DIFFERENTIAL COHOMOLOGY SEMINAR 2

TALK BY MATTHIAS FRERICHS

In this lecture we want to learn the basics of ∞-category theory. For the ∞-categorical background, we broadly follow [Gro10] and a little [Lur09].

1. Basics on $(\infty, 1)$-categories

$(\infty, 1)$-categories have different models that capture its essence. The first model are *quasi-categories*.

Definition 1. The *simplex category* $\mathbb{\Delta}$ is the category whose objects are the non-negative integers and whose morphisms are the non-decreasing maps.

Note, we have in particular maps $d^i\colon [n-1] \to [n]$ for $0 \leq k \leq n$ and $s^i\colon [n] \to [n+1]$, which satisfy the *cosimplicial identities.*

Definition 2. A *simplicial set* is a functor $X\colon \mathbb{\Delta}^{op} \to \mathcal{S}\mathrm{et}$, i.e. a contravariant functor from the simplex category to the category of sets. The category is denoted $s\mathcal{S}\mathrm{et}$.

We can depict such a simplicial set as

$$\begin{tikzcd} X_0 \arrow[r, shift left] & X_1 \arrow[l, shift left] \arrow[l, shift left=2] \arrow[r, shift left] \arrow[r, shift left=2] & X_2 \arrow[l, shift left] \arrow[l, shift left=2] \arrow[l, shift left=3] \arrow[r, shift left] \arrow[r, shift left=2] \arrow[r, shift left=3] & \cdots \arrow[l] \end{tikzcd}$$

In order to define ∞-categories in this context, we need horns. Let us denote by $\Delta^n\colon \mathbb{\Delta}^{op} \to \mathcal{S}\mathrm{et}$ the functor $\Delta^n_m = \mathrm{Hom}_{\mathbb{\Delta}}([m],[n])$.

Definition 3. The k-th horn in Δ^n is defined as a sub-simplicial set $\Lambda^n_k \subseteq \Delta^n$ defined as follows. A morphism $\varphi\colon [m] \to [n]$ in Δ^n_m is in $(\Lambda^n_k)_m$ precisely when φ is not surjective and if the image does not include k, it must also not include some other element.

Definition 4. A *quasi-category* is a simplicial set X such that the following conditions hold:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] \end{tikzcd}$$

for $0 < k < n$.

Definition 5. An ∞-*groupoid* is a simplicial set X such that the following conditions hold:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] \end{tikzcd}$$

for $0 \leq k \leq n$.

Example 6. Let $\mathcal{C}$ be a category. Let $N\mathcal{C}$ (the *nerve of* $\mathcal{C}$) be the simplicial set given by $N\mathcal{C}_n = \mathrm{Fun}([n], \mathcal{C})$. This defines a functor $N\colon \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}$.

Proposition 7. *$N\mathcal{C}$ is a quasi-category.*

Proof. Straightforward combinatorics (For example lift of $\Lambda^2_1 \to \Delta^2$ is the composition). $\square$

Remark 8. Note Λ^2_0 does not generally admit lifts, unless the relevant morphism is invertible.

Here is an important result, the proof of which is postponed.

Date: 06.05.2025 & 13.05.2025.

Proposition 9. *$N\colon \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}_\infty$ is fully faithful.*

The nerve functor has a left adjoint, the *fundamental category* or *homotopy category*. Here we observe some general facts about simplicial sets. Namely for every functor $\mathbb{\Delta} \to \mathcal{C}$, where $\mathcal{C}$ cocomplete, we have an adjunction between $s\mathcal{S}\mathrm{et}$ and $\mathcal{C}$.

Example 10. Let $\mathcal{C} = \mathcal{T}\mathrm{op}$ and $\mathbb{\Delta} \to \mathcal{T}\mathrm{op}$ the functor picking the geometric simplicial complex $|\Delta^n|$. Then the induced adjunction is the classical geometric realization functor $|\cdot|\colon s\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$ and singularity functor.

The nerve is indeed the right adjoint of the adjunctions induced via the functor $\mathbb{\Delta} \to \mathcal{C}\mathrm{at}$, which sends $[n]$ to the category $[n]$. This means we also have a left adjoint.

Definition 11. The *homotopy category* $h\colon s\mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at}$ is the left adjoint to the nerve functor.

Remark 12. If the simplicial set is a quasi-category, then the homotopy category can be defined directly as the usual homotopy category of an ∞-category (i.e. we take equivalence classes of morphisms).

We do have significantly more complicated examples of such adjunctions.

Example 13. Let $\mathfrak{C}\colon \mathbb{\Delta} \to s\mathcal{C}\mathrm{at}$ be the functor defined in [Lur09, Definition 1.1.5.1]. Then we get the adjunction $(\mathfrak{C}, N_\Delta)$ called the homotopy coherent categorification, homotopy coherent nerve.

This adjunction incudes an equivalence of ∞-categories, if one phrases those notions correctly. In particular, if $\mathcal{C}$ is a Kan-enriched category, then the homotopy coherent nerve is a quasi-category.

Definition 14. Let $\mathcal{K}\mathrm{an}$ denote the Kan-enriched category of Kan complexes. The quasi-category of spaces is defined as $N_\Delta(\mathcal{K}\mathrm{an})$.

2. Accessible and Presentable Categories

Note, here we benefited from the fact that $\mathcal{C}\mathrm{at}, s\mathcal{S}\mathrm{et}, \mathcal{T}\mathrm{op}, s\mathcal{C}\mathrm{at}$ are all cocomplete categories, hence admitting such adjunctions. We now want to focus on a class of categories where we similarly can construct adjunctions in such straightforward ways.

Definition 15. A category $\mathcal{C}$ is *locally presentable* if it is cocomplete and κ-accessible for some κ.

Definition 16. A category $\mathcal{C}$ is κ-accessible if there exists a set of κ-compact objects $\mathcal{C}^0$ in $\mathcal{C}$, such that every object in $\mathcal{C}$ is a κ-filtered colimit of objects in $\mathcal{C}^0$.

Definition 17. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *κ-accessible* if it preserves κ-filtered colimits.

Theorem 18. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) $\mathcal{C}$ is locally presentable.
(2) There exists a small category $\mathcal{C}^0$ and a fully faithful accessible right adjoint $\mathcal{C} \to \mathrm{Fun}((\mathcal{C}^0)^{op}, \mathcal{S}\mathrm{et})$.

Theorem 19. *Let $\mathcal{C}, \mathcal{D}$ be locally presentable categories.*

(1) $F\colon \mathcal{C} \to \mathcal{D}$ is a left adjoint if and only if it preserves colimits.
(2) $F\colon \mathcal{C} \to \mathcal{D}$ is a right adjoint if and only if it preserves limits and is accessible.

We now generalize this to ∞-categories.

Definition 20. Let K be a quasi-category. The quasi-category of simplicial presheaves is defined as $\mathrm{Fun}(K^{op}, \mathcal{S})$.

Theorem 21 (Yoneda)**.** *For a given quasi-category K, there is a functor $K \to \mathrm{Fun}(K^{op}, \mathcal{S})$ given by $x \mapsto \mathrm{Fun}(-, x)$, which is fully faithful. Every colimit preserving functor out of $\mathrm{Fun}(K^{op}, \mathcal{S})$ is equivalent to a functor out of K.*

Theorem 22. *Let K be a quasi-category.*

(1) There exists a set of κ-compact objects $\mathcal{C}^0$ in K, such that every object in K is a κ-filtered colimit of objects in $\mathcal{C}^0$.
(2) There exists a small category $\mathcal{C}^0$ and a fully faithful accessible right adjoint $\mathcal{C} \to \mathrm{Fun}((\mathcal{C}^0)^{op}, \mathcal{S}\mathrm{et})$.

In those cases we say K is presentable. We now again have the adjoint functor theorem.

Theorem 23. *Let $\mathcal{C}, \mathcal{D}$ be presentable ∞-categories.*

(1) $F\colon \mathcal{C} \to \mathcal{D}$ is a left adjoint if and only if it preserves colimits.
(2) $F\colon \mathcal{C} \to \mathcal{D}$ is a right adjoint if and only if it preserves limits and is accessible.

Note in particular the ∞-category of sheaves is a presentable ∞-category.

3. Stable ∞-Categories and Spectra

We now use the ∞-categorical framework to study spectra. Let us recall some facts about spectra, to motivate the story. The *Freudenthal suspension theorem* states that the suspension functor $\Sigma\colon \mathcal{T}\mathrm{op} \to \mathcal{T}\mathrm{op}$ stabilizes the homotopy type. More explicitly, the map

$$\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to ...$$

stabilizes for k large enough, if X satisfies some connectivity condition. This defined the stable homotopy groups $\pi_n^S(X)$ as the stabilization of this sequence.

There is significant interest in computing these stable homotopy groups, in particular in the case where X is a sphere, given that it helps us understand many phenomena in algebraic topology.

We now want a setting where these stable homotopy groups naturally live and can be studied. We know that (Σ, Ω) induces an adjunction on the category of pointed topological spaces. What we want is an adjustment of this definition such that the adjunction (Σ, Ω) is an equivalence.

We now take a ∞-categorical perspective on this and use it to study such stable phenomena.

Definition 24. Let $\mathcal{C}$ be an ∞-category with initial and terminal object. $\mathcal{C}$ has a 0-object if they are equivalent.

Example 25. Let $\mathcal{C}$ be a 1-category. Then $\mathcal{C}$ is pointed as a 1-category if and only if it is pointed as an ∞-category.

Example 26. Notice $\mathcal{S}$ is not pointed, we hence can define $\mathcal{S}_*$ as the slices under the terminal space, i.e. $\mathcal{S}_* = \mathcal{S}_{*/}$. This ∞-category is then pointed by construction.

Note $\mathcal{S}_*$ is not just some pointed ∞-category, it is in some sense the universal one.

Proposition 27. *Let $\mathcal{D}$ be a pointed ∞-category. Then the functor*

$$\mathrm{ev}_{S^0}\colon \mathrm{Fun}^L(\mathcal{S}_*, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}$$

that evaluates at S^0 is an equivalence.

We now generalize from there and define triangles in $\mathcal{S}_*$.

Definition 28. Let $\mathcal{C}$ be a pointed ∞-category. A *triangle* in $\mathcal{C}$ is a diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

where X, X, and Z are objects in $\mathcal{C}$.

Definition 29. We say a triangle is a *exact* if it is a pullback square and *coexact* if it is a pushout square.

Definition 30. Let $\mathcal{C}$ be a pointed ∞-category. Let $\mathcal{C}^\Sigma$ be the full subcategory of $\mathrm{Fun}([1] \times [1], \mathcal{C})$ with objects coexact triangles of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd},$$

meaning Y is the suspension of X.

Definition 31. Let $\mathcal{C}$ be a pointed ∞-category. Let $\mathcal{C}^{\Omega}$ be the full subcategory of $\mathrm{Fun}([1]\times[1],\mathcal{C})$ with objects exact triangles of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd},$$

meaning Y is the loop object of X.

Proposition 32. *If $\mathcal{C}$ is a pointed ∞-category with finite (co)limits. Then there exists functors*

$$\Sigma\colon \mathcal{C} \to \mathcal{C}^{\Sigma} \to \mathcal{C}$$

$$\Omega\colon \mathcal{C} \to \mathcal{C}^{\Omega} \to \mathcal{C}$$

Theorem 33. *Let $\mathcal{C}$ be a pointed ∞-category with finite (co)limits. The following are equivalent:*

(1) A triangle is exact if and only if it is coexact.
(2) The functors Σ and Ω are equivalences and the inverses of each other.
(3) A square is a pullback square if and only if it is a pushout square.

Definition 34. A pointed ∞-category $\mathcal{C}$ is *stable* if it satisfies one of the three equivalent conditions above.

Recall that before the rise of ∞-categories, *triangulated categories* were used to study stable homotopy theory. Hence, it is unsurprising that we can relate stable ∞-categories to triangulated categories.

Proposition 35. *If $\mathcal{C}$ is a stable ∞-category, then the homotopy category $h\mathcal{C}$ is a triangulated category.*

Of course arbitrary pointed ∞-categories will not be stable. We hence want a procedure that stabilizes them. There are several approaches. One approach, that is powerful from a theoretical perspective, is given via reduced 1-excisive functors out of finite pointed spaces. Here, we focus on explicit spectrum objects, as there are characterized more explicitly. For a comparison of these two approaches see [Lur17].

Definition 36. Let $\mathcal{C}$ be a pointed ∞-category. A *pre-spectrum object* in $\mathcal{C}$, is a functor $X\colon \mathbb{Z}\times\mathbb{Z}\to\mathcal{C}$ such that $X(i,j)=0$ for $i\neq j$ and all squares are pushout squares. Let $PSp(\mathcal{C})$ be the ∞-category of pre-spectrum objects in $\mathcal{C}$.

For a given pre-spectrum object X, let $\alpha_{m-1}\colon \Sigma X_{m-1}\to X_m$ and $\beta_m\colon X_m\to\Omega X_{m+1}=\Omega\Sigma X_m$.

Definition 37. Let $\mathcal{C}$ be a pointed ∞-category. A *spectrum object* in $\mathcal{C}$ is a pre-spectrum object in X, such that α_{m-1} and β_m are equivalences for all m. Let $\mathcal{S}\mathrm{p}(\mathcal{C})$ be the ∞-category of spectrum objects in $\mathcal{C}$.

Definition 38. Let $\mathcal{C}$ be a pointed ∞-category. The stabilization of $\mathcal{C}$ is the ∞-category $\mathcal{S}\mathrm{p}(\mathcal{C})$ of spectrum objects in $\mathcal{C}$.

Of course $\mathcal{C}$ and $\mathcal{S}\mathrm{p}(\mathcal{C})$ are suitably related.

Theorem 39. *For a given pointed ∞-category $\mathcal{C}$, there is an adjunction*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\Sigma"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathrm{p}(\mathcal{C}) \arrow[l, shift left, "\Omega"] \end{tikzcd}$$

Moreover, $\mathcal{S}\mathrm{p}(\mathcal{C})$ is in some sense the universal stabilization of $\mathcal{C}$.

Theorem 40. *Let $\mathcal{C}$ be a pointed ∞-category and $\mathcal{D}$ a stable ∞-category. Then Σ^{∞} induces an equivalence of ∞-categories*

$$(\Sigma^{\infty})^*\colon \mathrm{Fun}^L(\mathcal{S}\mathrm{p}(\mathcal{C}),\mathcal{D})\simeq\mathrm{Fun}^L(\mathcal{C},\mathcal{D})$$

Let us now focus on the case $\mathcal{C}=\mathcal{S}_*$.

Example 41. The stabilization of $\mathcal{S}_*$ is the ∞-category of spectra, denoted $\mathcal{S}\mathrm{p}$.

Similar to $\mathcal{S}_*$, $\mathcal{S}\mathrm{p}$ is also the universal stable ∞-category, as a special instance of the result above.

Theorem 42. *If $\mathcal{D}$ is a stable ∞-category. Then the functor*

$$\mathrm{ev}_{\mathbb{S}}\colon \mathrm{Fun}^L(\mathcal{S}\mathrm{p},\mathcal{D})\xrightarrow{\simeq}\mathcal{D}$$

that evaluates at $\mathbb{S}$ is an equivalence.

4. Generalized Cohomology Theories

Cohomology theories were traditionally defined in the context of topological spaces. However, now that we have the tools of ∞-categories and stable ∞-categories. We can significantly generalize those definitions. This last result follows work in [Lur17].

Definition 43. Let $\mathcal{C}$ be a pointed ∞-category with pushouts, and $\Sigma\colon \mathcal{C} \to \mathcal{C}$ and suspension functor. A *generalized cohomology theory* is a functor $H\colon h\mathcal{C}^{op} \to \mathcal{A}\mathrm{b}_{\mathbb{Z}}$, such that the following conditions hold:

- There is a natural isomorphism $H^\bullet \to H^{\bullet+1}\Sigma$
- Coexact sequences maps to exact sequences.
- Arbitrary coproducts map to products.

We now have the following major result that significantly generalizes the classical Brown representability theorem.

Theorem 44. *Let $\mathcal{C}$ be a nice ∞-category and $(H^\bullet, \delta)$ be a generalized cohomology theory. Then there exists a spectrum object E in $\mathcal{C}$, such that $H^\bullet(X) \cong \mathrm{Hom}_{h\mathcal{C}}(X, E^\bullet)$, where $\delta = (\beta_\bullet)_*$.*

Example 45. Unsurprisingly, $\mathcal{S}_*$ satisfies the niceness conditions, and so we can conclude that every generalized cohomology on $\mathcal{S}_*$ is given by a spectrum, recovering the original Brown representability theorem.

References

[Gro10] Moritz Groth. A short course on ∞-categories. *arXiv preprint*, 2010. arXiv:1007.2925.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur17] Jacob Lurie. Higher algebra. Available online, September 2017.