

# DIFFERENTIAL COHOMOLOGY SEMINAR 4 (DRAFT)

TALK BY ALESSANDRO NANTO

In the last talk we learned the definition of a differential cohomology theory, as a sheaf valued in spectra on the site of manifolds. This talk continues our journey through differential cohomology theories, and focuses on the following three topics:

- (1) We want to learn how to construct non-trivial examples out of sheaves valued in chain complexes.
- (2) We want to understand how we can extend classical cohomology operations to the setting of differential cohomology theories.
- (3) We want to introduce suitable analogues of fiber-wise integration.

## 1. ABELIAN GROUPS, SPECTRA AND THE HEART

Let us start by reviewing the relation between abelian groups, rings and spectra.

**Definition 1.1.** Let  $n \in \mathbb{Z}$  and  $X$  be a spectrum, define  $\pi_n(X) := \pi_0(\Omega^{\infty+n} X) = \pi_0(X_{-n})$ . We call  $\pi_n$  the  $n$ -th homotopy group of  $X$ .

*Remark 1.2.* Note that since  $X_n \simeq \Omega^2 X_{n+2}$ , for any  $n$ , the set  $\pi_0(X_n)$  underlies the structure of an abelian group.

The category  $\mathcal{S}p$  underlies the structure of a symmetric monoidal  $\infty$ -category ([Lur17, Corollary 4.8.2.19]). Following [Lur17], we denote by  $\otimes$  the tensor product on  $\mathcal{S}p$ .

**Definition 1.3.** A commutative algebra object in  $\mathcal{S}p$  is called an  $\mathbb{E}_\infty$ -ring spectrum, see [Lur17, Definition 7.1.0.1]. Given an  $\mathbb{E}_\infty$ -ring spectrum  $R$ , denote by  $\text{Mod}_R$  the corresponding category of left  $R$ -module spectra, see [Lur17, Definition 7.1.1.2].

*Remark 1.4.* The sphere spectrum  $\mathbb{S}$  acts as the monoidal unit of  $\mathcal{S}p$ , therefore it is a  $\mathbb{E}_\infty$ -ring spectrum. The category  $\text{Mod}_{\mathbb{S}}$  is canonically equivalent to  $\mathcal{S}p$ .

**Definition 1.5.** Denote by  $\mathcal{S}p_{\geq 0} \subseteq \mathcal{S}p$  the full sub-category generated by *connective spectra*, i.e. spectra  $X$  such that  $\pi_n(X) \simeq 0$ , for all  $n < 0$ . Denote by  $\mathcal{S}p^\heartsuit \subseteq \mathcal{S}p_{\geq 0}$  the *heart of spectra*, i.e. the full sub-category generated by spectra  $X$  such that  $\pi_n(X) \simeq 0$ , for all  $n > 0$ .

We have the following result relating connective spectra and the heart, which follow immediately.

**Lemma 1.6.** *Let  $X$  be a connective spectrum. The following are equivalent:*

- (1)  $X$  is in the heart.
- (2)  $\pi_n(\Omega^\infty X) = 0$ , for all  $n > 0$ .
- (3)  $\text{Hom}_{\mathcal{S}_*}(S, \Omega^\infty X) \simeq 0$ , for all connected, pointed spaces  $S$ .
- (4)  $X$  is local with respect to the class of maps  $\Sigma^\infty S \rightarrow 0$ , for every connected pointed space  $S$ .

The category  $\mathcal{S}p_{\geq 0}$  is presentable and  $\pi_0$  induces an equivalence between the heart and  $\mathcal{A}b$  ([Lur17, Proposition 1.4.3.6]). The heart is a sub-category of local objects of connective spectra, therefore the inclusion  $\mathcal{A}b \simeq \mathcal{S}p^\heartsuit \subseteq \mathcal{S}p_{\geq 0}$  is a right adjoint. The category  $\mathcal{S}p_{\geq 0}$  is closed under  $\otimes$  and, given  $X, Y$  connective spectra,

$$(1.7) \quad \pi_0(X \otimes Y) \simeq \pi_0(X) \otimes \pi_0(Y)$$

see [Dav24, Theorem 2.3.28]

**Definition 1.8.** Given an abelian group  $A$ , denote by  $HA$  the (unique up to equivalence) spectrum of the heart such that  $\pi_0(HA) \simeq A$ . We call  $HA$  the *Eilenberg-Mac Lane spectrum* of  $A$ .

Using [Equation \(1.7\)](#), one can prove  $H$ , viewed as a functor  $\mathcal{A}b \rightarrow \mathcal{S}p$ , is lax monoidal. In particular, if  $R$  is a commutative ring, then  $HR$  is a connective  $\mathbb{E}_\infty$ -ring spectrum. On the other hand, if  $R$  is a connective  $\mathbb{E}_\infty$ -ring spectrum and  $M$  a connective module, then  $\pi_0(M)$  is a  $\pi_0(R)$ -module.

**Definition 1.9.** Given a commutative ring  $R$ , denote by  $\mathrm{Ch}(R) = \mathrm{Ch}(\mathrm{Mod}_R)$  the ordinary category of unbounded chain complexes. Let  $\mathcal{D}(R)$  be the  $\infty$ -localization of  $\mathrm{Ch}(R)$  at the class of quasi-isomorphisms.

Similar to the heart of spectra, given an  $\mathbb{E}_\infty$ -ring spectrum  $R$ , denote by  $\mathrm{Mod}_R^\heartsuit \subseteq \mathrm{Mod}_R$  the full subcategory generated by  $R$ -modules such that the underlying spectrum belongs to the heart of spectra.

**Theorem 1.10** (Stable Dold-Kan Correspondence). *Let  $R$  be a commutative ring.*

- (1)  $\mathrm{Mod}_R \simeq \mathrm{Mod}_{HR}^\heartsuit$  via taking Eilenberg-Mac Lane spectra.
- (2) The equivalence in (1) extends to an equivalence  $H : \mathcal{D}(R) \simeq \mathrm{Mod}_{HR}$  of symmetric monoidal  $\infty$ -categories.

*Proof.* (1) is [[Lur17](#), Proposition 7.1.1.13], while (2) is [[Lur17](#), Theorem 7.1.2.13]. □

An interesting consequence of [Equation \(1.10\)](#) is the following:

**Corollary 1.11.** *Given  $F \in \mathcal{D}(R)$ , then  $\pi_n(HF) \simeq H_n(F)$ , for all  $n \in \mathbb{Z}$ .*

*Proof.*

$$\begin{aligned}
 \pi_n(HF) &= \pi_0(\Omega^{\infty+n} HF) \\
 &\stackrel{\textcircled{1}}{\simeq} \pi_0(\mathrm{Hom}_{\mathcal{S}p}(\Sigma^n \mathbb{S}, HF)) \\
 &\stackrel{\textcircled{2}}{\simeq} \pi_0(\mathrm{Hom}_{\mathrm{Mod}_{HR}}(\Sigma^n HR, HF)) \\
 &\stackrel{\textcircled{3}}{\simeq} \pi_0(\mathrm{Hom}_{\mathcal{D}(R)}(R[n], F)) \\
 &\stackrel{\textcircled{4}}{\simeq} H_n(F)
 \end{aligned}$$

- ① The functor  $\Omega^{\infty+n}$  is corepresented by the shifted sphere spectrum  $\Sigma^n \mathbb{S}$ .
- ② The forgetful functor  $\mathrm{Mod}_{HR} \rightarrow \mathrm{Mod}_{\mathbb{S}} \simeq \mathcal{S}p$  is right adjoint to tensoring by  $HR$  and  $HR \otimes (\Sigma^n \mathbb{S}) \simeq \Sigma^n HR$ .
- ③ [Equation \(1.10\)](#)
- ④  $\pi_0$  of the mapping space  $\mathrm{Hom}_{\mathcal{D}(R)}(R[n], F)$  is equivalent to the mapping space  $R[n] \rightarrow F$  in the ordinary derived category of  $R$ , i.e. homotopy classes of maps  $R[n] \rightarrow F$ , which correspond exactly to classes in  $H_n(F)$ . □

## 2. MORE $\infty$ -CATEGORICAL BAGGAGE

Let  $\mathcal{C}$  be a presentable  $\infty$ -category. The  $\infty$ -categorical background given in previous talks allows to conclude the existence of a number of functors. Here we give a (somewhat) explicit formula for one.

*Remark 2.1.* Recall  $\mathcal{E}uc$ , the full sub-category of  $\mathcal{M}fd$  generated by Euclidean manifolds  $\mathbb{R}^n$ , for every  $n \geq 0$ . Denote by  $j$  the inclusion functor  $\mathcal{E}uc \subseteq \mathcal{M}fd$ . Recall that the restriction along  $j$  induces an equivalence  $\mathrm{Shv}(\mathcal{M}fd, \mathcal{C}) \simeq \mathrm{Shv}(\mathcal{E}uc, \mathcal{C})$ , see [[ADH21](#), Corollary A.5.6].

Evaluation at  $\{0\}$  induces an adjunction  $(\mathrm{Lconst}, \Gamma) : \mathcal{C} \rightarrow \mathrm{Shv}(\mathcal{M}fd, \mathcal{C})$ , where the functor  $\Gamma$  is evaluation at  $\{0\}$ , while the left adjoint  $\mathrm{Lconst}$  maps  $C \in \mathcal{C}$  to the sheafification of the constant pre-sheaf with value  $C$ .

*Remark 2.2.* Every presentable  $\infty$ -category  $\mathcal{C}$  is uniquely *cotensored over*  $\mathcal{S}$ , see [[Lur09](#), Remark 5.5.2.6]. More explicitly, for every space  $S$  and object  $C$ , there is an object  $C^S$  together with a natural equivalence

$$\mathrm{Hom}_{\mathcal{S}}(S, \mathrm{Hom}_{\mathcal{C}}(-, C)) \simeq \mathrm{Hom}_{\mathcal{C}}(-, C^S)$$

**Definition 2.3.** Denote by  $\mathrm{Sing}$  the functor  $\mathcal{M}fd \rightarrow \mathcal{S}$  mapping a manifold to its underlying space. Given a presentable  $\infty$ -category  $\mathcal{C}$ , denote by  $\flat$  the composition  $\mathcal{C} \rightarrow \mathrm{Fun}(\mathcal{S}^{op}, \mathcal{C}) \rightarrow \mathrm{Fun}(\mathcal{M}fd^{op}, \mathcal{C})$ , the first functor coming from [Equation \(2.2\)](#), the second being pre-composition with  $\mathrm{Sing}^{op}$ .

Explicitly, given an object  $C \in \mathcal{C}$ , the associated pre-sheaf  $\flat C$  maps a manifold  $M$  to  $C^{\mathrm{Sing}(M)}$ .

**Lemma 2.4** ([[BG21](#), Corollary 6.46]).  *$\flat$  factors through  $\mathrm{Shv}(\mathcal{M}fd, \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{M}fd^{op}, \mathcal{C})$ .*

**Equation (2.4)** is the direct consequence of a weaker version of a generalized version of Seifert-van Kampen theorem, namely [Lur17, Proposition A.3.2], stating that, given a topological space  $X$  and a covering sieve  $\mathcal{O}$ , the space  $\text{Sing}(X)$  is the colimit of  $\text{Sing}(U)$  over  $U \in \mathcal{O}$ .

**Theorem 2.5.**  $\flat : \mathcal{C} \rightarrow \text{Shv}(\text{Mfd}, \mathcal{C})$  is left adjoint to  $\Gamma$ .

*Proof.* The composition  $\mathcal{C} \xrightarrow{\flat} \text{Shv}(\text{Mfd}, \mathcal{C}) \xrightarrow{j_*} \text{Shv}(\text{Euc}, \mathcal{C})$  maps an object  $C$  to the sheaf  $\flat C$  restricted to Euclidean spaces. Since  $\mathbb{R}^n$  is contractible,  $(\flat C)(\mathbb{R}^n) = C^{\text{Sing}(\mathbb{R}^n)} \simeq C$  and so  $\flat$  restricted to  $\text{Euc}$  is equivalent to  $\text{Const}$ , the functor taking  $C$  to the pre-sheaf with constant value  $C$ , which is left adjoint to  $\Gamma$  restricted to  $\text{Euc}$ .  $\square$

### 3. SHEAVES OF COMPLEXES AND SPECTRA

The stable Dold-Kan correspondence allows us to move freely between sheaves of  $H\mathbb{Z}$ -module spectras and sheaves valued in  $\mathcal{D}(\mathbb{Z})$ .

**Remark 3.1.** We identify the category of cochain complexes with  $\text{Ch}(R)$  by reversing grading. Namely, given a cochain  $V^*$ , we are implicitly identifying it with the chain complex  $V_n = V^{-n}$ .

**Definition 3.2** ([BNV16, Definition 7.14]). Given  $n \in \mathbb{Z}$ , denote by  $\tau^{\geq n}$ , resp.  $\tau^{\leq n}$ , the *naive truncation functors*, mapping a cochain complex  $V^*$  to

$$\cdots \rightarrow 0 \rightarrow V^n \rightarrow V^{n+1} \rightarrow \cdots, \quad \text{resp.} \quad \cdots \rightarrow V^{n-1} \rightarrow V^n \rightarrow 0 \rightarrow \cdots$$

Given  $F : \text{Mfd}^{op} \rightarrow \text{Ch}(\mathbb{Z})$ , denote by  $F^{\geq n}$  the composite  $\text{Mfd}^{op} \xrightarrow{F} \text{Ch}(\mathbb{Z}) \xrightarrow{\tau^{\geq n}} \text{Ch}(\mathbb{Z})$ , and similarly we define  $F^{\leq n}$ . If  $F$  is a sheaf, then so are its truncations.

**Lemma 3.3** ([BNV16, Lemma 7.12]). Let  $F : \text{Mfd}^{op} \rightarrow \text{Ch}(\mathbb{Z})$  a sheaf of chain complexes of  $C^\infty$ -modules, then  $\text{Mfd}^{op} \xrightarrow{F} \text{Ch}(\mathbb{Z}) \rightarrow \mathcal{D}(\mathbb{Z})$  is a sheaf.

**Definition 3.4.** Denote by  $\Omega^*$  the sheaf  $\text{Mfd}^{op} \rightarrow \text{Ch}(\mathbb{Z})$  mapping a manifold to its de Rham complex.

**Equation (3.3)** ensures that the sheaf in **Equation (3.4)** and the corresponding naive truncations remain sheaves after post-composition with the localization functor  $\text{Ch}(\mathbb{Z}) \rightarrow \mathcal{D}(\mathbb{Z})$ .

**Definition 3.5.** Given a sheaf  $F : \text{Mfd}^{op} \rightarrow \mathcal{D}(\mathbb{Z})$ , denote by  $HF$  the *Eilenberg-Mac Lane sheaf* of  $H\mathbb{Z}$ -module spectra obtained by applying point-wise the equivalence of **Equation (1.10)**.

Recall now the machinery set-up in **Section 2**.

**Remark 3.6.** Since  $\mathcal{D}(\mathbb{Z})$  is presentable, we know that they is cotensored over  $\mathcal{S}$ . Given a space  $S$  and a chain complex  $M_*$ , the cotensor  $M_*^S$  is the chain complex of graded linear maps  $C_*(S, \mathbb{Z}) \rightarrow M_*$ , from the singular chain complex of  $S$  to  $M_*$ , see [Lur17, Definition 1.3.2.1]. In particular, let  $M_* = M$  be concentrated in degree 0, then  $M_*^S$  is the singular cochain complex of  $S$  with values in  $M$ .

Applying **Equation (3.6)**, we see that  $(\flat A)(M)$  is the singular cochain complex of (the underlying topological space of)  $M$  with values in  $A$ . In particular for  $A = \mathbb{R}$ , consider the natural map  $\Omega^*(M) \rightarrow C^*(M, \mathbb{R})$  taking a form  $\omega \in \Omega^n(M)$  to the linear map  $\int \omega : C_n(M, \mathbb{Z}) \rightarrow \mathbb{R}$ . We call the induced natural transformation  $dR : \Omega^* \rightarrow \flat \mathbb{R}$  the *de Rham homomorphism*.

**Lemma 3.7.**  $dR$  is an equivalence.

*Proof.* The statement is equivalent to  $dR$  being an equivalence after restriction to the Euclidean site  $\text{Euc}$ , the conclusion then follows from Poincaré lemma.  $\square$

### 4. DELIGNE COHOMOLOGY

Finally, we have enough machinery to talk about Deligne cohomology.

**Definition 4.1.** Given  $\ell \in \mathbb{N}$ , define  $\hat{\mathbb{Z}}(\ell) : \text{Mfd}^{op} \rightarrow \mathcal{D}(\mathbb{Z})$  as the limit of

$$\begin{array}{ccc} & \Omega^{\geq \ell} & \\ & \downarrow & \\ \flat \mathbb{Z} & \longrightarrow & \flat \mathbb{R} \end{array}$$

The vertical morphism being the composition  $\Omega^{\geq \ell} \subseteq \Omega^* \xrightarrow{\mathrm{dR}} \mathfrak{b}\mathbb{R}$ . We call the corresponding sheaf of  $H\mathbb{Z}$ -modules spectra  $\hat{\mathbb{Z}}(\ell)$  the  $\ell$ -th *Deligne sheaf*.

*Remark 4.2* (Model A, see [HS05, §3.2]). Let  $\dot{C}(\ell)^n(M) \subseteq C^n(M, \mathbb{Z}) \oplus C^{n-1}(M, \mathbb{R}) \oplus \Omega^n(M)$  consist of triples  $(c, h, \omega)$  for which  $\omega = 0$  if  $n < \ell$ , with differential  $\delta(c, h, \omega) = (\delta c, \mathrm{dR}(\omega) - c - \delta h, d\omega)$ . This complex  $\dot{C}^*(\ell)(M)$  fits into a diagram

$$\begin{array}{ccc} \dot{C}^*(\ell)(M) & \longrightarrow & \Omega^{\geq \ell}(M) \\ \downarrow & & \downarrow \\ C^*(M, \mathbb{Z}) & \longrightarrow & C^*(M, \mathbb{R}) \end{array}$$

which commutes up to homotopy given by the projections  $\dot{C}^n(\ell)(M) \rightarrow C^{n-1}(M, \mathbb{R})$ . The diagram above model the homotopy pullback of Equation (4.1), hence  $\dot{C}^*(\ell)(M)$  is a model for  $\hat{\mathbb{Z}}(\ell)(M)$ .

*Remark 4.3* (Model B). Recall that  $\mathfrak{b}$  preserves cofiber sequences, since it is left adjoint, and that fiber sequences are the same a cofiber sequences in stable  $\infty$ -categories. Consider the diagram

$$\begin{array}{ccc} \hat{\mathbb{Z}}(\ell) & \longrightarrow & \Omega^{\geq \ell} \\ \downarrow & & \downarrow \\ \mathfrak{b}\mathbb{Z} & \longrightarrow & \mathfrak{b}\mathbb{R} \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{\text{red}} & \mathfrak{b}(\mathbb{R}/\mathbb{Z}) \end{array}$$

Since the bottom square is an homotopy pullback,  $\hat{\mathbb{Z}}(\ell)(M)$  is equivalent to the homotopy pullback of the diagram in red. Let  $\check{C}^n(\ell)(M) \subseteq C^{n-1}(M, \mathbb{R}/\mathbb{Z}) \oplus \Omega^n(M)$  consist of pairs  $(\chi, \omega)$  for which  $\omega = 0$  if  $n < \ell$ , with differential  $\delta(\chi, \omega) = (e^{2\pi i \mathrm{dR}(\omega)} - \delta\chi, d\omega)$ . Similar to Equation (4.2), the complex  $\check{C}^*(\ell)(M)$  fits into the above diagram, so that the outer square is an homotopy pullback, hence it is equivalent to  $\hat{\mathbb{Z}}(\ell)(M)$ .

Take an  $n$ -cocycle ( $n \geq \ell$ ) in the model from Equation (4.3), i.e.  $(\chi, \omega) \in C^{n-1}(M, \mathbb{R}/\mathbb{Z}) \oplus \Omega^n(M)$  such that  $d\omega = 0$  and  $\delta\chi = e^{2\pi i \mathrm{dR}(\omega)}$ . Such a cocycle determines a differential character of degree  $n - 1$  for  $M$ , in the sense of the following definition:

**Definition 4.4** ([HS05, Definition 3.4], see also [BB14, Chapter 5]). Consider a manifold  $M$ , a *differential character* of degree  $n$  consists of a character  $\chi : Z_n^\infty(M, \mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z}$  on the group of smooth  $n$ -cycles of  $M$  together with a  $n$ -form  $\omega \in \Omega^{n+1}(M)$ , such that, for every smooth  $(n + 1)$ -chain  $c$ ,

$$\chi(\partial c) = e^{2\pi i \int_c \omega}$$

*Remark 4.5* (Model C, see [ADH21, Lemma 7.3.4]). Consider the following diagram in the category of sheaves on  $\mathcal{E}\mathrm{uc}$

$$\begin{array}{ccccc} j_*\hat{\mathbb{Z}}(\ell) & \longrightarrow & \Omega^{\geq \ell} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \text{red} \\ \mathbb{Z} & \longrightarrow & \Omega^* & \longrightarrow & \Omega^{\leq \ell-1} \end{array}$$

The left square is the pullback square of  $\hat{\mathbb{Z}}(\ell)$  restricted to  $\mathcal{E}\mathrm{uc}$ . Since the right square is a pullback,  $j_*\hat{\mathbb{Z}}(\ell)$  is equivalent to the pullback of the diagram in red. Let  $\check{C}^*(\ell)$  be the sheaf of chain complexes  $\mathbb{Z} \rightarrow \Omega^0 \rightarrow \dots \rightarrow \Omega^{\ell-1} \rightarrow 0 \rightarrow \dots$ , where  $\mathbb{Z}$  is in degree 0. The complex  $\check{C}^*(\ell)$  fits into the above diagram, so that the outer square is an homotopy pullback, and thus  $j_*\hat{\mathbb{Z}}(\ell) \simeq \check{C}^*(\ell)$  and  $j^*\check{C}^*(\ell) \simeq \hat{\mathbb{Z}}(\ell)$ .

Given a manifold  $M$ , let  $\mathcal{O}$  be a good open cover and  $\mathcal{J}(\mathcal{O})$  the closure of  $\mathcal{O}$  under finite intersections, then

$$\hat{\mathbb{Z}}(\ell)(M) \simeq \lim_{U \in \mathcal{J}(\mathcal{O})} \check{C}^*(\ell)(U) \simeq \lim_{n \in \Delta} \prod_{U_1, \dots, U_n \in \mathcal{O}} \check{C}^*(\ell)(U_1 \cap \dots \cap U_n)$$

Finally, we can apply [BNV16, Lemma 7.10] to calculate the last limit as a the total complex functor applied to the bicomplex

$$\check{C}^{m,n}(\ell)(\mathcal{O}) := \prod_{U_1, \dots, U_n \in \mathcal{O}} \check{C}^m(\ell)(U_1 \cap \dots \cap U_n)$$

## 5. UNFOLDING THE FRACTURE SQUARE OF DELIGNE COHOMOLOGY

### REFERENCES

- [ADH21] Araminta Amabel, Arun Debray, and Peter J. Haine. Differential cohomology: Categories, characteristic classes, and connections. *arXiv preprint*, 2021. [arXiv:2109.12250](#).
- [BB14] Christian Bär and Christian Becker. *Differential Characters and Geometric Chains*, pages 1–90. Springer International Publishing, Cham, 2014.
- [BG21] Ulrich Bunke and David Gepner. Differential function spectra, the differential becker-gottlieb transfer, and applications to differential algebraic  $k$ -theory. *arXiv preprint*, 2021. [arXiv:1306.0247](#).
- [BNV16] Ulrich Bunke, Thomas Nikolaus, and Michael Völkl. Differential cohomology theories as sheaves of spectra. *J. Homotopy Relat. Struct.*, 11(1):1–66, 2016.
- [Dav24] Jack Davies. V4d2 - Algebraic Topology II So24 (stable and chromatic homotopy theory). *Lecture notes*, 2024. [Unpublished](#).
- [HS05] M. J. Hopkins and I. M. Singer. Quadratic functions in geometry, topology, and M-theory. *J. Differential Geom.*, 70(3):329–452, 2005.
- [Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.
- [Lur17] Jacob Lurie. Higher algebra. [Available online](#), September 2017.