

DIFFERENTIAL COHOMOLOGY SEMINAR 4 (DRAFT)

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In the last talk we learned the definition of a differential cohomology theory, as a sheaf valued in spectra on the site of manifolds. This talk continues our journey through differential cohomology theories, and focuses on the following three topics:

- (1) We want to learn how to construct non-trivial examples out of sheaves valued in chain complexes.
- (2) We want to understand how we can extend classical cohomology operations to the setting of differential cohomology theories.
- (3) We want to introduce suitable analogues of fiber-wise integration.

1. ABELIAN GROUPS, SPECTRA AND THE HEART

Let us start by reviewing the relation between abelian groups, rings and spectra.

Definition 2. Let $n \in \mathbb{Z}$ and X be a spectrum, define $\pi_n(X) := \pi_0(\Omega^{\infty+n}X) = \pi_0(X_{-n})$. We call π_n the n -th homotopy group of X .

Remark 3. Note that since $X_n \simeq \Omega^2 X_{n+2}$, for any n , the set $\pi_0(X_n)$ underlies the structure of an abelian group.

The category Sp underlies the structure of a symmetric monoidal ∞ -category ([Lur17, Corollary 4.8.2.19]). Following [Lur17], we denote by \otimes the tensor product on Sp .

Definition 4. A commutative algebra object in Sp is called an \mathbb{E}_∞ -ring spectrum, see [Lur17, Definition 7.1.0.1]. Given an \mathbb{E}_∞ -ring spectrum R , denote by Mod_R the corresponding category of left R -module spectra, see [Lur17, Definition 7.1.1.2].

Remark 5. The sphere spectrum \mathbb{S} acts as the monoidal unit of Sp , therefore it is a \mathbb{E}_∞ -ring spectrum. The category $\mathsf{Mod}_{\mathbb{S}}$ is canonically equivalent to Sp .

Definition 6. Denote by $\mathsf{Sp}_{\geq 0} \subseteq \mathsf{Sp}$ the full sub-category generated by *connective spectra*, i.e. spectra X such that $\pi_n(X) \simeq 0$, for all $n < 0$. Denote by $\mathsf{Sp}^\heartsuit \subseteq \mathsf{Sp}_{\geq 0}$ the *heart of spectra*, i.e. the full sub-category generated by spectra X such that $\pi_n(X) \simeq 0$, for all $n > 0$.

We have the following result relating connective spectra and the heart, which follow immediately.

Lemma 7. *Let X be a connective spectrum. The following are equivalent:*

- (1) X is in the heart.
- (2) $\pi_n(\Omega^\infty X) = 0$, for all $n > 0$.
- (3) $\mathrm{Hom}_{\mathbb{S}_*}(S, \Omega^\infty X) \simeq 0$, for all connected, pointed spaces S .
- (4) X is local with respect to the class of maps $\Sigma^\infty S \rightarrow 0$, for every connected pointed space S .

The category $\mathsf{Sp}_{\geq 0}$ is presentable and π_0 induces an equivalence between the heart and Ab ([Lur17, Proposition 1.4.3.6]). The heart is a sub-category of local objects of connective spectra, therefore the inclusion $\mathsf{Ab} \simeq \mathsf{Sp}^\heartsuit \subseteq \mathsf{Sp}_{\geq 0}$ is a right adjoint. The category $\mathsf{Sp}_{\geq 0}$ is closed under \otimes and, given X, Y connective spectra,

$$(1) \quad \pi_0(X \otimes Y) \simeq \pi_0(X) \otimes \pi_0(Y)$$

see [Dav24, Theorem 2.3.28]

Definition 8. Given an abelian group A , denote by HA the (unique up to equivalence) spectrum of the heart such that $\pi_0(HA) \simeq A$. We call HA the *Eilenberg-Mac Lane spectrum* of A .

Using [Equation \(1\)](#), one can prove H , viewed as a functor $\mathcal{A}b \rightarrow \mathcal{S}p$, is lax monoidal. In particular, if R is a commutative ring, then HR is a connective \mathbb{E}_∞ -ring spectrum. On the other hand, if R is a connective \mathbb{E}_∞ -ring spectrum and M a connective module, then $\pi_0(M)$ is a $\pi_0(R)$ -module.

Definition 9. Given a commutative ring R , denote by $\text{Ch}(R) = \text{Ch}(\text{Mod}_R)$ the ordinary category of unbounded chain complexes. Let $\mathcal{D}(R)$ be the ∞ -localization of $\text{Ch}(R)$ at the class of quasi-isomorphisms.

Similar to the heart of spectra, given an \mathbb{E}_∞ -ring spectrum R , denote by $\text{Mod}_R^\heartsuit \subseteq \text{Mod}_R$ the full subcategory generated by R -modules such that the underlying spectrum belongs to the heart of spectra.

Theorem 10. *Let R be a commutative ring.*

- (1) $\text{Mod}_R \simeq \text{Mod}_{HR}^\heartsuit$ *via taking Eilenberg-Mac Lane spectra.*
- (2) *The equivalence in (1) extends to an equivalence $\mathcal{D}(R) \simeq \text{Mod}_{HR}$ of symmetric monoidal, stable ∞ -categories.*

Proof. (1) is [[Lur17](#), Proposition 7.1.1.13], while (2) is [[Lur17](#), Theorem 7.1.2.13]. □

11. FROM CHAIN COMPLEXES TO SPECTRA VIA STABLE DOLD-KAN

We now use our newly gained understanding of spectra to construct differential cohomology theories out of sheaves valued in chain complexes, via the stable Dold-Kan correspondence.

Remark 12. We identify the category of cochain complexes with $\text{Ch}(R)$ by reversing grading. Namely, given a cochain V^* , we are implicitly identifying it with the chain complex $V_n = V^{-n}$.

Definition 13 ([[BNV16](#), Definition 7.14]). Given $n \in \mathbb{Z}$, denote by $\sigma^{\geq n}$, resp. $\sigma^{\leq n}$, the *naive truncation functors*, mapping a cochain complex V^* to

$$\cdots \rightarrow 0 \rightarrow V^n \rightarrow V^{n+1} \rightarrow \cdots$$

resp.

$$\cdots \rightarrow V^{n-1} \rightarrow V^n \rightarrow 0 \rightarrow \cdots$$

Recall that a *sheaf of C^∞ -modules* is a sheaf of abelian groups F such that $F(M)$ is a $C^\infty(M)$ -module, naturally in M .

Lemma 14 ([[BNV16](#), Lemma 7.12]). *Let $F : \text{Mfd}^{op} \rightarrow \text{Ch}(\mathbb{Z})$ a sheaf of chain complex of C^∞ -modules, then $\text{Mfd}^{op} \xrightarrow{F} \text{Ch}(\mathbb{Z}) \rightarrow \mathcal{D}(\mathbb{Z})$ is a sheaf.*

Definition 15. Denote by Ω^* the sheaf $\text{Mfd}^{op} \rightarrow \text{Ch}(\mathbb{Z})$ mapping a manifold to its de Rham complex. Given $n \in \mathbb{Z}$, let $\Omega^{\geq n} := \sigma^{\geq n}\Omega^*$, resp. $\Omega^{\leq n} := \sigma^{\leq n}\Omega^*$.

[Section 14](#) ensures that the sheaves in [Section 15](#) remain sheaves after post-composition with the localization functor $\text{Ch}(\mathbb{Z}) \rightarrow \mathcal{D}(\mathbb{Z})$.

Definition 16. Given a sheaf $F : \text{Mfd}^{op} \rightarrow \mathcal{D}(\mathbb{Z})$, denote by HF the *Eilenberg-Mac Lane sheaf* of $H\mathbb{Z}$ -module spectra obtained by applying point-wise the equivalence of [Section 10](#).

17. DELIGNE COHOMOLOGY AS A DIFFERENTIAL COHOMOLOGY THEORY

Now equipped with [Section 15](#), we can finally define Deligne cohomology as a differential cohomology theory.

Definition 18. Let $k \geq 0$. The *Deligne cohomology sheaf* $\mathcal{E}(k)$ is defined via the following pullback square in $\text{Shv}(\text{Mfd}; \mathcal{S}p)$:

$$\begin{array}{ccc} \mathcal{E}(k) & \longrightarrow & H(\Omega_{dR}^{\leq k}) \\ \downarrow & & \downarrow \\ H\mathbb{Z} & \longrightarrow & H\mathbb{R} \end{array}$$

Here H is the Eilenberg-MacLane sheaf.

Remark 19. If we take $k = \infty$, then the Hom $H(\Omega_{dR}) \rightarrow H\mathbb{R}$ is an equivalence, meaning $\mathcal{E}(\infty)$ is equivalent to $H\mathbb{Z}$ i.e. singular cohomology. On the other side, the individual $\mathcal{E}(k)$ are highly non-trivial and help classify many geometric invariants of interest (as we saw in the first talk). So, the $\mathcal{E}(k)$ are a non-trivial filtration of $H\mathbb{Z}$ by differential cohomology theories, in the sense that there are Hom $\mathcal{E}(k+1) \rightarrow \mathcal{E}(k)$, the limit of which is $H\mathbb{Z}$.

20. COHOMOLOGY OPERATIONS FOR DELIGNE COHOMOLOGY

Now that we have a rigorous definition of Deligne cohomology, we can start to think about operations on it. First of all, we need a suitable monoidal structure.

Definition 21. Let F, G be two differential cohomology theories. The *monoidal product* $F \otimes G$ is defined as the sheafification of the presheaf $F \wedge G$, which is the point-wise wedge product of spectra.

It is expected that sheafification is necessary, but example is missing.

Now, recall there is a Hom of differential forms

$$\Omega^{\leq k} \otimes \Omega^{\leq m} \rightarrow \Omega^{\leq k+m},$$

which induces a Hom of differential cohomology theories

$$\mathcal{E}(k) \otimes \mathcal{E}(m) \rightarrow \mathcal{E}(k+m)$$

Ideally, we would like to describe such an operation in a very explicit manner, however, in the realm of spectra this can be very challenging. This suggests an alternative perspective.

Definition 22. Let $\mathcal{L}(k)$ be the sheaf of chain complexes defined as the pullback in $\text{Shv}(\text{Mfd}, D(\mathbb{Z}))$ of the following diagram

$$\begin{array}{ccc} \mathcal{L}(k) & \longrightarrow & \Omega^{\leq k} \\ \downarrow & & \downarrow dR \\ \mathbb{Z} & \longrightarrow & \mathbb{R} \end{array}$$

where \mathbb{Z} is the functor $M \mapsto C^\bullet(M, \mathbb{Z})$ and \mathbb{R} is the functor $M \mapsto C^\bullet(M, \mathbb{R})$

Remark 23. We can explicitly describe the chain complex $\mathcal{L}(k)$ as follows.

$$\mathcal{L}(k)^n = \{(c, \omega, h) \in C^n(-\mathbb{Z}) \oplus \Omega^n(-) \oplus C^{n-1}(-\mathbb{R}) \mid \omega = 0 \text{ if } n > k \text{ and } c - dR(\omega) = dh\}$$

Remark 24. We expect that $H\mathcal{L}(k)$ in fact recovers $\mathcal{E}(k)$, meaning operations on $\mathcal{L}(k)$ help us understand operations on Deligne cohomology.

This needs to be checked.

Using the explicit description from [Section 23](#), we can define an operation on $\mathcal{L}(k)$ as follows:

$$(c_1, \omega_1, h_1) \otimes (c_2, \omega_2, h_2) = (c_1 \cup c_2, \omega_1 \wedge \omega_2, (-1)^{|c_1|} c_1 \cup h_2 + h_1 \cup \omega_2 + B(\omega_1, \omega_2)),$$

where

$$dR(\omega_1) \cup dR(\omega_2) = -dR(\omega_1 \wedge \omega_2) = dB(\omega_1, \omega_2)$$

Remark 25. Intuitively $B(\omega_1, \omega_2)$ measures the failure of dR taking \wedge to \cup .

Is there a reasonable way to pick $B(\omega_1, \omega_2)$?

Remark 26. Ideally we would expect this formula to be well-defined, meaning $(c_1, \omega_1, h_1) \otimes (c_2, \omega_2, h_2)$ should satisfy the conditions in [Section 23](#). In general, this is only true if c_1, ω_2 satisfy $dc_1 = d\omega_2 = 0$. In particular, it is well-defined at the level of cohomology classes, as any element is closed therein.

REFERENCES

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- [Lur17] Jacob Lurie. Higher algebra. [Available online](#), September 2017.