DIFFERENTIAL COHOMOLOGY SEMINAR 10

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Today we look at the construction of pushforward maps in twisted K-theory due to Carey and Wang [CW08].

1. The Classical Case

Let us do a quick review of the classical case. Given a map $f\colon X \to Y$ and induced map on K-theory $f^*\colon K(Y) \to K(X)$, we would like to construct a pushforward map $f_!\colon K(X) \to K(Y)$. In the classical case, this requires $w_2(X) = f^* w_2(Y)$. This precisely corresponds to choosing a spin^c-structure on the virtual bundle $TX - f^*TY$. This is the necessary condition to define this map.

2. Reviewing twisted K-Theory

Before we proceed to generalize the pushforward map to twisted K-theory, we first review the definition of twisted K-theory. Recall that

$$H^3(M, \mathbb{Z}) \cong \mathcal{PU}(\mathcal{H}) - \mathtt{Bun}(M)/\sim$$

where $\mathcal{H}$ is a separable Hilbert space and $\mathcal{PU}(\mathcal{H})$ is the projective unitary group. Choosing P a principal $\mathcal{PU}(\mathcal{H})$-bundle over M representing a class in $H^3(M,\mathbb{Z})$, we can form the associated bundle of Fredholm operators $\mathcal{F}\mathrm{red}(P) := P \times_{\mathcal{PU}(\mathcal{H})} \mathcal{F}\mathrm{red}(\mathcal{H})$. Then the 0-th twisted K-theory is defined as

$$K^0(M, P) := \pi_0(\mathrm{Map}(M, P \times_{\mathcal{PU}(\mathcal{H})} \mathcal{F}\mathrm{red}(\mathcal{H}))) \cong \pi_0(\mathrm{Map}(P, \mathcal{F}\mathrm{red}(\mathcal{H})))_{\mathcal{PU}(\mathcal{H})}.$$

More generally

$$K^n(M, P) := \pi_0(\mathrm{Map}(P, \Omega^n \mathcal{F}\mathrm{red}(\mathcal{H})))_{\mathcal{PU}(\mathcal{H})}.$$

In the coming sections we consider the pushforward map in two cases: the torsion case and the non-torsion case.

3. Understanding the Torsion Case

Let us consider the case when the bundle is torsion. Then P is a principal $\mathcal{PU}(n)$-bundle, using $U(n) \hookrightarrow U(\mathcal{H})$. In this case we can obtain a bundle gerbe $\mathcal{L}_P$, which has the following form

$$\begin{tikzcd}
U(n) \arrow[d] & \Gamma_P \arrow[l] \arrow[d] & E \arrow[d] \\
\mathcal{PU}(n) & P \times_M P \arrow[l] \arrow[r, shift left] \arrow[r, shift right] & P \arrow[d] \\
 & & M
\end{tikzcd}$$

This structure gives us a map

$$\phi\colon (\Gamma_P|_{p_1,p_2} \times_{U(1)} \mathbb{C}) \otimes E_{p_2} \to E_{p_1}$$

meaning (E, ϕ) is a $\mathcal{L}_P$-twisted vector bundle. We now can prove the following theorem.

Theorem 3.1. *Let P be a principal $\mathcal{PU}(n)$-bundle. Then, we have an isomorphism $K(\mathcal{L} - \mathcal{M}\mathrm{od}) \cong K_{U(n),\mathrm{scal}}(P)$.*

Here $K_{U(n),\mathrm{scal}}(P)$ is the $U(n)$-equivariant K-theory of P with scalar action of the center $U(1) \subset U(n)$. The proof proceeds by mapping a pair (E, ϕ) to the data $(U(n) \times E \to E)$ that maps $(A, v) \mapsto \phi(A, v)$.

We now have the following second theorem, relating twisted K-theory to twisted vector bundles.

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Theorem 3.2. *Let P be a principal $\mathcal{PU}(n)$-bundle. Then we have an isomorphism*

$$K^0(M,P) \cong K(\mathcal{L}_P - \mathcal{M}\mathrm{od}).$$

Here we are implicitly using the $\mathcal{PU}(\mathcal{H})$-bundle $\tilde{P} = P \times_{\mathcal{PU}(n)} \mathcal{PU}(\mathcal{H})$, meaning

$$K^0(M,P) = K^0(M,\tilde{P}) = \pi_0(\mathrm{Map}(\tilde{P}, \mathcal{F}\mathrm{red}(\mathcal{H})))_{\mathcal{PU}(\mathcal{H})}.$$

Now we also have $\mathcal{F}\mathrm{red}(\mathcal{H}) \cong \mathcal{F}\mathrm{red}_{U(n)-cts}(\mathcal{H})$, meaning it is the Fredholm operators with continuous $U(n)$-action.

Now before we proceed, recall that we have a an equivariant version of Atiyah-Jänich theorem (which holds if certain conditions are satisfied), meaning

$$[M, \mathcal{F}\mathrm{red}_{G-cts}(\mathcal{H})]_G \cong K^0(\mathcal{V}\mathrm{ect}_G(M)).$$

We can now restrict this bijection to a bijection of subsets

$$[M, \mathcal{F}\mathrm{red}_{U(n)-cts}(\mathcal{H}^{\mathrm{scal}})]_G \cong K^0(\mathcal{V}\mathrm{ect}_{U(n),\mathrm{scal}}(M)),$$

where $\mathcal{H}^{\mathrm{scal}}$ is the Hilbert space with scalar action of the center $U(1) \subset U(n)$.

The construction proceeds now by replacing $\mathcal{H}$ with $\mathcal{H}^{\mathrm{scal}}$ in the definition of twisted K-theory, meaning we go back to the first step and do

$$K^0(M,P) = K^0(M,\tilde{P}) = \pi_0(\mathrm{Map}(\tilde{P}, \mathcal{F}\mathrm{red}(\mathcal{H}^{\mathrm{scal}})))_{\mathcal{PU}(\mathcal{H})}.$$

which via the bijection above is isomorphic to $K_{U(n),\mathrm{scal}}(P)$, which by the previous theorem is isomorphic to $K(\mathcal{L}_P - \mathcal{M}\mathrm{od})$.

Remark 3.3. Due to recent work, we know that if P is torsion, then $\mathcal{L}_P \cong \mathcal{A}_P = P \times_{\mathcal{PU}(n)} \mathbb{C}^{n\times n}$, which is an isomorphism of 2-vector bundles, and induces an isomorphism $\mathcal{L}_P - \mathcal{M}\mathrm{od} \cong \mathcal{A}_P - \mathcal{M}\mathrm{od}$.

4. Thom Isomorphism in the Torsion Case

Let $V \to M$ be a $\mathbb{R}$-vector bundle. Then we get $Tr(V) \to M$, the associated $SO(n)$-bundle, which comes with a sub-bundle with a $\mathrm{spin}^c(n)$-structure. This gives us a bijection $\mathcal{L}_{Tr(V)} \cong CL(V)$, where $CL(V)$ is the complex Clifford bundle of V.

Here we can use the result by Karoubi.

Theorem 4.1. *We have an isomorphism $K(CL(V) - \mathcal{M}\mathrm{od}) \cong K(\mathrm{Th}(V))$.*

We now want to generalize this to more general twists, which should be the following result.

Theorem 4.2. *There is an isomorphism*

$$K^0(M, P + W_3(V)) \cong K^0(\mathrm{Th}(V), \pi^*P).$$

5. Non-Torsion Case

We now aim to generalize these results to the non-torsion case. Concretely, we want prove an analogue of the following theorem:

$$K(\mathcal{L}_P - \mathcal{M}\mathrm{od}) \cong K_{U(n),\mathrm{scal}}(P)$$

We can try to reproduce the same diagram

$$\begin{tikzcd} \Gamma_P \arrow[d] & E \arrow[d] \\ P \times_M P \arrow[r, shift left] \arrow[r, shift right] & P \arrow[d] \\ & M \end{tikzcd},$$

but we cannot compare to $U(n)$ anymore, since P is not a $\mathcal{PU}(n)$-bundle. So, we do not get the original result, but rather a restricted version.

Theorem 5.1. $K^0(M,P) \cong K(\mathcal{L}_P - \mathcal{M}\mathrm{od}^{U_2})$.

Here $U_2 \subseteq U(\mathcal{H})$ is the group of unitary objects that differ from the identity by a Hilbert-Schmidt operator, and $\mathcal{L}_P - \mathcal{M}\mathrm{od}^{U_2}$ is the category of $\mathcal{L}_P$-twisted vector bundles whose transition functions take values in U_2.

Here U^2 can also be described via colimits, as $U^2 \cong \mathrm{colim}_{n\to\infty} U(n)$. So this result can be interpreted as the fact that filtered colimits preserve the original theorem.

Now we analogously have an isomorphism

$$\mathcal{L}_P - \mathcal{M}\mathrm{od}^{U_2} \cong \mathcal{A}_p - \mathcal{M}\mathrm{od}^{U_2}.$$

Remark 5.2. Similar to above, we anticipate an isomorphism of 2-vector bundles $\mathcal{L}_P \cong P \times_{\mathcal{PU}(\mathcal{H})} \mathcal{HS}(\mathcal{H})$, that induces the isomorphism above as a special case. This would require a suitable Morita category for these Hilbert-Schmidt operators, which is currently unknown.

6. Twisted Pushforward

As a last step we can deduce the Thom isomorphism in this non-torsion case. Let $f\colon X \to Y$. Then we have $W_3(f) := W_3(TX \oplus f^*TY)$. This gives us a pushforward map

$$f_!\colon K^*(X, W_3(f) + f^*P) \to K^{(*+(\dim X - \dim Y) \mod 2)}(Y, P),$$

for $P \to Y$ some $\mathcal{PU}(\mathcal{H})$-bundle.

The construction of this map proceeds as follows: We can factor $X \to Y$ into $X \hookrightarrow Y \times S^n \to Y$. The first step uses the Thom isomorphism, and the second step uses C^*-algebra techniques.

Remark 6.1. If Y is trivial, then this construction gives us a version of an index.

7. D-Branes

There is an application of these methods to physics. Given a $U(n)$-bundle $\mathcal{G}$ over X with sub objects Q, such that the restriction along Q has a $\mathcal{G}|_Q$-module $\mathcal{E}$.

It is claimed by Witten that these d-branes are classified by some version of K-theory. This can be made precise as follows (with some adjustments). Given $\mathcal{E}$ and $\mathcal{G}|_Q \otimes \mathcal{L}_f$-module we get an element in $K^0(Q, \mathcal{G}|_Q \otimes \mathcal{L}_f)$. Using the pushforward map, we have a map

$$K^0(Q, \mathcal{G}|_Q \otimes \mathcal{L}_f) \to K^*(X, \mathcal{G})$$

giving us an element in $K^*(X, \mathcal{G})$, as desired.

We can in particular consider the case when $X = \mathcal{G}$, which is known as the WZW-model in physics. In this case, Q are conjugacy classes C_g of some element g in the group and for $f\colon C_g \to \mathcal{G}$, we have

$$W_3(f) = (f^*\mathcal{G}_{bas})^{\check{c}}$$

This is a non-trivial observation. Indeed comparing with the Freed-Hopkins-Teleman approach [FHT11], we see that

$$K_{\mathcal{G}}(\mathcal{G}, \mathcal{G}^k_{bas}) \cong \mathrm{Rep}^{k+\check{c}}(L\mathcal{G}),$$

meaning this $\check{c}$ Coxeter shift appears naturally in the representation theory of the loop group.

Hence this approach suggests a possible geometric interpretation of these constructions.

References

[CW08] Alan L. Carey and Bai-Ling Wang. Thom isomorphism and push-forward map in twisted K-theory. *J. K-Theory*, 1(2):357–393, 2008.

[FHT11] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman. Loop groups and twisted K-theory I. *J. Topol.*, 4(4):737–798, 2011.