DIFFERENTIAL COHOMOLOGY SEMINAR 2

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In this lecture we want to learn the basics of ∞-category theory. For the ∞-categorical background, we broadly follow [Gro10] and a little [Lur09].

1. Basics on $(\infty, 1)$-categories

$(\infty, 1)$-categories have different models that capture its essence. The first model are *quasi-categories*.

Definition 1. Given a natural number n, let $\langle n \rangle$ denote the linearly ordered set $\{0, \cdots, n\}$. The *simplex category* $\mathbf{\Delta}$ is the category of finite linerly ordered sets $\langle n \rangle$, for every n, and monotone functions.

Definition 2. Given $0 \leqslant i \leqslant n$, the *i-face map* is the unique injective map $\delta_n^i : \langle n-1 \rangle \to \langle n \rangle$ missing i. The *i-degeneracy map* is the unique surjective map $\sigma_n^i : \langle n+1 \rangle \to \langle n \rangle$ such that i and $i+1$ have the same image.

Theorem 3. *As a category, $\mathbf{\Delta}$ is generated from the face and degeneracy maps subject to the* simplicial identities*, i.e.*

$$\delta_{n+1}^i \delta_n^j = \delta_{n+1}^{j+1} \delta_n^i, \qquad i \leqslant j \tag{4}$$

$$\sigma_{n-1}^j \sigma_n^i = \sigma_{n-1}^i \sigma_n^{j+1}, \qquad i \leqslant j \tag{5}$$

$$\sigma_n^j \delta_{n+1}^i = \begin{cases} \delta_n^i \sigma_{n-1}^{j-1}, & i < j \\ 1, & i = j \\ \delta_n^{i-1} \sigma_{n-1}^j, & i > j \end{cases} \tag{6}$$

Proof. Omitted. □

Definition 7. A *simplicial set* is a contravariant functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. Denote by $s\mathcal{S}\mathrm{et}$ the category of simplicial sets. $X_n := X(\langle n \rangle)$ is the set of *n-simplices*.

By only representing the face maps, we can depict a simplicial set as follows:

$$\begin{tikzcd} X_0 & X_1 \arrow[l, shift left] \arrow[l, shift right] & X_2 \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift left=3] \arrow[l, shift right=3] \end{tikzcd} \tag{8}$$

∞-categories are then defined in terms of a lifting condition, for which we need to define horns.

Definition 9. Let Δ^n denote the representable functor associated to $\langle n \rangle$. The face map δ_n^i induces a map of simplicial sets $d_n^i : \Delta^{n-1} \to \Delta^n$. The image of d_n^i is called the *i-face*. The *i-horn* $\Lambda^{i,n}$ is the union of all faces, except the i-face.

Remark 10. Another characterization of $\Lambda^{i,n}$ is the following: A t-simplex $f : \langle t \rangle \to \langle n \rangle$ is a t-simplex for $\Lambda^{i,n}$ if and only if there is $j \neq i$ not in the image of f.

Definition 11. A simplicial set X is a *quasi-category* if, for every $0 < i < n$ and solid diagram

$$\begin{tikzcd} \Lambda^{i,n} \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{12}$$

there is a dashed arrow rendering the diagram commutative. If the above condition holds for every $0 \leqslant i \leqslant n$, we call X a *Kan complex*.

Example 13. Let $\mathcal{C}$ be a category. The *nerve of* $\mathcal{C}$, denoted $\mathrm{N}\mathcal{C}$, is the simplicial set where the n-simplices are $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\langle n \rangle, \mathcal{C})$. This defines a functor $\mathrm{N} : \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}$.

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Proposition 14. *$\mathrm{N}\mathcal{C}$ is a quasi-category, and a Kan complex if and only if $\mathcal{C}$ is a groupoid.*

Proof. Straightforward combinatorics. ☐

Remark 15. The nerve is a special case of the following construction: Let $\mathcal{C}$ be a category, $\Gamma : \mathbf{\Delta} \to \mathcal{C}$ a functor, then define N_Γ as the composition

$$\mathcal{C} \longrightarrow \mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et}) \longrightarrow s\mathcal{S}\mathrm{et} \tag{16}$$

where the first functor is Yoneda, while the second is pre-composition with Γ^{op}. In the case of the nerve, Γ is the functor sending $\langle n \rangle$ to the linerly ordered set viewed as a category. On the other hand, assuming $\mathcal{C}$ is cocomplete, we can left Kan extend Γ along the Yoneda functor $\mathbf{\Delta} \to s\mathcal{S}\mathrm{et}$, we denote by Ho_Γ the resulting functor $s\mathcal{S}\mathrm{et} \to \mathcal{C}$.

Proposition 17. *The pair $(\mathrm{Ho}_\Gamma, \mathrm{N}_\Gamma) : \mathcal{C} \to s\mathcal{S}\mathrm{et}$ is an adjoint pair. In the case of $\mathrm{N} : \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}$, the functor is fully faithful.*

Proof. Abstract nonsense about left Kan extensions. Full faithfulness can be checked directly. ☐

Remark 18. $\mathrm{Ho} : s\mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at}$ is called the *homotopy category* functor. If X is a quasi-category, $\mathrm{Ho}X$ has X_0 as set of objects and homotopy classes of maps as morphism, see [Lan21, 1.2.5].

Remark 19. Denote by $s\mathcal{C}\mathrm{at}$ the category of simplicially enriched categories. In [Lur09, 1.1.5.1], Lurie constructs a cosimplicial object $\mathbf{\Delta} \to s\mathcal{C}\mathrm{at}$. The resulting nerve functor $\mathrm{N}_\Delta : s\mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}$ is called *homotopy coherent nerve*. If $\mathcal{C}$ is a category enriched over ∞-groupoids, its homotopy coherent nerve is a ∞-category. The induced right adjoint is denoted $\mathfrak{C}$, the adjoint pair $(\mathfrak{C}, \mathrm{N}_\Delta)$ underlies a Quillen equivalence.

Denote by $\mathcal{K}\mathrm{an}$ the category of Kan complexes. One can show that $\mathcal{K}\mathrm{an}$ is self-enriched, which motivates, together with Equation (14), the following definition:

Definition 20. $\mathcal{S} := \mathrm{N}_\Delta(\mathcal{K}\mathrm{an})$ is called the *quasi-category of ∞-groupoids*.

2. Accessible and Presentable Categories

In general, a limit, resp. colimit, preserving functor need not have a left, resp. right, adjoint. Here we wish to introduce a rather large class of quasi-categories for which the previous statement holds. Let κ denote a regular cardinal.

Definition 21. A category $\mathcal{I}$ is *κ-filtered* if, for every $\mathcal{J}$ with $< \kappa$ many morphisms, every diagram $\mathcal{J} \to \mathcal{I}$ has a cocone. A functor $F : \mathcal{C} \to \mathcal{D}$ is *κ-accessible* if it preserves κ-filtered colimits. Given a category $\mathcal{C}$, an object X is *κ-compact* if $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{S}\mathrm{et}$ is κ-accessible.

Definition 22. A category $\mathcal{C}$ is *κ-accessible* if there exists a set $S \subseteq \mathcal{C}_0$ of κ-compact objects that generate $\mathcal{C}$ under κ-filtered colimits. A category is *accessible* if it is κ-accessible, for some regular cardinal κ.

Definition 23. A category $\mathcal{C}$ that is accessible and cocomplete is called *locally presentable*.

Theorem 24. *Let $\mathcal{C}$ be a category, then $\mathcal{C}$ is locally presentable if and only if there exists a small category S such that the induced functor $\mathcal{C} \to \mathcal{PS}\mathrm{hv}(S)$ is a fully faithful, accessible right adjoint.*

Theorem 25. *Let $\mathcal{C}, \mathcal{D}$ be locally presentable categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is a left, resp. right, adjoint if and only if it preserves colimits, resp. it preserves limits and is accessible.*

We now generalize this to quasi-categories.

Definition 26. Given a simplicial set X, denote by X^{op} the simplicial set obtained by reversing the structure maps: $X^{\mathrm{op}}_n = X_n$, for all n, and

$$(d_i : X^{\mathrm{op}}_n \to X^{\mathrm{op}}_{n-1}) = (d_{n-i} : X_n \to X_{n-1})$$
$$(s_i : X^{\mathrm{op}}_n \to X^{\mathrm{op}}_{n+1}) = (s_{n-i} : X_n \to X_{n+1})$$

If X is a quasi-category, so is X^{op}.

Definition 27. Let X be a quasi-category. The *quasi-category of simplicial presheaves* is defined as $\mathcal{F}\mathrm{un}(X^{\mathrm{op}}, \mathcal{S}) := \mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(X^{\mathrm{op}}, \mathcal{S})$.

Theorem 28 (Yoneda)**.** *Given a quasi-category X, the mapping space functor $X^{\mathrm{op}} \times X \to \mathcal{S}$ adjoints to a functor $y : X \to \hat{X} := \mathcal{F}\mathrm{un}(X^{\mathrm{op}}, \mathcal{S})$, called the* Yoneda embedding*. Given a cocomplete quasi-category $\mathcal{C}$, pre-composition by y induces an equivalence*

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(\hat{X}, \mathcal{C}) \longrightarrow \mathcal{F}\mathrm{un}(X, \mathcal{C}) \tag{29}$$

where $\mathcal{F}\mathrm{un}^{\mathrm{L}}(\hat{X}, \mathcal{C})$ denotes the category of colimit preserving functors $\hat{X} \to \mathcal{C}$.

In other words, $\hat{X}$ is the free cocompletion of X. The inverse equivalence if given by taking left Kan extensions. The definition of accessible category transfers directly to the ∞-categorical setting.

Theorem 30. *A quasi-category X is locally presentable (cocomplete and accessible) if and only if there is a small sub-quasi-category S such that $X \to \mathcal{F}\mathrm{un}(S^{\mathrm{op}}, \mathcal{S})$ is a fully faithful, accessible right adjoint.*

Remark 31. In view of Equation (30), one can define a category to be *locally presentable* if it is the accessible right localization of a pre-sheaf category for some small quasi-category S. In particular, every pre-sheaf category is locally presentable.

Theorem 32. *Let X, Y be presentable ∞-categories, then a functor $f : X \to Y$ is a left, resp. right, adjoint if and only if it preserves colimits, resp. it preserves limits and is accessible.*

3. Stable ∞-Categories and Spectra

We now use the ∞-categorical framework to study spectra. The study of spectra originates from the study of *stable phenomena*, i.e. patterns appearing after repeated application of the suspension functor $\Sigma : \mathcal{T}\mathrm{op}_* \to \mathcal{T}\mathrm{op}_*$.

Example 33. Let $(\Sigma, \Omega) : \mathcal{T}\mathrm{op}_{*/} \to \mathcal{T}\mathrm{op}_{*/}$ be the suspension-loop adjunction on pointed topological spaces. *Freudenthal Suspension Theorem* states that, if X is a n-connected space, the adjunction unit $X \to \Omega\Sigma X$ is $2n$-connected. If X is connected, $S^n \wedge X$ is n-connected, so $\Sigma^n X \to \Omega\Sigma^{n+1} X$ is $2n$-connected. In particular, $\pi_i(\Sigma^n X) \to \pi_{i+1}(\Sigma^{n+1} X)$ is an isomorphism, for all $i < 2n$. The group $\pi_i(\Sigma^n X)$ is denoted $\pi^s_{n-i}(X)$, called the $(n-i)$*-stable homotopy group of* X.

More examples?

Definition 34. Let $\mathcal{C}$ be an ∞-category, an object 0 that is both initial and terminal is called *zero object*. A category $\mathcal{C}$ with a zero object is called a *pointed category*.

Example 35. Let $1 \in \mathcal{C}$ be a terminal object, then the identity of 1 is the zero object in the slice category $\mathcal{C}_{1/}$ of objects under 1. In particular, the category $\mathcal{S}_* = \mathcal{S}_{*/}$ of pointed spaces is pointed.

Proposition 36. *Let $\mathcal{D}$ be a pointed ∞-category. Evaluation at the 0-sphere S^0 induces an equivalence*

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{S}_*, \mathcal{C}) \longrightarrow \mathcal{C} \tag{37}$$

where $\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{S}_*, \mathcal{C})$ denotes the category of colimit preserving functors $\mathcal{S}_* \to \mathcal{C}$.

We now introduce the notion of a triangle.

Definition 38. Let $\mathcal{C}$ be a pointed ∞-category. A *triangle* in $\mathcal{C}$ is a commutative diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd} \tag{39}$$

A triangle is *exact*, resp. *coexact*, if it is a pullback, resp. pushout, square.

Definition 40. Let $\mathcal{C}$ be a pointed ∞-category. Denote by $\mathcal{C}^{\Sigma}$, resp. $\mathcal{C}^{\Omega}$, the full sub-category of $\mathcal{F}\mathrm{un}(\Delta^1 \times \Delta^1, \mathcal{C})$ of coexact, resp. exact, triangles of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}, \tag{41}$$

If $\mathcal{C}$ is finitely cocomplete, resp. complete, for every object X, resp. Y, there is a contractible space of coexact, resp. exact, triangles as Equation (41). In particular, $\mathcal{C} \simeq \mathcal{C}^{\Sigma}$ and $\mathcal{C} \simeq \mathcal{C}^{\Omega}$.

Proposition 42. *If $\mathcal{C}$ is a finitely complete and cocomplete pointed ∞-category, define the following functors Then the functors*

$$\Sigma : \mathcal{C} \longrightarrow \mathcal{C}^{\Sigma} \xrightarrow{\mathrm{ev}_{(1,1)}} \mathcal{C} \qquad \Omega : \mathcal{C} \longrightarrow \mathcal{C}^{\Omega} \xrightarrow{\mathrm{ev}_{(0,0)}} \mathcal{C} \tag{43}$$

are adjoint (Σ is left adjoint to Ω).

Theorem 44. *Let $\mathcal{C}$ be a finitely bicomplete pointed ∞-category. The following are equivalent:*

(1) A triangle is exact if and only if it is coexact.
(2) (Σ, Ω) is an adjoint equivalence.
(3) A commutative square is a pullback if and only if it is a pushout.

Definition 45. A finite bicomplete pointed ∞-category $\mathcal{C}$ satisfying any of the equivalent conditions in Equation (44) is called *stable*.

In homological algebra, the derived category $\mathcal{D}(\mathbf{A})$ of an abelian category $\mathbf{A}$ underlies the structure of a triangulated category. In higher category theory, the derived category $\mathcal{D}_{\infty}(\mathbf{A})$ is constructed as the higher categorical localization of chain complexes at quasi-isomorphisms, then $\mathrm{Ho}\mathcal{D}_{\infty}(\mathbf{A}) \simeq \mathcal{D}(\mathbf{A})$ and the triangulated structure on $\mathcal{D}(\mathbf{A})$ is a reflection of $\mathcal{D}_{\infty}(\mathbf{A})$ being a stable ∞-category.

Proposition 46 ([Lur17, 3.11])**.** *If $\mathcal{C}$ is a stable ∞-category, then* $\mathrm{Ho}\mathcal{C}$ *has a canonical structure of a triangulated category.*

To construct the stabilization of a pointed ∞-category, there are several approaches, such as reduced excisive functors on $\mathcal{S}_*^{\mathrm{fin}}$, the category of pointed, finite spaces, see [Lur17, 1.4.2.8]. Here we consider the more explicit approach using spectrum objects.

Definition 47. Let $\mathcal{C}$ be a pointed ∞-category. A *pre-spectrum object in C* consists of a functor $E : \mathbb{Z}\times\mathbb{Z} \to \mathcal{C}$ such that $E(n,m) \simeq 0$, for all $n \neq m$. Denote by $\mathcal{PS}\mathrm{p}(\mathcal{C})$ the category of pre-spectrum objects. The functor $\Omega^{\infty-n} : \mathcal{PS}\mathrm{p}(\mathcal{C}) \to \mathcal{C}$ is defined as evaluation at (n,n).

For every n, the diagram

$$\begin{tikzcd} E(n,n) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & E(n+1,n+1) \end{tikzcd} \tag{48}$$

determines a pair of adjoint morphisms

$$\alpha_n : \Sigma E(n,n) \to E(n+1,n+1), \qquad \beta_n : E(n,n) \to \Omega E(n+1,n+1)$$

Definition 49. Let $\mathcal{C}$ be a pointed ∞-category. A *spectrum object in* $\mathcal{C}$ consists of a pre-spectrum object E such that β_n is an equivalence, for all n. Denote by $\mathcal{S}\mathrm{p}(\mathcal{C}) \subseteq \mathcal{PS}\mathrm{p}(\mathcal{C})$ the full sub-category of spectrum objects.

Theorem 50. *Let $\mathcal{C}$ be a presentable pointed ∞-category, then $\Omega^{\infty-n} : \mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\Sigma^{\infty-n} : \mathcal{C} \to \mathcal{S}\mathrm{p}(\mathcal{C})$, for every n.*

In particular, $\Sigma^{\infty} : \mathcal{C} \to \mathcal{S}\mathrm{p}(\mathcal{C})$ has the following universal property.

Theorem 51. *Let $\mathcal{C}$ be a presentable pointed ∞-category. Given a stable ∞-category $\mathcal{D}$, pre-composition by Σ^{∞} induces an equivalence*

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{S}\mathrm{p}(\mathcal{C}), \mathcal{D}) \longrightarrow \mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \tag{52}$$

In particular, for $\mathcal{C} = \mathcal{S}_*$, evaluation at the sphere spectrum $\mathbb{S} = \Sigma^{\infty} S^0$ *induces an equivalence*

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{S}\mathrm{p}(\mathcal{S}_*), \mathcal{D}) \longrightarrow \mathcal{D} \tag{53}$$

Definition 54. The *∞-category of spectra* is the category of spectrum objects in pointed spaces.

4. Generalized Cohomology Theories

We shall now use the language of ∞-categories to reformulate the concept of generalized cohomology theory à-là Eilenberg-Steenrod. In this new context, we recall a representability theorem for cohomology theories by spectrum object.

Remark 55. Denote by $\mathcal{S}\mathrm{et}^{\mathbb{Z}}$ the category of $\mathbb{Z}$-indexes families of sets. Given an object S and $n \in \mathbb{Z}$, denote by $\Sigma^n S$ the shifted family $(\Sigma^n S)_i = S_{i-n}$.

Definition 56 ([Lur17, 1.4.1.6])**.** Let $\mathcal{C}$ be a finitely cocomplete pointed ∞-category, $\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ the induced suspension functor. A *generalized cohomology theory* is a functor $H : \mathrm{Ho}\mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}^{\mathbb{Z}}$ together with a natural isomorphism $\partial : \Sigma H \to H\Sigma_{\mathcal{C}}$ such that:

- H preserves arbitrary products. In particular, $H^n(0)$ is the one-point set. Given an object X, the unique morphism $X \to 0$ induces an element $* \simeq H^n(0) \to H^n(X)$, which we denote by 0.
- Given a coexact triangle $X' \to X \to X''$, if $\eta \in H^n(X)$ has image $0 \in H^n(X'')$, then it lies in the image of $H^n(X') \to H^n(X)$.

Theorem 57 ([Lur17, 1.4.1.10])**.** *Let $\mathcal{C}$ be a nice ∞-category and (H, ∂) a generalized cohomology theory, then, for every n, the functor H^n is representable by an object $E(n)$.*

The natural isomorphism ∂ translates into an equivalence $E(n) \simeq \Omega E(n+1)$, which is then used to construct a spectrum object representing the cohomology theory H^n, see [Lur17, 1.4.1.11].

Remark 58. For $\mathcal{C} = \mathcal{S}_*$, the above definition of cohomology theory reduces to the classical Eilenberg-Steenrod definition. Since $\mathcal{S}_*$ is nice, we thus recover the classical *Brown representability theorem*.

References

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