DIFFERENTIAL COHOMOLOGY SEMINAR 4 (DRAFT)

TALK BY ALESSANDRO NANTO

In the last talk we learned the definition of a differential cohomology theory, as a sheaf valued in spectra on the site of manifolds. This talk continues our journey through differential cohomology theories, and focuses on the following three topics:

(1) We want to learn how to construct non-trivial examples out of sheaves valued in chain complexes.
(2) We want to understand how we can extend classical cohomology operations to the setting of differential cohomology theories.
(3) We want to introduce suitable analogues of fiber-wise integration.

1. Abelian Groups, Spectra and the Heart

Let us start by reviewing the relation between abelian groups, rings and spectra.

Definition 1.1. Let $n \in \mathbb{Z}$ and X be a spectrum, define $\pi_n(X) := \pi_0(\Omega^{\infty+n}X) = \pi_0(X_{-n})$. We call π_n the *n-th homotopy group* of X.

Remark 1.2. Note that since $X_n \simeq \Omega^2 X_{n+2}$, for any n, the set $\pi_0(X_n)$ underlies the structure of an abelian group.

The category $\mathcal{S}\mathrm{p}$ underlies the structure of a symmetric monoidal ∞-category ([Lur17, Corollary 4.8.2.19]). Following [Lur17], we denote by $\otimes$ the tensor product on $\mathcal{S}\mathrm{p}$.

Definition 1.3. A commutative algebra object in $\mathcal{S}\mathrm{p}$ is called an *$\mathbb{E}_\infty$-ring spectrum*, see [Lur17, Definition 7.1.0.1]. Given an $\mathbb{E}_\infty$-ring spectrum R, denote by Mod_R the corresponding category of *left R-module spectra*, see [Lur17, Definition 7.1.1.2].

Remark 1.4. The sphere spectrum $\mathbb{S}$ acts as the monoidal unit of $\mathcal{S}\mathrm{p}$, therefore it is a $\mathbb{E}_\infty$-ring spectrum. The category $\mathrm{Mod}_{\mathbb{S}}$ is canonically equivalent to $\mathcal{S}\mathrm{p}$.

Definition 1.5. Denote by $\mathcal{S}\mathrm{p}_{\geq 0} \subseteq \mathcal{S}\mathrm{p}$ the full sub-category generated by *connective spectra*, i.e. spectra X such that $\pi_n(X) \simeq 0$, for all $n < 0$. Denote by $\mathcal{S}\mathrm{p}^\heartsuit \subseteq \mathcal{S}\mathrm{p}_{\geq 0}$ the *heart of spectra*, i.e. the full sub-category generated by spectra X such that $\pi_n(X) \simeq 0$, for all $n > 0$.

We have the following result relating connective spectra and the heart, which follow immediately.

Lemma 1.6. *Let X be a connective spectrum. The following are equivalent:*

(1) X is in the heart.
(2) $\pi_n(\Omega^\infty X) = 0$, for all $n > 0$.
(3) $\mathrm{Hom}_{\mathcal{S}_*}(S, \Omega^\infty X) \simeq 0$, for all connected, pointed spaces S.
(4) X is local with respect to the class of maps $\Sigma^\infty S \to 0$, for every connected pointed space S.

The category $\mathcal{S}\mathrm{p}_{\geq 0}$ is presentable and π_0 induces an equivalence between the heart and $\mathcal{A}\mathrm{b}$ ([Lur17, Proposition 1.4.3.6]). The heart is a sub-category of local objects of connective spectra, therefore the inclusion $\mathcal{A}\mathrm{b} \simeq \mathcal{S}\mathrm{p}^\heartsuit \subseteq \mathcal{S}\mathrm{p}_{\geq 0}$ is a right adjoint. The category $\mathcal{S}\mathrm{p}_{\geq 0}$ is closed under $\otimes$ and, given X, Y connective spectra,

$$\pi_0(X \otimes Y) \simeq \pi_0(X) \otimes \pi_0(Y) \tag{1.7}$$

see [Dav24, Theorem 2.3.28]

Definition 1.8. Given an abelian group A, denote by HA the (unique up to equivalence) spectrum of the heart such that $\pi_0(HA) \simeq A$. We call HA the *Eilenberg-Mac Lane spectrum* of A.

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Using Equation (1.7), one can prove H, viewed as a functor $\mathcal{A}\mathrm{b} \to \mathcal{S}\mathrm{p}$, is lax monoidal. In particular, if R is a commutative ring, then HR is a connective $\mathbb{E}_\infty$-ring spectrum. On the other hand, if R is a connective $\mathbb{E}_\infty$-ring spectrum and M a connective module, then $\pi_0(M)$ is a $\pi_0(R)$-module.

Definition 1.9. Given a commutative ring R, denote by $\mathrm{Ch}(R) = \mathrm{Ch}(\mathrm{Mod}_R)$ the ordinary category of unbounded chain complexes. Let $\mathcal{D}(R)$ be the ∞-localization of $\mathrm{Ch}(R)$ at the class of quasi-isomorphisms.

Similar to the heart of spectra, given an $\mathbb{E}_\infty$-ring spectrum R, denote by $\mathrm{Mod}_R^\heartsuit \subseteq \mathrm{Mod}_R$ the full sub-category generated by R-modules such that the underlying spectrum belongs to the heart of spectra.

Theorem 1.10 (Stable Dold-Kan Correspondence)**.** *Let R be a commutative ring.*

(1) $\mathrm{Mod}_R \simeq \mathrm{Mod}_{HR}^\heartsuit$ via taking Eilenberg-Mac Lane spectra.
(2) The equivalence in (1) extends to an equivalence $H : \mathcal{D}(R) \simeq \mathrm{Mod}_{HR}$ of symmetric monoidal ∞-categories.

Proof. (1) is [Lur17, Proposition 7.1.1.13], while (2) is [Lur17, Theorem 7.1.2.13]. □

An interesting consequence of Theorem 1.10 is the following:

Corollary 1.11. *Given $F \in \mathcal{D}(R)$, then $\pi_n(HF) \simeq H_n(F)$, for all $n \in \mathbb{Z}$.*

Proof.

$$\begin{aligned}
\pi_n(HF) &= \pi_0(\Omega^{\infty+n} HF) \\
&\overset{①}{\simeq} \pi_0(\mathrm{Hom}_{\mathcal{S}\mathrm{p}}(\Sigma^n \mathbb{S}, HF)) \\
&\overset{②}{\simeq} \pi_0(\mathrm{Hom}_{\mathrm{Mod}_{HR}}(\Sigma^n HR, HF)) \\
&\overset{③}{\simeq} \pi_0(\mathrm{Hom}_{\mathcal{D}(R)}(R[n], F)) \\
&\overset{④}{\simeq} H_n(F)
\end{aligned}$$

① The functor $\Omega^{\infty+n}$ is corepresented by the shifted sphere spectrum $\Sigma^n\mathbb{S}$. ② The forgetful functor $\mathrm{Mod}_{HR} \to \mathrm{Mod}_{\mathbb{S}} \simeq \mathcal{S}\mathrm{p}$ is right adjoint to tensoring by HR and $HR \otimes (\Sigma^n\mathbb{S}) \simeq \Sigma^n HR$. ③ Theorem 1.10 ④ π_0 of the mapping space $\mathrm{Hom}_{\mathcal{D}(R)}(R[n], F)$ is equivalent to the mapping space $R[n] \to F$ in the *ordinary* derived category of R, i.e. homotopy classes of maps $R[n] \to F$, which correspond exactly to classes in $H_n(F)$. □

2. More ∞-categorical baggage

Let $\mathcal{C}$ be a presentable ∞-category. The ∞-categorical background given in previous talks allows to conclude the existence of a number of functors. Here we give (somewhat) explicit formulas for one.

Remark 2.1. Recall $\mathcal{E}\mathrm{uc}$, the full sub-category of $\mathcal{M}\mathrm{fd}$ generated by Euclidean manfiolds $\mathbb{R}^n$, for every $n \geq 0$. Denote by j the inclusion functor $\mathcal{E}\mathrm{uc} \subseteq \mathcal{M}\mathrm{fd}$. Recall that the restriction along j induces an equivalence $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, \mathcal{C}) \simeq \mathcal{S}\mathrm{hv}(\mathcal{E}\mathrm{uc}, \mathcal{C})$, see [ADH21, Corollary A.5.6].

Evaluation at $\{0\}$ induces an adjunction $(\mathrm{Lconst}, \Gamma) : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, \mathcal{C})$, where the functor Γ is evaluation at $\{0\}$, while the left adjoint Lconst maps $C \in \mathcal{C}$ to the sheafification of the constant pre-sheaf with value C.

Remark 2.2. Every presentable ∞-category $\mathcal{C}$ is uniquely *cotensored over* $\mathcal{S}$, see [Lur09, Remark 5.5.2.6]. More explicitly, for every space S and object C, there is an object C^S together with a natural equivalence

$$\mathrm{Hom}_{\mathcal{S}}(S, \mathrm{Hom}_{\mathcal{C}}(-, C)) \simeq \mathrm{Hom}_{\mathcal{C}}(-, C^S)$$

Definition 2.3. Denote by Sing the functor $\mathcal{M}\mathrm{fd} \to \mathcal{S}$ mapping a manifold to its underlying space. Given a presentable ∞-category $\mathcal{C}$, denote by $\flat$ the composition $\mathcal{C} \to \mathrm{Fun}(\mathcal{S}^{op}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{M}\mathrm{fd}^{op}, \mathcal{C})$, the first functor coming from Remark 2.2, the second being pre-composition with Sing^{op}.

Explicitly, given an object $C \in \mathcal{C}$, the associated pre-sheaf $\flat C$ maps a manifold M to $C^{\mathrm{Sing}(M)}$.

Lemma 2.4 ([BG21, Corollary 6.46])**.** *$\flat$ factors through $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{M}\mathrm{fd}^{op}, \mathcal{C})$.*

Lemma 2.4 is the direct consequence of a weaker version of a generalized version of Seifert-van Kampen theorem, namely [Lur17, Proposition A.3.2], stating that, given a topological space X and a covering sieve $\mathcal{O}$, the space $\mathrm{Sing}(X)$ is the colimit of $\mathrm{Sing}(U)$ over $U \in \mathcal{O}$.

Theorem 2.5. *$\flat : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, \mathcal{C})$ is left adjoint to Γ.*

Proof. The composition $\mathcal{C} \xrightarrow{\flat} \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, \mathcal{C}) \xrightarrow{j_*} \mathcal{S}\mathrm{hv}(\mathcal{E}\mathrm{uc}, \mathcal{C})$ maps an object C to the sheaf $\flat C$ restricted to Euclidean spaces. Since $\mathbb{R}^n$ is contractible, $(\flat C)(\mathbb{R}^n) = C^{\mathrm{Sing}(\mathbb{R}^n)} \simeq C$ and so $\flat$ restricted to $\mathcal{E}\mathrm{uc}$ is equivalent to Const, the functor taking C to the the pre-sheaf with constant value C, which is left adjoint to Γ restricted to $\mathcal{E}\mathrm{uc}$. □

3. Sheaves of complexes and spectra

The stable Dold-Kan correspondence allowes us to move freely between sheaves of $H\mathbb{Z}$-module spectras and sheaves valued in $\mathcal{D}(\mathbb{Z})$.

Remark 3.1. We identify the category of cochain complexes with $\mathrm{Ch}(R)$ by reversing grading. Namely, given a cochain V^*, we are implicitly identifying it with the chain complex $V_n = V^{-n}$.

Definition 3.2 ([BNV16, Definition 7.14])**.** Given $n \in \mathbb{Z}$, denote by $\tau^{\geq n}$, resp. $\tau^{\leq n}$, the *stupid truncation functors*, mapping a cochain complex V^* to

$$\cdots \to 0 \to V^n \to V^{n+1} \to \cdots, \qquad \text{resp. } \cdots \to V^{n-1} \to V^n \to 0 \to \cdots$$

Given $F : \mathcal{M}\mathrm{fd}^{op} \to \mathrm{Ch}(\mathbb{Z})$, denote by $F^{\geq n}$ the composite $\mathcal{M}\mathrm{fd}^{op} \xrightarrow{F} \mathrm{Ch}(\mathbb{Z}) \xrightarrow{\tau^{\geq n}} \mathrm{Ch}(\mathbb{Z})$, and similarly we define $F^{\leq n}$. If F is a sheaf, then so are its truncations.

Lemma 3.3 ([BNV16, Lemma 7.12])**.** *Let $F : \mathcal{M}\mathrm{fd}^{op} \to \mathrm{Ch}(\mathbb{Z})$ a sheaf of chain complexes of C^∞-modules, then $\mathcal{M}\mathrm{fd}^{op} \xrightarrow{F} \mathrm{Ch}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$ is a sheaf.*

Definition 3.4. Denote by Ω^* the sheaf $\mathcal{M}\mathrm{fd}^{op} \to \mathrm{Ch}(\mathbb{Z})$ mapping a manifold to its de Rham complex.

Lemma 3.3 ensures that the sheaf in Definition 3.4 and the corresponding naive truncations remain sheaves after post-composition with the localization functor $\mathrm{Ch}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$.

Definition 3.5. Given a sheaf $F : \mathcal{M}\mathrm{fd}^{op} \to \mathcal{D}(\mathbb{Z})$, denote by HF the *Eilenberg-Mac Lane sheaf* of $H\mathbb{Z}$-module spectra obtained by applying point-wise the equivalence of Theorem 1.10.

Recall now the machinery set-up in Section 2.

Remark 3.6. Since $\mathcal{D}(\mathbb{Z})$ is presentable, we know that they is cotensored over $\mathcal{S}$. Given a space S and a chain complex M_*, the cotensor ${M_*}^A$ is the chain complex of graded linear maps $C_*(S, \mathbb{Z}) \to M_*$, from the singular chain complex of S to M_*, see [Lur17, Definition 1.3.2.1]. In particular, let $M_* = M$ be concentrated in degree 0, then ${M_*}^S = C^{-*}(S, M)$, the singular cochain complex with values in M.

Lemma 3.7. *Let* $\mathrm{dR} : \Omega^* \to \flat\mathbb{R}$ *be the* de Rham homomorphism, *defined on a manifold M by mapping a form $\omega \in \Omega^n(M)$ to the cochain $\int_{\_} \omega : C_n(M, \mathbb{Z}) \to \mathbb{R}$. The homomorphism* dR *is an equivalence of sheaves.*

Proof. The statement is equivalent to dR being an equivalence after restriction to the Euclidean site $\mathcal{E}\mathrm{uc}$, the conclusion then follows from Poincaré lemma. □

4. Deligne Cohomology

Finally, we have enough machinery to talk about Deligne cohomology.

Definition 4.1. Given $\ell \in \mathbb{N}$, define $\widehat{\mathbb{Z}}(\ell) : \mathcal{M}\mathrm{fd}^{op} \to \mathcal{D}(\mathbb{Z})$ by the pullback

$$\begin{tikzcd} \widehat{\mathbb{Z}}(\ell) \arrow[r] \arrow[d] & \Omega^{\geq \ell} \arrow[d] \\ \flat\mathbb{Z} \arrow[r] & \flat\mathbb{R} \end{tikzcd}$$

We call the corresponding sheaf of $H\mathbb{Z}$-modules spectra $H\widehat{\mathbb{Z}}(\ell)$ the *ℓ-th Deligne sheaf.*

The morphism $\Omega^{\geq \ell} \to \flat\mathbb{R}$ is the composition of the inclusion $\Omega^{\geq \ell} \subseteq \Omega^*$ followed by the de Rham homomorphism of Lemma 3.3. Given a manifold M, we give some models for $\widehat{\mathbb{Z}}(\ell)$ evaluated at M.

5. Unfolding the fracture square of Deligne Cohomology

Now that we have a rigorous definition of Deligne cohomology, we can start to think about operations on it. First of all, we need a suitable monoidal structure.

Definition 5.1. Let F, G be two differential cohomology theories. The *monoidal product* $F \otimes G$ is defined as the sheafification of the presheaf $F \wedge G$, which is the point-wise wedge product of spectra.

It is expected that sheafification is necessary, but example is missing.

Now, recall there is a Hom of differential forms

$$\Omega^{\leq k} \otimes \Omega^{\leq m} \to \Omega^{\leq k+m},$$

which induces a Hom of differential cohomology theories

$$\mathcal{E}(k) \otimes \mathcal{E}(m) \to \mathcal{E}(k+m)$$

Ideally, we would like to describe such an operation in a very explicit manner, however, in the realm of spectra this can be very challenging. This suggests an alternative perspective.

Definition 5.2. Let $\mathcal{L}(k)$ be the sheaf of chain complexes defined as the pullback in $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, D(\mathbb{Z}))$ of the following diagram

$$\begin{tikzcd} \mathcal{L}(k) \arrow[r] \arrow[d] & \Omega^{\leq k} \arrow[d, "dR"] \\ \mathbb{Z} \arrow[r] & \mathbb{R} \end{tikzcd},$$

where $\mathbb{Z}$ is the functor $M \mapsto C^\bullet(M, \mathbb{Z})$ and $\mathbb{R}$ is the functor $M \mapsto C^\bullet(M, \mathbb{R})$

Remark 5.3. We can explicitly describe the chain complex $\mathcal{L}(k)$ as follows.

$$\mathcal{L}(k)^n = \{(c, \omega, h) \in C^n(-\mathbb{Z}) \oplus \Omega^n(-) \oplus C^{n-1}(-\mathbb{R}) | \omega = 0 if n > k and c - dR(\omega) = dh\}$$

Remark 5.4. We expect that $H\mathcal{L}(k)$ in fact recovers $\mathcal{E}(k)$, meaning operations on $\mathcal{L}(k)$ help us understand operations on Deligne cohomology.

This needs to be checked.

Using the explicit description from Remark 5.3, we can define an operation on $\mathcal{L}(k)$ as follows:

$$(c_1, \omega_1, h_1) \otimes (c_2, \omega_2, h_2) = (c_1 \cup c_2, \omega_1 \wedge \omega_2, (-1)^{|c_1|} c_1 \cup h_2 + h_1 \cup \omega_2 + B(\omega_1, \omega_2)),$$

where

$$dR(\omega_1) \cup dR(\omega_2) = -dR(\omega_1 \wedge \omega_2) = dB(\omega_1, \omega_2)$$

Remark 5.5. Intuitively $B(\omega_1, \omega_2)$ measures the failure of dR taking $\wedge$ to $\cup$.

Is there a reasonable way to pick $B(\omega_1, \omega_2)$?

Remark 5.6. Ideally we would expect this formula to be well-defined, meaning $(c_1, \omega_1, h_1) \otimes (c_2, \omega_2, h_2)$ should satisfy the conditions in Remark 5.3. In general, this is only true if c_1, ω_2 satisfy $dc_1 = d\omega_2 = 0$. In particular, it is well-defined at the level of cohomology classes, as any element is closed therein.

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