

LECTURE NOTES ON CATEGORY THEORY (IN PROGRESS)

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Category theory is a branch of mathematics that commenced with the work of Eilenberg and Mac Lane in the 1940s [EM45], motivated by research questions in algebra and topology.

1. DEFINITION OF CATEGORY

We directly commence with the definition of a category.

Definition 1.1. A category \mathcal{C} consists of the following data:

- A class of objects $\text{Obj}(\mathcal{C})$.
- For every pair of objects $X, Y \in \text{Obj}(\mathcal{C})$, a set of morphisms $\text{Hom}_{\mathcal{C}}(X, Y)$.
- For every triple of objects $X, Y, Z \in \text{Obj}(\mathcal{C})$, a composition map
$$- \circ -: \text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$$
- An identity morphism $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$ for every object $X \in \text{Obj}(\mathcal{C})$.
- Composition is unital: for every $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, we have $\text{id}_Y \circ f = f$ and $f \circ \text{id}_X = f$.
- Composition is associative: for every $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$ and $h \in \text{Hom}_{\mathcal{C}}(Z, W)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

If $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, we often write $f: X \rightarrow Y$ and call X the *domain* and Y the *codomain* of f .

Exercise 1.2. If e_X, e'_X are two identity morphisms for an object X in a category \mathcal{C} , then $e_X = e'_X$.

2. EXAMPLES OF CATEGORIES

This is a very heavy definition, so let's look at some examples!

Example 2.1. Let $[0]$ be the category with the following specification:

- $\text{Obj}([0]) = \{0\}$.
- $\text{Hom}_{[0]}(0, 0) = \{\text{id}_0\}$.
- $\text{id}_0 \circ \text{id}_0 = \text{id}_0$.

Graphically, we can represent this category as a single point!

Identities are not expressed in the graphical representations.

Example 2.2. Let $[1]$ be the category with the following specification:

- $\text{Obj}([1]) = \{0, 1\}$.
- $\text{Hom}_{[1]}(0, 0) = \{\text{id}_0\}$, $\text{Hom}_{[1]}(1, 1) = \{\text{id}_1\}$, $\text{Hom}_{[1]}(0, 1) = \{01\}$, and $\text{Hom}_{[1]}(1, 0) = \emptyset$.
- $\text{id}_0 \circ \text{id}_0 = \text{id}_0$, $\text{id}_1 \circ \text{id}_1 = \text{id}_1$, $01 \circ \text{id}_0 = 01$, and $01 \circ \text{id}_1 = 01$.

Graphically, we can represent this category as follows:

$$0 \xrightarrow{01} 1$$

We say it is a category with two objects and one non-identity morphism.

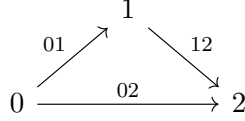
This is fun, so let's keep going!

Example 2.3. Let $[2]$ be the category with the following specification:

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- $\text{Obj}([2]) = \{0, 1, 2\}$.
- $\text{Hom}_{[2]}(0, 0) = \{\text{id}_0\}$, $\text{Hom}_{[2]}(1, 1) = \{\text{id}_1\}$, $\text{Hom}_{[2]}(2, 2) = \{\text{id}_2\}$, $\text{Hom}_{[2]}(0, 1) = \{01\}$, $\text{Hom}_{[2]}(1, 2) = \{12\}$, $\text{Hom}_{[2]}(0, 2) = \{02\}$, and all other hom-sets are empty.
- The composition is given by the identities and $12 \circ 01 = 02$.

Graphically, we can represent this category as follows:

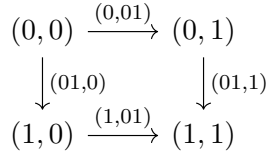


We say it is a category with three objects and three non-identity morphisms.

Example 2.4. Let $[1] \times [1]$ be the category with the following specification:

- $\text{Obj}([1] \times [1]) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.
- The hom-sets are given by
 - All hom-sets with domain and codomain equal only contain the identity.
 - $\text{Hom}((0, 0), (0, 1)) = \{(0, 01)\}$, $\text{Hom}((0, 0), (1, 0)) = \{(01, 0)\}$, $\text{Hom}((0, 1), (1, 1)) = \{(01, 1)\}$, and $\text{Hom}((1, 0), (1, 1)) = \{(1, 01)\}$,
 - $\text{Hom}((0, 0), (1, 1)) = \{(01, 01)\}$,
 - All other hom-sets are empty.
- The composition with identity are as before, and additionally we have
 - $(01, 1) \circ (0, 01) = (01, 01)$
 - $(1, 01) \circ (01, 0) = (01, 01)$

Graphically, we can represent this category as follows:



We say it is a category with four objects and five non-identity morphisms.

Definition 2.5. A preorder (P, \leq) is a set P along with a binary relation \leq that is reflexive and transitive.

Definition 2.6. Let P be a preorder. We define the category \mathcal{C}_P with the following specification:

- $\text{Obj}(\mathcal{C}_P) = P$.
- For $x, y \in P$, $\text{Hom}_{\mathcal{C}_P}(x, y) = \{*\}$ if $x \leq y$ and $\text{Hom}_{\mathcal{C}_P}(x, y) = \emptyset$ otherwise.
- The composition is given by the unique element in the hom-set if it exists, which exists by transitivity.
- The identity is given by the unique element in the hom-set, which exists by reflexivity.
- $\{0\}$ gives us the category $[0]$.
- $\{0 \leq 1\}$ gives us the category $[1]$.
- $\{0 \leq 1 \leq 2\}$ gives us the category $[2]$.
- $\{0 \leq 1\} \times \{0 \leq 1\}$ gives us the category $[1] \times [1]$.

Example 2.7. Let $(M, +, 0)$ be a monoid. We define the category BM with the following specification:

- $\text{Obj}(BM) = \{*\}$.
- $\text{Hom}_{BM}(*, *) = M$.
- The composition is given by the monoid operation, i.e. for $m, n \in M$, we have $m \circ n = m + n$.
- The identity is given by the monoid identity, i.e. $\text{id}_* = 0$.

Let us some examples of this example.

Example 2.8. Let $(\mathbb{N}, +, 0)$ be the monoid of natural numbers under addition. The category \mathbf{BN} has one object and morphisms given by natural numbers, with composition given by addition. It is also called the free endomorphism.

Example 2.9. Let $(\mathbb{Z}, +, 0)$ be the monoid of integers under addition. The category \mathbf{BZ} has one object and morphisms given by integers, with composition given by addition. It is also called the free automorphism.

Example 2.10. Let \mathbf{Set} be the category with the following specification:

- $\text{Obj}(\mathbf{Set}) = \{X \mid X \text{ is a set}\}.$
- For $X, Y \in \text{Obj}(\mathbf{Set})$, $\text{Hom}_{\mathbf{Set}}(X, Y)$ is the set of all functions from X to Y .
- The composition is given by the usual composition of functions.
- The identity is given by the usual identity function.

Example 2.11. Let \mathbf{Grp} be the category with the following specification:

- $\text{Obj}(\mathbf{Grp})$ is the class of all groups.
- For $G, H \in \text{Obj}(\mathbf{Grp})$, $\text{Hom}_{\mathbf{Grp}}(G, H)$ is the set of all group homomorphisms from G to H .
- The composition is given by the usual composition of functions.
- The identity is given by the usual identity function.

Example 2.12. Let \mathbf{Top} be the category with the following specification:

- $\text{Obj}(\mathbf{Top})$ is the class of all topological spaces.
- For $X, Y \in \text{Obj}(\mathbf{Top})$, $\text{Hom}_{\mathbf{Top}}(X, Y)$ is the set of all continuous functions from X to Y .
- The composition is given by the usual composition of functions.
- The identity is given by the usual identity function.

These last three examples illustrate why category became so important in mathematics: Every type of mathematical object (sets, groups, topological spaces, ...) form a category.

- Set theory is the study of the category \mathbf{Set} .
- Group theory is the study of the category \mathbf{Grp} .
- Topology is the study of the category \mathbf{Top} .
- ...

And there are many many other categories in mathematics, for any type of mathematical object (rings, varieties, vector spaces, modules, manifolds, ...).

3. GROUPOIDS

In the case of \mathbf{Set} , \mathbf{Grp} and \mathbf{Top} we have morphisms which have an inverse. Such morphisms are known as bijections, isomorphisms and homeomorphisms, respectively. This motivates the following definition.

Definition 3.1. Let \mathcal{C} be a category. A morphism $f: X \rightarrow Y$ is an isomorphism if it has an inverse.

Definition 3.2. Let I be the category with the following specification:

- $\text{Obj}(I) = \{0, 1\}.$
- $\text{Hom}_I(0, 0) = \{\text{id}_0\}$, $\text{Hom}_I(1, 1) = \{\text{id}_1\}$, $\text{Hom}_I(0, 1) = \{01\}$, and $\text{Hom}_I(1, 0) = \{10\}.$
- The composition is given by the identities and $10 \circ 01 = \text{id}_0$ and $01 \circ 10 = \text{id}_1.$
- Graphically, we can represent this category as follows:

$$0 \begin{array}{c} \xrightarrow{01} \\ \xleftarrow{10} \end{array} 1$$

We call this category the free isomorphism.

Definition 3.3. A morphism f with equal domain and codomain is called an endomorphism. An endomorphism that is an isomorphism is called an automorphism.

Definition 3.4. A groupoid is a category where every morphism is an isomorphism.

Example 3.5. $[0]$ is a groupoid. Every morphism is the identity!

Example 3.6. $[1]$ is not a groupoid. The morphism $01: 0 \rightarrow 1$ does not have an inverse. Similarly $[2]$.

Example 3.7. For a monoid M , BM is a groupoid if and only if M is a group. So $B\mathbb{N}$ is not a groupoid, but $B\mathbb{Z}$ is.

Example 3.8. Set , Grp and Top are not groupoids.

4. FUNCTORS

If categories are like different theories of mathematics, then functors are ways to translate between these theories. Functors are for categories what functions are for sets.

Definition 4.1. Let \mathcal{C}, \mathcal{D} be two categories. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of the following data:

- A function $F: \text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{D})$.
- For every pair of objects $X, Y \in \text{Obj}(\mathcal{C})$, a function $F: \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$.
- For every object $X \in \text{Obj}(\mathcal{C})$, we have $F(\text{id}_X) = \text{id}_{F(X)}$.
- For every triple of objects $X, Y, Z \in \text{Obj}(\mathcal{C})$, and every $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$, we have $F(g \circ f) = F(g) \circ F(f)$.

Example 4.2. Let \mathcal{C} be a category. The identity functor $\text{id}_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$ is given by the identity on objects and morphisms.

Exercise 4.3. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be three categories and $F: \mathcal{C} \rightarrow \mathcal{D}$, $G: \mathcal{D} \rightarrow \mathcal{E}$ be two functors. Show there is a composition functor $G \circ F: \mathcal{C} \rightarrow \mathcal{E}$.

Example 4.4. Let Cat be the category with the following specification:

- $\text{Obj}(\text{Cat})$ is the class of all categories.
- For $\mathcal{C}, \mathcal{D} \in \text{Obj}(\text{Cat})$, $\text{Hom}_{\text{Cat}}(\mathcal{C}, \mathcal{D})$ is the set of all functors from \mathcal{C} to \mathcal{D} .
- The composition is given by the usual composition of functions.
- The identity is given by the usual identity function.

Example 4.5. Let \mathcal{C} be a category and c an object. There is a functor $\text{Hom}_{\mathcal{C}}(c, -): \mathcal{C} \rightarrow \text{Set}$, which maps an object d to the set $\text{Hom}_{\mathcal{C}}(c, d)$ and a morphism $f: d \rightarrow d'$ to the function

$$\text{Hom}_{\mathcal{C}}(c, f): \text{Hom}_{\mathcal{C}}(c, d) \rightarrow \text{Hom}_{\mathcal{C}}(c, d')$$

given by $g \mapsto f \circ g$.

Exercise 4.6. Let \mathcal{C} be a category. Prove there is a bijection of sets $\text{Hom}_{\text{Cat}}([0], \mathcal{C}) \cong \text{Obj}_{\mathcal{C}}$.

Exercise 4.7. Let \mathcal{C} be a category. Prove there is a bijection of sets

$$\text{Hom}_{\text{Cat}}([1], \mathcal{C}) \cong \{\text{morphisms in } \mathcal{C}\}.$$

Exercise 4.8. Let \mathcal{C} be a category. Prove there is a bijection of sets

$$\text{Hom}_{\text{Cat}}([2], \mathcal{C}) \cong \{\text{two composable morphisms in } \mathcal{C}\}.$$

Exercise 4.9. Let \mathcal{C} be a category. Prove there is a bijection of sets

$$\text{Hom}_{\text{Cat}}([1] \times [1], \mathcal{C}) \cong \{\text{commutative squares in } \mathcal{C}\}.$$

Exercise 4.10. Let \mathcal{C} be a category. Prove there is a bijection of sets

$$\text{Hom}_{\text{Cat}}(B\mathbb{N}, \mathcal{C}) \cong \{\text{endomorphisms in } \mathcal{C}\}.$$

Exercise 4.11. Let \mathcal{C} be a category. Prove there is a bijection of sets

$$\mathrm{Hom}_{\mathrm{Cat}}(I, \mathcal{C}) \cong \{\text{isomorphisms in } \mathcal{C}\}.$$

Exercise 4.12. Let \mathcal{C} be a category. Prove there is a bijection of sets

$$\mathrm{Hom}_{\mathrm{Cat}}(B\mathbb{Z}, \mathcal{C}) \cong \{\text{automorphisms in } \mathcal{C}\}.$$

Exercise 4.13. Prove that the assignment $U: \mathrm{Grp} \rightarrow \mathrm{Set}$ that takes a group to its underlying set, and group homomorphism to the underlying function, is a functor.

Exercise 4.14. Prove that the assignment $F: \mathrm{Set} \rightarrow \mathrm{Grp}$ that takes a set to the free group on that set, and similarly on functions, is a functor.

Exercise 4.15. Let \mathcal{C}, \mathcal{D} be two categories, and d an object in \mathcal{D} . Prove the assignment $\Delta_d: \mathcal{C} \rightarrow \mathcal{D}$ that takes every object in \mathcal{C} to d and every morphism in \mathcal{C} to id_d is a functor.

Exercise 4.16. Let S be a set. Prove that $S \times -: \mathrm{Set} \rightarrow \mathrm{Set}$ that takes a set X to $S \times X$ and a function $f: X \rightarrow Y$ to the function $S \times f: S \times X \rightarrow S \times Y$ is a functor.

Exercise 4.17. Let $(P, \leq), (Q, \leq)$ be two preorders and $f: P \rightarrow Q$ be an order-preserving function. Prove that the assignment $\mathcal{C}_f: \mathcal{C}_P \rightarrow \mathcal{C}_Q$, which on objects is just given by f , lifts to a functor.

Exercise 4.18. Let M, N be two monoids. Let $f: M \rightarrow N$ be a monoid homomorphism. Prove $Bf: BM \rightarrow BN$ defined as follows is a functor

- The identity $\{*\} \rightarrow \{*\}$.
- The function $f: \mathrm{Hom}_{BM}(*, *) = M \rightarrow N = \mathrm{Hom}_{BN}(*, *)$.

Exercise 4.19. Let Set denote the category of sets, let Cat denote the category of small categories.

- (1) Let $\mathrm{Obj}: \mathrm{Cat} \rightarrow \mathrm{Set}$ be the assignment that takes a category to its set of objects, and a functor to the underlying function on objects. Prove this is a functor.
- (2) Let $D: \mathrm{Set} \rightarrow \mathrm{Cat}$ be the assignment that takes a set S to the discrete category on S (i.e. the category with objects S and only identity morphisms), and a function to the evident functor between discrete categories. Prove this is a functor.
- (3) Let $C: \mathrm{Set} \rightarrow \mathrm{Cat}$ be the assignment that takes a set S to the codiscrete category on S (i.e. the category with objects S and a unique morphism between any two objects), and a function to the evident functor between codiscrete categories. Prove this is a functor.

Functors can be used to define isomorphisms between categories.

Definition 4.20. A functor that has an inverse functor is called an isomorphism of categories.

None of the functors described until now are isomorphisms of categories! Let's describe one.

Definition 4.21. Let \mathcal{C} be a category. The opposite category \mathcal{C}^{op} is defined as follows:

- $\mathrm{Obj}(\mathcal{C}^{op}) = \mathrm{Obj}(\mathcal{C})$
- For objects X, Y in \mathcal{C}^{op} , $\mathrm{Hom}_{\mathcal{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$.
- The composition and identity are defined as in \mathcal{C} , but with the direction of morphisms reversed.

Definition 4.22. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. The opposite functor $F^{op}: \mathcal{C}^{op} \rightarrow \mathcal{D}^{op}$ is defined as follows:

- On objects, F^{op} is given by F .
- On morphisms, for $f: X \rightarrow Y$ in \mathcal{C}^{op} (i.e. $f: Y \rightarrow X$ in \mathcal{C}), we have $F^{op}(f) = F(f): F(Y) \rightarrow F(X)$ in \mathcal{D}^{op} .

Exercise 4.23. Prove that $(-)^{op}: \mathrm{Cat} \rightarrow \mathrm{Cat}$ is a functor and that this functor is in fact an isomorphism.

Why do we care about opposite categories? Here is a cool example.

Example 4.24. Let \mathcal{C} be a category and c an object. There is a functor $\text{Hom}_{\mathcal{C}}(-, c): \mathcal{C}^{op} \rightarrow \text{Set}$, which maps an object d to the set $\text{Hom}_{\mathcal{C}}(d, c)$ and a morphism $f: d' \rightarrow d$ to the function

$$\text{Hom}_{\mathcal{C}}(d, c) \rightarrow \text{Hom}_{\mathcal{C}}(d', c), \quad g \mapsto g \circ f.$$

Notice the direction of the arrows indeed flips!

5. NATURAL TRANSFORMATIONS

Functors are morphisms between categories. Natural transformations are morphisms between functors.

Definition 5.1. Let \mathcal{C}, \mathcal{D} be two categories and let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be two functors. A natural transformation $\eta: F \rightarrow G$ consists of the following data:

- For every object $X \in \text{Obj}(\mathcal{C})$, a morphism $\eta_X: F(X) \rightarrow G(X)$ in \mathcal{D} .
- For every morphism $f: X \rightarrow Y$ in \mathcal{C} , the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \eta_X \downarrow & & \downarrow \eta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

Exercise 5.2. Let \mathcal{C}, \mathcal{D} be two categories and let $F, G, H: \mathcal{C} \rightarrow \mathcal{D}$ be three functors. Let $\eta: F \rightarrow G$ and $\theta: G \rightarrow H$ be two natural transformations. Show there is a composition natural transformation $\theta \circ \eta: F \rightarrow H$.

Definition 5.3. Let \mathcal{C}, \mathcal{D} be two categories. We define the category $\text{Fun}(\mathcal{C}, \mathcal{D})$ with the following specification:

- $\text{Obj}(\text{Fun}(\mathcal{C}, \mathcal{D}))$ are functors from \mathcal{C} to \mathcal{D} .
- For $F, G \in \text{Obj}(\text{Fun}(\mathcal{C}, \mathcal{D}))$, we have $\text{Hom}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(F, G) = \text{Nat}(F, G)$.

Exercise 5.4. Prove there is an isomorphism of categories $\text{Fun}([0], \mathcal{C}) \cong \mathcal{C}$.

Definition 5.5. Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be two functors. A natural transformation $\eta: F \rightarrow G$ is a natural isomorphism if every component $\eta_X: F(X) \rightarrow G(X)$ is an isomorphism in \mathcal{D} .

6. EQUIVALENCES

Isomorphisms are not always what we want.

7. YONEDA LEMMA

Let us see one cool result in category theory.

REFERENCES

- [EM45] Samuel Eilenberg and Saunders MacLane. General theory of natural equivalences. *Trans. Amer. Math. Soc.*, 58:231–294, 1945.