are slack for for all types i, then the preferences of the agents in the economy can be represented by a representative household with preferences

$$U = \frac{1-\theta}{\theta}\left(\frac{C_0}{1-\theta} + b\right)^\theta + \beta\frac{1-\theta}{\theta}\left(\frac{C_1}{1-\theta} + b\right)^\theta,$$

where $C_t = \sum_{i=1}^{I} \pi_i c_t^i$ denotes the aggregate consumptions in $t = 0, 1$.

[1] For $\theta = 1$, we could interpret the preferences to be $u(c) = \ln(c + b)$.