DIFFERENTIAL COHOMOLOGY SEMINAR 6

TALK BY NIMA RASEKH

This talk is a quick review and plans for this semester.

1. Summary

In our last talk we already did a rather detailed summary of what we covered until now [Ras25], so here do a quick bullet point summary.

- Differential cohomology theories should be defined as sheaves valued in the ∞-category of spectra valued on the site of manifolds.
- We can use abstract ∞-topos theoretic properties to prove that every differential cohomology theory decomposes into an $\mathbb{R}$-invariant part (a purely homotopical differential cohomology) and a pure part (a purely geometric differential cohomology), via a pullback square called the *fracture square*.
- This fracture square relates the original definition to the “hexagon approach” going back to Simons–Sullivan and others [SS08, BS09].
- Using this abstract approach we can construct examples of interest, such as Deligne cohomology, which refines integral homology.

2. Talk Topics

Here is an overview of the possible talk topics we can cover this semester. There is some dependency, but many talks are independent of each other giving us significant flexibility, if we want to rearrange things.

Topic	**Section**	**Exp. vs. Research**	**# Talks**
Further Examples of Differential Cohomology Theories	Section 2.1	Expository	1
Twisted (differential) cohomology and applications	Section 2.2	Expository	2
Differential Cohomology in Cohesive ∞-Topoi	Section 2.3	Expository	2
Differential Cohomology of Lie Groupoids	Section 2.4	Research	1
Differential Refinements of Loop Group Representations	Section 2.5	Research	2
Cohomology Groups out of Differential Cohomology Theories	Section 2.6	Research	2
Abstract Proof and further Applications of the Fracture Square	Section 2.7	Research	2

Here are further details on each topic.

2.1. Further Examples of Differential Cohomology Theories.

We already saw how to can use the methods of the fracture square to construct $\hat{\mathbb{Z}}[l]$. However, the literature contains many other examples, constructed with different methods. How do these examples fit into our framework? Concrete examples are:

(1) Differential K-theory, as a differential refinement of ku [HS05].
(2) *differential algebraic K-theory* as a differential refinement of algebraic K-theory [BG21]
(3) *differential complex cobordism* as a differential refinement of complex cobordisms [BSSW09]

Beyond these already considered examples, it would be interesting if we could find some content about differential refinements of tmf.

2.2. Twisted (differential) cohomology and applications.

Twisted cohomology theories further generalize cohomology theories. They have been refined to a differential version. Given their myriad applications, both twisted cohomology and its differential version merit a careful analysis.

This topics would involve two parts (and hence probably two talks):

(1) The development of twisted cohomology theory, and its applications in geometry and physics [Ros24].
(2) The differential refinement due to Bunke–Nikolaus, and its applications [BN19].

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2.3. **Differential Cohomology in Cohesive ∞-Topoi.** Schreiber has introduced a notion of differential cohomology in the abstract concept of a cohesive ∞-topos [Sch13]. This has then be applied to physical settings [FSS24]. This suggest a 2-part talk with the following topics:

- Background on ∞-topos theory and cohesion, introducing differential cohomology in this setting, comparison to our approach [Lur09, Sch13]
- Applications to physics [FSS24]

2.4. **Differential Cohomology of Lie Groupoids.** While differential cohomology theories of manifolds are well-defined and studied extensively, the same cannot be said for Lie groupoids. Concretely, we can imagine several definitions in this context.

(1) We can embed Lie groupoids into sheaves of manifolds, and then define cohomology via left Kan extension.
(2) We can think of a Lie groupoid as a special kind of simplicial manifold, and then define differential cohomology point-wise and then geometric realization.
(3) In certain cases, such as differential K-theory, we can use classical methods to define differential cohomology (via vector bundles with connection).

Currently, it is unknown how these definitions relate to each other, and what properties they have, suggesting a talk analyzing these questions.

2.5. **Differential Refinements of Loop Group Representations.** Building on work of Freed and Hopkins, twisted K-theory has known connections to the concept of *loop group representations*. As a follow up to the talks in Section 2.2, we can explore the following question:

Question 2.1. Is there a differential analogue of loop group representations and their connection to twisted differential K-theory?

2.6. **Cohomology Groups out of Differential Cohomology Theories.** Recall that the Deligne cohomology groups do not arise as the homotopy groups of a single differential cohomology theory. Instead, there is a collection of differential cohomology theories, one for each degree, whose homotopy groups give the Deligne cohomology groups.

On the other hand, there are alternative methods to extract cohomology groups out of a single differential cohomology theory, as proposed by Bunke–Gepner [BG21, Definition 2.23]. This naturally results in the following question and possible talk topic:

Question 2.2. Can we use the methods from [BG21] to extract Deligne cohomology out of $\hat{\mathbb{Z}}[1]$?

Understanding this approach can help us understand ways to extract cohomological data in a non-formal way that is more helpful in geometrically motivated applications.

2.7. **Abstract Proof and further Applications of the Fracture Square.** One question is whether we want to explicitly go through the proof of the fracture square, and the fact that $\mathbb{R}$-invariant sheaves are precisely spectra. This involves understanding advanced aspects of sheaf theory, such as *recollements*.

The essential step in the proof is the following very technical result.

Proposition 2.3. *Assume we have the following data and assumptions:*

(1) A Grothendieck site $(\mathcal{C}, J)$, such that $\mathcal{C}$ has a terminal object.
(2) A stable ∞-category $\mathcal{T}$.
(3) The inclusion functor $\Delta\colon \mathcal{T} \to \mathrm{Shv}_{\mathcal{T}}(\mathcal{C}, J)$ is fully faithful and admits a left adjoint L_{const}.

Then the following holds:

(1) The full subcategory of $\mathrm{Shv}_{\mathcal{T}}(\mathcal{C}, J)$ consisting of sheaves P, such that $P(*)$ is the point admits a left adjoint $L^{\perp}\colon \mathrm{Shv}_{\mathcal{T}}(\mathcal{C}, J) \to \mathrm{Shv}_{\mathcal{T}}(\mathcal{C}, J)^{\perp}$.
(2) For every sheaf P in $\mathrm{Shv}_{\mathcal{T}}(\mathcal{C}, J)$, there is a pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & L^{\perp}P \arrow[d] \\ L_{const}P \arrow[r] & L_{const}L^{\perp}P \end{tikzcd},$$

inside $\mathrm{Shv}_{\mathcal{T}}(\mathcal{C}, J)$, where the inclusions are left implicit.

This is a very general result. We use it for the following specific case:

Lemma 2.4. *Let* $(\mathcal{E}\mathrm{uc}, J)$ *be the Euclidean site and* $\mathcal{S}\mathrm{p}$ *the stable* ∞*-category of spectra. Then the inclusion functor* $\Delta\colon \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{p}}(\mathcal{E}\mathrm{uc}, J)$ *is given by the constant presheaf functor.*

In other words, the constant presheaf is already a sheaf. This directly implies that Δ is fully faithful and that it admits a left adjoint via colimit. So we can directly apply the result above to get the fracture square.

Beyond looking at the proof, we can also look at manifestations of this general result in the context of other sites, such as:

(1) The site of topological spaces.
(2) The site of coarse spaces.
(3) The site of diffeological spaces.
(4) The site of Lie groupoids.

Concretely, we can pursue the following questions:

(1) Do these Grothendieck sites satify the assumptions of the proposition?
(2) If yes, what does the existence of a fracture square imply in these cases?
(3) If not, what are are the obstructions inherent to the theory?

3. Talk Dates

Here are some first talk date plans

	Date	**Speaker**	**Topic**
(0)	14.10.2025	Nima Rasekh	Review and Talk Distribution
(1)	29.10.2025	Matthias Ludewig	Further Examples
(2)	05.11.2025	Hannes Berkenhagen	Twisted Cohomology Theories
(3)	12.11.2025	Alessandro Nanto	Twisted Diff. Cohomology Theories
(4)	19.11.2025	Matthias Frerichs	∞-Topoi and Cohesion
(5)	26.11.2025	Nima Rasekh	Differential Cohomology and Cohesion
(6)	03.12.2025	Christian Becker	Differential Cohomology of Lie Groupoids

References

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