DIFFERENTIAL COHOMOLOGY SEMINAR 8

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The aim of this talk is to review twisted cohomology theory with the aim of later discussing twisted differential cohomology theories [BG16]. For these talks the main source is [ABG+14, ABG18].

1. Twisted Cohomology

Let R be a ring spectrum, meaning a monoid object in the ∞-category of spectra $\mathcal{S}\mathrm{p}$. From this we get a presentable stable ∞-category $\mathcal{M}\mathrm{od}_R$ of left R-module spectra. Objects therein are morphisms of the form $R \wedge M \to M$ satisfying the usual associativity and unit conditions up to coherent homotopies.

Note we have an adjunction diagram

$$\begin{tikzcd} \mathcal{S}\mathrm{p} \arrow[r, shift left, "R\wedge -"] \arrow[r, phantom, "\perp"] & \mathcal{M}\mathrm{od}_R \arrow[l, shift left, "\mathrm{Hom}_R(R,-)"] \end{tikzcd},$$

where the right adjoint is in fact the forgetful functor. This in particular means $\mathcal{M}\mathrm{od}_R$ has a distinguished object R i.e. the free R-module of rank 1. We now refine these constructions.

Definition 1.1. Let R be a ring spectrum. An *R-line* is an R-module L such that $L \simeq R$.

Definition 1.2. Let $\mathcal{L}\mathrm{ine}_R$ be the full sub-∞-groupoid of $\mathcal{M}\mathrm{od}_R$ spanned by the R-lines.

Note by construction this is equivalent to $BGL_1(R)$, the ∞-group of R-linear automorphisms of R.

Definition 1.3. Let X be a space. Then we denote by $\mathcal{M}\mathrm{od}_R(X)$ the ∞-category of R-module spectra parametrized over X, i.e. the functor category $\mathrm{Fun}(X^{op}, \mathcal{M}\mathrm{od}_R)$.

Definition 1.4. Let X be a space. Then we denote by $\mathcal{L}\mathrm{ine}_R(X)$ the full sub-∞-groupoid of $\mathcal{M}\mathrm{od}_R(X)$ spanned by those functors $L\colon X^{op} \to \mathcal{L}\mathrm{ine}_R$, meaning it is $\mathrm{Fun}(X^{op}, \mathcal{L}\mathrm{ine}_R)$.

Recall this generalization of local systems of abelian groups, which map out of groupoids instead of ∞-groupoids.

Example 1.5. Let $R_X\colon X^{op} \to \mathcal{L}\mathrm{ine}_R$ be the constant functor, then this is an object in $\mathcal{L}\mathrm{ine}_R(X)$.

We now proceed to the Thom construction.

Definition 1.6 (Thom spectrum)**.** The *Thom R-module spectrum* is the functor

$$M\colon \mathcal{G}\mathrm{rpd}_\infty^{op}/\mathcal{L}\mathrm{ine}_R \to \mathcal{M}\mathrm{od}_R,$$

which sends $f\colon X^{op} \to \mathcal{L}\mathrm{ine}_R$ to the R-module spectrum $\mathrm{colim}(X^{op} \xrightarrow{f} \mathcal{L}\mathrm{ine}_R \to \mathcal{M}\mathrm{od}_R)$.

Let us note an alternative characterization that will be important later.

Remark 1.7. For a given map $f\colon X \to Y$, we get a map $f^*\colon \mathcal{M}\mathrm{od}_R(Y) \to \mathcal{M}\mathrm{od}_R(X)$ by precomposition with f^{op}. By the adjoint functor theorem, this functor admits a left adjoint $f_!\colon \mathcal{M}\mathrm{od}_R(X) \to \mathcal{M}\mathrm{od}_R(Y)$ and a right adjoint $f_*\colon \mathcal{M}\mathrm{od}_R(X) \to \mathcal{M}\mathrm{od}_R(Y)$. Now for $p\colon X \to *$, we have

$$Mf \simeq p_!(i \circ f).$$

Remark 1.8 ([ABG10, 3.6]). Let $\mathcal{T}\mathrm{riv}_R$ be the slice groupoid $\mathcal{L}\mathrm{ine}_R/R$, together with the canonical projection $\pi : \mathcal{T}\mathrm{riv}_R \to \mathcal{L}\mathrm{ine}_R$. An object of $\mathcal{T}\mathrm{riv}_R$ is a pair (L, ϕ) of a R-line L and an equivalence $\phi : L \to R$. Consider then the following:

(1) $\mathcal{T}\mathrm{riv}_R$ is a slice ∞-groupoid, hence contractible, and π is a Kan fibration.
(2) $GL_1(R)$ is equivalent to the fiber of π over $R \in \mathcal{L}\mathrm{ine}_R$ and acts freely on the fibers of π.

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These observations imply $\mathcal{L}\mathrm{ine}_R$ is equivalent to $BGL_1(R)$, the classifying space for R-line bundles.

We now proceed to define twisted cohomology theories.

Definition 1.9 (Twisted cohomology)**.** Let R be a ring spectrum, X be a space and let $p\colon X \to *$ be the projection map, and let $f\colon X^{op} \to \mathcal{L}\mathrm{ine}_R$ be an R-line bundle over X. The *f-twisted R-cohomology of X* is defined as the mapping spectrum

$$R_f(X) := \mathrm{Map}_{\mathcal{M}\mathrm{od}_R}(Mf, R) \simeq \mathrm{Map}_{\mathcal{M}\mathrm{od}_R(X)}(f, p^*R) \simeq \mathrm{Map}_{\mathcal{M}\mathrm{od}_R(X)}(f, R_X).$$

Similarly, we have *f-twisted R-homology of X* defined as

$$R^f(X) := \mathrm{Map}_{\mathcal{M}\mathrm{od}_R}(R, Mf) \simeq Mf.$$

In particular we can define the twisted cohomology groups of a space X with twist given by a map $f\colon X \to \mathcal{L}\mathrm{ine}_R$, by

$$R^n_f(X) := \pi_0(\mathrm{Map}_{\mathcal{M}\mathrm{od}_R}(Mf, \Sigma^n R)) \cong \pi_{-n}(\mathrm{Map}_{\mathcal{M}\mathrm{od}_R(X)}(f, R_X)).$$

Similarly we define the twisted homology groups by

$$R^f_n(X) := \pi_0(\mathrm{Map}_{\mathcal{M}\mathrm{od}_R}(\Sigma^n R, Mf)) \cong \pi_n(Mf).$$

Remark 1.10. Note that if f is the constant map with value R, then we recover ordinary R-cohomology and R-homology. Concretely, we have

$$Mf \simeq R \wedge \Sigma^\infty_+ X$$

and so it follows that

$$R_f(X) \simeq \mathrm{Map}_{\mathcal{M}\mathrm{od}_R}(R \wedge \Sigma^\infty_+ X, R) \simeq \mathrm{Map}_{\mathcal{S}\mathrm{p}}(\Sigma^\infty_+ X, R)$$

Applying π_{-n} this recovers the regular R-cohomology groups of X, and analogous for homology.

Let us look at some examples of twisted cohomology theories.

Example 1.11. Let $BO(n)$ be the topological groupoid of n-dimensional, inner product spaces and orthogonal morphisms, viewed as an ∞-groupoid. Consider the composition

$$f_n\colon BO(n) \xrightarrow{\mathrm{Th}_n} \mathcal{S}_* \xrightarrow{\Sigma^{\infty-n}} \mathcal{S}\mathrm{p}$$

where the first functor maps an inner product space V to its one-point compactification. Choosing an orthonormal basis for V, we get an isomorphism $\mathbb{R}^n \simeq V$, which induces an equivalence of spectra

$$\Sigma^{\infty-n}\mathrm{Th}(V) \simeq \Sigma^{\infty-n}\mathrm{Th}(\mathbb{R}^n) \simeq \Sigma^{\infty-n} S^n \simeq \mathbb{S}.$$

In particular, f_n factors through $\mathcal{L}\mathrm{ine}_{\mathbb{S}}$. Let $p : BO(n) \to *$, then

$$Mf_n \simeq p_!\Sigma^{\infty-n}\mathrm{Th}_n \simeq \Sigma^{\infty-n} p_!\mathrm{Th}_n \simeq \Sigma^{\infty-n}\mathrm{Th}(E_n),$$

where $E_n \to BO(n)$ is the universal n-dimensional vector bundle and $\mathrm{Th}(E_n)$ is the one-point compactification of the total space E_n. To prove the last equivalence, consider the following facts:

1. Given a inner product space V, the Thom space of V is homotopy equivalent to the homotopy cofiber of the inclusion $V \setminus \{0\} \subseteq V$.
2. Th_n factors as the cofiber functor $C : \mathrm{Arr}(\mathcal{S}) \to \mathcal{S}_*$ following the functor $F : BO(n) \to \mathrm{Arr}(\mathcal{S})$ mapping V to the inclusion $V \setminus \{0\} \subseteq V$. Here, $\mathrm{Arr}(\mathcal{S})$ denotes the ∞-category of arrows in $\mathcal{S}$.
3. $p_!$ commutes with C, since this last functor is left adjoint to the inclusion $\mathcal{S}_* \subseteq \mathrm{Arr}(\mathcal{S})$.
4. By straightening-unstraightening, a functor $f : BO(n) \to \mathcal{S}$ is equivalent to a Kan fibration $E^f = BO(n) \times_{\mathcal{S}} \mathcal{S}_* \to BO(n)$ and $p_!(f) \simeq E^f$.
5. Let F_1, resp. F_0, be the target and source components of F, then $E^{F_1} \simeq E_n$ and $E^{F_0} \simeq E_n \setminus \zeta_n$, where ζ_n is the zero section.
6. Finally, $p_!\mathrm{Th}_n$ is equivalent to the cofiber of $E_n \setminus \zeta_n \subseteq E_n$, which is equivalent to the one-point compactification of the total space E_n.

Example 1.12. Let O be the stable orthogonal group. The map J_n induced by f_n from the automorphisms of $\mathbb{R}$ to $\mathrm{Aut}(\mathbb{S})$ sends $\phi : \mathbb{R}^n \to \mathbb{R}^n$ to $\Sigma^{\infty-n}\mathrm{Th}(\phi) : \mathbb{S} \to \mathbb{S}$. Consider $\mathbb{R} \oplus - : O(n) \to O(n+1)$, then $\mathrm{Th}(\mathbb{R} \oplus \phi) = \mathrm{Th}(\mathbb{R}) \wedge \mathrm{Th}(\phi) = \Sigma\mathrm{Th}(\phi)$, therefore the diagram

$$\begin{tikzcd} O(n) \arrow[r, "\mathbb{R}\oplus -"] \arrow[dr, "J_n"'] & O(n+1) \arrow[d, "J_{n+1}"] \\ & GL_1(\mathbb{S}) \end{tikzcd}$$

commutes, and the maps J_n induced a group homomorphism $J : O \to GL_1(\mathbb{S})$, called *J-homomorphism.* The delooped map $BJ : BO \to \mathcal{L}\mathrm{ine}_{\mathbb{S}}$ is equivalent to the colimit of the maps f_n. The Thom spectrum of BJ is denoted MO, called the *real bordism spectrum.*

Example 1.13. The decomplexification map $U \to O$ induces a twisting over $X = BU$ by post-composition with BJ. The Thom spectrum of $BU \to BO \to \mathcal{L}\mathrm{ine}_{\mathbb{S}}$ is denoted MU, called the *complex bordism spectrum.* In general, a group homomorphism $\xi : G \to O$ induces a twisting over $X = BG$ by post-composing with BJ. By taking $G = SO, Spin$ or $String$, we obtain the *oriented, spin* and *string bordism spectra.*

Example 1.14. Recall that we have a diagram characterizing $BString$

$$K(\mathbb{Z}, 3) \to BString \to BSpin \to BGL_1(\mathbb{S}) = \mathcal{L}\mathrm{ine}_{\mathbb{S}}$$

then applying the Thom construction to this diagram we get

$$\Sigma^\infty_+ K(\mathbb{Z}, 3) \to MString \to MSpin.$$

Here we use the fact that the map $K(\mathbb{Z}, 3) \to \mathcal{L}\mathrm{ine}_{\mathbb{S}}$ factors through the point, and hence its Thom spectrum is $\Sigma^\infty_+ K(\mathbb{Z}, 3)$.

As a next step we have a string orientation map $MString \to \mathrm{tmf}$, where tmf is the spectrum of topological modular forms, following the computation in [AHR10]. Using the fact that Σ^∞_+ is monoidal, we know that the units of $K(\mathbb{Z}, 3)$ include $K(\mathbb{Z}, 3)$ itself, meaning we get a map $K(\mathbb{Z}, 3) \to GL_1(\mathrm{tmf})$. Applying B gives us a map $f : K(\mathbb{Z}, 4) \to BGL_1(\mathrm{tmf}) = \mathcal{L}\mathrm{ine}_{\mathrm{tmf}}$, classifying the twist of tmf-cohomology. Further details can be found in [ABG10].

Let us try a more feasible example.

Example 1.15. Let

$$K(1, \mathbb{Z}/2\mathbb{Z}) \to BSpin \to BSO$$

be the fiber sequence and let

$$K(\mathbb{Z}, 2) \to BSpin^c \to BSO \to BGL_1\mathbb{S} = \mathcal{L}\mathrm{ine}_{\mathbb{S}}.$$

Applying the Thom construction to the second sequence and again using the fact that $K(\mathbb{Z}, 2) \to \mathcal{L}\mathrm{ine}_{\mathbb{S}}$ factors through the point, we get the sequence

$$\Sigma^\infty_+ K(\mathbb{Z}, 2) \to MSpin^c \to MSO.$$

Now, using Atiyah–Bott–Shapiro orientation $MSpin^c \to KU$ [ABS64], we get a map $K(\mathbb{Z}, 2) \to GL_1 KU$, which induces a map $K(\mathbb{Z}, 3) \to \mathcal{L}\mathrm{ine}_{KU}$. This gives us twisted KU-cohomology of $K(\mathbb{Z}, 3)$. As $K(\mathbb{Z}, 3)$ classifies bundle gerbes, this is the twist of KU-theory by bundle gerbes.

2. Twists via Picard Groupoids and Grading

This section requires some some further details. Up until now we defined everything via $\mathcal{L}\mathrm{ine}_R$, however for many applications we need to work with $\mathcal{P}\mathrm{ic}_R$ instead.

Definition 2.1. Let R be a ring spectrum. The *Picard* ∞*-groupoid* $\mathcal{P}\mathrm{ic}_R$ is the full sub-∞-groupoid of $\mathcal{M}\mathrm{od}_R$ spanned by the invertible R-modules, i.e. those R-modules M such that there exists an R-module N with $M \wedge_R N \simeq R$.

Remark 2.2. It is a common fact that the ∞-groupoid $\mathcal{P}\mathrm{ic}_R$ is equivalent to $\pi_0(R) \times \mathcal{L}\mathrm{ine}_R$, meaning that the invertible R-modules are given by the R-lines together with a shift by an element in $\pi_0(R)$. Here it is important to note that this equivalence does not respect the monoidal structure is only an equivalence of underlying ∞-groupoids.

We can now generalize Definition 1.6 to $\mathcal{P}\mathrm{ic}_R$.

Definition 2.3 (Thom spectrum)**.** The *Thom R-module spectrum* is the functor

$$M\colon \mathcal{G}\mathrm{rpd}_\infty^{op}/\mathcal{P}\mathrm{ic}_R \to \mathcal{M}\mathrm{od}_R,$$

which sends $f\colon X^{op} \to \mathcal{P}\mathrm{ic}_R$ to the R-module spectrum $\mathrm{colim}(X^{op} \xrightarrow{f} \mathcal{P}\mathrm{ic}_R \to \mathcal{M}\mathrm{od}_R)$.

We can now generalize Definition 1.9 to $\mathcal{P}\mathrm{ic}_R$.

Definition 2.4 (Twisted cohomology)**.** Let R be a ring spectrum, X be a space and let $p\colon X \to *$ be the projection map, and let $f\colon X^{op} \to \mathcal{P}\mathrm{ic}_R$. The *$f$-twisted R-cohomology of X* is defined as the mapping spectrum

$$R_f(X) \coloneqq \mathrm{Map}_{\mathcal{M}\mathrm{od}_R}(Mf, R) \simeq \mathrm{Map}_{\mathcal{M}\mathrm{od}_R(X)}(f, p^*R) \simeq \mathrm{Map}_{\mathcal{M}\mathrm{od}_R(X)}(f, R_x).$$

Similarly, we have *f-twisted R-homology of X* defined as

$$R^f(X) \coloneqq \mathrm{Map}_{\mathcal{M}\mathrm{od}_R}(R, Mf) \simeq Mf.$$

Let us see how this additional generality helps us. Let $f\colon X \to \mathcal{L}\mathrm{ine}_R$, then we get a map $\Sigma^n f\colon \Sigma^n X \to \mathcal{P}\mathrm{ic}_R$, which induces a cohomology $R^{*+n}(X)$. Now we have

$$R_{\Sigma^n f}(X) \simeq \mathrm{Map}(M\Sigma^n f, R) \simeq \mathrm{Map}(\Sigma^n f, R_x) \simeq \mathrm{Map}(f, \Omega^n R_x) \simeq \Omega^n \mathrm{Map}(f, R_x) \simeq \Omega^n R_f(X)$$

This equivalence can only work via $\mathcal{P}\mathrm{ic}_R$. Indeed even though R is in $\mathcal{L}\mathrm{ine}_R$, $\Sigma^n R$ is not in $\mathcal{L}\mathrm{ine}_R$. However, it is in $\mathcal{P}\mathrm{ic}_R$, with inverse $\Sigma^{-n}R$.

This show that gradings can be recovered via twisted cohomology theories using $\mathcal{P}\mathrm{ic}_R$ as twists.

References

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