

DIFFERENTIAL COHOMOLOGY SEMINAR 4 (DRAFT)

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In the last talk we learned the definition of a differential cohomology theory, as a sheaf valued in spectra on the site of manifolds. This talk continues our journey through differential cohomology theories, and focuses on the following three topics:

- (1) We want to learn how to construct non-trivial examples out of sheaves valued in chain complexes.
- (2) We want to understand how we can extend classical cohomology operations to the setting of differential cohomology theories.
- (3) We want to introduce suitable analogues of fiber-wise integration.

1. ABELIAN GROUPS, SPECTRA AND THE HEART

Let us start by reviewing the relation between abelian groups, rings and spectra.

Definition 1.1. Let $n \in \mathbb{Z}$ and X be a spectrum, define $\pi_n(X) := \pi_0(\Omega^{\infty+n} X) = \pi_0(X_{-n})$. We call π_n the n -th homotopy group of X .

Remark 1.2. Note that since $X_n \simeq \Omega^2 X_{n+2}$, for any n , the set $\pi_0(X_n)$ underlies the structure of an abelian group.

The category $\mathcal{S}p$ underlies the structure of a symmetric monoidal ∞ -category ([Lur17, Corollary 4.8.2.19]). Following [Lur17], we denote by \otimes the tensor product on $\mathcal{S}p$.

Definition 1.3. A commutative algebra object in $\mathcal{S}p$ is called an \mathbb{E}_∞ -ring spectrum, see [Lur17, Definition 7.1.0.1]. Given an \mathbb{E}_∞ -ring spectrum R , denote by Mod_R the corresponding category of left R -module spectra, see [Lur17, Definition 7.1.1.2].

Remark 1.4. The sphere spectrum \mathbb{S} acts as the monoidal unit of $\mathcal{S}p$, therefore it is a \mathbb{E}_∞ -ring spectrum. The category $\text{Mod}_{\mathbb{S}}$ is canonically equivalent to $\mathcal{S}p$.

Definition 1.5. Denote by $\mathcal{S}p_{\geq 0} \subseteq \mathcal{S}p$ the full sub-category generated by *connective spectra*, i.e. spectra X such that $\pi_n(X) \simeq 0$, for all $n < 0$. Denote by $\mathcal{S}p^\heartsuit \subseteq \mathcal{S}p_{\geq 0}$ the *heart of spectra*, i.e. the full sub-category generated by spectra X such that $\pi_n(X) \simeq 0$, for all $n > 0$.

We have the following result relating connective spectra and the heart, which follow immediately.

Lemma 1.6. *Let X be a connective spectrum. The following are equivalent:*

- (1) X is in the heart.
- (2) $\pi_n(\Omega^\infty X) = 0$, for all $n > 0$.
- (3) $\text{Hom}_{\mathcal{S}_*}(S, \Omega^\infty X) \simeq 0$, for all connected, pointed spaces S .
- (4) X is local with respect to the class of maps $\Sigma^\infty S \rightarrow 0$, for every connected pointed space S .

The category $\mathcal{S}p_{\geq 0}$ is presentable and π_0 induces an equivalence between the heart and $\mathcal{A}b$ ([Lur17, Proposition 1.4.3.6]). The heart is a sub-category of local objects of connective spectra, therefore the inclusion $\mathcal{A}b \simeq \mathcal{S}p^\heartsuit \subseteq \mathcal{S}p_{\geq 0}$ is a right adjoint. The category $\mathcal{S}p_{\geq 0}$ is closed under \otimes and, given X, Y connective spectra,

$$(1.7) \quad \pi_0(X \otimes Y) \simeq \pi_0(X) \otimes \pi_0(Y)$$

see [Dav24, Theorem 2.3.28]

Definition 1.8. Given an abelian group A , denote by HA the (unique up to equivalence) spectrum of the heart such that $\pi_0(HA) \simeq A$. We call HA the *Eilenberg-Mac Lane spectrum* of A .

Using [Equation \(1.7\)](#), one can prove H , viewed as a functor $\mathcal{A}b \rightarrow \mathcal{S}p$, is lax monoidal. In particular, if R is a commutative ring, then HR is a connective \mathbb{E}_∞ -ring spectrum. On the other hand, if R is a connective \mathbb{E}_∞ -ring spectrum and M a connective module, then $\pi_0(M)$ is a $\pi_0(R)$ -module.

Definition 1.9. Given a commutative ring R , denote by $\mathrm{Ch}(R) = \mathrm{Ch}(\mathrm{Mod}_R)$ the ordinary category of unbounded chain complexes. Let $\mathcal{D}(R)$ be the ∞ -localization of $\mathrm{Ch}(R)$ at the class of quasi-isomorphisms.

Similar to the heart of spectra, given an \mathbb{E}_∞ -ring spectrum R , denote by $\mathrm{Mod}_R^\heartsuit \subseteq \mathrm{Mod}_R$ the full subcategory generated by R -modules such that the underlying spectrum belongs to the heart of spectra.

Theorem 1.10 (Stable Dold-Kan Correspondence). *Let R be a commutative ring.*

- (1) $\mathrm{Mod}_R \simeq \mathrm{Mod}_{HR}^\heartsuit$ via taking Eilenberg-Mac Lane spectra.
- (2) The equivalence in (1) extends to an equivalence $H : \mathcal{D}(R) \simeq \mathrm{Mod}_{HR}$ of symmetric monoidal ∞ -categories.

Proof. (1) is [[Lur17](#), Proposition 7.1.1.13], while (2) is [[Lur17](#), Theorem 7.1.2.13]. □

An interesting consequence of [Equation \(1.10\)](#) is the following:

Corollary 1.11. *Given $F \in \mathcal{D}(R)$, then $\pi_n(HF) \simeq H_n(F)$, for all $n \in \mathbb{Z}$.*

Proof.

$$\begin{aligned}
 \pi_n(HF) &= \pi_0(\Omega^{\infty+n} HF) \\
 &\stackrel{\textcircled{1}}{\simeq} \pi_0(\mathrm{Hom}_{\mathcal{S}p}(\Sigma^n \mathbb{S}, HF)) \\
 &\stackrel{\textcircled{2}}{\simeq} \pi_0(\mathrm{Hom}_{\mathrm{Mod}_{HR}}(\Sigma^n HR, HF)) \\
 &\stackrel{\textcircled{3}}{\simeq} \pi_0(\mathrm{Hom}_{\mathcal{D}(R)}(R[n], F)) \\
 &\stackrel{\textcircled{4}}{\simeq} H_n(F)
 \end{aligned}$$

- ① The functor $\Omega^{\infty+n}$ is corepresented by the shifted sphere spectrum $\Sigma^n \mathbb{S}$.
- ② The forgetful functor $\mathrm{Mod}_{HR} \rightarrow \mathrm{Mod}_{\mathbb{S}} \simeq \mathcal{S}p$ is right adjoint to tensoring by HR and $HR \otimes (\Sigma^n \mathbb{S}) \simeq \Sigma^n HR$.
- ③ [Equation \(1.10\)](#)
- ④ π_0 of the mapping space $\mathrm{Hom}_{\mathcal{D}(R)}(R[n], F)$ is equivalent to the mapping space $R[n] \rightarrow F$ in the ordinary derived category of R , i.e. homotopy classes of maps $R[n] \rightarrow F$, which correspond exactly to classes in $H_n(F)$. □

2. MORE ∞ -CATEGORICAL BAGGAGE

Let \mathcal{C} be a presentable ∞ -category. The ∞ -categorical background given in previous talks allows to conclude the existence of a number of functors. Here we give a (somewhat) explicit formula for one.

Remark 2.1. Recall $\mathcal{E}uc$, the full sub-category of $\mathcal{M}fd$ generated by Euclidean manifolds \mathbb{R}^n , for every $n \geq 0$. Denote by j the inclusion functor $\mathcal{E}uc \subseteq \mathcal{M}fd$. Recall that the restriction along j induces an equivalence $\mathrm{Shv}(\mathcal{M}fd, \mathcal{C}) \simeq \mathrm{Shv}(\mathcal{E}uc, \mathcal{C})$, see [[ADH21](#), Corollary A.5.6].

Evaluation at $\{0\}$ induces an adjunction $(\mathrm{Lconst}, \Gamma) : \mathcal{C} \rightarrow \mathrm{Shv}(\mathcal{M}fd, \mathcal{C})$, where the functor Γ is evaluation at $\{0\}$, while the left adjoint Lconst maps $C \in \mathcal{C}$ to the sheafification of the constant pre-sheaf with value C .

Remark 2.2. Every presentable ∞ -category \mathcal{C} is uniquely *cotensored over* \mathcal{S} , see [[Lur09](#), Remark 5.5.2.6]. More explicitly, for every space S and object C , there is an object C^S together with a natural equivalence

$$\mathrm{Hom}_{\mathcal{S}}(S, \mathrm{Hom}_{\mathcal{C}}(-, C)) \simeq \mathrm{Hom}_{\mathcal{C}}(-, C^S)$$

Definition 2.3. Denote by Sing the functor $\mathcal{M}fd \rightarrow \mathcal{S}$ mapping a manifold to its underlying space. Given a presentable ∞ -category \mathcal{C} , denote by \flat the composition $\mathcal{C} \rightarrow \mathrm{Fun}(\mathcal{S}^{op}, \mathcal{C}) \rightarrow \mathrm{Fun}(\mathcal{M}fd^{op}, \mathcal{C})$, the first functor coming from [Equation \(2.2\)](#), the second being pre-composition with Sing^{op} .

Explicitly, given an object $C \in \mathcal{C}$, the associated pre-sheaf $\flat C$ maps a manifold M to $C^{\mathrm{Sing}(M)}$.

Lemma 2.4 ([[BG21](#), Corollary 6.46]). *\flat factors through $\mathrm{Shv}(\mathcal{M}fd, \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{M}fd^{op}, \mathcal{C})$.*

Equation (2.4) is the direct consequence of a weaker version of a generalized version of Seifert-van Kampen theorem, namely [Lur17, Proposition A.3.2], stating that, given a topological space X and a covering sieve \mathcal{O} , the space $\text{Sing}(X)$ is the colimit of $\text{Sing}(U)$ over $U \in \mathcal{O}$.

Theorem 2.5. $\flat : \mathcal{C} \rightarrow \text{Shv}(\text{Mfd}, \mathcal{C})$ is left adjoint to Γ .

Proof. The composition $\mathcal{C} \xrightarrow{\flat} \text{Shv}(\text{Mfd}, \mathcal{C}) \xrightarrow{j_*} \text{Shv}(\text{Euc}, \mathcal{C})$ maps an object C to the sheaf $\flat C$ restricted to Euclidean spaces. Since \mathbb{R}^n is contractible, $(\flat C)(\mathbb{R}^n) = C^{\text{Sing}(\mathbb{R}^n)} \simeq C$ and so \flat restricted to Euc is equivalent to Const , the functor taking C to the pre-sheaf with constant value C , which is left adjoint to Γ restricted to Euc . \square

3. SHEAVES OF COMPLEXES AND SPECTRA

The stable Dold-Kan correspondence allows us to move freely between sheaves of $H\mathbb{Z}$ -module spectras and sheaves valued in $\mathcal{D}(\mathbb{Z})$.

Remark 3.1. We identify the category of cochain complexes with $\text{Ch}(R)$ by reversing grading. Namely, given a cochain V^* , we are implicitly identifying it with the chain complex $V_n = V^{-n}$.

Definition 3.2 ([BNV16, Definition 7.14]). Given $n \in \mathbb{Z}$, denote by $\tau^{\geq n}$, resp. $\tau^{\leq n}$, the *naive truncation functors*, mapping a cochain complex V^* to

$$\cdots \rightarrow 0 \rightarrow V^n \rightarrow V^{n+1} \rightarrow \cdots, \quad \text{resp.} \quad \cdots \rightarrow V^{n-1} \rightarrow V^n \rightarrow 0 \rightarrow \cdots$$

Given $F : \text{Mfd}^{op} \rightarrow \text{Ch}(\mathbb{Z})$, denote by $F^{\geq n}$ the composite $\text{Mfd}^{op} \xrightarrow{F} \text{Ch}(\mathbb{Z}) \xrightarrow{\tau^{\geq n}} \text{Ch}(\mathbb{Z})$, and similarly we define $F^{\leq n}$. If F is a sheaf, then so are its truncations.

Lemma 3.3 ([BNV16, Lemma 7.12]). Let $F : \text{Mfd}^{op} \rightarrow \text{Ch}(\mathbb{Z})$ a sheaf of chain complexes of C^∞ -modules, then $\text{Mfd}^{op} \xrightarrow{F} \text{Ch}(\mathbb{Z}) \rightarrow \mathcal{D}(\mathbb{Z})$ is a sheaf.

Definition 3.4. Denote by Ω^* the sheaf $\text{Mfd}^{op} \rightarrow \text{Ch}(\mathbb{Z})$ mapping a manifold to its de Rham complex.

Equation (3.3) ensures that the sheaf in **Equation (3.4)** and the corresponding naive truncations remain sheaves after post-composition with the localization functor $\text{Ch}(\mathbb{Z}) \rightarrow \mathcal{D}(\mathbb{Z})$.

Definition 3.5. Given a sheaf $F : \text{Mfd}^{op} \rightarrow \mathcal{D}(\mathbb{Z})$, denote by HF the *Eilenberg-Mac Lane sheaf* of $H\mathbb{Z}$ -module spectra obtained by applying point-wise the equivalence of **Equation (1.10)**.

Recall now the machinery set-up in **Section 2**.

Remark 3.6. Since $\mathcal{D}(\mathbb{Z})$ is presentable, we know that they is cotensored over \mathcal{S} . Given a space S and a chain complex M_* , the cotensor M_*^A is the chain complex of graded linear maps $C_*(S, \mathbb{Z}) \rightarrow M_*$, from the singular chain complex of S to M_* , see [Lur17, Definition 1.3.2.1]. In particular, let $M_* = M$ be concentrated in degree 0, then M_*^S is the singular cochain complex of S with values in M .

Applying **Equation (3.6)**, we see that $(\flat A)(M)$ is the singular cochain complex of (the underlying topological space of) M with values in A . In particular for $A = \mathbb{R}$, consider the natural map $\Omega^*(M) \rightarrow C^*(M, \mathbb{R})$ taking a form $\omega \in \Omega^n(M)$ to $\int \omega : C_n(M, \mathbb{Z}) \rightarrow \mathbb{R}$. We call the induced natural transformation $dR : \Omega^* \rightarrow \flat \mathbb{R}$ the *de Rham homomorphism*.

Lemma 3.7. dR is an equivalence.

Proof. The statement is equivalent to dR being an equivalence after restriction to the Euclidean site Euc , the conclusion then follows from Poincaré lemma. \square

4. DELIGNE COHOMOLOGY

Finally, we have enough machinery to talk about Deligne cohomology.

Definition 4.1. Given $\ell \in \mathbb{N}$, define $\hat{\mathbb{Z}}(\ell) : \text{Mfd}^{op} \rightarrow \mathcal{D}(\mathbb{Z})$ as the limit of

$$\begin{array}{ccc} & \Omega^{\geq \ell} & \\ & \downarrow & \\ \flat \mathbb{Z} & \longrightarrow & \flat \mathbb{R} \end{array}$$

The vertical morphism being the composition $\Omega^{\geq \ell} \subseteq \Omega^* \xrightarrow{\mathrm{dR}} \mathfrak{b}\mathbb{R}$. We call the corresponding sheaf of $H\mathbb{Z}$ -modules spectra $\hat{\mathbb{Z}}(\ell)$ the ℓ -th *Deligne sheaf*.

Remark 4.2 (Model A, see [HS05, §3.2]). Let $\check{C}(\ell)^n(M) \subseteq C^n(M, \mathbb{Z}) \oplus C^{n-1}(M, \mathbb{R}) \oplus \Omega^n(M)$ consist of triples (c, h, ω) for which $\omega = 0$ if $n < \ell$, with differential $\delta(c, h, \omega) = (\delta c, \mathrm{dR}(\omega) - c - \delta h, d\omega)$. This complex $\check{C}^*(\ell)(M)$ fits into a diagram

$$\begin{array}{ccc} \check{C}^*(\ell)(M) & \longrightarrow & \Omega^{\geq \ell}(M) \\ \downarrow & & \downarrow \\ C^*(M, \mathbb{Z}) & \longrightarrow & C^*(M, \mathbb{R}) \end{array}$$

which commutes up to homotopy given by the projections $\check{C}^n(\ell)(M) \rightarrow C^{n-1}(M, \mathbb{R})$. The diagram above model the homotopy pullback of Equation (4.1), hence $\check{C}^*(\ell)(M)$ is a model for $\hat{\mathbb{Z}}(\ell)(M)$.

Remark 4.3 (Model B). Recall that \mathfrak{b} preserves cofiber sequences, since it is left adjoint, and that fiber sequences are the same a cofiber sequences in stable ∞ -categories. Consider the diagram

$$\begin{array}{ccc} & \Omega^{\geq \ell} & \\ & \downarrow & \\ \mathfrak{b}\mathbb{Z} & \longrightarrow & \mathfrak{b}\mathbb{R} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{b}(\mathbb{R}/\mathbb{Z}) \end{array}$$

(A red curved arrow points from $\Omega^{\geq \ell}$ to $\mathfrak{b}(\mathbb{R}/\mathbb{Z})$.)

Since the square is an homotopy pullback, $\hat{\mathbb{Z}}(\ell)(M)$ is equivalent to the homotopy pullback of the diagram in red. Let $\check{C}^n(\ell)(M) \subseteq C^{n-1}(M, \mathbb{R}/\mathbb{Z}) \oplus \Omega^n(M)$ consist of pairs (χ, ω) for which $\omega = 0$ if $n < \ell$, with differential $\delta(\chi, \omega) = (e^{2\pi i \mathrm{dR}(\omega)} - \delta\chi, d\omega)$. Similar to Equation (4.2), the complex $\check{C}^*(\ell)(M)$ fits into the above diagram, so that the outer square is an homotopy pullback.

Take an n -cocycle ($n \geq \ell$) in the model from Equation (4.3), i.e. $(\chi, \omega) \in C^{n-1}(M, \mathbb{R}/\mathbb{Z}) \oplus \Omega^n(M)$ such that $d\omega = 0$ and $\delta\chi = e^{2\pi i \mathrm{dR}(\omega)}$. Such a cocycle determines a differential character of degree $n - 1$ for M , in the sense of the following definition:

Definition 4.4 ([HS05, Definition 3.4], see also [BB14, Chapter 5]). Consider a manifold M , a *differential character* of degree n consists of a character $\chi : Z_n^\infty(M, \mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z}$ on the group of smooth n -cycles of M together with a n -form $\omega \in \Omega^{n+1}(M)$, such that, for every smooth $(n + 1)$ -chain c ,

$$\chi(\partial c) = e^{2\pi i \int_c \omega}$$

Remark 4.5 (Model C, see [ADH21, Lemma 7.3.4]). Consider the following diagram

$$\begin{array}{ccccc} \Omega^{\geq \ell} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \Omega^* & \longrightarrow & \Omega^{\leq \ell-1} \end{array}$$

(A red curved arrow points from \mathbb{Z} to $\Omega^{\leq \ell-1}$.)

in the category of sheaves on $\mathcal{E}\mathrm{uc}$ valued in chain complexes. Similar to Equation (4.3), we construct the homotopy pullback of the diagram in red. Let $\check{C}^*(\ell)$ be the sheaf of chain complexes $\mathbb{Z} \rightarrow \Omega^0 \rightarrow \dots \rightarrow \Omega^{\ell-1} \rightarrow 0 \rightarrow \dots$, where \mathbb{Z} is in degree 0. The complex $\check{C}^*(\ell)$ fits into the above diagram, so that the outer square is an homotopy pullback. Finally, the extension of $\check{C}^*(\ell)$ to the full site of manifolds is also a model for $\hat{\mathbb{Z}}(\ell)$.

To evaluate $\hat{\mathbb{Z}}(\ell)$ at a manifold M using $\check{C}^*(\ell)$, let \mathcal{O} be a good open cover and $\mathcal{I}(\mathcal{O})$ the closure of \mathcal{O} under finite intersections, then

$$\hat{\mathbb{Z}}(\ell)(M) \simeq \lim_{U \in \mathcal{I}(\mathcal{O})} \check{C}^*(\ell)(U) \simeq \lim_{n \in \Delta} \prod_{U_1, \dots, U_n \in \mathcal{O}} \check{C}^*(\ell)(U_1 \cap \dots \cap U_n)$$

Finally, we can apply [BNV16, Lemma 7.10] to calculate the last limit as a the total complex functor applied to the bicomplex $\check{C}^{m,n}(\ell)(\mathcal{O}) := \prod_{U_1, \dots, U_n \in \mathcal{O}} \check{C}^m(\ell)(U_1 \cap \dots \cap U_n)$.

5. UNFOLDING THE FRACTURE SQUARE OF DELIGNE COHOMOLOGY

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