

# DIFFERENTIAL COHOMOLOGY SEMINAR 8

TALK BY HANNES BERKENHAGEN

The aim of this talk is to review twisted cohomology theory with the aim of later discussing twisted differential cohomology theories [BG16]. For these talks the main source is [ABG<sup>+</sup>14, ABG18].

## 1. TWISTED COHOMOLOGY

Let  $R$  be a ring spectrum, meaning a monoid object in the  $\infty$ -category of spectra  $\mathcal{S}p$ . From this we get a presentable stable  $\infty$ -category  $\mathcal{M}od_R$  of left  $R$ -module spectra. Objects therein are morphisms of the form  $R \wedge M \rightarrow M$  satisfying the usual associativity and unit conditions up to coherent homotopies.

Note we have an adjunction diagram

$$\mathcal{S}p \begin{array}{c} \xrightarrow{R \wedge -} \\ \xleftarrow[\text{Map}_{\mathcal{M}od_R}(R, -)]{\perp} \end{array} \mathcal{M}od_R ,$$

where the right adjoint is in fact the forgetful functor. This in particular means  $\mathcal{M}od_R$  has a distinguished object  $R$  i.e. the free  $R$ -module of rank 1. We now refine these constructions.

**Definition 1.1.** Let  $R$  be a ring spectrum. An  $R$ -line is an  $R$ -module  $L$  such that  $L \simeq R$ .

**Definition 1.2.** Let  $\mathcal{L}ine_R$  be the full sub- $\infty$ -groupoid of  $\mathcal{M}od_R$  spanned by the  $R$ -lines.

By construction,  $\mathcal{L}ine_R$  is equivalent to the category with a single object  $R$  and hom-space  $GL_1 R$ , the  $\infty$ -group of  $R$ -linear automorphisms of  $R$ . Notice that  $GL_1(R) \subseteq \text{Map}_{\mathcal{M}od_R}(R, R) \simeq \text{Map}(\mathbb{S}, R) = \Omega^\infty R$ .

**Lemma 1.3.**  $GL_1(R)$  fits into the pullback square

$$\begin{array}{ccc} GL_1(R) & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ \pi_0(R)^\times & \longrightarrow & \pi_0(R) \end{array}$$

In particular, the inclusion  $GL_1(R) \rightarrow \Omega^\infty R$  induces an isomorphism on  $n$ -homotopy groups, for all  $n \geq 1$ .

*Proof.*  $\pi_0(R) \simeq \text{Map}_{\text{ho}\mathcal{M}od_R}(R, R)$ , where  $\text{ho}\mathcal{M}od_R$  is the homotopy category of  $R$ -modules, the right-vertical arrow corresponds to  $\text{Map}_{\mathcal{M}od_R}(R, R) \rightarrow \text{Map}_{\text{ho}\mathcal{M}od_R}(R, R)$  mapping a morphism to its homotopy class, and  $\pi_0(R)^\times \subseteq \text{Map}_{\text{ho}\mathcal{M}od_R}(R, R)$  is the set of isomorphisms. Finally, a morphism in a  $\infty$ -category  $\mathcal{C}$  is an equivalence if and only if its homotopy class is an isomorphism in  $\text{ho}\mathcal{C}$ .  $\square$

**Definition 1.4.** Let  $X$  be a space. Denote by  $\mathcal{M}od_R(X)$  the  $\infty$ -category of  $R$ -module spectra parametrized over  $X$ , i.e. the functor category  $\text{Fun}(X^{op}, \mathcal{M}od_R)$ .

**Definition 1.5.** Let  $X$  be a space. Denote by  $\mathcal{L}ine_R(X)$  the  $\infty$ -groupoid of  $R$ -line spectra parametrized over  $X$ , i.e. the  $\text{Fun}(X^{op}, \mathcal{L}ine_R)$ .

Since  $\mathcal{L}ine_R \simeq BGL_1(R)$ , functors  $X^{op} \rightarrow \mathcal{L}ine_R$  are generalization of local systems.

**Example 1.6.** Let  $R_X: X^{op} \rightarrow * \rightarrow \mathcal{L}ine_R$  be the constant functor with value  $R$ .

We now proceed to the Thom construction.

**Definition 1.7** (Thom spectrum). The *Thom  $R$ -module spectrum* is the functor

$$M: \mathcal{G}rpd_\infty^{op} / \mathcal{L}ine_R \rightarrow \mathcal{M}od_R,$$

which sends  $f: X^{op} \rightarrow \mathcal{L}ine_R$  to the  $R$ -module spectrum  $\text{colim}(X^{op} \xrightarrow{f} \mathcal{L}ine_R \xrightarrow{i} \mathcal{M}od_R)$ .

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*Remark 1.8.* Notice that the definition of Thom  $R$ -module spectrum make sense for any functor  $X^{op} \rightarrow R\text{Mod}$ .

Let us note an alternative characterization that will be important later.

*Remark 1.9.* For a given map  $f: X \rightarrow Y$ , we get a map  $f^*: \text{Mod}_R(Y) \rightarrow \text{Mod}_R(X)$  by precomposition with  $f^{op}$ . Since  $f^*$  preserves both limits and colimits, we construct a left adjoint  $f_!: \text{Mod}_R(X) \rightarrow \text{Mod}_R(Y)$  and a right adjoint  $f_*: \text{Mod}_R(X) \rightarrow \text{Mod}_R(Y)$ , using left and right Kan extension along  $f^{op}$ . Let  $f = p: X \rightarrow *$  be the terminal functor, then left Kan extension along  $p^{op}$  is exactly taking colimit, therefore

$$Mf \simeq p_!(i \circ f).$$

*Remark 1.10.* The equivalence  $\mathcal{L}ine_R \simeq BGL_1(R)$  comes from the fact that  $\mathcal{L}ine_R$  is a pointed (by  $R: * \rightarrow \mathcal{L}ine_R$ ), connected  $\infty$ -groupoid and  $* \times_{\mathcal{L}ine_R} * \simeq \Omega \mathcal{L}ine_R \simeq GL_1(R)$ .

We now proceed to define twisted cohomology theories.

**Definition 1.11** (Twisted cohomology). Let  $R$  be a ring spectrum,  $X$  be a space,  $p: X \rightarrow *$  the terminal functor, and  $f: X^{op} \rightarrow \mathcal{L}ine_R$  a  $R$ -line bundle over  $X$ . The  $f$ -twisted  $R$ -cohomology of  $X$  is defined as the mapping spectrum

$$R_f(X) := \text{Map}_{\text{Mod}_R}(Mf, R) \simeq \text{Map}_{\text{Mod}_R(X)}(f, p^*R) \simeq \text{Map}_{\text{Mod}_R(X)}(f, R_X).$$

Similarly, the  $f$ -twisted  $R$ -homology of  $X$  is defined as

$$R^f(X) := \text{Map}_{\text{Mod}_R}(R, Mf) \simeq Mf.$$

The  $f$ -twisted  $R$ -cohomology groups of  $X$  are defined as the homotopy groups of  $R_f(X)$ , i.e.

$$R_f^n(X) := \pi_0(\text{Map}_{\text{Mod}_R}(Mf, \Sigma^n R)) \cong \pi_{-n}(\text{Map}_{\text{Mod}_R(X)}(f, R_X)).$$

Similarly, the  $f$ -twisted  $R$ -homology groups of  $X$  are defined as the homotopy groups of  $R^f(X)$ , i.e.

$$R_n^f(X) := \pi_0(\text{Map}_{\text{Mod}_R}(\Sigma^n R, Mf)) \cong \pi_n(Mf).$$

**Example 1.12** (Trivial twist). If  $f: X \rightarrow \mathcal{L}ine_R$  factors through  $*$ , then  $f$  factors as the  $X^{op} \rightarrow * \rightarrow \mathcal{S}$ , the constant factor with value  $*$ , and  $R \wedge \Sigma_+^\infty(-): \mathcal{S} \rightarrow \text{Mod}_R$ . The latter functor commutes with colimits, being a left adjoint, while the colimit of the latter is  $X$  itself, then  $Mf \simeq R \wedge \Sigma_+^\infty X$ . In particular,  $f$ -twisted  $R$ -cohomology and  $R$ -homology of  $X$  reduce to ordinary (untwisted)  $R$ -cohomology and  $R$ -homology of  $X$ .

**Definition 1.13.** Given a vector bundle  $\pi: E \rightarrow B$ , define the *Thom space* of  $\pi$ , denoted  $\text{Th}(E)$ , to be the homotopy cofiber of  $E_0 \subseteq E$ , where  $E_0$  is the complement of the zero section.

## 2. EXAMPLES OF TWISTED COHOMOLOGY

We now proceed to analyze several examples of twisted cohomology theories. This requires some preliminary lemmas.

**Lemma 2.1.** Consider a space  $X$  and the  $\infty$ -categorical Yoneda's embedding  $y: X \rightarrow \text{Fun}(X^{op}, \mathcal{S})$ . The colimit of  $y$  is the terminal pre-sheaf on  $X$ , i.e. the pre-sheaf with constant value the one-point space.

*Proof.* Let  $S$  be a pre-sheaf on  $X$ , consider then the slice category  $X_{/S}$  of pairs  $(x, \phi)$ , where  $x$  is an object of  $X$  and  $\phi: y(x) \rightarrow S$ . The density theorem for  $\infty$ -categories states that  $S$  is equivalent to the colimit of  $X_{/S} \rightarrow X \xrightarrow{y} \text{Fun}(X^{op}, \mathcal{S})$ , the first map being the canonical projection. Take  $S = *$ , then  $X_{/*} \rightarrow X$  is an equivalence, hence the claim.  $\square$

Let  $G$  be a topological group and  $BG$  the  $\infty$ -groupoid with a single object 1 and hom-space  $G$ . The category  $\mathcal{S}_G := \text{Fun}(BG, \mathcal{S})$  is equivalent to the category of  $G$ -spaces.

**Lemma 2.2.** Consider  $X = BG$ , a  $G$ -space  $f: X \rightarrow \mathcal{S}$  and its left Kan extension  $f_!: \mathcal{S}_G \rightarrow \mathcal{S}$ , then  $f_! \simeq (- \times E)/G$ , where  $E = f(1)$ .

*Proof.* Evaluate at 1, then  $f_!(y(1)) = E$ , by definition, and  $y(1) \simeq G$ , as  $G$ -spaces, hence  $(y(1) \times E)/G \simeq (G \times E)/G \simeq E$ . Since  $f_!$  and  $(- \times E)/G$  agree on representables and are colimit-preserving, they are equivalent.  $\square$

**Example 2.3.** Take the space  $BO(n)$  and  $f_n : BO(n) \rightarrow \mathbb{S}_*$  the  $n$ -sphere  $S^n$  with  $O(n)$ -action coming from the one-point compactification of the regular action on  $\mathbb{R}^n$ . Let  $\alpha_n = \Sigma^{\infty-n} f_n : BO(n) \rightarrow \mathbb{S}_p$ , then  $\alpha_n(1) = \Sigma^{\infty-n} f_n(1) = \Sigma^{\infty-n} S^n \simeq \mathbb{S}$ , so  $\alpha_n$  factors through  $\mathcal{L}ine_{\mathbb{S}}$ . Let  $X = BO(n)^{op}$  and  $p : X \rightarrow *$  the terminal functor, then

$$M\alpha_n = p_! \Sigma^{\infty-n} f_n \simeq \Sigma^{\infty-n} p_!(f_n)_! y \simeq \Sigma^{\infty-n} (f_n)_! \underbrace{p_!(y)}_{\simeq *} \simeq \Sigma^{\infty-n} (* \times S^n)/O(n)$$

Let  $P = EO(n)$  be the universal  $O(n)$ -bundle and  $M$  a  $O(n)$ -space, then  $* \times_{O(n)} M$  is modelled by the *strict* quotient  $(P \times M)/O(n)$ , then

$$S^n/O(n) = \text{cofib}(\mathbb{R}_0^n \subseteq \mathbb{R}^n)/O(n) \simeq \text{cofib}(\underbrace{* \times_{O(n)} \mathbb{R}_0^n}_{\simeq E_0^n} \subseteq \underbrace{* \times_{O(n)} \mathbb{R}^n}_{\simeq E^n}) = \text{Th}(E^n)$$

where  $E^n = P \times_{O(n)} \mathbb{R}^n \rightarrow BO(n)$  is the universal  $n$ -dimensional vector bundle, hence  $M\alpha_n \simeq \Sigma^{\infty-n} \text{Th}(E^n)$ .

The functor  $BO(n) \rightarrow \mathcal{L}ine_{\mathbb{S}}$  induces a  $\infty$ -group homomorphism  $j_n : O(n) \rightarrow GL_1(\mathbb{S})$ , mapping  $\phi$  to  $\Sigma^{\infty-n} \text{Th}(\phi)$ . Consider the suspension morphism  $s_n = \mathbb{R} \oplus - : O(n) \rightarrow O(1+n)$ , then

$$j_n(\mathbb{R} \oplus \phi) = \Sigma^{\infty-n-1} \text{Th}(\mathbb{R} \oplus \phi) \simeq \Sigma^{\infty-n-1} \underbrace{\text{Th}(\mathbb{R})}_{\simeq S^1} \wedge \text{Th}(\phi) \simeq \Sigma^{\infty-n} \text{Th}(\phi) = j_n$$

Recall that the colimit over the suspension morphisms  $s_n$  is the stable orthogonal group  $O$ .

**Definition 2.4.** Denote by  $j$  the induced group homomorphism  $O \rightarrow GL_1(\mathbb{S})$ , called the *J-homomorphism*.

**Example 2.5.** Let  $X = O^{op}$  and take  $Bj : BO \rightarrow \mathcal{L}ine_{\mathbb{S}}$ , then  $Mj$  is denoted  $MO$  and called the *real bordism spectrum*.

Denote by  $M$  the extended Thom spectrum functor  $\text{Grp}_{\infty}^{op}/\text{Mod}_R \rightarrow \text{Mod}_R$ , this is a left adjoint to the functor  $\mathcal{O}$  sending a  $R$ -module to the functor  $* \rightarrow \text{Mod}_R$  picking out  $M$ . In particular,  $M$  preserves colimits and  $Bj \simeq \text{colim}_n Bj_n$ , therefore we have the following:

**Theorem 2.6.**  $MO \simeq \text{colim}_n MO(n) = \text{colim}_n \Sigma^{\infty-n} \text{Th}(E^n)$ .

**Example 2.7.** A group homomorphism  $\xi : G \rightarrow O$  induces a functor  $f : BG \rightarrow \mathcal{L}ine_{\mathbb{S}}$ . The Thom spectrum  $Mf$  is denoted  $MG$  or  $M\xi$ , and called *G-bordism spectrum*. For  $G = U, SO, Spin$ , and *String*, we obtain the *complex, oriented, spin, and string bordism spectra*.

*Remark 2.8.* In [Equation \(2.7\)](#) we might take  $G = \{*\}$ , the one-point group, then  $MG \simeq \mathbb{S}$ , which, if it didn't have a name, might be called *framed bordism spectrum*, following the naming convention in [Equation \(2.7\)](#) and in line with the theorem that  $\pi_*(\mathbb{S}) \simeq \Omega_*^{\text{fr}}$ , the bordism ring of framed (trivialized tangent bundle) smooth manifolds.

Let  $R$  be a ring in sets, then  $R$  is a  $A_{\infty}$ -ring spectrum (actually,  $E_{\infty}$ ),  $\Omega^{\infty} R$  is equivalent to  $R$  with discrete topology ( $\pi_0(\Omega^{\infty} R) \simeq R$ , as sets, and every other homotopy group vanish). In particular,  $GL_1(R)$  is simply  $R^{\times}$  with discrete topology. Consider then the fiber sequence  $SO \rightarrow O \rightarrow \mathbb{Z}^{\times} \simeq GL_1(\mathbb{Z})$ .

**Example 2.9.**  $X = BO^{op}$  and  $\alpha = w_1 : BO \rightarrow \mathcal{L}ine_{\mathbb{Z}}$ , the 1st Stiefel-Whitney class (delooping of the determinant  $O \rightarrow GL_1(\mathbb{Z})$ ), then  $Mw_1$  is a  $\mathbb{Z}$ -module spectra. Let  $i : SO \subseteq O$ , then  $w_1 i$  factors through the point, so  $M(w_1 i) \simeq \mathbb{Z} \wedge \Sigma_+^{\infty} SO$ .

Given  $f : X^{op} \rightarrow \mathcal{L}ine_R$  and a sequence  $F \xrightarrow{i} X \xrightarrow{\pi} Y$ , there is an induced sequence of Thom  $R$ -module spectra  $MF \rightarrow MX \rightarrow MY$ . If  $\pi i$  factors through the point,  $MF \simeq R \wedge \Sigma_+^{\infty} F$ .

**Lemma 2.10.** Let  $R$  be a ring spectrum and  $X$  a connected monoidal  $\infty$ -groupoid, then

$$\text{Map}_{\text{Mon}(\mathbb{S}_p)}(\Sigma_+^{\infty} X, R) \simeq \text{Map}_{\text{Mon}(\mathbb{S})}(X, GL_1(R))$$

*Proof.* Since  $X$  is connected, the space of homomorphisms  $X \rightarrow GL_1(R)$  is equivalent to the space of homomorphisms  $X \rightarrow \Omega^{\infty} R$ , then use that  $(\Sigma^{\infty}, \Omega^{\infty})$  is a monoidal adjunction (The monoidal structure on spectra is such that  $\Sigma^{\infty}$  is strong monoidal).  $\square$

*Remark 2.11.* Notice that we can weaken the result. Namely, if  $X$  is 1-connected (pointed and connected), then the space of functors (of  $\infty$ -groupoids)  $X \rightarrow GL_1(R)$  is equivalent to the space of functors  $X \rightarrow \Omega^\infty R$  such that  $* \rightarrow X \rightarrow \Omega^\infty R$  is an equivalence. This last space is equivalent, via the  $(\Sigma_+^\infty, \Omega^\infty)$  adjunction, to the space of morphisms of spectra  $\Sigma_+^\infty X \rightarrow R$ , such that  $\mathbb{S} \rightarrow \Sigma_+^\infty X \rightarrow R$  represents a unit in  $\pi_0(R)$ .

*Remark 2.12.* Notice that we can also strengthen the result. Namely, if  $X$  is a connected, commutative monoid object and  $R$  is a commutative ring spectrum, then  $\Omega^\infty R$  and  $GL_1(R)$  are also commutative monoid objects. Using the same argument, together with the fact that  $(\Sigma^\infty, \Omega^\infty)$  is actually a *symmetric* monoidal adjunction, we conclude that

$$\mathrm{Map}_{\mathrm{CMon}(\mathrm{Sp})}(\Sigma_+^\infty X, R) \simeq \mathrm{Map}_{\mathrm{CMon}(\mathrm{S})}(X, GL_1(R))$$

*Remark 2.13.* Let  $\mathcal{D}$  be a monoidal  $\infty$ -category. Consider  $\mathrm{Cat}_\infty/\mathcal{D}$ , the  $\infty$ -category of functors into  $\mathcal{D}$ , with monoidal structure given by

$$(F : \mathcal{A} \rightarrow \mathcal{D}, G : \mathcal{B} \rightarrow \mathcal{D}) \longmapsto (\mathcal{A} \times \mathcal{B} \xrightarrow{F \times G} \mathcal{D} \times \mathcal{D} \xrightarrow{\otimes} \mathcal{D})$$

The monoidal unit is the functor  $* \rightarrow \mathcal{D}$  picking out the monoidal unit of  $\mathcal{D}$ . If  $\mathcal{D}$  is symmetric monoidal, then so is  $\mathrm{Cat}_\infty/\mathcal{D}$ . A (commutative) monoid object in  $\mathrm{Cat}_\infty/\mathcal{D}$  is given by a (symmetric) monoidal category  $\mathcal{C}$  and a (symmetric) monoidal functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ .

In view of [Equation \(2.13\)](#), let  $R$  be commutative ring spectrum, then  $\mathrm{Mod}_R$  is a symmetric monoidal  $\infty$ -category and  $\mathrm{Line}_R$  is a symmetric monoidal  $\infty$ -groupoid. The category  $\mathrm{Grpd}_\infty^{\mathrm{op}}/\mathrm{Line}_R$  is then symmetric monoidal and (commutative) monoid objects are given by (symmetric) monoidal  $\infty$ -groupoids  $X^{\mathrm{op}}$  a (symmetric) monoidal functors  $X^{\mathrm{op}} \rightarrow \mathrm{Line}_R$ . One can then check that  $M$  is a symmetric monoidal functor, so that (commutative) monoid objects are sent to (commutative) monoid objects in  $\mathrm{Mod}_R$ , i.e. (commutative)  $R$ -algebras.

**Example 2.14.** Let  $\mathrm{Tmf}$  be the commutative ring spectrum of topological modular forms (see [Equation \(2.15\)](#)) and  $\sigma : MString \rightarrow \mathrm{tmf}$  the *String*-orientation of  $\mathrm{tmf}$ . In the sequence

$$BString \longrightarrow BO \xrightarrow{Bj} \mathrm{Line}_{\mathbb{S}}$$

all functors are symmetric monoidal, so that  $MString$  is a commutative  $\mathbb{S}$ -algebra, i.e. a commutative ring. The *String*-orientation of  $\mathrm{tmf}$  is also a commutative ring homomorphism. In the fiber sequence  $K(\mathbb{Z}, 3) \rightarrow BString \rightarrow BSO$ , the fiber map  $i : K(\mathbb{Z}, 3) \rightarrow BString$  is also a symmetric monoidal, so the composition

$$\Sigma_+^\infty K(\mathbb{Z}, 3) \xrightarrow{Mi} MString \xrightarrow{\sigma} \mathrm{tmf}$$

is a commutative ring homomorphism. Using [Equation \(2.10\)](#), we conclude that the induced homomorphism  $K(\mathbb{Z}, 3) \rightarrow \Omega^\infty \mathrm{tmf}$  (which is a homomorphism of commutative monoid objects, given [Equation \(2.12\)](#)) factors through  $GL_1(\mathrm{tmf})$ , and so it induces a (symmetric monoidal) functor  $K(\mathbb{Z}, 4) \rightarrow \mathrm{Line}_{\mathrm{tmf}}$ , i.e. *2-bundle gerbes twist*  $\mathrm{tmf}$ .

*Remark 2.15.* The spectrum of topological modular forms comes in three main flavors, namely:

- (1)  $\mathrm{TMF}$ , i.e. the global sections of the spectral structure sheaf  $\mathcal{O}^{\mathrm{top}} : (\mathrm{Aff}/\mathcal{M}_{\mathrm{ell}})^{\mathrm{op}} \rightarrow \mathrm{CMon}(\mathrm{Sp})$  on the (étale site of the) moduli stack of elliptic curves.
- (2)  $\mathrm{Tmf}$ , i.e. the global sections of the spectral structure sheaf  $\bar{\mathcal{O}}^{\mathrm{top}} : (\mathrm{Aff}/\bar{\mathcal{M}}_{\mathrm{ell}})^{\mathrm{op}} \rightarrow \mathrm{CMon}(\mathrm{Sp})$  on the (étale site of the) *compactified* moduli stack of elliptic curves. The inclusion  $\mathcal{M}_{\mathrm{ell}} \hookrightarrow \bar{\mathcal{M}}_{\mathrm{ell}}$  induces a commutative ring homomorphism  $\mathrm{Tmf} \rightarrow \mathrm{TMF}$ .
- (3)  $\mathrm{tmf}$ , i.e. the connective cover of  $\mathrm{Tmf}$ . By definition, there is a commutative ring homomorphism  $\mathrm{tmf} \rightarrow \mathrm{Tmf}$ .

In [\[AHR10\]](#),  $\mathrm{tmf}$  is used to denote our  $\mathrm{TMF}$ , in [\[Goe09\]](#),  $\mathrm{tmf}$  is used to denote our  $\mathrm{Tmf}$ , and [\[DFHH14\]](#) has the same notation as us. In [Equation \(2.14\)](#), we use  $\mathrm{tmf}$  to mean the connective cover of  $\mathrm{Tmf}$ .

Let us go down one step in the chromatic ladder.

**Example 2.16.** Recall the fiber sequences for  $Spin$  and  $Spin^c$ :

$$\mathbb{Z}_2 \rightarrow Spin \rightarrow SO, \quad S^1 \rightarrow Spin^c \rightarrow SO$$

All the spaces involved are commutative groups. Applying the Thom spectrum functor to the delooped sequences, we get

$$\Sigma_+^\infty K(\mathbb{Z}_2, 1) \rightarrow MSpin \rightarrow MSO, \quad \Sigma_+^\infty K(\mathbb{Z}, 2) \rightarrow MSpin^c \rightarrow MSO$$

Let  $\sigma : MSpin \rightarrow KO$  and  $\sigma^c : MSpin^c \rightarrow KU$  be the Atiyah-Bott-Shapiro orientation of real and complex  $K$ -theory (see [ABS64]). Similar to Equation (2.14), we get homomorphisms

$$\Sigma_+^\infty K(\mathbb{Z}_2, 1) \longrightarrow MSpin \xrightarrow{\sigma} KO, \quad \Sigma_+^\infty K(\mathbb{Z}, 2) \longrightarrow MSpin^c \xrightarrow{\sigma^c} KU$$

Using Equation (2.10) again and delooping, we obtain functors  $K(\mathbb{Z}_2, 2) \rightarrow \mathcal{L}ine_{KO}$  and  $K(\mathbb{Z}, 3) \rightarrow \mathcal{L}ine_{KU}$ , i.e. *real, resp. complex, bundle gerbes twist real, resp. complex,  $K$ -theory*.

### 3. TWISTS VIA PICARD GROUPOIDS AND GRADING

This section requires some further details. Up until now we defined everything via  $\mathcal{L}ine_R$ , however for many applications we need to work with  $\mathcal{P}ic_R$  instead.

**Definition 3.1.** Given a monoidal  $\infty$ -category  $(\mathcal{C}, \otimes, 1)$ , an object  $M$  is *invertible* if there is an object  $N$  such that  $N \otimes M \simeq M \otimes N \simeq 1$ . The *Picard  $\infty$ -groupoid* of  $\mathcal{C}$  is the sub- $\infty$ -groupoid generated by invertible modules.

*Remark 3.2.* A monoidal category  $(\mathcal{C}, \otimes, 1)$  is *closed* if, for every object  $M$ , the *left tensoring with  $M$*  functor  $M^\otimes : \mathcal{C} \rightarrow \mathcal{C}$  admits a right adjoint  $F(M, -)$ . If  $M$  is invertible, with inverse  $N$ , then  $M^\otimes$  is an equivalence with  $N^\otimes$  as inverse. In particular, we can promote  $(M^\otimes, N^\otimes)$  to an adjoint equivalence. By uniqueness of adjoint functors  $F(M, -) \simeq N^\otimes$ , and so

$$N \simeq N \otimes 1 = N^\otimes(1) \simeq F(M, 1) =: DM$$

**Definition 3.3.** Given a closed monoidal category  $(\mathcal{C}, \otimes, 1)$ , the functor  $D := F(-, 1) : \mathcal{C}^{op} \rightarrow \mathcal{C}$  will be called *duality*. Given an object  $M$ , the object  $DM$  is called the *dual of  $M$* .

**Definition 3.4.** If  $R$  is a ring spectrum,  $\mathcal{M}od_R$  is closed monoidal. Denote by  $\mathcal{P}ic_R$  the Picard  $\infty$ -groupoid of  $\mathcal{M}od_R$ .

*Remark 3.5.*  $\Sigma^n R$  is invertible, with inverse  $\Sigma^{-n} R$ . In particular, there is a map  $\mathbb{Z} \times \mathcal{L}ine_R \rightarrow \mathcal{P}ic_R$ . However, this map need not be neither injective (if  $R$  is  $n$ -periodic), nor surjective (see [HM17]).

As mentioned in Equation (1.8), the Thom spectrum functor makes sense for every functor  $f : X^{op} \rightarrow \mathcal{M}od_R$ . However, all examples of twists encountered so far came from functors into  $\mathcal{L}ine_R$ . An example of twist that is not the result of a  $R$ -line bundle is the *degree shift*.

**Definition 3.6.** Denote by  $M$  the *Thom  $R$ -module spectrum* functor

$$\mathcal{G}rpd_\infty^{op}/\mathcal{M}od_R \rightarrow \mathcal{M}od_R$$

sending a functor  $f : X^{op} \rightarrow \mathcal{M}od_R$  to its colimit.

**Example 3.7.** Let  $f : X^{op} \rightarrow \mathcal{L}ine_R$  be a twist. Denote by  $\Sigma^n f$  the composition of  $f$  with the shift functor  $\Sigma^n : \mathcal{L}ine_R \rightarrow \mathcal{P}ic_R$ . Since  $\Sigma^n$  is an equivalence, it commutes with colimits, so

$$M\Sigma^n f \simeq \Sigma^n Mf$$

If  $f = R_X$ , then  $M\Sigma^n f \simeq \Sigma^n R \wedge \Sigma_+^\infty X$ , so  $\Sigma^n f$ -twisted  $R$ -cohomology and  $R$ -homology correspond to normal  $R$ -cohomology and  $R$ -homology with a degree shift by  $n$ .

### 4. UMKEHR MAP

We now proceed to the construction of the umkehr map in twisted cohomology theories. Here we follow [ABG18]. Let  $R$  be a ring spectrum, denote by  $D_R$  the duality of  $\mathcal{M}od_R$ . Given a invertible  $R$ -twist  $\alpha : X^{op} \rightarrow \mathcal{P}ic_R$ , denote by  $D_R \alpha$  the post-composition of  $\alpha^{op}$  with  $D_R$ .

*Remark 4.1.* Depending on which cohomological degrees we want, we sometimes use the following alternative definition of twisted cohomology:

$$R^\alpha(X) = \pi_0 \text{Map}_{\text{Mod}_R}(M(D_R\alpha), R)$$

meaning we use the dual/inverse of  $\alpha$ . To justify the use of  $D_R\alpha$ , consider the following: Let  $\beta = D_R\alpha$ , then

$$\text{Map}_{\text{Mod}_R}(M\beta, R) \simeq \text{Map}_{\text{Mod}_R(X)}(D_R\alpha, R_X) \simeq \text{Map}_{\text{Mod}_R(X)}(R_X, \alpha \otimes_{R_X} R_X) \simeq \text{Map}_{\text{Mod}_R(X)}(R_X, \alpha)$$

The second equivalence is a consequence of the fact that, if  $M$  be an invertible  $R$ -module, left tensoring with  $M$  is left adjoint to tensoring with  $D_RM$ , and  $D_RD_RM \simeq M$ , see [Equation \(3.2\)](#). If we think of  $\alpha$  as a bundle of invertible  $R$ -modules over  $X$ , then  $\text{Map}_{\text{Mod}_R(X)}(R_X, \alpha)$  is the spectrum of global sections of  $\alpha$ , which aligns with the idea that twisted cohomology is the homotopy groups of the global sections of a bundle of spectra.

If  $R = \mathbb{S}$ , we denote the duality  $D_R$  by simply  $D$ .

**Definition 4.2.** The *Spanier-Whitehead dual* of a space  $X$  is the dual spectrum of  $\Sigma_+^\infty X$  with respect to the sphere spectrum.

*Remark 4.3.* Recall that  $\mathbb{S}\mathfrak{p}$  is a closed symmetric monoidal category, with  $\text{Map}(E, -)$  being the right adjoint to the functor given by smash product with  $E$ . Let  $(\mathcal{C}, \otimes, 1)$  a closed monoidal category, where  $F(X, -)$  is the right adjoint to the functor given by left tensoring with  $X$ , then

$$\text{Map}_{\mathcal{C}}(-, F(1, E)) \simeq \text{Map}_{\mathcal{C}}(1 \otimes -, E) \simeq \text{Map}_{\mathcal{C}}(-, E)$$

Using Yoneda's lemma, we conclude that  $E \simeq F(1, E)$ , for all  $E$ .

**Example 4.4.** By [Equation \(4.3\)](#), the Spanier-Whitehead dual to  $*$  is the sphere spectrum, since

$$D* = \text{Map}(\Sigma_+^\infty *, \mathbb{S}) = \text{Map}(\Sigma^\infty S^0, \mathbb{S}) = \text{Map}(\mathbb{S}, \mathbb{S}) \simeq \mathbb{S}$$

**Definition 4.5.** Denote by  $\phi$  the functor  $\mathcal{S}^{op} \rightarrow \mathbb{S}/\mathbb{S}\mathfrak{p}$  from spaces to the category of spectra under  $\mathbb{S}$ , mapping  $X$  to the map of spectra

$$\phi(X) : \mathbb{S} \simeq D* \xrightarrow{Dp} DX$$

where  $p$  is the terminal map  $X \rightarrow *$ .

**Definition 4.6.** Denote by  $\widehat{\text{Mfd}}_\infty$  the topological groupoid of closed smooth manifolds and diffeomorphisms. The set of diffeomorphisms are topologized by the weak  $C^\infty$  topology, see [\[Hir94\]](#). Let  $\text{Mfd}_\infty$  be homotopy coherent nerve of  $\widehat{\text{Mfd}}_\infty$ .

Let  $B$  be a connected compact space and  $\pi : X \rightarrow B$  a continuous function such that, for every  $b \in B$ , the fiber  $\pi^{-1}(b) =: X_b$  is equipped with the structure of a closed smooth manifold, which varies continuously in  $B$ , in the sense of being classified by a functor

$$f : B \rightarrow \text{Mfd}_\infty$$

Given such a classifying map  $f$ , consider the following composition

$$B^{op} \longrightarrow \text{Mfd}_\infty^{op} \longrightarrow \mathcal{S}^{op} \xrightarrow{\phi} \mathbb{S}/\mathbb{S}\mathfrak{p}$$

where the middle functor is the one forgetting the smooth structure. The above composition is then an object of

$$\text{Fun}(B^{op}, \mathbb{S}/\mathbb{S}\mathfrak{p}) \simeq \mathbb{S}_B / \text{Fun}(B^{op}, \mathbb{S}\mathfrak{p})$$

i.e. it is a natural transformation

$$\phi_{X/B} : \mathbb{S}_B \rightarrow D_B(f)$$

where  $D_B$  denotes the Spanier-Whitehead duality applied to (the opposite of)  $f : B \rightarrow \mathbb{S}$  point-wise. Let  $p : B \rightarrow *$  be the terminal functor and  $p_! : \text{Fun}(B^{op}, \mathbb{S}\mathfrak{p}) \rightarrow \mathbb{S}\mathfrak{p}$  the left Kan extension along  $p$ , i.e. the colimit functor. Applying  $p_!$  to  $\phi_{X/B}$ , we obtain a morphism of spectra

$$(4.7) \quad \Sigma_+^\infty B \simeq p_! p^* \mathbb{S} = p_! \mathbb{S}_B \xrightarrow{p_!(\phi_{X/B})} p_! D_B(f)$$

Let  $T_{X/B} \rightarrow X$  be the vector bundle of fiber-wise tangent vectors, classified by  $X^{op} \rightarrow BO(n)$ , where  $n$  is the manifold dimension of the fibers<sup>1</sup>. Denote by  $\alpha$  the composition

$$(4.8) \quad X^{op} \xrightarrow{T_{X/B}} BO(n) \longrightarrow \mathcal{S}_* \xrightarrow{\Sigma^\infty} \mathcal{S}p$$

the middle arrow being the  $n$ -sphere  $S^n$  with  $O(n)$ -action coming from the one-point compactification of the regular action on  $\mathbb{R}^n$ . The above diagram takes values in the Picard  $\infty$ -groupoid of the sphere spectrum.

**Theorem 4.9.**  $p_! D_B(f) \simeq M(-\alpha) =: X^{-T_{X/B}}$ .

**Definition 4.10.** The morphism

$$PT(f) : \Sigma_+^\infty B \longrightarrow X^{-T_{X/B}}$$

is called *Pontryagin-Thom transfer map*.

Consider now a general situation. Let  $R$  be a ring spectrum and  $\alpha : X^{op} \rightarrow \mathcal{L}ine_R$  a  $R$ -line bundle.

**Definition 4.11.** An *orientation* of  $M\alpha$  is lift

$$\begin{array}{ccc} & & * \\ & \nearrow & \downarrow \\ X^{op} & \xrightarrow{\alpha} & \mathcal{L}ine_R \end{array}$$

Explicitly, it is an equivalence of  $R_X$ -modules  $\alpha \rightarrow R_X$ .

Applying  $p_!$ , we see that an equivalence  $t : \alpha \rightarrow R_X$  induces an equivalence

$$M\alpha \xrightarrow{p_!(t)} R \wedge \Sigma_+^\infty X$$

In our case,  $\alpha$  in Equation (4.8) is not valued in  $\mathcal{L}ine_{\mathcal{S}}$ , but  $\Sigma^{-n}\alpha$  is. An orientation of  $\Sigma^{-n}\alpha$  induces an equivalence

$$(4.12) \quad \Sigma^{-n} M\alpha \xrightarrow{\simeq} R \wedge \Sigma_+^\infty X$$

**Definition 4.13.** The isomorphism in cohomology induced by Equation (4.12) is called *Thom isomorphism*.

Finally, we have the following definition:

**Definition 4.14.** Assuming  $\Sigma^{-n}\alpha : X^{op} \rightarrow \mathcal{L}ine_R$  is orientable, the *Umkehr map* is the map:

$$R^*(\Sigma_+^\infty X) \xrightarrow{\text{Thom iso.}} R^{*-n}(X^{-T_{X/B}}) \xrightarrow{PT(f)^{*-n}} R^{*-n}(\Sigma_+^\infty B)$$

*Remark 4.15.* The name *Umkehr* comes from the map going in the opposite direction to  $R^*(\Sigma_+^\infty X) \rightarrow R^*(\Sigma_+^\infty B)$ , given by pulling back along  $\pi$ .

We now introduce the *twisted Umkehr map*. Given a twist  $\beta : B^{op} \rightarrow \mathcal{P}ic_R$ , we can smash to get a map

$$\beta \simeq \mathbb{S}_B \wedge_B \beta \longrightarrow D_B(f) \wedge_B \beta$$

Applying the functor  $p_!$  once again, we obtain a morphism

$$M\beta \longrightarrow p_!(D_B(f) \wedge_B \beta) \simeq X^{-T_f + \beta\pi}$$

where  $\beta\pi : X^{op} \rightarrow \mathcal{P}ic_R$  is the composition of  $\pi$  and the twist  $\beta$ .

*Remark 4.16.* The *twisted Pontryagin-Thom transfer map* is the morphism:

$$PT(f, \beta) : M\beta \longrightarrow X^{-T_f + \beta\pi}$$

If  $X^{-T_{X/B}}$  is orientable, as before, there is a *twisted Thom isomorphism*

$$R^{*- \beta\pi}(X) \xrightarrow{\simeq} R^{*-n - \beta\pi}(X)$$

The *twisted Umkehr map* is the map of twisted cohomology groups induced by  $PT(f, \beta)$ .

**Example 4.17.** Assume that  $X^{-T_{X/B} + \beta\pi}$  is orientable, i.e. the following diagram commutes

<sup>1</sup> $B$  is connected, so the fibers have constant dimension.

$$\begin{array}{ccc}
X & \xrightarrow{-T_f} & \mathcal{P}ic_{\mathbb{S}} \\
\downarrow & & \downarrow \\
B & \xrightarrow{\alpha} & \mathcal{P}ic_R
\end{array}$$

then  $X^{-T_{X/B}+\beta\pi} \simeq R \wedge \Sigma_+^\infty X$  and the twisted Umkehr map becomes:

$$R^*(\Sigma_+^\infty X) \longrightarrow R^{*- \beta}(B)$$

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