

DIFFERENTIAL COHOMOLOGY SEMINAR 7

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We now aim to learn about differential K -theory, which is a differential refinement of topological K -theory.

1. REVIEWING DIFFERENTIAL COHOMOLOGY

Before we proceed to the main objective, we recall the following example.

Example 1.1. Let C be a real chain complex. Then $\Omega \otimes_{\mathbb{R}} C$ is a sheaf defined as

$$(\Omega \otimes_{\mathbb{R}} C)^n(M) = \bigoplus_{p+q=n} \Omega^p(M) \otimes_{\mathbb{R}} C^q$$

We similarly have the following generalization.

Example 1.2. Let C be a real chain complex and m an integer. Then $(\Omega \otimes_{\mathbb{R}} C)^{\geq m}$ is the sheaf of m -truncated forms.

We now have the following result, where we use notation from the previous talks.

Theorem 1.3 ([BNV16, Lemma 4.4]). *For $E = (\Omega \otimes_{\mathbb{R}} C)^{\geq n}$, we have*

- $R_{hi}(E)(*) = C^{\geq n}$
- $L_{hi}(E)(*) = C$
- $L_{hi}\text{Cyc}(E)(*) = C^{\leq n-1}$
- $\text{Cyc}(E) = (\Omega \otimes_{\mathbb{R}} C^{\leq n-1})^{\geq n}$
- $\text{Def}(E) = \Sigma(\Omega \otimes_{\mathbb{R}} C)^{\leq n-1}$
- $\Phi: C \rightarrow C^{\leq n-1}$ is the truncation map.

Note that the sheaf E is uniquely determined by the computation $L_{hi}(E)(*) = C$, $L_{hi}\text{Cyc}(E)(*) = C^{\leq n-1}$, and the map $\Phi: C \rightarrow C^{\leq n-1}$.

2. LINE BUNDLES WITH CONNECTION

Having reviewed the general machinery, we will move towards differential K -theory by first considering the simpler case of line bundles with connection. For this we need some definitions.

Definition 2.1. Let $\mathcal{L}\text{ine}^{\nabla}$ be the sheaf valued in the symmetric monoidal groupoids of complex line bundles with connection, with monoidal structure given by tensor product.

Theorem 2.2. *We have*

- $R_{hi}(\mathcal{L}\text{ine}^{\nabla})(*) \simeq \Sigma^{-1}H\mathbb{C}_{\delta}^{\times}$
- $L_{hi}(\mathcal{L}\text{ine}^{\nabla})(*) \simeq \Sigma^{-1}H\mathbb{C}_{top}^{\times} \simeq \Sigma^{-2}H\mathbb{Z}$
- $L_{hi}\text{Cyc}(\mathcal{L}\text{ine}^{\nabla})(*) \simeq \Sigma^{-2}H\mathbb{C}_{\delta}$
- $\text{Cyc}(\mathcal{L}\text{ine}^{\nabla}) \simeq \Sigma^{-2}H(\Omega^{\geq 2} \otimes_{\mathbb{R}} \mathbb{C})$
- $\text{Def}(\mathcal{L}\text{ine}^{\nabla}) \simeq \Sigma^{-1}H(\Omega^{\leq 1} \otimes_{\mathbb{R}} \mathbb{C})$

Here \mathbb{C}_{δ} is the complex numbers with the discrete topology and \mathbb{C}_{top} is the complex numbers with the usual topology.

This theorem implies that we only need a map $\Sigma^{-2}H\mathbb{Z} \rightarrow \Sigma^{-2}H\mathbb{C}_{\delta}$ to define a differential refinement of these line bundles, which is given by the evident inclusion $\mathbb{Z} \rightarrow \mathbb{C}$.

Now at the level of π_0 we have the following hexagon diagram.

$$\begin{array}{ccccc}
 & H^1(M, \mathbb{C}^\times) & \xrightarrow{\quad} & H^2(M, \mathbb{Z}) & \\
 & \nearrow & \searrow & \nearrow \text{chernclass} & \\
 H^1(M, \mathbb{C}) & & \pi_0(\mathcal{L}ine^\nabla(M)) & & H^2(M, \mathbb{C}) \\
 & \searrow \beta & \nearrow \text{def} & \searrow \text{curv} & \\
 & \Omega^1(M, \mathbb{C})/imd & \xrightarrow{\quad \text{deRham} \quad} & \Omega_{cl}^2(M, \mathbb{C}) &
 \end{array}$$

This diagram has several implications:

- (1) $H^1(M, \mathbb{C}^\times)$ classifies line bundles with trivial curvature (flat line bundles).
- (2) c_1 is the usual first Chern class of a line bundle.
- (3) The fiber of c_1 are given by geometric deformations of line bundles with connection keeping the chern character fixed.

3. DIFFERENTIAL K -THEORY

We now want to generalize this perspective to differential K -theory. Here we face several choices that we need to consider. Here [HS05] take one approach and [BNV16] take another.

REFERENCES

- [BNV16] Ulrich Bunke, Thomas Nikolaus, and Michael Völkl. Differential cohomology theories as sheaves of spectra. *J. Homotopy Relat. Struct.*, 11(1):1–66, 2016.
- [HS05] M. J. Hopkins and I. M. Singer. Quadratic functions in geometry, topology, and M-theory. *J. Differential Geom.*, 70(3):329–452, 2005.