

# LECTURE NOTES ON CATEGORY THEORY (IN PROGRESS)

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Category theory is a branch of mathematics that commenced with the work of Eilenberg and Mac Lane in the 1940s [EM45], motivated by research questions in algebra and topology.

## 1. DEFINITION OF CATEGORY

We directly commence with the definition of a category.

**Definition 1.1.** A category  $\mathcal{C}$  consists of the following data:

- A class of objects  $\text{Obj}(\mathcal{C})$ .
- For every pair of objects  $X, Y \in \text{Obj}(\mathcal{C})$ , a set of morphisms  $\text{Hom}_{\mathcal{C}}(X, Y)$ .
- For every triple of objects  $X, Y, Z \in \text{Obj}(\mathcal{C})$ , a composition map
$$-\circ- : \text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$$
- An identity morphism  $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$  for every object  $X \in \text{Obj}(\mathcal{C})$ .
- Composition is unital: for every  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ , we have  $\text{id}_Y \circ f = f$  and  $f \circ \text{id}_X = f$ .
- Composition is associative: for every  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ ,  $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$  and  $h \in \text{Hom}_{\mathcal{C}}(Z, W)$ , we have  $h \circ (g \circ f) = (h \circ g) \circ f$ .

If  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ , we often write  $f: X \rightarrow Y$  and call  $X$  the *domain* and  $Y$  the *codomain* of  $f$ .

**Exercise 1.2.** If  $e_X, e'_X$  are two identity morphisms for an object  $X$  in a category  $\mathcal{C}$ , then  $e_X = e'_X$ .

## 2. EXAMPLES OF CATEGORIES

This is a very heavy definition, so let's look at some examples!

**Example 2.1.** Let  $[0]$  be the category with the following specification:

- $\text{Obj}([0]) = \{0\}$ .
- $\text{Hom}_{[0]}(0, 0) = \{\text{id}_0\}$ .
- $\text{id}_0 \circ \text{id}_0 = \text{id}_0$ .

Graphically, we can represent this category as a single point!

Identities are not expressed in the graphical representations.

**Example 2.2.** Let  $[1]$  be the category with the following specification:

- $\text{Obj}([1]) = \{0, 1\}$ .
- $\text{Hom}_{[1]}(0, 0) = \{\text{id}_0\}$ ,  $\text{Hom}_{[1]}(1, 1) = \{\text{id}_1\}$ ,  $\text{Hom}_{[1]}(0, 1) = \{01\}$ , and  $\text{Hom}_{[1]}(1, 0) = \emptyset$ .
- $\text{id}_0 \circ \text{id}_0 = \text{id}_0$ ,  $\text{id}_1 \circ \text{id}_1 = \text{id}_1$ ,  $01 \circ \text{id}_0 = 01$ , and  $01 \circ \text{id}_1 = 01$ .

Graphically, we can represent this category as follows:

$$0 \xrightarrow{01} 1$$

We say it is a category with two objects and one non-identity morphism.

This is fun, so let's keep going!

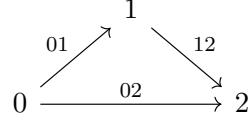
**Example 2.3.** Let  $[2]$  be the category with the following specification:

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- $\text{Obj}([2]) = \{0, 1, 2\}$ .
- $\text{Hom}_{[2]}(0, 0) = \{\text{id}_0\}$ ,  $\text{Hom}_{[2]}(1, 1) = \{\text{id}_1\}$ ,  $\text{Hom}_{[2]}(2, 2) = \{\text{id}_2\}$ ,  $\text{Hom}_{[2]}(0, 1) = \{01\}$ ,  $\text{Hom}_{[2]}(1, 2) = \{12\}$ ,  $\text{Hom}_{[2]}(0, 2) = \{02\}$ , and all other hom-sets are empty.
- The composition is given by the identities and  $12 \circ 01 = 02$ .

Graphically, we can represent this category as follows:



We say it is a category with three objects and three non-identity morphisms.

**Example 2.4.** Let  $[1] \times [1]$  be the category with the following specification:

- $\text{Obj}([1] \times [1]) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ .
- The hom-sets are given by
  - All hom-sets with domain and codomain equal only contain the identity.
  - $\text{Hom}((0, 0), (0, 1)) = \{(0, 01)\}$ ,  $\text{Hom}((0, 0), (1, 0)) = \{(01, 0)\}$ ,  $\text{Hom}((0, 1), (1, 1)) = \{(01, 1)\}$ , and  $\text{Hom}((1, 0), (1, 1)) = \{(1, 01)\}$ ,
  - $\text{Hom}((0, 0), (1, 1)) = \{(01, 01)\}$ ,
  - All other hom-sets are empty.
- The composition with identity are as before, and additionally we have
  - $(01, 1) \circ (0, 01) = (01, 01)$
  - $(1, 01) \circ (01, 0) = (01, 01)$

Graphically, we can represent this category as follows:

$$\begin{array}{ccc} (0, 0) & \xrightarrow{(0, 01)} & (0, 1) \\ \downarrow (01, 0) & & \downarrow (01, 1) \\ (1, 0) & \xrightarrow{(1, 01)} & (1, 1) \end{array}$$

We say it is a category with four objects and five non-identity morphisms.

**Definition 2.5.** A preorder  $(P, \leq)$  is a set  $P$  along with a binary relation  $\leq$  that is reflexive and transitive.

**Definition 2.6.** Let  $P$  be a preorder. We define the category  $\mathcal{C}_P$  with the following specification:

- $\text{Obj}(\mathcal{C}_P) = P$ .
- For  $x, y \in P$ ,  $\text{Hom}_{\mathcal{C}_P}(x, y) = \{*\}$  if  $x \leq y$  and  $\text{Hom}_{\mathcal{C}_P}(x, y) = \emptyset$  otherwise.
- The composition is given by the unique element in the hom-set if it exists, which exists by transitivity.
- The identity is given by the unique element in the hom-set, which exists by reflexivity.
- $\{0\}$  gives us the category  $[0]$ .
- $\{0 \leq 1\}$  gives us the category  $[1]$ .
- $\{0 \leq 1 \leq 2\}$  gives us the category  $[2]$ .
- $\{0 \leq 1\} \times \{0 \leq 1\}$  gives us the category  $[1] \times [1]$ .

**Example 2.7.** Let  $(M, +, 0)$  be a monoid. We define the category  $BM$  with the following specification:

- $\text{Obj}(BM) = \{*\}$ .
- $\text{Hom}_{BM}(*, *) = M$ .
- The composition is given by the monoid operation, i.e. for  $m, n \in M$ , we have  $m \circ n = m + n$ .
- The identity is given by the monoid identity, i.e.  $\text{id}_* = 0$ .

Let us some examples of this example.

**Example 2.8.** Let  $(\mathbb{N}, +, 0)$  be the monoid of natural numbers under addition. The category  $B\mathbb{N}$  has one object and morphisms given by natural numbers, with composition given by addition. It is also called the free endomorphism.

**Example 2.9.** Let  $(\mathbb{Z}, +, 0)$  be the monoid of integers under addition. The category  $B\mathbb{Z}$  has one object and morphisms given by integers, with composition given by addition. It is also called the free automorphism.

**Example 2.10.** Let  $\text{Set}$  be the category with the following specification:

- $\text{Obj}(\text{Set}) = \{X \mid X \text{ is a set}\}$ .
- For  $X, Y \in \text{Obj}(\text{Set})$ ,  $\text{Hom}_{\text{Set}}(X, Y)$  is the set of all functions from  $X$  to  $Y$ .
- The composition is given by the usual composition of functions.
- The identity is given by the usual identity function.

**Example 2.11.** Let  $\text{Grp}$  be the category with the following specification:

- $\text{Obj}(\text{Grp})$  is the class of all groups.
- For  $G, H \in \text{Obj}(\text{Grp})$ ,  $\text{Hom}_{\text{Grp}}(G, H)$  is the set of all group homomorphisms from  $G$  to  $H$ .
- The composition is given by the usual composition of functions.
- The identity is given by the usual identity function.

**Example 2.12.** Let  $\text{Top}$  be the category with the following specification:

- $\text{Obj}(\text{Top})$  is the class of all topological spaces.
- For  $X, Y \in \text{Obj}(\text{Top})$ ,  $\text{Hom}_{\text{Top}}(X, Y)$  is the set of all continuous functions from  $X$  to  $Y$ .
- The composition is given by the usual composition of functions.
- The identity is given by the usual identity function.

These last three examples illustrate why category became so important in mathematics: Every type of mathematical object (sets, groups, topological spaces, ... ) form a category.

- Set theory is the study of the category  $\text{Set}$ .
- Group theory is the study of the category  $\text{Grp}$ .
- Topology is the study of the category  $\text{Top}$ .
- ...

And there are many many other categories in mathematics, for any type of mathematical object (rings, varieties, vector spaces, modules, manifolds, ... ).

### 3. GROUPOIDS

In the case of  $\text{Set}$ ,  $\text{Grp}$  and  $\text{Top}$  we have morphisms which have an inverse. Such morphisms are known as bijections, isomorphisms and homeomorphisms, respectively. This motivates the following definition.

**Definition 3.1.** Let  $\mathcal{C}$  be a category. A morphism  $f: X \rightarrow Y$  is an isomorphism if it has an inverse.

**Definition 3.2.** Let  $I$  be the category with the following specification:

- $\text{Obj}(I) = \{0, 1\}$ .
- $\text{Hom}_I(0, 0) = \{\text{id}_0\}$ ,  $\text{Hom}_I(1, 1) = \{\text{id}_1\}$ ,  $\text{Hom}_I(0, 1) = \{01\}$ , and  $\text{Hom}_I(1, 0) = \{10\}$ .
- The composition is given by the identities and  $10 \circ 01 = \text{id}_0$  and  $01 \circ 10 = \text{id}_1$ .
- Graphically, we can represent this category as follows:

$$0 \xrightleftharpoons[10]{01} 1$$

We call this category the free isomorphism.

**Definition 3.3.** A morphism  $f$  with equal domain and codomain is called an endomorphism. An endomorphism that is an isomorphism is called an automorphism.

**Definition 3.4.** A groupoid is a category where every morphism is an isomorphism.

**Example 3.5.**  $[0]$  is a groupoid. Every morphism is the identity!

**Example 3.6.**  $[1]$  is not a groupoid. The morphism  $01: 0 \rightarrow 1$  does not have an inverse. Similarly  $[2]$ .

**Example 3.7.** For a monoid  $M$ ,  $BM$  is a groupoid if and only if  $M$  is a group. So  $B\mathbb{N}$  is not a groupoid, but  $B\mathbb{Z}$  is.

**Example 3.8.** Set, Grp and Top are not groupoids.

#### 4. FUNCTORS

If categories are like different theories of mathematics, then functors are ways to translate between these theories. Functors are for categories what functions are for sets.

**Definition 4.1.** Let  $\mathcal{C}, \mathcal{D}$  be two categories. A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  consists of the following data:

- A function  $F: \text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{D})$ .
- For every pair of objects  $X, Y \in \text{Obj}(\mathcal{C})$ , a function  $F: \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ .
- For every object  $X \in \text{Obj}(\mathcal{C})$ , we have  $F(\text{id}_X) = \text{id}_{F(X)}$ .
- For every triple of objects  $X, Y, Z \in \text{Obj}(\mathcal{C})$ , and every  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$  and  $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$ , we have  $F(g \circ f) = F(g) \circ F(f)$ .

**Example 4.2.** Let  $\mathcal{C}$  be a category. The identity functor  $\text{id}_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$  is given by the identity on objects and morphisms.

**Exercise 4.3.** Let  $\mathcal{C}, \mathcal{D}, \mathcal{E}$  be three categories and  $F: \mathcal{C} \rightarrow \mathcal{D}$ ,  $G: \mathcal{D} \rightarrow \mathcal{E}$  be two functors. Show there is a composition functor  $G \circ F: \mathcal{C} \rightarrow \mathcal{E}$ .

**Example 4.4.** Let  $\text{Cat}$  be the category with the following specification:

- $\text{Obj}(\text{Cat})$  is the class of all categories.
- For  $\mathcal{C}, \mathcal{D} \in \text{Obj}(\text{Cat})$ ,  $\text{Hom}_{\text{Cat}}(\mathcal{C}, \mathcal{D})$  is the set of all functors from  $\mathcal{C}$  to  $\mathcal{D}$ .
- The composition is given by the usual composition of functions.
- The identity is given by the usual identity function.

**Example 4.5.** Let  $\mathcal{C}$  be a category and  $c$  an object. There is a functor  $\text{Hom}_{\mathcal{C}}(c, -): \mathcal{C} \rightarrow \text{Set}$ , which maps an object  $d$  to the set  $\text{Hom}_{\mathcal{C}}(c, d)$  and a morphism  $f: d \rightarrow d'$  to the function

$$\text{Hom}_{\mathcal{C}}(c, f): \text{Hom}_{\mathcal{C}}(c, d) \rightarrow \text{Hom}_{\mathcal{C}}(c, d')$$

given by  $g \mapsto f \circ g$ .

**Exercise 4.6.** Let  $\mathcal{C}$  be a category. Prove there is a bijection of sets  $\text{Hom}_{\text{Cat}}([0], \mathcal{C}) \cong \text{Obj}_{\mathcal{C}}$ .

**Exercise 4.7.** Let  $\mathcal{C}$  be a category. Prove there is a bijection of sets

$$\text{Hom}_{\text{Cat}}([1], \mathcal{C}) \cong \{\text{morphisms in } \mathcal{C}\}.$$

**Exercise 4.8.** Let  $\mathcal{C}$  be a category. Prove there is a bijection of sets

$$\text{Hom}_{\text{Cat}}([2], \mathcal{C}) \cong \{\text{two composable morphisms in } \mathcal{C}\}.$$

**Exercise 4.9.** Let  $\mathcal{C}$  be a category. Prove there is a bijection of sets

$$\text{Hom}_{\text{Cat}}([1] \times [1], \mathcal{C}) \cong \{\text{commutative squares in } \mathcal{C}\}.$$

**Exercise 4.10.** Let  $\mathcal{C}$  be a category. Prove there is a bijection of sets

$$\text{Hom}_{\text{Cat}}(B\mathbb{N}, \mathcal{C}) \cong \{\text{endomorphisms in } \mathcal{C}\}.$$

**Exercise 4.11.** Let  $\mathcal{C}$  be a category. Prove there is a bijection of sets

$$\text{Hom}_{\text{Cat}}(I, \mathcal{C}) \cong \{\text{isomorphisms in } \mathcal{C}\}.$$

**Exercise 4.12.** Let  $\mathcal{C}$  be a category. Prove there is a bijection of sets

$$\text{Hom}_{\text{Cat}}(B\mathbb{Z}, \mathcal{C}) \cong \{\text{automorphisms in } \mathcal{C}\}.$$

**Exercise 4.13.** Prove that the assignment  $U: \text{Grp} \rightarrow \text{Set}$  that takes a group to its underlying set, and group homomorphism to the underlying function, is a functor.

**Exercise 4.14.** Prove that the assignment  $F: \text{Set} \rightarrow \text{Grp}$  that takes a set to the free group on that set, and similarly on functions, is a functor.

**Exercise 4.15.** Let  $\mathcal{C}, \mathcal{D}$  be two categories, and  $d$  an object in  $\mathcal{D}$ . Prove the assignment  $\Delta_d: \mathcal{C} \rightarrow \mathcal{D}$  that takes every object in  $\mathcal{C}$  to  $d$  and every morphism in  $\mathcal{C}$  to  $\text{id}_d$  is a functor.

**Exercise 4.16.** Let  $S$  be a set. Prove that  $S \times -: \text{Set} \rightarrow \text{Set}$  that takes a set  $X$  to  $S \times X$  and a function  $f: X \rightarrow Y$  to the function  $S \times f: S \times X \rightarrow S \times Y$  is a functor.

**Exercise 4.17.** Let  $(P, \leq), (Q, \leq)$  be two preorders and  $f: P \rightarrow Q$  be an order-preserving function. Prove that the assignment  $\mathcal{C}_f: \mathcal{C}_P \rightarrow \mathcal{C}_Q$ , which on objects is just given by  $f$ , lifts to a functor.

**Exercise 4.18.** Let  $M, N$  be two monoids. Let  $f: M \rightarrow N$  be a monoid homomorphism. Prove  $Bf: BM \rightarrow BN$  defined as follows is a functor

- The identity  $\{*\} \rightarrow \{*\}$ .
- The function  $f: \text{Hom}_{BM}(*, *) = M \rightarrow N = \text{Hom}_{BN}(*, *)$ .

**Exercise 4.19.** Let  $\text{Set}$  denote the category of sets, let  $\text{Cat}$  denote the category of small categories.

- (1) Let  $\text{Obj}: \text{Cat} \rightarrow \text{Set}$  be the assignment that takes a category to its set of objects, and a functor to the underlying function on objects. Prove this is a functor.
- (2) Let  $D: \text{Set} \rightarrow \text{Cat}$  be the assignment that takes a set  $S$  to the discrete category on  $S$  (i.e. the category with objects  $S$  and only identity morphisms), and a function to the evident functor between discrete categories. Prove this is a functor.
- (3) Let  $C: \text{Set} \rightarrow \text{Cat}$  be the assignment that takes a set  $S$  to the codiscrete category on  $S$  (i.e. the category with objects  $S$  and a unique morphism between any two objects), and a function to the evident functor between codiscrete categories. Prove this is a functor.

Functors can be used to define isomorphisms between categories.

**Definition 4.20.** A functor that has an inverse functor is called an isomorphism of categories.

None of the functors described until now are isomorphisms of categories! Let's describe one.

**Definition 4.21.** Let  $\mathcal{C}$  be a category. The opposite category  $\mathcal{C}^{\text{op}}$  is defined as follows:

- $\text{Obj}(\mathcal{C}^{\text{op}}) = \text{Obj}(\mathcal{C})$
- For objects  $X, Y$  in  $\mathcal{C}^{\text{op}}$ ,  $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$ .
- The composition and identity are defined as in  $\mathcal{C}$ , but with the direction of morphisms reversed.

**Definition 4.22.** Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor. The opposite functor  $F^{\text{op}}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$  is defined as follows:

- On objects,  $F^{\text{op}}$  is given by  $F$ .
- On morphisms, for  $f: X \rightarrow Y$  in  $\mathcal{C}^{\text{op}}$  (i.e.  $f: Y \rightarrow X$  in  $\mathcal{C}$ ), we have  $F^{\text{op}}(f) = F(f): F(Y) \rightarrow F(X)$  in  $\mathcal{D}^{\text{op}}$ .

**Exercise 4.23.** Prove that  $(-)^{\text{op}}: \text{Cat} \rightarrow \text{Cat}$  is a functor and that this functor is in fact an isomorphism.

Why do we care about opposite categories? Here is a cool example.

**Example 4.24.** Let  $\mathcal{C}$  be a category and  $c$  an object. There is a functor  $\text{Hom}_{\mathcal{C}}(-, c): \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ , which maps an object  $d$  to the set  $\text{Hom}_{\mathcal{C}}(d, c)$  and a morphism  $f: d' \rightarrow d$  to the function

$$\text{Hom}_{\mathcal{C}}(d, c) \rightarrow \text{Hom}_{\mathcal{C}}(d', c), \quad g \mapsto g \circ f.$$

Notice the direction of the arrows indeed flips!

## 5. NATURAL TRANSFORMATIONS

Functors are morphisms between categories. Natural transformations are morphisms between functors.

**Definition 5.1.** Let  $\mathcal{C}, \mathcal{D}$  be two categories and let  $F, G: \mathcal{C} \rightarrow \mathcal{D}$  be two functors. A natural transformation  $\eta: F \rightarrow G$  consists of the following data:

- For every object  $X \in \text{Obj}(\mathcal{C})$ , a morphism  $\eta_X: F(X) \rightarrow G(X)$  in  $\mathcal{D}$ .
- For every morphism  $f: X \rightarrow Y$  in  $\mathcal{C}$ , the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \eta_X \downarrow & & \downarrow \eta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

**Exercise 5.2.** Let  $\mathcal{C}, \mathcal{D}$  be two categories and let  $F, G, H: \mathcal{C} \rightarrow \mathcal{D}$  be three functors. Let  $\eta: F \rightarrow G$  and  $\theta: G \rightarrow H$  be two natural transformations. Show there is a composition natural transformation  $\theta \circ \eta: F \rightarrow H$ .

**Definition 5.3.** Let  $\mathcal{C}, \mathcal{D}$  be two categories. We define the category  $\text{Fun}(\mathcal{C}, \mathcal{D})$  with the following specification:

- $\text{Obj}(\text{Fun}(\mathcal{C}, \mathcal{D}))$  are functors from  $\mathcal{C}$  to  $\mathcal{D}$ .
- For  $F, G \in \text{Obj}(\text{Fun}(\mathcal{C}, \mathcal{D}))$ , we have  $\text{Hom}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(F, G) = \text{Nat}(F, G)$ .

**Exercise 5.4.** Prove there is an isomorphism of categories  $\text{Fun}([0], \mathcal{C}) \cong \mathcal{C}$ .

**Definition 5.5.** Let  $F, G: \mathcal{C} \rightarrow \mathcal{D}$  be two functors. A natural transformation  $\eta: F \rightarrow G$  is a natural isomorphism if every component  $\eta_X: F(X) \rightarrow G(X)$  is an isomorphism in  $\mathcal{D}$ .

## 6. EQUIVALENCES

Isomorphisms are not always what we want.

## 7. YONEDA LEMMA

Let us see one cool result in category theory.

## REFERENCES

- [EM45] Samuel Eilenberg and Saunders MacLane. General theory of natural equivalences. *Trans. Amer. Math. Soc.*, 58:231–294, 1945.