DIFFERENTIAL COHOMOLOGY SEMINAR 3

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In this lecture we cover the basics of sheaf theory and learn about differential cohomology theories as sheaves of spectra.

1. Sheaves in Category Theory

Let us review sheaves in classical category theory. A good reference remains [MLM94]. We start with the case of sheaves on a topological space.

Definition 1. Let X be a topological space. A *presheaf* on X is a contravariant functor $\mathcal{F} : (\mathcal{O}\mathrm{pen}_X)^{op} \to \mathcal{S}\mathrm{et}$. A presheaf is a *sheaf* if for every open cover U_i of an open subset U, $F(U)$ is the limit of the diagram $F(U_i)$, meaning there is an equalizer diagram

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

We can now easily generalize this definition to the case of categories. Naively, we can try the following: Let $\mathcal{C}$ be a category with finite limits. A *presheaf* on $\mathcal{C}$ is a contravariant functor $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$. A presheaf is a *sheaf* if for every (suitably defined notion of) covering family U_i of an object U, $F(U)$ is similarly the limit of the diagram $F(U_i)$. Here $U_i \to U$ is a covering $K(C)$. It is given axiomatically for every object C in $\mathcal{C}$.

More explicitly, for a given pullback square (assuming $\mathcal{C}$ has pullbacks)

$$\begin{tikzcd} C_i \times_C C_j \arrow[r] \arrow[d] & C_i \arrow[d] \\ C_j \arrow[r] & C \end{tikzcd}$$

the induced square via F is a pullback square

$$\begin{tikzcd} F(C) \arrow[r] \arrow[d] & F(C_i) \arrow[d] \\ F(C_j) \arrow[r] & F(C_i \times_C C_j) \end{tikzcd}$$

Hence, it suitably determines the elements in $F(C)$. For this naive characterization to work, the covering needs to have some nice properties.

This means we need to take a step back and carefully define a correct notion of covering families in arbitrary categories.

2. Sheaves via Grothendieck Topologies

We now generalize from sheaves on a topological space to sheaves on a category. The idea is to use Grothendieck topologies, which allow us to define a notion of covering families in an arbitrary category. Let us start with the definition of a *sieve*.

Definition 2. Let $\mathcal{C}$ be a category. A *sieve* on an object C is a full sub-functor of $\mathrm{Hom}(-, C)\colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$.

Definition 3. Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is a collection of sieves $J(C)$ on every object C in $\mathcal{C}$ satisfying the following axioms:

(1) The maximal sieve is a covering sieve.

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(2) If S is a covering sieve and $f\colon C' \to C$ is a morphism in $\mathcal{C}$, then the pullback sieve f^*S is also a covering sieve.
(3) If S is a sieve and for some covering sieve S', for all f^*S is a covering sieve for all $f \in S'$, then S is a covering sieve.

Definition 4. A *Grothendieck site* is a pair $(\mathcal{C}, J)$ where $\mathcal{C}$ is a category and J is a Grothendieck topology on $\mathcal{C}$.

Example 5. For every category $\mathcal{C}$, there is a Grothendieck topology J on $\mathcal{C}$ given by the collection of all sieves. This is called the *discrete Grothendieck topology.*

Example 6. For every category $\mathcal{C}$, there is a Grothendieck topology J on $\mathcal{C}$ given by only maximal sieves. This is known as the *indiscrete Grothendieck topology.*

Example 7. Let $\mathcal{T}\mathrm{op}$ be the category of topological spaces and continuous maps. There is a covering sieve given by the open covers.

Example 8. Let $\mathcal{M}\mathrm{fd}$ be the category of manifolds and smooth maps. There is a covering sieve given by the open embeddings, such that their image is an open cover.

Example 9. Let X be a topological space. There is a covering sieve of U is given by a collection of open sets that are a covering.

We are now ready to define sheaves on a Grothendieck site.

Definition 10. Let $(\mathcal{C}, J)$ be a Grothendieck site. A *presheaf* on $\mathcal{C}$ is a contravariant functor $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$. A presheaf is a *sheaf* if for every covering sieve S of an object C, the canonical morphism

$$F(C) \to \lim_{f_i\colon C_i \to C \in S} F(C_i)$$

is an isomorphism. More explicitly, an element in $\lim_{f_i\colon c_i \to c \in S} F(c_i)$ is a collection of elements $x_i \in F(c_i)$ such that for every morphism $f_{ij}\colon c_i \to c_j$ in S, $F(f_{ij})x_j = x_i$ and such a data should correspond uniquely to an element in $F(c)$.

We can equivalently characterize this limit as a natural transformation $S \to F$, where we here consider S as a sub-functor of $\mathrm{Hom}(-, c)$. This immediately gives us the following lemma.

Lemma 11. *A presheaf $\mathcal{F}$ on a Grothendieck site $(\mathcal{C}, J)$ is a sheaf if and only if for every covering sieve S of an object C, the canonical morphism*

$$F(C) = \mathrm{Nat}_{\mathcal{C}}(\mathrm{Hom}(-, c), F) \to \mathrm{Nat}_{\mathcal{C}}(S, F)$$

is an isomorphism.

3. Sheaves on ∞-Categories

Now, that we know hwo to define sheaves on 1-categories, we now generalize the definition of sheaves to the case of ∞-categories. Here we largely follow [Lur09].

Definition 12. Let $\mathcal{C}$ be an ∞-category. A sieve on $\mathcal{C}$ is precisely a sieve on $\mathrm{Ho}(\mathcal{C})$, the homotopy category of $\mathcal{C}$.

Unwinding this, a sieve for an ∞-category $\mathcal{C}$ on an object C, is a full sub-∞-category of $\mathcal{C}_{/C}$, closed under pre-composition.

Definition 13. Let $\mathcal{C}$ be an ∞-category. A *Grothendieck topology* on $\mathcal{C}$ is a collection of sieves $J(C)$ on every object C in $\mathcal{C}$ satisfying the following axioms:

(1) The maximal sieve is a covering sieve.
(2) If S is a covering sieve and $f\colon C' \to C$ is a morphism in $\mathcal{C}$, then the pullback sieve f^*S is also a covering sieve.
(3) If S is a sieve and for some covering sieve S', for all f^*S is a covering sieve for all $f \in S'$, then S is a covering sieve.

Notice, we have the following compatibility observation.

Lemma 14. *Let $\mathcal{C}$ be a 1-category. A Grothendieck topology on $\mathcal{C}$ is precisely a site on $\mathcal{C}$ seen as an ∞-category.*

Definition 15. Let $\mathcal{C}$ be an ∞-category. A *presheaf* is a functor $F : \mathcal{C}^{op} \to \mathcal{S}$. A presheaf is a *sheaf* if for every covering sieve S of an object C, the canonical morphism

$$F(C) = \mathrm{Map}(\mathrm{Map}_{\mathcal{C}}(-, c), F) \to \mathrm{Map}(S, F)$$

is an equivalence. Denote by $\mathcal{S}\mathrm{hv}(\mathcal{C}, J)$ the full sub-∞-category of $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ spanned by the sheaves on $(\mathcal{C}, J)$.

Note we defined sheaves as local objects in a presentable ∞-category. Hence, using the formalism of presentable ∞-categories, we immediately have the following result.

Proposition 16. *Then the ∞-category of sheaves on $\mathcal{S}\mathrm{hv}(\mathcal{C}, J)$ is presentable.*

4. Sheaves with arbitrary Values

We started with sheaves valued in sets. Then generalized to ∞-categorical sheaves valued in spaces. However, we want ∞-categorical sheaves valued in spectra. Hence the next step is to generalize the values of our ∞-categorical sheaves. Abstractly we obtain such sheaves via the *tensor product* of presentable ∞-categories, which we review now.

Definition 17. Let $\mathcal{C}, \mathcal{D}$ be presentable ∞-categories. Then there exists a presentable ∞-category $\mathcal{C} \otimes \mathcal{D}$, given via universal property

$$\mathrm{Fun}^L(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{L,L}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

Here Fun^L denotes the ∞-category of colimit preserving functors (the L stands for left adjoints, which is an equivalent to colimit preserving for presentable ∞-categories). Similarly, $\mathrm{Fun}^{L,L}$ are colimit preserving functors in both variables.

We can now use that to define sheaves with values in $\mathcal{D}$.

Definition 18. Let (C, J) be $\mathcal{C}$ be a Grothendieck site, and $\mathcal{D}$ be a presentable ∞-category. A sheave on $(\mathcal{C}, J)$ with values in $\mathcal{D}$ is a limit preserving functor $\mathcal{F} \colon \mathcal{S}\mathrm{hv}(\mathcal{C}, J) \to \mathcal{D}$, which corresponds to an object in $\mathcal{S}\mathrm{hv}(\mathcal{C}, J) \otimes \mathcal{D}$.

Of course, this description is very abstract and ideally we want a more explicit description that we can use in the future. That is the aim of the next sections.

5. An Explicit Description of Sheaves valued in Spectra

We have given a formal definition of sheaves valued in spectra, via the tensor product of presentable ∞-categories. We now want a more explicit description thereof. For this we make use of the following result.

Proposition 19 ([Lur17], see also [GGN15])**.** *The inclusion $\mathrm{Pr}^L_{st} \to \mathrm{Pr}^L$ of stable presentable ∞-categories into the category of presentable ∞-categories admits a left adjoint, which is explicitly given by $\mathcal{S}\mathrm{p} \otimes -$.*

We can recall from our previous talk that a stabilization of a presentable ∞-category $\mathcal{C}$ is given by the ∞-category of spectrum objects in it $\mathcal{S}\mathrm{p}(\mathcal{C})$. This means we have an equivalence of ∞-categories $\mathcal{S}\mathrm{p} \otimes \mathcal{C} \simeq \mathcal{S}\mathrm{p}(\mathcal{C})$. So, we can characterize an object in $\mathcal{S}\mathrm{p} \otimes \mathcal{S}\mathrm{hv}(\mathcal{C}, J)$ as a spectrum object in $\mathcal{S}\mathrm{hv}(\mathcal{C}, J)$. However, the inclusion $\mathcal{S}\mathrm{hv}(\mathcal{C}, J) \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ preserves limits, and in particular pullbacks. Hence, a spectrum object in sheaves remains a spectrum object in presheaves. Combining this we have the following result.

Proposition 20. *A spectrum object in sheaves on the Grothendieck site $(\mathcal{C}, J)$ is given by a spectrum object in presheaves on $(\mathcal{C}, J)$, that is point-wise a sheaf.*

This gives us the following explicit description of sheaves valued in spectra.

Theorem 21. *An object in $\mathcal{S}\mathrm{p} \otimes \mathcal{S}\mathrm{hv}(\mathcal{C}, J)$ is given by a presheaf $F \colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{p}$, such that $\Omega^{\infty - n} \colon \mathcal{C}^{op} \to \mathcal{S}$ is a sheaf of spaces.*

Hence, the sheaf condition on spectra is given level-wise.

6. An Explicit Description of the Sheaf Condition via Limits

Recall that we defined a sheave in spaces as a presheaf that is local with respect to maps $S \to \mathcal{Y}\mathrm{on}(C)$. We now want a more explicit characterization of this sheaf condition. In particular we want to connect it to a limit description we might be familiar with. For this we review the density theorem for presheaves.

Proposition 22. *Let $F\colon \mathcal{C}^{op} \to \mathcal{S}$ be a presheaf. Then*

$$F \simeq \operatorname{colim}(\mathcal{C}_{/F} \to \mathcal{C} \to \mathcal{PS}\mathrm{h}(\mathcal{C}))$$

We can now use this to simplify the sheaf condition. Let $F\colon \mathcal{C}^{op} \to \mathcal{S}$ be a sheaf. Then we have

$$\begin{aligned}
F(c) &\simeq \mathrm{Map}(\mathcal{Y}\mathrm{on}(c), F) && \text{Yoneda lemma}\\
&\simeq \mathrm{Map}(S, F) && \text{sheaf condition}\\
&\simeq \mathrm{Map}(\operatorname{colim}_{\mathcal{Y}\mathrm{on}(c')\to S}\mathcal{Y}\mathrm{on}(c'), F) && \text{density}\\
&\simeq \lim_{\mathcal{Y}\mathrm{on}(c')\to S} \mathrm{Map}(\mathcal{Y}\mathrm{on}(c), F) && \text{Map commutes with colimits}\\
&\simeq \lim_{\mathcal{Y}\mathrm{on}(c')\to S} F(c') && \text{Yoneda lemma}
\end{aligned}$$

7. An Explicit Description of Sheaves on the Site of Manifolds

We continue our analysis of sheaves with the aim of providing explicit descriptions. Now we focus on the case of sheaves on the site of manifolds. Concretely, we now have the following explicit description.

Proposition 23. *A presheaf $F\colon \mathcal{M}\mathrm{fd}^{op} \to \mathcal{S}\mathrm{p}$ is a sheaf if and only if for every manifold M and every open cover $U = \{U_i\}$ of M, the canonical morphism*

$$F(M) \xrightarrow{\simeq} \lim_{U_i \in \mathcal{I}(U)} F(U_i)$$

is an equivalence. Here $\mathcal{I}(U)$ is the poset of open subsets consisting of finite intersections of the U_i.

In fact we can check the sheaf condition for sheaves on the site of manifolds with a very specific type of covering families. Here we will state this result without proof.

Proposition 24. *A presheaf $F\colon \mathcal{M}\mathrm{fd}^{op} \to \mathcal{S}\mathrm{p}$ is a sheaf if and only if*

(1) $F(\emptyset)$ is terminal
(2) For every $M = U \cup V$, there is a pullback square

$$\begin{tikzcd}
F(M) \arrow[r] \arrow[d] & F(U) \arrow[d] \\
F(V) \arrow[r] & F(U \cap V)
\end{tikzcd}$$

(3) For an increasing union $M = U_1 \cup U_2 \cup \cdots$, the canonical morphism $F(M) \to \lim_i F(U_i)$ is an equivalence.

8. Equivalences of Sheaves

We now have etsablished a very solid understanding of sheaves on the site of manifolds with arbitrary values as presheaves with suitable limits conditions. We now want to use these explicit descriptions to better understand equivalences of sheaves on the site of manifolds.

Lemma 25. *Let $f\colon F \to F'$ be a morphism of sheaves on the site of manifolds. Then f is an equivalence if and only if for every $n \geq 0$, $f(\mathbb{R}^n)$ is an equivalence.*

In fact we can refine this statement.

Theorem 26. *Let $j\colon \mathcal{E}\mathrm{uc} \to \mathcal{M}\mathrm{fd}$ be the inclusion of the category of Euclidean spaces into the category of manifolds. Then the induced adjunction*

$$Fun(\mathcal{M}\mathrm{fd}^{op}, \mathcal{S}\mathrm{p}) \underset{j_*}{\overset{j^*}{\rightrightarrows}} Fun(\mathcal{E}\mathrm{uc}^{op}, \mathcal{S}\mathrm{p})$$

has the following properties:

1. *The functors j^* and j_* preserve the sheaf condition.*
2. *The restricted adjunction is an equivalence of sheaves.*

Let us see one more way to understand equivalences of sheaves via stalks.

Definition 27. Let $x \in M$ be a point and denote by $\mathcal{O}\mathrm{pen}_x(M)$ the poset of open neighborhoods of M around x. The *stalk* of a presheaf F at x is defined as the filtered colimit

$$x^*(F) = \operatorname*{colim}_{U \in \mathcal{O}\mathrm{pen}_x(M)} F(U).$$

We now have the following theorem.

Theorem 28. *A morphism $f\colon F \to F'$ of sheaves on the site of manifolds is an equivalence if and only if for every manifold M and for every point $x \in M$, the induced map $x^*(f)\colon x^*(F) \to x^*(F')$ is an equivalence.*

Note, the general condition above is equivalent to the following simpler condition.

Corollary 29. *A morphism $f\colon F \to F'$ of sheaves on the site of manifolds is an equivalence if and only if for all $n \geq 0$, the induced map $(0_n)^*(f)\colon (0_n)^*(F) \to (0_n)^*(F')$ is an equivalence. Here 0_n is the origin in $\mathbb{R}^n$.*

References

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