DIFFERENTIAL COHOMOLOGY SEMINAR 3

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In this lecture we cover the basics of sheaf theory and learn about differential cohomology theories as sheaves of spectra.

1. Sheaves in Category Theory

Let us review sheaves in classical category theory. A good reference remains [MLM94]. We start with the case of sheaves on a topological space.

Definition 1.1. Let $\mathcal{C}$ be a category. A *pre-sheaf* on $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$.

Definition 1.2. Let X be a topological space, we denote by $\mathcal{O}\mathrm{pen}_X$ the partial order of open subsets of X, viewed as a category. We call a pre-sheaf on $\mathcal{O}\mathrm{pen}_X$ a *pre-sheaf on X*. Given $F : \mathcal{O}\mathrm{pen}_X{}^{op} \to \mathcal{S}\mathrm{et}$ and $u \in F(U)$, for every $V \subseteq U$, we write $u|_V$ for the image of u under the map $F(U) \to F(V)$.

Definition 1.3. Let X be a topological space. A pre-sheaf on X is a *sheaf* if, for every open $U \subseteq X$ and open cover $\{U_i \subseteq U\}_i$, the following is an equalizer diagram:

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

Now, we wish to extend the notion of sheaf to pre-sheaves on a generic category $\mathcal{C}$. To do so, we first generalize the concept of cover, for examples by specifying for each object C a class of families of maps $\{f_i : C_i \to C\}$ that cover C. Finally, in the sheaf condition, we replace intersections $U_i \cap U_j$ with fiber products (assuming $\mathcal{C}$ has the necessary limits). Then a pre-sheaf $F : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$ is a sheaf if, for every object C and any covering family $\{f_i : C_i \to C\}_i$, the following is an equalizer diagram:

$$F(C) \longrightarrow \prod_i F(C_i) \rightrightarrows \prod_{i,j} F(C_i \times_C C_j)$$

The above idea leads to the notion of *coverage*, here we will use another language, namely that of *sieves*.

2. Sheaves via Grothendieck Topologies

We now generalize from sheaves on a topological space to sheaves on a category via the additional data of a Grothendieck topology. We begin by recalling a few definitions. Denote by y the *Yoneda embedding* $\mathcal{C} \to \mathcal{P}\mathcal{S}\mathrm{h}(\mathcal{C})$, mapping C to the pre-sheaf $y(C) := \mathrm{Hom}_{\mathcal{C}}(-, C) \colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$.

Definition 2.1. A *sub-functor* to a pre-sheaf $F \colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$ consists of a pre-sheaf $S : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$ such that $S(C) \subseteq F(C)$, for all objects C, and the subset inclusions define a natural transformation $S \to F$. The last condition is equivalent to the following: Given $f \colon D \to C$ and $u \in S(C)$, then $F(f)(u) \in S(D)$.

Definition 2.2. Given a pre-sheaf $F \colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$, denote by $\mathcal{C}_{/F}$ the category of pairs (C, u), consisting of an object C and an element $u \in F(C)$. A morphism $(D, v) \to (C, u)$ consists of a morphism $f \colon D \to C$ such that $F(f)(u) = v$.

By Yoneda's lemma, an element $c \in F(C)$ corresponds to a natural transformation $y(C) \to F$. Taking the colimit over $(C, u) \in \mathcal{C}_{/F}$ of the pre-sheaves $y(C)$, we get a natural transformation $\mathrm{colim}_{(C,u)\in\mathcal{C}_{/F}} y(C) \to F$. The following theorem is called *density theorem*:

Theorem 2.3. *For every pre-sheaf F, the map $\mathrm{colim}_{(C,u)\in\mathcal{C}_{/F}} y(C) \to F$ is a natural isomorphism.*

Definition 2.4. Let $\mathcal{C}$ be a category. A *sieve* on an object C is a sub-functor of $y(C)$, with $y(C)$ being the *maximal sieve*. Given a morphism $f \colon D \to C$ and a sieve S on C, denote by $f^*(S)$ the *pullback sieve*, mapping E to the set of arrows $g \colon E \to D$ such that $fg \in S(E)$.

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Definition 2.5. Let $\mathcal{F}$ be a set of morphisms with codomain $C \in \mathcal{C}$. Define the *sieve generated by* $\mathcal{F}$ as the sieve $S_{\mathcal{F}} \subseteq \mathrm{Hom}_{\mathcal{C}}(-, C)$ mapping D to the set of morphisms $D \to C$ that factor through some element of $\mathcal{F}$.

Definition 2.6. Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is a collection $J(C)$ of sieves, called *covering sieves*, on every object C in $\mathcal{C}$, satisfying the following axioms:

(1) The maximal sieve is a covering sieve.
(2) Given a covering sieve S and a morphism $f\colon D \to C$, the pullback sieve $f^*(S)$ is a covering sieve on D.
(3) A sieve S on C is a covering sieve if, for some covering sieve S' on C and every map $f \in S'$, the pullback sieve $f^*(S)$ is also a covering sieve.

A *Grothendieck site* is a pair $(\mathcal{C}, J)$ where $\mathcal{C}$ is a category and J is a Grothendieck topology on $\mathcal{C}$.

Example 2.7. If every sieve is a covering sieve, we obtain the *discrete Grothendieck topology*. If the only covering sieves are the maximal ones, we obtain the *indiscrete Grothendieck topology*.

Example 2.8. Let X be a topological space. A sieve on an object U of $\mathcal{O}\mathrm{pen}_X$ corresponds to a collection of open subsets of U. Define a sieve to be a covering sieve on U if the corresponding family of subsets covers U in the classical sense.

Example 2.9. Let $\mathcal{T}\mathrm{op}$ be the category of topological spaces and continuous maps. Define a sieve S to be a covering sieve if generated, in the sense of Equation (2.5), by a family $\mathcal{O}$ of jointly surjective, open embeddings.

Example 2.10. Let $\mathcal{M}\mathrm{fd}$ be the category of smooth manifolds and smooth maps. We can define a topology in the same way as Equation (2.9), using families of jointly surjective, smooth open embeddings.

Definition 2.11. Let $(\mathcal{C}, J)$ be a site and $\mathcal{D} \subseteq \mathcal{C}$ a full sub-category. The *sub-category Grothendieck topology* on $\mathcal{D}$ assigns to each object $C \in \mathcal{D}$ the collection of sieves $S \subseteq \mathrm{Hom}_{\mathcal{D}}(-, C)$ that are the restriction to $\mathcal{D}$ of covering sieves of C in $\mathcal{C}$.

We are now ready to define sheaves on a Grothendieck site. Given a sieve S on C, notice that the category $\mathcal{C}_{/S}$ can be identified with the full sub-category of $\mathcal{C}_{/C}$ spanned by morphisms $f \in S$. On the other hand, a full sub-category $\mathcal{D} \subseteq \mathcal{C}_{/C}$ determines a sieve if $f \in \mathcal{D}$ implies $fg \in \mathcal{D}$, for every g composable with f.

Definition 2.12. Let $(\mathcal{C}, J)$ be a site and $C \in \mathcal{C}$. Given $f : D \to C$, there is a canonical identification $\alpha_f : (\mathcal{C}_{/C})_{/f} \simeq \mathcal{C}_{/D}$. The *slice Grothendieck topology* on $\mathcal{C}_{/C}$ assigns to each object $f : D \to C$ the collection of sieves $\mathcal{D} \subseteq (\mathcal{C}_{/C})_{/f}$ that correspond to covering sieves of D under α_f.

Definition 2.13. Given a pre-sheaf $F\colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$, denote by $\mathrm{Map}(\mathcal{C}_{/S}, F)$ the limit of the following diagram:

$$(\mathcal{C}_{/S})^{op} \longrightarrow \mathcal{C}^{op} \xrightarrow{F} \mathcal{S}\mathrm{et}$$

The inclusion $\mathcal{C}_{/S} \subseteq \mathcal{C}_{/C}$ induces a natural comparison map $\mathrm{Map}(\mathcal{C}_{/C}, F) \to \mathrm{Map}(\mathcal{C}_{/S}, F)$.

Notice that $\mathrm{Map}(\mathcal{C}_{/C}, F)$ is the limit of a diagram indexed by the category $(\mathcal{C}_{/C})^{op}$, which has an initial object, the identity of C, therefore the limit is canonically isomorphic to $F(C)$, the value of the diagram at the initial object.

Definition 2.14. Let $(\mathcal{C}, J)$ be a Grothendieck site. A pre-sheaf $F : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$ is a *sheaf* if, for every object C and covering sieve S of C, the canonical morphism

$$F(C) \cong \mathrm{Map}(\mathcal{C}_{/C}, F) \to \mathrm{Map}(\mathcal{C}_{/S}, F)$$

is an isomorphism. Explicitly, an object in $\mathrm{Map}(\mathcal{C}_{/S}, F)$ consists of an element $x_f \in F(C')$, for every map $f \in S$ (where C' is the domain of f), such that $x_{fg} = F(g)(x_f)$, for every $g\colon C'' \to C'$. The canonical morphism takes $x \in F(C)$ to the tuple $x_f := F(f)(x)$.

Using the density theorem and Yoneda's lemma, we have the following chain of natural isomorphisms:

$$\begin{aligned} \mathrm{Map}(\mathcal{C}_{/S}, F) &= \lim_{(C',c') \in \mathcal{C}_{/S}} F(C') \\ &\simeq \lim_{(C',c') \in \mathcal{C}_{/S}} \mathrm{Nat}_{\mathcal{C}}(y(C'), F) \\ &\simeq \mathrm{Nat}_{\mathcal{C}}(\mathrm{colim}_{(C',c') \in \mathcal{C}_{/S}} y(C'), F) \\ &\simeq \mathrm{Nat}_{\mathcal{C}}(S, F) \end{aligned}$$

One can then check that under the above isomorphism, the comparison map $\mathrm{Map}(\mathcal{C}_{/C}, F) \to \mathrm{Map}(\mathcal{C}_{/S}, F)$ corresponds to the restriction map $\mathrm{Nat}_{\mathcal{C}}(y(C), F) \to \mathrm{Nat}_{\mathcal{C}}(S, F)$ induced by the inclusion $S \subseteq y(C)$. In particular, we can rephrase the sheaf condition as follows:

Corollary 2.15. *A pre-sheaf F on a Grothendieck site $(\mathcal{C}, J)$ is a sheaf if and only if for every object C and covering sieve S of C, the canonical morphism*

$$F(C) = \mathrm{Nat}_{\mathcal{C}}(y(c), F) \to \mathrm{Nat}_{\mathcal{C}}(S, F)$$

is an isomorphism.

3. Sheaves on ∞-Categories

Now that we know how to define sheaves on 1-categories, we generalize the definition of sheaves to the case of ∞-categories. Here we largely follow [Lur09]. Recall that a sieve on C determines a full sub-category of $\mathcal{C}_{/C}$. Conversely, a full sub-category $\mathcal{D}$ of $\mathcal{C}_{/C}$ determines a sieve if and only if $f \in \mathcal{D}$ implies $fg \in \mathcal{D}$, for every g (composable with f).

Definition 3.1. Let $\mathcal{C}$ be a ∞-category and $\mathcal{S}$ the ∞-category of spaces. A *pre-sheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to \mathcal{S}$.

Definition 3.2. Let $\mathcal{C}$ be an ∞-category and C an object of $\mathcal{C}$. A *sieve* on C is a full sub-∞-category $\mathcal{D}$ of $\mathcal{C}_{/C}$ such that $f \in \mathcal{D}$ implies $fg \in \mathcal{D}$, for every g (composable with f).

A sieve $\mathcal{D} \subseteq \mathcal{C}_{/C}$ induces a sieve $\mathcal{D}' \subseteq \mathrm{Ho}(\mathcal{C})_{/C}$ by taking homotopy classes of 1-morphisms in $\mathcal{D}$.

Definition 3.3. Let $\mathcal{C}$ be an ∞-category. A *Grothendieck topology* on $\mathcal{C}$ is a collection $J(C)$ of sieves for every object C, such that it induces a Grothendieck topology on $\mathrm{Ho}(\mathcal{C})$.

Notice, we have the following compatibility observation.

Lemma 3.4. *Let $\mathcal{C}$ be a 1-category. A Grothendieck topology on $\mathcal{C}$ is precisely a Grothendieck topology on $\mathcal{C}$ seen as an ∞-category.*

Definition 3.5. Let $\mathcal{C}$ be a ∞-category and C an object of $\mathcal{C}$. Given a sieve $\mathcal{D} \subseteq \mathcal{C}_{/C}$ and a pre-sheaf $F : \mathcal{C}^{op} \to \mathcal{S}$, denote by $\mathrm{Map}(\mathcal{D}, F)$ the limit of the diagram

$$\mathcal{D}^{op} \longrightarrow \mathcal{C}^{op} \xrightarrow{F} \mathcal{S}$$

where the first functor is the opposite of the canonical projection $\mathcal{C}_{/C} \to \mathcal{C}$ restricted to $\mathcal{D}$. Using the same argument as in the 1-categorical case, there is a comparison map $\mathrm{Map}(\mathcal{C}_{/C}, F) \to \mathrm{Map}(\mathcal{D}, F)$ and a canonical equivalence $\mathrm{Map}(\mathcal{C}_{/C}, F) \simeq F(C)$.

Definition 3.6. Let $\mathcal{C}$ be an ∞-category. A pre-sheaf $F\colon \mathcal{C}^{op} \to \mathcal{S}$ is a *sheaf* if for every object C and covering sieve $\mathcal{D}$ of C, the canonical morphism

$$F(C) \simeq \mathrm{Map}(\mathcal{C}_{/C}, F) \to \mathrm{Map}(\mathcal{D}, F)$$

is an equivalence. Denote by $\mathcal{S}\mathrm{hv}(\mathcal{C}, J)$ the full sub-∞-category of $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ spanned by the sheaves on $(\mathcal{C}, J)$.

Note we defined sheaves as local objects in a presentable ∞-category. Hence, using the formalism of presentable ∞-categories, we immediately have the following result.

Proposition 3.7. *The ∞-category of sheaves on $\mathcal{S}\mathrm{hv}(\mathcal{C}, J)$ is presentable.*

4. Sheaves with arbitrary Values

We started with sheaves valued in sets. Then generalized to ∞-categorical sheaves valued in spaces. However, we want ∞-categorical sheaves valued in spectra. Hence the next step is to generalize the values of our ∞-categorical sheaves. Abstractly we obtain such sheaves via the *tensor product* of presentable ∞-categories, which we review now.

Theorem 4.1 ([Lur17, Proposition 4.8.1.15])**.** *Let $\mathcal{C}, \mathcal{D}$ be presentable ∞-categories. Then there exists a presentable ∞-category $\mathcal{C} \otimes \mathcal{D}$ together with a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ such that*

(1) The functor F preserves colimits component-wise.

(2) For any presentable ∞-category $\mathcal{E}$, pre-composition with F induces an equivalence between the sub-∞-category $\mathrm{Fun}^L(\mathcal{C}\otimes\mathcal{D},\mathcal{E})\subseteq\mathrm{Fun}(\mathcal{C}\otimes\mathcal{D},\mathcal{E})$ of colimit preserving functors, and the sub-∞-category $\mathrm{Fun}^{L,L}(\mathcal{C}\times\mathcal{D},\mathcal{E})\subseteq\mathrm{Fun}(\mathcal{C}\times\mathcal{D},\mathcal{E})$ of functors preserving colimits in each component.

Theorem 4.2 ([Lur17, Proposition 4.8.1.17])**.** *For $\mathcal{C}$ and $\mathcal{D}$ presentable ∞-categories, there is a canonical equivalence between $\mathcal{C}\otimes\mathcal{D}$ and $\mathrm{Fun}^R(\mathcal{C}^{op},\mathcal{D})$, of limit preserving functors $\mathcal{C}^{op}\to\mathcal{D}$.*

Definition 4.3. Let (C,J) be $\mathcal{C}$ be a Grothendieck site, and $\mathcal{D}$ be a presentable ∞-category. A *sheaf on* $(\mathcal{C},J)$ *with values in* $\mathcal{D}$ is a limit preserving functor $F\colon \mathcal{S}\mathrm{hv}(\mathcal{C},J)^{op}\to\mathcal{D}$, which corresponds to an object in $\mathcal{S}\mathrm{hv}(\mathcal{C},J)\otimes\mathcal{D}$.

Of course, this description is very abstract and ideally we want a more explicit description that we can use in the future. That is the aim of the next sections.

5. An Explicit Description of Sheaves valued in Spectra

We have given a formal definition of sheaves valued in spectra, via the tensor product of presentable ∞-categories. We now want a more explicit description thereof. For this we make use of the following result.

Proposition 5.1 ([Lur17], see also [GGN15])**.** *The inclusion $\mathrm{Pr}^L_{st}\to\mathrm{Pr}^L$ of stable presentable ∞-categories into the category of presentable ∞-categories admits a left adjoint, which is explicitly given by $-\otimes\mathcal{S}\mathrm{p}$.*

We can recall from our previous talk that the stabilization of a presentable ∞-category $\mathcal{C}$ is given by the ∞-category $\mathcal{S}\mathrm{p}(\mathcal{C})$ of spectrum objects in $\mathcal{C}$, therefore $\mathcal{C}\otimes\mathcal{S}\mathrm{p}\simeq\mathcal{S}\mathrm{p}(\mathcal{C})$, for every presentable ∞-category $\mathcal{C}$. In particular, the ∞-category $\mathcal{S}\mathrm{hv}(\mathcal{C},J)\otimes\mathcal{S}\mathrm{p}$ is equivalent to the category of spectrum objects in $\mathcal{S}\mathrm{hv}(\mathcal{C},J)$. Finally, since the inclusion $\mathcal{S}\mathrm{hv}(\mathcal{C},J)\to\mathrm{Fun}(\mathcal{C}^{op},\mathcal{S})$ preserves and reflects limits, a commutative square in $\mathcal{S}\mathrm{hv}(\mathcal{C},J)$ is a pullback square if and only if it is a pullback square in $\mathrm{Fun}(\mathcal{C}^{op},\mathcal{S})$.

Proposition 5.2. *A spectrum object in sheaves on the Grothendieck site $(\mathcal{C},J)$ is given by a spectrum object in pre-sheaves on $(\mathcal{C},J)$ that is point-wise a sheaf.*

This gives us the following explicit description of sheaves valued in spectra. Recall that $\Omega^{\infty-n}\colon\mathcal{S}\mathrm{p}\to\mathcal{S}$ is the functor mapping a spectrum object into its n-th component.

Theorem 5.3. *There is a canonical equivalence between $\mathcal{S}\mathrm{hv}(\mathcal{C},J)\otimes\mathcal{S}\mathrm{p}$ and the category of functors $F\colon\mathcal{C}^{op}\to\mathcal{S}\mathrm{p}$ such that $\Omega^{\infty-n}F\colon\mathcal{C}^{op}\to\mathcal{S}$ is a sheaf of spaces, for every n.*

6. An Explicit Description of Sheaves on the Site of Manifolds

We continue our analysis of sheaves with the aim of providing explicit descriptions. Now we focus on the case of sheaves on the site of manifolds. Recall that a sieve S on a manifold M is a covering sieve if an open cover $\mathcal{O}$ of M exists, such that every morphism in S factors through the inclusion $U\subseteq M$ of some element $U\in\mathcal{O}$. From the definition of the topology on $\mathcal{M}\mathrm{fd}$ we have the following: Given a covering sieve S generated by an open cover $\mathcal{O}$. Denote by $\mathcal{I}(\mathcal{O})$ the poset (viewed as a category) consisting of finite intersections of elements of $\mathcal{O}$.

Lemma 6.1. *The functor $i:\mathcal{I}(\mathcal{O})\to\mathcal{M}\mathrm{fd}_{/S}$ is cofinal.*

In particular, since $\mathrm{Map}(\mathcal{M}\mathrm{fd}_{/S},F)$ is defined as the limit $(\mathcal{M}\mathrm{fd}_{/S})^{op}\to\mathcal{M}\mathrm{fd}^{op}\xrightarrow{F}\mathcal{S}$ and i^{op} is *final*, there is a canonical equivalence

$$\mathrm{Map}(\mathcal{M}\mathrm{fd}_{/S},F)\simeq\lim(\mathcal{I}(\mathcal{O})^{op}\to\mathcal{M}\mathrm{fd}^{op}\xrightarrow{F}\mathcal{S})=\lim_{U\in\mathcal{I}(\mathcal{O})}F(U)$$

Proposition 6.2. *A pre-sheaf $F\colon\mathcal{M}\mathrm{fd}^{op}\to\mathcal{S}\mathrm{p}$ is a sheaf if and only if for every manifold M and every open cover $\mathcal{O}$ of M, the canonical morphism $F(M)\to\lim_{U\in\mathcal{I}(\mathcal{O})}F(U)$ is an equivalence.*

In fact we can check the sheaf condition for sheaves on the site of manifolds with a very specific type of covering families. For a proof of the following result, we refer to [ADH21, Proposition 3.6.6].

Proposition 6.3. *A pre-sheaf $F\colon\mathcal{M}\mathrm{fd}^{op}\to\mathcal{S}\mathrm{p}$ is a sheaf if and only if*

(1) $F(\emptyset)$ is terminal.

(2) *For every $M = U \cup V$, the canonical square*

$$\begin{tikzcd} F(M) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ F(V) \arrow[r] & F(U \cap V) \end{tikzcd}$$

is a pullback square.

(3) *For a sequential family of opens $U_1 \subseteq U_2 \subseteq \cdots$ covering M, the canonical map $F(M) \to \lim_n F(U_n)$ is an equivalence.*

7. Enough Points and Equivalences of Sheaves

We now have established a very solid understanding of sheaves on the site of manifolds with arbitrary values as pre-sheaves with suitable limit conditions. We now want to use these explicit descriptions to better understand equivalences of sheaves on the site of manifolds.

Definition 7.1 ([Lur09, Remark 6.5.4.8])**.** Given a ∞-topos $\mathcal{X}$, a *point* of $\mathcal{X}$ is a functor $p : \mathcal{S} \to \mathcal{X}$ admitting a left exact left adjoint p^*. We say $\mathcal{X}$ has *enough points* if the following holds: Given a morphism f, if $p^*(f)$ is an equivalence for every point p, then f is an equivalence.

Here we are mainly interested in ∞-topoi of sheaves, i.e. $\mathcal{X} = \mathcal{S}\mathrm{hv}(\mathcal{C})$, for some site $\mathcal{C}$.

Remark 7.2. A functor $p : \mathcal{S} \to \mathcal{X}$ admitting a left exact left adjoint is called a *geometric morphism*, see [Lur09, Definition 6.3.1.1].

Definition 7.3. Let X be a topological space and $\mathcal{C}$ a presentable category. Given a point $x \in X$, denote by x^* the functor $\mathcal{S}\mathrm{hv}(X, \mathcal{C}) \to \mathcal{C}$ given by $x^*(F) = \mathrm{colim}_{x \in U} F(U)$. The functor x^* is called *stalk at x* and is left adjoint to the *skyscraper at x* functor, denoted as x_* and given by $x_*(S)(U) = S$, if $x \in U$, and $= 1$, otherwise (where $1 \in \mathcal{C}$ is the terminal object).

Remark 7.4. Using that filtered colimits commute with finite limits (see [Lur09, Example 7.3.4.7]), one can prove that x^* is left exact and x_* is a point of $\mathcal{S}\mathrm{hv}(X)$ according to Equation (7.1). In particular, every point $x \in X$ induces a point of $\mathcal{S}\mathrm{hv}(X)$.

Theorem 7.5. *Let X be a smooth manifold, then $\mathcal{S}\mathrm{hv}(X)$ has enough points.*

Proof. [Lur09, Corollary 7.2.1.17] and [Lur09, Theorem 7.2.3.6] reduce the claim to X having finite Lebesgue's covering dimension (see [Lur09, Definition 7.2.3.1]). The Fundamental Urysohn's Identity states that, for separable metric spaces (such as smooth paracompact manifolds), Lebesgue's covering dimension coincides with the *small inductive dimension*, or SID, for short. The SID of a space X, denoted by $\mathrm{ind}X$, is defined inductively as follows:

(1) $\mathrm{ind}\emptyset = -1$.
(2) $\mathrm{Ind}X = n$ is the smallest natural number such that every point has a neighborhood basis of open sets U such that $\mathrm{ind}\partial U \leq n - 1$. If no such natural number n exists, we set $\mathrm{ind}X = \infty$.

One can check by induction that $\mathrm{ind}X = n$, for every n-dimensional topological manifold X. □

Remark 7.6. The proof of [Lur09, Corollary 7.2.1.17] actually shows that the collection of points $x_* : \mathcal{S} \to \mathcal{S}\mathrm{hv}(X)$, for all $x \in X$, is enough to detect equivalences, namely f is an equivalence if $x^*(f)$ is an equivalence, for every $x \in X$.

Theorem 7.7. *The ∞-category $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd})$ has enough points.*

Proof. Let M be a smooth manifold, there is a faithful functor $\mathcal{O}\mathrm{pen}_M \to \mathcal{M}\mathrm{fd}$ that is a morphism of sites (see [Pst22, Definition A.10]) and satisfies the *covering lifting property* (see [Pst22, Definition A.12]), therefore the restriction functor $-|_M : \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}) \to \mathcal{S}\mathrm{hv}(M)$ is a left adjoint (see [Pst22, Proposition A.13]) and a right adjoint (hence left exact). In particular, a point $p : \mathcal{S} \to \mathcal{S}\mathrm{hv}(M)$, for any manifold M, induces a point of $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd})$ as the composition of $p : \mathcal{S} \to \mathcal{S}\mathrm{hv}(M)$ with the right adjoint $\mathcal{S}\mathrm{hv}(M) \to \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd})$. The left adjoint is then $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}) \xrightarrow{-|_M} \mathcal{S}\mathrm{hv}(M) \xrightarrow{p^*} \mathcal{S}$.

Let f be a morphism such that $p^*(f)$ is an equivalence for every point p of $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd})$. In particular, $p^*(f|_M)$ is an equivalence for every point of $\mathcal{S}\mathrm{hv}(M)$, hence $f|_M$ is an equivalence in $\mathcal{S}\mathrm{hv}(M)$, by Equation (7.5). Finally, if $f|_M$ is an equivalence, for every M, then f is an equivalence. □

The proof of Equation (7.7) together with Equation (7.6) implies the following:

Corollary 7.8. *A morphism f in $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd})$ is an equivalence if and only if for every manifold M and every point $x \in M$, the map on stalks $x^*(f)$ is an equivalence.*

Note, for every n-manifold M and $x \in M$, there is an open embedding $x' : \mathbb{R}^n \to M$ such that $x'(0) = x$. The induced map $\mathcal{O}\mathrm{pen}_{\mathbb{R}^n,0}{}^{op} \to \mathcal{O}\mathrm{pen}_{M,x}{}^{op}$ is final (since $\{x'(U) | 0 \in U \subseteq \mathbb{R}^n\}$ forms a neighborhood basis for x), therefore it induces a natural equivalence $0^* \simeq x^*$. Denote by 0_n the origin of $\mathbb{R}^n$, the previous argument together with Equation (7.8) implies the following:

Corollary 7.9. *A morphism f of sheaves on $\mathcal{M}\mathrm{fd}$ is an equivalence if and only if for all $n \geq 0$, the induced map on stalks $(0_n)^*(f)$ is an equivalence.*

8. Sheaves of Manifolds via the Euclidean site

According to Equation (2.11), every full sub-category $\mathcal{B} \subseteq \mathcal{C}$ of a site $\mathcal{C}$ inherits a Grothendieck topology. If $\mathcal{B}$ is *dense* (see [Joh02, Definition 2.2.1]) then the restriction functor $\mathcal{S}\mathrm{hv}(\mathcal{C}, \mathcal{S}\mathrm{et}) \to \mathcal{S}\mathrm{hv}(\mathcal{B}, \mathcal{S}\mathrm{et})$ is an equivalence, this is the *comparison lemma* ([Joh02, Theorem 2.2.3]). However, this is no longer true with sheaves valued in ∞-categories, i.e. a dense sub-site $\mathcal{B} \subseteq \mathcal{C}$ does not induce an equivalence between ∞-categories of sheaves (see [Lur18, Counterexample 20.4.0.1]). But, the comparison lemma from ∞-sheaves *does* hold for $\mathcal{C} = \mathcal{M}\mathrm{fd}$ because of *hypercompleteness*, as we'll show. This comparison lemma allows us to relate statements about sheaves on dense sub-categories of $\mathcal{M}\mathrm{fd}$ to statements about sheaves on all manifolds.

Let $\mathcal{C}$ be a 1-site and $\mathcal{P}_\Delta(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, s\mathcal{S}\mathrm{et})$ the ordinary category of simplicial pre-sheaves on $\mathcal{C}$. Given a pre-sheaf $F : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$, denote by F^δ the simplicial pre-sheaf mapping C to the constant simplicial set with value $F(C)$.

Definition 8.1. Given a morphism of pre-sheaves of sets $f : F \to \mathrm{Hom}_\mathcal{C}(-, C)$, the *image of f* is the sieve S_f mapping D to the image of the map of sets $F(D) \to \mathrm{Hom}_C(D, C)$. If $C \in \mathcal{C}$ and $\mathcal{C}$ is a site, we call f a *local epimorphism* if S_f is a covering sieve of C. A morphism of pre-sheaves $F \to G$ is a local epimorphism if, for every $y(C) \to G$, the projection $y(C) \times_G F \to y(C)$ is a local epimorphism.

Definition 8.2 ([DHI03, Lemma 4.9])**.** Given $C \in \mathcal{C}$, an *hypercover* of C consists of a simplicial pre-sheaf $U : \mathcal{C}^{op} \to s\mathcal{S}\mathrm{et}$ together with a morphism $U \to y(C)^\delta$, called *augmentation*, such that the maps $U_0 \to y(C)$ and $U_n \simeq \mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\Delta^n, U) \to \mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\partial\Delta^n, U)$, for every n, are local epimorphisms, see Equation (8.1). An hypercover has *height k* if $U_n \to \mathrm{Hom}_{s\mathcal{S}\mathrm{et}}(\partial\Delta^n, U)$ is an isomorphism, for all $n \geq k$.

Remark 8.3. An hypercover has height 0 if and only if it is the Čech nerve of the morphism $U_0 \to y(C)$.

Definition 8.4. Denote by $\widehat{\mathcal{S}\mathrm{hv}}(\mathcal{C}) \subseteq \mathcal{S}\mathrm{hv}(\mathcal{C})$ the full sub-∞-category of sheaves satisfying *hyperdescent*, meaning that, for every object C and hypercover U of C, the augmentation induced an equivalence

$$F(C) \simeq \mathrm{Nat}_\mathcal{C}(y(C), F) \to \lim_n \mathrm{Nat}_\mathcal{C}(U_n, F)$$

$\mathcal{S}\mathrm{hv}(\mathcal{C})$ is called *hypercomplete* if every sheaf satisfies hyperdescent.

Definition 8.5 ([BGH20, Definition 3.12.2])**.** Let $\mathcal{C}$ be a site. A full sub-category $\mathcal{B} \subseteq \mathcal{C}$ is *dense* if for every object $C \in \mathcal{C}$ there is a family $\mathcal{F}$ of morphisms with codomain C and domain in $\mathcal{B}$, such that $S_\mathcal{F}$ (see Equation (2.5)) is a covering sieve.

Theorem 8.6. *$\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd})$ is hypercomplete. If $i : \mathcal{D} \subseteq \mathcal{M}\mathrm{fd}$ is a full, dense sub-site, then $\mathcal{S}\mathrm{hv}(\mathcal{D})$ is also hypercomplete. Moreover, the adjunction $(i_*, i^*) : \mathcal{S}\mathrm{hv}(\mathcal{D}) \to \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd})$ is an equivalence.*

Proof. Hypercompleteness of $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd})$ follows from Equation (7.7), since having enough points implies hypercompleteness, see [BGH20, Example 3.11.10]. The second and third claims are a direct consequence of [BGH20, Corollary 3.12.13] and the first claim. By [BGH20, Proposition 3.12.11], pre-sheves restriction i^* preserves sheaves satisfying hyperdescent. □

Recall that, given a presentable category $\mathcal{E}$, there is a canonical equivalence $\mathcal{S}\mathrm{hv}(\mathcal{C}, \mathcal{E}) \simeq \mathcal{S}\mathrm{hv}(\mathcal{C}) \otimes \mathcal{E}$. In particular, the following is a simple corollary of Equation (8.6):

Corollary 8.7. *For every presentable category $\mathcal{E}$ and full, dense sub-site $\mathcal{D} \subseteq \mathcal{M}\mathrm{fd}$, the restriction functor $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, \mathcal{E}) \to \mathcal{S}\mathrm{hv}(\mathcal{D}, \mathcal{E})$ is an equivalence.*

We will mainly focus on the following sub-site of $\mathcal{M}\mathrm{fd}$.

Definition 8.8. Denote by $\mathcal{E}\mathrm{uc}$ the full sub-category of $\mathcal{M}\mathrm{fd}$ generated by Euclidean manifolds, i.e. $\mathbb{R}^n$, for every n. $\mathcal{E}\mathrm{uc} \subseteq \mathcal{M}\mathrm{fd}$ is dense in the sense of Equation (8.5).

Related to the notion of hypercompleteness is the notion of *Postnikov completeness*. A space S is n-*truncated*, $n \geq 0$, if $\pi_i(S) \simeq 0$, for all $i > n$. We say S is (-1)-*truncated* if S if it's either empty or contractible. Denote by $\tau_{\leq n}\mathcal{S} \subseteq \mathcal{S}$ the full sub-category of n-truncated spaces. The inclusion functor admits a left adjoint $\tau_{\leq n}$, and so every space has a natural map $S \to \tau_{\leq n} S$. A classical result of algebraic geometry is that $S \simeq \lim_n \tau_{\leq n} S$. The notion of truncated object makes sense in a general ∞-category.

Definition 8.9 ([Lur09, Definition 5.5.6.1])**.** Let $n \geq -1$, an object $C \in \mathcal{C}$ is n-*truncated* if $\mathrm{Hom}_{\mathcal{C}}(D, C)$ is a n-truncated space, for every $D \in \mathcal{C}$. Denote by $\tau_{\leq n}\mathcal{C}$ the full sub-category of n-truncated objects. A morphism $f : C \to D$ *exhibit* D *as the* n*-truncation of* C if D is n-truncated and the induced morphism $\tau_{\leq n} C \to D$ is an equivalence.

If $\mathcal{C}$ is a presentable ∞-category, $\tau_{\leq n}\mathcal{C}$ can be characterized as a class of local objects (see [Lur09, Proposition 5.5.6.18]). In particular, the inclusion functor $\tau_{\leq n}\mathcal{C} \subseteq \mathcal{C}$ admits a left adjoint $\tau_{\leq n} : \mathcal{C} \to \tau_{\leq n}\mathcal{C}$. Given an object $C \in \mathcal{C}$, its truncations form a sequence

$$\cdots \to \tau_{\leq 2} C \to \tau_{\leq 1} C \to \tau_{\leq 0} C \tag{8.10}$$

where $\tau_{\leq n+1} C \to \tau_{\leq n} C$ is the $(n+1)$-truncation of the map $C \to \tau_{\leq n} C$ together with the fact that $\tau_{\leq n+1} E \simeq E$ naturally, for every n-truncated E. Let $\mathbb{Z}_{\geq 0}$ be the partial order of natural numbers.

Definition 8.11 ([Lur18, Definition A.7.2.1] and [Lur09, Definition 5.5.6.23])**.** A *Postnikov pre-tower* is a functor $X : {\mathbb{Z}_{\geq 0}}^{op} \to \mathcal{C}$ such that $X_{n+1} \to X_n$ exhibits X_n as the n-truncation of X_{n+1}, for every $n \geq 0$. Denote by $\mathrm{Post}(\mathcal{C})$ the category of Postnikov pre-towers, then there is a comparison functor $\pi : \mathcal{C} \to \mathrm{Post}(\mathcal{C})$. We say $\mathcal{C}$ is *Postnikov complete* if π is an equivalence.

One can show that π is left left adjoint to the limit functor $\lim : \mathrm{Fun}({\mathbb{Z}_{\geq 0}}^{op}, \mathcal{C}) \to \mathcal{C}$ restricted to Postnikov pre-towers. In particular, if $\mathcal{C}$ is Postnikov complete, then $C \simeq \lim_n \tau_{\leq n} C$. Let $\mathcal{X} = \mathcal{S}\mathrm{hv}(\mathcal{C})$, for some site $\mathcal{C}$, and $\widehat{\mathcal{X}} = \widehat{\mathcal{S}\mathrm{hv}(\mathcal{C})}$, then $\tau_{\leq n}\mathcal{X} \subseteq \widehat{\mathcal{X}}$, for every n, see [Lur09, Lemma 6.5.2.9]. Moreover, if $\mathcal{X}$ is Postnikov complete, then every object C is equivalent to the limit of its truncations, which are hypercomplete, which implies C is hypercomplete ($\widehat{\mathcal{X}} \subseteq \mathcal{X}$ is a right adjoint, since $\widehat{\mathcal{X}}$ is a category of local objects, which implies $\widehat{\mathcal{X}}$ is closed under limits in $\mathcal{X}$).

Theorem 8.12. *If* $\mathcal{S}\mathrm{hv}(\mathcal{C})$ *is Postnikov complete, then it is hypercomplete.*

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