DIFFERENTIAL COHOMOLOGY SEMINAR 2

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In this lecture we want to learn the basics of ∞-category theory. For the ∞-categorical background, we broadly follow [Gro10] and a little [Lur09].

1. Basics on $(\infty, 1)$-categories

$(\infty, 1)$-categories have different models that capture its essence. The first model are *quasi-categories*.

Definition 1.1. Given a natural number n, let $\langle n \rangle$ denote the linearly ordered set $\{0, \cdots, n\}$. The *simplex category* $\mathbf{\Delta}$ is the category of finite linerly ordered sets $\langle n \rangle$, for every n, and monotone functions.

Definition 1.2. Given $0 \leq i \leq n$, the *i-face map* is the unique injective map $\delta_n^i : \langle n-1 \rangle \to \langle n \rangle$ missing i. The *i-degeneracy map* is the unique surjective map $\sigma_n^i : \langle n+1 \rangle \to \langle n \rangle$ such that i and $i+1$ have the same image.

Theorem 1.3. *As a category, $\mathbf{\Delta}$ is generated from the face and degeneracy maps subject to the* simplicial identities *(see* [GJ99, I.4]*).*

Definition 1.4. A *simplicial set* is a contravariant functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. Denote by $\mathrm{s}\mathcal{S}\mathrm{et}$ the category of simplicial sets. The representable simplicial set $\mathrm{Hom}_{\mathbf{\Delta}}(-, \langle n \rangle)$ is denoted as Δ^n. The set of *n-simplices*, denoted by X_n, is the image of $\langle n \rangle$. By Yoneda lemma, we identify a n-simplex with the corresponding map of simplicial sets $\Delta^n \to X$. Let $d_i^n : X_n \to X_{n-1}$ be the map of sets induced by the i-face map δ_n^i, which we also call *i-face map*.

By only representing the face maps, we can depict a simplicial set as follows:

$$X_0 \leftleftarrows X_1 \, \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow} \, X_2 \, \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow} \, \cdots \tag{1.5}$$

∞-categories are then defined in terms of a lifting condition, for which we need to define horns.

Definition 1.6. Given $i \in \langle n \rangle$, the i-horn is the simplicial sub-set $\Lambda_i^n \subseteq \Delta^n$ defined as follows: A monotone map $f : \langle k \rangle \to \langle n \rangle$ is a k-simplex of Λ_i^n if there is $j \neq i$ that does not belong in the image of f.

Definition 1.7. A simplicial set X is called a *quasi-category* if every inner horn $\Lambda_i^n \to X$, i.e. $0 < i < n$, can be extended to a n-simplex $\Delta^n \to X$. If every horn $\Lambda_i^n \to X$ admits an extension, X is called a *∞-groupoid*.

Remark 1.8. The nerve of a category $\mathcal{C}$ is a quasi-category (one can simply check the horn filling condition directly). Moreover, the nerve is a ∞-groupoid if and only if $\mathcal{C}$ is a groupoid. The nerve functor $\mathrm{N} : \mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{S}\mathrm{et}$ has a left adjoint, the *homotopy category* functor Ho. If X is a quasi-category, $\mathrm{Ho}(X)$ has an explicit construction where objects are the 0-simplices of X and morphisms are homotopy classes of 1-simplices, see [Lan21, 1.2.5] and [Gro10, Proposition 1.15].

Definition 1.9. Let X be a quasi-category, a 1-simplex $f : x \to y$ is an *equivalence* if there are 2-simplices of the following form:

$$\begin{tikzcd} & x \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow"] & \\ y \arrow[ur] \arrow[rr, equal] & {} & y \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ x \arrow[ur, "f"] \arrow[rr, equal] & {} & x \end{tikzcd} \tag{1.10}$$

where the bottom 1-simplices are the image of y and x, respectively, under the function $X_0 \to X_1$ induced by σ_0^0.

The following theorem shows that Kan complex have the same relation with quasi-categories, as groupoids have with categories.

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Theorem 1.11. *A quasi-category X is a ∞-groupoid if and only if every 1-simplex is an equivalence.*

Remark 1.12. [Gro10] approaches ∞-groupoids in the other direction, namely defining a ∞-groupoid as a quasi-category where every 1-simplex is an equivalence, then characterizing ∞-groupoids as quasi-categories satisfying the horn filling condition for every horn.

A strict model for ∞-categories is given by simplicially enriched categories.

Definition 1.13. Denote by $\mathrm{s}\mathcal{C}\mathrm{at}$ the category of simplicially enriched categories.

Remark 1.14. Lurie constructs in [Lur09, 1.1.5.1] a functor $\mathrm{N}_\Delta : \mathrm{s}\mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{S}\mathrm{et}$, called the *homotopy coherent nerve*. Similarly to the ordinary nerve, it admits a left adjoint, denoted $\mathfrak{C}$ and called *rigidification*, see [DS11].

Theorem 1.15. $\mathrm{s}\mathcal{S}\mathrm{et}$ *and* $\mathrm{s}\mathcal{C}\mathrm{at}$ *underlie model structures, called* Joyal *and* Dwyer-Kan-Bergner*, respectively, such that fibrant-cofibrant objects are quasi-categories and categories enriched over ∞-groupoids, respectively, and the adjunction*

$$\mathrm{s}\mathcal{C}\mathrm{at} \underset{\mathfrak{C}}{\overset{\mathrm{N}_\Delta}{\rightleftarrows}} \mathrm{s}\mathcal{S}\mathrm{et} \tag{1.16}$$

lifts to a Quillen equivalence.

Definition 1.17. The category $\mathcal{K}\mathrm{an}$ of ∞-groupoids is a self-enriched category, so $\mathcal{S} := \mathrm{N}_\Delta(\mathcal{K}\mathrm{an})$ is a quasi-category, namely the *quasi-category of ∞-groupoids*.

Let $\mathcal{T}\mathrm{op}$ be the category of compactly generated topological spaces.

Theorem 1.18. $\mathrm{s}\mathcal{S}\mathrm{et}$ *and* $\mathcal{T}\mathrm{op}$ *underlie model structures, both called* Quillen*, such that fibrant-cofibrant are ∞-groupoids and retracts of cellular complexes, respectively, and the adjunction*

$$\mathcal{T}\mathrm{op} \overset{\mathrm{Sing}}{\rightleftarrows} \mathrm{s}\mathcal{S}\mathrm{et} \tag{1.19}$$

lifts to a Quillen equivalence.

2. Basic constructions

Remark 2.1. Recall that $\mathrm{s}\mathcal{S}\mathrm{et}$ is self-enriched. More specifically, given simplicial sets X, Y, the simplicial set $\mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}(X, Y)$ is characterized as follows: A n-simplex is a map of simplicial sets $\Delta^n \times X \to Y$.

Lemma 2.2. *If X is a simplicial set and $\mathcal{C}$ a ∞-category, then $\mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}(X, \mathcal{C})$ is a ∞-category.*

Definition 2.3. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, denote by $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$ the *∞-category of functors* $\mathcal{C} \to \mathcal{D}$.

Definition of opposite ∞-category, slice ∞-category

3. Accessible and Presentable Categories

In general, a limit, resp. colimit, preserving functor need not have a left, resp. right, adjoint. Here we wish to introduce a rather large class of ∞-categories for which the previous statement holds. We begin by recalling the definition of locally presentable 1-categories. Let κ denote a regular cardinal.

Definition 3.1. A category $\mathcal{I}$ is *κ-filtered* if, for every $\mathcal{J}$ with $< \kappa$ many morphisms, every diagram $\mathcal{J} \to \mathcal{I}$ has a cocone. A functor $F : \mathcal{C} \to \mathcal{D}$ is *κ-accessible* if it preserves κ-filtered colimits. Given a category $\mathcal{C}$, an object X is *κ-compact* if $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{S}\mathrm{et}$ is κ-accessible.

Definition 3.2. A category $\mathcal{C}$ is *κ-accessible* if there exists a set $S \subseteq \mathcal{C}_0$ of κ-compact objects that generate $\mathcal{C}$ under κ-filtered colimits. A category is *accessible* if it is κ-accessible, for some regular cardinal κ.

Definition 3.3. A category $\mathcal{C}$ that is accessible and cocomplete is called *locally presentable*.

Theorem 3.4. *Let $\mathcal{C}$ be a category, then $\mathcal{C}$ is locally presentable if and only if there exists a small category S such that the induced functor $\mathcal{C} \to \mathcal{P}(S)$ is a fully faithful, accessible right adjoint.*

Theorem 3.5. *Let $\mathcal{C}, \mathcal{D}$ be locally presentable categories. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is a left, resp. right, adjoint if and only if it preserves colimits, resp. it preserves limits and is accessible.*

Definition 3.6. Let X be a ∞-category. The *∞-category of pre-sheaves of spaces* is defined as $\mathcal{P}(X) := \mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}(X^{\mathrm{op}}, \mathcal{S})$.

Theorem 3.7 (Yoneda)**.** *Given a ∞-category X, there is a fully faithful functor $y : X \to \mathcal{P}(X)$, called the* Yoneda embedding*, such that: Given a cocomplete ∞-category Y, pre-composition by y induces an equivalence*

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{P}(X), Y) \longrightarrow \mathcal{F}\mathrm{un}(X, Y) \tag{3.8}$$

where $\mathcal{F}\mathrm{un}^{\mathrm{L}}$ denotes the category of colimit preserving functors.

The definition of accessible category transfers directly to the ∞-categorical setting.

Theorem 3.9. *A ∞-category X is locally presentable (cocomplete and accessible) if and only if there is a small sub-∞-category S such that $X \to \mathcal{F}\mathrm{un}(S^{\mathrm{op}}, \mathcal{S})$ is a fully faithful, accessible right adjoint.*

Remark 3.10. In view of Equation (3.9), one can define a category to be *locally presentable* if it is the accessible right localization of a pre-sheaf category for some small ∞-category S. In particular, every pre-sheaf category is locally presentable.

Theorem 3.11. *Let X, Y be presentable ∞-categories, then a functor $f : X \to Y$ is a left, resp. right, adjoint if and only if it preserves colimits, resp. it preserves limits and is accessible.*

4. Stable ∞-Categories and Spectra

We now use the ∞-categorical framework to study spectra. The study of spectra originates from the study of *stable phenomena*, i.e. patterns appearing after repeated application of the suspension functor $\Sigma : \mathcal{T}\mathrm{op}_* \to \mathcal{T}\mathrm{op}_*$.

Example 4.1. Let $(\Sigma, \Omega) : \mathcal{T}\mathrm{op}_{*/} \to \mathcal{T}\mathrm{op}_{*/}$ be the suspension-loop adjunction on pointed topological spaces. *Freudenthal Suspension Theorem* states that, if X is a n-connected space, the adjunction unit $X \to \Omega\Sigma X$ is $2n$-connected. If X is connected, $S^n \wedge X$ is n-connected, so $\Sigma^n X \to \Omega\Sigma^{n+1} X$ is $2n$-connected. In particular, $\pi_i(\Sigma^n X) \to \pi_{i+1}(\Sigma^{n+1} X)$ is an isomorphism, for all $i < 2n$. The group $\pi_i(\Sigma^n X)$ is denoted $\pi^s_{n-i}(X)$, called the $(n-i)$*-stable homotopy group of* X.

More examples?

Definition 4.2. Let $\mathcal{C}$ be an ∞-category, an object 0 that is both initial and terminal is called *zero object*. A category $\mathcal{C}$ with a zero object is called a *pointed category*.

Example 4.3. Let $1 \in \mathcal{C}$ be a terminal object, then the identity of 1 is the zero object in the slice category $\mathcal{C}_{1/}$ of objects under 1. In particular, the category $\mathcal{S}_* = \mathcal{S}_{*/}$ of pointed spaces is pointed.

Proposition 4.4. *Let $\mathcal{D}$ be a pointed ∞-category. Evaluation at the 0-sphere S^0 induces an equivalence*

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{S}_*, \mathcal{C}) \longrightarrow \mathcal{C} \tag{4.5}$$

We now introduce the notion of a triangle.

Definition 4.6. Let $\mathcal{C}$ be a pointed ∞-category. A *triangle* in $\mathcal{C}$ is a commutative diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd} \tag{4.7}$$

A triangle is *exact*, resp. *coexact*, if it is a pullback, resp. pushout, square.

Definition 4.8. Let $\mathcal{C}$ be a pointed ∞-category. Denote by $\mathcal{C}^{\Sigma}$, resp. $\mathcal{C}^{\Omega}$, the full sub-category of $\mathcal{F}\mathrm{un}(\Delta^1 \times \Delta^1, \mathcal{C})$ of coexact, resp. exact, triangles of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}, \tag{4.9}$$

If $\mathcal{C}$ is finitely cocomplete, resp. complete, for every object X, resp. Y, there is a contractible space of coexact, resp. exact, triangles as Equation (4.9). In particular, $\mathcal{C} \simeq \mathcal{C}^{\Sigma}$ and $\mathcal{C} \simeq \mathcal{C}^{\Omega}$.

Proposition 4.10. *If $\mathcal{C}$ is a finitely complete and cocomplete pointed ∞-category, define the following functors Then the functors*

$$\Sigma : \mathcal{C} \longrightarrow \mathcal{C}^{\Sigma} \xrightarrow{\mathrm{ev}_{(1,1)}} \mathcal{C} \qquad\qquad \Omega : \mathcal{C} \longrightarrow \mathcal{C}^{\Omega} \xrightarrow{\mathrm{ev}_{(0,0)}} \mathcal{C} \tag{4.11}$$

are adjoint (Σ is left adjoint to Ω).

Theorem 4.12. *Let $\mathcal{C}$ be a finitely bicomplete pointed ∞-category. The following are equivalent:*

1. *A triangle is exact if and only if it is coexact.*
2. *(Σ, Ω) is an adjoint equivalence.*
3. *A commutative square is a pullback if and only if it is a pushout.*

Definition 4.13. A finite bicomplete pointed ∞-category $\mathcal{C}$ satisfying any of the equivalent conditions in Equation (4.12) is called *stable*.

If $\mathbb{A}$ denotes a nice abelian category, there is a *derived ∞-category* of $\mathbb{A}$, denoted $\mathcal{D}(\mathbb{A})$ such that $\mathrm{Ho}\mathcal{D}(\mathbb{A}) = D(\mathbb{A})$ is the ordinary derived category of $\mathbb{A}$, i.e. the localization of chain complex at quasi-isomorphisms. From homological algebra, it is known that $D(\mathbb{A})$ underlies the structure of a triangulated category, which turns out to be the 1-categorical reflection of $\mathcal{D}(\mathbb{A})$ being stable.

Proposition 4.14 ([Lur17, 3.11])**.** *If $\mathcal{C}$ is a stable ∞-category, then* $\mathrm{Ho}\mathcal{C}$ *has a canonical structure of a triangulated category.*

To construct the stabilization of a pointed ∞-category, there are several approaches, such as reduced excisive functors on $\mathcal{S}_*^{\mathrm{fin}}$, the category of pointed, finite spaces, see [Lur17, 1.4.2.8]. Here we consider the more explicit approach using spectrum objects.

Definition 4.15. Let $\mathcal{C}$ be a pointed ∞-category. A *pre-spectrum object in C* consists of a functor $E : \mathbb{Z} \times \mathbb{Z} \to \mathcal{C}$ such that $E(n,m) \simeq 0$, for all $n \neq m$. Denote by $\mathcal{PS}\mathrm{p}(\mathcal{C})$ the category of pre-spectrum objects. The functor $\Omega^{\infty-n} : \mathcal{PS}\mathrm{p}(\mathcal{C}) \to \mathcal{C}$ is defined as evaluation at (n,n).

For every n, the diagram

$$\begin{tikzcd} E(n,n) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & E(n+1,n+1) \end{tikzcd} \tag{4.16}$$

determines a pair of adjoint morphisms

$$\alpha_n : \Sigma E(n,n) \to E(n+1,n+1), \qquad \beta_n : E(n,n) \to \Omega E(n+1,n+1)$$

Definition 4.17. Let $\mathcal{C}$ be a pointed ∞-category. A *spectrum object in* $\mathcal{C}$ consists of a pre-spectrum object E such that β_n is an equivalence, for all n. Denote by $\mathcal{S}\mathrm{p}(\mathcal{C}) \subseteq \mathcal{PS}\mathrm{p}(\mathcal{C})$ the full sub-category of spectrum objects.

Theorem 4.18. *Let $\mathcal{C}$ be a presentable pointed ∞-category, then $\Omega^{\infty-n} : \mathcal{S}\mathrm{p}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\Sigma^{\infty-n} : \mathcal{C} \to \mathcal{S}\mathrm{p}(\mathcal{C})$, for every n.*

In particular, $\Sigma^{\infty} : \mathcal{C} \to \mathcal{S}\mathrm{p}(\mathcal{C})$ has the following universal property.

Theorem 4.19. *Let $\mathcal{C}$ be a presentable pointed ∞-category. Given a stable ∞-category $\mathcal{D}$, pre-composition by Σ^{∞} induces an equivalence*

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{S}\mathrm{p}(\mathcal{C}), \mathcal{D}) \longrightarrow \mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \tag{4.20}$$

In particular, for $\mathcal{C} = \mathcal{S}_*$, evaluation at the sphere spectrum *$\mathbb{S} = \Sigma^{\infty} S^0$ induces an equivalence*

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{S}\mathrm{p}(\mathcal{S}_*), \mathcal{D}) \longrightarrow \mathcal{D} \tag{4.21}$$

Definition 4.22. The *∞-category of spectra* is the category of spectrum objects in pointed spaces.

5. Generalized Cohomology Theories

We shall now use the language of ∞-categories to reformulate the concept of generalized cohomology theory à-là Eilenberg-Steenrod. In this new context, we recall a representability theorem for cohomology theories by spectrum object.

Remark 5.1. Denote by $\mathcal{S}\mathrm{et}^{\mathbb{Z}}$ the category of $\mathbb{Z}$-indexes families of sets. Given an object S and $n \in \mathbb{Z}$, denote by $\Sigma^n S$ the shifted family $(\Sigma^n S)_i = S_{i-n}$.

Definition 5.2 ([Lur17, 1.4.1.6])**.** Let $\mathcal{C}$ be a finitely cocomplete pointed ∞-category, $\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ the induced suspension functor. A *generalized cohomology theory* is a functor $H : \mathrm{Ho}\mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}^{\mathbb{Z}}$ together with a natural isomorphism $\partial : \Sigma H \to H\Sigma_{\mathcal{C}}$ such that:

- H preserves arbitrary products. In particular, $H^n(0)$ is the one-point set. Given an object X, the unique morphism $X \to 0$ induces an element $* \simeq H^n(0) \to H^n(X)$, which we denote by 0.
- Given a coexact triangle $X' \to X \to X''$, if $\eta \in H^n(X)$ has image $0 \in H^n(X'')$, then it lies in the image of $H^n(X') \to H^n(X)$.

Theorem 5.3 ([Lur17, 1.4.1.10])**.** *Let $\mathcal{C}$ be a nice ∞-category and (H, ∂) a generalized cohomology theory, then, for every n, the functor H^n is representable by an object $E(n)$.*

The natural isomorphism ∂ translates into an equivalence $E(n) \simeq \Omega E(n+1)$, which is then used to construct a spectrum object representing the cohomology theory H^n, see [Lur17, 1.4.1.11].

Remark 5.4. For $\mathcal{C} = \mathcal{S}_*$, the above definition of cohomology theory reduces to the classical Eilenberg-Steenrod definition. Since $\mathcal{S}_*$ is nice, we thus recover the classical *Brown representability theorem*.

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