DIFFERENTIAL COHOMOLOGY SEMINAR 9

TALK BY ALESSANDRO NANTO

The goal of this talk is to compare different models of twisted K-theory. Concretely, we learn about the comparison between the original approach to twisted K-theory by Atiyah–Segal [AS04] and the modern approach in [ABG+14] that builds on the abstract approach to twisted cohomology via ∞-categorical methods. There is a third approach by Freed–Hopkins–Teleman [FHT11], that we will not consider here.

Remark 0.1. Throughout this talk we adopt the following notational conventions:

- For an ∞-category $\mathcal{C}$ and objects $X, Y \in \mathcal{C}$, we denote by $\mathrm{Map}_{\mathcal{C}}(X,Y)$ the mapping *space* from X to Y in $\mathcal{C}$.
- If an ∞-category $\mathcal{C}$ is enriched over spectra (e.g. if $\mathcal{C}$ is stable), denote the *mapping spectrum* from X to Y by $\underline{\mathrm{Map}}_{\mathcal{C}}(X,Y)$.
- Note, in this case we have the relationship $\Omega^\infty \underline{\mathrm{Map}}_{\mathcal{C}}(X,Y) \simeq \mathrm{Map}_{\mathcal{C}}(X,Y)$, i.e., the underlying space of this mapping spectrum is then equivalent to the mapping space $\mathrm{Map}_{\mathcal{C}}(X,Y)$.

1. Reviewing the ∞-categorical Approach to twisted Cohomology

Let us recall the modern approach again. Given a ring spectrum R, one can consider its space of invertible R-modules $\mathcal{L}\mathrm{ine}_R$. Now, given a space X, a twist of R on X is a map $\alpha\colon X \to \mathcal{L}\mathrm{ine}_R$. Then we define twisted homology as

$$X^\alpha := M\alpha = \mathrm{colim}(X \xrightarrow{\alpha} \mathcal{L}\mathrm{ine}_R \to \mathcal{M}\mathrm{od}_R)$$

where the last map is the inclusion of $\mathcal{L}\mathrm{ine}_R$ into $\mathcal{M}\mathrm{od}_R$. The R-cohomology groups twisted by α are then defined as the homotopy groups of the mapping spectrum $\underline{\mathrm{Map}}_R(X^\alpha, R)$, which is the spectrum given by the mapping spaces of maps from X^α to $\Sigma^n R$ in $\mathcal{M}\mathrm{od}_R$.

2. The ∞-categorical Approach to twisted Cohomology via Sections

Eventually we want to compare this definition with the classical approach. In this section, as a first step, we translate the definition into a more suitable form via sections, which will then allow us to compare it to the classical approach.

Recall that the objects in $\mathcal{L}\mathrm{ine}_R$ are invertible R-modules. Let $-\alpha$ denote the map

$$-\alpha := X \xrightarrow{\alpha} \mathcal{L}\mathrm{ine}_R \xrightarrow{\mathrm{inv}} \mathcal{L}\mathrm{ine}_R,$$

We now have the following simple lemma.

Lemma 2.1. *Let M be an invertible R-module. Then there is a natural equivalence of mapping spectra*

$$\underline{\mathrm{Map}}(-M, R) \simeq \underline{\mathrm{Map}}(R, M)$$

This in particular means that the inverse of M is given by $\underline{\mathrm{Map}}(M,R)$.

Proof. We know that $\underline{\mathrm{Map}}(M,-)$ is the right adjoint to $M \otimes_R -$. But $M^{-1} \otimes_R -$ is also a right adjoint to $M \otimes_R -$, so they must be equivalent, meaning $M^{-1} \otimes R \simeq \underline{\mathrm{Map}}(M,R)$. $\square$

Based on this lemma we have the following chain of equivalences

$$\underline{\mathrm{Map}}_R(X^{-\alpha}, R) \simeq \underline{\mathrm{Map}}(-\alpha, R_X) \simeq \underline{\mathrm{Map}}(R_X, \alpha)$$

Here in the last step we used the previous lemma point-wise. This proves the following proposition.

Proposition 2.2. *The twisted R-cohomology spectrum twisted by $-\alpha$ is given by the mapping spectrum $\underline{\mathrm{Map}}(R_X, \alpha)$.*

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Let us now recall that every ring spectrum R comes with a unique ring map $\mathbb{S} \to R$, which induces an adjunction between $\mathbb{S}$-modules (i.e. spectra) and R-modules:

$$\begin{tikzcd} \mathcal{M}\mathrm{od}_{\mathbb{S}} \arrow[r, shift left, "R\otimes_{\mathbb{S}}-"] \arrow[r, phantom, "\perp"] & \mathcal{M}\mathrm{od}_R \arrow[l, shift left, "\mathrm{Forget}"] \end{tikzcd} .$$

If we apply this adjunction point-wise to α we immediately get the following lemma.

Lemma 2.3. *There is an equivalence of mapping spectra*

$$\underline{\mathrm{Map}}_R(R_X, \alpha) \simeq \underline{\mathrm{Map}}(\mathbb{S}_X, \alpha).$$

Finally, let $\Omega^{\infty-n}\alpha\colon X \to \mathcal{L}\mathrm{ine}_R \to \mathcal{S}$ be the composition of α with $\Omega^\infty\colon \mathcal{L}\mathrm{ine}_R \to \mathcal{S}\mathrm{p} \to \mathcal{S}$. As $\mathbb{S} \simeq \Sigma^\infty_+ *$, the adjunction between Σ^∞_+ and Ω^∞ gives us the following equivalence of spaces:

$$\mathrm{Map}_{\mathcal{S}\mathrm{p}}(\mathbb{S}_X, \alpha) \simeq \mathrm{Map}_{\mathcal{S}}(\mathrm{const}_X, \Omega^\infty\alpha).$$

Here const_X is the constant functor at the point. The data of the right hand side is exactly a section. Hence combining these equivalences we get the following theorem.

Theorem 2.4. *A point in the underlying space of the twisted R-cohomology spectrum $\Omega^\infty\underline{\mathrm{Map}}_R(X^{-\alpha}, R)$ is equivalent to a section of the bundle over X classified by the map $\Omega^\infty\alpha\colon X \to \mathcal{S}$.*

Moreover, by similar analysis we have the following more general theorem.

Theorem 2.5. *A point in the n-th underlying space of the twisted R-cohomology spectrum $\Omega^{\infty-n}\underline{\mathrm{Map}}_R(X^{-\alpha}, R)$ is equivalent to a section of the bundle over X classified by the map $\Omega^{\infty-n}\alpha\colon X \to \mathcal{S}$.*

3. K-Theory à la Atiyah

Before we can comprehend how Atiyah–Segal defined twisted K-theory, we first need to review how Atiyah defined classical K-theory via Fredholm operators in [Ati67] (also in [Jän65]).

Remark 3.1. Recall that every infinite-dimensional separable Hilbert space is isomorphic to each other. We will hence fix one such choice $\mathcal{H}$ throughout these two sections.

We now recall the notion of a Fredholm operator.

Definition 3.2. A Fredholm operator on $\mathcal{H}$ is a bounded linear operator $F\colon \mathcal{H} \to \mathcal{H}$ such that both its kernel and cokernel are finite-dimensional. The space of Fredholm operators is denoted $\mathcal{F}\mathrm{red}(\mathcal{H})$ and is topologized as a subspace of the space of bounded linear operators with the norm topology.

Remark 3.3. Fredholm operators admit many alternative characterizations. For example, it is equivalent to being invertible modulo compact operators i.e.

$$\mathcal{F}\mathrm{red}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : AB - 1, BA - 1 \in \mathcal{K} \text{ for some } B \in \mathcal{B}(\mathcal{H})\}$$

We now define the spectrum K_n, given level-wise by the space of Fredholm operators on $\mathcal{H}$:

$$K_n = \mathcal{F}\mathrm{red}^{(n)}(\mathcal{H})$$

Here $\mathcal{F}\mathrm{red}^{(0)}(\mathcal{H})$ is simply $\mathcal{F}\mathrm{red}(\mathcal{H})$, and the other can be characterized similarly.
We will claim the following result regarding these spaces that we shall not prove.

Lemma 3.4. *The spaces K_n assemble into a spectrum, meaning there are maps*

$$K^{(n)}(\mathcal{H}) \to \Omega\mathcal{F}\mathrm{red}^{(n+1)}(\mathcal{H}).$$

How does this construction relate to K-theory? We have the following remark that can give us some intuition.

Remark 3.5. For a compact space X, and given map $X \to \mathcal{F}\mathrm{red}^{(0)}(\mathcal{H})$, we can associate the formal difference of vector bundles given by the kernel and cokernel of the associated family of Fredholm operators (which are by assumption finite-dimensional). This associated to every element in $[X, \mathcal{F}\mathrm{red}^{(0)}(\mathcal{H})]$ an element in $K^0(X)$.

This map is not just coincidental and in fact we have the following major result.

Theorem 3.6 (Atiyah–Jänich)**.** *The map from Remark 3.5 is an isomorphism.*

This relates classical K-theory to Fredholm operators, meaning the spectrum above is indeed coincides with K-theory.

4. Atiyah–Segal's Approach

We now proceed to review the original approach to twisted K-theory by Atiyah–Segal [AS04]. This approach generalizes the Fredholm approach to K-theory from the regular to the twisted setting.

Definition 4.1. Fix a topological space X. Let $PU(\mathcal{H})$ be the projective unitary group of $\mathcal{H}$, i.e. the quotient of the unitary group $U(\mathcal{H})$ by its center S^1. Let $P \to X$ be a principal $PU(\mathcal{H})$-bundle over X. Then we define the τ-twisted K-theory group as the space of sections

$$K^0_\tau \coloneqq \pi_0(\Gamma(X, P \times_{PU(\mathcal{H})} K)).$$

Remark 4.2. Here, $P \times_{PU(\mathcal{H})} KU_0(\mathcal{H})$ denotes the associated bundle of spectra over X with fibre $KU_0(\mathcal{H})$. The associated bundle can be described more explicitly as the bundle whose fibre over $x \in X$ is given by the $P_x \times_{PU(\mathcal{H})} K_0$, meaning we take the products of the fibers and quotient out the simultaneous action of $PU(\mathcal{H})$ on the fibers P_x and K_0.

Remark 4.3. As we have $BPU(\mathcal{H}) \simeq K(\mathbb{Z}, 3)$, the bundle $P \to X$ is classified via a map $\tau\colon X \to BPU(\mathcal{H}) \simeq K(\mathbb{Z}, 3)$.

5. Comparing Classical and Modern

We are now ready to compare the classical and modern approaches to twisted K-theory. Let X be a topological space and $P \to X$ be a principal $PU(\mathcal{H})$-bundle over X. Following Remark 4.3 this is classified by a map $\tau\colon X \to K(\mathbb{Z}, 3)$.

For such a map we now have the following explicit commutative diagram

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[dr, "\tau"'] & K(\mathbb{Z},3) \arrow[r, "?"] \arrow[d, "\mathrm{id}"] & BGL_1(KU) \arrow[r] \arrow[d] & \mathcal{L}\mathrm{ine}_{KU} \arrow[d, "\Omega^{\infty-n}"] \\ & K(\mathbb{Z},3) \arrow[r] & BAut(K_n) \arrow[r] & \mathcal{S} \end{tikzcd},$$

which we now explain in more detail.

1. The top row corresponds to the modern approach, as it takes a map into $K(\mathbb{Z}, 3)$ to a KU-line bundle. Following Theorem 2.5, taking $\Omega^{\infty-n}$ and then sections of the resulting bundle correspond precisely to classes of τ-twisted KU-cohomology.
2. The bottom row corresponds to the classical approach. Indeed, post-composing τ with the map $K(\mathbb{Z}, 3) \to BAut(K_n)$ corresponds to the constructing the associated bundle. In this case, it is by definition (Definition 4.1) the case that sections of this bundle give us elements in twisted K-theory.
3. Hence compatibility between these two approaches corresponds to these diagrams commuting, which is a direct observation.

6. Some Comments regarding Freed–Hopkins–Teleman

Ideally we could add some comments regarding the Freed–Hopkins–Teleman approach here, but that might turn out to be beyond existing capabilities.

References

[ABG+14] Matthew Ando, Andrew J. Blumberg, David Gepner, Michael J. Hopkins, and Charles Rezk. An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology. *J. Topol.*, 7(3):869–893, 2014.

[AS04] Michael Atiyah and Graeme Segal. Twisted K-theory. *Ukr. Mat. Visn.*, 1(3):287–330, 2004.

[Ati67] M. F. Atiyah. *K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1967. Lecture notes by D. W. Anderson.

[FHT11] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman. Loop groups and twisted K-theory I. *J. Topol.*, 4(4):737–798, 2011.

[Jän65] Klaus Jänich. Vektorraumbündel und der Raum der Fredholm-Operatoren. *Math. Ann.*, 161:129–142, 1965.