LECTURE NOTES ON CATEGORY THEORY

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Category theory is a branch of mathematics that commenced with the work of Eilenberg and Mac Lane in the 1940s [EM45], motivated by research questions in algebra and topology.

1. Definition of Category

We directly commence with the definition of a category.

Definition 1.1. *A* category $\mathcal{C}$ *consists of the following data:*

- *A class of objects* $\mathrm{Obj}(\mathcal{C})$*.*
- *For every pair of objects* $X, Y \in \mathrm{Obj}(\mathcal{C})$*, a set of morphisms* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$*.*
- *For every triple of objects* $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$*, a composition map*

$$- \circ -\colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

- *An identity morphism* $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ *for every object* $X \in \mathrm{Obj}(\mathcal{C})$*.*
- *Composition is unital: for every* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$*, we have* $\mathrm{id}_Y \circ f = f$ *and* $f \circ \mathrm{id}_X = f$*.*
- *Composition is associative: for every* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$*,* $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ *and* $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$*, we have* $h \circ (g \circ f) = (h \circ g) \circ f$*.*

If $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we often write $f\colon X \to Y$ and call X the *domain* and Y the *codomain* of f.

Exercise 1.2. *If* e_X, e'_X *are two identity morphisms for an object* X *in a category* $\mathcal{C}$*, then* $e_X = e'_X$*.*

2. Examples of Categories

This is a very heavy definition, so let's look at some examples!

Example 2.1. *Let* $[0]$ *be the category with the following specification:*

- $\mathrm{Obj}([0]) = \{0\}$.
- $\mathrm{Hom}_{[0]}(0, 0) = \{\mathrm{id}_0\}$.
- $\mathrm{id}_0 \circ \mathrm{id}_0 = \mathrm{id}_0$.

Graphically, we can represent this category as a single point!

Identities are not expressed in the graphical representations.

Example 2.2. *Let* $[1]$ *be the category with the following specification:*

- $\mathrm{Obj}([1]) = \{0, 1\}$.
- $\mathrm{Hom}_{[1]}(0, 0) = \{\mathrm{id}_0\}$, $\mathrm{Hom}_{[1]}(1, 1) = \{\mathrm{id}_1\}$, $\mathrm{Hom}_{[1]}(0, 1) = \{01\}$, *and* $\mathrm{Hom}_{[1]}(1, 0) = \emptyset$.
- $\mathrm{id}_0 \circ \mathrm{id}_0 = \mathrm{id}_0$, $\mathrm{id}_1 \circ \mathrm{id}_1 = \mathrm{id}_1$, $01 \circ \mathrm{id}_0 = 01$, *and* $01 \circ \mathrm{id}_1 = 01$.

Graphically, we can represent this category as follows:

$$0 \xrightarrow{01} 1$$

We say it is a category with two objects and one non-identity morphism.

This is fun, so let's keep going!

Example 2.3. *Let* $[2]$ *be the category with the following specification:*

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- $\mathrm{Obj}([2]) = \{0, 1, 2\}$.
- $\mathrm{Hom}_{[2]}(0, 0) = \{\mathrm{id}_0\}$, $\mathrm{Hom}_{[2]}(1, 1) = \{\mathrm{id}_1\}$, $\mathrm{Hom}_{[2]}(2, 2) = \{\mathrm{id}_2\}$, $\mathrm{Hom}_{[2]}(0, 1) = \{01\}$, $\mathrm{Hom}_{[2]}(1, 2) = \{12\}$, $\mathrm{Hom}_{[2]}(0, 2) = \{02\}$, *and all other hom-sets are empty.*
- *The composition is given by the identities and* $12 \circ 01 = 02$.

Graphically, we can represent this category as follows:

$$\begin{tikzcd} & 1 \arrow[dr, "12"] & \\ 0 \arrow[ur, "01"] \arrow[rr, "02"] & & 2 \end{tikzcd}$$

We say it is a category with three objects and three non-identity morphisms.

Example 2.4. *Let* $[1] \times [1]$ *be the category with the following specification:*

- $\mathrm{Obj}([1] \times [1]) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.
- *The hom-sets are given by*
  - *All hom-sets with domain and codomain equal only contain the identity.*
  - $\mathrm{Hom}((0, 0), (0, 1)) = \{(0, 01)\}$, $\mathrm{Hom}((0, 0), (1, 0)) = \{(01, 0)\}$, $\mathrm{Hom}((0, 1), (1, 1)) = \{(01, 1)\}$, *and* $\mathrm{Hom}((1, 0), (1, 1)) = \{(1, 01)\}$,
  - $\mathrm{Hom}((0, 0), (1, 1)) = \{(01, 01)\}$,
  - *All other hom-sets are empty.*
- *The composition with identity are as before, and additionally we have*
  - $(01, 1) \circ (0, 01) = (01, 01)$
  - $(1, 01) \circ (01, 0) = (01, 01)$

Graphically, we can represent this category as follows:

$$\begin{tikzcd} (0,0) \arrow[r, "{(0,01)}"] \arrow[d, "{(01,0)}"] & (0,1) \arrow[d, "{(01,1)}"] \\ (1,0) \arrow[r, "{(1,01)}"] & (1,1) \end{tikzcd}$$

We say it is a category with four objects and five non-identity morphisms.

Definition 2.5. *A preorder* $(P, \leq)$ *is a set* P *along with a binary relation* $\leq$ *that is reflexive and transitive.*

Definition 2.6. *Let* P *be a preorder. We define the category* $\mathcal{C}_P$ *with the following specification:*

- $\mathrm{Obj}(\mathcal{C}_P) = P$.
- *For* $x, y \in P$, $\mathrm{Hom}_{\mathcal{C}_P}(x, y) = \{*\}$ *if* $x \leq y$ *and* $\mathrm{Hom}_{\mathcal{C}_P}(x, y) = \emptyset$ *otherwise.*
- *The composition is given by the unique element in the hom-set if it exists, which exists by transitivity.*
- *The identity is given by the unique element in the hom-set, which exists by reflexivity.*

- $\{0\}$ gives us the category $[0]$.
- $\{0 \leq 1\}$ gives us the category $[1]$.
- $\{0 \leq 1 \leq 2\}$ gives us the category $[2]$.
- $\{0 \leq 1\} \times \{0 \leq 1\}$ gives us the category $[1] \times [1]$.

Example 2.7. *Let* $(M, +, 0)$ *be a monoid. We define the category* BM *with the following specification:*

- $\mathrm{Obj}(BM) = \{*\}$.
- $\mathrm{Hom}_{BM}(*, *) = M$.
- *The composition is given by the monoid operation, i.e. for* $m, n \in M$, *we have* $m \circ n = m + n$.
- *The identity is given by the monoid identity, i.e.* $\mathrm{id}_* = 0$.

Let us some examples of this example.

Example 2.8. *Let* $(\mathbb{N}, +, 0)$ *be the monoid of natural numbers under addition. The category* $B\mathbb{N}$ *has one object and morphisms given by natural numbers, with composition given by addition. It is also called the* free endomorphism*.*

Example 2.9. *Let* $(\mathbb{Z}, +, 0)$ *be the monoid of integers under addition. The category* $B\mathbb{Z}$ *has one object and morphisms given by integers, with composition given by addition. It is also called the* free automorphism*.*

Example 2.10. *Let* $\mathcal{S}\mathrm{et}$ *be the category with the following specification:*

- $\mathrm{Obj}(\mathcal{S}\mathrm{et}) = \{X \mid X \text{ is a set}\}$.
- *For* $X, Y \in \mathrm{Obj}(\mathcal{S}\mathrm{et})$, $\mathrm{Hom}_{\mathcal{S}\mathrm{et}}(X, Y)$ *is the set of all functions from* X *to* Y.
- *The composition is given by the usual composition of functions.*
- *The identity is given by the usual identity function.*

Example 2.11. *Let* $\mathcal{G}\mathrm{rp}$ *be the category with the following specification:*

- $\mathrm{Obj}(\mathcal{G}\mathrm{rp})$ *is the class of all groups.*
- *For* $G, H \in \mathrm{Obj}(\mathcal{G}\mathrm{rp})$, $\mathrm{Hom}_{\mathcal{G}\mathrm{rp}}(G, H)$ *is the set of all group homomorphisms from* G *to* H.
- *The composition is given by the usual composition of functions.*
- *The identity is given by the usual identity function.*

Example 2.12. *Let* $\mathcal{T}\mathrm{op}$ *be the category with the following specification:*

- $\mathrm{Obj}(\mathcal{T}\mathrm{op})$ *is the class of all topological spaces.*
- *For* $X, Y \in \mathrm{Obj}(\mathcal{T}\mathrm{op})$, $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, Y)$ *is the set of all continuous functions from* X *to* Y.
- *The composition is given by the usual composition of functions.*
- *The identity is given by the usual identity function.*

These last three examples illustrate why category became so important in mathematics: Every type of mathematical object (sets, groups, topological spaces, ...) form a category.

- Set theory is the study of the category $\mathcal{S}\mathrm{et}$.
- Group theory is the study of the category $\mathcal{G}\mathrm{rp}$.
- Topology is the study of the category $\mathcal{T}\mathrm{op}$.
- ...

And there are many many other categories in mathematics, for any type of mathematical object (rings, varieties, vector spaces, modules, manifolds, ...).

3. Groupoids

In the case of $\mathcal{S}\mathrm{et}$, $\mathcal{G}\mathrm{rp}$ and $\mathcal{T}\mathrm{op}$ we have morphisms which have an inverse. Such morphisms are known as bijections, isomorphisms and homeomorphisms, respectively. This motivates the following definition.

Definition 3.1. *Let* $\mathcal{C}$ *be a category. A morphism* $f\colon X \to Y$ *is an* isomorphism *if it has an inverse.*

Definition 3.2. *Let* I *be the category with the following specification:*

- $\mathrm{Obj}(I) = \{0, 1\}$.
- $\mathrm{Hom}_I(0, 0) = \{\mathrm{id}_0\}$, $\mathrm{Hom}_I(1, 1) = \{\mathrm{id}_1\}$, $\mathrm{Hom}_I(0, 1) = \{01\}$, *and* $\mathrm{Hom}_I(1, 0) = \{10\}$.
- *The composition is given by the identities and* $10 \circ 01 = \mathrm{id}_0$ *and* $01 \circ 10 = \mathrm{id}_1$.
- *Graphically, we can represent this category as follows:*

$$0 \underset{10}{\overset{01}{\rightleftarrows}} 1$$

We call this category the free isomorphism*.*

Definition 3.3. *A morphism f with equal domain and codomain is called an* endomorphism*. An endomorphism that is an isomorphism is called an* automorphism*.*

Definition 3.4. *A* groupoid *is a category where every morphism is an isomorphism.*

Example 3.5. $[0]$ *is a groupoid. Every morphism is the identity!*

Example 3.6. $[1]$ *is not a groupoid. The morphism $01\colon 0 \to 1$ does not have an inverse. Similarly* $[2]$.

Example 3.7. *For a monoid M, BM is a groupoid if and only if M is a group. So $B\mathbb{N}$ is not a groupoid, but $B\mathbb{Z}$ is.*

Example 3.8. $\mathcal{S}\mathrm{et}$, $\mathcal{G}\mathrm{rp}$ *and* $\mathcal{T}\mathrm{op}$ *are not groupoids.*

4. Functors

If categories are like different theories of mathematics, then functors are ways to translate between these theories. Functors are for categories what functions are for sets.

Definition 4.1. *Let $\mathcal{C}, \mathcal{D}$ be two categories. A* functor *$F\colon \mathcal{C} \to \mathcal{D}$ consists of the following data:*

- *A function $F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$.*
- *For every pair of objects $X, Y \in \mathrm{Obj}(\mathcal{C})$, a function $F\colon \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$.*
- *For every object $X \in \mathrm{Obj}(\mathcal{C})$, we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.*
- *For every triple of objects $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, and every $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, we have $F(g \circ f) = F(g) \circ F(f)$.*

Example 4.2. *Let $\mathcal{C}$ be a category. The* identity functor *$\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ is given by the identity on objects and morphisms.*

Exercise 4.3. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be three categories and $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{E}$ be two functors. Show there is a composition functor $G \circ F\colon \mathcal{C} \to \mathcal{E}$.*

Example 4.4. *Let* $\mathcal{C}\mathrm{at}$ *be the category with the following specification:*

- $\mathrm{Obj}(\mathcal{C}\mathrm{at})$ *is the class of all categories.*
- *For $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathcal{C}\mathrm{at})$, $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C},\mathcal{D})$ is the set of all functors from $\mathcal{C}$ to $\mathcal{D}$.*
- *The composition is given by the usual composition of functions.*
- *The identity is given by the usual identity function.*

Example 4.5. *Let $\mathcal{C}$ be a category and c an object. There is a functor $\mathrm{Hom}_{\mathcal{C}}(c,-)\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$, which maps an object d to the set $\mathrm{Hom}_{\mathcal{C}}(c,d)$ and a morphism $f\colon d \to d'$ to the function*

$$\mathrm{Hom}_{\mathcal{C}}(c,f)\colon \mathrm{Hom}_{\mathcal{C}}(c,d) \to \mathrm{Hom}_{\mathcal{C}}(c,d')$$

given by $g \mapsto f \circ g$.

Exercise 4.6. *Let $\mathcal{C}$ be a category. Prove there is a bijection of sets $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}([0],\mathcal{C}) \cong \mathrm{Obj}_{\mathcal{C}}$.*

Exercise 4.7. *Let $\mathcal{C}$ be a category. Prove there is a bijection of sets*

$$\mathrm{Hom}_{\mathcal{C}\mathrm{at}}([1],\mathcal{C}) \cong \{\textit{morphisms in } \mathcal{C}\}.$$

Exercise 4.8. *Let $\mathcal{C}$ be a category. Prove there is a bijection of sets*

$$\mathrm{Hom}_{\mathcal{C}\mathrm{at}}([2],\mathcal{C}) \cong \{\textit{two composable morphisms in } \mathcal{C}\}.$$

Exercise 4.9. *Let $\mathcal{C}$ be a category. Prove there is a bijection of sets*

$$\mathrm{Hom}_{\mathcal{C}\mathrm{at}}([1] \times [1],\mathcal{C}) \cong \{\textit{commutative squares in } \mathcal{C}\}.$$

Exercise 4.10. *Let $\mathcal{C}$ be a category. Prove there is a bijection of sets*

$$\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(B\mathbb{N},\mathcal{C}) \cong \{\textit{endomorphisms in } \mathcal{C}\}.$$

Exercise 4.11. *Let $\mathcal{C}$ be a category. Prove there is a bijection of sets*
$$\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(I, \mathcal{C}) \cong \{\textit{isomorphisms in } \mathcal{C}\}.$$

Exercise 4.12. *Let $\mathcal{C}$ be a category. Prove there is a bijection of sets*
$$\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(B\mathbb{Z}, \mathcal{C}) \cong \{\textit{automorphisms in } \mathcal{C}\}.$$

Exercise 4.13. *Prove that the assignment $U\colon \mathcal{G}\mathrm{rp} \to \mathcal{S}\mathrm{et}$ that takes a group to its underlying set, and group homomorphism to the underlying function, is a functor.*

Exercise 4.14. *Prove that the assignment $F\colon \mathcal{S}\mathrm{et} \to \mathcal{G}\mathrm{rp}$ that takes a set to the free group on that set, and similarly on functions, is a functor.*

Exercise 4.15. *Let $\mathcal{C}, \mathcal{D}$ be two categories, and d an object in $\mathcal{D}$. Prove the assignment $\Delta_d\colon \mathcal{C} \to \mathcal{D}$ that takes every object in $\mathcal{C}$ to d and every morphism in $\mathcal{C}$ to id_d is a functor.*

Exercise 4.16. *Let S be a set. Prove that $S \times -\colon \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$ that takes a set X to $S \times X$ and a function $f\colon X \to Y$ to the function $S \times f\colon S \times X \to S \times Y$ is a functor.*

Exercise 4.17. *Let $(P, \leq), (Q, \leq)$ be two preorders and $f\colon P \to Q$ be an order-preserving function. Prove that the assignment $\mathcal{C}_f\colon \mathcal{C}_P \to \mathcal{C}_Q$, which on objects is just given by f, lifts to a functor.*

Exercise 4.18. *Let $(\mathbb{N}, |)$ be the preorder of natural numbers under divisibility, and define $(\mathbb{N}\times\mathbb{N}, |)$ analogously. Prove that the assignment $\mathcal{C}_{gcd}\colon \mathcal{C}_{(\mathbb{N}\times\mathbb{N},|)} \to \mathcal{C}_{(\mathbb{N},|)}$ defined on objects as $\mathrm{gcd}(n,m)$ lifts to a functor.*

Exercise 4.19. *Let M, N be two monoids. Let $f\colon M \to N$ be a monoid homomorphism. Prove $Bf\colon BM \to BN$ defined as follows is a functor*
- *The identity $\{*\} \to \{*\}$.*
- *The function $f\colon \mathrm{Hom}_{BM}(*,*) = M \to N = \mathrm{Hom}_{BN}(*,*)$.*

Exercise 4.20. *Let $\mathcal{S}\mathrm{et}$ denote the category of sets, let $\mathcal{C}\mathrm{at}$ denote the category of small categories.*
1. *Let $\mathrm{Obj}\colon \mathcal{C}\mathrm{at} \to \mathcal{S}\mathrm{et}$ be the assignment that takes a category to its set of objects, and a functor to the underlying function on objects. Prove this is a functor.*
2. *Let $D\colon \mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at}$ be the assignment that takes a set S to the discrete category on S (i.e. the category with objects S and only identity morphisms), and a function to the evident functor between discrete categories. Prove this is a functor.*
3. *Let $C\colon \mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at}$ be the assignment that takes a set S to the codiscrete category on S (i.e. the category with objects S and a unique morphism between any two objects), and a function to the evident functor between codiscrete categories. Prove this is a functor.*

Functors can be used to define isomorphisms between categories.

Definition 4.21. *A functor that has an inverse functor is called an* isomorphism of categories.

None of the functors described until now are isomorphisms of categories! Let's describe one.

Definition 4.22. *Let $\mathcal{C}$ be a category. The* opposite category *$\mathcal{C}^{op}$ is defined as follows:*
- $\mathrm{Obj}(\mathcal{C}^{op}) = \mathrm{Obj}(\mathcal{C})$
- *For objects X, Y in $\mathcal{C}^{op}$, $\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$.*
- *The composition and identity are defined as in $\mathcal{C}$, but with the direction of morphisms reversed.*

Definition 4.23. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. The* opposite functor *$F^{op}\colon \mathcal{C}^{op} \to \mathcal{D}^{op}$ is defined as follows:*
- *On objects, F^{op} is given by F.*

- *On morphisms, for* $f : X \to Y$ *in* $\mathcal{C}^{op}$ *(i.e.* $f : Y \to X$ *in* $\mathcal{C}$*), we have* $F^{op}(f) = F(f) : F(Y) \to F(X)$ *in* $\mathcal{D}^{op}$*.*

Exercise 4.24. *Prove that* $(-)^{op} : \mathfrak{Cat} \to \mathfrak{Cat}$ *is a functor and that this functor is in fact an isomorphism.*

Why do we care about opposite categories? Here is a cool example.

Example 4.25. *Let* $\mathcal{C}$ *be a category and* c *an object. There is a functor* $\mathrm{Hom}_{\mathcal{C}}(-, c) : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$*, which maps an object* d *to the set* $\mathrm{Hom}_{\mathcal{C}}(d, c)$ *and a morphism* $f : d' \to d$ *to the function*

$$\mathrm{Hom}_{\mathcal{C}}(d, c) \to \mathrm{Hom}_{\mathcal{C}}(d', c), \quad g \mapsto g \circ f.$$

Notice the direction of the arrows indeed flips!

5. Natural Transformations

Functors are morphisms between categories. Natural transformations are morphisms between functors.

Definition 5.1. *Let* $\mathcal{C}, \mathcal{D}$ *be two categories and let* $F, G : \mathcal{C} \to \mathcal{D}$ *be two functors. A* natural transformation $\eta : F \to G$ *consists of the following data:*

- *For every object* $X \in \mathrm{Obj}(\mathcal{C})$*, a morphism* $\eta_X : F(X) \to G(X)$ *in* $\mathcal{D}$*.*
- *For every morphism* $f : X \to Y$ *in* $\mathcal{C}$*, the following diagram commutes:*

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

Example 5.2. *Let* $\mathcal{C}, \mathcal{D}$ *be two categories and* d, d' *two objects in* $\mathcal{D}$*. A natural transformation* $\eta : \Delta_d \to \Delta_{d'}$ *between the constant functors* $\Delta_d, \Delta_{d'} : \mathcal{C} \to \mathcal{D}$ *consists of the following data:*

- *For every object* $X \in \mathrm{Obj}(\mathcal{C})$*, a morphism* $\eta_X : d \to d'$ *in* $\mathcal{D}$*.*
- *For every morphism* $f : X \to Y$ *in* $\mathcal{C}$*, the following diagram commutes:*

$$\begin{tikzcd} \Delta_d(c) = d \arrow[r, "\mathrm{id}_d"] \arrow[d, "\eta_c"'] & d = \Delta_d(c') \arrow[d, "\eta_{c'}"] \\ \Delta_{d'}(c) = d' \arrow[r, "\mathrm{id}_{d'}"'] & d' = \Delta_{d'}(c') \end{tikzcd}$$

Thus, a natural transformation between constant functors is just a single morphism $\eta : d \to d'$ *in* $\mathcal{D}$*.*

Example 5.3. *Let* $\mathrm{inc} : D \to C$ *be the natural transformation between the functors* $D, C : \mathcal{S}\mathrm{et} \to \mathfrak{Cat}$ *defined as follows:*

- *On objects it is the identity.*
- *On morphisms it preserves the identity morphisms.*

Exercise 5.4. *Let* $\mathcal{C}, \mathcal{D}$ *be two categories and let* $F, G, H : \mathcal{C} \to \mathcal{D}$ *be three functors. Let* $\eta : F \to G$ *and* $\theta : G \to H$ *be two natural transformations. Show there is a composition natural transformation* $\theta \circ \eta : F \to H$*.*

Definition 5.5. *Let* $\mathcal{C}, \mathcal{D}$ *be two categories. We define the category* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *with the following specification:*

- $\mathrm{Obj}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ *are functors from* $\mathcal{C}$ *to* $\mathcal{D}$*.*
- *For* $F, G \in \mathrm{Obj}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$*, we have* $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) = \mathrm{Nat}(F, G)$*.*

Functor categories are themselves functorial.

Exercise 5.6. *Let $F\colon \mathcal{D} \to \mathcal{E}$ be a functor. Show there is a functor $F_*\colon \mathrm{Fun}(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\mathcal{C},\mathcal{E})$ defined as follows:*

- *On objects, $F_*(G) = F \circ G$ for $G \in \mathrm{Obj}(\mathrm{Fun}(\mathcal{C},\mathcal{D}))$.*
- *On morphisms, given a natural transformation $\eta\colon G \to H$ for $G, H \in \mathrm{Obj}(\mathrm{Fun}(\mathcal{C},\mathcal{D}))$, define $F_*(\eta)\colon F_*(G) \to F_*(H)$ by*

$$F_*(\eta)_X = F(\eta_X)\colon F(G(X)) \to F(H(X)).$$

Exercise 5.7. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Show there is a functor $F^*\colon \mathrm{Fun}(\mathcal{D},\mathcal{E}) \to \mathrm{Fun}(\mathcal{C},\mathcal{E})$ defined as follows:*

- *On objects, $F^*(G) = G \circ F$ for $G \in \mathrm{Obj}(\mathrm{Fun}(\mathcal{D},\mathcal{E}))$.*
- *On morphisms, given a natural transformation $\eta\colon G \to H$ for $G, H \in \mathrm{Obj}(\mathrm{Fun}(\mathcal{D},\mathcal{E}))$, define $F^*(\eta)\colon F^*(G) \to F^*(H)$ by*

$$F^*(\eta)_X = \eta_{F(X)}\colon G(F(X)) \to H(F(X)).$$

Exercise 5.8. *Prove there is an isomorphism of categories $\mathrm{Fun}([0],\mathcal{C}) \cong \mathcal{C}$.*

Definition 5.9. *Let G be a group. Let $\mathcal{S}\mathrm{et}_G$ be the category with the following specification:*

- $\mathrm{Obj}(\mathcal{S}\mathrm{et}_G)$ *is the class of all sets with a G-action.*
- *For $X, Y \in \mathrm{Obj}(\mathcal{S}\mathrm{et}_G)$, $\mathrm{Hom}_{\mathcal{S}\mathrm{et}_G}(X,Y)$ is the set of all G-equivariant functions from X to Y.*
- *The composition is given by the usual composition of functions.*

Exercise 5.10. *Let G be a group. Prove there is an isomorphism of categories $\mathrm{Fun}(BG, \mathcal{S}\mathrm{et}) \cong \mathcal{S}\mathrm{et}_G$.*

Definition 5.11. *Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\eta\colon F \to G$ is a* natural isomorphism *if every component $\eta_X\colon F(X) \to G(X)$ is an isomorphism in $\mathcal{D}$.*

6. Yoneda Lemma

Let us see one major result in category theory. Let $\mathcal{C}$ be a category and c an object in $\mathcal{C}$. We saw that we have functors $\mathrm{Hom}_{\mathcal{C}}(c,-)\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$.

Now, assume there is a natural transformation $\alpha\colon \mathrm{Hom}_{\mathcal{C}}(c,-) \to F$. Then we can construct one element in $F(c)$, namely $\alpha_c(\mathrm{id}_c) \in F(c)$. This gives us a morphism

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(c,-), F) \to F(c), \quad \alpha \mapsto \alpha_c(\mathrm{id}_c).$$

Turns out this map is very important.

Theorem 6.1 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category, c an object in $\mathcal{C}$, and $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$ a functor. The map*

$$\mathcal{Y}\mathrm{on}\colon \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(c,-), F) \to F(c), \quad \alpha \mapsto \alpha_c(\mathrm{id}_c)$$

is a bijection of sets.

Proof. We construct the inverse map. Let $x \in F(c)$. We define a natural transformation $\alpha^x\colon \mathrm{Hom}_{\mathcal{C}}(c,-) \to F$ as follows: for every object $d \in \mathrm{Obj}(\mathcal{C})$, define

$$\alpha^x_d\colon \mathrm{Hom}_{\mathcal{C}}(c,d) \to F(d), \quad f \mapsto F(f)(x).$$

We need to check this is indeed a natural transformation. Let $g\colon d \to d'$ be a morphism in $\mathcal{C}$. We need to show the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(c,d) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(c,g)"] \arrow[d, "\alpha^x_d"'] & \mathrm{Hom}_{\mathcal{C}}(c,d') \arrow[d, "\alpha^x_{d'}"] \\ F(d) \arrow[r, "F(g)"'] & F(d') \end{tikzcd}$$

Take $f \in \mathrm{Hom}_{\mathcal{C}}(c,d)$. Then we have

$$F(g)(\alpha^x_d(f)) = F(g)(F(f)(x)) = F(g \circ f)(x) = \alpha^x_{d'}(g \circ f) = \alpha^x_{d'}(\mathrm{Hom}_{\mathcal{C}}(c,g)(f)).$$

So the diagram commutes. Thus, we have defined a natural transformation α^x. This gives us a map

$$F(c) \to \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(c,-),F), \quad x \mapsto \alpha^x.$$

It is straightforward to check that these two maps are inverses of each other. $\square$

Here are some applications.

Exercise 6.2. *Let G be a group. Prove that* $\mathrm{Hom}_{BG}(*,-)\colon BG \to \mathcal{S}\mathrm{et}$ *corresponds to the G-set with underlying set G and action given by left multiplication.*

Now, the Yoneda lemma immediately gives us the following result.

Corollary 6.3. *There is a bijection between G-equivariant functions from G to a G-set X and elements in X.*

In particular, taking $X = G$ with the action given by right multiplication, we get the following result.

Corollary 6.4 (Cayley's Theorem)**.** *There is a bijection between G and G-equivariant bijections $G \to G$. Hence there is an injection G to the group of permutations of the set G.*

7. Equivalences

Let $\mathrm{Vect}_{\mathbb{R}}$ be the category of real vector spaces and linear transformations. Let $\mathrm{Vect}^{fd}_{\mathbb{R}}$ be the category of finite-dimensional real vector spaces and linear transformations. Now here is a different category: Let $\mathrm{Mat}_{\mathbb{R}}$ be the category with the following specification:

- $\mathrm{Obj}(\mathrm{Mat}_{\mathbb{R}}) = \{\mathbb{R}^n \mid n \in \mathbb{N}\}$.
- For $\mathbb{R}^n, \mathbb{R}^m \in \mathrm{Obj}(\mathrm{Mat}_{\mathbb{R}})$, $\mathrm{Hom}_{\mathrm{Mat}_{\mathbb{R}}}(\mathbb{R}^n, \mathbb{R}^m)$ is the set of all $m \times n$ real matrices.
- The composition is given by matrix multiplication.

There is a functor $F\colon \mathrm{Mat}_{\mathbb{R}} \to \mathrm{Vect}^{fd}_{\mathbb{R}}$ defined as follows:

- On objects, $F(\mathbb{R}^n) = \mathbb{R}^n$.
- On morphisms, for $A \in \mathrm{Hom}_{\mathrm{Mat}_{\mathbb{R}}}(\mathbb{R}^n, \mathbb{R}^m)$, $F(A)\colon \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation defined by A.

On the other side, let

$$\mathcal{B}\mathrm{asis} = \{(V, \beta_V) | V \in \mathrm{Obj}(\mathrm{Vect}^{fd}_{\mathbb{R}}), \beta \text{ Basis of } V\}$$

and fix a set function $\mathcal{B}\colon \mathrm{Obj}(\mathrm{Vect}^{fd}_{\mathbb{R}}) \to \mathcal{B}\mathrm{asis}$ that is the identity on the first component, and that maps $\mathbb{R}^n$ to the standard basis.

Define the functor $G\colon \mathrm{Vect}^{fd}_{\mathbb{R}} \mathrm{Mat}_{\mathbb{R}}$ as follows:

- On objects $G(V) = \mathbb{R}^{\dim(V)}$
- On morphisms $G(T\colon V \to W) = [T]^{\mathcal{B}(W)}_{\mathcal{B}(V)}$

- Functoriality is a standard result in linear algebra: $[T_2 \circ T_1]^{\mathcal{B}(V_3)}_{\mathcal{B}(V_1)} = [T_2]^{\mathcal{B}(V_3)}_{\mathcal{B}(V_2)}[T_1]^{\mathcal{B}(V_2)}_{\mathcal{B}(V_1)}$.
- Identity is also a standard feature of linear algebra: $[\mathrm{id}_V]^{\mathcal{B}(V)}_{\mathcal{B}(V)} = I_{\dim(V)}$.

We can now obtain two composition functors:

- $G \circ F \colon \mathrm{Mat}_{\mathbb{R}} \to \mathrm{Mat}_{\mathbb{R}}$.
- $F \circ G \colon \mathrm{Vect}^{fd}_{\mathbb{R}} \to \mathrm{Vect}^{fd}_{\mathbb{R}}$.

What is the functor $G \circ F$? On objects it is the identity. On morphisms, for $A \in \mathrm{Hom}_{\mathrm{Mat}_{\mathbb{R}}}(\mathbb{R}^n, \mathbb{R}^m)$, we have

$$G(F(A)) = [F(A)]^{\mathcal{B}(\mathbb{R}^m)}_{\mathcal{B}(\mathbb{R}^n)} = [F(A)]^{\mathrm{st}}_{\mathrm{st}} = A,$$

which is again a classical result in linear algebra. So $G \circ F = \mathrm{id}_{\mathrm{Mat}_{\mathbb{R}}}$.

How about the functor $F \circ G$? On objects, for $V \in \mathrm{Obj}(\mathrm{Vect}^{fd}_{\mathbb{R}})$, we have

$$F(G(V)) = F(\mathbb{R}^{\dim(V)}) = \mathbb{R}^{\dim(V)},$$

which is not necessarily equal to V. However, they are isomorphic vector spaces. On morphisms, for $T \colon V \to W$, we have

$$F(G(T)) = F([T]^{\mathcal{B}(W)}_{\mathcal{B}(V)}) = \tilde{T},$$

where $\tilde{T} \colon \mathbb{R}^{\dim(V)} \to \mathbb{R}^{\dim(W)}$ is the linear transformation defined by the matrix $[T]^{\mathcal{B}(W)}_{\mathcal{B}(V)}$. Again, $\tilde{T}$ is not necessarily equal to T, but the following diagram commutes:

$$\begin{tikzcd} \mathbb{R}^{\dim(V)} \arrow[r, "\tilde{T}"'] \arrow[d, "\cong"'] & \mathbb{R}^{\dim(W)} \arrow[d, "\cong"] \\ V \arrow[r, "T"] & W \end{tikzcd},$$

where the vertical arrows send the standard basis to the chosen basis. This is again a very standard exercise in linear algebra. So $F \circ G$ is not equal to the identity but it is naturally isomorphic to $\mathrm{id}_{\mathrm{Vect}^{fd}_{\mathbb{R}}}$.

This motivates the following fundamental definition in category theory.

Definition 7.1. *Let $\mathcal{C}, \mathcal{D}$ be two categories. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is an* equivalence of categories *if there exists a functor $G \colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ is naturally isomorphic to $\mathrm{id}_{\mathcal{C}}$ and $F \circ G$ is naturally isomorphic to $\mathrm{id}_{\mathcal{D}}$.*

From the perspective of this result we can summarize a significant part of early linear algebra as the fact that $\mathrm{Mat}_{\mathbb{R}}$ is equivalent to $\mathrm{Vect}^{fd}_{\mathbb{R}}$.

Recall that when we have a set function it might not always be easy to prove it is bijective, but constructing an inverse. Instead we use the fact that a function has an inverse if it is injective and surjective. In category theory we have the following similar result.

Theorem 7.2. *Let $\mathcal{C}, \mathcal{D}$ be two categories and $F \colon \mathcal{C} \to \mathcal{D}$ be a functor. The following are equivalent:*

- *F is an equivalence of categories.*
- *F has the following two properties:*
  - ***Fully faithful:** For every pair of objects $X, Y \in \mathrm{Obj}(\mathcal{C})$, the map*

    $$F \colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

    is a bijection of sets.
  - ***Essentially surjective:** For every object $d \in \mathrm{Obj}(\mathcal{D})$, there exists an object $c \in \mathrm{Obj}(\mathcal{C})$ such that $F(c)$ is isomorphic to d in $\mathcal{D}$.*

8. Adjunction

We have now seen isomorphisms and equivalences of categories, but how about weaker conditions that are still interesting? Here is an easy example:

Definition 8.1. *Let* $\mathcal{S}\mathrm{et}_*$ *denote the category with*

- *Objects are pairs* (X, x) *where* X *is a set and* $x \in X$*, these are called* pointed sets*.*
- *Morphisms* $(X, x) \to (Y, y)$ *are functions* $f \colon X \to Y$*, such that* $f(x) = y$*. These are called* pointed maps*.*

Can we think of an evident functor $U \colon \mathcal{S}\mathrm{et}_* \to \mathcal{S}\mathrm{et}$? Yes, of course. We can just map a pair (X, x) to the set X and keep the same morphisms. How about the other direction? Given a set X we want to associate in some easy way a set with a specific element. Here the easiest choice is to define $F(X) = (X \coprod *, *)$, and $F(f) = f \coprod \mathrm{id}_*$.

Now, what happens when I do $UF(X)$? I get $UF(X) = X \coprod *$. On the other side $FU(X) = (X \coprod *, *)$. Evidently $UF(X)$ will never be isomorphic to X, natural or otherwise, but we do have the following fact:

$$\epsilon_X \colon FU(X, x) \to (X, x), \epsilon(x') = x', \epsilon(*) = x$$

$$\eta_X \colon X \to UF(X), \eta(x') = x'$$

Moreover, for a given set X and pointed set (Y, y), we have two maps

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}_*}(FX, (Y, y)) \xrightarrow{U} \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(UFX, U(Y, y)) \xrightarrow{\eta_X} \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(X, U(Y, y))$$

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}}(X, U(Y, y)) \xrightarrow{F} \mathrm{Hom}_{\mathcal{S}\mathrm{et}_*}(FX, FU(Y, y)) \xrightarrow{\eta_X} \mathrm{Hom}_{\mathcal{S}\mathrm{et}_*}(FX, (Y, y))$$

which are inverses of each other!

So a map from $FX \to (Y, y)$ is the same thing as a map $X \to UY$. In this case this is not a particularly deep statement, but such scenarios happen a lot throughout mathematics, and so merit their definition.

Definition 8.2. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories. An adjunction is a pair of functors* $F \colon \mathcal{C} \to \mathcal{D}$*,* $G \colon \mathcal{D} \to \mathcal{C}$ *along with natural transformations* η *from the identity to* GF *and* ϵ *from* FG *to the identity. Such that the induced map*

$$\mathrm{Hom}_{\mathcal{D}}(FX, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, GY)$$

is a bijection.

Every isomorphism gives us an equivalence, and every equivalence gives us an adjunction.

Adjunctions are very ubiquitous throughout mathematics. Including in algebra where the there is often a forgetful functor, and the left adjoint is the "free construction"

9. Universal Properties

A universal property is an effort to describe an object not by what it is, but rather by what it does! Here is a sample:

The person standing in this room.

This uniquely characterizes me in contrast to everybody else, without ever mentioning who I am.

Let's try an example:

Example 9.1. *What is the set that admits a unique map from every other objects? It's the one element set. Notice here the choice is in fact not unique, but only unique up to isomorphism.*

Example 9.2. *What is the set that admits a unique map to every other set? It's the empty set!*

These type of notions are so prevalent they merit a definition.

Definition 9.3. *Let $\mathcal{C}$ be a category. An object I is* initial *if there is a unique map from I to any other objects.*

Definition 9.4. *Let $\mathcal{C}$ be a category. An object T is* terminal *if there is a unique map from every other object to T.*

Notice, we did not assume these are unique, but we can straightforwardly check.

Exercise 9.5. *Let $\mathcal{C}$ be a category and T_1, T_2 two terminal objects. Then $T_1 \cong T_2$. Similarly for initial objects.*

Initial and terminal objects are examples of *limits* and *colimits*. These are used very broadly to generate a wide variety of universal properties.

References

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