

## DIFFERENTIAL COHOMOLOGY SEMINAR 8

TALK BY HANNES BERKENHAGEN

The aim of this talk is to review twisted cohomology theory with the aim of later discussing twisted differential cohomology theories [BG16]. For these talks the main source is [ABG<sup>+</sup>14, ABG18].

### 1. TWISTED COHOMOLOGY

Let  $R$  be a ring spectrum, meaning a monoid object in the  $\infty$ -category of spectra  $\mathcal{S}p$ . From this we get a presentable stable  $\infty$ -category  $\mathcal{M}od_R$  of left  $R$ -module spectra. Objects therein are morphisms of the form  $R \wedge M \rightarrow M$  satisfying the usual associativity and unit conditions up to coherent homotopies.

Note we have an adjunction diagram

$$\begin{array}{ccc} \mathcal{S}p & \xrightleftharpoons[\text{Hom}_R(R, -)]{R \wedge -} & \mathcal{M}od_R , \end{array}$$

where the right adjoint is in fact the forgetful functor. This in particular means  $\mathcal{M}od_R$  has a distinguished object  $R$  i.e. the free  $R$ -module of rank 1. We now refine these constructions.

**Definition 1.1.** Let  $R$  be a ring spectrum. An  $R$ -line is an  $R$ -module  $L$  such that  $L \simeq R$ .

**Definition 1.2.** Let  $\mathcal{L}ine_R$  be the full sub- $\infty$ -groupoid of  $\mathcal{M}od_R$  spanned by the  $R$ -lines.

By construction,  $\mathcal{L}ine_R$  is equivalent to the category with a single object  $R$  and hom-space  $GL_1 R$ , the  $\infty$ -group of  $R$ -linear automorphisms of  $R$ . Notice that  $GL_1(R) \subseteq \text{Hom}_{\mathcal{M}od_R}(R, R) \simeq \text{Hom}(\mathbb{S}, R) = \Omega^\infty R$ .

**Lemma 1.3.**  $GL_1(R)$  fits into the pullback square

$$\begin{array}{ccc} GL_1(R) & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ \pi_0(R)^\times & \longrightarrow & \pi_0(R) \end{array}$$

In particular, the inclusion  $GL_1(R) \rightarrow \Omega^\infty R$  induces an isomorphism on  $n$ -homotopy groups, for all  $n \geq 1$ .

*Proof.*  $\pi_0(R) \simeq \text{Hom}_{\text{ho}\mathcal{M}od_R}(R, R)$ , where  $\text{ho}\mathcal{M}od_R$  is the homotopy category of  $R$ -modules, the right-vertical arrow corresponds to  $\text{Hom}_{\mathcal{M}od_R}(R, R) \rightarrow \text{Hom}_{\text{ho}\mathcal{M}od_R}(R, R)$  mapping a morphism to its homotopy class, and  $\pi_0(R)^\times \subseteq \text{Hom}_{\text{ho}\mathcal{M}od_R}(R, R)$  is the set of isomorphisms. Finally, a morphism in a  $\infty$ -category  $\mathcal{C}$  is an equivalence if and only if its homotopy class is an isomorphism in  $\text{ho}\mathcal{C}$ .  $\square$

**Definition 1.4.** Let  $X$  be a space. Denote by  $\mathcal{M}od_R(X)$  the  $\infty$ -category of  $R$ -module spectra parametrized over  $X$ , i.e. the functor category  $\text{Fun}(X^{op}, \mathcal{M}od_R)$ .

**Definition 1.5.** Let  $X$  be a space. Denote by  $\mathcal{L}ine_R(X)$  the  $\infty$ -groupoid of  $R$ -line spectra parametrized over  $X$ , i.e. the  $\text{Fun}(X^{op}, \mathcal{L}ine_R)$ .

Since  $\mathcal{L}ine_R \simeq BGL_1(R)$ , functors  $X^{op} \rightarrow \mathcal{L}ine_R$  are generalization of local systems.

**Example 1.6.** Let  $R_X: X^{op} \rightarrow * \rightarrow \mathcal{L}ine_R$  be the constant functor with value  $R$ .

We now proceed to the Thom construction.

**Definition 1.7** (Thom spectrum). The *Thom R-module spectrum* is the functor

$$M: \mathcal{S}rp\mathcal{d}_\infty^{op}/\mathcal{L}ine_R \rightarrow \mathcal{M}od_R,$$

which sends  $f: X^{op} \rightarrow \mathcal{L}ine_R$  to the  $R$ -module spectrum  $\text{colim}(X^{op} \xrightarrow{f} \mathcal{L}ine_R \xrightarrow{i} \mathcal{M}od_R)$ .

*Remark 1.8.* Notice that the definition of Thom  $R$ -module spectrum make sense for any functor  $X^{op} \rightarrow R\text{Mod}$ .

Let us note an alternative characterization that will be important later.

*Remark 1.9.* For a given map  $f: X \rightarrow Y$ , we get a map  $f^*: \text{Mod}_R(Y) \rightarrow \text{Mod}_R(X)$  by precomposition with  $f^{op}$ . Since  $f^*$  preserves both limits and colimits, we construct a left adjoint  $f_!: \text{Mod}_R(X) \rightarrow \text{Mod}_R(Y)$  and a right adjoint  $f_*: \text{Mod}_R(X) \rightarrow \text{Mod}_R(Y)$ , using left and right Kan extension along  $f^{op}$ . Let  $f = p: X \rightarrow *$  be the terminal functor, then left Kan extension along  $p^{op}$  is exactly taking colimit, therefore

$$Mf \simeq p_!(i \circ f).$$

*Remark 1.10.* The equivalence  $\mathcal{L}\text{ine}_R \simeq BGL_1(R)$  comes from the fact that  $\mathcal{L}\text{ine}_R$  is a pointed (by  $R: * \rightarrow \mathcal{L}\text{ine}_R$ ), connected  $\infty$ -groupoid and  $* \times_{\mathcal{L}\text{ine}_R} * \simeq \Omega\mathcal{L}\text{ine}_R \simeq GL_1(R)$ .

We now proceed to define twisted cohomology theories.

**Definition 1.11** (Twisted cohomology). Let  $R$  be a ring spectrum,  $X$  be a space,  $p: X \rightarrow *$  the terminal functor, and  $f: X^{op} \rightarrow \mathcal{L}\text{ine}_R$  a  $R$ -line bundle over  $X$ . The  $f$ -twisted  $R$ -cohomology of  $X$  is defined as the mapping spectrum

$$R_f(X) := \text{Map}_{\text{Mod}_R}(Mf, R) \simeq \text{Map}_{\text{Mod}_R(X)}(f, p^*R) \simeq \text{Map}_{\text{Mod}_R(X)}(f, R_X).$$

Similarly, the  $f$ -twisted  $R$ -homology of  $X$  is defined as

$$R^f(X) := \text{Map}_{\text{Mod}_R}(R, Mf) \simeq Mf.$$

The  $f$ -twisted  $R$ -cohomology groups of  $X$  are defined as the homotopy groups of  $R_f(X)$ , i.e.

$$R_f^n(X) := \pi_0(\text{Map}_{\text{Mod}_R}(Mf, \Sigma^n R)) \cong \pi_{-n}(\text{Map}_{\text{Mod}_R(X)}(f, R_X)).$$

Similarly, the  $f$ -twisted  $R$ -homology groups of  $X$  are defined as the homotopy groups of  $R^f(X)$ , i.e.

$$R_n^f(X) := \pi_0(\text{Map}_{\text{Mod}_R}(\Sigma^n R, Mf)) \cong \pi_n(Mf).$$

**Example 1.12** (Trivial twist). If  $f: X \rightarrow \mathcal{L}\text{ine}_R$  factors through  $*$ , then  $f$  factors as the  $X^{op} \rightarrow * \rightarrow S$ , the constant factor with value  $*$ , and  $R \wedge \Sigma_+^\infty(-): S \rightarrow \text{Mod}_R$ . The latter functor commutes with colimits, being a left adjoint, while the colimit of the latter is  $X$  itself, then  $Mf \simeq R \wedge \Sigma_+^\infty X$ . In particular,  $f$ -twisted  $R$ -cohomology and  $R$ -homology of  $X$  reduce to ordinary (untwisted)  $R$ -cohomology and  $R$ -homology of  $X$ .

**Definition 1.13.** Given a vector bundle  $\pi: E \rightarrow B$ , define the *Thom space* of  $\pi$ , denoted  $\text{Th}(E)$ , to be the homotopy cofiber of  $E_0 \subseteq E$ , where  $E_0$  is the complement of the zero section.

## 2. EXAMPLES OF TWISTED COHOMOLOGY

We now proceed to analyze several examples of twisted cohomology theories. This requires some preliminary lemmas.

**Lemma 2.1.** Consider a space  $X$  and the  $\infty$ -categorical Yoneda's embedding  $y: X \rightarrow \text{Fun}(X^{op}, S)$ . The colimit of  $y$  is the terminal pre-sheaf on  $X$ , i.e. the pre-sheaf with constant value the one-point space.

*Proof.* Let  $S$  be a pre-sheaf on  $X$ , consider then the slice category  $X/S$  of pairs  $(x, \phi)$ , where  $x$  is an object of  $X$  and  $\phi: y(x) \rightarrow S$ . The density theorem for  $\infty$ -categories states that  $S$  is equivalent to the colimit of  $X/S \rightarrow X \xrightarrow{y} \text{Fun}(X^{op}, S)$ , the first map being the canonical projection. Take  $S = *$ , then  $X/_* \rightarrow X$  is an equivalence, hence the claim.  $\square$

Let  $G$  be a topological group and  $BG$  the  $\infty$ -groupoid with a single object  $1$  and hom-space  $G$ . The category  $\mathcal{S}_G := \text{Fun}(BG, S)$  is equivalent to the category of  $G$ -spaces.

**Lemma 2.2.** Consider  $X = BG$ , a  $G$ -space  $f: X \rightarrow S$  and its left Kan extension  $f_!: \mathcal{S}_G \rightarrow S$ , then  $f_! \simeq (- \times E)/G$ , where  $E = f(1)$ .

*Proof.* Evaluate at  $1$ , then  $f_!(y(1)) = E$ , by definition, and  $y(1) \simeq G$ , as  $G$ -spaces, hence  $(y(1) \times E)/G \simeq (G \times E)/G \simeq E$ . Since  $f_!$  and  $(- \times E)/G$  agree on representables and are colimit-preserving, they are equivalent.  $\square$

**Example 2.3.** Take the space  $BO(n)$  and  $f_n : BO(n) \rightarrow \mathcal{S}_*$  the  $n$ -sphere  $S^n$  with  $O(n)$ -action coming from the one-point compactification of the regular action on  $\mathbb{R}^n$ . Let  $\alpha_n = \Sigma^{\infty-n} f_n : BO(n) \rightarrow \mathcal{S}\mathbf{p}$ , then  $\alpha_n(1) = \Sigma^{\infty-n} f_n(1) = \Sigma^{\infty-n} S^n \simeq \mathbb{S}$ , so  $\alpha_n$  factors through  $\mathcal{L}\text{ines}_{\mathbb{S}}$ . Let  $X = BO(n)^{op}$  and  $p : X \rightarrow *$  the terminal functor, then

$$M\alpha_n = p_! \Sigma^{\infty-n} f_n \simeq \Sigma^{\infty-n} p_!(f_n)_! y \simeq \Sigma^{\infty-n} (f_n)_! \underbrace{p_!(y)}_{\simeq *} \simeq \Sigma^{\infty-n} (* \times S^n) / O(n)$$

Let  $P = EO(n)$  be the universal  $O(n)$ -bundle and  $M$  a  $O(n)$ -space, then  $* \times_{O(n)} M$  is modelled by the *strict* quotient  $(P \times M) / O(n)$ , then

$$S^n / O(n) = \text{cofib}(\mathbb{R}_0^n \subseteq \mathbb{R}^n) / O(n) \simeq \text{cofib}(\underbrace{* \times_{O(n)} \mathbb{R}_0^n}_{\simeq E_0^n} \subseteq \underbrace{* \times_{O(n)} \mathbb{R}^n}_{\simeq E^n}) = \text{Th}(E^n)$$

where  $E^n = P \times_{O(n)} \mathbb{R}^n \rightarrow BO(n)$  is the universal  $n$ -dimensional vector bundle, hence  $M\alpha_n \simeq \Sigma^{\infty-n} \text{Th}(E^n)$ .

The functor  $BO(n) \rightarrow \mathcal{L}\text{ines}_{\mathbb{S}}$  induces a  $\infty$ -group homomorphism  $j_n : O(n) \rightarrow GL_1(\mathbb{S})$ , mapping  $\phi$  to  $\Sigma^{\infty-n} \text{Th}(\phi)$ . Consider the suspension morphism  $s_n = \mathbb{R} \oplus - : O(n) \rightarrow O(1+n)$ , then

$$j_n(\mathbb{R} \oplus \phi) = \Sigma^{\infty-n-1} \text{Th}(\mathbb{R} \oplus \phi) \simeq \Sigma^{\infty-n-1} \underbrace{\text{Th}(\mathbb{R}) \wedge \text{Th}(\phi)}_{\simeq S^1} \simeq \Sigma^{\infty-n} \text{Th}(\phi) = j_n$$

Recall that the colimit over the suspension morphisms  $s_n$  is the stable orthogonal group  $O$ .

**Definition 2.4.** Denote by  $j$  the induced group homomorphism  $O \rightarrow GL_1(\mathbb{S})$ , called the *J-homomorphism*.

**Example 2.5.** Let  $X = O^{op}$  and take  $Bj : BO \rightarrow \mathcal{L}\text{ines}_{\mathbb{S}}$ , then  $Mj$  is denoted  $MO$  and called the *real bordism spectrum*.

Denote by  $M$  the extended Thom spectrum functor  $\mathcal{G}\text{rp}_{\infty}^{op}/\mathcal{M}\text{od}_R \rightarrow \mathcal{M}\text{od}_R$ , this is a left adjoint to the functor  $\mathcal{O}$  sending a  $R$ -module to the functor  $* \rightarrow \mathcal{M}\text{od}_R$  picking out  $M$ . In particular,  $M$  preserves colimits and  $Bj \simeq \text{colim}_n Bj_n$ , therefore we have the following:

**Theorem 2.6.**  $MO \simeq \text{colim}_n MO(n) = \text{colim}_n \Sigma^{\infty-n} \text{Th}(E^n)$ .

**Example 2.7.** A group homomorphism  $\xi : G \rightarrow O$  induces a functor  $f : BG \rightarrow \mathcal{L}\text{ines}_{\mathbb{S}}$ . The Thom spectrum  $Mf$  is denoted  $MG$  or  $M\xi$ , and called *G-bordism spectrum*. For  $G = U, SO, Spin$ , and *String*, we obtain the *complex, oriented, spin, and string bordism spectra*.

**Remark 2.8.** In [Example 2.7](#) we might take  $G = \{*\}$ , the one-point group, then  $MG \simeq \mathbb{S}$ , which, if it didn't have a name, might be called *framed bordism spectrum*, following the naming convention in [Example 2.7](#) and in line with the theorem that  $\pi_*(\mathbb{S}) \simeq \Omega_*^{\text{fr}}$ , the bordism ring of framed (trivialized tangent bundle) smooth manifolds.

Let  $R$  be a ring in sets, then  $R$  is a  $A_{\infty}$ -ring spectrum (actually,  $E_{\infty}$ ),  $\Omega^{\infty} R$  is equivalent to  $R$  with discrete topology ( $\pi_0(\Omega^{\infty} R) \simeq R$ , as sets, and every other homotopy group vanish). In particular,  $GL_1(R)$  is simply  $R^{\times}$  with discrete topology. Consider then the fiber sequence  $SO \rightarrow O \rightarrow \mathbb{Z}^{\times} \simeq GL_1(\mathbb{Z})$ .

**Example 2.9.**  $X = BO^{op}$  and  $\alpha = w_1 : BO \rightarrow \mathcal{L}\text{ine}_{\mathbb{Z}}$ , the 1st Stiefel-Whitney class (delooping of the determinant  $O \rightarrow GL_1(\mathbb{Z})$ ), then  $Mw_1$  is a  $\mathbb{Z}$ -module spectra. Let  $i : SO \subseteq O$ , then  $w_1 i$  factors through the point, so  $M(w_1 i) \simeq \mathbb{Z} \wedge \Sigma_+^{\infty} SO$ .

Given  $f : X^{op} \rightarrow \mathcal{L}\text{ine}_R$  and a sequence  $F \xrightarrow{i} X \xrightarrow{\pi} Y$ , there is an induced sequence of Thom  $R$ -module spectra  $MF \rightarrow MX \rightarrow MY$ . If  $\pi i$  factors through the point,  $MF \simeq R \wedge \Sigma_+^{\infty} F$ .

**Lemma 2.10.** Let  $R$  be a ring spectrum and  $X$  a connected monoidal  $\infty$ -groupoid, then

$$\text{Hom}_{\text{Mon}(\mathcal{S}\mathbf{p})}(\Sigma_+^{\infty} X, R) \simeq \text{Hom}_{\text{Mon}(\mathbb{S})}(X, GL_1(R))$$

*Proof.* Since  $X$  is connected, the space of homomorphisms  $X \rightarrow GL_1(R)$  is equivalent to the space of homomorphisms  $X \rightarrow \Omega^{\infty} R$ , then use that  $(\Sigma^{\infty}, \Omega^{\infty})$  is a monoidal adjunction (The monoidal structure on spectra is such that  $\Sigma^{\infty}$  is strong monoidal).  $\square$

*Remark 2.11.* Notice that we can weaken the result. Namely, if  $X$  is 1-connected (pointed and connected), then the space of functors (of  $\infty$ -groupoids)  $X \rightarrow GL_1(R)$  is equivalent to the space of functors  $X \rightarrow \Omega^\infty R$  such that  $* \rightarrow X \rightarrow \Omega^\infty R$  is an equivalence. This last space is equivalent, via the  $(\Sigma_+^\infty, \Omega^\infty)$  adjunction, to the space of morphisms of spectra  $\Sigma_+^\infty X \rightarrow R$ , such that  $\mathbb{S} \rightarrow \Sigma_+^\infty X \rightarrow R$  represents a unit in  $\pi_0(R)$ .

*Remark 2.12.* Notice that we can also strengthen the result. Namely, if  $X$  is a connected, commutative monoid object and  $R$  is a commutative ring spectrum, then  $\Omega^\infty R$  and  $GL_1(R)$  are also commutative monoid objects. Using the same argument, together with the fact that  $(\Sigma^\infty, \Omega^\infty)$  is actually a *symmetric* monoidal adjunction, we conclude that

$$\mathrm{Hom}_{\mathrm{CMon}(\mathrm{Sp})}(\Sigma_+^\infty X, R) \simeq \mathrm{Hom}_{\mathrm{CMon}(\mathcal{S})}(X, GL_1(R))$$

*Remark 2.13.* Let  $\mathcal{D}$  be a monoidal  $\infty$ -category. Consider  $\mathrm{Cat}_\infty/\mathcal{D}$ , the  $\infty$ -category of functors into  $\mathcal{D}$ , with monoidal structure given by

$$(F : \mathcal{A} \rightarrow \mathcal{D}, G : \mathcal{B} \rightarrow \mathcal{D}) \longmapsto (\mathcal{A} \times \mathcal{B} \xrightarrow{F \times G} \mathcal{D} \times \mathcal{D} \xrightarrow{\otimes} \mathcal{D})$$

The monoidal unit is the functor  $* \rightarrow \mathcal{D}$  picking out the monoidal unit of  $\mathcal{D}$ . If  $\mathcal{D}$  is symmetric monoidal, then so is  $\mathrm{Cat}_\infty/\mathcal{D}$ . A (commutative) monoid object in  $\mathrm{Cat}_\infty/\mathcal{D}$  is given by a (symmetric) monoidal category  $\mathcal{C}$  and a (symmetric) monoidal functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ .

In view of [Remark 2.13](#), let  $R$  be commutative ring spectrum, then  $\mathrm{Mod}_R$  is a symmetric monoidal  $\infty$ -category and  $\mathcal{L}\mathrm{ine}_R$  is a symmetric monoidal  $\infty$ -groupoid. The category  $\mathrm{Grpd}_\infty^{op}/\mathcal{L}\mathrm{ine}_R$  is then symmetric monoidal and (commutative) monoid objects are given by (symmetric) monoidal  $\infty$ -groupoids  $X^{op}$  a (symmetric) monoidal functors  $X^{op} \rightarrow \mathcal{L}\mathrm{ine}_R$ . One can then check that  $M$  is a symmetric monoidal functor, so that (commutative) monoid objects are sent to (commutative) monoid objects in  $\mathrm{Mod}_R$ , i.e. (commutative)  $R$ -algebras.

**Example 2.14.** Let  $\mathrm{Tmf}$  be the commutative ring spectrum of topological modular forms (see [Remark 2.15](#)) and  $\sigma : MString \rightarrow \mathrm{tmf}$  the *String*-orientation of  $\mathrm{tmf}$ . In the sequence

$$BString \longrightarrow BO \xrightarrow{Bj} \mathcal{L}\mathrm{ine}_S$$

all functors are symmetric monoidal, so that  $MString$  is a commutative  $\mathbb{S}$ -algebra, i.e. a commutative ring. The *String*-orientation of  $\mathrm{tmf}$  is also a commutative ring homomorphism. In the fiber sequence  $K(\mathbb{Z}, 3) \rightarrow BString \rightarrow BSO$ , the fiber map  $i : K(\mathbb{Z}, 3) \rightarrow BString$  is also a symmetric monoidal, so the composition

$$\Sigma_+^\infty K(\mathbb{Z}, 3) \xrightarrow{Mi} MString \xrightarrow{\sigma} \mathrm{tmf}$$

is a commutative ring homomorphism. Using [Lemma 2.10](#), we conclude that the induced homomorphism  $K(\mathbb{Z}, 3) \rightarrow \Omega^\infty \mathrm{tmf}$  (which is a homomorphism of commutative monoid objects, given [Remark 2.12](#)) factors through  $GL_1(\mathrm{tmf})$ , and so it induces a (symmetric monoidal) functor  $K(\mathbb{Z}, 4) \rightarrow \mathcal{L}\mathrm{ine}_{\mathrm{tmf}}$ , i.e. *2-bundle gerbes twist tmf*.

*Remark 2.15.* The spectrum of topological modular forms comes in three main flavors, namely:

- (1) TMF, i.e. the global sections of the spectral structure sheaf  $\mathcal{O}^{top} : (\mathrm{Aff}/\mathcal{M}_{ell})^{op} \rightarrow \mathrm{CMon}(\mathrm{Sp})$  on the (étale site of the) moduli stack of elliptic curves.
- (2) Tmf, i.e. the global sections of the spectral structure sheaf  $\bar{\mathcal{O}}^{top} : (\mathrm{Aff}/\bar{\mathcal{M}}_{ell})^{op} \rightarrow \mathrm{CMon}(\mathrm{Sp})$  on the (étale site of the) *compactified* moduli stack of elliptic curves. The inclusion  $\mathcal{M}_{ell} \hookrightarrow \bar{\mathcal{M}}_{ell}$  induces a commutative ring homomorphism  $\mathrm{Tmf} \rightarrow \mathrm{TMF}$ .
- (3) tmf, i.e. the connective cover of Tmf. By definition, there is a commutative ring homomorphism  $\mathrm{tmf} \rightarrow \mathrm{Tmf}$ .

In [\[AHR10\]](#), tmf is used to denote our TMF, in [\[Goe09\]](#), tmf is used to denote our Tmf, and [\[DFHH14\]](#) has the same notation as us. In [Example 2.14](#), we use tmf to mean the connective cover of Tmf.

Let us go down one step in the chromatic ladder.

**Example 2.16.** Recall the fiber sequences for *Spin* and *Spin*<sup>c</sup>:

$$\mathbb{Z}_2 \rightarrow Spin \rightarrow SO, \quad S^1 \rightarrow Spin^c \rightarrow SO$$

All the spaces involved are commutative groups. Applying the Thom spectrum functor to the delooped sequences, we get

$$\Sigma_+^\infty K(\mathbb{Z}_2, 1) \rightarrow MSpin \rightarrow MSO, \quad \Sigma_+^\infty K(\mathbb{Z}, 2) \rightarrow MSpin^c \rightarrow MSO$$

Let  $\sigma : MSpin \rightarrow KO$  and  $\sigma^c : MSpin^c \rightarrow KU$  be the Atiyah-Bott-Shapiro orientation of real and complex  $K$ -theory (see [ABS64]). Similar to [Example 2.14](#), we get homomorphisms

$$\Sigma_+^\infty K(\mathbb{Z}_2, 1) \longrightarrow MSpin \xrightarrow{\sigma} KO, \quad \Sigma_+^\infty K(\mathbb{Z}, 2) \longrightarrow MSpin^c \xrightarrow{\sigma^c} KU$$

Using [Lemma 2.10](#) again and delooping, we obtain functors  $K(\mathbb{Z}_2, 2) \rightarrow \mathcal{L}\text{ine}_{KO}$  and  $K(\mathbb{Z}, 3) \rightarrow \mathcal{L}\text{ine}_{KU}$ , i.e. *real, resp. complex, bundle gerbes twist real, resp. complex,  $K$ -theory*.

### 3. TWISTS VIA PICARD GROUPOIDS AND GRADING

This section requires some further details. Up until now we defined everything via  $\mathcal{L}\text{ine}_R$ , however for many applications we need to work with  $\mathcal{P}\text{ic}_R$  instead.

**Definition 3.1.** Given a monoidal  $\infty$ -category  $(\mathcal{C}, \otimes, 1)$ , an object  $M$  is *invertible* if there is an object  $D$  such that  $D \otimes M \simeq M \otimes D \simeq 1$ . The *Picard  $\infty$ -groupoid* of  $\mathcal{C}$  is the sub- $\infty$ -groupoid generated by invertible modules.

**Definition 3.2.** If  $R$  is a ring spectrum,  $\mathcal{M}\text{od}_R$  is monoidal. Denote by  $\mathcal{P}\text{ic}_R$  the Picard groupoid of  $\mathcal{M}\text{od}_R$ .

**Remark 3.3.**  $\mathcal{P}\text{ic}_R$  splits as the disjoint union of  $\pi_0(\mathcal{P}\text{ic}_R)$ -many sub-groupoids. Moreover, if  $M \simeq N$ , then  $R \simeq M^{-1} \otimes N$ , so  $M^{-1} \otimes N$  is a  $R$ -line. In particular, every connected component of  $\mathcal{P}\text{ic}_R$  is equivalent to  $\mathcal{L}\text{ine}_R$ , so  $\mathcal{P}\text{ic}_R \simeq \pi_0(\mathcal{P}\text{ic}_R) \times \mathcal{L}\text{ine}_R$ . However, this is not a monoidal equivalence for general ring spectra.

**Remark 3.4.**  $\Sigma^n R$  is invertible, with inverse  $\Sigma^{-n} R$ . In particular, there is a map  $\mathbb{Z} \times \mathcal{L}\text{ine}_R \rightarrow \mathcal{P}\text{ic}_R$ . However, this map need not be neither injective (if  $R$  is  $n$ -periodic), nor surjective (see [HM17]).

As mentioned in [Remark 1.8](#), the Thom spectrum functor makes sense for every functor  $f : X^{op} \rightarrow \mathcal{M}\text{od}_R$ . However, all examples of twists encountered so far came from functors into  $\mathcal{L}\text{ine}_R$ . An example of twist that is not the result of a  $R$ -line bundle is the *degree shift*.

**Definition 3.5.** Denote by  $M$  the *Thom  $R$ -module spectrum* functor

$$\mathcal{G}\text{rp}_\infty^{op}/\mathcal{M}\text{od}_R \rightarrow \mathcal{M}\text{od}_R$$

sending a functor  $f : X^{op} \rightarrow \mathcal{M}\text{od}_R$  to its colimit.

**Example 3.6.** Let  $f : X^{op} \rightarrow \mathcal{L}\text{ine}_R$  be a twist. Denote by  $\Sigma^n f$  the composition of  $f$  with the shift functor  $\Sigma^n : \mathcal{L}\text{ine}_R \rightarrow \mathcal{P}\text{ic}_R$ . Since  $\Sigma^n$  is an equivalence, it commutes with colimits, so

$$M\Sigma^n f \simeq \Sigma^n Mf$$

If  $f = R_X$ , then  $M\Sigma^n f \simeq \Sigma^n R \wedge \Sigma_+^\infty X$ , so  $\Sigma^n f$ -twisted  $R$ -cohomology and  $R$ -homology correspond to normal  $R$ -cohomology and  $R$ -homology with a degree shift by  $n$ .

### 4. UMKEHR MAP

We now proceed to the construction of the umkehr map in twisted cohomology theories. Here we follow [ABG18]

**Remark 4.1.** Depending on which cohomological degrees we want, we sometimes use the following alternative definition of twisted cohomology:

$$R^{*+\alpha}(X) = \pi_* \text{Map}_R(M(\alpha^{-1}), R)$$

meaning we use the inverse of  $\alpha$  to get a more standard degree convention at the level of actual cohomology groups.

For  $X$  a space, we will write

$$DX := \text{Map}(\Sigma_+^\infty X, \mathbb{S})$$

for the Spanier-Whitehead dual of  $X$ . Notice  $D$  is evidently given as the composition functor

$$D : \mathcal{S}^{op} \xrightarrow{\Sigma_+^\infty} \mathcal{S}\text{p}^{op} \xrightarrow{\text{Map}(-, \mathbb{S})} \mathcal{S}\text{p}.$$

Applying  $D$  to the unique map  $X \rightarrow *$  gives a map of spectra

$$\phi(X): \mathbb{S} \rightarrow DX$$

and we regard this as a functor

$$\phi: \mathcal{S}^{op} \rightarrow \mathcal{S}\text{p}_{\mathbb{S}/}$$

i.e. the induced functor on the slice category.

As part of this explanation it is helpful to recall the following facts.

*Remark 4.2.* Given a spectrum  $E$ , the mapping spectrum  $\text{Map}(\mathbb{S}, E)$  is equivalent to  $E$  itself. This is because  $\mathbb{S}$  is the unit for the smash product of spectra, and the mapping spectrum from the unit to any spectrum recovers that spectrum. In particular  $\text{Map}(\mathbb{S}, \mathbb{S}) \simeq \mathbb{S}$ .

Suppose we are given  $f: X \rightarrow B$ , a smooth and proper family of manifolds over  $B$ ,  $f^{-1}(b) = X_b$  is a smooth and proper manifold. Where proper means a compact closed manifold. This should vary continuously over  $B$  in the sense that  $f$  is classified by a functor

$$B \rightarrow \mathcal{Mfd}$$

where  $\mathcal{Mfd}$  is the  $\infty$ -category of smooth and proper manifolds, where the  $\infty$ -categorical structure is given by the usual topological enrichment. Notice it comes with an evident forgetful functor to  $\mathcal{S}$ , the  $\infty$ -category of spaces (i.e.  $\infty$ -groupoids).

Given such a functor  $F$  we have the composition functor

$$B^{op} \xrightarrow{F^{op}} \mathcal{Mfd}^{op} \rightarrow \mathcal{S}^{op} \xrightarrow{\phi} \mathcal{S}\text{p}_{\mathbb{S}/},$$

which is an object in  $\text{Fun}(B^{op}, \mathcal{S}\text{p}_{\mathbb{S}/})$ . Notice, there is an evident equivalence of  $\infty$ -categories

$$\text{Fun}(B^{op}, \mathcal{S}\text{p}_{\mathbb{S}/}) \simeq (\text{Fun}(B^{op}, \mathcal{S}))_{\mathbb{S}_B/}$$

where  $\mathbb{S}_B$  is the constant functor with value  $\mathbb{S}$ . Thus, our composition above, corresponds to a natural transformation

$$\phi_{X/B}: \mathbb{S}_B \rightarrow D_B(X)$$

This just unwinds to a natural map of spectra

$$\mathbb{S} \rightarrow D(X_b)$$

We can now take the left Kan extension along the terminal map  $p: B \rightarrow *$  (i.e. take the colimit of these diagrams), to get a map of spectra

$$p_! \phi_{X/B}: p_! \mathbb{S}_B \rightarrow p_! D_B(X)$$

Using the fact that

$$p_! \mathbb{S}_B \simeq p_! p^* \mathbb{S}_B \simeq \Sigma^\infty_+ B,$$

we get a map of spectra

$$\Sigma^\infty_+ B \rightarrow p_! D_B(X).$$

On the right hand side

$$p_! D_B(X) \simeq X^{-T_f} \simeq \Sigma^\infty Th(-T_f)$$

where  $T_f$  is the bundle of tangents along the fibers of  $f$  and  $X^{-T_f}$ , giving us the Thom spectrum. This last equivalence is a complicated computation.

*Remark 4.3.* Here argument involves proving that the classical Thom space  $X^{-T_f}$  is equivalent to the colimit of the functor  $D_B: B \rightarrow \mathcal{S}$

*Remark 4.4.* Let us justify this notation. Let  $X \rightarrow BO(d)$  be the classifying map classifying the tangent bundle (as we have fiber-wise tangent bundles, which assemble into a map). Then we get the induced map

$$X \rightarrow BO(d) \xrightarrow{J} BGL_1(\mathbb{S}) \rightarrow BGL_1(R) \rightarrow \mathcal{P}ic(R)$$

where  $J$  is the  $J$ -homomorphism, and here  $R$  is an arbitrary ring spectrum, which we want to use as our cohomology coefficients.

These computations give us the *Pontryagin-Thom transfer map*:

$$PT(f): \Sigma_+^\infty B \rightarrow X^{-T_f}.$$

Now, given a choice of  $R$ -orientation, we get a Thom isomorphism

$$R^*(\Sigma_+^\infty X) \xrightarrow{\sim} R^{*-d}(X^{-T_f})$$

We now have the following definition.

**Definition 4.5.** With the assumptions as above, the *Umkehr map* is the map:

$$R^*(\Sigma_+^\infty X) \simeq R^{*-d}(X^{-T_f}) \xrightarrow{PT(f)^{*,-d}} R^{*-d}(\Sigma_+^\infty B).$$

Here it is called the Umkehr map, as it goes the other way from the regular map  $R^*f: R^*B \rightarrow R^*X$ .

Recall here the following definition of orientation that we used here.

**Definition 4.6.** Given a twist  $\alpha: X \rightarrow \mathcal{L}\text{ine}_R$ , an orientation of  $X^\alpha$ , is a lift

$$\begin{array}{ccc} & \mathcal{T}\text{riv}_R & \\ & \nearrow & \downarrow \pi \\ X & \xrightarrow{\alpha} & \mathcal{L}\text{ine}_R \end{array}$$

We now define a twisted variant of the Umkehr map.

Given a twist  $\alpha: B \rightarrow \mathcal{P}\text{ic}_R$ , we can smash to get a map

$$\mathbb{S}_B \wedge_B \alpha \rightarrow D_B X \wedge \alpha$$

The left hand side will compute to  $B^\alpha$ . The right hand side is equivalent to  $X^{-T_f + \alpha f}$ , where  $\alpha f$  is the composition of  $\alpha$  and  $f: X \rightarrow B$ .

Now again assuming we have a chosen an  $R$ -orientation of  $T_f$ , we get a Thom isomorphism

$$R^{*+d}(X) \simeq R^*(X^{-T_f + \alpha f}),$$

which gives us

$$R^{*+d}(X) \simeq R^*(X^{-T_f + \alpha f}) \rightarrow R^*(B^\alpha) = R^{*-\alpha-d}(B)$$

This map is called the *twisted Umkehr map*.

Let us look at some examples. Assume the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{-T_f} & \mathcal{P}\text{ic}_{\mathbb{S}} \\ \downarrow & & \downarrow \\ B & \xrightarrow{\alpha} & \mathcal{P}\text{ic}_R \end{array}$$

Then we have  $X^{-T_f + \alpha f} \simeq \Sigma_+^\infty X \wedge R$ . Then we have an induced map  $B^\alpha \rightarrow \Sigma_+^\infty X \wedge R$ , but then applying  $R^*$  we get the Umkehr map

$$R^*(X) \rightarrow R^{*-\alpha}(B)$$

#### REFERENCES

- [ABG<sup>+</sup>14] Matthew Ando, Andrew J. Blumberg, David Gepner, Michael J. Hopkins, and Charles Rezk. An  $\infty$ -categorical approach to  $R$ -line bundles,  $R$ -module Thom spectra, and twisted  $R$ -homology. *J. Topol.*, 7(3):869–893, 2014.
- [ABG18] Matthew Ando, Andrew J. Blumberg, and David Gepner. Parametrized spectra, multiplicative Thom spectra and the twisted Umkehr map. *Geom. Topol.*, 22(7):3761–3825, 2018.
- [ABS64] M. F. Atiyah, R. Bott, and A. Shapiro. Clifford modules. *Topology*, 3(suppl):3–38, 1964.
- [AHR10] Matthew Ando, Michael J Hopkins, and Charles Rezk. Multiplicative orientations of  $KO$ -theory and of the spectrum of topological modular forms. *preprint*, 3:72, 2010.
- [BG16] Ulrich Bunke and David Gepner. Differential function spectra, the differential Becker-Gottlieb transfer, and applications to differential algebraic  $K$ -theory. [arXiv:1306.0247](#), 2016.
- [DFHH14] Christopher L Douglas, John Francis, André G Henriques, and Michael A Hill. *Topological modular forms*, volume 201. American Mathematical Soc., 2014.
- [Goe09] Paul G. Goerss. Topological modular forms (after Hopkins, Miller, and Lurie), 2009.
- [HM17] Michael Hill and Lennart Meier. The  $C_2$ -spectrum  $\mathrm{Tmf}_1(3)$  and its invertible modules. *Algebraic & Geometric Topology*, 17(4), 2017.