DIFFERENTIAL COHOMOLOGY SEMINAR 4 (DRAFT)

TALK BY ALESSANDRO NANTO

In the last talk we learned the definition of a differential cohomology theory, as a sheaf valued in spectra on the site of manifolds. This talk continues our journey through differential cohomology theories, and focuses on the following three topics:

(1) We want to learn how to construct non-trivial examples out of sheaves valued in chain complexes.
(2) We want to understand how we can extend classical cohomology operations to the setting of differential cohomology theories.
(3) We want to introduce suitable analogues of fiber-wise integration.

1. Abelian Groups, Spectra and the Heart

Let us start by reviewing the relation between abelian groups and spectra.

Definition 1. Let $n \in \mathbb{Z}$ and X be a spectrum, define $\pi_n(X) := \pi_0(\Omega^{\infty+n}X) = \pi_0(X_{-n})$. We call π_n the *n-th homotopy group*[1] of X. Denote by $\mathcal{S}\mathrm{p}_{\geq 0} \subseteq \mathcal{S}\mathrm{p}$ the full sub-category generated by *connective* spectra, i.e. spectra X such that $\pi_n(X) \simeq 0$, for all $n < 0$. Finally, denote by $\mathcal{S}\mathrm{p}^{\heartsuit} \subseteq \mathcal{S}\mathrm{p}_{\geq 0}$ the *heart of spectra*, the full sub-category generated by connective spectra X such that $\pi_n(X) \simeq 0$, for all $n > 0$.

By [Lur17, Proposition 1.4.3.6], the category $\mathcal{S}\mathrm{p}_{\geq 0}$ is presentable and the heart is equivalent via π_0 to the category of abelian groups.

Definition 2. The functor $\mathcal{A}\mathrm{b} \to \mathcal{S}\mathrm{p}^{\heartsuit}$ inverse to π_0 is called *Eilenberg-Mac Lane spectra*.

Lemma 3. *A connective spectrum X belongs to the heart if and only if it is local with respect to $0 \to \Sigma^{\infty+1}S$, for every pointed space S. In particular, $\mathcal{S}\mathrm{p}^{\heartsuit}$ is a reflexive localization of $\mathcal{S}\mathrm{p}_{\geq 0}$.*

Proof. X belongs to the heart if and only if $\Omega^{\infty}X$ is homotopically discrete, i.e. $\pi_0(\Omega^{\infty}X) \simeq 0$, for all $n > 0$. Since $\pi_n(\Omega^{\infty+1}X) \simeq \pi_{n+1}(\Omega^{\infty}X)$, for all $n \geq 0$, the condition that $\Omega^{\infty}X$ is homotopically discrete is equivalent to $\Omega^{\infty+1}X$ being contractible. Finally, $\Omega^{\infty+1}X$ being contractible is equivalent to $\mathrm{Map}_{\mathcal{S}_*}(S, \Omega^{\infty+1}X) \simeq 0$, for every pointed space S, which is equivalent to $\mathrm{Map}_{\mathcal{S}\mathrm{p}}(\Sigma^{\infty+1}S, X) \simeq 0$ (using the adjunction between $\Sigma^{\infty+1}$ and $\Omega^{\infty+1}$). □

Remark 4. Lemma 3 together with the equivalence $\mathcal{S}\mathrm{p}^{\heartsuit} \simeq \mathcal{A}\mathrm{b}$, imply that the Eilenberg-Mac Lane spectra functor, viewed as a functor $\mathcal{A}\mathrm{b} \to \mathcal{S}\mathrm{p}_{\geq 0}$, is a fully faithful left adjoint.

The category of spectra $\mathcal{S}\mathrm{p}$ underlies the structure of a symmetric monoidal category $\mathcal{S}\mathrm{p}^{\wedge}$.

Proposition 5. *The functor $H\colon \mathcal{A}\mathrm{b}^{\otimes} \to \mathcal{S}\mathrm{p}^{\wedge}$ is symmetric lax monoidal. In particular, if R is a (commutative) monoid in $\mathcal{A}\mathrm{b}^{\otimes}$, i.e. a (commutative) ring, then HR is a (commutative) monoid in $\mathcal{S}\mathrm{p}^{\wedge}$, i.e. a (commutativity) ring spectra. Moreover, H induces a functor $R-\mathcal{M}\mathrm{od} \to HR-\mathcal{M}\mathrm{od}$.*

Remark 6. Since $\mathcal{S}\mathrm{p}^{\wedge}$ is a symmetric monoidal ∞-category, a commutative ring spectrum is commutative *up to coherent homotopies*, in the sense of E_{∞}-algebras.

2. From Chain Complexes to Spectra via stable Dold-Kan

Let $\Omega^{\bullet}_{dR}\colon \mathcal{M}\mathrm{fd} \to \mathcal{C}\mathrm{h}(R-\mathcal{M}\mathrm{od})$ be the de Rham chain complex of a manifold, which is indeed a sheaf on the site of manifolds. Moreover, by **??**, the ring map $\mathbb{Z} \to \mathbb{R}$, gives us a map of ring spectra $H\mathbb{Z} \to H\mathbb{R}$. Ideally Deligne cohomology should be characterized as the pullback of some sort of truncated deRham complex along the map $H\mathbb{Z} \to H\mathbb{R}$. This requires a precise definition of the spectrum associated to the k-truncated de Rham complex $\Omega^{\leq k}$. For this we use advanced result from stable homotopy theory.

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[1] Since $X_n \simeq \Omega^2 X_{n+2}$, for any n, the set $\pi_0(X_n)$ underlies the structure of an abelian group.

Theorem 7. *Let R be a ring. The functor $H\colon \mathcal{M}\mathrm{od}_R \to \mathcal{M}\mathrm{od}_{H(R)}$ lifts*

$$\begin{tikzcd} \mathcal{M}\mathrm{od}_R \arrow[r, "H"] \arrow[d] & \mathcal{M}\mathrm{od}_{H(R)} \\ \mathcal{D}(R) \arrow[ur, dashed] & \end{tikzcd}$$

where $\mathcal{D}(R)$ is the derived category of R-modules.

Recall that a $\mathbb{Z}$-module is just an abelian group. Hence, applying this result to $R = \mathbb{Z}$, we get the following corollary.

Corollary 8. *The functor $H\colon \mathcal{M}\mathrm{od}_{\mathbb{Z}} \to \mathcal{M}\mathrm{od}_{H\mathbb{Z}}$ lifts to a functor*

$$\begin{tikzcd} \mathcal{M}\mathrm{od}_{\mathbb{Z}} \arrow[r, "H"] \arrow[d] & \mathcal{M}\mathrm{od}_{H\mathbb{Z}} \\ D(\mathbb{Z}) \arrow[ur, dashed, "DK_{st}"'] & \end{tikzcd}$$

We call this lift $D(\mathbb{Z}) \to \mathcal{M}\mathrm{od}_{H\mathbb{Z}}$ the stable Dold-Kan correspondence*.*

One thing one might wonder is how this relates to the more classical Dold-Kan correspondence, which relates chain complexes of abelian groups to simplicial abelian groups. Let $\mathcal{C}\mathrm{h}^+$ be the category of bounded below chain complexes of abelian groups. The classical Dold-Kan correspondence gives us a functor

$$DK\colon \mathcal{C}\mathrm{h}^+ \to s\mathcal{A}\mathrm{b}$$

from bounded below chain complexes of abelian groups to simplicial abelian groups. However, every simplicial abelian group comes with an abelian group structure on a simplicial set, meaning it is in particular an E_∞-group in spaces. This means we have a functor

What do we know about the properties of this functor?

$$s\mathcal{A}\mathrm{b} \to \mathcal{G}\mathrm{rp}_{E_\infty}(\mathcal{S})$$

However, $\mathcal{G}\mathrm{rp}_{E_\infty}(\mathcal{S})$ fully faithfully embeds in $\mathcal{S}\mathrm{p}$ as connected spectra. Composing all these functors, we get a functor

$$DK\colon \mathcal{C}\mathrm{h}^+ \to \mathcal{S}\mathrm{p},$$

Why?

which is fully faithful and recovers the classical Dold-Kan correspondence. The stable Dold-Kan correspondence is a lift of this functor to $\mathcal{D}(\mathbb{Z})$ i.e.

$$\begin{tikzcd} \mathcal{C}\mathrm{h}^+ \arrow[r, "DK"] \arrow[d, hook] & \mathcal{S}\mathrm{p} \\ D(\mathbb{Z}) \arrow[ur, "DK_{st}"'] & \end{tikzcd}$$

How?

relates to the stable Dold-Kan correspondence. Finally, we can now use stable Dold-Kan to get a functor of sheaves.

Definition 9. Let

$$H\colon \mathcal{S}\mathrm{hv}(Mfd; \mathcal{D}(\mathbb{Z})) \to \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}; \mathcal{S}\mathrm{p})$$

denote the functor that post-composes a sheaf of chain complexes on manifolds with the stable Dold-Kan correspondence and then sheafifies. For a given sheaf of chain complexes F, we call the image the associated *Eilenberg-MacLane sheaf.*

Do we really need sheafification? This might need some checking.

3. Deligne Cohomology as a Differential Cohomology Theory

Now equipped with Definition 9, we can finally define Deligne cohomology as a differential cohomology theory.

Definition 10. Let $k \geq 0$. The *Deligne cohomology sheaf* $\mathcal{E}(k)$ is defined via the following pullback square in $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}; \mathcal{S}\mathrm{p})$:

$$\begin{array}{ccc} \mathcal{E}(k) & \longrightarrow & H(\Omega^{\leq k}_{dR}) \\ \downarrow & & \downarrow \\ H\mathbb{Z} & \longrightarrow & H\mathbb{R} \end{array}$$

Here H is the Eilenberg-MacLane sheaf.

Remark 11. If we take $k = \infty$, then the map $H(\Omega_{dR}) \to H\mathbb{R}$ is an equivalence, meaning $\mathcal{E}(\infty)$ is equivalent to $H\mathbb{Z}$ i.e. singular cohomology. On the other side, the individual $\mathcal{E}(k)$ are highly non-trivial and help classify many geometric invariants of interest (as we saw in the first talk). So, the $\mathcal{E}(k)$ are a non-trivial filtration of $H\mathbb{Z}$ by differential cohomology theories, in the sense that there are map $\mathcal{E}(k+1) \to \mathcal{E}(k)$, the limit of which is $H\mathbb{Z}$.

4. Cohomology Operations for Deligne Cohomology

Now that we have a rigorous definition of Deligne cohomology, we can start to think about operations on it. First of all, we need a suitable monoidal structure.

Definition 12. Let F, G be two differential cohomology theories. The *monoidal product* $F \otimes G$ is defined as the sheafification of the presheaf $F \wedge G$, which is the point-wise wedge product of spectra.

It is expected that sheafification is necessary, but example is missing.

Now, recall there is a map of differential forms

$$\Omega^{\leq k} \otimes \Omega^{\leq m} \to \Omega^{\leq k+m},$$

which induces a map of differential cohomology theories

$$\mathcal{E}(k) \otimes \mathcal{E}(m) \to \mathcal{E}(k+m)$$

Ideally, we would like to describe such an operation in a very explicit manner, however, in the realm of spectra this can be very challenging. This suggests an alternative perspective.

Definition 13. Let $\mathcal{L}(k)$ be the sheaf of chain complexes defined as the pullback in $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, D(\mathbb{Z}))$ of the following diagram

$$\begin{array}{ccc} \mathcal{L}(k) & \longrightarrow & \Omega^{\leq k} \\ \downarrow & & \downarrow{\scriptstyle dR} \\ \mathbb{Z} & \longrightarrow & \mathbb{R} \end{array},$$

where $\mathbb{Z}$ is the functor $M \mapsto C^\bullet(M, \mathbb{Z})$ and $\mathbb{R}$ is the functor $M \mapsto C^\bullet(M, \mathbb{R})$

Remark 14. We can explicitly describe the chain complex $\mathcal{L}(k)$ as follows.

$$\mathcal{L}(k)^n = \{(c, \omega, h) \in C^n(-\mathbb{Z}) \oplus \Omega^n(-) \oplus C^{n-1}(-\mathbb{R}) | \omega = 0 if n > k and c - dR(\omega) = dh\}$$

Remark 15. We expect that $H\mathcal{L}(k)$ in fact recovers $\mathcal{E}(k)$, meaning operations on $\mathcal{L}(k)$ help us understand operations on Deligne cohomology.

This needs to be checked.

Using the explicit description from Remark 14, we can define an operation on $\mathcal{L}(k)$ as follows:

$$(c_1, \omega_1, h_1) \otimes (c_2, \omega_2, h_2) = (c_1 \cup c_2, \omega_1 \wedge \omega_2, (-1)^{|c_1|} c_1 \cup h_2 + h_1 \cup \omega_2 + B(\omega_1, \omega_2)),$$

where

$$dR(\omega_1) \cup dR(\omega_2) = -dR(\omega_1 \wedge \omega_2) = dB(\omega_1, \omega_2)$$

Remark 16. Intuitively $B(\omega_1, \omega_2)$ measures the failure of dR taking $\wedge$ to $\cup$.

Is there a reasonable way to pick $B(\omega_1, \omega_2)$?

Remark 17. Ideally we would expect this formula to be well-defined, meaning $(c_1, \omega_1, h_1) \otimes (c_2, \omega_2, h_2)$ should satisfy the conditions in Remark 14. In general, this is only true if c_1, ω_2 satisfy $dc_1 = d\omega_2 = 0$. In particular, it is well-defined at the level of cohomology classes, as any element is closed therein.

References

[Lur17] Jacob Lurie. Higher algebra. Available online, September 2017.