DIFFERENTIAL COHOMOLOGY SEMINAR 5

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In this talk we summarize what we covered until now and discuss possible future directions.

1. Summary

We established that in the modern point of view a differential cohomology theory is an ∞-categorical sheaf on the site of manifolds valued in the ∞-category of spectra. This approach naturally leads to several relevant questions, that were the focus of our talks:

(1) What kind of theoretical framework is needed to work with such a theory?
(2) How does this framework relate to other approaches to differential cohomology?
(3) How can we construct differential cohomology theories in this framework?
There are also questions, we did not address, but could be focus of future talks:
(4) What are concrete benefits of this approach in contrast to others?

2. Theoretical framework

We saw that the theoretical framework is fundamentally that of presentable ∞-categories, stable ∞-categories, and ∞-categorical sheaves, as developed by Lurie [Lur09], among others.

As it primarily a background, for what follows we will take it for granted, and refer to the relevant sources.

3. Relation to other approaches

A lot of historical development of differential cohomology theories has focused on constructing them via specific data, and concretely the input often consists of an ordinary cohomology theory and some geometric data. This perspective is completely absent in the definition we just gave, so how can we reconcile them? This is the central theme of the "fracture square". This already came up so let us quickly summarize.

Definition 3.1. A sheaf is called $\mathbb{R}$-*invariant* if for all manifolds M, $M \times \mathbb{R}^1 \to M$ is mapped to an equivalence of spectra. We denote the full subcategory of $\mathbb{R}$-invariant sheaves by $\mathcal{S}\mathrm{hv}_{\mathbb{R}}(\mathcal{M}\mathrm{fd})$.

Note $\mathbb{R}$-invariant sheaves are closed under limits and colimits. Hence, due to the abstract theory of presentable ∞-categories, we have the following result:

Theorem 3.2. *There is a diagram of adjunctions*

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}_{\mathbb{R}}(\mathcal{M}\mathrm{fd}) \arrow[r, hook] & \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}) \arrow[l, shift right=2, "L_{hi}"'] \arrow[l, shift left=2, "R_{hi}"] \end{tikzcd}$$

Definition 3.3. Let $\mathrm{Def}\colon \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}) \to \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd})$ be the functor that takes E to the fiber over $E \to L_{hi}E$.

Definition 3.4. A sheaf is *pure* if the value at the point is the terminal spectrum. We denote the full subcategory of pure sheaves by $\mathcal{S}\mathrm{hv}_{\mathrm{pure}}(\mathcal{M}\mathrm{fd})$.

Similarly, the category of pure sheaves is closed under limits and colimits, so we have the following result:

Theorem 3.5. *There is a diagram of adjunctions*

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}_{\mathrm{pure}}(\mathcal{M}\mathrm{fd}) \arrow[r, shift left=2, hook] \arrow[r, shift right=2, hook] & \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}) \arrow[l, shift right=4, "\mathrm{Def}"'] \arrow[l, shift left=4, "\mathrm{Cyc}"] \end{tikzcd}$$

Remark 3.6. The notational choice is not coincidental. The left adjoint to Cyc is precisely the functor Def, where the domain is restricted to pure sheaves.

Notice we obviously have the following result.

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Lemma 3.7. *A sheaf is* trivial *if it is $\mathbb{R}$-invariant and pure.*

So, pure and $\mathbb{R}$-invariant sheaves are "disjoint". Even better, they cover everything, which is the gist of the fracture square.

Remark 3.8. Notice in both adjunction diagrams there is one fully faithful functor that is not named and we will abuse notation and directly consider objects in the full subcategory as objects in the larger category.

Theorem 3.9. *Let E be a differential cohomology theory. Then the following is a pullback square*

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathrm{Cyc}E \arrow[d] \\ L_{hi}E \arrow[r] & L_{hi}\mathrm{Cyc}E \end{tikzcd}$$

In fact we can further expand this pullback square to several other pullback squares:

$$\begin{tikzcd} \Sigma^{-1}L_{hi}\mathrm{Cyc}E \arrow[r] \arrow[d] & R_{hi}E \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathrm{Def}E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \mathrm{Cyc}E \arrow[d] \\ 0 \arrow[r] & L_{hi}E \arrow[r] & L_{hi}\mathrm{Cyc}E \end{tikzcd}$$

We can restructure this diagram to more resemble the fracture square, which is the following:

$$\begin{tikzcd} & R_{hi}(E) \arrow[rr] \arrow[dr] & & L_{hi}(E) \arrow[dr] & \\ \Sigma^{-1}L_{hi}\mathrm{Cyc}E \arrow[ur] \arrow[dr] & & E \arrow[ur] \arrow[dr] & & L_{hi}\mathrm{Cyc}E \\ & \mathrm{Def}(E) \arrow[rr] \arrow[ur] & & \mathrm{Cyc}E \arrow[ur] & \end{tikzcd}$$

The proof of all these results has as of yet been postponed.

4. Construction of differential cohomology theories

Having advanced theory is of course interesting, but not enough, we also want explicit examples. Indeed we saw in past weeks several ways to explicitly construct differential cohomology theories.

Example 4.1. Given a spectrum (cohomology theory), we can define the constant sheaf, which is by definition a differential cohomology theory. Even better, it is an $\mathbb{R}$-invariant sheaf, and every $\mathbb{R}$-invariant sheaf is obtained this way.

This means we have a very good understanding of $\mathbb{R}$-invariant sheaves, using classical algebraic topology. However, of course these examples have no geometric content, so we want to go beyond them.

Example 4.2. Given a sheaf of abelian groups, we can construct the differential cohomology theory given by post-composing with the functor that takes an abelian group A to the associated Eilenberg-MacLane spectrum $H(A)$.

Example 4.3. There is a non-trivial way to extend a functor from abelian groups to spectra to one from chain complexes to spectra, which is called the *stable Dold-Kan embedding*. Hence, again via post-composition, every sheaf of chain complexes gives rise to a differential cohomology theory.

Example 4.4. An important example of the previous example are truncated forms. Given $k \geq 0$, $\Omega^{\geq k}$ is a sheaf that associates to a manifold M the chain complex of k-truncated forms on M. Using the previous example we hence get a differential cohomology theory which we also denote $\Omega^{\geq k}$.

Notice, if $k > 0$ then $\Omega^{\geq k}$ is in fact pure, and also not trivial, which means it cannot be $\mathbb{R}$-invariant.

The approach we used until now helps us generate examples that either pure or $\mathbb{R}$-invariant, but we would also like to mix them. Here we use the fracture square. The fracture square is not only an abstract theoretical tool, it also gives us very explicit ways to construct differential cohomology theories, using methods closer to the historical approaches. Concretely, we now have the following results:

Corollary 4.5. *A differential cohomology theory is uniquely determined by the following three pieces of data:*

(1) An $\mathbb{R}$-invariant sheaf $E_{\mathbb{R}}$, which is equivalently a spectrum, or equivalently a cohomology theory.
(2) A pure sheaf E_{pure}.
(3) A morphism of spectra $E_{\mathbb{R}} \to L_{hi}E_{\text{pure}}$.

More specifically, we often face the situation that we have a fixed cohomology theory E, and want to find a lift of sorts.

Definition 4.6. Let E be a cohomology theory. A *differential refinement of* E is a differential cohomology $\hat{E}$, such that $L_{hi}\hat{E} \simeq E$.

Putting the definition and corollary together, we get the final result to generate examples of interest.

Corollary 4.7. *Let E be a cohomology theory (spectrum). The differential refinement $\hat{E}$ is uniquely determined by the following pieces of data:*

(1) A pure sheaf E_{pure}.
(2) A morphism of spectra $E \to L_{hi}E_{\text{pure}}$.

These results suggests the following algorithm for constructing differential cohomology theories of interest:

(1) Pick a pure sheaf E_{pure}, that encodes geometric data of interest.
(2) Compute the spectrum $L_{hi}E_{\text{pure}}$.
(3) Then for an arbitrary spectrum E, every map of spectra $E \to L_{hi}E_{\text{pure}}$ gives rise to a differential refinement of E.

Let us implement this algorithm in practice. As we saw, the key example a pure sheaf is $\Omega^{k\geq}$, the sheaf of k-truncated forms. We now have the following result due to [BNV16].

Theorem 4.8. *Let $\Omega^{k\geq}$ be the pure sheaf of k-truncated forms. Then, $L_{hi}\Omega^{k\geq} \simeq H\mathbb{R}$.*

Example 4.9. Let E be a spectrum with a map of spectra $E \to H\mathbb{R}$. Then there is a differential refinement, $\hat{E}$, given via the pullback square

$$\begin{tikzcd}
\hat{E} \arrow[r] \arrow[d] & \Omega^{k\geq} \arrow[d] \\
E \arrow[r] & H\mathbb{R}
\end{tikzcd}$$

Example 4.10. Let us see an example of the example. Let $E = H\mathbb{Z}$, i.e. singular cohomology, and $H\mathbb{Z} \to H\mathbb{R}$ the evident inclusion. Then the resulting differential refinement of singular cohomology recovers *Deligne cohomology* $\hat{\mathbb{Z}}[l]$. It has the property that the k-th sheaf cohomology group recovers the classical Deligne cohomology group $\hat{H}^k(M)$, however at other degrees we get trivial groups. This means the classical Deligne cohomology groups sit inside short exact sequences.

So, in hindsight the fact that historically people have been looking at maps into $H\mathbb{R}$ is not a coincidence, every differential cohomology theory whose pure part is $\Omega^{k\geq}$ is obtained this way.

Question 5. Given the example, here is a natural question one might wonder. Is there a single differential cohomology theory, such that its k-th sheaf cohomology group recovers the classical Deligne cohomology group $\hat{H}^k(M)$ for all $k \geq 0$? It seems the answer would have to be no, because one single cohomology theory would not decompose into a collection of short exact sequences.

6. Future Directions

We can (at least) look at the following topics:

6.1. **Proofs.** One question is whether we want to explicitly go through the proof of the fracture square, and the fact that $\mathbb{R}$-invariant sheaves are precisely spectra. This involves understanding advanced aspects of sheaf theory, such as *recollements*.

The essential step in the proof is the following very technical result.

Proposition 6.1. *Assume we have the following data and assumptions:*

(1) A Grothendieck site $(\mathcal{C}, J)$, such that $\mathcal{C}$ has a terminal object.
(2) A stable ∞-category $\mathcal{T}$.
(3) The inclusion functor $\Delta\colon \mathcal{T} \to \mathcal{S}\mathrm{hv}_{\mathcal{T}}(\mathcal{C}, J)$ is fully faithful and admits a left adjoint L_{const}.

Then the following holds:

(1) The full subcategory of $\mathcal{S}\mathrm{hv}_{\mathcal{T}}(\mathcal{C}, J)$ consisting of sheaves P, such that $P(*)$ is the point admits a left adjoint $L^{\perp}\colon \mathcal{S}\mathrm{hv}_{\mathcal{T}}(\mathcal{C}, J) \to \mathcal{S}\mathrm{hv}_{\mathcal{T}}(\mathcal{C}, J)^{\perp}$.
(2) For every sheaf P in $\mathcal{S}\mathrm{hv}_{\mathcal{T}}(\mathcal{C}, J)$, there is a pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & L^{\perp}P \arrow[d] \\ L_{const}P \arrow[r] & L_{const}L^{\perp}P \end{tikzcd},$$

inside $\mathcal{S}\mathrm{hv}_{\mathcal{T}}(\mathcal{C}, J)$, where the inclusions are left implicit.

This is a very general result. We use it for the following specific case:

Lemma 6.2. *Let $(\mathcal{E}\mathrm{uc}, J)$ be the Euclidean site and $\mathcal{S}\mathrm{p}$ the stable ∞-category of spectra. Then the inclusion functor $\Delta\colon \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{p}}(\mathcal{E}\mathrm{uc}, J)$ is given by the constant presheaf functor.*

In other words, the constant presheaf is already a sheaf. This directly implies that Δ is fully faithful and that it admits a left adjoint via colimit. So we can directly apply the result above to get the fracture square.

6.2. **Examples.** One particular case are further examples of interest:

(1) We could look at differential K-theory. It should be a differential refinement of ku, the K-theory spectrum. Following the algorithm we want:
  - A pure sheaf $\Omega^{\geq k}(-; \mathbb{C}[u^{\pm 1}])$.
  - The computation $L_{hi}\Omega^{\geq k}(-; \mathbb{C}[u^{\pm 1}]) \simeq H\mathbb{C}[u^{\pm 1}]$.
  - A map of spectra $ku \to H\mathbb{C}[u^{\pm 1}]$, which is the *Chern character*.

  This was studied with classical means by Hopkins–Singer [HS05].
(2) There is the example of twisted differential cohomology due to Bunke–Nikolaus [BN19].
(3) There are various examples of differential cohomologies in the work of Schreiber et al., with suggested applications in physics [FSS24].
(4) There appears to be other versions of differential cohomology, such as *differential algebraic K-theory* [BG21] or *differential complex cobordism* [BSSW09], however, they seem to have been studied before the sheaf-theoretic framework was developed.

6.3. **Applications to geometry and/or physics.** Can we recover classical results in this setting using the modern framework? The book [ADH21] does provide some ideas and references, Examples include *Chern-Weil theory*, classifying (equivariant) bundles with connection, differential characteristic classes, relation to invertible field theories, ...

However, those should be explored based on interest and feasibility.

References

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