DIFFERENTIAL COHOMOLOGY SEMINAR 4 (DRAFT)

TALK BY ALESSANDRO NANTO

In the last talk we learned the definition of a differential cohomology theory, as a sheaf valued in spectra on the site of manifolds. This talk continues our journey through differential cohomology theories, and focuses on the following three topics:

(1) We want to learn how to construct non-trivial examples out of sheaves valued in chain complexes.
(2) We want to understand how we can extend classical cohomology operations to the setting of differential cohomology theories.
(3) We want to introduce suitable analogues of fiber-wise integration.

1. Abelian Groups, Spectra and the Heart

Let us start by reviewing the relation between abelian groups, rings and spectra.

Definition 1.1. Let $n \in \mathbb{Z}$ and X be a spectrum, define $\pi_n(X) := \pi_0(\Omega^{\infty+n}X) = \pi_0(X_{-n})$. We call π_n the *n-th homotopy group* of X.

Remark 1.2. Note that since $X_n \simeq \Omega^2 X_{n+2}$, for any n, the set $\pi_0(X_n)$ underlies the structure of an abelian group.

The category $\mathcal{S}\mathrm{p}$ underlies the structure of a symmetric monoidal ∞-category ([Lur17, Corollary 4.8.2.19]). Following [Lur17], we denote by $\otimes$ the tensor product on $\mathcal{S}\mathrm{p}$.

Definition 1.3. A (commutative) algebra object in $\mathcal{S}\mathrm{p}$ is called an ($\mathbb{E}_\infty$-)*ring spectrum*, see [Lur17, Definition 7.1.0.1]. Given a ring spectrum R, denote by Mod_R the corresponding category of *left R-module spectra*, see [Lur17, Definition 7.1.1.2].

Remark 1.4. The sphere spectrum $\mathbb{S}$ acts as the monoidal unit of $\mathcal{S}\mathrm{p}$, therefore it is a $\mathbb{E}_\infty$-ring spectrum. The category $\mathrm{Mod}_{\mathbb{S}}$ is canonically equivalent to $\mathcal{S}\mathrm{p}$.

Definition 1.5. Denote by $\mathcal{S}\mathrm{p}_{\geq 0} \subseteq \mathcal{S}\mathrm{p}$ the full sub-category generated by *connective spectra*, i.e. spectra X such that $\pi_n(X) \simeq 0$, for all $n < 0$. Denote by $\mathcal{S}\mathrm{p}^\heartsuit \subseteq \mathcal{S}\mathrm{p}_{\geq 0}$ the *heart of spectra*, i.e. the full sub-category generated by spectra X such that $\pi_n(X) \simeq 0$, for all $n > 0$.

We have the following result relating connective spectra and the heart, which follow immediately.

Lemma 1.6. *Let X be a connective spectrum. The following are equivalent:*

(1) X is in the heart.
(2) $\pi_n(\Omega^\infty X) = 0$, for all $n > 0$.
(3) $\mathrm{Hom}_{\mathcal{S}_*}(S, \Omega^\infty X) \simeq 0$, for all connected, pointed spaces S.
(4) X is local with respect to the class of maps $\Sigma^\infty S \to 0$, for every connected pointed space S.

The category $\mathcal{S}\mathrm{p}_{\geq 0}$ is presentable and π_0 induces an equivalence between the heart and $\mathcal{A}\mathrm{b}$ ([Lur17, Proposition 1.4.3.6]). The heart is a sub-category of local objects of connective spectra, therefore the inclusion $\mathcal{A}\mathrm{b} \simeq \mathcal{S}\mathrm{p}^\heartsuit \subseteq \mathcal{S}\mathrm{p}_{\geq 0}$ is a right adjoint. The category $\mathcal{S}\mathrm{p}_{\geq 0}$ is closed under $\otimes$ and, given X, Y connective spectra,

$$\pi_0(X \otimes Y) \simeq \pi_0(X) \otimes \pi_0(Y) \tag{1.7}$$

see [Dav24, Theorem 2.3.28]

Definition 1.8. Given an abelian group A, denote by HA the (unique up to equivalence) spectrum of the heart such that $\pi_0(HA) \simeq A$. We call HA the *Eilenberg-Mac Lane spectrum* of A.

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Using Equation (1.7) and the adjunction between H and π_0, one can prove H, viewed as a functor $\mathcal{A}\mathrm{b} \to \mathcal{S}\mathrm{p}$, is lax monoidal. In particular, if R is a commutative ring, then HR is a connective $\mathbb{E}_\infty$-ring spectrum. On the other hand, if R is a connective $\mathbb{E}_\infty$-ring spectrum and M a connective module, then $\pi_0(M)$ is a $\pi_0(R)$-module.

Definition 1.9. Given a commutative ring R, denote by $\mathrm{Ch}(R) = \mathrm{Ch}(\mathrm{Mod}_R)$ the ordinary category of unbounded chain complexes. Let $\mathcal{D}(R)$ be the ∞-localization of $\mathrm{Ch}(R)$ at the class of quasi-isomorphisms.

Similar to the heart of spectra, given an $\mathbb{E}_\infty$-ring spectrum R, denote by $\mathrm{Mod}_R^\heartsuit \subseteq \mathrm{Mod}_R$ the full subcategory generated by R-modules such that the underlying spectrum belongs to the heart of spectra.

Theorem 1.10 (Stable Dold-Kan Correspondence)**.** *Let R be a commutative ring.*

(1) $\mathrm{Mod}_R \simeq \mathrm{Mod}_{HR}^\heartsuit$ *via taking Eilenberg-Mac Lane spectra.*

(2) The equivalence in (1) extends to an equivalence $H : \mathcal{D}(R) \simeq \mathrm{Mod}_{HR}$ *of symmetric monoidal ∞-categories.*

Proof. (1) is [Lur17, Proposition 7.1.1.13], while (2) is [Lur17, Theorem 7.1.2.13]. □

An interesting consequence of Theorem 1.10 is the following:

Corollary 1.11. *Given $F \in \mathcal{D}(R)$, then $\pi_n(HF) \simeq H_n(F)$, for all $n \in \mathbb{Z}$.*

Proof.

$$\begin{aligned}
\pi_n(HF) &= \pi_0(\Omega^{\infty+n}HF) \\
&\overset{①}{\simeq} \pi_0(\mathrm{Hom}_{\mathcal{S}\mathrm{p}}(\Sigma^n\mathbb{S}, HF)) \\
&\overset{②}{\simeq} \pi_0(\mathrm{Hom}_{\mathrm{Mod}_{HR}}(\Sigma^n HR, HF)) \\
&\overset{③}{\simeq} \pi_0(\mathrm{Hom}_{\mathcal{D}(R)}(R[n], F)) \\
&\overset{④}{\simeq} H_n(F)
\end{aligned}$$

① The functor $\Omega^{\infty+n}$ is corepresented by the shifted sphere spectrum $\Sigma^n\mathbb{S}$. ② The forgetful functor $\mathrm{Mod}_{HR} \to \mathrm{Mod}_{\mathbb{S}} \simeq \mathcal{S}\mathrm{p}$ is right adjoint to tensoring with HR and $HR \otimes (\Sigma^n\mathbb{S}) \simeq \Sigma^n HR$. ③ Theorem 1.10 ④ π_0 of the mapping space $\mathrm{Hom}_{\mathcal{D}(R)}(R[n], F)$ is equivalent to the mapping space $R[n] \to F$ in the *ordinary* derived category of R, i.e. homotopy classes of maps $R[n] \to F$, which correspond exactly to classes in $H_n(F)$. □

2. Locally constant sheaves on manifolds

Let $\mathcal{C}$ be a presentable ∞-category. The ∞-categorical background given in previous talks allows to conclude the existence of a number of functors. Here we give a (somewhat) explicit formula for one.

Remark 2.1. Let $\mathcal{B}$ be a full, dense sub-site of $\mathcal{M}\mathrm{fd}$, recall then that, for every presentable category $\mathcal{C}$, the restriction functor induces an equivalence $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, \mathcal{C}) \simeq \mathcal{S}\mathrm{hv}(\mathcal{B}, \mathcal{C})$.

Evaluation at $\{0\}$ induces an adjunction $(\mathrm{L}, \Gamma) : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, \mathcal{C})$, where the functor Γ is evaluation at $\{0\}$, while the left adjoint L maps $C \in \mathcal{C}$ to the sheafification of the constant pre-sheaf with value C.

Remark 2.2. Every presentable ∞-category $\mathcal{C}$ is uniquely *cotensored over* $\mathcal{S}$, see [Lur09, Remark 5.5.2.6]. More explicitly, for every space S and object C, there is an object C^S together with a natural equivalence

$$\mathrm{Hom}_{\mathcal{S}}(S, \mathrm{Hom}_{\mathcal{C}}(-, C)) \simeq \mathrm{Hom}_{\mathcal{C}}(-, C^S)$$

Definition 2.3. Denote by Sing the functor $\mathcal{M}\mathrm{fd} \to \mathcal{S}$ mapping a manifold to its underlying space of *smooth* simplexes. Given a presentable ∞-category $\mathcal{C}$, denote by $\flat$ the composition $\mathcal{C} \to \mathrm{Fun}(\mathcal{S}^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{M}\mathrm{fd}^{\mathrm{op}}, \mathcal{C})$, the first functor coming from Remark 2.2, the second being pre-composition with $\mathrm{Sing}^{\mathrm{op}}$.

Explicitly, given an object $C \in \mathcal{C}$, the associated pre-sheaf $\flat C$ maps a manifold M to $C^{\mathrm{Sing}(M)}$.

Remark 2.4. Given a topological space X, denote by $\mathrm{Sing}^{top}(X)$ the corresponding space, i.e. its singular simplicial set. If $X = M$ is a smooth manifold, then $\mathrm{Sing}(M) \subseteq \mathrm{Sing}^{top}(M)$ and the inclusion is a homotopy equivalence by Whitney's Approximation Theorem, see [Zuo21, Theorem 1.6].

Lemma 2.5 ([BG16, Corollary 6.46])**.** *Sing is a cosheaf. In particular, $\flat$ factors through $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{M}\mathrm{fd}^{\mathrm{op}}, \mathcal{C})$.*

Lemma 2.5 is essentially the consequence of a generalized version of Seifert-van Kampen theorem, namely [Lur17, Proposition A.3.2], stating that, given a topological space X and a covering sieve $\mathcal{O}$, the space $\mathrm{Sing}^{top}(X)$ is the colimit of $\mathrm{Sing}^{top}(U)$, over $U \in \mathcal{I}(\mathcal{O})$.

Theorem 2.6. *$\flat : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, \mathcal{C})$ is left adjoint to Γ.*

Proof. The composition $\mathcal{C} \xrightarrow{\flat} \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, \mathcal{C}) \xrightarrow{|_{\mathcal{E}\mathrm{uc}}} \mathcal{S}\mathrm{hv}(\mathcal{E}\mathrm{uc}, \mathcal{C})$ maps an object C to the sheaf $\flat C$ restricted to Euclidean spaces. Since $\mathbb{R}^n$ is contractible, $(\flat C)(\mathbb{R}^n) = C^{\mathrm{Sing}(\mathbb{R}^n)} \simeq C$ and so $\flat$ restricted to $\mathcal{E}\mathrm{uc}$ is equivalent to L, the functor taking C to the the pre-sheaf with constant value C, which is left adjoint to Γ restricted to $\mathcal{E}\mathrm{uc}$. □

Remark 2.7. The proof of Theorem 2.6 shows that, given an object $C \in \mathcal{C}$ and a dense sub-site $\mathcal{B} \subseteq \mathcal{M}\mathrm{fd}$ of *contractible* manifolds, the constant pre-sheaf with value C is equivalent to the sheaf $\flat C$ restricted to $\mathcal{B}$, hence it is already a sheaf.

3. Chain complexes and sheaves

The stable Dold-Kan correspondence allows us to move freely between sheaves of $H\mathbb{Z}$-module spectras and sheaves valued in the derived category $\mathcal{D}(\mathbb{Z})$ of abelian groups. In this section, we introduce definition and some technical lemmas regarding sheaves on manifolds valued in the derived category.

Remark 3.1. All chain complexes are $\mathbb{Z}$-graded. Given a ring R, we identify chain complexes with cochain complexes. More specifically, we identify a cochain complex V^* with the chain complex $V_n = V^{-n}$. Viceversa, we identify V_* with the cochain complex $V^n = V_{-n}$. In particular, operations defined on cochain complexes are extended to chain complexes via this identification.

Definition 3.2 ([BNV16, Definition 7.14])**.** Given $n \in \mathbb{Z}$ and a cochain complex V^*, denote by $V^{\geq n}$, resp. $V^{\leq n}$, the *naive truncations*

$$\cdots \to 0 \to V^n \to V^{n+1} \to \cdots, \qquad \text{resp. } \cdots \to V^{n-1} \to V^n \to 0 \to \cdots$$

Given $F : \mathcal{M}\mathrm{fd}^{\mathrm{op}} \to \mathrm{Ch}(\mathbb{Z})$, denote by $F^{\geq n}$ and $F^{\leq n}$ the point-wise naive truncations of F. If F is a sheaf of chain complexes, then so are its truncations.

Lemma 3.3 ([BNV16, Lemma 7.12])**.** *Let $F : \mathcal{M}\mathrm{fd}^{\mathrm{op}} \to \mathrm{Ch}(\mathbb{Z})$ a sheaf of chain complexes of C^∞-modules, then $\mathcal{M}\mathrm{fd}^{\mathrm{op}} \xrightarrow{F} \mathrm{Ch}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$ is a sheaf.*

Definition 3.4. Denote by Ω^* the sheaf $\mathcal{M}\mathrm{fd}^{\mathrm{op}} \to \mathrm{Ch}(\mathbb{Z})$ mapping a manifold to its de Rham complex.

Lemma 3.3 ensures that the sheaf in Definition 3.4 and the corresponding naive truncations remain sheaves after post-composition with the localization functor $\mathrm{Ch}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$.

Definition 3.5. Given a sheaf $F : \mathcal{M}\mathrm{fd}^{\mathrm{op}} \to \mathcal{D}(\mathbb{Z})$, denote by HF the *Eilenberg-Mac Lane sheaf* of $H\mathbb{Z}$-module spectra obtained by applying point-wise the equivalence of Theorem 1.10.

4. Higher de Rham Theorem

The classical de Rham theorem gives an explicit ring isomorphism between the de Rham cohomology of a manifold M and its singular cohomology with real coefficients. Using the modern perspective we can recover de Rham theorem as a corollary of a more general $\mathbb{A}_\infty$-quasi-isomorphism of differential graded algebras. An $\mathbb{A}_\infty$-*morphism* of DGAs $\mu : R \to S$ consists of a sequence of graded linear maps $\{\mu_n : R^{\otimes n} \to S\}_{n \geq 1}$ satisfying a sequence of coherence conditions, see [LV12, Proposition 10.2.12]. One such condition is that μ_1 is a chain map, then a $\mathbb{A}_\infty$-*quasi-isomorphism* is an $\mathbb{A}_\infty$-morphism μ such that μ_1 is a quasi-isomorphism. Here we use the machinery set-up in Section 2.

Remark 4.1. Since $\mathcal{D}(\mathbb{Z})$ is presentable, we know that it is cotensored over $\mathcal{S}$. Given a space S and a chain complex M_*, the cotensor ${M_*}^S$ is the chain complex of graded linear maps $C_*(S,\mathbb{Z}) \to M_*$, from the (normalized) singular chain complex of S to M_*, see [Lur17, Definition 1.3.2.1]. In particular, let $M_* = M$ be concentrated in degree 0, then ${M_*}^S$ is the singular cochain complex of S with values in M.

Remark 4.2. Given a smooth manifold M, denote by $C_*^{\mathrm{sm}}(M,\mathbb{Z})$ the *smooth* singular chain complex generated by the smooth singular simplicial set $\mathrm{Sing}(M)$. Given an abelian group A, denote by $C^*_{\mathrm{sm}}(M,A)$ the cochain complex valued in A associated to $C_*^{\mathrm{sm}}(M,\mathbb{Z})$.

Definition 4.3. Consider the morphism $\Omega^*(M) \to (\flat\mathbb{R})(M) = C^*_{\mathrm{sm}}(M,\mathbb{R})$ taking a form $\omega \in \Omega^n(M)$ to the linear map $\int \omega : C_n^{\mathrm{sm}}(M,\mathbb{Z}) \to \mathbb{R}$. We call the induced transformation $\mathrm{dR} : \Omega^* \to \flat\mathbb{R}$ the *de Rham morphism*.

We can now state the main theorem.

Theorem 4.4 ([Aa10, Theorem 3.25])**.** $\mathrm{dR} : \Omega^* \to \flat\mathbb{R}$ *lifts to an $\mathbb{A}_\infty$-quasi-isomorphism of sheaves of DGAs.*

5. Deligne Cohomology

In this section, we give the definition of the ℓ-th Deligne sheaf as the Eilenberg-Mac Lane spectrum (see Definition 3.5) associated to a sheaf $F : \mathcal{M}\mathrm{fd}^{\mathrm{op}} \to \mathcal{D}(\mathbb{Z})$. Following that, we give explicit cochain complexes that are quasi-isomorphic to the value of F at a manifold M.

Definition 5.1. Given $\ell \in \mathbb{N}$, define $\hat{\mathbb{Z}}(\ell) : \mathcal{M}\mathrm{fd}^{\mathrm{op}} \to \mathcal{D}(\mathbb{Z})$ as the pullback

$$\begin{tikzcd}
\hat{\mathbb{Z}}(\ell) \arrow[r] \arrow[d] & \Omega^{\geq \ell} \arrow[d] \\
\flat\mathbb{Z} \arrow[r] & \flat\mathbb{R}
\end{tikzcd}$$

The vertical morphism being the composition $\Omega^{\geq \ell} \hookrightarrow \Omega^* \xrightarrow{\mathrm{dR}} \flat\mathbb{R}$. We call the corresponding Eilenberg-Mac Lane spectrum $H\hat{\mathbb{Z}}(\ell)$ the *ℓ-th Deligne sheaf.*

Remark 5.2 (Model A, see [HS05, §3.2]). Let $\acute{C}(\ell)^n(M) \subseteq C^n_{\mathrm{sm}}(M,\mathbb{Z}) \oplus C^{n-1}_{\mathrm{sm}}(M,\mathbb{R}) \oplus \Omega^n(M)$ consist of triples (c,h,ω) for which $\omega = 0$ if $n < \ell$, with differential $\delta(c,h,\omega) = (\delta c, \mathrm{dR}(\omega) - c - \delta h, d\omega)$. This complex $\acute{C}^*(\ell)(M)$ fits into a diagram

$$\begin{tikzcd}
\acute{C}^*(\ell)(M) \arrow[r] \arrow[d] & \Omega^{\geq \ell}(M) \arrow[d] \\
C^*_{\mathrm{sm}}(M,\mathbb{Z}) \arrow[r] & C^*_{\mathrm{sm}}(M,\mathbb{R})
\end{tikzcd}$$

which commutes up to homotopy given by the projections $\acute{C}^n(\ell)(M) \to C^{n-1}_{\mathrm{sm}}(M,\mathbb{R})$. The diagram above model the pullback of Definition 5.1, hence $\acute{C}^*(\ell)(M) \simeq \hat{\mathbb{Z}}(\ell)(M)$.

Remark 5.3 (Model B). Recall that $\flat$ preserves cofiber sequences, since it is left adjoint, and that fiber sequences are the same a cofiber sequences in stable ∞-categories. Consider the diagram

$$\begin{tikzcd}
\hat{\mathbb{Z}}(\ell) \arrow[r] \arrow[d] & \Omega^{\geq \ell} \arrow[d] \arrow[dd, bend left] \\
\flat\mathbb{Z} \arrow[r] \arrow[d] & \flat\mathbb{R} \arrow[d] \\
0 \arrow[r] & \flat(\mathbb{R}/\mathbb{Z})
\end{tikzcd}$$

Since the bottom square is an pullback, $\hat{\mathbb{Z}}(\ell)(M)$ is equivalent to the pullback of the diagram in red. Let $\breve{C}^n(\ell)(M) \subseteq C^{n-1}_{\mathrm{sm}}(M,\mathbb{R}/\mathbb{Z}) \oplus \Omega^n(M)$ consist of pairs (χ,ω) for which $\omega = 0$ if $n < \ell$, with differential $\delta(\chi,\omega) = (e^{2\pi i \mathrm{dR}(\omega)} - \delta\chi, d\omega)$. Similar to Remark 5.2, the complex $\breve{C}^*(\ell)(M)$ fits into the above diagram so that the outer square is an pullback, hence it is equivalent to $\hat{\mathbb{Z}}(\ell)(M)$.

Take an n-cocycle ($n \geq \ell$) in the model from Remark 5.3, i.e. $(\chi, \omega) \in C^{n-1}_{\mathrm{sm}}(M, \mathbb{R}/\mathbb{Z}) \oplus \Omega^n(M)$ such that $d\omega = 0$ and $\delta\chi = e^{2\pi i \mathrm{dR}(\omega)}$. Such a cocycle determines a differential character of degree $n-1$ for M, in the sense of the following definition:

Definition 5.4 ([HS05, Definition 3.4], see also [BB14, Chapter 5])**.** Consider a manifold M, a *differential character* of degree n consists of a character $\chi : Z^{\mathrm{sm}}_n(M, \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ on the group of smooth n-cycles of M together with a n-form $\omega \in \Omega^{n+1}(M)$, such that, for every smooth $(n+1)$-chain c,

$$\chi(\partial c) = e^{2\pi i \int_c \omega}$$

Definition 5.5. Given a full sub-site $\mathcal{B} \subseteq \mathcal{M}\mathrm{fd}$ and a manifold M, a $\mathcal{B}$-*good open cover* of M consists of an open cover $\mathcal{O}$ such that every finite intersection of elements in $\mathcal{O}$ is either empty or diffeomorphic to an object of $\mathcal{B}$. If $\mathcal{B} = \mathcal{E}\mathrm{uc}$ the sub-site of Euclidean spaces, we call a $\mathcal{B}$-good open cover a *differentiably good open cover*, see [FSS11, Definition 6.3.9].

If $\mathcal{B}$ is the sub-site of contractible manifolds, we recover the notion of a *good open cover*. In our next model, we will construct a sheaf F equivalent to $\hat{\mathbb{Z}}(\ell)$ on a dense sub-site $\mathcal{B}$. Given a generic manifold M, we can evaluate $\hat{\mathbb{Z}}(\ell)(M)$ using F, if we assume the existence of a $\mathcal{B}$-good open cover. Here we care mainly about $\mathcal{B} = \mathcal{E}\mathrm{uc}$.

Theorem 5.6 ([FSS11, Theorem A.1])**.** *Every paracompact smooth manifold admits a differentiably good open cover.*

Remark 5.7 (Model C, see [ADH21, Lemma 7.3.4]). Let $\mathcal{B} \subseteq \mathcal{M}\mathrm{fd}$ be a dense sub-site of contractible manifolds, consider the following diagram in the category of sheaves on $\mathcal{B}$

$$\begin{tikzcd}
\hat{\mathbb{Z}}(\ell)|_{\mathcal{B}} \arrow[r] \arrow[d] & \Omega^{\geq \ell}|_{\mathcal{B}} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r] \arrow[rr, bend right] & \Omega^{*}|_{\mathcal{B}} \arrow[r] & \Omega^{\leq \ell-1}|_{\mathcal{B}}
\end{tikzcd}$$

The left square is obtained by: ① Apply the restriction to $\mathcal{B}$ functor to the pullback diagram of Definition 5.1 ② Substitute $\flat\mathbb{R}|_{\mathcal{B}}$ with $\Omega^*|_{\mathcal{B}}$ (see Theorem 4.4) and $\flat\mathbb{Z}|_{\mathcal{B}}$ with $\mathbb{Z}$ (see Remark 2.7). Since the right square is a pullback, $\hat{\mathbb{Z}}(\ell)|_{\mathcal{B}}$ is equivalent to the pullback of the diagram in red. Let $\check{C}^*(\ell)$ be the sheaf of chain complexes $\mathbb{Z} \to \Omega^0 \to \cdots \to \Omega^{\ell-1} \to 0 \to \cdots$, where $\mathbb{Z}$ is in degree 0 includes into Ω^0 as constant functions. The complex $\check{C}^*(\ell)$ fits into the above diagram, so that the outer square is an pullback, therefore $\hat{\mathbb{Z}}(\ell)|_{\mathcal{B}} \simeq \check{C}^*(\ell)$.

Remark 5.8 (Model C, Continued). Since $\hat{\mathbb{Z}}(\ell)$ is equivalent to $\check{C}^*(\ell)$ restricted to $\mathcal{B}$, consider a manifold M having a $\mathcal{B}$-good open cover $\mathcal{O}$ and let $\mathcal{I}(\mathcal{O})$ the closure of $\mathcal{O}$ under finite intersections, then recall

$$\hat{\mathbb{Z}}(\ell)(M) \simeq \lim_{U \in \mathcal{I}(\mathcal{O})} \check{C}^*(\ell)(U)$$

By rearranging the limit, we see it is equivalent to the limit of the cosimplicial object taking n to the product over all tuples $(U_0, \cdots, U_n) \in \mathcal{O}^{n+1}$ of $\check{C}^*(\ell)(U_0 \cap \cdots \cap U_n)$. We can then apply [BNV16, Lemma 7.10] to calculate the limit of this cosimplicial cochain complex as the total cochain complex of

$$\check{C}^{n,m}(\ell) := \prod_{\mathcal{O}^{n+1}} \check{C}^m(\ell)(U_0 \cap \cdots \cap U_n)$$

The differential in the m-direction simply comes from $\check{C}^*(\ell)$. The differential in the n-direction is the alternating sum of the morphisms $\delta^i : \check{C}^{n,m}(\ell) \to \check{C}^{n+1,m}(\ell)$, for $0 \leq i \leq n+1$, described in components as follows: Given $\mu \in \check{C}^{n,m}(\ell)$, the $(U_0, \cdots, U_{n+1})$-component of $\delta^i\mu$ is the $(U_0, \cdots, \widehat{U_i}, \cdots, U_{n+1})$-component of μ restricted to $U_0 \cap \cdots \cap U_{n+1}$, where the hat under U_i means we're removing it from the tuple.

6. Unfolding the fracture square of Deligne Cohomology

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