LECTURE NOTES ON CATEGORY THEORY

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Category theory is a branch of mathematics that commenced with the work of Eilenberg and Mac Lane in the 1940s [EM45], motivated by research questions in algebra and topology.

1. Definition of Category

We directly commence with the definition of a category.

Definition 1.1. *A* category $\mathcal{C}$ *consists of the following data:*
- *A class of objects* $\mathrm{Obj}(\mathcal{C})$.
- *For every pair of objects* $X, Y \in \mathrm{Obj}(\mathcal{C})$, *a set of morphisms* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.
- *For every triple of objects* $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, *a composition map*

$$- \circ - : \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

- *An identity morphism* $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ *for every object* $X \in \mathrm{Obj}(\mathcal{C})$.
- *Composition is unital: for every* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, *we have* $\mathrm{id}_Y \circ f = f$ *and* $f \circ \mathrm{id}_X = f$.
- *Composition is associative: for every* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ *and* $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$, *we have* $h \circ (g \circ f) = (h \circ g) \circ f$.

If $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we often write $f \colon X \to Y$ and call X the *domain* and Y the *codomain* of f.

Exercise 1.2. *If* e_X, e'_X *are two identity morphisms for an object* X *in a category* $\mathcal{C}$, *then* $e_X = e'_X$.

2. Examples of Categories

This is a very heavy definition, so let's look at some examples!

Example 2.1. *Let* $[0]$ *be the category with the following specification:*
- $\mathrm{Obj}([0]) = \{0\}$.
- $\mathrm{Hom}_{[0]}(0, 0) = \{\mathrm{id}_0\}$.
- $\mathrm{id}_0 \circ \mathrm{id}_0 = \mathrm{id}_0$.

Graphically, we can represent this category as a single point!

Identities are not expressed in the graphical representations.

Example 2.2. *Let* $[1]$ *be the category with the following specification:*
- $\mathrm{Obj}([1]) = \{0, 1\}$.
- $\mathrm{Hom}_{[1]}(0, 0) = \{\mathrm{id}_0\}$, $\mathrm{Hom}_{[1]}(1, 1) = \{\mathrm{id}_1\}$, $\mathrm{Hom}_{[1]}(0, 1) = \{01\}$, *and* $\mathrm{Hom}_{[1]}(1, 0) = \emptyset$.
- $\mathrm{id}_0 \circ \mathrm{id}_0 = \mathrm{id}_0$, $\mathrm{id}_1 \circ \mathrm{id}_1 = \mathrm{id}_1$, $01 \circ \mathrm{id}_0 = 01$, *and* $01 \circ \mathrm{id}_1 = 01$.

Graphically, we can represent this category as follows:

$$0 \xrightarrow{\;\;01\;\;} 1$$

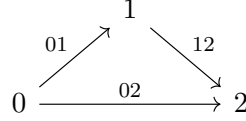*We say it is a category with two objects and one non-identity morphism.*

This is fun, so let's keep going!

Example 2.3. *Let* $[2]$ *be the category with the following specification:*

- $\mathrm{Obj}([2]) = \{0, 1, 2\}$.
- $\mathrm{Hom}_{[2]}(0,0) = \{\mathrm{id}_0\}$, $\mathrm{Hom}_{[2]}(1,1) = \{\mathrm{id}_1\}$, $\mathrm{Hom}_{[2]}(2,2) = \{\mathrm{id}_2\}$, $\mathrm{Hom}_{[2]}(0,1) = \{01\}$, $\mathrm{Hom}_{[2]}(1,2) = \{12\}$, $\mathrm{Hom}_{[2]}(0,2) = \{02\}$, *and all other hom-sets are empty.*
- *The composition is given by the identities and* $12 \circ 01 = 02$.

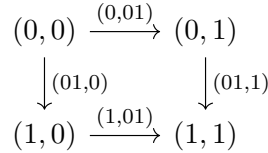*Graphically, we can represent this category as follows:*

$$
\begin{array}{ccc}
 & 1 & \\
{}^{01}\nearrow & & \searrow^{12} \\
0 \xrightarrow{\quad 02 \quad} & & 2
\end{array}
$$

We say it is a category with three objects and three non-identity morphisms.

Example 2.4. *Let* $[1] \times [1]$ *be the category with the following specification:*

- $\mathrm{Obj}([1] \times [1]) = \{(0,0), (0,1), (1,0), (1,1)\}$.
- *The hom-sets are given by*
  - *All hom-sets with domain and codomain equal only contain the identity.*
  - $\mathrm{Hom}((0,0),(0,1)) = \{(0,01)\}$, $\mathrm{Hom}((0,0),(1,0)) = \{(01,0)\}$, $\mathrm{Hom}((0,1),(1,1)) = \{(01,1)\}$, *and* $\mathrm{Hom}((1,0),(1,1)) = \{(1,01)\}$,
  - $\mathrm{Hom}((0,0),(1,1)) = \{(01,01)\}$,
  - *All other hom-sets are empty.*
- *The composition with identity are as before, and additionally we have*
  - $(01,1) \circ (0,01) = (01,01)$
  - $(1,01) \circ (01,0) = (01,01)$

Graphically, we can represent this category as follows:

$$
\begin{array}{ccc}
(0,0) & \xrightarrow{(0,01)} & (0,1) \\
{\scriptstyle(01,0)}\downarrow & & \downarrow{\scriptstyle(01,1)} \\
(1,0) & \xrightarrow{(1,01)} & (1,1)
\end{array}
$$

We say it is a category with four objects and five non-identity morphisms.

Definition 2.5. *A preorder* $(P, \leq)$ *is a set* P *along with a binary relation* $\leq$ *that is reflexive and transitive.*

Definition 2.6. *Let* P *be a preorder. We define the category* $\mathcal{C}_P$ *with the following specification:*

- $\mathrm{Obj}(\mathcal{C}_P) = P$.
- *For* $x, y \in P$, $\mathrm{Hom}_{\mathcal{C}_P}(x,y) = \{*\}$ *if* $x \leq y$ *and* $\mathrm{Hom}_{\mathcal{C}_P}(x,y) = \emptyset$ *otherwise.*
- *The composition is given by the unique element in the hom-set if it exists, which exists by transitivity.*
- *The identity is given by the unique element in the hom-set, which exists by reflexivity.*

- $\{0\}$ gives us the category $[0]$.
- $\{0 \leq 1\}$ gives us the category $[1]$.
- $\{0 \leq 1 \leq 2\}$ gives us the category $[2]$.
- $\{0 \leq 1\} \times \{0 \leq 1\}$ gives us the category $[1] \times [1]$.

Example 2.7. *Let* $(M, +, 0)$ *be a monoid. We define the category* BM *with the following specification:*

- $\mathrm{Obj}(BM) = \{*\}$.
- $\mathrm{Hom}_{BM}(*, *) = M$.
- *The composition is given by the monoid operation, i.e. for* $m, n \in M$, *we have* $m \circ n = m + n$.
- *The identity is given by the monoid identity, i.e.* $\mathrm{id}_* = 0$.

Let us some examples of this example.

Example 2.8. *Let* $(\mathbb{N}, +, 0)$ *be the monoid of natural numbers under addition. The category* $B\mathbb{N}$ *has one object and morphisms given by natural numbers, with composition given by addition. It is also called the* free endomorphism.

Example 2.9. *Let* $(\mathbb{Z}, +, 0)$ *be the monoid of integers under addition. The category* $B\mathbb{Z}$ *has one object and morphisms given by integers, with composition given by addition. It is also called the* free automorphism.

Example 2.10. *Let* $\mathcal{S}\mathrm{et}$ *be the category with the following specification:*
- $\mathrm{Obj}(\mathcal{S}\mathrm{et}) = \{X \mid X \text{ is a set}\}$.
- *For* $X, Y \in \mathrm{Obj}(\mathcal{S}\mathrm{et})$, $\mathrm{Hom}_{\mathcal{S}\mathrm{et}}(X, Y)$ *is the set of all functions from* X *to* Y.
- *The composition is given by the usual composition of functions.*
- *The identity is given by the usual identity function.*

Example 2.11. *Let* $\mathcal{G}\mathrm{rp}$ *be the category with the following specification:*
- $\mathrm{Obj}(\mathcal{G}\mathrm{rp})$ *is the class of all groups.*
- *For* $G, H \in \mathrm{Obj}(\mathcal{G}\mathrm{rp})$, $\mathrm{Hom}_{\mathcal{G}\mathrm{rp}}(G, H)$ *is the set of all group homomorphisms from* G *to* H.
- *The composition is given by the usual composition of functions.*
- *The identity is given by the usual identity function.*

Example 2.12. *Let* $\mathcal{T}\mathrm{op}$ *be the category with the following specification:*
- $\mathrm{Obj}(\mathcal{T}\mathrm{op})$ *is the class of all topological spaces.*
- *For* $X, Y \in \mathrm{Obj}(\mathcal{T}\mathrm{op})$, $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, Y)$ *is the set of all continuous functions from* X *to* Y.
- *The composition is given by the usual composition of functions.*
- *The identity is given by the usual identity function.*

These last three examples illustrate why category became so important in mathematics: Every type of mathematical object (sets, groups, topological spaces, ...) form a category.
- Set theory is the study of the category $\mathcal{S}\mathrm{et}$.
- Group theory is the study of the category $\mathcal{G}\mathrm{rp}$.
- Topology is the study of the category $\mathcal{T}\mathrm{op}$.
- ...

And there are many many other categories in mathematics, for any type of mathematical object (rings, varieties, vector spaces, modules, manifolds, ...).

3. Groupoids

In the case of $\mathcal{S}\mathrm{et}$, $\mathcal{G}\mathrm{rp}$ and $\mathcal{T}\mathrm{op}$ we have morphisms which have an inverse. Such morphisms are known as bijections, isomorphisms and homeomorphisms, respectively. This motivates the following definition.

Definition 3.1. *Let* $\mathcal{C}$ *be a category. A morphism* $f\colon X \to Y$ *is an* isomorphism *if it has an inverse.*

Definition 3.2. *Let* I *be the category with the following specification:*
- $\mathrm{Obj}(I) = \{0, 1\}$.
- $\mathrm{Hom}_I(0, 0) = \{\mathrm{id}_0\}$, $\mathrm{Hom}_I(1, 1) = \{\mathrm{id}_1\}$, $\mathrm{Hom}_I(0, 1) = \{01\}$, *and* $\mathrm{Hom}_I(1, 0) = \{10\}$.
- *The composition is given by the identities and* $10 \circ 01 = \mathrm{id}_0$ *and* $01 \circ 10 = \mathrm{id}_1$.
- *Graphically, we can represent this category as follows:*

$$0 \underset{10}{\overset{01}{\rightleftarrows}} 1$$

We call this category the free isomorphism.

Definition 3.3. *A morphism f with equal domain and codomain is called an* endomorphism. *An endomorphism that is an isomorphism is called an* automorphism.

Definition 3.4. *A* groupoid *is a category where every morphism is an isomorphism.*

Example 3.5. $[0]$ *is a groupoid. Every morphism is the identity!*

Example 3.6. $[1]$ *is not a groupoid. The morphism $01\colon 0 \to 1$ does not have an inverse. Similarly* $[2]$.

Example 3.7. *For a monoid M, BM is a groupoid if and only if M is a group. So $B\mathbb{N}$ is not a groupoid, but $B\mathbb{Z}$ is.*

Example 3.8. $\mathcal{S}\mathrm{et}$, $\mathcal{G}\mathrm{rp}$ *and* $\mathcal{T}\mathrm{op}$ *are not groupoids.*

4. Functors

If categories are like different theories of mathematics, then functors are ways to translate between these theories. Functors are for categories what functions are for sets.

Definition 4.1. *Let $\mathcal{C}, \mathcal{D}$ be two categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of the following data:*
- *A function $F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$.*
- *For every pair of objects $X, Y \in \mathrm{Obj}(\mathcal{C})$, a function $F\colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$.*
- *For every object $X \in \mathrm{Obj}(\mathcal{C})$, we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.*
- *For every triple of objects $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, and every $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, we have $F(g \circ f) = F(g) \circ F(f)$.*

Example 4.2. *Let $\mathcal{C}$ be a category. The* identity functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ *is given by the identity on objects and morphisms.*

Exercise 4.3. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be three categories and $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{E}$ be two functors. Show there is a composition functor $G \circ F\colon \mathcal{C} \to \mathcal{E}$.*

Example 4.4. *Let $\mathcal{C}\mathrm{at}$ be the category with the following specification:*
- $\mathrm{Obj}(\mathcal{C}\mathrm{at})$ *is the class of all categories.*
- *For $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathcal{C}\mathrm{at})$, $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}, \mathcal{D})$ is the set of all functors from $\mathcal{C}$ to $\mathcal{D}$.*
- *The composition is given by the usual composition of functions.*
- *The identity is given by the usual identity function.*

Example 4.5. *Let $\mathcal{C}$ be a category and c an object. There is a functor $\mathrm{Hom}_{\mathcal{C}}(c, -)\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$, which maps an object d to the set $\mathrm{Hom}_{\mathcal{C}}(c, d)$ and a morphism $f\colon d \to d'$ to the function*

$$\mathrm{Hom}_{\mathcal{C}}(c, f)\colon \mathrm{Hom}_{\mathcal{C}}(c, d) \to \mathrm{Hom}_{\mathcal{C}}(c, d')$$

given by $g \mapsto f \circ g$.

Exercise 4.6. *Let $\mathcal{C}$ be a category. Prove there is a bijection of sets $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}([0], \mathcal{C}) \cong \mathrm{Obj}_{\mathcal{C}}$.*

Exercise 4.7. *Let $\mathcal{C}$ be a category. Prove there is a bijection of sets*

$$\mathrm{Hom}_{\mathcal{C}\mathrm{at}}([1], \mathcal{C}) \cong \{\text{morphisms in } \mathcal{C}\}.$$

Exercise 4.8. *Let $\mathcal{C}$ be a category. Prove there is a bijection of sets*

$$\mathrm{Hom}_{\mathcal{C}\mathrm{at}}([2], \mathcal{C}) \cong \{\text{two composable morphisms in } \mathcal{C}\}.$$

Exercise 4.9. *Let $\mathcal{C}$ be a category. Prove there is a bijection of sets*

$$\mathrm{Hom}_{\mathcal{C}\mathrm{at}}([1] \times [1], \mathcal{C}) \cong \{\text{commutative squares in } \mathcal{C}\}.$$

Exercise 4.10. *Let $\mathcal{C}$ be a category. Prove there is a bijection of sets*

$$\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(B\mathbb{N}, \mathcal{C}) \cong \{\text{endomorphisms in } \mathcal{C}\}.$$

Exercise 4.11. *Let $\mathcal{C}$ be a category. Prove there is a bijection of sets*

$$\mathrm{Hom}_{\mathfrak{Cat}}(I, \mathcal{C}) \cong \{isomorphisms\ in\ \mathcal{C}\}.$$

Exercise 4.12. *Let $\mathcal{C}$ be a category. Prove there is a bijection of sets*

$$\mathrm{Hom}_{\mathfrak{Cat}}(B\mathbb{Z}, \mathcal{C}) \cong \{automorphisms\ in\ \mathcal{C}\}.$$

Exercise 4.13. *Prove that the assignment $U\colon \mathfrak{Grp} \to \mathfrak{Set}$ that takes a group to its underlying set, and group homomorphism to the underlying function, is a functor.*

Exercise 4.14. *Prove that the assignment $F\colon \mathfrak{Set} \to \mathfrak{Grp}$ that takes a set to the free group on that set, and similarly on functions, is a functor.*

Exercise 4.15. *Let $\mathcal{C}, \mathcal{D}$ be two categories, and d an object in $\mathcal{D}$. Prove the assignment $\Delta_d\colon \mathcal{C} \to \mathcal{D}$ that takes every object in $\mathcal{C}$ to d and every morphism in $\mathcal{C}$ to id_d is a functor.*

Exercise 4.16. *Let S be a set. Prove that $S \times -\colon \mathfrak{Set} \to \mathfrak{Set}$ that takes a set X to $S \times X$ and a function $f\colon X \to Y$ to the function $S \times f\colon S \times X \to S \times Y$ is a functor.*

Exercise 4.17. *Let $(P, \leq), (Q, \leq)$ be two preorders and $f\colon P \to Q$ be an order-preserving function. Prove that the assignment $\mathcal{C}_f\colon \mathcal{C}_P \to \mathcal{C}_Q$, which on objects is just given by f, lifts to a functor.*

Exercise 4.18. *Let M, N be two monoids. Let $f\colon M \to N$ be a monoid homomorphism. Prove $Bf\colon BM \to BN$ defined as follows is a functor*
- *The identity $\{*\} \to \{*\}$.*
- *The function $f\colon \mathrm{Hom}_{BM}(*, *) = M \to N = \mathrm{Hom}_{BN}(*, *)$.*

Exercise 4.19. *Let $\mathfrak{Set}$ denote the category of sets, let $\mathfrak{Cat}$ denote the category of small categories.*
- *(1) Let $\mathrm{Obj}\colon \mathfrak{Cat} \to \mathfrak{Set}$ be the assignment that takes a category to its set of objects, and a functor to the underlying function on objects. Prove this is a functor.*
- *(2) Let $D\colon \mathfrak{Set} \to \mathfrak{Cat}$ be the assignment that takes a set S to the discrete category on S (i.e. the category with objects S and only identity morphisms), and a function to the evident functor between discrete categories. Prove this is a functor.*
- *(3) Let $C\colon \mathfrak{Set} \to \mathfrak{Cat}$ be the assignment that takes a set S to the codiscrete category on S (i.e. the category with objects S and a unique morphism between any two objects), and a function to the evident functor between codiscrete categories. Prove this is a functor.*

Functors can be used to define isomorphisms between categories.

Definition 4.20. *A functor that has an inverse functor is called an* isomorphism of categories.

None of the functors described until now are isomorphisms of categories! Let's describe one.

Definition 4.21. *Let $\mathcal{C}$ be a category. The* opposite category $\mathcal{C}^{op}$ *is defined as follows:*
- $\mathrm{Obj}(\mathcal{C}^{op}) = \mathrm{Obj}(\mathcal{C})$
- *For objects X, Y in $\mathcal{C}^{op}$, $\mathrm{Hom}_{\mathcal{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$.*
- *The composition and identity are defined as in $\mathcal{C}$, but with the direction of morphisms reversed.*

Definition 4.22. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. The* opposite functor $F^{op}\colon \mathcal{C}^{op} \to \mathcal{D}^{op}$ *is defined as follows:*
- *On objects, F^{op} is given by F.*
- *On morphisms, for $f\colon X \to Y$ in $\mathcal{C}^{op}$ (i.e. $f\colon Y \to X$ in $\mathcal{C}$), we have $F^{op}(f) = F(f)\colon F(Y) \to F(X)$ in $\mathcal{D}^{op}$.*

Exercise 4.23. *Prove that $(-)^{op}\colon \mathfrak{Cat} \to \mathfrak{Cat}$ is a functor and that this functor is in fact an isomorphism.*

Why do we care about opposite categories? Here is a cool example.

Example 4.24. *Let $\mathcal{C}$ be a category and c an object. There is a functor $\mathrm{Hom}_{\mathcal{C}}(-,c)\colon \mathcal{C}^{op} \to \mathrm{Set}$, which maps an object d to the set $\mathrm{Hom}_{\mathcal{C}}(d,c)$ and a morphism $f\colon d' \to d$ to the function*

$$\mathrm{Hom}_{\mathcal{C}}(d,c) \to \mathrm{Hom}_{\mathcal{C}}(d',c), \quad g \mapsto g \circ f.$$

Notice the direction of the arrows indeed flips!

5. Natural Transformations

Functors are morphisms between categories. Natural transformations are morphisms between functors.

Definition 5.1. *Let $\mathcal{C}, \mathcal{D}$ be two categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\eta\colon F \to G$ consists of the following data:*

- *For every object $X \in \mathrm{Obj}(\mathcal{C})$, a morphism $\eta_X\colon F(X) \to G(X)$ in $\mathcal{D}$.*
- *For every morphism $f\colon X \to Y$ in $\mathcal{C}$, the following diagram commutes:*

$$
\begin{array}{ccc}
F(X) & \xrightarrow{\ F(f)\ } & F(Y) \\
\eta_X \downarrow & & \downarrow \eta_Y \\
G(X) & \xrightarrow[\ G(f)\ ]{} & G(Y)
\end{array}
$$

Exercise 5.2. *Let $\mathcal{C}, \mathcal{D}$ be two categories and let $F, G, H\colon \mathcal{C} \to \mathcal{D}$ be three functors. Let $\eta\colon F \to G$ and $\theta\colon G \to H$ be two natural transformations. Show there is a composition natural transformation $\theta \circ \eta\colon F \to H$.*

Definition 5.3. *Let $\mathcal{C}, \mathcal{D}$ be two categories. We define the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with the following specification:*

- $\mathrm{Obj}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ *are functors from $\mathcal{C}$ to $\mathcal{D}$.*
- *For $F, G \in \mathrm{Obj}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$, we have $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) = \mathrm{Nat}(F, G)$.*

Functor categories are themselves functorial.

Exercise 5.4. *Let $F\colon \mathcal{D} \to \mathcal{E}$ be a functor. Show there is a functor $F_*\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ defined as follows:*

- *On objects, $F_*(G) = F \circ G$ for $G \in \mathrm{Obj}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$.*
- *On morphisms, given a natural transformation $\eta\colon G \to H$ for $G, H \in \mathrm{Obj}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$, define $F_*(\eta)\colon F_*(G) \to F_*(H)$ by*

$$F_*(\eta)_X = F(\eta_X)\colon F(G(X)) \to F(H(X)).$$

Exercise 5.5. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Show there is a functor $F^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ defined as follows:*

- *On objects, $F^*(G) = G \circ F$ for $G \in \mathrm{Obj}(\mathrm{Fun}(\mathcal{D}, \mathcal{E}))$.*
- *On morphisms, given a natural transformation $\eta\colon G \to H$ for $G, H \in \mathrm{Obj}(\mathrm{Fun}(\mathcal{D}, \mathcal{E}))$, define $F^*(\eta)\colon F^*(G) \to F^*(H)$ by*

$$F^*(\eta)_X = \eta_{F(X)}\colon G(F(X)) \to H(F(X)).$$

Exercise 5.6. *Prove there is an isomorphism of categories $\mathrm{Fun}([0], \mathcal{C}) \cong \mathcal{C}$.*

Definition 5.7. *Let G be a group. Let Set_G be the category with the following specification:*

- $\mathrm{Obj}(\mathrm{Set}_G)$ *is the class of all sets with a G-action.*
- *For $X, Y \in \mathrm{Obj}(\mathrm{Set}_G)$, $\mathrm{Hom}_{\mathrm{Set}_G}(X, Y)$ is the set of all G-equivariant functions from X to Y.*

- *The composition is given by the usual composition of functions.*

Exercise 5.8. *Let G be a group. Prove there is an isomorphism of categories $\mathrm{Fun}(BG, \mathrm{Set}) \cong \mathrm{Set}_G$.*

Definition 5.9. *Let $F, G \colon \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\eta \colon F \to G$ is a* natural isomorphism *if every component $\eta_X \colon F(X) \to G(X)$ is an isomorphism in $\mathcal{D}$.*

6. Yoneda Lemma

Let us see one major result in category theory. Let $\mathcal{C}$ be a category and c an object in $\mathcal{C}$. We saw that we have functors $\mathrm{Hom}_{\mathcal{C}}(c, -) \colon \mathcal{C} \to \mathrm{Set}$.

Now, assume there is a natural transformation $\alpha \colon \mathrm{Hom}_{\mathcal{C}}(c, -) \to F$. Then we can construct one element in $F(c)$, namely $\alpha_c(\mathrm{id}_c) \in F(c)$. This gives us a morphism

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(c, -), F) \to F(c), \quad \alpha \mapsto \alpha_c(\mathrm{id}_c).$$

Turns out this map is very important.

Theorem 6.1 (Yoneda Lemma). *Let $\mathcal{C}$ be a category, c an object in $\mathcal{C}$, and $F \colon \mathcal{C} \to \mathrm{Set}$ a functor. The map*

$$\mathcal{Y}\mathrm{on} \colon \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(c, -), F) \to F(c), \quad \alpha \mapsto \alpha_c(\mathrm{id}_c)$$

is a bijection of sets.

Proof. We construct the inverse map. Let $x \in F(c)$. We define a natural transformation $\alpha^x \colon \mathrm{Hom}_{\mathcal{C}}(c, -) \to F$ as follows: for every object $d \in \mathrm{Obj}(\mathcal{C})$, define

$$\alpha_d^x \colon \mathrm{Hom}_{\mathcal{C}}(c, d) \to F(d), \quad f \mapsto F(f)(x).$$

We need to check this is indeed a natural transformation. Let $g \colon d \to d'$ be a morphism in $\mathcal{C}$. We need to show the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(c, d) & \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(c, g)} & \mathrm{Hom}_{\mathcal{C}}(c, d') \\
\downarrow{\scriptstyle \alpha_d^x} & & \downarrow{\scriptstyle \alpha_{d'}^x} \\
F(d) & \xrightarrow[F(g)]{} & F(d')
\end{array}
$$

Take $f \in \mathrm{Hom}_{\mathcal{C}}(c, d)$. Then we have

$$F(g)(\alpha_d^x(f)) = F(g)(F(f)(x)) = F(g \circ f)(x) = \alpha_{d'}^x(g \circ f) = \alpha_{d'}^x(\mathrm{Hom}_{\mathcal{C}}(c, g)(f)).$$

So the diagram commutes. Thus, we have defined a natural transformation α^x. This gives us a map

$$F(c) \to \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(c, -), F), \quad x \mapsto \alpha^x.$$

It is straightforward to check that these two maps are inverses of each other. $\qquad \square$

Here are some applications.

Exercise 6.2. *Let G be a group. Prove that $\mathrm{Hom}_{BG}(*, -) \colon BG \to \mathrm{Set}$ corresponds to the G-set with underlying set G and action given by left multiplication.*

Now, the Yoneda lemma immediately gives us the following result.

Corollary 6.3. *There is a bijection between G-equivariant functions from G to a G-set X and elements in X.*

In particular, taking $X = G$ with the action given by right multiplication, we get the following result.

Corollary 6.4 (Cayley's Theorem). *There is a bijection between G and G-equivariant bijections $G \to G$. Hence there is an injection G to the group of permutations of the set G.*

7. Equivalences

Let $\text{Vect}_{\mathbb{R}}$ be the category of real vector spaces and linear transformations. Let $\text{Vect}_{\mathbb{R}}^{fd}$ be the category of finite-dimensional real vector spaces and linear transformations. Now here is a different category: Let $\text{Mat}_{\mathbb{R}}$ be the category with the following specification:

- $\text{Obj}(\text{Mat}_{\mathbb{R}}) = \{\mathbb{R}^n \mid n \in \mathbb{N}\}$.
- For $\mathbb{R}^n, \mathbb{R}^m \in \text{Obj}(\text{Mat}_{\mathbb{R}})$, $\text{Hom}_{\text{Mat}_{\mathbb{R}}}(\mathbb{R}^n, \mathbb{R}^m)$ is the set of all $m \times n$ real matrices.
- The composition is given by matrix multiplication.

There is a functor $F \colon \text{Mat}_{\mathbb{R}} \to \text{Vect}_{\mathbb{R}}^{fd}$ defined as follows:

- On objects, $F(\mathbb{R}^n) = \mathbb{R}^n$.
- On morphisms, for $A \in \text{Hom}_{\text{Mat}_{\mathbb{R}}}(\mathbb{R}^n, \mathbb{R}^m)$, $F(A) \colon \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation defined by A.

What we learned in linear algebra is that every object in $\text{Vect}_{\mathbb{R}}^{fd}$ is isomorphic to an object in the image of F, and that F induces bijections between hom-sets. So every information about objects and morphisms in $\text{Vect}_{\mathbb{R}}^{fd}$ is already contained in $\text{Mat}_{\mathbb{R}}$, and this is the major result of linear algebra.

However, F is not an isomorphism of categories, since there are many objects in $\text{Vect}_{\mathbb{R}}^{fd}$ which are not in the image of F (e.g. the vector space of polynomials). What we need is a notion of "equivalence of categories".

Definition 7.1. *Let $\mathcal{C}, \mathcal{D}$ be two categories. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if there exists a functor $G \colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ is naturally isomorphic to $\text{id}_{\mathcal{C}}$ and $F \circ G$ is naturally isomorphic to $\text{id}_{\mathcal{D}}$.*

How is our example above an equivalence of categories? For this we have the following result.

Theorem 7.2. *Let $\mathcal{C}, \mathcal{D}$ be two categories and $F \colon \mathcal{C} \to \mathcal{D}$ be a functor. The following are equivalent:*

- *F is an equivalence of categories.*
- *F has the following two properties:*
  - ***Fully faithful:** For every pair of objects $X, Y \in \text{Obj}(\mathcal{C})$, the map*

  $$F \colon \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

  is a bijection of sets.
  - ***Essentially surjective:** For every object $d \in \text{Obj}(\mathcal{D})$, there exists an object $c \in \text{Obj}(\mathcal{C})$ such that $F(c)$ is isomorphic to d in $\mathcal{D}$.*

Example 7.3. *The functor $F \colon \text{Mat}_{\mathbb{R}} \to \text{Vect}_{\mathbb{R}}^{fd}$ described above is an equivalence of categories, that is not an isomorphism. By the theorem, there exists a functor $G \colon \text{Vect}_{\mathbb{R}}^{fd} \to \text{Mat}_{\mathbb{R}}$ such that $G \circ F = \text{id}_{\text{Mat}_{\mathbb{R}}}$, and $F \circ G$ is naturally isomorphic to $\text{id}_{\text{Vect}_{\mathbb{R}}^{fd}}$.*

References

[EM45] Samuel Eilenberg and Saunders MacLane. General theory of natural equivalences. *Trans. Amer. Math. Soc.*, 58:231–294, 1945.