DIFFERENTIAL COHOMOLOGY SEMINAR 12

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Today we want discuss twisted differential cohomology theory following [BG21].

1. Reviewing Twisted Cohomology

Let us first recall the definition of twisted cohomology theories, as discussed in a previous talk [Ber25]. Let X be a topological spaces, R be a commutative ring spectrum. We denote by $\mathcal{M}\mathrm{od}_R$ the ∞-category of R-module spectra. The Picard ∞-groupoid of R is defined as

$$\mathcal{P}\mathrm{ic}_R := \mathrm{Pic}(\mathcal{M}\mathrm{od}_R) \subset \mathcal{M}\mathrm{od}_R$$

the full sub- ∞-groupoid of invertible R-modules and equivalences.

Definition 1.1. A *twist* of R over X is a map of spaces (∞-groupoids)

$$\alpha : X \to \mathcal{P}\mathrm{ic}_R.$$

Definition 1.2. Given a twist $\alpha : X \to \mathcal{P}\mathrm{ic}_R$, the *$\alpha$-twisted R-cohomology of X* is defined as

$$R^\alpha(X) := \mathrm{Map}_{\mathcal{M}\mathrm{od}_R}(R, \alpha(X)).$$

We can now compute the twisted cohomology groups as homotopy groups of the mapping spectrum.

$$R^{k+\alpha} = \pi_k(R^\alpha(X)) = \pi_k\left(\mathrm{Map}_{\mathcal{M}\mathrm{od}_R}(R, \alpha(X))\right).$$

This coincides with the n-homotopy group of $\Gamma(X, \alpha)$ where $\Gamma(X, \alpha)$ is the spectrum of sections of the bundle of spectra over X associated to the twist α.

2. Reviewing Differential Cohomology

We now want to recall the definition of differential cohomology theories. Here we don't follow the more modern approach via pure and $\mathbb{R}$-invariant sheaves of spectra, as studied in [BNV16], but rather the more classical approach [HS05]. This approach is also the one used in previous talks to study differential K-theory [Lud25].

We first review some notations and definition. Let $\mathcal{C}\mathrm{h}(\mathbb{R})$ be the 1-category of chain complexes of real vector spaces. We denote by $\mathcal{D}(\mathbb{R})$ the derived ∞-category obtained from $\mathcal{C}\mathrm{h}(\mathbb{R})$ by inverting quasi-isomorphisms. Finally, let C be an object in an ∞-category $\mathcal{C}$. Then $\underline{C}$ denotes the constant sheaf with value C.

Definition 2.1. Let X be a spectrum. A *differential refinement of* X is a triple $(V, c\colon R \wedge X \to HV)$, where

- V is a chain complex of real vector spaces,
- HV is the Eilenberg-MacLane spectrum associated to V, via the stable Dold-Kan correspondence,
- α is an equivalence of spectra.

Before we can proceed, we need to introduce further notation. Let V be a chain complex of real vector spaces. Let $\Omega^* V$ be the sheaf of differential forms with values in V.

Definition 2.2. Let $\mathcal{M}$ be a sheaf of chain complexes on the site of manifolds. The *naive truncation* $\mathcal{M}^{\geq n}$ is defined as $\mathcal{M}$ if $k > n$ and 0 otherwise.

We can now get higher categorical sheaves out of sheaves of truncated chain complexes via localizations.

Lemma 2.3. *Let $\mathcal{M}$ be a sheaf of C^∞-modules, then post-composition with the localization functor $\mathcal{C}\mathrm{h}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ preserves limits, and hence preserves the sheaf condition.*

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Definition 2.4. Let X be a (X, V, c) be a differential refinement of a spectrum X, and $n \in \mathbb{Z}$. Let $F^n(X, V, c)$, the *differential function spectrum*, be defined as the pullback

$$\begin{tikzcd} F^n(X,V,c) \arrow[rrr] \arrow[d] & & & H(\Omega^\bullet V^{\geq n}) \arrow[d, "c"] \\ \underline{X} \arrow[r, "1\wedge X"] & \underline{H\mathbb{R}\wedge X} \arrow[r, "c"] & \underline{HV} \arrow[r] & H(\Omega^\bullet V) \end{tikzcd}$$

Finally we can use the differential function spectrum to define differential cohomology groups.

Definition 2.5. Let X be a spectrum, (X, V, c) be a differential refinement of X, and $n \in \mathbb{Z}$. Then

$$\hat{X}^n(M) \coloneqq \pi_{-n}(F^n(X, V, c)(M))$$

is the *n-th differential X-cohomology group* of the manifold M.

3. Towards Twisted Differential Cohomology

We can now combine the two previous definitions to define twisted differential cohomology theories. Let R be a commutative ring spectrum, M a manifold. We now want to define sheaf theoretic analogue of a twist, generalizing our previous definitions. Here we start using non-trivial definitions and results of [BG21].

Definition 3.1. Let $\mathcal{M}\mathrm{od}_{\underline{R}}(M)$ be the ∞-category of sheaves of $\underline{R}$-module spectra on M (i.e. sheaves valued in $\mathcal{M}\mathrm{od}_R$). Define $\mathcal{P}\mathrm{ic}^{loc}_{\underline{R}}(M)$ as the full sub-∞-groupoid of $\mathcal{M}\mathrm{od}_{\underline{R}}(M)$ spanned by the locally constant $\underline{R}$-modules. The objects of $\mathcal{P}\mathrm{ic}^{loc}_{\underline{R}}(M)$ are called *topological R-twists* of M.

This name is motivated by the following observation. There is an equivalence

$$\mathcal{F}\mathrm{un}(M^{top}, \mathcal{P}\mathrm{ic}_R) \to \mathcal{P}\mathrm{ic}^{loc}_{\underline{R}}(M) \hookrightarrow \mathcal{S}\mathrm{hv}((\mathcal{M}\mathrm{fd}_{/M})^{op}, \mathcal{M}\mathrm{od}_R),$$

which sends f to the sheaf that sends $g\colon N \to M$ to the R-module of global sections of $f \circ g$. Here M^{top} is the underlying topological space of the manifold M. Here we also used the fact that $\mathcal{S}\mathrm{hv}((\mathcal{M}\mathrm{fd}_{/M})^{op}, \mathcal{M}\mathrm{od}_R)$ is equivalent to the $\mathcal{S}\mathrm{hv}((\mathcal{M}\mathrm{fd})^{op}, \mathcal{M}\mathrm{od}_R)$ objects over the locally constant sheaf $\underline{M}$. So, the topological R-twists of M are precisely the twists that come from a functor $M \to \mathcal{P}\mathrm{ic}_R$, which is what a twist should really mean.

We now use this notion to define twisted differential cohomology.

Remark 3.2. Recall that the localization functor $\mathcal{C}\mathrm{h}(\mathbb{R})\mathcal{M}\mathrm{od}_{H\mathbb{R}}$ is lax monoidal, meaning it preserves commutative algebra objects.

The previous remark implies that the composition of the stable Dold-Kan equivalence $\mathcal{C}\mathrm{h}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ with the localization $\mathcal{C}\mathrm{h}(\mathbb{R})\mathcal{M}\mathrm{od}_{H\mathbb{R}}$ is lax monoidal and hence preserves algebra objects. This means we get a functor

$$\mathcal{CM}\mathrm{on}(\mathcal{C}\mathrm{h}(\mathbb{R})) \to \mathcal{CM}\mathrm{on}(\mathcal{M}\mathrm{od}_{H\mathbb{R}})$$

Here the left hand side is the category of differential graded commutative algebras (CDGAs) over $\mathbb{R}$, while the right hand side is the category of commutative $H\mathbb{R}$-algebras.

Definition 3.3. Let R be a commutative ring spectrum, A *differential (ring) refinement* of R is a triple (A, c, R) where

- A is a CDGA over $\mathbb{R}$,
- $c\colon R \wedge H\mathbb{R} \to HA$ is an equivalence of commutative $H\mathbb{R}$-algebras.

We can in fact make this definition more categorical.

Definition 3.4. We define the ∞-category $\widehat{\mathrm{Ring}}$ of differential ring refinements as the pullback

$$\begin{tikzcd} \widehat{\mathrm{Ring}} \arrow[r] \arrow[d] & \mathrm{DGCAlg}arrow[d, "H"] \\ \mathcal{CM}\mathrm{on}(\mathcal{S}\mathrm{p}) \arrow[r, "-\wedge H\mathbb{R}"] & \mathcal{CM}\mathrm{on}(\mathcal{M}\mathrm{od}_{H\mathbb{R}}) \end{tikzcd}$$

Notice we can use the above to definition to again define a new sheaf of ring spectra via pullback.

Definition 3.5. Let (R, A, c) be a differential refinement of R. Then the *differential function ring spectrum* $\hat{R}$ is defined as the pullback

$$\begin{tikzcd}
\hat{R} \arrow[rrr] \arrow[d] & & & H(\Omega^\bullet A)^{\geq n} \arrow[d] \\
\underline{R} \arrow[r] & \underline{R \wedge H\mathbb{R}} \arrow[r, "c"] & \underline{HA} \arrow[r] & H(\Omega^\bullet A)
\end{tikzcd}$$

We now want to define a notion of "differential twists" and "differential module refinements" of topological twists. For this we need to introduce some more notation and elaborate properties.

Definition 3.6. Let A be a CDGA over $\mathbb{R}$. Let $\mathcal{P}\mathrm{ic}_A^{wloc,fl}(M)$ be the full sub-∞-groupoid of $\mathcal{M}\mathrm{od}_{\Omega^\bullet A}(M)$ spanned by the sheaves $\mathcal{M}$ of $\Omega^\bullet A$-modules which are

- *weakly locally constant*, i.e. locally equivalent to a constant sheaf of $\Omega^\bullet A$-modules, as a sheaf valued in $\mathcal{D}(\mathbb{R})$,
- *K-flat*, i.e. such that the functor $\mathcal{M} \otimes_{\Omega^\bullet A} -$ preserves quasi-isomorphisms,
- *invertible*, i.e. there exists $\mathcal{N}$ such that $\mathcal{M} \otimes_{\Omega^\bullet A} \mathcal{N} \cong \Omega^\bullet A$ as sheaves valued in $\mathcal{C}\mathrm{h}(\mathbb{R})$.

We are now ready to give the main definition of twisted differential cohomology.

Definition 3.7. Let E be a topological twist in $\mathcal{P}\mathrm{ic}_{\underline{R}}^{loc}(M)$, a *differential module refinement of E* is a triple $(E, \mathcal{M}, c)$ where

- $\mathcal{M} \in \mathcal{P}\mathrm{ic}_A^{wloc,fl}(M)$,
- $c\colon E \wedge H\mathbb{R} \to H\mathcal{M}$ is an equivalence in $\mathcal{P}\mathrm{ic}_{\underline{HA}}^{loc}(M)$.

Definition 3.8. Let $(E, \mathcal{M}, c)$ be a differential module refinement of a topological twist $E \in \mathcal{P}\mathrm{ic}_{\underline{R}}^{loc}(M)$. Let $F(E, \mathcal{M}, c)$ be the pullback

$$\begin{tikzcd}
F(E,\mathcal{M},c) \arrow[rr] \arrow[d] & & H(\mathcal{M}^{\geq 0}) \arrow[d] \\
E \arrow[r] & \underline{H\mathbb{R}} \wedge E \arrow[r, "c"] & \underline{H\mathcal{M}}
\end{tikzcd}$$

Denote by $\hat{R}^{(E,\mathcal{M},c)}(M) := \pi_0(F(E,\mathcal{M},c)(M))$ the *twisted differential R-cohomology group* of M associated to the differential refinement $(E, \mathcal{M}, c)$ of the topological twist E.

Definition 3.9. Let $\mathrm{Tw}_{\hat{R}}(M)$, the *∞-category of differential twists*, be the ∞-category defined as the pullback

$$\begin{tikzcd}
\mathrm{Tw}_{\hat{R}}(M) \arrow[r] \arrow[d] & \mathcal{P}\mathrm{ic}_{\Omega^\bullet A}^{wloc,fl}(M) \arrow[d, "H"] \\
\mathcal{P}\mathrm{ic}_{\underline{R}}^{loc}(M) \arrow[r, "-\wedge H\mathbb{R}"] & \mathcal{P}\mathrm{ic}_{\underline{HA}}^{loc}(M)
\end{tikzcd}$$

As a next step we can explore applications and examples of this abstract setup, in particular in the context of differential twisted K-theory.

References

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