

DIFFERENTIAL COHOMOLOGY SEMINAR 4 (DRAFT)

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In the last talk we learned the definition of a differential cohomology theory, as a sheaf valued in spectra on the site of manifolds. This talk continues our journey through differential cohomology theories, and focuses on the following three topics:

- (1) We want to learn how to construct non-trivial examples out of sheaves valued in chain complexes.
- (2) We want to understand how we can extend classical cohomology operations to the setting of differential cohomology theories.
- (3) We want to introduce suitable analogues of fiber-wise integration.

1. ABELIAN GROUPS, SPECTRA AND THE HEART

Let us start by reviewing the relation between abelian groups, rings and spectra.

Definition 2. Let $n \in \mathbb{Z}$ and X be a spectrum, define $\pi_n(X) := \pi_0(\Omega^{\infty+n}X) = \pi_0(X_{-n})$. We call π_n the n -th homotopy group of X .

Remark 3. Note that since $X_n \simeq \Omega^2 X_{n+2}$, for any n , the set $\pi_0(X_n)$ underlies the structure of an abelian group.

The category Sp underlies the structure of a symmetric monoidal ∞ -category ([Lur17, Corollary 4.8.2.19]). Following [Lur17], we denote by \otimes the tensor product on Sp .

Definition 4. A commutative algebra object in Sp is called an \mathbb{E}_∞ -ring spectrum, see [Lur17, Definition 7.1.0.1]. Given an \mathbb{E}_∞ -ring spectrum R , denote by Mod_R the corresponding category of left R -module spectra, see [Lur17, Definition 7.1.1.2].

Remark 5. The sphere spectrum \mathbb{S} acts as the monoidal unit of Sp , therefore it is a \mathbb{E}_∞ -ring spectrum. The category $\mathsf{Mod}_{\mathbb{S}}$ is canonically equivalent to Sp .

Definition 6. Denote by $\mathsf{Sp}_{\geq 0} \subseteq \mathsf{Sp}$ the full sub-category generated by *connective spectra*, i.e. spectra X such that $\pi_n(X) \simeq 0$, for all $n < 0$. Denote by $\mathsf{Sp}^\heartsuit \subseteq \mathsf{Sp}_{\geq 0}$ the *heart of spectra*, i.e. the full sub-category generated by spectra X such that $\pi_n(X) \simeq 0$, for all $n > 0$.

We have the following result relating connective spectra and the heart, which follow immediately.

Lemma 7. *Let X be a connective spectrum. The following are equivalent:*

- (1) X is in the heart.
- (2) $\pi_n(\Omega^\infty X) = 0$, for all $n > 0$.
- (3) $\mathrm{Hom}_{\mathbb{S}_*}(S, \Omega^\infty X) \simeq 0$, for all connected, pointed spaces S .
- (4) X is local with respect to the class of maps $\Sigma^\infty S \rightarrow 0$, for every connected pointed space S .

The category $\mathsf{Sp}_{\geq 0}$ is presentable and π_0 induces an equivalence between the heart and Ab ([Lur17, Proposition 1.4.3.6]). The heart is a sub-category of local objects of connective spectra, therefore the inclusion $\mathsf{Ab} \simeq \mathsf{Sp}^\heartsuit \subseteq \mathsf{Sp}_{\geq 0}$ is a right adjoint. The category $\mathsf{Sp}_{\geq 0}$ is closed under \otimes and, given X, Y connective spectra,

$$(1) \quad \pi_0(X \otimes Y) \simeq \pi_0(X) \otimes \pi_0(Y)$$

see [Dav24, Theorem 2.3.28]

Definition 8. Given an abelian group A , denote by HA the (unique up to equivalence) spectrum of the heart such that $\pi_0(HA) \simeq A$. We call HA the *Eilenberg-Mac Lane spectrum* of A .

Using [Equation \(1\)](#), one can prove H , viewed as a functor $\mathcal{A}b \rightarrow \mathcal{S}p$, is lax monoidal. In particular, if R is a commutative ring, then HR is a connective \mathbb{E}_∞ -ring spectrum. On the other hand, if R is a connective \mathbb{E}_∞ -ring spectrum and M a connective module, then $\pi_0(M)$ is a $\pi_0(R)$ -module.

Definition 9. Given a commutative ring R , denote by $\text{Ch}(R) = \text{Ch}(\text{Mod}_R)$ the ordinary category of unbounded chain complexes. Let $\mathcal{D}(R)$ be the ∞ -localization of $\text{Ch}(R)$ at the class of quasi-isomorphisms.

Similar to the heart of spectra, given an \mathbb{E}_∞ -ring spectrum R , denote by $\text{Mod}_R^\heartsuit \subseteq \text{Mod}_R$ the full subcategory generated by R -modules such that the underlying spectrum belongs to the heart of spectra.

Theorem 10 (Stable Dold-Kan Correspondence). *Let R be a commutative ring.*

- (1) $\text{Mod}_R \simeq \text{Mod}_{HR}^\heartsuit$ via taking Eilenberg-Mac Lane spectra.
- (2) The equivalence in (1) extends to an equivalence $H : \mathcal{D}(R) \simeq \text{Mod}_{HR}$ of symmetric monoidal ∞ -categories.

Proof. (1) is [[Lur17](#), Proposition 7.1.1.13], while (2) is [[Lur17](#), Theorem 7.1.2.13]. □

An interesting consequence of [Theorem 10](#) is the following:

Corollary 11. *Given $F \in \mathcal{D}(R)$, then $\pi_n(HF) \simeq H_n(F)$, for all $n \in \mathbb{Z}$.*

Proof.

$$\begin{aligned}
 \pi_n(HF) &= \pi_0(\Omega^{\infty+n} HF) \\
 &\stackrel{\textcircled{1}}{\simeq} \pi_0(\text{Hom}_{\mathcal{S}p}(\Sigma^n \mathbb{S}, HF)) \\
 &\stackrel{\textcircled{2}}{\simeq} \pi_0(\text{Hom}_{\text{Mod}_{HR}}(\Sigma^n HR, HF)) \\
 &\stackrel{\textcircled{3}}{\simeq} \pi_0(\text{Hom}_{\mathcal{D}(R)}(R[n], F)) \\
 &\stackrel{\textcircled{4}}{\simeq} H_n(F)
 \end{aligned}$$

① The functor $\Omega^{\infty+n}$ is corepresented by the shifted sphere spectrum $\Sigma^n \mathbb{S}$. ② The forgetful functor $\text{Mod}_{HR} \rightarrow \text{Mod}_{\mathbb{S}} \simeq \mathcal{S}p$ is right adjoint to tensoring by HR and $HR \otimes (\Sigma^n \mathbb{S}) \simeq \Sigma^n HR$. ③ H is an equivalence of stable categories. ④ π_0 of the mapping space $\text{Hom}_{\mathcal{D}(R)}(R[n], F)$ is equivalent to the mapping space $R[n] \rightarrow F$ in the *ordinary* derived category of R , i.e. homotopy classes of maps $R[n] \rightarrow F$, which correspond exactly to classes in $H_n(F)$. □

12. FROM CHAIN COMPLEXES TO SPECTRA VIA STABLE DOLD-KAN

We now use our newly gained understanding of spectra to construct differential cohomology theories out of sheaves valued in chain complexes, via the stable Dold-Kan correspondence.

Remark 13. We identify the category of cochain complexes with $\text{Ch}(R)$ by reversing grading. Namely, given a cochain V^* , we are implicitly identifying it with the chain complex $V_n = V^{-n}$.

Definition 14 ([[BNV16](#), Definition 7.14]). Given $n \in \mathbb{Z}$, denote by $\sigma^{\geq n}$, resp. $\sigma^{\leq n}$, the *naive truncation functors*, mapping a cochain complex V^* to

$$\dots \rightarrow 0 \rightarrow V^n \rightarrow V^{n+1} \rightarrow \dots$$

resp.

$$\dots \rightarrow V^{n-1} \rightarrow V^n \rightarrow 0 \rightarrow \dots$$

Given $F : \text{Mfd}^{op} \rightarrow \text{Ch}(\mathbb{Z})$ and $\sharp \in \{\geq n, \leq n\}$, denote by F^\sharp the composite $\text{Mfd}^{op} \rightarrow \text{Ch}(\mathbb{Z}) \xrightarrow{\sigma^\sharp} \text{Ch}(\mathbb{Z})$. Notice that if F is a sheaf, then F^\sharp is also a sheaf.

Lemma 15 ([[BNV16](#), Lemma 7.12]). *Let $F : \text{Mfd}^{op} \rightarrow \text{Ch}(\mathbb{Z})$ a sheaf of chain complexes of C^∞ -modules, then $\text{Mfd}^{op} \xrightarrow{F} \text{Ch}(\mathbb{Z}) \rightarrow \mathcal{D}(\mathbb{Z})$ is a sheaf.*

Definition 16. Denote by Ω^* the sheaf $\text{Mfd}^{op} \rightarrow \text{Ch}(\mathbb{Z})$ mapping a manifold to its de Rham complex.

Lemma 15 ensures that the sheaf in **Definition 16** and the corresponding naive truncations remain sheaves after post-composition with the localization functor $\text{Ch}(\mathbb{Z}) \rightarrow \mathcal{D}(\mathbb{Z})$.

Definition 17. Given a sheaf $F : \text{Mfd}^{op} \rightarrow \mathcal{D}(\mathbb{Z})$, denote by HF the *Eilenberg-Mac Lane sheaf* of $H\mathbb{Z}$ -module spectra obtained by applying point-wise the equivalence of **Theorem 10**.

18. DELIGNE COHOMOLOGY AS A DIFFERENTIAL COHOMOLOGY THEORY

Now equipped with **Section 12**, we can finally define Deligne cohomology as a differential cohomology theory.

Definition 19. Let $k \geq 0$. The *Deligne cohomology sheaf* $\mathcal{E}(k)$ is defined via the following pullback square in $\text{Shv}(\text{Mfd}; \mathbb{S}\text{p})$:

$$\begin{array}{ccc} \mathcal{E}(k) & \longrightarrow & H(\Omega_{dR}^{\leq k}) \\ \downarrow & & \downarrow \\ H\mathbb{Z} & \longrightarrow & H\mathbb{R} \end{array}$$

Here H is the Eilenberg-MacLane sheaf.

Remark 20. If we take $k = \infty$, then the Hom $H(\Omega_{dR}) \rightarrow H\mathbb{R}$ is an equivalence, meaning $\mathcal{E}(\infty)$ is equivalent to $H\mathbb{Z}$ i.e. singular cohomology. On the other side, the individual $\mathcal{E}(k)$ are highly non-trivial and help classify many geometric invariants of interest (as we saw in the first talk). So, the $\mathcal{E}(k)$ are a non-trivial filtration of $H\mathbb{Z}$ by differential cohomology theories, in the sense that there are Hom $\mathcal{E}(k+1) \rightarrow \mathcal{E}(k)$, the limit of which is $H\mathbb{Z}$.

21. COHOMOLOGY OPERATIONS FOR DELIGNE COHOMOLOGY

Now that we have a rigorous definition of Deligne cohomology, we can start to think about operations on it. First of all, we need a suitable monoidal structure.

Definition 22. Let F, G be two differential cohomology theories. The *monoidal product* $F \otimes G$ is defined as the sheafification of the presheaf $F \wedge G$, which is the point-wise wedge product of spectra.

Now, recall there is a Hom of differential forms

$$\Omega^{\leq k} \otimes \Omega^{\leq m} \rightarrow \Omega^{\leq k+m},$$

which induces a Hom of differential cohomology theories

$$\mathcal{E}(k) \otimes \mathcal{E}(m) \rightarrow \mathcal{E}(k+m)$$

Ideally, we would like to describe such an operation in a very explicit manner, however, in the realm of spectra this can be very challenging. This suggests an alternative perspective.

Definition 23. Let $\mathcal{L}(k)$ be the sheaf of chain complexes defined as the pullback in $\text{Shv}(\text{Mfd}, D(\mathbb{Z}))$ of the following diagram

$$\begin{array}{ccc} \mathcal{L}(k) & \longrightarrow & \Omega^{\leq k} \\ \downarrow & & \downarrow dR \\ \mathbb{Z} & \longrightarrow & \mathbb{R} \end{array},$$

where \mathbb{Z} is the functor $M \mapsto C^\bullet(M, \mathbb{Z})$ and \mathbb{R} is the functor $M \mapsto C^\bullet(M, \mathbb{R})$

Remark 24. We can explicitly describe the chain complex $\mathcal{L}(k)$ as follows.

$$\mathcal{L}(k)^n = \{(c, \omega, h) \in C^n(-\mathbb{Z}) \oplus \Omega^n(-) \oplus C^{n-1}(-\mathbb{R}) \mid \omega = 0 \text{ if } n > k \text{ and } c - dR(\omega) = dh\}$$

Remark 25. We expect that $H\mathcal{L}(k)$ in fact recovers $\mathcal{E}(k)$, meaning operations on $\mathcal{L}(k)$ help us understand operations on Deligne cohomology.

Using the explicit description from **Remark 24**, we can define an operation on $\mathcal{L}(k)$ as follows:

$$(c_1, \omega_1, h_1) \otimes (c_2, \omega_2, h_2) = (c_1 \cup c_2, \omega_1 \wedge \omega_2, (-1)^{|c_1|} c_1 \cup h_2 + h_1 \cup \omega_2 + B(\omega_1, \omega_2)),$$

where

$$dR(\omega_1) \cup dR(\omega_2) = -dR(\omega_1 \wedge \omega_2) = dB(\omega_1, \omega_2)$$

It is expected that sheafification is necessary, but example is missing.

This needs to be checked.

Is there a
reasonable
way to pick
 $B(\omega_1, \omega_2)$?

Remark 26. Intuitively $B(\omega_1, \omega_2)$ measures the failure of dR taking \wedge to \cup .

Remark 27. Ideally we would expect this formula to be well-defined, meaning $(c_1, \omega_1, h_1) \otimes (c_2, \omega_2, h_2)$ should satisfy the conditions in [Remark 24](#). In general, this is only true if c_1, ω_2 satisfy $dc_1 = d\omega_2 = 0$. In particular, it is well-defined at the level of cohomology classes, as any element is closed therein.

REFERENCES

- [BNV16] Ulrich Bunke, Thomas Nikolaus, and Michael Völkl. Differential cohomology theories as sheaves of spectra. *J. Homotopy Relat. Struct.*, 11(1):1–66, 2016.
- [Dav24] Jack Davies. V4d2 - Algebraic Topology II So24 (stable and chromatic homotopy theory). *Lecture notes*, 2024. [Unpublished](#).
- [Lur17] Jacob Lurie. Higher algebra. [Available online](#), September 2017.