DIFFERENTIAL COHOMOLOGY SEMINAR 3

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In this lecture we cover the basics of sheaf theory and learn about differential cohomology theories as sheaves of spectra.

1. Sheaves in Category Theory

Let us review sheaves in classical category theory. A good reference remains [MLM94]. We start with the case of sheaves on a topological space.

Definition 1.1. Let X be a topological space, denote by $\mathcal{O}\mathrm{pen}_X$ the poset of open subsets of X, viewed as a category. A *presheaf* on X is a functor $F : (\mathcal{O}\mathrm{pen}_X)^{op} \to \mathcal{S}\mathrm{et}$.

Definition 1.2. Let X be a topological space. A presheaf is a *sheaf* if for every open $U \subseteq M$ and open cover $\{U_\alpha \subseteq U\}_\alpha$, the following is an equalizer diagram:

$$F(U) \longrightarrow \prod_\alpha F(U_\alpha) \Longrightarrow \prod_{\alpha,\beta} F(U_\alpha \cap U_\beta)$$

Definition 1.3. Let $\mathcal{C}$ be a category. A *presheaf* on $\mathcal{C}$ consists of a contravariant functor $F \colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$.

Now, for every object C we specify a class of maps $\{f_\alpha : C_\alpha \to C\}$ that *cover* C. Finally, in the sheaf condition, we replace intersections $U_\alpha \cap U_\beta$ with fiber products (assuming $\mathcal{C}$ has the necessary limits):

$$\begin{array}{ccc} C_\alpha \times_C C_\beta & \longrightarrow & C_\alpha \\ \downarrow & & \downarrow \\ C_\beta & \longrightarrow & C \end{array}$$

Similar to presheaves on a topological space, we have a similar diagram as before:

$$F(C) \longrightarrow \prod_\alpha F(C_\alpha) \Longrightarrow \prod_{\alpha,\beta} F(C_\alpha \times_C C_\beta)$$

We can then require the above diagram to be an equalizer diagram, for every object C and every covering family of C. We will use the language of *sieves* to polish the above idea into the formal notion of a *Grothendieck topology*.

2. Sheaves via Grothendieck Topologies

We now generalize from sheaves on a topological space to sheaves on a category via the additional data of a Grothendieck topology. We begin by recalling a few definitions. Denote by y the *Yoneda embedding* $\mathcal{C} \to \mathcal{P}\mathcal{S}\mathrm{h}(\mathcal{C})$, mapping C to the presheaf $y(C) := \mathrm{Hom}(-, C) \colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$.

Definition 2.1. Given pre-sheaves $G, F \colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$, we say G is a *sub-functor* of F if $G(C) \subseteq F(C)$, for all objects C, and the subset inclusions define a natural transformation $G \to F$. The last condition is equivalent to the following: Given $f \colon D \to C$ and $s \in G(C)$, then $F(f)(s) \in G(D)$.

Definition 2.2. Given a presheaf $F \colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$, denote by $\mathcal{C}_{/F}$ the category of pairs consisting of an object C and an element $x \in F(C)$. A morphism $(C', c') \to (C, c)$ consists of a morphism $f \colon C' \to C$ such that $F(f)(c) = c'$.

By Yoneda's lemma, an element $c \in F(C)$ corresponds to a natural transformation $y(C) \to F$. Taking the colimit over $(C, c) \in \mathcal{C}_{/F}$ of the pre-sheaves $y(C)$, we get a natural transformation $\mathrm{colim}_{(C,c)\in\mathcal{C}_{/F}} y(C) \to F$. The following theorem is called *density theorem*:

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Theorem 2.3. *For every pre-sheaf F, the map $\mathrm{colim}_{(C,c)\in \mathcal{C}_{/F}} y(C) \to F$ is a natural isomorphism.*

Definition 2.4. Let $\mathcal{C}$ be a category. A *sieve* on an object C is a sub-functor of $y(C)$, with $y(C)$ being the *maximal sieve*. Given a morphism $f: C' \to C$ and a sieve S on C, denote by $f^*(S)$ the *pullback sieve*, mapping D to the set of arrows $g: D \to C'$ such that $f \circ g \in S(D)$.

Definition 2.5. Let $\mathcal{F}$ be a set of morphisms with codomain $C \in \mathcal{C}$. Define the *sieve generated by $\mathcal{F}$* as the sieve $S_{\mathcal{F}} \subseteq \mathrm{Hom}_{\mathcal{C}}(-, C)$ mapping D to the set of morphisms $D \to C$ that factor through some element of $\mathcal{F}$.

Definition 2.6. Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is a collection of sieves $J(C)$, called *covering sieves*, on every object C in $\mathcal{C}$, satisfying the following axioms:

(1) The maximal sieve is a covering sieve.
(2) If S is a covering sieve and a morphism $f: C' \to C$, the pullback sieve $f^*(S)$ is a covering sieve on C'.
(3) A sieve S on C is a covering sieve if, for some covering sieve S' on C and every map $f \in S'$, the pullback sieve f^*S is also a covering sieve.

A *Grothendieck site* is a pair $(\mathcal{C}, J)$ where $\mathcal{C}$ is a category and J is a Grothendieck topology on $\mathcal{C}$.

Example 2.7. If every sieve is a covering sieve, we obtain the *discrete Grothendieck topology*. If the only covering sieves are the maximal ones, we obtain the *indiscrete Grothendieck topology*.

Example 2.8. Let X be a topological space. A sieve on an object U of $\mathcal{O}\mathrm{pen}_X$ corresponds to a collection of open subsets of U. Define a sieve to be a covering sieve on U if the corresponding family of subsets covers U in the classical sense.

Example 2.9. Let $\mathcal{T}\mathrm{op}$ be the category of topological spaces and continuous maps. Define a sieve S to be a covering sieve if generated, in the sense of Definition 2.5, by a family $\mathcal{O}$ of jointly surjective, open embeddings.

Example 2.10. Let $\mathcal{M}\mathrm{fd}$ be the category of smooth manifolds and smooth maps. We can define a topology in the same way as Example 2.9, using families of jointly surjective, smooth open embeddings.

Definition 2.11. Let $(\mathcal{C}, J)$ be a site and $\mathcal{D} \subseteq \mathcal{C}$ a full sub-category. The *induced Grothendieck topology* on $\mathcal{D}$ assigns to each object $C \in \mathcal{D}$ the collection of sieves $S \subseteq \mathrm{Hom}_{\mathcal{D}}(-, C)$ that are the restriction to $\mathcal{D}$ of covering sieves of C in $\mathcal{C}$.

We are now ready to define sheaves on a Grothendieck site. Given a sieve S on C, notice that the category $\mathcal{C}_{/S}$ can be identified with the full sub-category of $\mathcal{C}_{/C}$ spanned by morphisms $f \in S$.

Definition 2.12. Given a presheaf $F: \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$, denote by $\mathrm{Map}(\mathcal{C}_{/S}, F)$ the limit of the following diagram:

$$(\mathcal{C}_{/S})^{op} \longrightarrow \mathcal{C}^{op} \xrightarrow{\ F\ } \mathcal{S}\mathrm{et}$$

The inclusion $\mathcal{C}_{/S} \subseteq \mathcal{C}_{/C}$ induces a natural comparison map $\mathrm{Map}(\mathcal{C}_{/C}, F) \to \mathrm{Map}(\mathcal{C}_{/S}, F)$.

Notice that $\mathrm{Map}(\mathcal{C}_{/C}, F)$ is the limit of a diagram indexed by the category $(\mathcal{C}_{/C})^{op}$, which has an initial object, the identity of C, therefore the limit is canonically isomorphic to $F(C)$, the value of the diagram at the initial object.

Definition 2.13. Let $(\mathcal{C}, J)$ be a Grothendieck site. A presheaf $F: \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$ is a *sheaf* if, for every object C and covering sieve S of C, the canonical morphism

$$F(C) \cong \mathrm{Map}(\mathcal{C}_{/C}, F) \to \mathrm{Map}(\mathcal{C}_{/S}, F)$$

is an isomorphism. Explicitly, an object in $\mathrm{Map}(\mathcal{C}_{/S}, F)$ consists of an element $x_f \in F(C')$, for every map $f \in S$ (where C' is the domain of f), such that $x_{fg} = F(g)(x_f)$, for every $g: C'' \to C'$. The canonical morphism takes $x \in F(C)$ to the tuple $x_f := F(f)(x)$.

Using the density theorem and Yoneda's lemma, we have the following chain of natural isomorphisms:

$$\begin{aligned}
\mathrm{Map}(\mathcal{C}_{/S}, F) &= \lim_{(C',c')\in \mathcal{C}_{/S}} F(C') \\
&\simeq \lim_{(C',c')\in \mathcal{C}_{/S}} \mathrm{Nat}_{\mathcal{C}}(y(C'), F) \\
&\simeq \mathrm{Nat}_{\mathcal{C}}(\mathrm{colim}_{(C',c')\in \mathcal{C}_{/S}} y(C'), F) \\
&\simeq \mathrm{Nat}_{\mathcal{C}}(S, F)
\end{aligned}$$

One can then check that under the above isomorphism, the comparison map $\text{Map}(\mathcal{C}_{/C}, F) \to \text{Map}(\mathcal{C}_{/S}, F)$ corresponds to the restriction map $\text{Nat}_{\mathcal{C}}(y(C), F) \to \text{Nat}_{\mathcal{C}}(S, F)$ induced by the inclusion $S \subseteq y(C)$. In particular, we can rephrase the sheaf condition as follows:

Corollary 2.14. *A presheaf F on a Grothendieck site $(\mathcal{C}, J)$ is a sheaf if and only if for every object C and covering sieve S of C, the canonical morphism*

$$F(C) = \text{Nat}_{\mathcal{C}}(y(c), F) \to \text{Nat}_{\mathcal{C}}(S, F)$$

is an isomorphism.

3. Sheaves on ∞-Categories

Now that we know how to define sheaves on 1-categories, we generalize the definition of sheaves to the case of ∞-categories. Here we largely follow [Lur09]. Recall that a sieve on C determines a full sub-category of $\mathcal{C}_{/C}$. Conversely, a full sub-category $\mathcal{D}$ of $\mathcal{C}_{/C}$ determines a sieve if and only if $f \in \mathcal{D}$ implies $fg \in \mathcal{D}$, for every g (composable with f).

Definition 3.1. Let $\mathcal{C}$ be a ∞-category and $\mathcal{S}$ the ∞-category of spaces. A *pre-sheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to \mathcal{S}$.

Definition 3.2. Let $\mathcal{C}$ be an ∞-category and C an object of $\mathcal{C}$. A *sieve* on C is a full sub-∞-category $\mathcal{D}$ of $\mathcal{C}_{/C}$ such that $f \in \mathcal{D}$ implies $fg \in \mathcal{D}$, for every g (composable with f).

A sieve $\mathcal{D} \subseteq \mathcal{C}_{/C}$ induces a sieve $\mathcal{D}' \subseteq \text{Ho}(\mathcal{C})_{/C}$ by taking homotopy classes of 1-morphisms in $\mathcal{D}$.

Definition 3.3. Let $\mathcal{C}$ be an ∞-category. A *Grothendieck topology* on $\mathcal{C}$ is a collection $J(C)$ of sieves for every object C, such that it induces a Grothendieck topology on $\text{Ho}(\mathcal{C})$.

Notice, we have the following compatibility observation.

Lemma 3.4. *Let $\mathcal{C}$ be a 1-category. A Grothendieck topology on $\mathcal{C}$ is precisely a Grothendieck topology on $\mathcal{C}$ seen as an ∞-category.*

Definition 3.5. Let $\mathcal{C}$ be a ∞-category and C an object of $\mathcal{C}$. Given a sieve $\mathcal{D} \subseteq \mathcal{C}_{/C}$ and a pre-sheaf $F : \mathcal{C}^{op} \to \mathcal{S}$, denote by $\text{Map}(\mathcal{D}, F)$ the limit of the diagram

$$\mathcal{D}^{op} \longrightarrow \mathcal{C}^{op} \xrightarrow{\ F\ } \mathcal{S}$$

where the first functor is the opposite of the canonical projection $\mathcal{C}_{/C} \to \mathcal{C}$ restricted to $\mathcal{D}$. Using the same argument as in the 1-categorical case, there is a comparison map $\text{Map}(\mathcal{C}_{/C}, F) \to \text{Map}(\mathcal{D}, F)$ and a canonical equivalence $\text{Map}(\mathcal{C}_{/C}, F) \simeq F(C)$.

Definition 3.6. Let $\mathcal{C}$ be an ∞-category. A presheaf $F \colon \mathcal{C}^{op} \to \mathcal{S}$ is a *sheaf* if for every object C and covering sieve $\mathcal{D}$ of C, the canonical morphism

$$F(C) \simeq \text{Map}(\mathcal{C}_{/C}, F) \to \text{Map}(\mathcal{D}, F)$$

is an equivalence. Denote by $\text{Shv}(\mathcal{C}, J)$ the full sub-∞-category of $\text{Fun}(\mathcal{C}^{op}, \mathcal{S})$ spanned by the sheaves on $(\mathcal{C}, J)$.

Note we defined sheaves as local objects in a presentable ∞-category. Hence, using the formalism of presentable ∞-categories, we immediately have the following result.

Proposition 3.7. *The ∞-category of sheaves on $\text{Shv}(\mathcal{C}, J)$ is presentable.*

4. Sheaves with arbitrary Values

We started with sheaves valued in sets. Then generalized to ∞-categorical sheaves valued in spaces. However, we want ∞-categorical sheaves valued in spectra. Hence the next step is to generalize the values of our ∞-categorical sheaves. Abstractly we obtain such sheaves via the *tensor product* of presentable ∞-categories, which we review now.

Theorem 4.1 ([Lur17, Proposition 4.8.1.15])**.** *Let $\mathcal{C}, \mathcal{D}$ be presentable ∞-categories. Then there exists a presentable ∞-category $\mathcal{C} \otimes \mathcal{D}$ together with a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ such that*

(1) The functor F preserves colimits component-wise.

(2) For any presentable ∞-category $\mathcal{E}$, pre-composition with F induces an equivalence between the sub-∞-category $\mathrm{Fun}^L(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \subseteq \mathrm{Fun}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})$ of colimit preserving functors, and the sub-∞-category $\mathrm{Fun}^{L,L}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \subseteq \mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ of functors preserving colimits in each component.

Theorem 4.2 ([Lur17, Proposition 4.8.1.17])**.** *For $\mathcal{C}$ and $\mathcal{D}$ presentable ∞-categories, there is a canonical equivalence between $\mathcal{C} \otimes \mathcal{D}$ and $\mathrm{Fun}^R(\mathcal{C}^{op}, \mathcal{D})$, of limit preserving functors $\mathcal{C}^{op} \to \mathcal{D}$.*

Definition 4.3. Let (C, J) be $\mathcal{C}$ be a Grothendieck site, and $\mathcal{D}$ be a presentable ∞-category. A *sheaf on* $(\mathcal{C}, J)$ *with values in* $\mathcal{D}$ is a limit preserving functor $F \colon \mathrm{Shv}(\mathcal{C}, J)^{op} \to \mathcal{D}$, which corresponds to an object in $\mathrm{Shv}(\mathcal{C}, J) \otimes \mathcal{D}$.

Of course, this description is very abstract and ideally we want a more explicit description that we can use in the future. That is the aim of the next sections.

5. An Explicit Description of Sheaves valued in Spectra

We have given a formal definition of sheaves valued in spectra, via the tensor product of presentable ∞-categories. We now want a more explicit description thereof. For this we make use of the following result.

Proposition 5.1 ([Lur17], see also [GGN15])**.** *The inclusion $\mathrm{Pr}^L_{st} \to \mathrm{Pr}^L$ of stable presentable ∞-categories into the category of presentable ∞-categories admits a left adjoint, which is explicitly given by $- \otimes \mathrm{Sp}$.*

We can recall from our previous talk that the stabilization of a presentable ∞-category $\mathcal{C}$ is given by the ∞-category $\mathrm{Sp}(\mathcal{C})$ of spectrum objects in $\mathcal{C}$, therefore $\mathcal{C} \otimes \mathrm{Sp} \simeq \mathrm{Sp}(\mathcal{C})$, for every presentable ∞-category $\mathcal{C}$. In particular, the ∞-category $\mathrm{Shv}(\mathcal{C}, J) \otimes \mathrm{Sp}$ is equivalent to the category of spectrum objects in $\mathrm{Shv}(\mathcal{C}, J)$. Finally, since the inclusion $\mathrm{Shv}(\mathcal{C}, J) \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ preserves and reflects limits, a commutative square in $\mathrm{Shv}(\mathcal{C}, J)$ is a pullback square if and only if it is a pullback square in $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$.

Proposition 5.2. *A spectrum object in sheaves on the Grothendieck site $(\mathcal{C}, J)$ is given by a spectrum object in presheaves on $(\mathcal{C}, J)$ that is point-wise a sheaf.*

This gives us the following explicit description of sheaves valued in spectra. Recall that $\Omega^{\infty-n} \colon \mathrm{Sp} \to \mathcal{S}$ is the functor mapping a spectrum object into its n-th component.

Theorem 5.3. *There is a canonical equivalence between $\mathrm{Shv}(\mathcal{C}, J) \otimes \mathrm{Sp}$ and the category of functors $F \colon \mathcal{C}^{op} \to \mathrm{Sp}$ such that $\Omega^{\infty-n} F \colon \mathcal{C}^{op} \to \mathcal{S}$ is a sheaf of spaces, for every n.*

6. An Explicit Description of Sheaves on the Site of Manifolds

We continue our analysis of sheaves with the aim of providing explicit descriptions. Now we focus on the case of sheaves on the site of manifolds. Recall that a sieve S on a manifold M is a covering sieve if an open cover $\mathcal{O}$ of M exists, such that every morphism in S factors through the inclusion $U \subseteq M$ of some element $U \in \mathcal{O}$. From the definition of the topology on Mfd we have the following: Given a covering sieve S generated by an open cover $\mathcal{O}$. Denote by $\mathcal{I}(\mathcal{O})$ the poset (viewed as a category) consisting of finite intersections of elements of $\mathcal{O}$.

Lemma 6.1. *The functor $i \colon \mathcal{I}(\mathcal{O}) \to \mathrm{Mfd}_{/S}$ is cofinal.*

In particular, since $\mathrm{Map}(\mathrm{Mfd}_{/S}, F)$ is defined as the limit $(\mathrm{Mfd}_{/S})^{op} \to \mathrm{Mfd}^{op} \xrightarrow{F} \mathcal{S}$ and i^{op} is *final*, there is a canonical equivalence

$$\mathrm{Map}(\mathrm{Mfd}_{/S}, F) \simeq \lim(\mathcal{I}(\mathcal{O})^{op} \to \mathrm{Mfd}^{op} \xrightarrow{F} \mathcal{S}) = \lim_{U \in \mathcal{I}(\mathcal{O})} F(U)$$

Proposition 6.2. *A presheaf $F \colon \mathrm{Mfd}^{op} \to \mathrm{Sp}$ is a sheaf if and only if for every manifold M and every open cover $\mathcal{O}$ of M, the canonical morphism $F(M) \to \lim_{U \in \mathcal{I}(\mathcal{O})} F(U)$ is an equivalence.*

In fact we can check the sheaf condition for sheaves on the site of manifolds with a very specific type of covering families. For a proof of the following result, we refer to [ADH21, Proposition 3.6.6].

Proposition 6.3. *A presheaf $F \colon \mathrm{Mfd}^{op} \to \mathrm{Sp}$ is a sheaf if and only if*

(1) $F(\emptyset)$ *is terminal.*

(2) *For every $M = U \cup V$, the canonical square*

$$
\begin{array}{ccc}
F(M) & \longrightarrow & F(U) \\
\downarrow & & \downarrow \\
F(V) & \longrightarrow & F(U \cap V)
\end{array}
$$

   is a pullback square.

(3) *For a sequential family of opens $U_1 \subseteq U_2 \subseteq \cdots$ covering M, the canonical map $F(M) \to \lim_n F(U_n)$ is an equivalence.*

7. Equivalences of Sheaves

We now have established a very solid understanding of sheaves on the site of manifolds with arbitrary values as presheaves with suitable limits conditions. We now want to use these explicit descriptions to better understand equivalences of sheaves on the site of manifolds.

Definition 7.1 ([Lur09, Remark 6.5.4.8])**.** Given a site $\mathcal{C}$, a *point* of $\mathcal{S}hv(\mathcal{C})$ is a functor $p : \mathcal{S} \to \mathcal{S}hv(\mathcal{C})$ admitting a left exact left adjoint p^*. We say $\mathcal{S}hv(\mathcal{C})$ has *enough points* if the following holds: Given a morphism f, if $p^*(f)$ is an equivalence for every point p, then f is an equivalence.

Remark 7.2. A functor $p : \mathcal{S} \to \mathcal{S}hv(\mathcal{C})$ admitting a left exact left adjoint is called a *geometric morphism*, see [Lur09, Definition 6.3.1.1].

Definition 7.3. Let X be a topological space and $\mathcal{C}$ a presentable category. Given a point $x \in X$, denote by x^* the functor $\mathcal{S}hv(X, \mathcal{C}) \to \mathcal{C}$ given by $x^*(F) = \mathrm{colim}_{x \in U} F(U)$. The functor x^* is called *stalk at x* and is left adjoint to the *skyscraper at x* functor x_* given by $x_*(S)(U) = S$, if $x \in U$, and $= 1$, otherwise (where $1 \in \mathcal{C}$ is the terminal object).

Remark 7.4. Using that filtered colimits commute with finite limits (see [Lur09, Example 7.3.4.7]), one can prove that x^* is left exact and x_* is a point of $\mathcal{S}hv(X)$ according to Definition 7.1. In particular, every point $x \in X$ induces a point of $\mathcal{S}hv(X)$.

Theorem 7.5. *Let X be a smooth manifold, then $\mathcal{S}hv(X)$ has enough points.*

Proof. [Lur09, Corollary 7.2.1.17] and [Lur09, Theorem 7.2.3.6] reduce the claim to X having finite Lebesgue covering dimension. $\qquad\square$

Remark 7.6. The proof of [Lur09, Corollary 7.2.1.17] actually shows that the collection of points $x_* : \mathcal{S} \to \mathcal{S}hv(X)$, for all $x \in X$, are enough to detect equivalences, namely f is an equivalence if $x^*(f)$ is an equivalence, for every $x \in X$.

Theorem 7.7. *The ∞-category $\mathcal{S}hv(\mathcal{M}fd)$ has enough points.*

Proof. Let M be a smooth manifold, there is a clear faithful functor $\mathcal{O}pen_M \to \mathcal{M}fd$ that is a morphism of sites (see [Pst22, Definition A.10]) and satisfies the *covering lifting property* (see [Pst22, Definition A.12]), therefore the restriction functor $-|_M : \mathcal{S}hv(\mathcal{M}fd) \to \mathcal{S}hv(M)$ is a right adjoint (hence left exact) and right adjoint (see [Pst22, Proposition A.13]). In particular, a point $p : \mathcal{S} \to \mathcal{S}hv(M)$, for any manifold M, induces a point of $\mathcal{S}hv(\mathcal{M}fd)$ as the composition of $p : \mathcal{S} \to \mathcal{S}hv(M)$ with the right adjoint $\mathcal{S}hv(M) \to \mathcal{S}hv(\mathcal{M}fd)$. The left adjoint is then $p^*(-|_M)$.

Let f be a morphism such that $p^*(f)$ is an equivalence for every point p of $\mathcal{S}hv(\mathcal{M}fd)$. In particular, $p^*(f|_M)$ is an equivalence for every point of $\mathcal{S}hv(M)$, hence $f|_M$ is an equivalence in $\mathcal{S}hv(M)$, by Theorem 7.5. Finally, if $f|_M$ is an equivalence, for every M, then f is an equivalence. $\qquad\square$

The proof of Theorem 7.7 together with Remark 7.6 implies the following:

Corollary 7.8. *A morphism f in $\mathcal{S}hv(\mathcal{M}fd)$ is an equivalence if and only if for every manifold M and every point $x \in M$, the map on stalks $x^*(f)$ is an equivalence.*

Note, the general condition above is equivalent to the following simpler condition. Denote by 0_n the origin of $\mathbb{R}^n$.

Corollary 7.9. *A morphism f of sheaves on Mfd is an equivalence if and only if for all $n \geq 0$, the induced map on stalks $(0_n)^*(f)$ is an equivalence.*

Definition 7.10. Denote by $\mathcal{E}\mathrm{uc}$ the full sub-category of Mfd generated by the Euclidean manifolds $\mathbb{R}^n$, for every $n \geq 0$. The category $\mathcal{E}\mathrm{uc}$ inherits a Grothendieck topology from Mfd according to Definition 2.11.

Definition 7.11 ([BGH20, Definition 3.12.2]). Let $\mathcal{C}$ be a site. A full sub-category $\mathcal{B} \subseteq \mathcal{C}$ is *dense* if for every object $C \in \mathcal{C}$ there is a family $\mathcal{F}$ of morphisms with codomain C and domain in $\mathcal{B}$, such that $S_{\mathcal{F}}$ (see Definition 2.5) is a covering sieve.

$\mathcal{E}\mathrm{uc} \subseteq \mathrm{Mfd}$ is dense in the sense of Definition 7.11.

Theorem 7.12. *Let $j : \mathcal{E}\mathrm{uc} \subseteq \mathrm{Mfd}$ be the inclusion functor. Let $\mathcal{C}$ be a presentable ∞-category, the following hold:*

(1) The functor $j_* : \mathrm{Fun}(\mathcal{E}\mathrm{uc}^{op}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Mfd}^{op}, \mathcal{C})$ preserves sheaves.

(2) $\mathrm{Shv}(\mathrm{Mfd})$ and $\mathrm{Shv}(\mathcal{E}\mathrm{uc})$ are hypercomplete (see [Lur09, §§6.5.4]) and $j_* : \mathrm{Shv}(\mathcal{E}\mathrm{uc}, \mathcal{C}) \to \mathrm{Shv}(\mathrm{Mfd}, \mathcal{C})$ is an equivalence.

(3) The functor $j^* : \mathrm{Fun}(\mathrm{Mfd}^{op}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{E}\mathrm{uc}^{op}, \mathcal{C})$ right adjoint to j_*, preserves sheaves. In particular, $(j_*, j^*) : \mathrm{Shv}(\mathcal{E}\mathrm{uc}, \mathcal{C}) \to \mathrm{Shv}(\mathrm{Mfd}, \mathcal{C})$ is an adjoint equivalence.

Proof.   (1) [BGH20, Lemma 3.12.7].

(2) Hypercompleteness of $\mathrm{Shv}(\mathrm{Mfd})$ is a consequence of Theorem 7.7, since having enough points imply hypercompleteness, see [BGH20, Example 3.11.10]. [BGH20, Corollary 3.12.13] implies Hypercompleteness of $\mathrm{Shv}(\mathcal{E}\mathrm{uc})$ and that j_* induces an equivalence $\mathrm{Shv}(\mathcal{E}\mathrm{uc}, \mathcal{S}) \to \mathrm{Shv}(\mathrm{Mfd}, \mathcal{S})$ (to generalize from $\mathcal{S}$ to any presentable ∞-category $\mathcal{C}$, simply tensor with $\mathcal{C}$)

(3) Hypercompleteness implies every sheaf is an hypercomplete sheaf and by [BGH20, Proposition 3.12.11], j^* preserves hypercomplete sheaves.

$\square$

Remark 7.13. Some background on hypercovers and hypercompleteness can be found in [Lur09, §§6.5.3] and [DHI03].

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