DIFFERENTIAL COHOMOLOGY SEMINAR 4 (DRAFT)

TALK BY ALESSANDRO NANTO

In the last talk we learned the definition of a differential cohomology theory, as a sheaf valued in spectra on the site of manifolds. This talk continues our journey through differential cohomology theories, and focuses on the following three topics:

(1) We want to learn how to construct non-trivial examples out of sheaves valued in chain complexes.
(2) We want to understand how we can extend classical cohomology operations to the setting of differential cohomology theories.
(3) We want to introduce suitable analogues of fiber-wise integration.

1. Abelian Groups, Spectra and the Heart

Let us start by reviewing the relation between abelian groups, rings and spectra.

Definition 1. Let $n \in \mathbb{Z}$ and X be a spectrum, define $\pi_n(X) := \pi_0(\Omega^{\infty+n}X) = \pi_0(X_{-n})$. We call π_n the *n-th homotopy group*[1] of X.

The category $\mathcal{S}\mathrm{p}$ underlies the structure of a symmetric monoidal ∞-category ([Lur17, Corollary 4.8.2.19]). Following [Lur17], we denote by $\otimes$ the tensor product on $\mathcal{S}\mathrm{p}$.

Definition 2. A commutative algebra object in $\mathcal{S}\mathrm{p}$ is called an *$\mathbb{E}_\infty$-ring spectrum*, see [Lur17, Definition 7.1.0.1]. Given an $\mathbb{E}_\infty$-ring spectrum R, denote by Mod_R the corresponding category of *left R-module spectra*, see [Lur17, Definition 7.1.1.2].

Remark 3. The sphere spectrum $\mathbb{S}$ acts as the monoidal unit of $\mathcal{S}\mathrm{p}$, therefore it is a $\mathbb{E}_\infty$-ring spectrum. The category $\mathrm{Mod}_\mathbb{S}$ is canonically equivalent to $\mathcal{S}\mathrm{p}$.

Definition 4. Denote by $\mathcal{S}\mathrm{p}_{\geq 0} \subseteq \mathcal{S}\mathrm{p}$ the full sub-category generated by *connective spectra*, i.e. spectra X such that $\pi_n(X) \simeq 0$, for all $n < 0$. Denote by $\mathcal{S}\mathrm{p}^\heartsuit \subseteq \mathcal{S}\mathrm{p}_{\geq 0}$ the *heart of spectra*, i.e. the full sub-category generated by spectra X such that $\pi_n(X) \simeq 0$, for all $n > 0$.

The category $\mathcal{S}\mathrm{p}_{\leq 0}$ is presentable and π_0 induces an equivalence between the heart and $\mathcal{A}\mathrm{b}$ ([Lur17, Proposition 1.4.3.6]). The heart is a sub-category of local objects of connective spectra[2], therefore the inclusion $\mathcal{A}\mathrm{b} \simeq \mathcal{S}\mathrm{p}^\heartsuit \subseteq \mathcal{S}\mathrm{p}_{\geq 0}$ is a right adjoint. The category $\mathcal{S}\mathrm{p}_{\geq 0}$ is closed under $\otimes$ and, given X, Y connective spectra,

$$\pi_0(X \otimes Y) \simeq \pi_0(X) \otimes \pi_0(Y) \tag{5}$$

(see these notes by Jack Davies, Theorem 2.3.28).

Definition 6. Given an abelian group A, denote by HA the (unique up to equivalence) spectrum of the heart such that $\pi_0(HA) \simeq A$. We call HA the *Eilenberg-Mac Lane spectrum* of A.

Using Equation (5), one can prove H, viewed as a functor $\mathcal{A}\mathrm{b} \to \mathcal{S}\mathrm{p}$, is lax monoidal. In particular, if R is a commutative ring, then HR is a connective $\mathbb{E}_\infty$-ring spectrum.

Definition 7. Given a commutative ring R, denote by $\mathrm{Ch}(R) = \mathrm{Ch}(\mathrm{Mod}_R)$ the ordinary category of unbounded chain complexes. Let $\mathcal{D}(R)$ be the ∞-localization of $\mathrm{Ch}(R)$ at the class of quasi-isomorphisms.

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[1] Since $X_n \simeq \Omega^2 X_{n+2}$, for any n, the set $\pi_0(X_n)$ underlies the structure of an abelian group.

[2] A connective spectrum X belongs to the heart if and only if $\pi_n(\Omega^\infty X) = 0$, for all $n > 0$, which is equivalent to $\mathrm{Map}_{\mathcal{S}_*}(S, \Omega^\infty X) \simeq 0$, for all connected, pointed spaces S. Using the adjunction $(\Sigma^\infty, \Omega^\infty)$, we can conclude X belongs to the heart if and only if X is local with respect to class of maps $\Sigma^\infty S \to 0$, for every S connected, pointed space.

Theorem 8. *Let R be a commutative ring. The functor π_0 induces an equivalence between Mod_R and the heart of HR-module spectra, i.e. HR-modules M such that the underlying spectrum belongs to the heart. This equivalence extends to a symmetric monoidal equivalence between $\mathcal{D}(R)$ and Mod_{HR}.*

Proof. The first part is [Lur17, Proposition 7.1.1.13]. The second is [Lur17, Theorem 7.1.2.13]. □

2. From Chain Complexes to Spectra via stable Dold-Kan

Definition 9. Let Ω^* be the sheaf $\mathcal{M}\mathrm{fd}^{op} \to \mathrm{Ch}(\mathbb{Z})$ mapping a manifold M to its de Rham complex.

Moreover, by **??**, the ring map $\mathbb{Z} \to \mathbb{R}$, gives us a map of ring spectra $H\mathbb{Z} \to H\mathbb{R}$. Ideally Deligne cohomology should be characterized as the pullback of some sort of truncated deRham complex along the map $H\mathbb{Z} \to H\mathbb{R}$. This requires a precise definition of the spectrum associated to the k-truncated de Rham complex $\Omega^{\leq k}$. For this we use advanced result from stable homotopy theory.

Theorem 10. *Let R be a ring. The functor $H\colon \mathrm{Mod}_R \to \mathrm{Mod}_{H(R)}$ lifts*

$$\begin{tikzcd} \mathrm{Mod}_R \arrow[r, "H"] \arrow[d] & \mathrm{Mod}_{H(R)} \\ \mathcal{D}(R) \arrow[ur, dashed] & \end{tikzcd}$$

where $\mathcal{D}(R)$ is the derived category of R-modules.

Recall that a $\mathbb{Z}$-module is just an abelian group. Hence, applying this result to $R = \mathbb{Z}$, we get the following corollary.

Corollary 11. *The functor $H\colon \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{H\mathbb{Z}}$ lifts to a functor*

$$\begin{tikzcd} \mathrm{Mod}_{\mathbb{Z}} \arrow[r, "H"] \arrow[d] & \mathrm{Mod}_{H\mathbb{Z}} \\ D(\mathbb{Z}) \arrow[ur, dashed, "DK_{st}"'] & \end{tikzcd}$$

We call this lift $D(\mathbb{Z}) \to \mathrm{Mod}_{H\mathbb{Z}}$ the stable Dold-Kan correspondence.

One thing one might wonder is how this relates to the more classical Dold-Kan correspondence, which relates chain complexes of abelian groups to simplicial abelian groups. Let Ch^+ be the category of bounded below chain complexes of abelian groups. The classical Dold-Kan correspondence gives us a functor

$$DK\colon \mathrm{Ch}^+ \to s\mathcal{A}\mathrm{b}$$

from bounded below chain complexes of abelian groups to simplicial abelian groups. However, every simplicial abelian group comes with an abelian group structure on a simplicial set, meaning it is in particular an E_∞-group in spaces. This means we have a functor

What do we know about the properties of this functor?

$$s\mathcal{A}\mathrm{b} \to \mathcal{G}\mathrm{rp}_{E_\infty}(\mathcal{S})$$

However, $\mathcal{G}\mathrm{rp}_{E_\infty}(\mathcal{S})$ fully faithfully embeds in $\mathcal{S}\mathrm{p}$ as connected spectra. Composing all these functors, we get a functor

$$DK\colon \mathrm{Ch}^+ \to \mathcal{S}\mathrm{p},$$

Why?

which is fully faithful and recovers the classical Dold-Kan correspondence. The stable Dold-Kan correspondence is a lift of this functor to $\mathcal{D}(\mathbb{Z})$ i.e.

$$\begin{tikzcd} \mathrm{Ch}^+ \arrow[r, "DK"] \arrow[d, hook] & \mathcal{S}\mathrm{p} \\ D(\mathbb{Z}) \arrow[ur, "DK_{st}"'] & \end{tikzcd}$$

How?

relates to the stable Dold-Kan correspondence. Finally, we can now use stable Dold-Kan to get a functor of sheaves.

Definition 12. Let

$$H\colon \mathcal{S}\mathrm{hv}(Mfd;\mathcal{D}(\mathbb{Z}))\to \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd};\mathcal{S}\mathrm{p})$$

denote the functor that post-composes a sheaf of chain complexes on manifolds with the stable Dold-Kan correspondence and then sheafifies. For a given sheaf of chain complexes F, we call the image the associated *Eilenberg-MacLane sheaf.*

Do we really need sheafification? This might need some checking.

3. Deligne Cohomology as a Differential Cohomology Theory

Now equipped with Definition 12, we can finally define Deligne cohomology as a differential cohomology theory.

Definition 13. Let $k \geq 0$. The *Deligne cohomology sheaf* $\mathcal{E}(k)$ is defined via the following pullback square in $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd};\mathcal{S}\mathrm{p})$:

$$\begin{tikzcd} \mathcal{E}(k) \arrow[r] \arrow[d] & H(\Omega^{\leq k}_{dR}) \arrow[d] \\ H\mathbb{Z} \arrow[r] & H\mathbb{R} \end{tikzcd}$$

Here H is the Eilenberg-MacLane sheaf.

Remark 14. If we take $k=\infty$, then the map $H(\Omega_{dR})\to H\mathbb{R}$ is an equivalence, meaning $\mathcal{E}(\infty)$ is equivalent to $H\mathbb{Z}$ i.e. singular cohomology. On the other side, the individual $\mathcal{E}(k)$ are highly non-trivial and help classify many geometric invariants of interest (as we saw in the first talk). So, the $\mathcal{E}(k)$ are a non-trivial filtration of $H\mathbb{Z}$ by differential cohomology theories, in the sense that there are map $\mathcal{E}(k+1)\to\mathcal{E}(k)$, the limit of which is $H\mathbb{Z}$.

4. Cohomology Operations for Deligne Cohomology

Now that we have a rigorous definition of Deligne cohomology, we can start to think about operations on it. First of all, we need a suitable monoidal structure.

Definition 15. Let F,G be two differential cohomology theories. The *monoidal product* $F\otimes G$ is defined as the sheafification of the presheaf $F\wedge G$, which is the point-wise wedge product of spectra.

It is expected that sheafification is necessary, but example is missing.

Now, recall there is a map of differential forms

$$\Omega^{\leq k}\otimes\Omega^{\leq m}\to\Omega^{\leq k+m},$$

which induces a map of differential cohomology theories

$$\mathcal{E}(k)\otimes\mathcal{E}(m)\to\mathcal{E}(k+m)$$

Ideally, we would like to describe such an operation in a very explicit manner, however, in the realm of spectra this can be very challenging. This suggests an alternative perspective.

Definition 16. Let $\mathcal{L}(k)$ be the sheaf of chain complexes defined as the pullback in $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, D(\mathbb{Z}))$ of the following diagram

$$\begin{tikzcd} \mathcal{L}(k) \arrow[r] \arrow[d] & \Omega^{\leq k} \arrow[d, "dR"] \\ \mathbb{Z} \arrow[r] & \mathbb{R} \end{tikzcd},$$

where $\mathbb{Z}$ is the functor $M\mapsto C^\bullet(M,\mathbb{Z})$ and $\mathbb{R}$ is the functor $M\mapsto C^\bullet(M,\mathbb{R})$

Remark 17. We can explicitly describe the chain complex $\mathcal{L}(k)$ as follows.

$$\mathcal{L}(k)^n=\{(c,\omega,h)\in C^n(-\mathbb{Z})\oplus\Omega^n(-)\oplus C^{n-1}(-\mathbb{R})|\omega=0 if n>k and c-dR(\omega)=dh\}$$

Remark 18. We expect that $H\mathcal{L}(k)$ in fact recovers $\mathcal{E}(k)$, meaning operations on $\mathcal{L}(k)$ help us understand operations on Deligne cohomology.

This needs to be checked.

Using the explicit description from Remark 17, we can define an operation on $\mathcal{L}(k)$ as follows:

$$(c_1, \omega_1, h_1) \otimes (c_2, \omega_2, h_2) = (c_1 \cup c_2, \omega_1 \wedge \omega_2, (-1)^{|c_1|} c_1 \cup h_2 + h_1 \cup \omega_2 + B(\omega_1, \omega_2)),$$

where

$$dR(\omega_1) \cup dR(\omega_2) = -dR(\omega_1 \wedge \omega_2) = dB(\omega_1, \omega_2)$$

Is there a reasonable way to pick $B(\omega_1, \omega_2)$?

Remark 19. Intuitively $B(\omega_1, \omega_2)$ measures the failure of dR taking $\wedge$ to $\cup$.

Remark 20. Ideally we would expect this formula to be well-defined, meaning $(c_1, \omega_1, h_1) \otimes (c_2, \omega_2, h_2)$ should satisfy the conditions in Remark 17. In general, this is only true if c_1, ω_2 satisfy $dc_1 = d\omega_2 = 0$. In particular, it is well-defined at the level of cohomology classes, as any element is closed therein.

References

[Lur17] Jacob Lurie. Higher algebra. Available online, September 2017.