

DIFFERENTIAL COHOMOLOGY SEMINAR 2

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In this lecture we want to learn the basics of ∞ -category theory. For the ∞ -categorical background, we broadly follow [Gro10] and a little [Lur09].

1. BASICS ON $(\infty, 1)$ -CATEGORIES

$(\infty, 1)$ -categories have different models that capture its essence. The first model are *quasi-categories*.

Definition 1. Given a natural number n , let $\langle n \rangle$ denote the linearly ordered set $\{0, \dots, n\}$. The *simplex category* Δ is the category of finite linearly ordered sets $\langle n \rangle$, for every n , and monotone functions.

Definition 2. Given $0 \leq i \leq n$, the *i -face map* is the unique injective map $\delta_n^i : \langle n-1 \rangle \rightarrow \langle n \rangle$ missing i . The *i -degeneracy map* is the unique surjective map $\sigma_n^i : \langle n+1 \rangle \rightarrow \langle n \rangle$ such that i and $i+1$ have the same image.

Theorem 3. As a category, Δ is generated from the face and degeneracy maps subject to the simplicial identities, i.e.

$$(4) \quad \delta_{n+1}^i \delta_n^j = \delta_{n+1}^{j+1} \delta_n^i, \quad i \leq j$$

$$(5) \quad \sigma_{n-1}^j \sigma_n^i = \sigma_{n-1}^i \sigma_n^{j+1}, \quad i \leq j$$

$$(6) \quad \sigma_n^j \delta_{n+1}^i = \begin{cases} \delta_n^i \sigma_{n-1}^{j-1}, & i < j \\ 1, & i = j \\ \delta_n^{i-1} \sigma_{n-1}^j, & i > j \end{cases}$$

Proof. Omitted. □

Definition 7. A *simplicial set* is a contravariant functor $X : \Delta^{\text{op}} \rightarrow \text{Set}$. Denote by sSet the category of simplicial sets. $X_n := X(\langle n \rangle)$ is the set of n -simplices.

By only representing the face maps, we can depict a simplicial set as follows:

$$(8) \quad X_0 \xleftarrow{\quad} X_1 \xleftarrow{\quad} X_2 \xleftarrow{\quad} \dots$$

∞ -categories are then defined in terms of a lifting condition, for which we need to define horns.

Definition 9. Let Δ^n denote the representable functor associated to $\langle n \rangle$. The face map δ_n^i induces a map of simplicial sets $d_n^i : \Delta^{n-1} \rightarrow \Delta^n$. The image of d_n^i is called the *i -face*. The *i -horn* $\Lambda^{i,n}$ is the union of all faces, except the i -face.

Remark 10. Another characterization of $\Lambda^{i,n}$ is the following: A t -simplex $f : \langle t \rangle \rightarrow \langle n \rangle$ is a t -simplex for $\Lambda^{i,n}$ if and only if there is $j \neq i$ not in the image of f .

Definition 11. A simplicial set X is a *quasi-category* if, for every $0 < i < n$ and solid diagram

$$(12) \quad \begin{array}{ccc} \Lambda^{i,n} & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

there is a dashed arrow rendering the diagram commutative. If the above condition holds for every $0 \leq i \leq n$, we call X a *Kan complex*.

Example 13. Let \mathcal{C} be a category. The *nerve* of \mathcal{C} , denoted $N\mathcal{C}$, is the simplicial set where the n -simplices are $\text{Hom}_{\mathcal{C}\text{at}}(\langle n \rangle, \mathcal{C})$. This defines a functor $N : \mathcal{C}\text{at} \rightarrow \text{sSet}$.

Date: 06.05.2025 & 13.05.2025.

Proposition 14. $\mathcal{N}\mathcal{C}$ is a quasi-category, and a Kan complex if and only if \mathcal{C} is a groupoid.

Proof. Straightforward combinatorics. \square

Remark 15. The nerve is a special case of the following construction: Let \mathcal{C} be a category, $\Gamma : \Delta \rightarrow \mathcal{C}$ a functor, then define N_Γ as the composition

$$(16) \quad \mathcal{C} \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{C}\mathrm{at}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \longrightarrow \mathcal{S}\mathrm{et}$$

where the first functor is Yoneda, while the second is pre-composition with Γ^{op} . In the case of the nerve, Γ is the functor sending $\langle n \rangle$ to the linearly ordered set viewed as a category. On the other hand, assuming \mathcal{C} is cocomplete, we can left Kan extend Γ along the Yoneda functor $\Delta \rightarrow \mathcal{S}\mathrm{et}$, we denote by Ho_Γ the resulting functor $\mathcal{S}\mathrm{et} \rightarrow \mathcal{C}$.

Proposition 17. The pair $(\mathrm{Ho}_\Gamma, N_\Gamma) : \mathcal{C} \rightarrow \mathcal{S}\mathrm{et}$ is an adjoint pair. In the case of $N : \mathcal{C}\mathrm{at} \rightarrow \mathcal{S}\mathrm{et}$, the functor is fully faithful.

Proof. Abstract nonsense about left Kan extensions. Full faithfulness can be checked directly. \square

Remark 18. $\mathrm{Ho} : \mathcal{S}\mathrm{et} \rightarrow \mathcal{C}\mathrm{at}$ is called the *homotopy category* functor. If X is a quasi-category, $\mathrm{Ho}X$ has X_0 as set of objects and homotopy classes of maps as morphism, see [Lan21, 1.2.5].

Remark 19. Denote by $\mathcal{S}\mathcal{C}\mathrm{at}$ the category of simplicially enriched categories. In [Lur09, 1.1.5.1], Lurie constructs a cosimplicial object $\Delta \rightarrow \mathcal{S}\mathcal{C}\mathrm{at}$. The resulting nerve functor $N_\Delta : \mathcal{S}\mathcal{C}\mathrm{at} \rightarrow \mathcal{S}\mathrm{et}$ is called *homotopy coherent nerve*. If \mathcal{C} is a category enriched over ∞ -groupoids, its homotopy coherent nerve is a ∞ -category. The induced right adjoint is denoted \mathfrak{C} , the adjoint pair (\mathfrak{C}, N_Δ) underlies a Quillen equivalence.

Denote by $\mathcal{K}\mathrm{an}$ the category of Kan complexes. One can show that $\mathcal{K}\mathrm{an}$ is self-enriched, which motivates, together with Equation (14), the following definition:

Definition 20. $\mathcal{S} := N_\Delta(\mathcal{K}\mathrm{an})$ is called the *quasi-category of ∞ -groupoids*.

2. ACCESSIBLE AND PRESENTABLE CATEGORIES

In general, a limit, resp. colimit, preserving functor need not have a left, resp. right, adjoint. Here we wish to introduce a rather large class of quasi-categories for which the previous statement holds. Let κ denote a regular cardinal.

Definition 21. A category \mathcal{J} is κ -filtered if, for every \mathcal{J} with $< \kappa$ many morphisms, every diagram $\mathcal{J} \rightarrow \mathcal{J}$ has a cocone. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is κ -accessible if it preserves κ -filtered colimits. Given a category \mathcal{C} , an object X is κ -compact if $\mathcal{H}\mathrm{om}_{\mathcal{C}}(X, -) : \mathcal{C} \rightarrow \mathcal{S}\mathrm{et}$ is κ -accessible.

Definition 22. A category \mathcal{C} is κ -accessible if there exists a set $S \subseteq \mathcal{C}_0$ of κ -compact objects that generate \mathcal{C} under κ -filtered colimits. A category is *accessible* if it is κ -accessible, for some regular cardinal κ .

Definition 23. A category \mathcal{C} that is accessible and cocomplete is called *locally presentable*.

Theorem 24. Let \mathcal{C} be a category, then \mathcal{C} is locally presentable if and only if there exists a small category S such that the induced functor $\mathcal{C} \rightarrow \mathcal{P}(S)$ is a fully faithful, accessible right adjoint.

Theorem 25. Let \mathcal{C}, \mathcal{D} be locally presentable categories. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a left, resp. right, adjoint if and only if it preserves colimits, resp. it preserves limits and is accessible.

We now generalize this to quasi-categories.

Definition 26. Given a simplicial set X , denote by X^{op} the simplicial set obtained by reversing the structure maps: $X_n^{\mathrm{op}} = X_n$, for all n , and

$$\begin{aligned} (d_i : X_n^{\mathrm{op}} \rightarrow X_{n-1}^{\mathrm{op}}) &= (d_{n-i} : X_n \rightarrow X_{n-1}) \\ (s_i : X_n^{\mathrm{op}} \rightarrow X_{n+1}^{\mathrm{op}}) &= (s_{n-i} : X_n \rightarrow X_{n+1}) \end{aligned}$$

If X is a quasi-category, so is X^{op} .

Definition 27. Let X be a quasi-category. The *quasi-category of simplicial presheaves* is defined as $\mathcal{F}\mathrm{un}(X^{\mathrm{op}}, \mathcal{S}) := \mathcal{H}\mathrm{om}_{\mathcal{S}\mathrm{et}}(X^{\mathrm{op}}, \mathcal{S})$.

Theorem 28 (Yoneda). *Given a quasi-category X , the mapping space functor $X^{\text{op}} \times X \rightarrow \mathcal{S}$ adjoints to a functor $y : X \rightarrow \hat{X} := \mathcal{F}\text{un}(X^{\text{op}}, \mathcal{S})$, called the Yoneda embedding. Given a cocomplete quasi-category \mathcal{C} , pre-composition by y induces an equivalence*

$$(29) \quad \mathcal{F}\text{un}^{\text{L}}(\hat{X}, \mathcal{C}) \longrightarrow \mathcal{F}\text{un}(X, \mathcal{C})$$

where $\mathcal{F}\text{un}^{\text{L}}(\hat{X}, \mathcal{C})$ denotes the category of colimit preserving functors $\hat{X} \rightarrow \mathcal{C}$.

In other words, \hat{X} is the free cocompletion of X . The inverse equivalence is given by taking left Kan extensions. The definition of accessible category transfers directly to the ∞ -categorical setting.

Theorem 30. *A quasi-category X is locally presentable (cocomplete and accessible) if and only if there is a small sub-quasi-category S such that $X \rightarrow \mathcal{F}\text{un}(S^{\text{op}}, \mathcal{S})$ is a fully faithful, accessible right adjoint.*

Remark 31. In view of [Equation \(30\)](#), one can define a category to be *locally presentable* if it is the accessible right localization of a pre-sheaf category for some small quasi-category S . In particular, every pre-sheaf category is locally presentable.

Theorem 32. *Let X, Y be presentable ∞ -categories, then a functor $f : X \rightarrow Y$ is a left, resp. right, adjoint if and only if it preserves colimits, resp. it preserves limits and is accessible.*

3. STABLE ∞ -CATEGORIES AND SPECTRA

We now use the ∞ -categorical framework to study spectra. The study of spectra originates from the study of *stable phenomena*, i.e. patterns appearing after repeated application of the suspension functor $\Sigma : \mathcal{T}\text{op}_* \rightarrow \mathcal{T}\text{op}_*$.

Example 33. Let $(\Sigma, \Omega) : \mathcal{T}\text{op}_{*/} \rightarrow \mathcal{T}\text{op}_{*/}$ be the suspension-loop adjunction on pointed topological spaces. *Freudenthal Suspension Theorem* states that, if X is a n -connected space, the adjunction unit $X \rightarrow \Omega\Sigma X$ is $2n$ -connected. If X is connected, $S^n \wedge X$ is n -connected, so $\Sigma^n X \rightarrow \Omega\Sigma^{n+1} X$ is $2n$ -connected. In particular, $\pi_i(\Sigma^n X) \rightarrow \pi_{i+1}(\Sigma^{n+1} X)$ is an isomorphism, for all $i < 2n$. The group $\pi_i(\Sigma^n X)$ is denoted $\pi_{n-i}^s(X)$, called the $(n-i)$ -stable homotopy group of X .

More examples?

Definition 34. Let \mathcal{C} be an ∞ -category, an object 0 that is both initial and terminal is called *zero object*. A category \mathcal{C} with a zero object is called a *pointed category*.

Example 35. Let $1 \in \mathcal{C}$ be a terminal object, then the identity of 1 is the zero object in the slice category $\mathcal{C}_{1/}$ of objects under 1 . In particular, the category $\mathcal{S}_* = \mathcal{S}_{*/}$ of pointed spaces is pointed.

Proposition 36. *Let \mathcal{D} be a pointed ∞ -category. Evaluation at the 0-sphere S^0 induces an equivalence*

$$(37) \quad \mathcal{F}\text{un}^{\text{L}}(\mathcal{S}_*, \mathcal{C}) \longrightarrow \mathcal{C}$$

where $\mathcal{F}\text{un}^{\text{L}}(\mathcal{S}_*, \mathcal{C})$ denotes the category of colimit preserving functors $\mathcal{S}_* \rightarrow \mathcal{C}$.

We now introduce the notion of a triangle.

Definition 38. Let \mathcal{C} be a pointed ∞ -category. A *triangle* in \mathcal{C} is a commutative diagram of the form

$$(39) \quad \begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

A triangle is *exact*, resp. *coexact*, if it is a pullback, resp. pushout, square.

Definition 40. Let \mathcal{C} be a pointed ∞ -category. Denote by \mathcal{C}^{Σ} , resp. \mathcal{C}^{Ω} , the full sub-category of $\mathcal{F}\text{un}(\Delta^1 \times \Delta^1, \mathcal{C})$ of coexact, resp. exact, triangles of the form

$$(41) \quad \begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Y \end{array},$$

If \mathcal{C} is finitely cocomplete, resp. complete, for every object X , resp. Y , there is a contractible space of coexact, resp. exact, triangles as [Equation \(41\)](#). In particular, $\mathcal{C} \simeq \mathcal{C}^\Sigma$ and $\mathcal{C} \simeq \mathcal{C}^\Omega$.

Proposition 42. *If \mathcal{C} is a finitely complete and cocomplete pointed ∞ -category, define the following functors Then the functors*

$$(43) \quad \Sigma : \mathcal{C} \longrightarrow \mathcal{C}^\Sigma \xrightarrow{\text{ev}(1,1)} \mathcal{C} \quad \quad \quad \Omega : \mathcal{C} \longrightarrow \mathcal{C}^\Omega \xrightarrow{\text{ev}(0,0)} \mathcal{C}$$

are adjoint (Σ is left adjoint to Ω).

Theorem 44. *Let \mathcal{C} be a finitely bicomplete pointed ∞ -category. The following are equivalent:*

- (1) *A triangle is exact if and only if it is coexact.*
- (2) *(Σ, Ω) is an adjoint equivalence.*
- (3) *A commutative square is a pullback if and only if it is a pushout.*

Definition 45. A finite bicomplete pointed ∞ -category \mathcal{C} satisfying any of the equivalent conditions in [Equation \(44\)](#) is called *stable*.

In homological algebra, the derived category $\mathcal{D}(\mathbb{A})$ of an abelian category \mathbb{A} underlies the structure of a triangulated category. In higher category theory, the derived category $\mathcal{D}_\infty(\mathbb{A})$ is constructed as the higher categorical localization of chain complexes at quasi-isomorphisms, then $\text{Ho}\mathcal{D}_\infty(\mathbb{A}) \simeq \mathcal{D}(\mathbb{A})$ and the triangulated structure on $\mathcal{D}(\mathbb{A})$ is a reflection of $\mathcal{D}_\infty(\mathbb{A})$ being a stable ∞ -category.

Proposition 46 ([Lur17, 3.11]). *If \mathcal{C} is a stable ∞ -category, then $\text{Ho}\mathcal{C}$ has a canonical structure of a triangulated category.*

To construct the stabilization of a pointed ∞ -category, there are several approaches, such as reduced excisive functors on $\mathcal{S}_*^{\text{fin}}$, the category of pointed, finite spaces, see [Lur17, 1.4.2.8]. Here we consider the more explicit approach using spectrum objects.

Definition 47. Let \mathcal{C} be a pointed ∞ -category. A *pre-spectrum object in \mathcal{C}* consists of a functor $E : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathcal{C}$ such that $E(n, m) \simeq 0$, for all $n \neq m$. Denote by $\mathcal{P}(\mathcal{C})$ the category of pre-spectrum objects. The functor $\Omega^{\infty-n} : \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{C}$ is defined as evaluation at (n, n) .

For every n , the diagram

$$(48) \quad \begin{array}{ccc} E(n, n) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & E(n+1, n+1) \end{array}$$

determines a pair of adjoint morphisms

$$\alpha_n : \Sigma E(n, n) \rightarrow E(n+1, n+1), \quad \beta_n : E(n, n) \rightarrow \Omega E(n+1, n+1)$$

Definition 49. Let \mathcal{C} be a pointed ∞ -category. A *spectrum object in \mathcal{C}* consists of a pre-spectrum object E such that β_n is an equivalence, for all n . Denote by $\text{Sp}(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ the full sub-category of spectrum objects.

Theorem 50. *Let \mathcal{C} be a presentable pointed ∞ -category, then $\Omega^{\infty-n} : \text{Sp}(\mathcal{C}) \rightarrow \mathcal{C}$ admits a left adjoint $\Sigma^{\infty-n} : \mathcal{C} \rightarrow \text{Sp}(\mathcal{C})$, for every n .*

In particular, $\Sigma^\infty : \mathcal{C} \rightarrow \text{Sp}(\mathcal{C})$ has the following universal property.

Theorem 51. *Let \mathcal{C} be a presentable pointed ∞ -category. Given a stable ∞ -category \mathcal{D} , pre-composition by Σ^∞ induces an equivalence*

$$(52) \quad \mathcal{F}\text{un}^{\text{L}}(\text{Sp}(\mathcal{C}), \mathcal{D}) \longrightarrow \mathcal{F}\text{un}^{\text{L}}(\mathcal{C}, \mathcal{D})$$

In particular, for $\mathcal{C} = \mathcal{S}_*$, evaluation at the sphere spectrum $\mathbb{S} = \Sigma^\infty S^0$ induces an equivalence

$$(53) \quad \mathcal{F}\text{un}^{\text{L}}(\text{Sp}(\mathcal{S}_*), \mathcal{D}) \longrightarrow \mathcal{D}$$

Definition 54. The *∞ -category of spectra* is the category of spectrum objects in pointed spaces.

4. GENERALIZED COHOMOLOGY THEORIES

We shall now use the language of ∞ -categories to reformulate the concept of generalized cohomology theory à-la Eilenberg-Steenrod. In this new context, we recall a representability theorem for cohomology theories by spectrum object.

Remark 55. Denote by $\mathrm{Set}^{\mathbb{Z}}$ the category of \mathbb{Z} -indexes families of sets. Given an object S and $n \in \mathbb{Z}$, denote by $\Sigma^n S$ the shifted family $(\Sigma^n S)_i = S_{i-n}$.

Definition 56 ([Lur17, 1.4.1.6]). Let \mathcal{C} be a finitely cocomplete pointed ∞ -category, $\Sigma_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ the induced suspension functor. A *generalized cohomology theory* is a functor $H : \mathrm{Ho}\mathcal{C}^{\mathrm{op}} \rightarrow \mathrm{Set}^{\mathbb{Z}}$ together with a natural isomorphism $\partial : \Sigma H \rightarrow H\Sigma_{\mathcal{C}}$ such that:

- H preserves arbitrary products. In particular, $H^n(0)$ is the one-point set. Given an object X , the unique morphism $X \rightarrow 0$ induces an element $* \simeq H^n(0) \rightarrow H^n(X)$, which we denote by 0 .
- Given a coexact triangle $X' \rightarrow X \rightarrow X''$, if $\eta \in H^n(X)$ has image $0 \in H^n(X'')$, then it lies in the image of $H^n(X') \rightarrow H^n(X)$.

Theorem 57 ([Lur17, 1.4.1.10]). *Let \mathcal{C} be a nice ∞ -category and (H, ∂) a generalized cohomology theory, then, for every n , the functor H^n is representable by an object $E(n)$.*

The natural isomorphism ∂ translates into an equivalence $E(n) \simeq \Omega E(n+1)$, which is then used to construct a spectrum object representing the cohomology theory H^n , see [Lur17, 1.4.1.11].

Remark 58. For $\mathcal{C} = \mathcal{S}_*$, the above definition of cohomology theory reduces to the classical Eilenberg-Steenrod definition. Since \mathcal{S}_* is nice, we thus recover the classical *Brown representability theorem*.

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