DIFFERENTIAL COHOMOLOGY SEMINAR 4 (DRAFT)

TALK BY ALESSANDRO NANTO

In the last talk we learned the definition of a differential cohomology theory, as a sheaf valued in spectra on the site of manifolds. This talk continues our journey through differential cohomology theories, and focuses on the following three topics:

(1) We want to learn how to construct non-trivial examples out of sheaves valued in chain complexes.
(2) We want to understand how we can extend classical cohomology operations to the setting of differential cohomology theories.
(3) We want to introduce suitable analogues of fiber-wise integration.

1. Abelian Groups, Spectra and the Heart

Let us start by reviewing the relation between abelian groups, rings and spectra.

Definition 1.1. Let $n \in \mathbb{Z}$ and X be a spectrum, define $\pi_n(X) := \pi_0(\Omega^{\infty+n}X) = \pi_0(X_{-n})$. We call π_n the *n-th homotopy group* of X.

Remark 1.2. Note that since $X_n \simeq \Omega^2 X_{n+2}$, for any n, the set $\pi_0(X_n)$ underlies the structure of an abelian group.

The category $\mathcal{S}\mathrm{p}$ underlies the structure of a symmetric monoidal ∞-category ([Lur17, Corollary 4.8.2.19]). Following [Lur17], we denote by $\otimes$ the tensor product on $\mathcal{S}\mathrm{p}$.

Definition 1.3. A commutative algebra object in $\mathcal{S}\mathrm{p}$ is called an *$\mathbb{E}_\infty$-ring spectrum*, see [Lur17, Definition 7.1.0.1]. Given an $\mathbb{E}_\infty$-ring spectrum R, denote by Mod_R the corresponding category of *left R-module spectra*, see [Lur17, Definition 7.1.1.2].

Remark 1.4. The sphere spectrum $\mathbb{S}$ acts as the monoidal unit of $\mathcal{S}\mathrm{p}$, therefore it is a $\mathbb{E}_\infty$-ring spectrum. The category $\mathrm{Mod}_{\mathbb{S}}$ is canonically equivalent to $\mathcal{S}\mathrm{p}$.

Definition 1.5. Denote by $\mathcal{S}\mathrm{p}_{\geq 0} \subseteq \mathcal{S}\mathrm{p}$ the full sub-category generated by *connective spectra*, i.e. spectra X such that $\pi_n(X) \simeq 0$, for all $n < 0$. Denote by $\mathcal{S}\mathrm{p}^{\heartsuit} \subseteq \mathcal{S}\mathrm{p}_{\geq 0}$ the *heart of spectra*, i.e. the full sub-category generated by spectra X such that $\pi_n(X) \simeq 0$, for all $n > 0$.

We have the following result relating connective spectra and the heart, which follow immediately.

Lemma 1.6. *Let X be a connective spectrum. The following are equivalent:*

(1) X is in the heart.
(2) $\pi_n(\Omega^\infty X) = 0$, for all $n > 0$.
(3) $\mathrm{Hom}_{\mathcal{S}_*}(S, \Omega^\infty X) \simeq 0$, for all connected, pointed spaces S.
(4) X is local with respect to the class of maps $\Sigma^\infty S \to 0$, for every connected pointed space S.

The category $\mathcal{S}\mathrm{p}_{\geq 0}$ is presentable and π_0 induces an equivalence between the heart and $\mathcal{A}\mathrm{b}$ ([Lur17, Proposition 1.4.3.6]). The heart is a sub-category of local objects of connective spectra, therefore the inclusion $\mathcal{A}\mathrm{b} \simeq \mathcal{S}\mathrm{p}^{\heartsuit} \subseteq \mathcal{S}\mathrm{p}_{\geq 0}$ is a right adjoint. The category $\mathcal{S}\mathrm{p}_{\geq 0}$ is closed under $\otimes$ and, given X, Y connective spectra,

$$\pi_0(X \otimes Y) \simeq \pi_0(X) \otimes \pi_0(Y) \tag{1.7}$$

see [Dav24, Theorem 2.3.28]

Definition 1.8. Given an abelian group A, denote by HA the (unique up to equivalence) spectrum of the heart such that $\pi_0(HA) \simeq A$. We call HA the *Eilenberg-Mac Lane spectrum* of A.

Date: 03.06.2025 & 17.06.2025.

Using Equation (1.7), one can prove H, viewed as a functor $\mathcal{A}\mathrm{b} \to \mathcal{S}\mathrm{p}$, is lax monoidal. In particular, if R is a commutative ring, then HR is a connective $\mathbb{E}_\infty$-ring spectrum. On the other hand, if R is a connective $\mathbb{E}_\infty$-ring spectrum and M a connective module, then $\pi_0(M)$ is a $\pi_0(R)$-module.

Definition 1.9. Given a commutative ring R, denote by $\mathrm{Ch}(R) = \mathrm{Ch}(\mathrm{Mod}_R)$ the ordinary category of unbounded chain complexes. Let $\mathcal{D}(R)$ be the ∞-localization of $\mathrm{Ch}(R)$ at the class of quasi-isomorphisms.

Similar to the heart of spectra, given an $\mathbb{E}_\infty$-ring spectrum R, denote by $\mathrm{Mod}_R^\heartsuit \subseteq \mathrm{Mod}_R$ the full subcategory generated by R-modules such that the underlying spectrum belongs to the heart of spectra.

Theorem 1.10 (Stable Dold-Kan Correspondence)**.** *Let R be a commutative ring.*

(1) $\mathrm{Mod}_R \simeq \mathrm{Mod}_{HR}^\heartsuit$ via taking Eilenberg-Mac Lane spectra.
(2) The equivalence in (1) extends to an equivalence $H : \mathcal{D}(R) \simeq \mathrm{Mod}_{HR}$ of symmetric monoidal ∞-categories.

Proof. (1) is [Lur17, Proposition 7.1.1.13], while (2) is [Lur17, Theorem 7.1.2.13]. □

An interesting consequence of Theorem 1.10 is the following:

Corollary 1.11. *Given $F \in \mathcal{D}(R)$, then $\pi_n(HF) \simeq H_n(F)$, for all $n \in \mathbb{Z}$.*

Proof.

$$\begin{aligned}\pi_n(HF) &= \pi_0(\Omega^{\infty+n}HF)\\ &\overset{①}{\simeq} \pi_0(\mathrm{Hom}_{\mathcal{S}\mathrm{p}}(\Sigma^n\mathbb{S}, HF))\\ &\overset{②}{\simeq} \pi_0(\mathrm{Hom}_{\mathrm{Mod}_{HR}}(\Sigma^n HR, HF))\\ &\overset{③}{\simeq} \pi_0(\mathrm{Hom}_{\mathcal{D}(R)}(R[n], F))\\ &\overset{④}{\simeq} H_n(F)\end{aligned}$$

① The functor $\Omega^{\infty+n}$ is corepresented by the shifted sphere spectrum $\Sigma^n\mathbb{S}$. ② The forgetful functor $\mathrm{Mod}_{HR} \to \mathrm{Mod}_{\mathbb{S}} \simeq \mathcal{S}\mathrm{p}$ is right adjoint to tensoring by HR and $HR \otimes (\Sigma^n\mathbb{S}) \simeq \Sigma^n HR$. ③ H is an equivalence of stable categories. ④ π_0 of the mapping space $\mathrm{Hom}_{\mathcal{D}(R)}(R[n], F)$ is equivalent to the mapping space $R[n] \to F$ in the *ordinary* derived category of R, i.e. homotopy classes of maps $R[n] \to F$, which correspond exactly to classes in $H_n(F)$. □

2. From Chain Complexes to Spectra via stable Dold-Kan

We now use our newly gained understanding of spectra to construct differential cohomology theories out of sheaves valued in chain complexes, via the stable Dold-Kan correspondence.

Remark 2.1. We identify the category of cochain complexes with $\mathrm{Ch}(R)$ by reversing grading. Namely, given a cochain V^*, we are implicitly identifying it with the chain complex $V_n = V^{-n}$.

Definition 2.2 ([BNV16, Definition 7.14])**.** Given $n \in \mathbb{Z}$, denote by $\sigma^{\geq n}$, resp. $\sigma^{\leq n}$, the *naive truncation functors*, mapping a cochain complex V^* to

$$\cdots \to 0 \to V^n \to V^{n+1} \to \cdots$$

resp.

$$\cdots \to V^{n-1} \to V^n \to 0 \to \cdots$$

Given $F : \mathcal{M}\mathrm{fd}^{op} \to \mathrm{Ch}(\mathbb{Z})$ and $\sharp \in \{\geq n, \leq n\}$, denote by $F^\sharp$ the composite $\mathcal{M}\mathrm{fd}^{op} \to \mathrm{Ch}(\mathbb{Z}) \xrightarrow{\sigma^\sharp} \mathrm{Ch}(\mathbb{Z})$. Notice that if F is a sheaf, then $F^\sharp$ is also a sheaf.

Lemma 2.3 ([BNV16, Lemma 7.12])**.** *Let $F : \mathcal{M}\mathrm{fd}^{op} \to \mathrm{Ch}(\mathbb{Z})$ a sheaf of chain complexes of C^∞-modules, then $\mathcal{M}\mathrm{fd}^{op} \xrightarrow{F} \mathrm{Ch}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$ is a sheaf.*

Definition 2.4. Denote by Ω^* the sheaf $\mathcal{M}\mathrm{fd}^{op} \to \mathrm{Ch}(\mathbb{Z})$ mapping a manifold to its de Rham complex.

Lemma 2.3 ensures that the sheaf in Definition 2.4 and the corresponding naive truncations remain sheaves after post-composition with the localization functor $\mathrm{Ch}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$.

Definition 2.5. Given a sheaf $F : \mathcal{M}\mathrm{fd}^{op} \to \mathcal{D}(\mathbb{Z})$, denote by HF the *Eilenberg-Mac Lane sheaf* of $H\mathbb{Z}$-module spectra obtained by applying point-wise the equivalence of Theorem 1.10.

3. Deligne Cohomology as a Differential Cohomology Theory

Now equipped with Definition 2.5, we can finally define Deligne cohomology as a differential cohomology theory.

Definition 3.1. Let $k \geq 0$. The *Deligne cohomology sheaf* $\mathcal{E}(k)$ is defined via the following pullback square in $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}; \mathcal{S}\mathrm{p})$:

$$\begin{tikzcd} \mathcal{E}(k) \arrow[r] \arrow[d] & H(\Omega^{\leq k}_{dR}) \arrow[d] \\ H\mathbb{Z} \arrow[r] & H\mathbb{R} \end{tikzcd}$$

Here H is the Eilenberg-MacLane sheaf.

Remark 3.2. If we take $k = \infty$, then the Hom $H(\Omega_{dR}) \to H\mathbb{R}$ is an equivalence, meaning $\mathcal{E}(\infty)$ is equivalent to $H\mathbb{Z}$ i.e. singular cohomology. On the other side, the individual $\mathcal{E}(k)$ are highly non-trivial and help classify many geometric invariants of interest (as we saw in the first talk). So, the $\mathcal{E}(k)$ are a non-trivial filtration of $H\mathbb{Z}$ by differential cohomology theories, in the sense that there are Hom $\mathcal{E}(k+1) \to \mathcal{E}(k)$, the limit of which is $H\mathbb{Z}$.

4. Cohomology Operations for Deligne Cohomology

Now that we have a rigorous definition of Deligne cohomology, we can start to think about operations on it. First of all, we need a suitable monoidal structure.

Definition 4.1. Let F, G be two differential cohomology theories. The *monoidal product* $F \otimes G$ is defined as the sheafification of the presheaf $F \wedge G$, which is the point-wise wedge product of spectra. It is expected that sheafification is necessary, but example is missing.

Now, recall there is a Hom of differential forms

$$\Omega^{\leq k} \otimes \Omega^{\leq m} \to \Omega^{\leq k+m},$$

which induces a Hom of differential cohomology theories

$$\mathcal{E}(k) \otimes \mathcal{E}(m) \to \mathcal{E}(k+m)$$

Ideally, we would like to describe such an operation in a very explicit manner, however, in the realm of spectra this can be very challenging. This suggests an alternative perspective.

Definition 4.2. Let $\mathcal{L}(k)$ be the sheaf of chain complexes defined as the pullback in $\mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}, D(\mathbb{Z}))$ of the following diagram

$$\begin{tikzcd} \mathcal{L}(k) \arrow[r] \arrow[d] & \Omega^{\leq k} \arrow[d, "dR"] \\ \mathbb{Z} \arrow[r] & \mathbb{R} \end{tikzcd},$$

where $\mathbb{Z}$ is the functor $M \mapsto C^\bullet(M, \mathbb{Z})$ and $\mathbb{R}$ is the functor $M \mapsto C^\bullet(M, \mathbb{R})$

Remark 4.3. We can explicitly describe the chain complex $\mathcal{L}(k)$ as follows.

$$\mathcal{L}(k)^n = \{(c, \omega, h) \in C^n(-\mathbb{Z}) \oplus \Omega^n(-) \oplus C^{n-1}(-\mathbb{R}) | \omega = 0 if n > k and c - dR(\omega) = dh\}$$

Remark 4.4. We expect that $H\mathcal{L}(k)$ in fact recovers $\mathcal{E}(k)$, meaning operations on $\mathcal{L}(k)$ help us understand operations on Deligne cohomology. This needs to be checked.

Using the explicit description from Remark 4.3, we can define an operation on $\mathcal{L}(k)$ as follows:

$$(c_1, \omega_1, h_1) \otimes (c_2, \omega_2, h_2) = (c_1 \cup c_2, \omega_1 \wedge \omega_2, (-1)^{|c_1|} c_1 \cup h_2 + h_1 \cup \omega_2 + B(\omega_1, \omega_2)),$$

where

$$dR(\omega_1) \cup dR(\omega_2) = -dR(\omega_1 \wedge \omega_2) = dB(\omega_1, \omega_2)$$

Is there a reasonable way to pick $B(\omega_1, \omega_2)$?

Remark 4.5. Intuitively $B(\omega_1, \omega_2)$ measures the failure of dR taking $\wedge$ to $\cup$.

Remark 4.6. Ideally we would expect this formula to be well-defined, meaning $(c_1, \omega_1, h_1) \otimes (c_2, \omega_2, h_2)$ should satisfy the conditions in Remark 4.3. In general, this is only true if c_1, ω_2 satisfy $dc_1 = d\omega_2 = 0$. In particular, it is well-defined at the level of cohomology classes, as any element is closed therein.

References

[BNV16] Ulrich Bunke, Thomas Nikolaus, and Michael Völkl. Differential cohomology theories as sheaves of spectra. *J. Homotopy Relat. Struct.*, 11(1):1–66, 2016.

[Dav24] Jack Davies. V4d2 - Algebraic Topology II So24 (stable and chromatic homotopy theory). *Lecture notes*, 2024. Unpublished.

[Lur17] Jacob Lurie. Higher algebra. Available online, September 2017.