DIFFERENTIAL COHOMOLOGY SEMINAR 2

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In this lecture we want to learn the basics of ∞-category theory. For the ∞-categorical background, we broadly follow [Gro10] and a little [Lur09].

1. Basics on $(\infty, 1)$-categories

$(\infty, 1)$-categories have different models that capture its essence. The first model are *quasi-categories*.

Definition 1. Given a natural number n, let $\langle n \rangle$ denote the linearly ordered set $\{0, \cdots, n\}$. The *simplex category* $\mathbf{\Delta}$ is the category of finite linerly ordered sets $\langle n \rangle$, for every n, and monotone functions.

Definition 2. Given $0 \leqslant i \leqslant n$, the *i-face map* is the unique injective map $\delta_n^i : \langle n-1 \rangle \to \langle n \rangle$ missing i. The *i-degeneracy map* is the unique surjective map $\sigma_n^i : \langle n+1 \rangle \to \langle n \rangle$ such that i and $i+1$ have the same image.

Theorem 3. *As a category,* $\mathbf{\Delta}$ *is generated from the face and degeneracy maps subject to the* simplicial identities*, i.e.*

(4)
$$\delta_{n+1}^i \delta_n^j = \delta_{n+1}^{j+1} \delta_n^i, \qquad i \leqslant j$$

(5)
$$\sigma_{n-1}^j \sigma_n^i = \sigma_{n-1}^i \sigma_n^{j+1}, \qquad i \leqslant j$$

(6)
$$\sigma_n^j \delta_{n+1}^i = \begin{cases} \delta_n^i \sigma_{n-1}^{j-1}, & i < j \\ 1, & i = j \\ \delta_n^{i-1} \sigma_{n-1}^j, & i > j \end{cases}$$

Proof. Omitted. $\square$

Definition 7. A *simplicial set* is a contravariant functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. Denote by $s\mathcal{S}\mathrm{et}$ the category of simplicial sets. $X_n := X(\langle n \rangle)$ is the set of *n-simplices*.

By only representing the face maps, we can depict a simplicial set as follows:

(8)
$$\begin{tikzcd} X_0 & X_1 \arrow[l, shift left] \arrow[l, shift right] & X_2 \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \arrow[l, shift left] \arrow[l, shift right] \arrow[l, shift left=3] \arrow[l, shift right=3] \end{tikzcd}$$

∞-categories are then defined in terms of a lifting condition, for which we need to define horns.

Definition 9. Let Δ^n denote the representable functor associated to $\langle n \rangle$. The face map δ_n^i induces a map of simplicial sets $d_n^i : \Delta^{n-1} \to \Delta^n$. The image of d_n^i is called the *i-face*. The *i-horn* $\Lambda^{i,n}$ is the union of all faces, except the i-face.

Remark 10. Another characterization of $\Lambda^{i,n}$ is the following: A t-simplex $f : \langle t \rangle \to \langle n \rangle$ is a t-simplex for $\Lambda^{i,n}$ if and only if there is $j \neq i$ not in the image of f.

Definition 11. A simplicial set X is a *quasi-category* if, for every $0 < i < n$ and solid diagram

(12)
$$\begin{tikzcd} \Lambda^{i,n} \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

there is a dashed arrow rendering the diagram commutative. If the above condition holds for every $0 \leqslant i \leqslant n$, we call X a *Kan complex*.

Example 13. Let $\mathcal{C}$ be a category. The *nerve of* $\mathcal{C}$, denoted $\mathrm{N}\mathcal{C}$, is the simplicial set where the n-simplices are $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\langle n \rangle, \mathcal{C})$. This defines a functor $\mathrm{N} : \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}$.

Date: 06.05.2025 & 13.05.2025.

Proposition 14. $\mathrm{N}\mathcal{C}$ *is a quasi-category, and a Kan complex if and only if* $\mathcal{C}$ *is a groupoid.*

Proof. Straightforward combinatorics. □

Remark 15. The nerve is a special case of the following construction: Let $\mathcal{C}$ be a category, $\Gamma : \boldsymbol{\Delta} \to \mathcal{C}$ a functor, then define N_Γ as the composition

$$\mathcal{C} \longrightarrow \mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et}) \longrightarrow s\mathcal{S}\mathrm{et} \tag{16}$$

where the first functor is Yoneda, while the second is pre-composition with Γ^{op}. In the case of the nerve, Γ is the functor sending $\langle n \rangle$ to the linerly ordered set viewed as a category. On the other hand, assuming $\mathcal{C}$ is cocomplete, we can left Kan extend Γ along the Yoneda functor $\boldsymbol{\Delta} \to s\mathcal{S}\mathrm{et}$, we denote by Ho_Γ the resulting functor $s\mathcal{S}\mathrm{et} \to \mathcal{C}$.

Proposition 17. *The pair* $(\mathrm{Ho}_\Gamma, \mathrm{N}_\Gamma) : \mathcal{C} \to s\mathcal{S}\mathrm{et}$ *is an adjoint pair. In the case of* $\mathrm{N} : \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}$*, the functor is fully faithful.*

Proof. Abstract nonsense about left Kan extensions. Full faithfulness can be checked directly. □

Remark 18. $\mathrm{Ho} : s\mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at}$ is called the *homotopy category* functor. If X is a quasi-category, $\mathrm{Ho}X$ has X_0 as set of objects and homotopy classes of maps as morphism, see [Lan21, 1.2.5].

Remark 19. Denote by $s\mathcal{C}\mathrm{at}$ the category of simplicially enriched categories. In [Lur09, 1.1.5.1], Lurie constructs a cosimplicial object $\boldsymbol{\Delta} \to s\mathcal{C}\mathrm{at}$. The resulting nerve functor $\mathrm{N}_\Delta : s\mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}$ is called *homotopy coherent nerve*. If $\mathcal{C}$ is a category enriched over ∞-groupoids, its homotopy coherent nerve is a ∞-category. The induced right adjoint is denoted $\mathfrak{C}$, the adjoint pair $(\mathfrak{C}, \mathrm{N}_\Delta)$ underlies a Quillen equivalence.

Denote by $\mathcal{K}\mathrm{an}$ the category of Kan complexes. One can show that $\mathcal{K}\mathrm{an}$ is self-enriched, which motivates, together with Equation (14), the following definition:

Definition 20. $\mathcal{S} := \mathrm{N}_\Delta(\mathcal{K}\mathrm{an})$ is called the *quasi-category of* ∞*-groupoids.*

2. Accessible and Presentable Categories

In general, a limit, resp. colimit, preserving functor need not have a left, resp. right, adjoint. Here we wish to introduce a rather large class of quasi-categories for which the previous statement holds. Let κ denote a regular cardinal.

Definition 21. A category $\mathcal{I}$ is *κ-filtered* if, for every $\mathcal{J}$ with $< \kappa$ many morphisms, every diagram $\mathcal{J} \to \mathcal{I}$ has a cocone. A functor $F : \mathcal{C} \to \mathcal{D}$ is *κ-accessible* if it preserves κ-filtered colimits. Given a category $\mathcal{C}$, an object X is *κ-compact* if $\mathrm{Hom}_\mathcal{C}(X, -) : \mathcal{C} \to \mathcal{S}\mathrm{et}$ is κ-accessible.

Definition 22. A category $\mathcal{C}$ is *κ-accessible* if there exists a set $S \subseteq \mathcal{C}_0$ of κ-compact objects that generate $\mathcal{C}$ under κ-filtered colimits. A category is *accessible* if it is κ-accessible, for some regular cardinal κ.

Definition 23. A category $\mathcal{C}$ that is accessible and cocomplete is called *locally presentable.*

Theorem 24. *Let* $\mathcal{C}$ *be a category, then* $\mathcal{C}$ *is locally presentable if and only if there exists a small category* S *such that the induced functor* $\mathcal{C} \to \mathcal{PS}\mathrm{hv}(S)$ *is fully faithful, accessible, and a right adjoint.*

Theorem 25. *Let* $\mathcal{C}, \mathcal{D}$ *be locally presentable categories. A functor* $F : \mathcal{C} \to \mathcal{D}$ *is a left, resp. right, adjoint if and only if it preserves colimits, resp. limits and is accessible.*

We now generalize this to quasi-categories.

Definition 26. Given a simplicial set X, denote by X^{op} the simplicial set obtained by reversing the structure maps: $X_n^{\mathrm{op}} = X_n$, for all n, and

$$\begin{aligned}(d_i : X_n^{\mathrm{op}} \to X_{n-1}^{\mathrm{op}}) &= (d_{n-i} : X_n \to X_{n-1}) \\ (s_i : X_n^{\mathrm{op}} \to X_{n+1}^{\mathrm{op}}) &= (s_{n-i} : X_n \to X_{n+1})\end{aligned}$$

If X is a quasi-category, so is X^{op}.

Definition 27. Let X be a quasi-category. The *quasi-category of simplicial presheaves* is defined as $\mathcal{PS}\mathrm{hv}(X) := \mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(X^{\mathrm{op}}, \mathcal{S})$.

Theorem 28 (Yoneda)**.** *Given a quasi-category X, there is a fully faithful functor $X \to \mathcal{PS}\mathrm{hv}(X)$. The functor $X \to \mathcal{PS}\mathrm{hv}(X)$ presents the category of pre-sheaves as the free cocompletion of X.*

Theorem 29. *Let K be a quasi-category.*

(1) There exists a set of κ-compact objects $\mathcal{C}^0$ in K, such that every object in K is a κ-filtered colimit of objects in $\mathcal{C}^0$.
(2) There exists a small category $\mathcal{C}^0$ and a fully faithful accessible right adjoint $\mathcal{C} \to \mathrm{Fun}((\mathcal{C}^0)^{op}, \mathcal{S}\mathrm{et})$.

In those cases we say K is presentable. We now again have the adjoint functor theorem.

Theorem 30. *Let $\mathcal{C}, \mathcal{D}$ be presentable ∞-categories.*

(1) $F\colon \mathcal{C} \to \mathcal{D}$ is a left adjoint if and only if it preserves colimits.
(2) $F\colon \mathcal{C} \to \mathcal{D}$ is a right adjoint if and only if it preserves limits and is accessible.

Note in particular the ∞-category of sheaves is a presentable ∞-category.

3. Stable ∞-Categories and Spectra

We now use the ∞-categorical framework to study spectra. Let us recall some facts about spectra, to motivate the story. The *Freudenthal suspension theorem* states that the suspension functor $\Sigma\colon \mathcal{T}\mathrm{op} \to \mathcal{T}\mathrm{op}$ stabilizes the homotopy type. More explicitly, the map

$$\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to \ldots$$

stabilizes for k large enough, if X satisfies some connectivity condition. This defined the stable homotopy groups $\pi_n^S(X)$ as the stabilization of this sequence.

There is significant interest in computing these stable homotopy groups, in particular in the case where X is a sphere, given that it helps us understand many phenomena in algebraic topology.

We now want a setting where these stable homotopy groups naturally live and can be studied. We know that (Σ, Ω) induces an adjunction on the category of pointed topological spaces. What we want is an adjustment of this definition such that the adjunction (Σ, Ω) is an equivalence.

We now take a ∞-categorical perspective on this and use it to study such stable phenomena.

Definition 31. Let $\mathcal{C}$ be an ∞-category with initial and terminal object. $\mathcal{C}$ has a 0-object if they are equivalent.

Example 32. Let $\mathcal{C}$ be a 1-category. Then $\mathcal{C}$ is pointed as a 1-category if and only if it is pointed as an ∞-category.

Example 33. Notice $\mathcal{S}$ is not pointed, we hence can define $\mathcal{S}_*$ as the slices under the terminal space, i.e. $\mathcal{S}_* = \mathcal{S}_{*/}$. This ∞-category is then pointed by construction.

Note $\mathcal{S}_*$ is not just some pointed ∞-category, it is in some sense the universal one.

Proposition 34. *Let $\mathcal{D}$ be a pointed ∞-category. Then the functor*

$$\mathrm{ev}_{S^0}\colon \mathrm{Fun}^L(\mathcal{S}_*, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}$$

that evaluates at S^0 is an equivalence.

We now generalize from there and define triangles in $\mathcal{S}_*$.

Definition 35. Let $\mathcal{C}$ be a pointed ∞-category. A *triangle* in $\mathcal{C}$ is a diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

where X, X, and Z are objects in $\mathcal{C}$.

Definition 36. We say a triangle is a *exact* if it is a pullback square and *coexact* if it is a pushout square.

Definition 37. Let $\mathcal{C}$ be a pointed ∞-category. Let $\mathcal{C}^{\Sigma}$ be the full subcategory of $\mathrm{Fun}([1] \times [1], \mathcal{C})$ with objects coexact triangles of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd},$$

meaning Y is the suspension of X.

Definition 38. Let $\mathcal{C}$ be a pointed ∞-category. Let $\mathcal{C}^{\Omega}$ be the full subcategory of $\mathrm{Fun}([1] \times [1], \mathcal{C})$ with objects exact triangles of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd},$$

meaning Y is the loop object of X.

Proposition 39. *If $\mathcal{C}$ is a pointed ∞-category with finite (co)limits. Then there exists functors*

$$\Sigma \colon \mathcal{C} \to \mathcal{C}^{\Sigma} \to \mathcal{C}$$

$$\Omega \colon \mathcal{C} \to \mathcal{C}^{\Omega} \to \mathcal{C}$$

Theorem 40. *Let $\mathcal{C}$ be a pointed ∞-category with finite (co)limits. The following are equivalent:*

(1) A triangle is exact if and only if it is coexact.
(2) The functors Σ and Ω are equivalences and the inverses of each other.
(3) A square is a pullback square if and only if it is a pushout square.

Definition 41. A pointed ∞-category $\mathcal{C}$ is *stable* if it satisfies one of the three equivalent conditions above.

Recall that before the rise of ∞-categories, *triangulated categories* were used to study stable homotopy theory. Hence, it is unsurprising that we can relate stable ∞-categories to triangulated categories.

Proposition 42. *If $\mathcal{C}$ is a stable ∞-category, then the homotopy category $h\mathcal{C}$ is a triangulated category.*

Of course arbitrary pointed ∞-categories will not be stable. We hence want a procedure that stabilizes them. There are several approaches. One approach, that is powerful from a theoretical perspective, is given via reduced 1-excisive functors out of finite pointed spaces. Here, we focus on explicit spectrum objects, as there are characterized more explicitly. For a comparison of these two approaches see [Lur17].

Definition 43. Let $\mathcal{C}$ be a pointed ∞-category. A *pre-spectrum object* in $\mathcal{C}$, is a functor $X \colon \mathbb{Z} \times \mathbb{Z} \to \mathcal{C}$ such that $X(i, j) = 0$ for $i \neq j$ and all squares are pushout squares. Let $PSp(\mathcal{C})$ be the ∞-category of pre-spectrum objects in $\mathcal{C}$.

For a given pre-spectrum object X, let $\alpha_{m-1} \colon \Sigma X_{m-1} \to X_m$ and $\beta_m \colon X_m \to \Omega X_{m+1} = \Omega\Sigma X_m$.

Definition 44. Let $\mathcal{C}$ be a pointed ∞-category. A *spectrum object* in $\mathcal{C}$ is a pre-spectrum object in X, such that α_{m-1} and β_m are equivalences for all m. Let $\mathcal{S}\mathrm{p}(\mathcal{C})$ be the ∞-category of spectrum objects in $\mathcal{C}$.

Definition 45. Let $\mathcal{C}$ be a pointed ∞-category. The stabilization of $\mathcal{C}$ is the ∞-category $\mathcal{S}\mathrm{p}(\mathcal{C})$ of spectrum objects in $\mathcal{C}$.

Of course $\mathcal{C}$ and $\mathcal{S}\mathrm{p}(\mathcal{C})$ are suitably related.

Theorem 46. *For a given pointed ∞-category $\mathcal{C}$, there is an adjunction*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\Sigma"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathrm{p}(\mathcal{C}) \arrow[l, shift left, "\Omega"] \end{tikzcd}$$

Moreover, $\mathcal{S}\mathrm{p}(\mathcal{C})$ is in some sense the universal stabilization of $\mathcal{C}$.

Theorem 47. *Let $\mathcal{C}$ be a pointed ∞-category and $\mathcal{D}$ a stable ∞-category. Then Σ^{∞} induces an equivalence of ∞-categories*

$$(\Sigma^{\infty})^* \colon \mathrm{Fun}^L(\mathcal{S}\mathrm{p}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$$

Let us now focus on the case $\mathcal{C} = \mathcal{S}_*$.

Example 48. The stabilization of $\mathcal{S}_*$ is the ∞-category of spectra, denoted $\mathcal{S}\mathrm{p}$.

Similar to $\mathcal{S}_*$, $\mathcal{S}\mathrm{p}$ is also the universal stable ∞-category, as a special instance of the result above.

Theorem 49. *If $\mathcal{D}$ is a stable ∞-category. Then the functor*

$$\mathrm{ev}_{\mathbb{S}} \colon \mathrm{Fun}^L(\mathcal{S}\mathrm{p}, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}$$

that evaluates at $\mathbb{S}$ is an equivalence.

4. Generalized Cohomology Theories

Cohomology theories were traditionally defined in the context of topological spaces. However, now that we have the tools of ∞-categories and stable ∞-categories. We can significantly generalize those definitions. This last result follows work in [Lur17].

Definition 50. Let $\mathcal{C}$ be a pointed ∞-category with pushouts, and $\Sigma \colon \mathcal{C} \to \mathcal{C}$ and suspension functor. A *generalized cohomology theory* is a functor $H \colon h\mathcal{C}^{op} \to \mathcal{A}\mathrm{b}_{\mathbb{Z}}$, such that the following conditions hold:

- There is a natural isomorphism $H^\bullet \to H^{\bullet+1}\Sigma$
- Coexact sequences maps to exact sequences.
- Arbitrary coproducts map to products.

We now have the following major result that significantly generalizes the classical Brown representability theorem.

Theorem 51. *Let $\mathcal{C}$ be a nice ∞-category and $(H^\bullet, \delta)$ be a generalized cohomology theory. Then there exists a spectrum object E in $\mathcal{C}$, such that $H^\bullet(X) \cong \mathrm{Hom}_{h\mathcal{C}}(X, E^\bullet)$, where $\delta = (\beta_\bullet)_*$.*

Example 52. Unsurprisingly, $\mathcal{S}_*$ satisfies the niceness conditions, and so we can conclude that every generalized cohomology on $\mathcal{S}_*$ is given by a spectrum, recovering the original Brown representability theorem.

References

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