

DIFFERENTIAL COHOMOLOGY SEMINAR 11

TALK BY MATTHIAS FRERICHS

1. DIFFERENTIAL COHOMOLOGY IN A COHESIVE ∞ -TOPOS

1.1. Sheaves on manifolds are cohesive. We consider the essentially small category of manifolds with corners, denoted \mathcal{Mfd} . This category is endowed with the Grothendieck topology generated by surjective submersions with discrete fibers. This is also a dense subsite of the site of all smooth manifolds with jointly surjective open covers.

Definition 1.1. Let \mathcal{C} be a $(\infty, 1)$ -category. The category of presheaves on the site of manifolds with values in \mathcal{C} is denoted by $\mathrm{Fun}(\mathcal{Mfd}^{\mathrm{op}}, \mathcal{C})$.

We will only consider presentable $(\infty, 1)$ -categories here, even if [BNV16] treat the more general case too.

Definition 1.2. ([BNV16, Def. 2.3.], Sheaf on Manifolds) A functor $F \in \mathrm{Fun}(\mathcal{Mfd}^{\mathrm{op}}, \mathcal{C})$ is a sheaf if for any manifold M and covering $U \rightarrow M$ the canonical map

$$F(M) \rightarrow \lim_{\Delta} F(U^\bullet)$$

is an equivalence. Here U^\bullet is the Čech nerve of the covering and $F(U^\bullet)$ is the simplicial object in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ by applying F to the Čech nerve.

Denote the full subcategory of sheaves by $\mathrm{Shv}_{\mathcal{Mfd}}(\mathcal{C}) \subset \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ which is a reflective localization given by

$$L: \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) \rightleftarrows \mathrm{Shv}_{\mathcal{Mfd}}(\mathcal{C}): \text{inclusion.}$$

Remark 1.3. ([BNV16, Def. 2.3.], Homotopy-invariance) A presheaf $F \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ is called homotopy invariant, if for all manifolds $M \in \mathcal{Mfd}$ the canonical map $F(M) \rightarrow F(\Delta^1 \times M)$ induced by the projection is an equivalence.

There is an adjunction

$$\mathcal{H}^{\mathrm{pre}}: \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) \rightleftarrows \mathrm{Fun}^h(\Delta^{\mathrm{op}}, \mathcal{C}): \text{inclusion}$$

where $\mathcal{H}^{\mathrm{pre}}$ is called homotopification.

This functor commutes with sheaffication and restricts to sheaves yielding

$$\mathcal{H}: \mathrm{Shv}_{\mathcal{Mfd}}(\mathcal{C}) \rightleftarrows \mathrm{Shv}_{\mathcal{Mfd}}^h(\mathcal{C}): \text{inclusion.}$$

By [BNV16, Prop. 2.6.] this functor preserves finite products for the special case that \mathcal{C} is stable or the category of spaces. In our case we will only consider the category of spaces.

The category $\mathrm{Shv}_{\mathcal{Mfd}}^h(\mathcal{C})$ is equivalent to \mathcal{C} by sheaffication of the constant presheaf and evaluation at the point

$$\mathrm{const}: \mathcal{C} \rightleftarrows \mathrm{Shv}_{\mathcal{Mfd}}^h(\mathcal{C}): \mathrm{ev}_*.$$

In conclusion we have the following quadruple of adjoint functors.

Remark 1.4. ([BNV16, Rmk. 2.7.], Coherent Topos) There is an adjoint quadruple of functors

$$(\mathrm{ev}_* \circ \mathcal{H} \dashv \mathrm{const} \dashv \mathrm{ev}_* \dashv \mathcal{G}): \mathrm{Shv}_{\mathcal{Mfd}}(\mathcal{C}) \begin{array}{c} \xrightarrow{\mathrm{ev}_* \circ \mathcal{H}} \\ \xleftarrow{\mathrm{const}} \\ \xleftarrow{\mathrm{ev}_*} \\ \xleftarrow{\mathcal{G}} \end{array} \mathcal{C}$$

where $G: \mathcal{C} \rightarrow \mathrm{Shv}_{\mathcal{Mfd}}(\mathcal{C})$ is defined by $G(F)(M) := \mathrm{const}(F)(*)^{M_{\mathrm{disc}}} = F^{M_{\mathrm{disc}}}$. Here $\mathrm{ev}_* \circ \mathcal{H}$ preserves products ([BNV16, Prop. 2.6.]) and const and G are fully faithful.

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We will only consider cohesive topoi over the ∞ -category of spaces.

Definition 1.5. ([Sch13, Rmk. 3.4.2.]) A ∞ -topos \mathcal{X} is called cohesive if it admits a quadruple adjoint

$$(\Pi \dashv \text{Disc} \dashv \Gamma \dashv \text{coDisc}): \mathcal{X} \begin{array}{c} \xrightarrow{\Pi} \\ \xleftarrow{\text{Disc}} \\ \xrightarrow{\Gamma} \\ \xleftarrow{\text{coDisc}} \end{array} \mathcal{C}$$

such that Π preserves finite products and Disc and coDisc are fully faithfull.

Now we can state the definition of the intrinsic non-abelian cohomology in a cohesive ∞ -topos as in [Sch13, Def. 3.6.134.].

Definition 1.6. ([Sch13, Def. 3.6.134.], Cohomolgy in an ∞ -topos) For two objects $X, A \in \mathcal{X}$ we define the cohomology set of X with coefficients in A by

$$H(X, A) := \pi_0 \mathcal{X}(X, A)$$

where $\mathcal{X}(X, A)$ is the ∞ -groupoid of morphisms between X and A . Generally, if A has a p -fold delooping for $p \in \mathbb{N}$ we define

$$H^p(X, A) := \pi_0 \mathcal{X}(X, \mathbf{B}^p A)$$

The goal is now to write down a differential refinement of the cohomolgy in a cohesive ∞ -topos. For this we need to define a version of De Rham cohomolgy that will later correspond to a truncated complex of differential form as used in the classical case.

1.2. Cohesive De-Rham complex.

Definition 1.7. ([Sch13, Def. 3.9.12]) Let \mathcal{X} be a cohesive ∞ -topos. For $X \in \mathcal{X}$ we define the cohesive de Rham homotopy type $\Pi_{\text{dR}} X$ of X as the pushout

$$(1.8) \quad \begin{array}{ccc} X & \longrightarrow & * \\ \downarrow \eta & & \downarrow \\ \text{Disc} \circ \Pi(X) & \longrightarrow & \Pi_{\text{dR}} X \end{array}$$

where η is the counit of the $\Pi \dashv \text{Disc}$.

For a pointed object $* \rightarrow A$ in \mathcal{H} we define the de Rham coefficient object $\flat_{\text{dR}} A$ by the pullback

$$(1.9) \quad \begin{array}{ccc} \flat_{\text{dR}} A & \longrightarrow & \text{Disc} \circ \Gamma(A) \\ \downarrow & & \downarrow \epsilon \\ * & \longrightarrow & A \end{array}$$

where ϵ is the counit of the adjunction $\text{Disc} \dashv \Gamma$.

This also yields an adjunction

Proposition 1.10. ([Sch13, Prop. 3.9.13, Def 3.9.15]) *There is an adjunction*

$$(\Pi_{\text{dR}} \dashv \flat_{\text{dR}}): * / \mathcal{X} \begin{array}{c} \xleftarrow{\Pi_{\text{dR}}} \\ \xrightarrow{\flat_{\text{dR}}} \end{array} \mathcal{X}$$

where $* / \mathcal{X}$ is the category of pointed objects in \mathcal{X} . We write

$$H_{\text{dR}}(X, A) := H(\Pi_{\text{dR}} X, A) \cong H(X, \flat_{\text{dR}} A)$$

for the de Rham cohomology of X with coefficients in A .

Now it is sensible to check wether this yield the well known de Rham cohomolgy in the case that \mathcal{X} is a sheaf on a smooth manifold M .

Definition 1.11. Define the site of cartesian spaces CartSp as the category of cartesian spaces \mathbb{R}^n for $n \in \mathbb{N}$ with continuous smooth functions between them.

Consider the category of simplicial presheaves $[\text{CartSp}^{\text{op}}, \text{sSet}]$

Now we endow the category of simplicial sets with the Quillen model structure and localize at the maps

$$\check{C}(\{U_i \rightarrow U\}) \rightarrow U$$

where $\{U_i \rightarrow U\}$ is a covering family of U , \check{C} denotes the simplicial manifold defined by the Čech nerve which gives a simplicial presheaf by Yoneda embedding. Here U is considered as a simplicial presheaf by inserting the constant presheaf U in each degree, see [Dug00, Section 2]. We then obtain the category of ∞ -sheaves Shv_{Mfd} valued in spaces, see [Lur09, Prop. 6.5.2.14.] and [Sch13, Thm. 2.2.15.].

Proposition 1.12. ([Sch13, Prop. 4.4.22., Def. 4.4.21.]) *The simplicial presheaf*

$$U(1)[n] := [\cdots \rightarrow 0 \rightarrow \mathcal{C}^\infty(-, U(1)) \rightarrow 0 \rightarrow \cdots \rightarrow 0]$$

concentrated in degree n is a fibrant representative in Shv_{Mfd} of $\mathbf{B}^n U(1)$ under the Quillen equivalence of chain complexes with projective model structure and the Quillen model structure on simplicial sets, see [Sch13, Prop. 2.2.31.].

Proposition 1.13. ([Sch13, Prop. 4.4.49.]) *A fibrant representative in Shv_{Mfd} of the de Rham coefficient object $\flat_{\text{dR}} \mathbf{B}^n U(1)$ is given by the truncated ordinary de Rham complex of smooth differential forms*

$$\flat_{\text{dR}} \mathbf{B}^n U(1)_{\text{chn}} := \Psi[\Omega^1(-) \xrightarrow{\text{d}_{\text{dR}}} \Omega^2(-) \xrightarrow{\text{d}_{\text{dR}}} \cdots \xrightarrow{\text{d}_{\text{dR}}} \Omega^{n-1}(-) \xrightarrow{\text{d}_{\text{dR}}} \Omega^n_{\text{cl}}(-)]$$

where Ψ denotes the Quillen equivalence between simplicial sets with the Quillen model structure and the projective model structure on chain complexes, see [Sch13, Prop. 2.2.31.].

Proposition 1.14. ([Sch13, Prop. 4.4.50.]) *There are isomorphisms for $n \in \mathbb{N}$*

$$H_{\text{dR}}^n(X, \mathbf{B}^n U(1)) := \pi_0 \text{Shv}_{\text{Mfd}}(M, \flat_{\text{dR}} \mathbf{B}^n U(1)) \cong \begin{cases} H_{\text{dR}}^n(M) & n \geq 2 \\ \Omega_{\text{cl}}^1(M) & n = 1 \\ 0 & n = 0 \end{cases}$$

where a manifold considered as a constant sheaf.

1.3. Differential cohomology. Now we still need to define a characteristic map for the differential refinement we want to construct.

Definition 1.15. ([Sch13, Def. 3.9.29.]) For a group object $G \in \mathcal{X}$ in the cohesive ∞ -topos \mathcal{X} define the Maurer-Cartan Form

$$\theta: G \rightarrow \flat_{\text{dR}} \mathbf{B}G$$

via the pullback diagrams:

$$(1.16) \quad \begin{array}{ccc} G & \longrightarrow & * \\ \downarrow \theta & & \downarrow \\ \flat_{\text{dR}} \mathbf{B}G & \longrightarrow & \text{Disc} \circ \Gamma(\mathbf{B}G) \\ \downarrow & & \downarrow \epsilon \\ * & \longrightarrow & \mathbf{B}G \end{array}$$

where, again ϵ is the counit of the adjunction.

Now the general definition of differential cohomology with coefficients in a braided ∞ -group, that means the double delooping $\mathbf{B}^2 G$ exists, is given as follows:

Definition 1.17. ([Sch13, Def. 3.9.32., Rmk. 3.9.33.], General differential cohomology in a cohesive topos)

The differential cohomology with coefficients in $\mathbf{B}G$ is defined as the cohomology of the slice topos

$$\mathcal{X}/\flat_{\text{dR}} \mathbf{B}^2 G.$$

A domain object in this is an object $X \in \mathcal{X}$ with a de Rham cocycle $F: X \rightarrow \flat_{\text{dR}} \mathbf{B}^2 G$. An element in the corresponding cohomology group $\mathcal{X}/\flat_{\text{dR}} \mathbf{B}^2 G((X, F), \text{curv}_G) := \theta: \mathbf{B}G \rightarrow \flat_{\text{dR}} \mathbf{B}^2 G$ is then given by a transformation

$$(1.18) \quad \begin{array}{ccc} X & \xrightarrow{g} & \mathbf{B}G \\ & \searrow F \quad \swarrow \nabla & \downarrow \text{curv}_G \\ & & \flat_{\text{dR}} \mathbf{B}^2 G \end{array}$$

This is:

- a cocycle $g: X \rightarrow \mathbf{B}G$ for a G -principal bundle over X (in the cohesive sense [Sch13, Def 3.6.152.]
- an equivalence $g^* \text{curv}_G \rightarrow F$ interpreted as a connection on the G -principal bundle.

Restricting to the special case that G is an Eilenberg-MacLane object, i.e. $\mathbf{B}G \cong \mathbf{B}^n A$ for a 0-truncated abelian group object A , see [Lur09, Def. 5.5.6.1.], we finally look at a more familiar diagram of a differential refinement.

Definition 1.19. For $X \in \mathcal{X}$ and $n \geq 1$ write

$$\mathcal{X}_{\text{diff}}(X, \mathbf{B}^n A) := \mathcal{X}(X, \mathbf{B}^n A) \prod_{\mathcal{X}(X, \flat_{\text{dR}} \mathbf{B}^n A)} H_{\text{dR}}^{n+1}(X, A)$$

for the pullback of the diagram

$$(1.20) \quad \begin{array}{ccc} \mathcal{X}_{\text{diff}}(X, \mathbf{B}^n A) & \longrightarrow & H_{\text{dR}}^{n+1}(X, A) \\ \downarrow & & \downarrow \\ \mathcal{X}(X, \mathbf{B}^n A) & \xrightarrow{\text{curv}^*} & \mathcal{X}(X, \flat_{\text{dR}} \mathbf{B}^n A) \end{array}$$

where $\text{curv}: \mathbf{B}^n A \rightarrow \flat_{\text{dR}} \mathbf{B}^{n+1} A$ is the Maurer-Cartan Form, in this case called the curvature characteristic morphism and the right vertical map is the unique, up to equivalence morphism from $H_{\text{dR}}^{n+1}(X, A)$ seen as the 0-truncation of $\mathcal{X}(X, \flat_{\text{dR}} \mathbf{B}^n A)$.

We call

$$H_{\text{diff}}^n(X, A) := \pi_0 \mathcal{X}_{\text{diff}}(X, \mathbf{B}^n A)$$

the degree- n differential cohomology of X with coefficients in A .

One (not me) can also see that in the case of $A = U(1) \cong K(\mathbb{Z}, 1)$ this definition gives the classical definition of Hopkins and Singer, see [HS05].

Theorem 1.21. ([Sch13, Thm. 4.4.87.]) *For a smooth manifold M seen as a sheaf on CartSp we have that $H_{\text{diff}}^n(M, U(1))$ is given by the subset of Deligne cocycles that picks for each de Rham cohomology class of M a curvature form representative.*

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