# DIFFERENTIAL COHOMOLOGY SEMINAR 9

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The goal of this talk is to compare different models of twisted $K$-theory. Concretely, we learn about the comparison between the original approach to twisted $K$-theory by Atiyah–Segal [AS04] and the modern approach in [ABG$^+$14] that builds on the abstract approach to twisted cohomology via $\infty$-categorical methods. There is a third approach by Freed–Hopkins–Teleman [FHT11], that we will not consider here.

*Remark* 0.1. Throughout this talk we adopt the following notational conventions:

- For an $\infty$-category $\mathcal{C}$ and objects $X, Y \in \mathcal{C}$, we denote by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ the mapping *space* from $X$ to $Y$ in $\mathcal{C}$.
- If an $\infty$-category $\mathcal{C}$ is enriched over spectra (e.g. if $\mathcal{C}$ is stable), denote the *mapping spectrum* from $X$ to $Y$ by $\mathcal{H}\mathrm{om}_{\mathcal{C}}(X, Y)$.
- Note, in this case we have the relationship $\Omega^\infty \mathcal{H}\mathrm{om}_{\mathcal{C}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y)$, i.e., the underlying space of this mapping spectrum is then equivalent to the mapping space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

## 1. Reviewing the $\infty$-categorical Approach to twisted Cohomology

Let us recall the modern approach again. Given a ring spectrum $R$, one can consider its space of invertible $R$-modules $\mathcal{L}\mathrm{ine}_R$. Now, given a space $X$, a twist of $R$ on $X$ is a map $\alpha\colon X \to \mathcal{L}\mathrm{ine}_R$. Then we define twisted homology as

$$X^\alpha := M\alpha = \mathrm{colim}(X \xrightarrow{\alpha} \mathcal{L}\mathrm{ine}_R \to \mathcal{M}\mathrm{od}_R)$$

where the last map is the inclusion of $\mathcal{L}\mathrm{ine}_R$ into $\mathcal{M}\mathrm{od}_R$. The $R$-cohomology groups twisted by $\alpha$ are then defined as the homotopy groups of the mapping spectrum $\mathcal{H}\mathrm{om}_R(X^\alpha, R)$, which is the spectrum given by the mapping spaces of maps from $X^\alpha$ to $\Sigma^n R$ in $\mathcal{M}\mathrm{od}_R$.

## 2. The $\infty$-categorical Approach to twisted Cohomology via Sections

Eventually we want to compare this definition with the classical approach. In this section, as a first step, we translate the definition into a more suitable form via sections, which will then allow us to compare it to the classical approach.

Recall that the objects in $\mathcal{L}\mathrm{ine}_R$ are invertible $R$-modules. Let $-\alpha$ denote the map

$$-\alpha := X \xrightarrow{\alpha} \mathcal{L}\mathrm{ine}_R \xrightarrow{\mathrm{inv}} \mathcal{L}\mathrm{ine}_R,$$

We now have the following simple lemma.

**Lemma 2.1.** *Let $M$ be an invertible $R$-module. Then there is a natural equivalence of mapping spectra*

$$\mathcal{H}\mathrm{om}(-M, R) \simeq \mathcal{H}\mathrm{om}(R, M)$$

*This in particular means that the inverse of $M$ is given by $\mathcal{H}\mathrm{om}(M, R)$.*

*Proof.* We know that $\mathcal{H}\mathrm{om}(M, -)$ is the right adjoint to $M \otimes_R -$. But $M^{-1} \otimes_R -$ is also a right adjoint to $M \otimes_R -$, so they must be equivalent, meaning $M^{-1} \otimes R \simeq \mathcal{H}\mathrm{om}(M, R)$. □

Based on this lemma we have the following chain of equivalences

$$\mathcal{H}\mathrm{om}_R(X^{-\alpha}, R) \simeq \mathcal{H}\mathrm{om}(-\alpha, R_X) \simeq \mathcal{H}\mathrm{om}(R_X, \alpha)$$

Here in the last step we used the previous lemma point-wise. This proves the following proposition.

**Proposition 2.2.** *The twisted $R$-cohomology spectrum twisted by $-\alpha$ is given by the mapping spectrum $\mathcal{H}\mathrm{om}(R_X, \alpha)$.*

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Let us now recall that every ring spectrum $R$ comes with a unique ring map $\mathbb{S} \to R$, which induces an adjunction between $\mathbb{S}$-modules (i.e. spectra) and $R$-modules:

$$\begin{tikzcd} \mathcal{M}\mathrm{od}_{\mathbb{S}} \arrow[r, shift left, "R\otimes_{\mathbb{S}}-"] \arrow[r, phantom, "\perp"] & \mathcal{M}\mathrm{od}_R \arrow[l, shift left, "\text{Forget}"] \end{tikzcd} .$$

If we apply this adjunction point-wise to $\alpha$ we immediately get the following lemma.

**Lemma 2.3.** *There is an equivalence of mapping spectra*

$$\mathcal{H}\mathrm{om}_R(R_X, \alpha) \simeq \mathcal{H}\mathrm{om}(\mathbb{S}_X, \alpha).$$

Finally, let $\Omega^{\infty-n}\alpha \colon X \to \mathcal{L}\mathrm{ine}_R \to \mathcal{S}$ be the composition of $\alpha$ with $\Omega^\infty \colon \mathcal{L}\mathrm{ine}_R \to \mathcal{S}\mathrm{p} \to \mathcal{S}$. As $\mathbb{S} \simeq \Sigma^\infty_+ *$, the adjunction between $\Sigma^\infty_+$ and $\Omega^\infty$ gives us the following equivalence of spaces:

$$\mathrm{Hom}_{\mathcal{S}\mathrm{p}}(\mathbb{S}_X, \alpha) \simeq \mathrm{Hom}_{\mathcal{S}}(*_X, \Omega^\infty\alpha).$$

Here $*_X$ is the constant functor at the point. The data of the right hand side is exactly a section. Hence combining these equivalences we get the following theorem.

**Theorem 2.4.** *A point in the underlying space of the twisted $R$-cohomology spectrum $\Omega^\infty\mathcal{H}\mathrm{om}_R(X^{-\alpha}, R)$ is equivalent to a section of the bundle over $X$ classified by the map $\Omega^\infty\alpha \colon X \to \mathcal{S}$.*

Moreover, by similar analysis we have the following more general theorem.

**Theorem 2.5.** *A point in the $n$-th underlying space of the twisted $R$-cohomology spectrum $\Omega^{\infty-n}\mathcal{H}\mathrm{om}_R(X^{-\alpha}, R)$ is equivalent to a section of the bundle over $X$ classified by the map $\Omega^{\infty-n}\alpha \colon X \to \mathcal{S}$.*

## 3. $K$-Theory à la Atiyah

Before we can comprehend how Atiyah–Segal defined twisted $K$-theory, we first need to review how Atiyah defined classical $K$-theory via Fredholm operators in [Ati67] (also in [Jän65]).

*Remark* 3.1. Recall that every infinite-dimensional separable Hilbert space is isomorphic to each other. We will hence fix one such choice $\mathcal{H}$ throughout these two sections.

We now recall the notion of a Fredholm operator.

**Definition 3.2.** A Fredholm operator on $\mathcal{H}$ is a bounded linear operator $F \colon \mathcal{H} \to \mathcal{H}$ such that both its kernel and cokernel are finite-dimensional. The space of Fredholm operators is denoted $\mathcal{F}\mathrm{red}(\mathcal{H})$ and is topologized as a subspace of the space of bounded linear operators with the norm topology.

*Remark* 3.3. Fredholm operators admit many alternative characterizations. For example, it is equivalent to being invertible modulo compact operators i.e.

$$\mathcal{F}\mathrm{red}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : AB - 1, BA - 1 \in \mathcal{K} \text{ for some } B \in \mathcal{B}(\mathcal{H})\}$$

We now define the spectrum $K_n$, given level-wise by the space of Fredholm operators on $\mathcal{H}$:

$$K_n = \mathcal{F}\mathrm{red}^{(n)}(\mathcal{H})$$

Here $\mathcal{F}\mathrm{red}^{(0)}(\mathcal{H})$ is simply $\mathcal{F}\mathrm{red}(\mathcal{H})$, and the other can be characterized similarly.
We will claim the following result regarding these spaces that we shall not prove.

**Lemma 3.4.** *The spaces $K_n$ assemble into a spectrum, meaning there are maps*

$$K^{(n)}(\mathcal{H}) \to \Omega\mathcal{F}\mathrm{red}^{(n+1)}(\mathcal{H}).$$

How does this construction relate to $K$-theory? We have the following remark that can give us some intuition.

*Remark* 3.5. For a compact space $X$, and given map $X \to \mathcal{F}\mathrm{red}^{(0)}(\mathcal{H})$, we can associate the formal difference of vector bundles given by the kernel and cokernel of the associated family of Fredholm operators (which are by assumption finite-dimensional). This associated to every element in $[X, \mathcal{F}\mathrm{red}^{(0)}(\mathcal{H})]$ an element in $K^0(X)$.

This map is not just coincidental and in fact we have the following major result.

**Theorem 3.6** (Atiyah–Jänich)**.** *The map from Equation (3.5) is an isomorphism.*

This relates classical $K$-theory to Fredholm operators, meaning the spectrum above is indeed coincides with $K$-theory.

## 4. Atiyah–Segal's Approach

We now proceed to review the original approach to twisted $K$-theory by Atiyah–Segal [AS04]. This approach generalizes the Fredholm approach to $K$-theory from the regular to the twisted setting.

**Definition 4.1.** Fix a topological space $X$. Let $PU(\mathcal{H})$ be the projective unitary group of $\mathcal{H}$, i.e. the quotient of the unitary group $U(\mathcal{H})$ by its center $S^1$. Let $P \to X$ be a principal $PU(\mathcal{H})$-bundle over $X$. Then we define the $\tau$-twisted $K$-theory group as the space of sections

$$K^0_\tau \coloneqq \pi_0(\Gamma(X, P \times_{PU(\mathcal{H})} K)).$$

*Remark* 4.2. Here, $P \times_{PU(\mathcal{H})} KU_0(\mathcal{H})$ denotes the associated bundle of spectra over $X$ with fibre $KU_0(\mathcal{H})$. The associated bundle can be described more explicitly as the bundle whose fibre over $x \in X$ is given by the $P_x \times_{PU(\mathcal{H})} K_0$, meaning we take the products of the fibers and quotient out the simultaneous action of $PU(\mathcal{H})$ on the fibers $P_x$ and $K_0$.

*Remark* 4.3. As we have $BPU(\mathcal{H}) \simeq K(\mathbb{Z}, 3)$, the bundle $P \to X$ is classified via a map $\tau\colon X \to BPU(\mathcal{H}) \simeq K(\mathbb{Z}, 3)$.

## 5. Comparing Classical and Modern

We are now ready to compare the classical and modern approaches to twisted $K$-theory. Let $X$ be a topological space and $P \to X$ be a principal $PU(\mathcal{H})$-bundle over $X$. Following Equation (4.3) this is classified by a map $\tau\colon X \to K(\mathbb{Z}, 3)$.

For such a map we now have the following explicit commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\tau"] \arrow[dr, "\tau"'] & K(\mathbb{Z},3) \arrow[r, "?"] \arrow[d, "\mathrm{id}"] & BGL_1(KU) \arrow[r] \arrow[d] & \mathcal{L}\mathrm{ine}_{KU} \arrow[d, "\Omega^{\infty-n}"] \\
& K(\mathbb{Z},3) \arrow[r] & BAut(K_n) \arrow[r] & \mathcal{S}
\end{tikzcd},$$

which we now explain in more detail.

(1) The top row corresponds to the modern approach, as it takes a map into $K(\mathbb{Z}, 3)$ to a $KU$-line bundle. Following Equation (2.5), taking $\Omega^{\infty-n}$ and then sections of the resulting bundle correspond precisely to classes of $\tau$-twisted $KU$-cohomology.

(2) The bottom row corresponds to the classical approach. Indeed, post-composing $\tau$ with the map $K(\mathbb{Z}, 3) \to BAut(K_n)$ corresponds to the constructing the associated bundle. In this case, it is by definition (Equation (4.1)) the case that sections of this bundle give us elements in twisted $K$-theory.

(3) Hence compatibility between these two approaches corresponds to these diagrams commuting, which is a direct observation.

## 6. Some Comments regarding Freed–Hopkins–Teleman

Ideally we could add some comments regarding the Freed–Hopkins–Teleman approach here, but that might turn out to be beyond existing capabilities.

## References


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