LECTURE NOTES ON CATEGORY THEORY (DRAFT)

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Category theory is a branch of mathematics that commenced with the work of Eilenberg and Mac Lane in the 1940s [EM45], motivated by research questions in algebra and topology.

1. Definition of Category

We directly commence with the definition of a category.

Definition 1.1. *A* category $\mathcal{C}$ *consists of the following data:*

- *A class of objects* $\mathrm{Obj}(\mathcal{C})$*.*
- *For every pair of objects* $X, Y \in \mathrm{Obj}(\mathcal{C})$*, a set of morphisms* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$*.*
- *For every triple of objects* $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$*, a composition map*
$$- \circ - \colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$
- *An identity morphism* $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ *for every object* $X \in \mathrm{Obj}(\mathcal{C})$*.*
- *Composition is unital: for every* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$*, we have* $\mathrm{id}_Y \circ f = f$ *and* $f \circ \mathrm{id}_X = f$*.*
- *Composition is associative: for every* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$*,* $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ *and* $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$*, we have* $h \circ (g \circ f) = (h \circ g) \circ f$*.*

If $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we often write $f \colon X \to Y$ and call X the *domain* and Y the *codomain* of f.

Exercise 1.2. *If* e_X, e'_X *are two identity morphisms for an object* X *in a category* $\mathcal{C}$*, then* $e_X = e'_X$*.*

2. Examples of Categories

This is a very heavy definition, so let's look at some examples!

Example 2.1. *Let* $[0]$ *be the category with the following specification:*

- $\mathrm{Obj}([0]) = \{0\}$.
- $\mathrm{Hom}_{[0]}(0, 0) = \{\mathrm{id}_0\}$.
- $\mathrm{id}_0 \circ \mathrm{id}_0 = \mathrm{id}_0$.

Graphically, we can represent this category as a single point!

Identities are not expressed in the graphical representations.

Example 2.2. *Let* $[1]$ *be the category with the following specification:*

- $\mathrm{Obj}([1]) = \{0, 1\}$.
- $\mathrm{Hom}_{[1]}(0, 0) = \{\mathrm{id}_0\}$, $\mathrm{Hom}_{[1]}(1, 1) = \{\mathrm{id}_1\}$, $\mathrm{Hom}_{[1]}(0, 1) = \{01\}$, *and* $\mathrm{Hom}_{[1]}(1, 0) = \emptyset$.
- $\mathrm{id}_0 \circ \mathrm{id}_0 = \mathrm{id}_0$, $\mathrm{id}_1 \circ \mathrm{id}_1 = \mathrm{id}_1$, $01 \circ \mathrm{id}_0 = 01$, *and* $01 \circ \mathrm{id}_1 = 01$.

Graphically, we can represent this category as follows:

$$0 \xrightarrow{01} 1$$

We say it is a category with two objects and one non-identity morphism.

This is fun, so let's keep going!

Example 2.3. *Let* $[2]$ *be the category with the following specification:*

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- $\mathrm{Obj}([2]) = \{0, 1, 2\}$.
- $\mathrm{Hom}_{[2]}(0,0) = \{\mathrm{id}_0\}$, $\mathrm{Hom}_{[2]}(1,1) = \{\mathrm{id}_1\}$, $\mathrm{Hom}_{[2]}(2,2) = \{\mathrm{id}_2\}$, $\mathrm{Hom}_{[2]}(0,1) = \{01\}$, $\mathrm{Hom}_{[2]}(1,2) = \{12\}$, $\mathrm{Hom}_{[2]}(0,2) = \{02\}$, *and all other hom-sets are empty.*
- *The composition is given by the identities and* $12 \circ 01 = 02$.

Graphically, we can represent this category as follows:

$$\begin{tikzcd} & 1 \arrow[dr, "12"] & \\ 0 \arrow[ur, "01"] \arrow[rr, "02"] & & 2 \end{tikzcd}$$

We say it is a category with three objects and three non-identity morphisms.

Example 2.4. *Let* $[1] \times [1]$ *be the category with the following specification:*

- $\mathrm{Obj}([1] \times [1]) = \{(0,0), (0,1), (1,0), (1,1)\}$.
- *The hom-sets are given by*
  - *All hom-sets with domain and codomain equal only contain the identity.*
  - $\mathrm{Hom}((0,0),(0,1)) = \{(0,01)\}$, $\mathrm{Hom}((0,0),(1,0)) = \{(01,0)\}$, $\mathrm{Hom}((0,1),(1,1)) = \{(01,1)\}$, *and* $\mathrm{Hom}((1,0),(1,1)) = \{(1,01)\}$,
  - $\mathrm{Hom}((0,0),(1,1)) = \{(01,01)\}$,
  - *All other hom-sets are empty.*
- *The composition with identity are as before, and additionally we have*
  - $(01,1) \circ (0,01) = (01,01)$
  - $(1,01) \circ (01,0) = (01,01)$

Graphically, we can represent this category as follows:

$$\begin{tikzcd} (0,0) \arrow[r, "(0{,}01)"] \arrow[d, "(01{,}0)"] & (0,1) \arrow[d, "(01{,}1)"] \\ (1,0) \arrow[r, "(1{,}01)"] & (1,1) \end{tikzcd}$$

We say it is a category with four objects and five non-identity morphisms.

Definition 2.5. *A poset* $(P, \leq)$ *is a set* P *along with a binary relation* $\leq$ *that is reflexive, antisymmetric and transitive.*

Definition 2.6. *Let* P *be a poset. We define the category* $\mathcal{C}_P$ *with the following specification:*

- $\mathrm{Obj}(\mathcal{C}_P) = P$.
- *For* $x, y \in P$, $\mathrm{Hom}_{\mathcal{C}_P}(x,y) = \{*\}$ *if* $x \leq y$ *and* $\mathrm{Hom}_{\mathcal{C}_P}(x,y) = \emptyset$ *otherwise.*
- *The composition is given by the unique element in the hom-set if it exists.*
- *The identity is given by the unique element in the hom-set.*

- $\{0\}$ gives us the category $[0]$.
- $\{0 \leq 1\}$ gives us the category $[1]$.
- $\{0 \leq 1 \leq 2\}$ gives us the category $[2]$.
- $\{0 \leq 1\} \times \{0 \leq 1\}$ gives us the category $[1] \times [1]$.

Example 2.7. *Let* $(M, +, 0)$ *be a monoid. We define the category* BM *with the following specification:*

- $\mathrm{Obj}(BM) = \{*\}$.
- $\mathrm{Hom}_{BM}(*, *) = M$.
- *The composition is given by the monoid operation, i.e. for* $m, n \in M$, *we have* $m \circ n = m + n$.
- *The identity is given by the monoid identity, i.e.* $\mathrm{id}_* = 0$.

Example 2.8. *Let* $\mathcal{S}\mathrm{et}$ *be the category with the following specification:*

- $\mathrm{Obj}(\mathcal{S}\mathrm{et}) = \{X \mid X \text{ is a set}\}$.
- *For* $X, Y \in \mathrm{Obj}(\mathcal{S}\mathrm{et})$, $\mathrm{Hom}_{\mathcal{S}\mathrm{et}}(X, Y)$ *is the set of all functions from* X *to* Y.
- *The composition is given by the usual composition of functions.*
- *The identity is given by the usual identity function.*

Example 2.9. *Let* $\mathcal{G}\mathrm{rp}$ *be the category with the following specification:*

- $\mathrm{Obj}(\mathcal{G}\mathrm{rp})$ *is the class of all groups.*
- *For* $G, H \in \mathrm{Obj}(\mathcal{G}\mathrm{rp})$, $\mathrm{Hom}_{\mathcal{G}\mathrm{rp}}(G, H)$ *is the set of all group homomorphisms from* G *to* H.
- *The composition is given by the usual composition of functions.*
- *The identity is given by the usual identity function.*

Example 2.10. *Let* $\mathcal{T}\mathrm{op}$ *be the category with the following specification:*

- $\mathrm{Obj}(\mathcal{T}\mathrm{op})$ *is the class of all topological spaces.*
- *For* $X, Y \in \mathrm{Obj}(\mathcal{T}\mathrm{op})$, $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, Y)$ *is the set of all continuous functions from* X *to* Y.
- *The composition is given by the usual composition of functions.*
- *The identity is given by the usual identity function.*

These last three examples illustrate why category became so important in mathematics: Every type of mathematical object (sets, groups, topological spaces, ...) form a category.

- Set theory is the study of the category $\mathcal{S}\mathrm{et}$.
- Group theory is the study of the category $\mathcal{G}\mathrm{rp}$.
- Topology is the study of the category $\mathcal{T}\mathrm{op}$.
- ...

And there are many many other categories in mathematics, for any type of mathematical object (rings, varieties, vector spaces, modules, manifolds, ...).

3. Groupoids

In the case of $\mathcal{S}\mathrm{et}$, $\mathcal{G}\mathrm{rp}$ and $\mathcal{T}\mathrm{op}$ we have morphisms which have an inverse. Such morphisms are known as bijections, isomorphisms and homeomorphisms, respectively. This motivates the following definition.

Definition 3.1. *Let* $\mathcal{C}$ *be a category. A morphism* $f \colon X \to Y$ *is an* isomorphism *if it has an inverse.*

Definition 3.2. *A* groupoid *is a category where every morphism is an isomorphism.*

Example 3.3. $[0]$ *is a groupoid. Every morphism is the identity!*

Example 3.4. $[1]$ *is not a groupoid. The morphism* $01 \colon 0 \to 1$ *does not have an inverse. Similarly* $[2]$.

Example 3.5. *For a monoid* M, BM *is a groupoid if and only if* M *is a group.*

Example 3.6. $\mathcal{S}\mathrm{et}$, $\mathcal{G}\mathrm{rp}$ *and* $\mathcal{T}\mathrm{op}$ *are not groupoids.*

4. Functors

Functors are for categories what functions are for sets.

Definition 4.1. *Let* $\mathcal{C}, \mathcal{D}$ *be two categories. A* functor $F \colon \mathcal{C} \to \mathcal{D}$ *consists of the following data:*

- *A function* $F \colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$.
- *For every pair of objects* $X, Y \in \mathrm{Obj}(\mathcal{C})$, *a function* $F \colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$.
- *For every object* $X \in \mathrm{Obj}(\mathcal{C})$, *we have* $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
- *For every triple of objects* $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, *and every* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ *and* $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, *we have* $F(g \circ f) = F(g) \circ F(f)$.

We denote the set of functors by $\mathrm{Fun}(\mathcal{C},\mathcal{D})$*.*

Example 4.2. *Let* $\mathcal{C}$ *be a category. The* identity functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C}\to\mathcal{C}$ *is given by the identity on objects and morphisms.*

Example 4.3. *Let* M,N *be two monoids. Let* $f\colon M\to N$ *be a function. We have the data* $Bf\colon BM\to BN$ *given by*

- *The identity* $\{*\}\to\{*\}$*.*
- *The function* $f\colon \mathrm{Hom}_{BM}(*,*)=M\to N=\mathrm{Hom}_{BN}(*,*)$*.*

Exercise 4.4. *This assignment is a functor if and only if* f *is a monoid homomorphism.*

Exercise 4.5. *Let* $\mathcal{C},\mathcal{D},\mathcal{E}$ *be three categories and* $F\colon\mathcal{C}\to\mathcal{D}$*,* $G\colon\mathcal{D}\to\mathcal{E}$ *be two functors. Show there is a composition functor* $G\circ F\colon\mathcal{C}\to\mathcal{E}$*.*

Example 4.6. *Let* $\mathcal{C}\mathrm{at}$ *be the category with the following specification:*

- $\mathrm{Obj}(\mathcal{C}\mathrm{at})$ *is the class of all categories.*
- *For* $\mathcal{C},\mathcal{D}\in\mathrm{Obj}(\mathcal{C}\mathrm{at})$*,* $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C},\mathcal{D})$ *is the set of all functors from* $\mathcal{C}$ *to* $\mathcal{D}$*.*
- *The composition is given by the usual composition of functions.*
- *The identity is given by the usual identity function.*

Example 4.7. *Let* $\mathcal{C}$ *be a category and* c *an object. There is a functor* $\mathrm{Hom}_{\mathcal{C}}(c,-)\colon\mathcal{C}\to\mathcal{S}\mathrm{et}$*, which maps an object* d *to the set* $\mathrm{Hom}_{\mathcal{C}}(c,d)$ *and a morphism* $f\colon d\to d'$ *to the function*

$$\mathrm{Hom}_{\mathcal{C}}(c,f)\colon \mathrm{Hom}_{\mathcal{C}}(c,d)\to\mathrm{Hom}_{\mathcal{C}}(c,d')$$

given by $g\mapsto f\circ g$*.*

Exercise 4.8. *Let* $\mathcal{C}$ *be a category. Prove there is a bijection of sets* $\mathrm{Obj}_{\mathcal{C}}\cong\mathrm{Fun}([0],\mathcal{C})$*.*

Exercise 4.9. *Let* $\mathcal{C}$ *be a category. Prove there is a bijection of sets* $\mathrm{Fun}([1],\mathcal{C})\cong\{$*morphisms in* $\mathcal{C}\}$*.*

Exercise 4.10. *Prove that the assignment* $U\colon\mathcal{G}\mathrm{rp}\to\mathcal{S}\mathrm{et}$ *that takes a group to its underlying set is a functor.*

5. Natural Transformations

Functors are morphisms between categories. Natural transformations are morphisms between functors.

Definition 5.1. *Let* $\mathcal{C},\mathcal{D}$ *be two categories and let* $F,G\colon\mathcal{C}\to\mathcal{D}$ *be two functors. A* natural transformation $\eta\colon F\to G$ *consists of the following data:*

- *For every object* $X\in\mathrm{Obj}(\mathcal{C})$*, a morphism* $\eta_X\colon F(X)\to G(X)$ *in* $\mathcal{D}$*.*
- *For every morphism* $f\colon X\to Y$ *in* $\mathcal{C}$*, the following diagram commutes:*

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

Exercise 5.2. *Let* $\mathcal{C},\mathcal{D}$ *be two categories and let* $F,G,H\colon\mathcal{C}\to\mathcal{D}$ *be three functors. Let* $\eta\colon F\to G$ *and* $\theta\colon G\to H$ *be two natural transformations. Show there is a composition natural transformation* $\theta\circ\eta\colon F\to H$*.*

Definition 5.3. *Let* $\mathcal{C},\mathcal{D}$ *be two categories. We define the category* $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ *with the following specification:*

- $\mathrm{Obj}(\mathrm{Fun}(\mathcal{C},\mathcal{D}))$ *are functors from* $\mathcal{C}$ *to* $\mathcal{D}$*.*
- *For* $F,G\in\mathrm{Obj}(\mathrm{Fun}(\mathcal{C},\mathcal{D}))$*, we have* $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F,G)=\mathrm{Nat}(F,G)$*.*

6. Yoneda Lemma

Let us see one cool result in category theory.

References

[EM45] Samuel Eilenberg and Saunders MacLane. General theory of natural equivalences. *Trans. Amer. Math. Soc.*, 58:231–294, 1945.