DIFFERENTIAL COHOMOLOGY SEMINAR 4 (DRAFT)

TALK BY ALESSANDRO NANTO

In the last talk we learned the definition of a differential cohomology theory, as a sheaf valued in spectra on the site of manifolds. This talk continues our journey through differential cohomology theories, and focuses on the following three topics:

(1) We want to learn how to construct non-trivial examples out of sheaves valued in chain complexes.
(2) We want to understand how we can extend classical cohomology operations to the setting of differential cohomology theories.
(3) We want to introduce suitable analogues of fiber-wise integration.

1. Abelian Groups, Spectra and the Heart

Let us start by reviewing the relation between abelian groups and spectra.

Definition 1. Let A be an abelian group. The *Eilenberg-MacLane spectrum* $H(A)$ is defined as the spectrum whose n-th space is the Eilenberg-MacLane space $H(A, n)$.

Eilenberg-MacLane spectra assemble into a functor $H\colon \mathcal{A}\mathrm{b} \to \mathcal{S}\mathrm{p}$. What can we say about the image of this functor?

Definition 2. Let X be a spectrum. The n-th *homotopy group* of X is defined as

$$\pi_n(X) = \mathrm{colim}_k(X)\pi_{n+k}(X_k).$$

Remark 3. Note if X is a space. Then the homotopy groups of ΣX are precisely the stable homotopy groups of X, which are well-defined by the Freudenthal suspension theorem.

We now have the following basic result.

Definition 4. Let $\mathcal{S}\mathrm{p}^\heartsuit$ be the full subcategory of $\mathcal{S}\mathrm{p}$ consisting of spectra X such that $\pi_n(X) = 0$ for all $n \neq 0$.

We now have the following theorem, relating abelian groups and the heart.

Theorem 5. *The functor $H\colon \mathcal{A}\mathrm{b} \to \mathcal{S}\mathrm{p}$ is fully faithful, and its essential image is $\mathcal{S}\mathrm{p}^\heartsuit$. The restricted equivalence $H\colon \mathcal{A}\mathrm{b} \to \mathcal{S}\mathrm{p}^\heartsuit$ has as inverse $\pi_0\colon \mathcal{S}\mathrm{p}^\heartsuit \to \mathcal{A}\mathrm{b}$, which sends a spectrum X to its 0-th homotopy group $\pi_0(X)$.*

Note H has several important properties.

Proposition 6. *The functor $H\colon (\mathcal{A}\mathrm{b}, \otimes, \mathbb{Z}) \to (\mathcal{S}\mathrm{p}, \wedge, \mathbb{S})$ is a symmetric monoidal functor.*

The monoidality has the following evident implication.

Corollary 7. *H sends rings to ring spectra, and modules over a ring to modules over a ring spectrum, meaning for a given ring R, there is a functor $H\colon \mathcal{M}\mathrm{od}_R \to \mathcal{M}\mathrm{od}_{H(R)}$.*

2. From Chain Complexes to Spectra via stable Dold-Kan

Let $\Omega^\bullet_{dR}\colon \mathcal{M}\mathrm{fd} \to \mathcal{C}\mathrm{h}(R - \mathcal{M}\mathrm{od})$ be the de Rham chain complex of a manifold, which is indeed a sheaf on the site of manifolds. Moreover, by Corollary 7, the ring map $\mathbb{Z} \to \mathbb{R}$, gives us a map of ring spectra $H\mathbb{Z} \to H\mathbb{R}$. Ideally Deligne cohomology should be characterized as the pullback of some sort of truncated deRham complex along the map $H\mathbb{Z} \to H\mathbb{R}$. This requires a precise definition of the spectrum associated to the k-truncated de Rham complex $\Omega^{\leq k}$. For this we use advanced result from stable homotopy theory.

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Theorem 8. *Let R be a ring. The functor $H\colon \mathcal{M}\mathrm{od}_R \to \mathcal{M}\mathrm{od}_{H(R)}$ lifts*

$$\begin{tikzcd}
\mathcal{M}\mathrm{od}_R \arrow[r, "H"] \arrow[d] & \mathcal{M}\mathrm{od}_{H(R)} \\
\mathcal{D}(R) \arrow[ur, dashed] &
\end{tikzcd}$$

where $\mathcal{D}(R)$ is the derived category of R-modules.

Recall that a $\mathbb{Z}$-module is just an abelian group. Hence, applying this result to $R = \mathbb{Z}$, we get the following corollary.

Corollary 9. *The functor $H\colon \mathcal{M}\mathrm{od}_{\mathbb{Z}} \to \mathcal{M}\mathrm{od}_{H\mathbb{Z}}$ lifts to a functor*

$$\begin{tikzcd}
\mathcal{M}\mathrm{od}_{\mathbb{Z}} \arrow[r, "H"] \arrow[d] & \mathcal{M}\mathrm{od}_{H\mathbb{Z}} \\
D(\mathbb{Z}) \arrow[ur, dashed, "DK_{st}"'] &
\end{tikzcd}$$

We call this lift $D(\mathbb{Z}) \to \mathcal{M}\mathrm{od}_{H\mathbb{Z}}$ the stable Dold-Kan correspondence*.*

One thing one might wonder is how this relates to the more classical Dold-Kan correspondence, which relates chain complexes of abelian groups to simplicial abelian groups. Let $\mathcal{C}\mathrm{h}^+$ be the category of bounded below chain complexes of abelian groups. The classical Dold-Kan correspondence gives us a functor

$$DK\colon \mathcal{C}\mathrm{h}^+ \to s\mathcal{A}\mathrm{b}$$

from bounded below chain complexes of abelian groups to simplicial abelian groups. However, every simplicial abelian group comes with an abelian group structure on a simplicial set, meaning it is in particular an E_∞-group in spaces. This means we have a functor

What do we know about the properties of this functor?

$$s\mathcal{A}\mathrm{b} \to \mathcal{G}\mathrm{rp}_{E_\infty}(\mathcal{S})$$

However, $\mathcal{G}\mathrm{rp}_{E_\infty}(\mathcal{S})$ fully faithfully embeds in $\mathcal{S}\mathrm{p}$ as connected spectra. Composing all these functors, we get a functor

$$DK\colon \mathcal{C}\mathrm{h}^+ \to \mathcal{S}\mathrm{p},$$

Why?

which is fully faithful and recovers the classical Dold-Kan correspondence. The stable Dold-Kan correspondence is a lift of this functor to $\mathcal{D}(\mathbb{Z})$ i.e.

$$\begin{tikzcd}
\mathcal{C}\mathrm{h}^+ \arrow[r, "DK"] \arrow[d, hook] & \mathcal{S}\mathrm{p} \\
D(\mathbb{Z}) \arrow[ur, "DK_{st}"'] &
\end{tikzcd}$$

How?

relates to the stable Dold-Kan correspondence. Finally, we can now use stable Dold-Kan to get a functor of sheaves.

Definition 10. Let

$$H\colon \mathcal{S}\mathrm{hv}(Mfd; \mathcal{D}(\mathbb{Z})) \to \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}; \mathcal{S}\mathrm{p})$$

denote the functor that post-composes a sheaf of chain complexes on manifolds with the stable Dold-Kan correspondence and then sheafifies. For a given sheaf of chain complexes F, we call the image the associated *Eilenberg-MacLane sheaf.*

Do we really need sheafification? This might need some checking.

3. Deligne Cohomology as a Differential Cohomology Theory

Now equipped with Definition 10, we can finally define Deligne cohomology as a differential cohomology theory.

Definition 11. Let $k \geq 0$. The *Deligne cohomology sheaf* $\mathcal{E}(k)$ is defined via the following pullback square in $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}; \mathcal{S}\mathrm{p})$:

$$\begin{array}{ccc} \mathcal{E}(k) & \longrightarrow & H(\Omega_{dR}^{\leq k}) \\ \downarrow & & \downarrow \\ H\mathbb{Z} & \longrightarrow & H\mathbb{R} \end{array}$$

Here H is the Eilenberg-MacLane sheaf.

Remark 12. If we take $k = \infty$, then the map $H(\Omega_{dR}) \to H\mathbb{R}$ is an equivalence, meaning $\mathcal{E}(\infty)$ is equivalent to $H\mathbb{Z}$ i.e. singular cohomology. On the other side, the individual $\mathcal{E}(k)$ are highly non-trivial and help classify many geometric invariants of interest (as we saw in the first talk). So, the $\mathcal{E}(k)$ are a non-trivial filtration of $H\mathbb{Z}$ by differential cohomology theories, in the sense that there are map $\mathcal{E}(k+1) \to \mathcal{E}(k)$, the limit of which is $H\mathbb{Z}$.

4. Cohomology Operations for Deligne Cohomology

Now that we have a rigorous definition of Deligne cohomology, we can start to think about operations on it. First of all, we need a suitable monoidal structure.

Definition 13. Let F, G be two differential cohomology theories. The *monoidal product* $F \otimes G$ is defined as the sheafification of the presheaf $F \wedge G$, which is the point-wise wedge product of spectra.

It is expected that sheafification is necessary, but example is missing.

Now, recall there is a map of differential forms

$$\Omega^{\leq k} \otimes \Omega^{\leq m} \to \Omega^{\leq k+m},$$

which induces a map of differential cohomology theories

$$\mathcal{E}(k) \otimes \mathcal{E}(m) \to \mathcal{E}(k+m)$$

Ideally, we would like to describe such an operation in a very explicit manner, however, in the realm of spectra this can be very challenging. This suggests an alternative perspective.

Definition 14. Let $\mathcal{L}(k)$ be the sheaf of chain complexes defined as the pullback in $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, D(\mathbb{Z}))$ of the following diagram

$$\begin{array}{ccc} \mathcal{L}(k) & \longrightarrow & \Omega^{\leq k} \\ \downarrow & & \downarrow{\scriptstyle dR} \\ \mathbb{Z} & \longrightarrow & \mathbb{R} \end{array},$$

where $\mathbb{Z}$ is the functor $M \mapsto C^\bullet(M, \mathbb{Z})$ and $\mathbb{R}$ is the functor $M \mapsto C^\bullet(M, \mathbb{R})$

Remark 15. We can explicitly describe the chain complex $\mathcal{L}(k)$ as follows.

$$\mathcal{L}(k)^n = \{(c, \omega, h) \in C^n(-\mathbb{Z}) \oplus \Omega^n(-) \oplus C^{n-1}(-\mathbb{R}) | \omega = 0 if n > k and c - dR(\omega) = dh\}$$

Remark 16. We expect that $H\mathcal{L}(k)$ in fact recovers $\mathcal{E}(k)$, meaning operations on $\mathcal{L}(k)$ help us understand operations on Deligne cohomology.

This needs to be checked.

Using the explicit description from Remark 15, we can define an operation on $\mathcal{L}(k)$ as follows:

$$(c_1, \omega_1, h_1) \otimes (c_2, \omega_2, h_2) = (c_1 \cup c_2, \omega_1 \wedge \omega_2, (-1)^{|c_1|} c_1 \cup h_2 + h_1 \cup \omega_2 + B(\omega_1, \omega_2)),$$

where

$$dR(\omega_1) \cup dR(\omega_2) = -dR(\omega_1 \wedge \omega_2) = dB(\omega_1, \omega_2)$$

Remark 17. Intuitively $B(\omega_1, \omega_2)$ measures the failure of dR taking $\wedge$ to $\cup$.

Is there a reasonable way to pick $B(\omega_1, \omega_2)$?

Remark 18. Ideally we would expect this formula to be well-defined, meaning $(c_1, \omega_1, h_1) \otimes (c_2, \omega_2, h_2)$ should satisfy the conditions in Remark 15. In general, this is only true if c_1, ω_2 satisfy $dc_1 = d\omega_2 = 0$. In particular, it is well-defined at the level of cohomology classes, as any element is closed therein.