DIFFERENTIAL COHOMOLOGY SEMINAR 11

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Today we look at differential cohomology in the cohesive setting and in particular the generalization to non-abelian differential cohomology in the sense of Schreiber [Sch13, FSS24].

We saw previously that differential cohomology lifts order cohomology theories represented by spectra. However, spectra can only classify phenomena that are inherently stable, whereas certain applications require unstable structures (see Remark 1.5 for a more detailed discussion).

We hence discuss a "non-abelian" generalization of differential cohomology, that can incorporate such unstable phenomena. Concretely, instead of working with E_∞-groups (which correspond to connective spectra), we want to work with E_2-groups, meaning homotopy commutative group objects in spaces.

1. Non-Abelian Cohomology

Let us first define non-abelian cohomology in the classical setting. HEre the idea is to generalize from spectra to spaces.

Definition 1.1. Let X, A be ∞-groupoids. The *non-abelian cohomology* of X with coefficients in A is defined as

$$H^0(X; A) := \pi_0 \mathrm{Map}(X, A).$$

We can generalize to higher cohomology if A admits a delooping.

Definition 1.2. Let X be an ∞-groupoid and A an E_2-group with delooping $\mathrm{B}A$. The *non-abelian cohomology* of X with coefficients in A is defined as

$$H^1(X; A) := \pi_0 \mathrm{Map}(X, \mathrm{B}A).$$

Of course, if A admits further deloopings we can continue and obtain higher non-abelian cohomology groups.

Definition 1.3. Let X be an ∞-groupoid and A admits deloopings $\mathrm{B}^n A$. The *non-abelian cohomology* of X with coefficients in A is defined as

$$H^n(X; A) := \pi_0 \mathrm{Map}(X, \mathrm{B}^n A).$$

If A admits infinite deloopings, then A is precisely an E_∞-group and hence corresponds to a connective spectrum. In that case we recover the classical definition.

Remark 1.4. If A is a spectrum, then we have $\mathrm{B}^n A = \Omega^{\infty-n} A$ and we recover the ordinary cohomology groups

$$H^n(X; A) = \pi_0 \mathrm{Map}(X, \Omega^{\infty-n} A) \cong [X, \Omega^{\infty-n} A] \cong A^n(X).$$

Let us motivate this story.

Remark 1.5.
(1) The string bundle of a manifold X is classified by homotopy classes of maps $X \to B\mathrm{String}$, which coincides with $H^1(X; \mathrm{String})$, originally defined by Giraud via non-abelian Cech cohomology [Gir71]. Here String is an E_1-group (it is not abelian), so $B\mathrm{String}$ is just E_2.
(2) Another motivation comes from electro magnetics. Let P be a $U(1)$-principal fiber bundle on X, then $[X, BU(1)] \cong [X, K(\mathbb{Z}, 2)] \cong [X, \mathbb{CP}^\infty]$. We know that $\mathbb{CP}^\infty$ is a colimit of $\mathbb{CP}^n$, none of the intermediate stages are ∞-loop spaces. Generalizing the definition permits looking at $H(X, \mathbb{CP}^n)$, which should have relevance in physics.

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2. Differential Non-Abelian Cohomology

We now want to generalize this approach to differential cohomology. Concretely, we want to look at differential refinements of non-abelian cohomology. We will do so by looking at differential refinements of intrinsic cohomology of a cohesive ∞-topos.

Naively, given an ∞-topos $\mathcal{G}$ with cohesive structure and X, A objects in $\mathcal{G}$, we would like to define the non-abelian cohomology as

$$H^n(X;A) := \pi_0 \mathrm{Map}_{\mathcal{G}}(X, \mathrm{B}^n A).$$

assuming A admits a delooping $\mathrm{B}^n A$ in $\mathcal{G}$.

Our goal is now to extract the fracture square, analogous to [BNV16].

$$\begin{tikzcd} \hat{E} \arrow[r] \arrow[d] & Z \arrow[d, "\text{homotopification}"] \\ E \arrow[r, "\text{characteristic map}"] & \mathcal{H}(Z) \end{tikzcd}$$

How can we do this in the non-abelian case? We first review relevant definitions.

Definition 2.1. A cohesive ∞-topos $\mathcal{G}$ is an ∞-topos together with an adjoint quadruple

$$\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{Codisc},$$

where $\Gamma \colon \mathcal{G} \to \mathcal{S}$ is the global sections functor, such that

(1) Disc and Codisc are fully faithful,
(2) Π preserves finite products.

Example 2.2. Let $\mathcal{M}\mathrm{fd}$ be the category of smooth manifolds with the usual open cover topology. Then the ∞-topos of sheaves of ∞-groupoids on $\mathcal{M}\mathrm{fd}$ is cohesive, where

- Π is given by taking the fundamental ∞-groupoid,
- Disc is given by regarding an ∞-groupoid as a constant sheaf,
- Γ is given by taking global sections,
- Codisc is given by regarding an ∞-groupoid as a codiscrete sheaf.

We can now define an analogue of the de Rham complex in the setting of a cohesive ∞-topos.

Definition 2.3. Let $\mathcal{G}$ be a cohesive ∞-topos and $A \in \mathcal{G}$. The *de Rham complex* of A is defined as

$$\begin{tikzcd} A \arrow[r] \arrow[d] & * \arrow[d] \\ \mathrm{Disc} \circ \Pi(A) \arrow[r] & \Pi_{dR} A \end{tikzcd}$$

and for an object A the *coefficient object* of A

$$\begin{tikzcd} \flat_{dR} A \arrow[r] \arrow[d] & * \arrow[d] \\ \mathrm{Disc} \circ \Gamma(A) \arrow[r] & A \end{tikzcd}$$

Note here, there is an adjunction

$$\begin{tikzcd} \mathcal{G} \arrow[r, shift left, "\Pi_{dR}"] \arrow[r, phantom, "\perp"] & \mathcal{G} \arrow[l, shift left, "\flat_{dR}"] \end{tikzcd}$$

With that at hand, we can formulate the following definition.

Definition 2.4. For objects X, A in a cohesive ∞-topos $\mathcal{G}$ the *differential non-abelian cohomology* of X with coefficients in A is defined as the set

$$H_{dR}(X, A) := H(\Pi_{dR} X, A) \simeq H(X, \flat_{dR} A).$$

Notice this abstract definition recovers the classical case.

Proposition 2.5. *Let X be a sheaf of manifolds. Then*

$$\mathrm{Map}(X, \flat_{dR}\mathbf{B}^n U(1)) \simeq \begin{cases} H^n_{dR} & \text{for } n \geq 2, \\ \Omega^1_{cl} & \text{for } n = 1, \\ 0 & \text{for } n = 0. \end{cases}$$

Remark 2.6. Recall that in our original approach to differential cohomology we often used the pure sheaf as $\Omega^{\geq n}$. So, this definition does diverge slightly. However, it appears this difference does not play an adverse role.

We will not prove this result, however, we sketch the idea of the proof.

Idea. Fundamentally the result follows from the following equivalence:

$$\mathrm{Map}(X, \flat_{dR} B^n U(1)) \simeq \mathrm{Map}(C(\{U_i\}), \Phi(\Omega^1(-) \to \Omega^2(-) \to ...\Omega^n_{cl})),$$

where $C(\{U_i\})$ is the Cech nerve of a good open cover of X and Φ is the Dold-Kan correspondence.

This observation fundamentally relies on the computation that the sheaf associated to the chain complex

$$\Phi(\Omega^1(-) \to \Omega^2(-) \to ... \to \Omega^n_{cl}).$$

is given by $\flat_{dR} B^n U(1)$.

This allows unwinding the definition of cohomology in Proposition 2.5 via $\flat_{dR}$ into a computation with differential forms, which recovers the classical de Rham cohomology groups. □

We can now use this generalized definition to define curvature maps.

Definition 2.7. Let A be an object in $\mathcal{G}$, such that $B^{n+1}A$ exists. The *curvature map* is defined as the map pullbacks

$$\begin{tikzcd} B^n A \arrow[r] \arrow[d, "\mathrm{curv}"] & * \arrow[d] \\ \flat dR B^{n+1} A \arrow[r] \arrow[d] & \mathrm{Disc} \circ \Gamma B^{n+1} A \arrow[d] \\ * \arrow[r] & B^{n+1} \end{tikzcd}$$

With this curvature map we can finally state the fracture square.

Theorem 2.8. *Let A admit deloopings $B^n A$. Then we have a pullback square*

$$\begin{tikzcd} H_{diff}(X, B^n A) \arrow[r] \arrow[d] & H_{dR}(X, A) \arrow[d] \\ \mathrm{Map}_{sG}(X, B^n A) \arrow[r, "\mathrm{curv}_*"] & \mathrm{Map}_{sG}(X, \flat_{dR} B^{n+1} A) \end{tikzcd},$$

where the bottom map is induced by the curvature map from Definition 2.7.

The claim, which shall remain unproven, is that in the case of $A = U(1)$ this recovers the ordinary differential cohomology fracture square.

References

[BNV16] Ulrich Bunke, Thomas Nikolaus, and Michael Völkl. Differential cohomology theories as sheaves of spectra. *J. Homotopy Relat. Struct.*, 11(1):1–66, 2016.

[FSS24] Domenico Fiorenza, Hisham Sati, and Urs Schreiber. *The character map in non-abelian cohomology—twisted, differential, and generalized.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, [2024] ©2024.

[Gir71] Jean Giraud. *Cohomologie non abélienne*, volume Band 179 of *Die Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, Berlin-New York, 1971.

[Sch13] Urs Schreiber. Differential cohomology in a cohesive infinity-topos. *arXiv preprint*, 2013. arXiv:1310.7930.