DIFFERENTIAL COHOMOLOGY SEMINAR 8

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The aim of this talk is to review twisted cohomology theory with the aim of later discussing twisted differential cohomology theories [BG16]. For these talks the main source is [ABG+14, ABG18].

1. Twisted Cohomology

Let R be a ring spectrum, meaning a monoid object in the ∞-category of spectra $\mathcal{S}\mathrm{p}$. From this we get a presentable stable ∞-category $\mathcal{M}\mathrm{od}_R$ of left R-module spectra. Objects therein are morphisms of the form $R \wedge M \to M$ satisfying the usual associativity and unit conditions up to coherent homotopies.

Note we have an adjunction diagram

$$\begin{tikzcd} \mathcal{S}\mathrm{p} \arrow[r, shift left, "R\wedge -"] \arrow[r, phantom, "\perp"] & \mathcal{M}\mathrm{od}_R \arrow[l, shift left, "\mathrm{Hom}_R(R,-)"] \end{tikzcd},$$

where the right adjoint is in fact the forgetful functor. This in particular means $\mathcal{M}\mathrm{od}_R$ has a distinguished object R i.e. the free R-module of rank 1. We now refine these constructions.

Definition 1.1. Let R be a ring spectrum. An *R-line* is an R-module L such that $L \simeq R$.

Definition 1.2. Let $\mathcal{L}\mathrm{ine}_R$ be the full sub-∞-groupoid of $\mathcal{M}\mathrm{od}_R$ spanned by the R-lines.

By construction, $\mathcal{L}\mathrm{ine}_R$ is equivalent to the category with a single object R and hom-space GL_1R, the ∞-group of R-linear automorphisms of R. Notice that $GL_1(R) \subseteq \mathrm{Hom}_{\mathcal{M}\mathrm{od}_R}(R,R) \simeq \mathrm{Hom}(\mathbb{S}, R) = \Omega^\infty R$.

Lemma 1.3. *$GL_1(R)$ fits into the pullback square*

$$\begin{tikzcd} GL_1(R) \arrow[r] \arrow[d] & \Omega^\infty R \arrow[d] \\ \pi_0(R)^\times \arrow[r] & \pi_0(R) \end{tikzcd}$$

In particular, the inclusion $GL_1(R) \to \Omega^\infty R$ induces an isomorphism on n-homotopy groups, for all $n \geq 1$.

Proof. $\pi_0(R) \simeq \mathrm{Hom}_{\mathrm{ho}\mathcal{M}\mathrm{od}_R}(R,R)$, where $\mathrm{ho}\mathcal{M}\mathrm{od}_R$ is the homotopy category of R-modules, the right-vertical arrow corresponds to $\mathrm{Hom}_{\mathcal{M}\mathrm{od}_R}(R,R) \to \mathrm{Hom}_{\mathrm{ho}\mathcal{M}\mathrm{od}_R}(R,R)$ mapping a morphism to its homotopy class, and $\pi_0(R)^\times \subseteq \mathrm{Hom}_{\mathrm{ho}\mathcal{M}\mathrm{od}_R}(R,R)$ is the set of isomorphisms. Finally, a morphism in a ∞-category $\mathcal{C}$ is an equivalence if and only if its homotopy class is an isomorphism in $\mathrm{ho}\mathcal{C}$. $\square$

Definition 1.4. Let X be a space. Denote by $\mathcal{M}\mathrm{od}_R(X)$ the ∞-category of R-module spectra parametrized over X, i.e. the functor category $\mathrm{Fun}(X^{op}, \mathcal{M}\mathrm{od}_R)$.

Definition 1.5. Let X be a space. Denote by $\mathcal{L}\mathrm{ine}_R(X)$ the ∞-groupoid of R-line spectra parametrized over X, i.e. the $\mathrm{Fun}(X^{op}, \mathcal{L}\mathrm{ine}_R)$.

Since $\mathcal{L}\mathrm{ine}_R \simeq BGL_1(R)$, functors $X^{op} \to \mathcal{L}\mathrm{ine}_R$ are generalization of local systems.

Example 1.6. Let $R_X \colon X^{op} \to * \to \mathcal{L}\mathrm{ine}_R$ be the constant functor with value R.

We now proceed to the Thom construction.

Definition 1.7 (Thom spectrum)**.** The *Thom R-module spectrum* is the functor

$$M \colon \mathcal{G}\mathrm{rpd}_\infty^{op}/\mathcal{L}\mathrm{ine}_R \to \mathcal{M}\mathrm{od}_R,$$

which sends $f \colon X^{op} \to \mathcal{L}\mathrm{ine}_R$ to the R-module spectrum $\mathrm{colim}(X^{op} \xrightarrow{f} \mathcal{L}\mathrm{ine}_R \xrightarrow{i} \mathcal{M}\mathrm{od}_R)$.

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Remark 1.8. Notice that the definition of Thom R-module spectrum make sense for any functor $X^{op} \to R\mathcal{M}\mathrm{od}$.

Let us note an alternative characterization that will be important later.

Remark 1.9. For a given map $f\colon X \to Y$, we get a map $f^*\colon \mathcal{M}\mathrm{od}_R(Y) \to \mathcal{M}\mathrm{od}_R(X)$ by precomposition with f^{op}. Since f^* preserves both limits and colimits, we construct a left adjoint $f_!\colon \mathcal{M}\mathrm{od}_R(X) \to \mathcal{M}\mathrm{od}_R(Y)$ and a right adjoint $f_*\colon \mathcal{M}\mathrm{od}_R(X) \to \mathcal{M}\mathrm{od}_R(Y)$, using left and right Kan extension along f^{op}. Let $f = p : X \to *$ be the terminal functor, then left Kan extension along p^{op} is exactly taking colimit, therefore

$$Mf \simeq p_!(i \circ f).$$

Remark 1.10 ([ABG10, 3.6]). Let $\mathcal{T}\mathrm{riv}_R$ be the slice groupoid $\mathcal{L}\mathrm{ine}_R/R$, together with the canonical projection $\pi : \mathcal{T}\mathrm{riv}_R \to \mathcal{L}\mathrm{ine}_R$, then:

(1) $\mathcal{T}\mathrm{riv}_R$ is a slice ∞-groupoid, hence contractible, and π is a Kan fibration.
(2) $GL_1(R)$ is equivalent to the fiber of π over $R \in \mathcal{L}\mathrm{ine}_R$ and acts freely on the fibers of π.

These observations imply $\mathcal{L}\mathrm{ine}_R$ is the classifying space for $GL_1(R)$-bundles. Here we use the term $GL_1(R)$-bundle to mean a parametrized family of $GL_1(R)$-spaces with a free and transitive action.

We now proceed to define twisted cohomology theories.

Definition 1.11 (Twisted cohomology)**.** Let R be a ring spectrum, X be a space, $p\colon X \to *$ the terminal functor, and $f\colon X^{op} \to \mathcal{L}\mathrm{ine}_R$ a R-line bundle over X. The *f-twisted R-cohomology of X* is defined as the mapping spectrum

$$R_f(X) \coloneqq \mathrm{Map}_{\mathcal{M}\mathrm{od}_R}(Mf, R) \simeq \mathrm{Map}_{\mathcal{M}\mathrm{od}_R(X)}(f, p^*R) \simeq \mathrm{Map}_{\mathcal{M}\mathrm{od}_R(X)}(f, R_X).$$

Similarly, the *f-twisted R-homology of X* is defined as

$$R^f(X) \coloneqq \mathrm{Map}_{\mathcal{M}\mathrm{od}_R}(R, Mf) \simeq Mf.$$

The f-twisted R-cohomology groups of X are defined as the homotopy groups of $R_f(X)$, i.e.

$$R^n_f(X) \coloneqq \pi_0(\mathrm{Map}_{\mathcal{M}\mathrm{od}_R}(Mf, \Sigma^n R)) \cong \pi_{-n}(\mathrm{Map}_{\mathcal{M}\mathrm{od}_R(X)}(f, R_X)).$$

Similarly, the f-twisted R-homology groups of X are defined as the homotopy groups of $R^f(X)$, i.e.

$$R^f_n(X) \coloneqq \pi_0(\mathrm{Map}_{\mathcal{M}\mathrm{od}_R}(\Sigma^n R, Mf)) \cong \pi_n(Mf).$$

Example 1.12 (Trivial twist)**.** If $f : X \to \mathcal{L}\mathrm{ine}_R$ factors through $*$, then f factors as the $X^{op} \to * \to \mathcal{S}$, the constant factor with value $*$, and $R \wedge \Sigma^\infty_+(-) : \mathcal{S} \to \mathcal{M}\mathrm{od}_R$. The latter functor commutes with colimits, being a left adjoint, while the colimit of the latter is X itself, then $Mf \simeq R \wedge \Sigma^\infty_+ X$. In particular, f-twisted R-cohomology and R-homology of X reduce to ordinary (untwisted) R-cohomology and R-homology of X.

Definition 1.13. Given a vector bundle $\pi : E \to B$, define the *Thom space* of π, denoted $\mathrm{Th}(E)$, to be the homotopy cofiber of $E_0 \subseteq E$, where E_0 is the complement of the zero section.

2. Examples of Twisted Cohomology

We now proceed to analyze several examples of twisted cohomology theories. This requires some preliminary lemmas.

Lemma 2.1. *Consider a space X and the ∞-categorical Yoneda's embedding $y : X \to \mathrm{Fun}(X^{op}, \mathcal{S})$. The colimit of y is the terminal pre-sheaf on X, i.e. the pre-sheaf with constant value the one-point space.*

Proof. Let S be a pre-sheaf on X, consider then the slice category $X_{/S}$ of pairs (x, ϕ), where x is an object of X and $\phi : y(x) \to S$. The density theorem for ∞-categories states that S is equivalent to the colimit of $X_{/S} \to X \xrightarrow{y} \mathrm{Fun}(X^{op}, \mathcal{S})$, the first map being the canonical projection. Take $S = *$, then $X_{/*} \to X$ is an equivalence, hence the claim. □

Let G be a topological group and BG the ∞-groupoid with a single object 1 and hom-space G. The category $\mathcal{S}_G \coloneqq \mathrm{Fun}(BG, \mathcal{S})$ is equivalent to the category of G-spaces.

Lemma 2.2. *Consider $X = BG$, a G-space $f : X \to \mathcal{S}$ and its left Kan extension $f_! : \mathcal{S}_G \to \mathcal{S}$, then $f_! \simeq (- \times E)/G$, where $E = f(1)$.*

Proof. Evaluate at 1, then $f_!(y(1)) = E$, by definition, and $y(1) \simeq G$, as G-spaces, hence $(y(1) \times E)/G \simeq (G \times E)/G \simeq E$. Since $f_!$ and $(- \times E)/G$ agree on representables and are colimit-preserving, they are equivalent. □

Example 2.3. Take the space $BO(n)$ and $f_n : BO(n) \to \mathcal{S}_*$ the n-sphere S^n with $O(n)$-action coming from the one-point compactification of the regular action on $\mathbb{R}^n$. Let $\alpha_n = \Sigma^{\infty-n} f_n : BO(n) \to \mathcal{S}\mathrm{p}$, then $\alpha_n(1) = \Sigma^{\infty-n} f_n(1) = \Sigma^{\infty-n} S^n \simeq \mathbb{S}$, so α_n factors through $\mathcal{L}\mathrm{ine}_{\mathbb{S}}$. Let $X = BO(n)^{op}$ and $p : X \to *$ the terminal functor, then

$$M\alpha_n = p_!\Sigma^{\infty-n} f_n \simeq \Sigma^{\infty-n} p_!(f_n)_! y \simeq \Sigma^{\infty-n}(f_n)_! \underbrace{p_!(y)}_{\simeq *} \simeq \Sigma^{\infty-n}(* \times S^n)/O(n)$$

Let $P = EO(n)$ be the universal $O(n)$-bundle and M a $O(n)$-space, then $* \times_{O(n)} M$ is modelled by the *strict* quotient $(P \times M)/O(n)$, then

$$S^n/O(n) = \mathrm{cofib}(\mathbb{R}^n_0 \subseteq \mathbb{R}^n)/O(n) \simeq \mathrm{cofib}(\underbrace{* \times_{O(n)} \mathbb{R}^n_0}_{\simeq E^n_0} \subseteq \underbrace{* \times_{O(n)} \mathbb{R}^n}_{\simeq E^n}) = \mathrm{Th}(E^n)$$

where $E^n = P \times_{O(n)} \mathbb{R}^n \to BO(n)$ is the universal n-dimensional vector bundle, hence $M\alpha_n \simeq \Sigma^{\infty-n}\mathrm{Th}(E^n)$.

The functor $BO(n) \to \mathcal{L}\mathrm{ine}_{\mathbb{S}}$ induces a ∞-group homomorphism $j_n : O(n) \to GL_1(\mathbb{S})$, mapping ϕ to $\Sigma^{\infty-n}\mathrm{Th}(\phi)$. Consider the suspension morphism $s_n = \mathbb{R} \oplus - : O(n) \to O(1+n)$, then

$$j_n(\mathbb{R} \oplus \phi) = \Sigma^{\infty-n-1}\mathrm{Th}(\mathbb{R} \oplus \phi) \simeq \Sigma^{\infty-n-1} \underbrace{\mathrm{Th}(\mathbb{R})}_{\simeq S^1} \wedge \mathrm{Th}(\phi) \simeq \Sigma^{\infty-n}\mathrm{Th}(\phi) = j_n$$

Recall that the colimit over the suspension morphisms s_n is the stable orthogonal group O.

Definition 2.4. Denote by j the induced group homomorphism $O \to GL_1(\mathbb{S})$, called the *J-homomorphism*.

Example 2.5. Let $X = O^{op}$ and take $Bj : BO \to \mathcal{L}\mathrm{ine}_{\mathbb{S}}$, then Mj is denoted MO and called the *real bordism spectrum*.

Denote by M the extended Thom spectrum functor $\mathcal{G}\mathrm{rp}^{op}_{\infty}/\mathcal{M}\mathrm{od}_R \to \mathcal{M}\mathrm{od}_R$, this is a left adjoint to the functor $\mathcal{O}$ sending a R-module to the functor $* \to \mathcal{M}\mathrm{od}_R$ picking out M. In particular, M preserves colimits and $Bj \simeq \mathrm{colim}_n Bj_n$, therefore we have the following:

Theorem 2.6. $MO \simeq \mathrm{colim}_n MO(n) = \mathrm{colim}_n \Sigma^{\infty-n}\mathrm{Th}(E^n)$.

Example 2.7. A group homomorphism $\xi : G \to O$ induces a functor $f : BG \to \mathcal{L}\mathrm{ine}_{\mathbb{S}}$. The Thom spectrum Mf is denoted MG or $M\xi$, and called *G-bordism spectrum.* For $G = U, SO, Spin$, and $String$, we obtain the *complex, oriented, spin*, and *string bordism spectra.*

Remark 2.8. In Example 2.7 we might take $G = \{*\}$, the one-point group, then $MG \simeq \mathbb{S}$, which, if it didn't have a name, might be called *framed bordism spectrum*, following the naming convention in Example 2.7 and in line with the theorem that $\pi_*(\mathbb{S}) \simeq \Omega^{\mathrm{fr}}_*$, the bordism ring of framed (trivialized tangent bundle) smooth manifolds.

Let R be a ring in sets, then R is a A_∞-ring spectrum (actually, E_∞), $\Omega^\infty R$ is equivalent to R with discrete topology ($\pi_0(\Omega^\infty R) \simeq R$, as sets, and every other homotopy group vanish). In particular, $GL_1(R)$ is simply $R^\times$ with discrete topology. Consider then the fiber sequence $SO \to O \to \mathbb{Z}^\times \simeq GL_1(\mathbb{Z})$.

Example 2.9. $X = BO^{op}$ and $\alpha = w_1 : BO \to \mathcal{L}\mathrm{ine}_{\mathbb{Z}}$, the 1st Stiefel-Whitney class (delooping of the determinant $O \to GL_1(\mathbb{Z})$), then Mw_1 is a $\mathbb{Z}$-module spectra. Let $i : SO \subseteq O$, then $w_1 i$ factors through the point, so $M(w_1 i) \simeq \mathbb{Z} \wedge \Sigma^\infty_+ SO$.

Given $f : X^{op} \to \mathcal{L}\mathrm{ine}_R$ and a sequence $F \xrightarrow{i} X \xrightarrow{\pi} Y$, there is an induced sequence of Thom R-module spectra $MF \to MX \to MY$. If πi factors through the point, $MF \simeq R \wedge \Sigma^\infty_+ F$.

Lemma 2.10. *Let R be a ring spectrum and X a connected monoidal ∞-groupoid, then*

$$\mathrm{Hom}_{\mathrm{Mon}(\mathcal{S}\mathrm{p})}(\Sigma^\infty_+ X, R) \simeq \mathrm{Hom}_{\mathrm{Mon}(\mathcal{S})}(X, GL_1(R))$$

Proof. Since X is connected, the space of homomorphisms $X \to GL_1(R)$ is equivalent to the space of homomorphisms $X \to \Omega^\infty R$, then use that $(\Sigma^\infty, \Omega^\infty)$ is a monoidal adjunction (The monoidal structure on spectra is such that Σ^∞ is strong monoidal). ☐

Remark 2.11. Notice that we can weaken the result. Namely, if X is 1-connected (pointed and connected), then the space of functors (of ∞-groupoids) $X \to GL_1(R)$ is equivalent to the space of functors $X \to \Omega^\infty R$ such that $* \to X \to \Omega^\infty R$ is an equivalence. This last space is equivalent, via the $(\Sigma^\infty_+, \Omega^\infty)$ adjunction, to the space of morphisms of spectra $\Sigma^\infty_+ X \to R$, such that $\mathbb{S} \to \Sigma^\infty_+ X \to R$ represents a unit in $\pi_0(R)$.

Remark 2.12. Notice that we can also strengthen the result. Namely, if X is a connected, commutative monoid object and R is a commutative ring spectrum, then $\Omega^\infty R$ and $GL_1(R)$ are also commutative monoid objects. Using the same argument, together with the fact that $(\Sigma^\infty, \Omega^\infty)$ is actually a *symmetric* monoidal adjunction, we conclude that

$$\mathrm{Hom}_{\mathrm{CMon}(\mathcal{S}\mathrm{p})}(\Sigma^\infty_+ X, R) \simeq \mathrm{Hom}_{\mathrm{CMon}(\mathcal{S})}(X, GL_1(R))$$

Remark 2.13. Let $\mathcal{D}$ be a monoidal ∞-category. Consider $\mathcal{C}\mathrm{at}_\infty/\mathcal{D}$, the ∞-category of functors into $\mathcal{D}$, with monoidal structure given by

$$(F : \mathcal{A} \to \mathcal{D}, G : \mathcal{B} \to \mathcal{D}) \longmapsto (\mathcal{A} \times \mathcal{B} \xrightarrow{F \times G} \mathcal{D} \times \mathcal{D} \xrightarrow{\otimes} \mathcal{D})$$

The monoidal unit is the functor $* \to \mathcal{D}$ picking out the monoidal unit of $\mathcal{D}$. If $\mathcal{D}$ is symmetric monoidal, then so is $\mathcal{C}\mathrm{at}_\infty/\mathcal{D}$. A (commutative) monoid object in $\mathcal{C}\mathrm{at}_\infty/\mathcal{D}$ is given by a (symmetric) monoidal category $\mathcal{C}$ and a (symmetric) monoidal functor $F : \mathcal{C} \to \mathcal{D}$.

In view of Remark 2.13, let R be commutative ring spectrum, then $\mathcal{M}\mathrm{od}_R$ is a symmetric monoidal ∞-category and $\mathcal{L}\mathrm{ine}_R$ is a symmetric monoidal ∞-groupoid. The category $\mathcal{G}\mathrm{rpd}_\infty^{op}/\mathcal{L}\mathrm{ine}_R$ is then symmetric monoidal and (commutative) monoid objects are given by (symmetric) monoidal ∞-groupoids X^{op} a (symmetric) monoidal functors $X^{op} \to \mathcal{L}\mathrm{ine}_R$. One can then check that M is a symmetric monoidal functor, so that (commutative) monoid objects are sent to (commutative) monoid objects in $\mathcal{M}\mathrm{od}_R$, i.e. (commutative) R-algebras.

Example 2.14. Let Tmf be the commutative ring spectrum of topological modular forms (see Remark 2.15) and $\sigma : MString \to \mathrm{tmf}$ the *String*-orientation of tmf. In the sequence

$$BString \longrightarrow BO \xrightarrow{Bj} \mathcal{L}\mathrm{ine}_{\mathbb{S}}$$

all functors are symmetric monoidal, so that $MString$ is a commutative $\mathbb{S}$-algebra, i.e. a commutative ring. The *String*-orientation of tmf is also a commutative ring homomorphism. In the fiber sequence $K(\mathbb{Z}, 3) \to BString \to BSO$, the fiber map $i : K(\mathbb{Z}, 3) \to BString$ is also a symmetric monoidal, so the composition

$$\Sigma^\infty_+ K(\mathbb{Z}, 3) \xrightarrow{Mi} MString \xrightarrow{\sigma} \mathrm{tmf}$$

is a commutative ring homomorphism. Using Lemma 2.10, we conclude that the induced homomorphism $K(\mathbb{Z}, 3) \to \Omega^\infty\mathrm{tmf}$ (which is a homomorphism of commutative monoid objects, given Remark 2.12) factors through $GL_1(\mathrm{tmf})$, and so it induces a (symmetric monoidal) functor $K(\mathbb{Z}, 4) \to \mathcal{L}\mathrm{ine}_{\mathrm{tmf}}$, i.e. *2-bundle gerbes twist* tmf.

Remark 2.15. The spectrum of topological modular forms comes in three main flavors, namely:

(1) TMF, i.e. the global sections of the spectral structure sheaf $\mathcal{O}^{top} : (\mathrm{Aff}/\mathcal{M}_{ell})^{op} \to \mathrm{CMon}(\mathcal{S}\mathrm{p})$ on the (étale site of the) moduli stack of elliptic curves.
(2) Tmf, i.e. the global sections of the spectral structure sheaf $\bar{\mathcal{O}}^{top} : (\mathrm{Aff}/\bar{\mathcal{M}}_{ell})^{op} \to \mathrm{CMon}(\mathcal{S}\mathrm{p})$ on the (étale site of the) *compactified* moduli stack of elliptic curves. The inclusion $\mathcal{M}_{ell} \hookrightarrow \bar{\mathcal{M}}_{ell}$ induces a commutative ring homomorphism $\mathrm{Tmf} \to \mathrm{TMF}$.
(3) tmf, i.e. the connective cover of Tmf. By definition, there is a commutative ring homomorphism $\mathrm{tmf} \to \mathrm{Tmf}$.

In [AHR10], tmf is used to denote our TMF, in [Goe09], tmf is used to denoted our Tmf, and [DFHH14] has the same notation as us. In Example 2.14, we use tmf to mean the connective cover of Tmf.

Let us go down one step in the chromatic ladder.

Example 2.16. Recall the fiber sequences for $Spin$ and $Spin^c$:

$$\mathbb{Z}_2 \to Spin \to SO, \qquad S^1 \to Spin^c \to SO$$

All the spaces involved are commutative groups. Applying the Thom spectrum functor to the delooped sequences, we get

$$\Sigma^\infty_+ K(\mathbb{Z}_2, 1) \to MSpin \to MSO, \qquad \Sigma^\infty_+ K(\mathbb{Z}, 2) \to MSpin^c \to MSO$$

Let $\sigma : MSpin \to KO$ and $\sigma^c : MSpin^c \to KU$ be the Atiyah-Bott-Shapiro orientation of real and complex K-theory (see [ABS64]). Similar to Example 2.14, we get homomorphisms

$$\Sigma^\infty_+ K(\mathbb{Z}_2, 1) \longrightarrow MSpin \xrightarrow{\sigma} KO, \qquad \Sigma^\infty_+ K(\mathbb{Z}, 2) \longrightarrow MSpin^c \xrightarrow{\sigma^c} KU$$

Using Lemma 2.10 again and delooping, we obtain functors $K(\mathbb{Z}_2, 2) \to \mathcal{L}\mathrm{ine}_{KO}$ and $K(\mathbb{Z}, 3) \to \mathcal{L}\mathrm{ine}_{KU}$, i.e. *real, resp. complex, bundle gerbes twist real, resp. complex, K-theory.*

3. Twists via Picard Groupoids and Grading

This section requires some some further details. Up until now we defined everything via $\mathcal{L}\mathrm{ine}_R$, however for many applications we need to work with $\mathcal{P}\mathrm{ic}_R$ instead.

Definition 3.1. Given a monoidal ∞-category $(\mathcal{C}, \otimes, 1)$, an object M is *invertible* if there is an object D such that $D \otimes M \simeq M \otimes D \simeq 1$. The *Picard ∞-groupoid* of $\mathcal{C}$ is the sub-∞-groupoid generated by invertible modules.

Definition 3.2. If R is a ring spectrum, $\mathcal{M}\mathrm{od}_R$ is monoidal. Denote by $\mathcal{P}\mathrm{ic}_R$ the Picard groupoid of $\mathcal{M}\mathrm{od}_R$.

Remark 3.3. $\mathcal{P}\mathrm{ic}_R$ splits as the disjoint union of $\pi_0(\mathcal{P}\mathrm{ic}_R)$-many sub-groupoids. Moreover, if $M \simeq N$, then $R \simeq M^{-1} \otimes N$, so $M^{-1} \otimes N$ is a R-line. In particular, every connected component of $\mathcal{P}\mathrm{ic}_R$ is equivalent to $\mathcal{L}\mathrm{ine}_R$, so $\mathcal{P}\mathrm{ic}_R \simeq \pi_0(\mathcal{P}\mathrm{ic}_R) \times \mathcal{L}\mathrm{ine}_R$. However, this is not a monoidal equivalence for general ring spectra.

Remark 3.4. $\Sigma^n R$ is invertible, with inverse $\Sigma^{-n} R$. In particular, there is a map $\mathbb{Z} \times \mathcal{L}\mathrm{ine}_R \to \mathcal{P}\mathrm{ic}_R$. However, this map need not be neither injective (if R is n-periodic), nor surjective (see [HM17]).

As mentioned in Remark 1.8, the Thom spectrum functor makes sense for every functor $f : X^{op} \to \mathcal{M}\mathrm{od}_R$. However, all examples of twists encountered so far came from functors into $\mathcal{L}\mathrm{ine}_R$. An example of twist that is not the result of a R-line bundle is the *degree shift*.

Definition 3.5. Denote by M the *Thom R-module spectrum* functor

$$\mathcal{G}\mathrm{rpd}^{op}_\infty / \mathcal{M}\mathrm{od}_R \to \mathcal{M}\mathrm{od}_R$$

sending a functor $f : X^{op} \to \mathcal{M}\mathrm{od}_R$ to its colimit.

Example 3.6. Let $f : X^{op} \to \mathcal{L}\mathrm{ine}_R$ be a twist. Denote by $\Sigma^n f$ the composition of f with the shift functor $\Sigma^n : \mathcal{L}\mathrm{ine}_R \to \mathcal{P}\mathrm{ic}_R$. Since Σ^n is an equivalence, it commutes with colimits, so

$$M\Sigma^n f \simeq \Sigma^n Mf$$

If $f = R_X$, then $M\Sigma^n f \simeq \Sigma^n R \wedge \Sigma^\infty_+ X$, so $\Sigma^n f$-twisted R-cohomology and R-homology correspond to normal R-cohomology and R-homology with a degree shift by n.

4. Umkehr Map

We now proceed to the construction of the umkehr map in twisted cohomology theories. Here we follow [ABG18]

Remark 4.1. Depending on which cohomological degrees we want, we sometimes use the following alternative definition of twisted cohomology:

$$R^{*+\alpha}(X) = \pi_* \mathrm{Map}_R(M(\alpha^{-1}), R)$$

meaning we use the inverse of α to get a more standard degree convention at the level of actual cohomology groups.

For X a space, we will write

$$DX := \mathrm{Map}(\Sigma^\infty_+ X, \mathbb{S})$$

for the Spanier-Whitehead dual of X. Notice D is evidently given as the composition functor

$$D\colon \mathcal{S}^{op} \xrightarrow{\Sigma^\infty_+} \mathcal{S}\mathrm{p}^{op} \xrightarrow{\mathrm{Map}(-,\mathbb{S})} \mathcal{S}\mathrm{p}.$$

Applying D to the unique map $X \to *$ gives a map of spectra

$$\phi(X)\colon \mathbb{S} \to DX$$

and we regard this as a functor

$$\phi\colon \mathcal{S}^{op} \to \mathcal{S}\mathrm{p}_{\mathbb{S}/}$$

i.e. the induced functor on the slice category.

As part of this explanation it is helpful to recall the following facts.

Remark 4.2. Given a spectrum E, the mapping spectrum $\mathrm{Map}(\mathbb{S}, E)$ is equivalent to E itself. This is because $\mathbb{S}$ is the unit for the smash product of spectra, and the mapping spectrum from the unit to any spectrum recovers that spectrum. In particular $\mathrm{Map}(\mathbb{S}, \mathbb{S}) \simeq \mathbb{S}$.

Suppose we are given $f\colon X \to B$, a smooth and proper family of manifolds over B, $f^{-1}(b) = X_b$ is a smooth and proper manifold. Where proper means a compact closed manifold. This should vary continuously over B in the sense that f is classified by a functor

$$B \to \mathcal{M}\mathrm{fd}$$

where $\mathcal{M}\mathrm{fd}$ is the ∞-category of smooth and proper manifolds, where the ∞-categorical structure is given by the usual topological enrichment. Notice it comes with an evident forgetful functor to $\mathcal{S}$, the ∞-category of spaces (i.e. ∞-groupoids).

Given such a functor F we have the composition functor

$$B^{op} \xrightarrow{F^{op}} \mathcal{M}\mathrm{fd}^{op} \to \mathcal{S}^{op} \xrightarrow{\phi} \mathcal{S}\mathrm{p}_{\mathbb{S}/},$$

which is an object in $\mathrm{Fun}(B^{op}, \mathcal{S}\mathrm{p}_{\mathbb{S}/})$. Notice, there is an evident equivalence of ∞-categories

$$\mathrm{Fun}(B^{op}, \mathcal{S}\mathrm{p}_{\mathbb{S}/}) \simeq (\mathrm{Fun}(B^{op}, \mathcal{S}\mathrm{p}))_{\mathbb{S}_B/}$$

where $\mathbb{S}_B$ is the constant functor with value $\mathbb{S}$. Thus, our composition above, corresponds to a natural transformation

$$\phi_{X/B}\colon \mathbb{S}_B \to D_B(X)$$

This just unwinds to a natural map of spectra

$$\mathbb{S} \to D(X_b)$$

We can now take the left Kan extension along the terminal map $p\colon B \to *$ (i.e. take the colimit of these diagrams), to get a map of spectra

$$p_!\phi_{X/B}\colon p_!\mathbb{S}_B \to p_! D_B(X)$$

Using the fact that

$$p_!\mathbb{S}_B \simeq p_! p^*\mathbb{S}_B \simeq \Sigma^\infty_+ B,$$

we get a map of spectra

$$\Sigma^\infty_+ B \to p_! D_B(X).$$

On the right hand side

$$p_! D_B(X) \simeq X^{-T_f} \simeq \Sigma^\infty Th(_-T_f)$$

where T_f is the bundle of tangents along the fibers of f and X^{-T_f}, giving us the Thom spectrum. This last equivalence is a complicated computation.

Remark 4.3. Here argument involves proving that the classical Thom space X^{-T_f} is equivalent to the colimit of the functor $D_B\colon B \to \mathcal{S}\mathrm{p}$

Remark 4.4. Let us justify this notation. Let $X \to BO(d)$ be the classifying map classifying the tangent bundle (as we have fiber-wise tangent bundles, which assemble into a map). Then we get the induced map

$$X \to BO(d) \xrightarrow{J} BGL_1(\mathbb{S}) \to BGL_1(R) \to \mathcal{P}\mathrm{ic}(R)$$

where J is the J-homomorphism, and here R is an arbitrary ring spectrum, which we want to use as our cohomology coefficients.

These computations give us the *Pontryagin-Thom transfer map*:

$$PT(f)\colon \Sigma^\infty_+ B \to X^{-T_f}.$$

Now, given a choice of R-orientation, we get a Thom isomorphism

$$R^*(\Sigma^\infty_+ X) \xrightarrow{\simeq} R^{*-d}(X^{-T_f})$$

We now have the following definition.

Definition 4.5. With the assumptions as above, the *Umkehr map* is the map:

$$R^*(\Sigma^\infty_+ X) \simeq R^{*-d}(X^{-T_f}) \xrightarrow{PT(f)^{*-d}} R^{*-d}(\Sigma^\infty_+ B).$$

Here it is called the Umkehr map, as it goes the other way from the regular map $R^*f\colon R^*B \to R^*X$. Recall here the following definition of orientation that we used here.

Definition 4.6. Given a twist $\alpha\colon X \to \mathcal{L}\mathrm{ine}_R$, an orientation of X^α, is a lift

$$\begin{tikzcd} & \mathcal{T}\mathrm{riv}_R \arrow[d, "\pi"] \\ X \arrow[ur, dashed] \arrow[r, "\alpha"] & \mathcal{L}\mathrm{ine}_R \end{tikzcd}$$

We now define a twisted variant of the Umkehr map.
Given a twist $\alpha\colon B \to \mathcal{P}\mathrm{ic}_R$, we can smash to get a map

$$\mathbb{S}_B \wedge_B \alpha \to D_B X \wedge \alpha$$

The left hand side will compute to B^α. The right hand side is equivalent to $X^{-T_f + \alpha f}$, where αf is the composition of α and $f\colon X \to B$.

Now again assuming we have a chosen an R-orientation of T_f, we get a Thom isomorphism

$$R^{*+d}(X) \simeq R^*(X^{-T_f+\alpha f}),$$

which gives us

$$R^{*+d}(X) \simeq R^*(X^{-T_f+\alpha f}) \to R^*(B^\alpha) = R^{*-\alpha-d}(B)$$

This map is called the *twisted Umkehr map.*
Let us look at some examples. Assume the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "-T_f"] \arrow[d] & \mathcal{P}\mathrm{ic}_{\mathbb{S}} \arrow[d] \\ B \arrow[r, "\alpha"] & \mathcal{P}\mathrm{ic}_R \end{tikzcd}$$

Then we have $X^{-T_f+\alpha f} \simeq \Sigma^\infty_+ X \wedge R$. Then we have an induced map $B^\alpha \to \Sigma^\infty_+ X \wedge R$, but then applying R^* we get the Umkehr map

$$R^*(X) \to R^{*-\alpha}(B)$$

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