# DIFFERENTIAL COHOMOLOGY SEMINAR 2

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In this lecture we want to learn the basics of $\infty$-category theory. For the $\infty$-categorical background, we broadly follow [Gro10] and a little [Lur09].

## 1. Basics on $(\infty, 1)$-categories

$(\infty, 1)$-categories have different models that capture its essence. The first model are *quasi-categories*.

**Definition 1.** Given a natural number $n$, let $\langle n \rangle$ denote the linearly ordered set $\{0, \cdots, n\}$. The *simplex category* $\mathbf{\Delta}$ is the category of finite linerly ordered sets $\langle n \rangle$, for every $n$, and monotone functions.

**Definition 2.** Given $0 \leqslant i \leqslant n$, the *$i$-face map* is the unique injective map $\delta_n^i : \langle n-1 \rangle \to \langle n \rangle$ missing $i$. The *$i$-degeneracy map* is the unique surjective map $\sigma_n^i : \langle n+1 \rangle \to \langle n \rangle$ such that $i$ and $i+1$ have the same image.

**Theorem 3.** *As a category, $\mathbf{\Delta}$ is generated from the face and degeneracy maps subject to the* simplicial identities*, i.e.*

$$\delta_{n+1}^i \delta_n^j = \delta_{n+1}^{j+1} \delta_n^i, \qquad i \leqslant j \tag{4}$$

$$\sigma_{n-1}^j \sigma_n^i = \sigma_{n-1}^i \sigma_n^{j+1}, \qquad i \leqslant j \tag{5}$$

$$\sigma_n^j \delta_{n+1}^i = \begin{cases} \delta_n^i \sigma_{n-1}^{j-1}, & i < j \\ 1, & i = j \\ \delta_n^{i-1} \sigma_{n-1}^j, & i > j \end{cases} \tag{6}$$

*Proof.* Omitted. $\square$

**Definition 7.** A *simplicial set* is a contravariant functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. Denote by $s\mathcal{S}\mathrm{et}$ the category of simplicial sets. $X_n := X(\langle n \rangle)$ is the set of *$n$-simplices*.

By only representing the face maps, we can depict a simplicial set as follows:

$$X_0 \leftleftarrows X_1 \;\overleftarrow{\overleftarrow{\leftarrow}}\; X_2 \;\overleftarrow{\overleftarrow{\overleftarrow{\leftarrow}}}\; \cdots \tag{8}$$

$\infty$-categories are then defined in terms of a lifting condition, for which we need to define horns.

**Definition 9.** Let $\Delta^n$ denote the representable functor associated to $\langle n \rangle$. The face map $\delta_n^i$ induces a map of simplicial sets $d_n^i : \Delta^{n-1} \to \Delta^n$. The image of $d_n^i$ is called the *$i$-face*. The *$i$-horn* $\Lambda^{i,n}$ is the union of all faces, except the $i$-face.

*Remark* 10. Another characterization of $\Lambda^{i,n}$ is the following: A $t$-simplex $f : \langle t \rangle \to \langle n \rangle$ is a $t$-simplex for $\Lambda^{i,n}$ if and only if there is $j \neq i$ not in the image of $f$.

**Definition 11.** A simplicial set $X$ is a *$\infty$-category* if every map $\Lambda^{i,n} \to X$ can be extended to $\Delta^n$, for every $n$ and $0 < i < n$. If the extension condition holds for every $0 \leqslant i \leqslant n$, we call $X$ a *$\infty$-groupoid*.

*Remark* 12. There are different models for $\infty$-categories. To distinguish the one above from other models, a simplicial set satisfying the horn filling condition, for $0 < i < n$, is called a *quasi-category*.

**Example 13.** Let $\mathcal{C}$ be a category. The *nerve of* $\mathcal{C}$, denoted $\mathrm{N}\mathcal{C}$, is the simplicial set where the $n$-simplices are $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\langle n \rangle, \mathcal{C})$. This defines a functor $\mathrm{N} : \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}$.

**Proposition 14.** $\mathrm{N}\mathcal{C}$ *is a $\infty$-category, and an $\infty$-groupoid if and only if $\mathcal{C}$ is a groupoid.*

*Proof.* Straightforward combinatorics. $\square$

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Let $\infty\mathcal{C}\mathrm{at} \subseteq s\mathcal{S}\mathrm{et}$ be the full sub-category of quasi-categories.

**Proposition 15.** $\mathrm{N} : \mathcal{C}\mathrm{at} \to \infty\mathcal{C}\mathrm{at}$ *is fully faithful.*

The nerve functor has a left adjoint, the *fundamental category* or *homotopy category*. Here we observe some general facts about simplicial sets. Namely for every functor $\mathbf{\Delta} \to \mathcal{C}$, where $\mathcal{C}$ cocomplete, we have an adjunction between $s\mathcal{S}\mathrm{et}$ and $\mathcal{C}$.

**Example 16.** Let $\mathcal{C} = \mathcal{T}\mathrm{op}$ and $\mathbf{\Delta} \to \mathcal{T}\mathrm{op}$ the functor picking the geometric simplicial complex $|\Delta^n|$. Then the induced adjunction is the classical geometric realization functor $|\cdot|\colon s\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$ and singularity functor.

The nerve is indeed the right adjoint of the adjunctions induced via the functor $\mathbf{\Delta} \to \mathcal{C}\mathrm{at}$, which sends $[n]$ to the category $[n]$. This means we also have a left adjoint.

**Definition 17.** The *homotopy category* $h\colon s\mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at}$ is the left adjoint to the nerve functor.

*Remark* 18. If the simplicial set is a quasi-category, then the homotopy category can be defined directly as the usual homotopy category of an $\infty$-category (i.e. we take equivalence classes of morphisms).

We do have significantly more complicated examples of such adjunctions.

**Example 19.** Let $\mathfrak{C}\colon \mathbf{\Delta} \to s\mathcal{C}\mathrm{at}$ be the functor defined in [Lur09, Definition 1.1.5.1]. Then we get the adjunction $(\mathfrak{C}, N_\Delta)$ called the homotopy coherent categorification, homotopy coherent nerve.

This adjunction incudes an equivalence of $\infty$-categories, if one phrases those notions correctly. In particular, if $\mathcal{C}$ is a Kan-enriched category, then the homotopy coherent nerve is a quasi-category.

**Definition 20.** Let $\mathcal{K}\mathrm{an}$ denote the Kan-enriched category of Kan complexes. The quasi-category of spaces is defined as $N_\Delta(\mathcal{K}\mathrm{an})$.

## 2. Accessible and Presentable Categories

Note, here we benefited from the fact that $\mathcal{C}\mathrm{at}, s\mathcal{S}\mathrm{et}, \mathcal{T}\mathrm{op}, s\mathcal{C}\mathrm{at}$ are all cocomplete categories, hence admitting such adjunctions. We now want to focus on a class of categories where we similarly can construct adjunctions in such straightforward ways.

**Definition 21.** A category $\mathcal{C}$ is *locally presentable* if it is cocomplete and $\kappa$-accessible for some $\kappa$.

**Definition 22.** A category $\mathcal{C}$ is $\kappa$-accessible if there exists a set of $\kappa$-compact objects $\mathcal{C}^0$ in $\mathcal{C}$, such that every object in $\mathcal{C}$ is a $\kappa$-filtered colimit of objects in $\mathcal{C}^0$.

**Definition 23.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *$\kappa$-accessible* if it preserves $\kappa$-filtered colimits.

**Theorem 24.** *Let $\mathcal{C}$ be a category. The following are equivalent:*

*(1) $\mathcal{C}$ is locally presentable.*
*(2) There exists a small category $\mathcal{C}^0$ and a fully faithful accessible right adjoint $\mathcal{C} \to \mathrm{Fun}((\mathcal{C}^0)^{op}, \mathcal{S}\mathrm{et})$.*

**Theorem 25.** *Let $\mathcal{C}, \mathcal{D}$ be locally presentable categories.*

*(1) $F\colon \mathcal{C} \to \mathcal{D}$ is a left adjoint if and only if it preserves colimits.*
*(2) $F\colon \mathcal{C} \to \mathcal{D}$ is a right adjoint if and only if it preserves limits and is accessible.*

We now generalize this to $\infty$-categories.

**Definition 26.** Let $K$ be a quasi-category. The quasi-category of simplicial presheaves is defined as $\mathrm{Fun}(K^{op}, \mathcal{S})$.

**Theorem 27** (Yoneda)**.** *For a given quasi-category $K$, there is a functor $K \to \mathrm{Fun}(K^{op}, \mathcal{S})$ given by $x \mapsto \mathrm{Fun}(-, x)$, which is fully faithful. Every colimit preserving functor out of $\mathrm{Fun}(K^{op}, \mathcal{S})$ is equivalent to a functor out of $K$.*

**Theorem 28.** *Let $K$ be a quasi-category.*

*(1) There exists a set of $\kappa$-compact objects $\mathcal{C}^0$ in $K$, such that every object in $K$ is a $\kappa$-filtered colimit of objects in $\mathcal{C}^0$.*
*(2) There exists a small category $\mathcal{C}^0$ and a fully faithful accessible right adjoint $\mathcal{C} \to \mathrm{Fun}((\mathcal{C}^0)^{op}, \mathcal{S}\mathrm{et})$.*

In those cases we say $K$ is presentable. We now again have the adjoint functor theorem.

**Theorem 29.** *Let $\mathcal{C}, \mathcal{D}$ be presentable $\infty$-categories.*

*(1) $F\colon \mathcal{C} \to \mathcal{D}$ is a left adjoint if and only if it preserves colimits.*
*(2) $F\colon \mathcal{C} \to \mathcal{D}$ is a right adjoint if and only if it preserves limits and is accessible.*

Note in particular the $\infty$-category of sheaves is a presentable $\infty$-category.

## 3. Stable $\infty$-Categories and Spectra

We now use the $\infty$-categorical framework to study spectra. Let us recall some facts about spectra, to motivate the story. The *Freudenthal suspension theorem* states that the suspension functor $\Sigma\colon \mathcal{T}\mathrm{op} \to \mathcal{T}\mathrm{op}$ stabilizes the homotopy type. More explicitly, the map

$$\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to \ldots$$

stabilizes for $k$ large enough, if $X$ satisfies some connectivity condition. This defined the stable homotopy groups $\pi_n^S(X)$ as the stabilization of this sequence.

There is significant interest in computing these stable homotopy groups, in particular in the case where $X$ is a sphere, given that it helps us understand many phenomena in algebraic topology.

We now want a setting where these stable homotopy groups naturally live and can be studied. We know that $(\Sigma, \Omega)$ induces an adjunction on the category of pointed topological spaces. What we want is an adjustment of this definition such that the adjunction $(\Sigma, \Omega)$ is an equivalence.

We now take a $\infty$-categorical perspective on this and use it to study such stable phenomena.

**Definition 30.** Let $\mathcal{C}$ be an $\infty$-category with initial and terminal object. $\mathcal{C}$ has a 0-object if they are equivalent.

**Example 31.** Let $\mathcal{C}$ be a 1-category. Then $\mathcal{C}$ is pointed as a 1-category if and only if it is pointed as an $\infty$-category.

**Example 32.** Notice $\mathcal{S}$ is not pointed, we hence can define $\mathcal{S}_*$ as the slices under the terminal space, i.e. $\mathcal{S}_* = \mathcal{S}_{*/}$. This $\infty$-category is then pointed by construction.

Note $\mathcal{S}_*$ is not just some pointed $\infty$-category, it is in some sense the universal one.

**Proposition 33.** *Let $\mathcal{D}$ be a pointed $\infty$-category. Then the functor*

$$\mathrm{ev}_{S^0}\colon \mathrm{Fun}^L(\mathcal{S}_*, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}$$

*that evaluates at $S^0$ is an equivalence.*

We now generalize from there and define triangles in $\mathcal{S}_*$.

**Definition 34.** Let $\mathcal{C}$ be a pointed $\infty$-category. A *triangle* in $\mathcal{C}$ is a diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

where $X$, $X$, and $Z$ are objects in $\mathcal{C}$.

**Definition 35.** We say a triangle is a *exact* if it is a pullback square and *coexact* if it is a pushout square.

**Definition 36.** Let $\mathcal{C}$ be a pointed $\infty$-category. Let $\mathcal{C}^\Sigma$ be the full subcategory of $\mathrm{Fun}([1] \times [1], \mathcal{C})$ with objects coexact triangles of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd},$$

meaning $Y$ is the suspension of $X$.

**Definition 37.** Let $\mathcal{C}$ be a pointed $\infty$-category. Let $\mathcal{C}^{\Omega}$ be the full subcategory of $\mathrm{Fun}([1] \times [1], \mathcal{C})$ with objects exact triangles of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd},$$

meaning $Y$ is the loop object of $X$.

**Proposition 38.** *If $\mathcal{C}$ is a pointed $\infty$-category with finite (co)limits. Then there exists functors*

$$\Sigma\colon \mathcal{C} \to \mathcal{C}^{\Sigma} \to \mathcal{C}$$
$$\Omega\colon \mathcal{C} \to \mathcal{C}^{\Omega} \to \mathcal{C}$$

**Theorem 39.** *Let $\mathcal{C}$ be a pointed $\infty$-category with finite (co)limits. The following are equivalent:*

*(1) A triangle is exact if and only if it is coexact.*
*(2) The functors $\Sigma$ and $\Omega$ are equivalences and the inverses of each other.*
*(3) A square is a pullback square if and only if it is a pushout square.*

**Definition 40.** A pointed $\infty$-category $\mathcal{C}$ is *stable* if it satisfies one of the three equivalent conditions above.

Recall that before the rise of $\infty$-categories, *triangulated categories* were used to study stable homotopy theory. Hence, it is unsurprising that we can relate stable $\infty$-categories to triangulated categories.

**Proposition 41.** *If $\mathcal{C}$ is a stable $\infty$-category, then the homotopy category $h\mathcal{C}$ is a triangulated category.*

Of course arbitrary pointed $\infty$-categories will not be stable. We hence want a procedure that stabilizes them. There are several approaches. One approach, that is powerful from a theoretical perspective, is given via reduced 1-excisive functors out of finite pointed spaces. Here, we focus on explicit spectrum objects, as there are characterized more explicitly. For a comparison of these two approaches see [Lur17].

**Definition 42.** Let $\mathcal{C}$ be a pointed $\infty$-category. A *pre-spectrum object* in $\mathcal{C}$, is a functor $X\colon \mathbb{Z} \times \mathbb{Z} \to \mathcal{C}$ such that $X(i,j) = 0$ for $i \neq j$ and all squares are pushout squares. Let $PSp(\mathcal{C})$ be the $\infty$-category of pre-spectrum objects in $\mathcal{C}$.

For a given pre-spectrum object $X$, let $\alpha_{m-1}\colon \Sigma X_{m-1} \to X_m$ and $\beta_m\colon X_m \to \Omega X_{m+1} = \Omega\Sigma X_m$.

**Definition 43.** Let $\mathcal{C}$ be a pointed $\infty$-category. A *spectrum object* in $\mathcal{C}$ is a pre-spectrum object in $X$, such that $\alpha_{m-1}$ and $\beta_m$ are equivalences for all $m$. Let $\mathcal{S}\mathrm{p}(\mathcal{C})$ be the $\infty$-category of spectrum objects in $\mathcal{C}$.

**Definition 44.** Let $\mathcal{C}$ be a pointed $\infty$-category. The stabilization of $\mathcal{C}$ is the $\infty$-category $\mathcal{S}\mathrm{p}(\mathcal{C})$ of spectrum objects in $\mathcal{C}$.

Of course $\mathcal{C}$ and $\mathcal{S}\mathrm{p}(\mathcal{C})$ are suitably related.

**Theorem 45.** *For a given pointed $\infty$-category $\mathcal{C}$, there is an adjunction*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\Sigma"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathrm{p}(\mathcal{C}) \arrow[l, shift left, "\Omega"] \end{tikzcd}$$

Moreover, $\mathcal{S}\mathrm{p}(\mathcal{C})$ is in some sense the universal stabilization of $\mathcal{C}$.

**Theorem 46.** *Let $\mathcal{C}$ be a pointed $\infty$-category and $\mathcal{D}$ a stable $\infty$-category. Then $\Sigma^{\infty}$ induces an equivalence of $\infty$-categories*

$$(\Sigma^{\infty})^*\colon \mathrm{Fun}^L(\mathcal{S}\mathrm{p}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$$

Let us now focus on the case $\mathcal{C} = \mathcal{S}_*$.

**Example 47.** The stabilization of $\mathcal{S}_*$ is the $\infty$-category of spectra, denoted $\mathcal{S}\mathrm{p}$.

Similar to $\mathcal{S}_*$, $\mathcal{S}\mathrm{p}$ is also the universal stable $\infty$-category, as a special instance of the result above.

**Theorem 48.** *If $\mathcal{D}$ is a stable $\infty$-category. Then the functor*

$$\mathrm{ev}_{\mathbb{S}}\colon \mathrm{Fun}^L(\mathcal{S}\mathrm{p}, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}$$

*that evaluates at $\mathbb{S}$ is an equivalence.*

## 4. Generalized Cohomology Theories

Cohomology theories were traditionally defined in the context of topological spaces. However, now that we have the tools of $\infty$-categories and stable $\infty$-categories. We can significantly generalize those definitions. This last result follows work in [Lur17].

**Definition 49.** Let $\mathcal{C}$ be a pointed $\infty$-category with pushouts, and $\Sigma\colon \mathcal{C} \to \mathcal{C}$ and suspension functor. A *generalized cohomology theory* is a functor $H\colon h\mathcal{C}^{op} \to \mathcal{A}\mathrm{b}_{\mathbb{Z}}$, such that the following conditions hold:

- There is a natural isomorphism $H^\bullet \to H^{\bullet+1}\Sigma$
- Coexact sequences maps to exact sequences.
- Arbitrary coproducts map to products.

We now have the following major result that significantly generalizes the classical Brown representability theorem.

**Theorem 50.** *Let $\mathcal{C}$ be a nice $\infty$-category and $(H^\bullet, \delta)$ be a generalized cohomology theory. Then there exists a spectrum object $E$ in $\mathcal{C}$, such that $H^\bullet(X) \cong \mathrm{Hom}_{h\mathcal{C}}(X, E^\bullet)$, where $\delta = (\beta_\bullet)_*$.*

**Example 51.** Unsurprisingly, $\mathcal{S}_*$ satisfies the niceness conditions, and so we can conclude that every generalized cohomology on $\mathcal{S}_*$ is given by a spectrum, recovering the original Brown representability theorem.

## References

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[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

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