

DIFFERENTIAL COHOMOLOGY SEMINAR 8

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The aim of this talk is to review twisted cohomology theory with the aim of later discussing twisted differential cohomology theories [BG16]. For these talks the main source is [ABG⁺14, ABG18].

1. TWISTED COHOMOLOGY

Let R be a ring spectrum, meaning a monoid object in the ∞ -category of spectra $\mathcal{S}p$. From this we get a presentable stable ∞ -category $\mathcal{M}od_R$ of left R -module spectra. Objects therein are morphisms of the form $R \wedge M \rightarrow M$ satisfying the usual associativity and unit conditions up to coherent homotopies.

Note we have an adjunction diagram

$$\mathcal{S}p \begin{array}{c} \xrightarrow{R \wedge -} \\ \perp \\ \xleftarrow{\text{Hom}_R(R, -)} \end{array} \mathcal{M}od_R ,$$

where the right adjoint is in fact the forgetful functor. This in particular means $\mathcal{M}od_R$ has a distinguished object R i.e. the free R -module of rank 1. We now refine these constructions.

Definition 1.1. Let R be a ring spectrum. An R -line is an R -module L such that $L \simeq R$.

Definition 1.2. Let $\mathcal{L}ine_R$ be the full sub- ∞ -groupoid of $\mathcal{M}od_R$ spanned by the R -lines.

By construction, $\mathcal{L}ine_R$ is equivalent to the category with a single object R and hom-space $GL_1 R$, the ∞ -group of R -linear automorphisms of R . Notice that $GL_1(R) \subseteq \text{Hom}_{\mathcal{M}od_R}(R, R) \simeq \text{Hom}(\mathbb{S}, R) = \Omega^\infty R$.

Lemma 1.3. $GL_1(R)$ fits into the pullback square

$$\begin{array}{ccc} GL_1(R) & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ \pi_0(R)^\times & \longrightarrow & \pi_0(R) \end{array}$$

Proof. $\pi_0(R) \simeq \text{Hom}_{\text{ho}\mathcal{M}od_R}(R, R)$, where $\text{ho}\mathcal{M}od_R$ is the homotopy category of R -modules, the right-vertical arrow corresponds to $\text{Hom}_{\mathcal{M}od_R}(R, R) \rightarrow \text{Hom}_{\text{ho}\mathcal{M}od_R}(R, R)$ mapping a morphism to its homotopy class, and $\pi_0(R)^\times \subseteq \text{Hom}_{\text{ho}\mathcal{M}od_R}(R, R)$ is the set of isomorphisms. Finally, a morphism in a ∞ -category \mathcal{C} is an equivalence if and only if its homotopy class is an isomorphism in $\text{ho}\mathcal{C}$. \square

In particular, $GL_1(R)$ consists of the connected components of equivalences in Ω^∞ .

Definition 1.4. Let X be a space. Denote by $\mathcal{M}od_R(X)$ the ∞ -category of R -module spectra parametrized over X , i.e. the functor category $\text{Fun}(X^{op}, \mathcal{M}od_R)$.

Definition 1.5. Let X be a space. Denote by $\mathcal{L}ine_R(X)$ the ∞ -groupoid of R -line spectra parametrized over X , i.e. the $\text{Fun}(X^{op}, \mathcal{L}ine_R)$.

Since $\mathcal{L}ine_R \simeq BGL_1(R)$, functors $X^{op} \rightarrow \mathcal{L}ine_R$ are generalization of local systems.

Example 1.6. Let $R_X: X^{op} \rightarrow * \rightarrow \mathcal{L}ine_R$ be the constant functor with value R .

We now proceed to the Thom construction.

Definition 1.7 (Thom spectrum). The *Thom R-module spectrum* is the functor

$$M: \mathcal{S}rp\mathcal{d}_\infty^{op}/\mathcal{L}ine_R \rightarrow \mathcal{M}od_R,$$

which sends $f: X^{op} \rightarrow \mathcal{L}ine_R$ to the R -module spectrum $\text{colim}(X^{op} \xrightarrow{f} \mathcal{L}ine_R \xrightarrow{i} \mathcal{M}od_R)$.

Remark 1.8. Notice that the definition of Thom R -module spectrum make sense for any functor $X^{op} \rightarrow R\text{Mod}$.

Let us note an alternative characterization that will be important later.

Remark 1.9. For a given map $f: X \rightarrow Y$, we get a map $f^*: \text{Mod}_R(Y) \rightarrow \text{Mod}_R(X)$ by precomposition with f^{op} . Since f^* preserves both limits and colimits, we construct a left adjoint $f_!: \text{Mod}_R(X) \rightarrow \text{Mod}_R(Y)$ and a right adjoint $f_*: \text{Mod}_R(X) \rightarrow \text{Mod}_R(Y)$, using left and right Kan extension along f^{op} . Let $f = p: X \rightarrow *$ be the terminal functor, then left Kan extension along p^{op} is exactly taking colimit, therefore

$$Mf \simeq p_!(i \circ f).$$

Remark 1.10 ([ABG10, 3.6]). Let $\mathcal{T}\text{riv}_R$ be the slice groupoid $\mathcal{L}\text{ine}_R/R$, together with the canonical projection $\pi: \mathcal{T}\text{riv}_R \rightarrow \mathcal{L}\text{ine}_R$. An object of $\mathcal{T}\text{riv}_R$ is a pair (L, ϕ) of a R -line L and an equivalence $\phi: L \rightarrow R$. Consider then the following:

- (1) $\mathcal{T}\text{riv}_R$ is a slice ∞ -groupoid, hence contractible, and π is a Kan fibration.
- (2) $GL_1(R)$ is equivalent to the fiber of π over $R \in \mathcal{L}\text{ine}_R$ and acts freely on the fibers of π .

These observations imply $\mathcal{L}\text{ine}_R$ is the classifying space for $GL_1(R)$ -bundles. Here we use the term $GL_1(R)$ -bundle to mean a parametrized family of $GL_1(R)$ -spaces with a free and transitive action.

We now proceed to define twisted cohomology theories.

Definition 1.11 (Twisted cohomology). Let R be a ring spectrum, X be a space, $p: X \rightarrow *$ the terminal functor, and $f: X^{op} \rightarrow \mathcal{L}\text{ine}_R$ a R -line bundle over X . The f -twisted R -cohomology of X is defined as the mapping spectrum

$$R_f(X) := \text{Map}_{\text{Mod}_R}(Mf, R) \simeq \text{Map}_{\text{Mod}_R(X)}(f, p^*R) \simeq \text{Map}_{\text{Mod}_R(X)}(f, R_X).$$

Similarly, the f -twisted R -homology of X is defined as

$$R^f(X) := \text{Map}_{\text{Mod}_R}(R, Mf) \simeq Mf.$$

The f -twisted R -cohomology groups of X are defined as the homotopy groups of $R_f(X)$, i.e.

$$R_f^n(X) := \pi_0(\text{Map}_{\text{Mod}_R}(Mf, \Sigma^n R)) \cong \pi_{-n}(\text{Map}_{\text{Mod}_R(X)}(f, R_X)).$$

Similarly, the f -twisted R -homology groups of X are defined as the homotopy groups of $R^f(X)$, i.e.

$$R_n^f(X) := \pi_0(\text{Map}_{\text{Mod}_R}(\Sigma^n R, Mf)) \cong \pi_n(Mf).$$

Example 1.12 (Trivial twist). If $f: X \rightarrow \mathcal{L}\text{ine}_R$ factors through $*$, then f factors as the $X^{op} \rightarrow * \rightarrow S$, the constant factor with value $*$, and $R \wedge \Sigma_+^\infty(-): S \rightarrow \text{Mod}_R$. The latter functor commutes with colimits, being a left adjoint, while the colimit of the latter is X itself, then $Mf \simeq R \wedge \Sigma_+^\infty X$. In particular, f -twisted R -cohomology and R -homology of X reduce to ordinary (untwisted) R -cohomology and R -homology of X .

Definition 1.13. Given a vector bundle $\pi: E \rightarrow B$, define the *Thom space* of π , denoted $\text{Th}(E)$, to be the homotopy cofiber of $E_0 \subseteq E$, where E_0 is the complement of the zero section.

Lemma 1.14. Consider a space X and the ∞ -categorical Yoneda's embedding $y: X \rightarrow \text{Fun}(X^{op}, S)$. The colimit of y is the terminal pre-sheaf on X , i.e. the pre-sheaf with constant value the one-point space.

Proof. Let S be a pre-sheaf on X , consider then the slice category X/S of pairs (x, ϕ) , where x is an object of X and $\phi: y(x) \rightarrow S$. The density theorem for ∞ -categories states that S is equivalent to the colimit of $X/S \rightarrow X \xrightarrow{y} \text{Fun}(X^{op}, S)$, the first map being the canonical projection. Take $S = *$, then $X/* \rightarrow X$ is an equivalence, hence the claim. \square

Let G be a topological group and BG the ∞ -groupoid with a single object 1 and hom-space G . The category $\mathcal{S}_G := \text{Fun}(BG, S)$ is equivalent to the category of G -spaces.

Lemma 1.15. Consider $X = BG$, a G -space $f: X \rightarrow S$ and its left Kan extension $f_!: \mathcal{S}_G \rightarrow S$, then $f_! \simeq (- \times E)/G$, where $E = f(1)$.

Proof. Evaluate at 1 , then $f_!(y(1)) = E$, by definition, and $y(1) \simeq G$, as G -spaces, hence $(y(1) \times E)/G \simeq (G \times E)/G \simeq E$. Since $f_!$ and $(- \times E)/G$ agree on representables and are colimit-preserving, they are equivalent. \square

Example 1.16. Take the space $BO(n)$ and $f_n : BO(n) \rightarrow \mathbb{S}_*$ the n -sphere S^n with $O(n)$ -action coming from the one-point compactification of the regular action on \mathbb{R}^n . Let $\alpha_n = \Sigma^{\infty-n} f_n : BO(n) \rightarrow \mathcal{S}\mathrm{p}$, then $\alpha_n(1) = \Sigma^{\infty-n} f_n(1) = \Sigma^{\infty-n} S^n \simeq \mathbb{S}$, so α_n factors through $\mathcal{L}\mathrm{ine}_{\mathbb{S}}$. Let $X = BO(n)^{op}$ and $p : X \rightarrow *$ the terminal functor, then

$$M\alpha_n = p_! \Sigma^{\infty-n} f_n \simeq \Sigma^{\infty-n} p_!(f_n)_! y \simeq \Sigma^{\infty-n} (f_n)_! \underbrace{p_!(y)}_{\simeq *} \simeq \Sigma^{\infty-n} (* \times S^n) / O(n)$$

Let $P = EO(n)$ be the universal $O(n)$ -bundle and M a $O(n)$ -space, then $* \times_{O(n)} M$ is modelled by the *strict* quotient $(P \times M) / O(n)$, then

$$S^n / O(n) = \mathrm{cofib}(\mathbb{R}_0^n \subseteq \mathbb{R}^n) / O(n) \simeq \mathrm{cofib}(\underbrace{* \times_{O(n)} \mathbb{R}_0^n}_{\simeq E_0^n} \subseteq \underbrace{* \times_{O(n)} \mathbb{R}^n}_{\simeq E^n}) = \mathrm{Th}(E^n)$$

where $E^n = P \times_{O(n)} \mathbb{R}^n \rightarrow BO(n)$ is the universal n -dimensional vector bundle, hence $M\alpha_n \simeq \Sigma^{\infty-n} \mathrm{Th}(E^n)$.

The functor $BO(n) \rightarrow \mathcal{L}\mathrm{ine}_{\mathbb{S}}$ induces a ∞ -group homomorphism $j_n : O(n) \rightarrow GL_1(\mathbb{S})$, mapping ϕ to $\Sigma^{\infty-n} \mathrm{Th}(\phi)$. Consider the suspension morphism $s_n = \mathbb{R} \oplus - : O(n) \rightarrow O(1+n)$, then

$$j_n(\mathbb{R} \oplus \phi) = \Sigma^{\infty-n-1} \mathrm{Th}(\mathbb{R} \oplus \phi) \simeq \Sigma^{\infty-n-1} \underbrace{\mathrm{Th}(\mathbb{R}) \wedge \mathrm{Th}(\phi)}_{\simeq S^1} \simeq \Sigma^{\infty-n} \mathrm{Th}(\phi) = j_n$$

Recall that the colimit over the suspension morphisms s_n is the stable orthogonal group O .

Definition 1.17. Denote by j the induced ∞ -group homomorphism $O \rightarrow GL_1(\mathbb{S})$ induced by the sequence j_n , called the *J-homomorphism*.

Example 1.18. Let $X = O^{op}$ and take $Bj : BO \rightarrow \mathcal{L}\mathrm{ine}_{\mathbb{S}}$, then Mj is denoted MO and called the *real bordism spectrum*.

Denote by M the extended Thom spectrum functor $\mathrm{Grp}_{\infty}^{op}/\mathrm{Mod}_R \rightarrow \mathrm{Mod}_R$, this is a left adjoint to the functor \mathcal{O} sending a R -module to the functor $* \rightarrow \mathrm{Mod}_R$ picking out M . In particular, M preserves colimits and $Bj \simeq \mathrm{colim}_n Bj_n$, therefore we have the following:

Theorem 1.19. $MO \simeq \mathrm{colim}_n MO(n) = \mathrm{colim}_n \Sigma^{\infty-n} \mathrm{Th}(E^n)$.

Example 1.20. A group homomorphism $\xi : G \rightarrow O$ induces a functor $f : BG \rightarrow \mathcal{L}\mathrm{ine}_{\mathbb{S}}$. The Thom spectrum Mf is denoted MG or $M\xi$, and called *G-bordism spectrum*. For $G = U, SO, Spin$, and *String*, we obtain the *complex, oriented, spin, and string bordism spectra*.

Remark 1.21. In [Equation \(1.20\)](#) we might take $G = \{*\}$, the one-point group, then $MG \simeq \mathbb{S}$, which, if it didn't have a name, might be called *framed bordism spectrum*, following the naming convention in [Equation \(1.20\)](#) and in line with the theorem that $\pi_*(\mathbb{S}) \simeq \Omega_*^{\mathrm{fr}}$, the bordism ring of framed (trivialized tangent bundle) smooth manifolds.

Let R be a ring in sets, then R is a A_{∞} -ring spectrum (actually, E_{∞}), $\Omega^{\infty} R$ is equivalent to R with discrete topology ($\pi_0(\Omega^{\infty} R) \simeq R$, as sets, and every other homotopy group vanish). In particular, $GL_1(R)$ is simply R^{\times} with discrete topology. Consider then the fiber sequence $SO \rightarrow O \rightarrow \mathbb{Z}^{\times} \simeq GL_1(\mathbb{Z})$.

Example 1.22. $X = BO^{op}$ and $\alpha = w_1 : BO \rightarrow \mathcal{L}\mathrm{ine}_{\mathbb{Z}}$, the 1st Stiefel-Whitney class (delooping of the determinant $O \rightarrow GL_1(\mathbb{Z})$), then Mw_1 is a \mathbb{Z} -module spectra. Let $i : SO \subseteq O$, then $w_1 i$ factors through the point, so $M(w_1 i) \simeq \mathbb{Z} \wedge \Sigma_+^{\infty} SO$.

Given $f : X^{op} \rightarrow \mathrm{Mod}_R$ and a sequence $F \xrightarrow{i} X \xrightarrow{\pi} Y$, there is an induced sequence of Thom R -module spectra $MF \rightarrow MX \rightarrow MY$. If πi factors through the point, $MF \simeq R \wedge \Sigma_+^{\infty} F$.

Lemma 1.23. Let R be a ring spectrum and X a connected monoidal ∞ -groupoid, then

$$\mathrm{Hom}_{\mathrm{Mon}(\mathcal{S}\mathrm{p})}(\Sigma_+^{\infty} X, R) \simeq \mathrm{Hom}_{\mathrm{Mon}(\mathbb{S})}(X, GL_1(R))$$

Proof. Since X is connected, the space of homomorphisms $X \rightarrow GL_1(R)$ is equivalent to the space of homomorphisms $X \rightarrow \Omega^{\infty} R$, then use that $(\Sigma^{\infty}, \Omega^{\infty})$ is a monoidal adjunction (The monoidal structure on spectra is such that Σ^{∞} is strong monoidal). \square

Example 1.24. Let tmf be the E_∞ -ring spectrum of topological modular forms and $\sigma : MString \rightarrow \mathrm{tmf}$ the *String*-orientation of tmf . The group *String* is E_∞ -monoidal, so $MString$ is a E_∞ -ring spectrum and σ is a ring homomorphism. In the fiber sequence $K(\mathbb{Z}, 3) \rightarrow BString \rightarrow BSO$, the fiber map $i : K(\mathbb{Z}, 3) \rightarrow BString$ is also a monoid homomorphism, so the composition

$$\Sigma_+^\infty K(\mathbb{Z}, 3) \xrightarrow{Mi} MString \xrightarrow{\sigma} \mathrm{tmf}$$

is a homomorphism. Using [Equation \(1.23\)](#), we conclude that the induced homomorphism $K(\mathbb{Z}, 3) \rightarrow \Omega^\infty \mathrm{tmf}$ factors through $GL_1(\mathrm{tmf})$, or equivalently, $K(\mathbb{Z}, 4) \rightarrow \mathcal{L}\mathrm{ine}_{\mathrm{tmf}}$, i.e. *2-bundle gerbes twist tmf*.

Let us go down one step in the chromatic ladder.

Example 1.25. Recall the fiber sequences for *Spin* and *Spin*^c:

$$\mathbb{Z}_2 \rightarrow \mathrm{Spin} \rightarrow SO, \quad S^1 \rightarrow \mathrm{Spin}^c \rightarrow SO$$

All the spaces involved are E_∞ -groups. Applying the Thom spectrum functor to the delooped sequences, we get

$$\Sigma_+^\infty K(\mathbb{Z}_2, 1) \rightarrow M\mathrm{Spin} \rightarrow MSO, \quad \Sigma_+^\infty K(\mathbb{Z}, 2) \rightarrow M\mathrm{Spin}^c \rightarrow MSO$$

Let $\sigma : M\mathrm{Spin} \rightarrow KO$ and $\sigma^c : M\mathrm{Spin}^c \rightarrow KU$ be the Atiyah-Bott-Shapiro orientation of real and complex K -theory (see, [\[ABS64\]](#)). Similar to [Equation \(1.24\)](#), we get homomorphisms

$$\Sigma_+^\infty K(\mathbb{Z}_2, 1) \longrightarrow M\mathrm{Spin} \xrightarrow{\sigma} KO, \quad \Sigma_+^\infty K(\mathbb{Z}, 2) \longrightarrow M\mathrm{Spin}^c \xrightarrow{\sigma^c} KU$$

Using [Equation \(1.23\)](#) again and delooping, we obtain functors $K(\mathbb{Z}_2, 2) \rightarrow \mathcal{L}\mathrm{ine}_{KO}$ and $K(\mathbb{Z}, 3) \rightarrow \mathcal{L}\mathrm{ine}_{KU}$, i.e. *real, resp. complex, bundle gerbes twist real, resp. complex, K-theory*.

2. TWISTS VIA PICARD GROUPOIDS AND GRADING

This section requires some further details. Up until now we defined everything via $\mathcal{L}\mathrm{ine}_R$, however for many applications we need to work with $\mathcal{P}\mathrm{ic}_R$ instead.

Definition 2.1. Given a monoidal ∞ -category $(\mathcal{C}, \otimes, 1)$, an object M is *invertible* if there is an object D such that $D \otimes M \simeq M \otimes D \simeq 1$. The *Picard ∞ -groupoid* of \mathcal{C} is the sub- ∞ -groupoid generated by invertible modules.

Definition 2.2. If R is a ring spectrum, Mod_R is monoidal. Denote by $\mathcal{P}\mathrm{ic}_R$ the Picard groupoid of Mod_R .

Remark 2.3. It is a common fact that the ∞ -groupoid $\mathcal{P}\mathrm{ic}_R$ is equivalent to $\pi_0(R) \times \mathcal{L}\mathrm{ine}_R$, meaning that the invertible R -modules are given by the R -lines together with a shift by an element in $\pi_0(R)$. Here it is important to note that this equivalence does not respect the monoidal structure is only an equivalence of underlying ∞ -groupoids.

As mentioned in [Equation \(1.8\)](#), the Thom spectrum functor makes sense for every functor $f : X^{op} \rightarrow \mathrm{Mod}_R$. However, all examples of twists encountered so far came from functors into $\mathcal{L}\mathrm{ine}_R$. An example of twist that is not the result of a R -line bundle is the *degree shift*.

Definition 2.4. Denote by M the *Thom R-module spectrum* functor

$$\mathrm{Srp}_\infty^{op}/\mathrm{Mod}_R \rightarrow \mathrm{Mod}_R$$

sending a functor $f : X^{op} \rightarrow \mathrm{Mod}_R$ to its colimit.

If L is a R -line, then $\Sigma^n L$ is invertible, with inverse $\Sigma^{-n} L$. Moreover, $\Sigma^n L$ is a line if and only if R is n -periodic.

Example 2.5. Let $f : X^{op} \rightarrow \mathcal{L}\mathrm{ine}_R$ be a twist. Denote by $\Sigma^n f$ the composition of f with the shift functor $\Sigma^n : \mathcal{L}\mathrm{ine}_R \rightarrow \mathcal{P}\mathrm{ic}_R$. Since Σ^n is an equivalence, it commutes with colimits, so

$$M\Sigma^n f \simeq \Sigma^n Mf$$

If $f = R_X$, then $M\Sigma^n f \simeq \Sigma^n R \wedge \Sigma_+^\infty X$, so $\Sigma^n f$ -twisted R -cohomology and R -homology correspond to normal R -cohomology and R -homology with a degree shift by n .

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