# DIFFERENTIAL COHOMOLOGY SEMINAR 3

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In this lecture we cover the basics of sheaf theory and learn about differential cohomology theories as sheaves of spectra.

## 1. Sheaves in Category Theory

Let us review sheaves in classical category theory. A good reference remains [MLM94]. We start with the case of sheaves on a topological space.

**Definition 1.** Let $X$ be a topological space, denote by $\mathcal{O}\mathrm{pen}_X$ the poset of open subsets of $X$, viewed as a category. A *presheaf* on $X$ is a functor $F : (\mathcal{O}\mathrm{pen}_X)^{op} \to \mathcal{S}\mathrm{et}$.

**Definition 2.** Let $X$ be a topological space. A presheaf is a *sheaf* if for every open $U \subseteq M$ and open cover $\{U_\alpha \subseteq U\}_\alpha$, the following is an equalizer diagram:

$$F(U) \longrightarrow \prod_\alpha F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_\alpha \cap U_\beta)$$

**Definition 3.** Let $\mathcal{C}$ be a category. A *presheaf* on $\mathcal{C}$ consists of a contravariant functor $F\colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$.

Now, for every object $C$ we specify a class of maps $\{f_\alpha : C_\alpha \to C\}$ that *cover* $C$. Finally, in the sheaf condition, we replace intersections $U_\alpha \cap U_\beta$ with fiber products (assuming $\mathcal{C}$ has the necessary limits):

$$\begin{tikzcd} C_\alpha \times_C C_\beta \arrow[r] \arrow[d] & C_\alpha \arrow[d] \\ C_\beta \arrow[r] & C \end{tikzcd}$$

Similar to presheaves on a topological space, we have a similar diagram as before:

$$F(C) \longrightarrow \prod_\alpha F(C_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(C_\alpha \times_C C_\beta)$$

We can then require the above diagram to be an equalizer diagram, for every object $C$ and every covering family of $C$. We will use the language of *sieves* to polish the above idea into the formal notion of a *Grothendieck topology.*

## 2. Sheaves via Grothendieck Topologies

We now generalize from sheaves on a topological space to sheaves on a category via the additional data of a Grothendieck topology. We begin by recalling a few definitions. Denote by $y$ the *Yoneda embedding* $\mathcal{C} \to \mathcal{PS}\mathrm{h}(\mathcal{C})$, mapping $C$ to the presheaf $y(C) := \mathrm{Hom}(-, C)\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$.

**Definition 4.** Given pre-sheaves $G, F\colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$, we say $G$ is a *sub-functor* of $F$ if $G(C) \subseteq F(C)$, for all objects $C$, and the subset inclusions define a natural transformation $G \to F$. The last condition is equivalent to the following: Given $f\colon D \to C$ and $s \in G(C)$, then $F(f)(s) \in G(D)$.

**Definition 5.** Given a presheaf $F\colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$, denote by $\mathcal{C}_{/F}$ the category of pairs consisting of an object $C$ and an element $x \in F(C)$. A morphism $(C', c') \to (C, c)$ consists of a morphism $f\colon C' \to C$ such that $F(f)(c) = c'$.

By Yoneda's lemma, an element $c \in F(C)$ corresponds to a natural transformation $y(C) \to F$. Taking the colimit over $(C, c) \in \mathcal{C}_{/F}$ of the pre-sheaves $y(C)$, we get a natural transformation $\mathrm{colim}_{(C,c)\in\mathcal{C}_{/F}} y(C) \to F$. The following theorem is called *density theorem*:

*Date*: 21.05.2025 & 27.05.2025.

**Theorem 6.** *For every pre-sheaf $F$, the map* $\mathrm{colim}_{(C,c)\in\mathcal{C}_{/F}} y(C) \to F$ *is a natural isomorphism.*

**Definition 7.** Let $\mathcal{C}$ be a category. A *sieve* on an object $C$ is a sub-functor of $y(C)$, with $y(C)$ being the *maximal sieve*. Given a morphism $f\colon C' \to C$ and a sieve $S$ on $C$, denote by $f^*S$ the *pullback sieve*, mapping $D$ to the set of arrows $g\colon D \to C'$ such that $f \circ g \in S(D)$.

**Definition 8.** Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is a collection of sieves $J(C)$, called *covering sieves*, on every object $C$ in $\mathcal{C}$, satisfying the following axioms:

(1) The maximal sieve is a covering sieve.
(2) If $S$ is a covering sieve and a morphism $f\colon C' \to C$, the pullback sieve $f^*S$ is a covering sieve on $C'$.
(3) A sieve $S$ on $C$ is a covering sieve if, for some covering sieve $S'$ on $C$ and every map $f \in S'$, the pullback sieve $f^*S$ is also a covering sieve.

A *Grothendieck site* is a pair $(\mathcal{C}, J)$ where $\mathcal{C}$ is a category and $J$ is a Grothendieck topology on $\mathcal{C}$.

**Example 9.** If every sieve is a covering sieve, we obtain the *discrete Grothendieck topology*. If the only covering sieves are the maximal ones, we obtain the *indiscrete Grothendieck topology*.

**Example 10.** Let $X$ be a topological space. A sieve on an object $U$ of $\mathcal{O}\mathrm{pen}_X$ corresponds to a collection of open subsets of $U$. Define a sieve to be a covering sieve on $U$ if the corresponding family of subsets covers $U$ in the classical sense.

**Example 11.** Let $\mathcal{T}\mathrm{op}$ be the category of topological spaces and continuous maps. Given a family $\mathcal{O}$ open embeddings covering $X$, denote by $S_{\mathcal{O}}$ the sieve mapping $T$ to the set of maps $T \to X$ factoring through a map in $\mathcal{O}$. Define a sieve $S$ to be a covering sieve if of the form $S_{\mathcal{O}}$, for some family $\mathcal{O}$.

**Example 12.** Let $\mathcal{M}\mathrm{fd}$ be the category of smooth manifolds and smooth maps. We can define a Grothendieck topology on $\mathcal{M}\mathrm{fd}$ similar to the above Grothendieck topology on $\mathcal{T}\mathrm{op}$.

We are now ready to define sheaves on a Grothendieck site. Given a sieve $S$ on $C$, notice that the category $\mathcal{C}_{/S}$ can be identified with the full sub-category of $\mathcal{C}_{/C}$ spanned by morphisms $f \in S$.

**Definition 13.** Given a presheaf $F\colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$, denote by $\mathrm{Map}(\mathcal{C}_{/S}, F)$ the limit of the following diagram:

$$(\mathcal{C}_{/S})^{op} \longrightarrow \mathcal{C}^{op} \xrightarrow{F} \mathcal{S}\mathrm{et}$$

The inclusion $\mathcal{C}_{/S} \subseteq \mathcal{C}_{/C}$ induces a natural comparison map $\mathrm{Map}(\mathcal{C}_{/C}, F) \to \mathrm{Map}(\mathcal{C}_{/S}, F)$.

Notice that $\mathrm{Map}(\mathcal{C}_{/C}, F)$ is the limit of a diagram indexed by the category $(\mathcal{C}_{/C})^{op}$, which has an initial object, the identity of $C$, therefore the limit is canonically isomorphic to $F(C)$, the value of the diagram at the initial object.

**Definition 14.** Let $(\mathcal{C}, J)$ be a Grothendieck site. A presheaf $F : \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$ is a *sheaf* if, for every object $C$ and covering sieve $S$ of $C$, the canonical morphism

$$F(C) \cong \mathrm{Map}(\mathcal{C}_{/C}, F) \to \mathrm{Map}(\mathcal{C}_{/S}, F)$$

is an isomorphism. Explicitly, an object in $\mathrm{Map}(\mathcal{C}_{/S}, F)$ consists of an element $x_f \in F(C')$, for every map $f \in S$ (where $C'$ is the domain of $f$), such that $x_{fg} = F(g)(x_f)$, for every $g\colon C'' \to C'$. The canonical morphism takes $x \in F(C)$ to the tuple $x_f := F(f)(x)$.

Using the density theorem and Yoneda's lemma, we have the following chain of natural isomorphisms:

$$\begin{aligned}
\mathrm{Map}(\mathcal{C}_{/S}, F) &= \mathrm{colim}_{(C',c')\in\mathcal{C}_{/S}} F(C') \\
&\simeq \mathrm{colim}_{(C',c')\in\mathcal{C}_{/S}} \mathrm{Nat}_{\mathcal{C}}(y(C'), F) \\
&\simeq \mathrm{Nat}_{\mathcal{C}}(\mathrm{colim}_{(C',c')\in\mathcal{C}_{/S}} y(C'), F) \\
&\simeq \mathrm{Nat}_{\mathcal{C}}(S, F)
\end{aligned}$$

One can then check that under the above isomorphism, the comparison map $\mathrm{Map}(\mathcal{C}_{/C}, F) \to \mathrm{Map}(\mathcal{C}_{/S}, F)$ corresponds to the restriction map $\mathrm{Nat}_{\mathcal{C}}(y(C), F) \to \mathrm{Nat}_{\mathcal{C}}(S, F)$ induced by the inclusion $S \subseteq y(C)$. In particular, we can rephrase the sheaf condition as follows:

**Corollary 15.** *A presheaf $F$ on a Grothendieck site $(\mathcal{C}, J)$ is a sheaf if and only if for every object $C$ and covering sieve $S$ of $C$, the canonical morphism*

$$F(C) = \mathrm{Nat}_{\mathcal{C}}(y(c), F) \to \mathrm{Nat}_{\mathcal{C}}(S, F)$$

*is an isomorphism.*

## 3. Sheaves on $\infty$-Categories

Now that we know how to define sheaves on 1-categories, we generalize the definition of sheaves to the case of $\infty$-categories. Here we largely follow [Lur09]. Recall that a sieve on $C$ determines a full sub-category of $\mathcal{C}_{/C}$. Conversely, a full sub-category $\mathcal{D}$ of $\mathcal{C}_{/C}$ determines a sieve if $f \in \mathcal{D}$ implies $fg \in \mathcal{D}$, for every $g$ (pre-composable with $f$).

**Definition 16.** Let $\mathcal{C}$ be a $\infty$-category and $\mathcal{S}$ the $\infty$-category of spaces. A *pre-sheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to \mathcal{S}$.

**Definition 17.** Let $\mathcal{C}$ be an $\infty$-category and $C$ an object of $\mathcal{C}$. A *sieve* on $C$ is a full sub-$\infty$-category $\mathcal{D}$ of $\mathcal{C}_{/C}$ such that $\mathrm{Ho}(\mathcal{D}) \subseteq \mathrm{Ho}(\mathcal{C}_{/C}) \simeq \mathrm{Ho}(\mathcal{C})_{/C}$ is a sieve over $C$ in the homotopy category.

**Definition 18.** Let $\mathcal{C}$ be an $\infty$-category. A *Grothendieck topology* on $\mathcal{C}$ is a collection $J(C)$ of sieves for every object $C$, such that it induces a Grothendieck topology on $\mathrm{Ho}(\mathcal{C})$.

Notice, we have the following compatibility observation.

**Lemma 19.** *Let $\mathcal{C}$ be a 1-category. A Grothendieck topology on $\mathcal{C}$ is precisely a Grothendieck topology on $\mathcal{C}$ seen as an $\infty$-category.*

**Definition 20.** Let $\mathcal{C}$ be a $\infty$-category and $C$ an object of $\mathcal{C}$. Given a sieve $\mathcal{D} \subseteq \mathcal{C}_{/C}$ and a pre-sheaf $F : \mathcal{C}^{op} \to \mathcal{S}$, denote by $\mathrm{Map}(\mathcal{D}, F)$ the limit of the diagram

$$\mathcal{D}^{op} \longrightarrow \mathcal{C}^{op} \xrightarrow{F} \mathcal{S}$$

where the first functor is the opposite of the canonical projection $\mathcal{C}_{/C} \to \mathcal{C}$ restricted to $\mathcal{D}$. Using the same argument as in the 1-categorical case, there is a comparison map $\mathrm{Map}(\mathcal{C}_{/C}, F) \to \mathrm{Map}(\mathcal{D}, F)$ and a canonical equivalence $\mathrm{Map}(\mathcal{C}_{/C}, F) \simeq F(C)$.

**Definition 21.** Let $\mathcal{C}$ be an $\infty$-category. A presheaf $F \colon \mathcal{C}^{op} \to \mathcal{S}$ is a *sheaf* if for every object $C$ and covering sieve $\mathcal{D}$ of $C$, the canonical morphism

$$F(C) \simeq \mathrm{Map}(C_{/C}, F) \to \mathrm{Map}(\mathcal{D}, F)$$

is an equivalence. Denote by $\mathcal{S}\mathrm{hv}(\mathcal{C}, J)$ the full sub-$\infty$-category of $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ spanned by the sheaves on $(\mathcal{C}, J)$.

Note we defined sheaves as local objects in a presentable $\infty$-category. Hence, using the formalism of presentable $\infty$-categories, we immediately have the following result.

**Proposition 22.** *The $\infty$-category of sheaves on $\mathcal{S}\mathrm{hv}(\mathcal{C}, J)$ is presentable.*

## 4. Sheaves with arbitrary Values

We started with sheaves valued in sets. Then generalized to $\infty$-categorical sheaves valued in spaces. However, we want $\infty$-categorical sheaves valued in spectra. Hence the next step is to generalize the values of our $\infty$-categorical sheaves. Abstractly we obtain such sheaves via the *tensor product* of presentable $\infty$-categories, which we review now.

**Definition 23.** Let $\mathcal{C}, \mathcal{D}$ be presentable $\infty$-categories. Then there exists a presentable $\infty$-category $\mathcal{C} \otimes \mathcal{D}$ together with a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ such that

(*) The functor $F$ preserves colimits component-wise.
(*) For any presentable $\infty$-category $\mathcal{E}$, pre-composition with $F$ induces an equivalence between the sub-$\infty$-category $\mathrm{Fun}^L(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \subseteq \mathrm{Fun}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})$ of colimit preserving functors, and the sub-$\infty$-category $\mathrm{Fun}^{L,L}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \subseteq \mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ of functors preserving colimits component-wise.

**Theorem 24** ([Lur17, Proposition 4.8.1.17])**.** *For $\mathcal{C}$ and $\mathcal{D}$ presentable $\infty$-categories, there is a canonical equivalence between $\mathcal{C} \otimes \mathcal{D}$ and $\mathrm{Fun}^R(\mathcal{C}^{op}, \mathcal{D})$, of limit preserving functors $\mathcal{C}^{op} \to \mathcal{D}$.*

**Definition 25.** Let $(C, J)$ be $\mathcal{C}$ be a Grothendieck site, and $\mathcal{D}$ be a presentable $\infty$-category. A *sheaf on* $(\mathcal{C}, J)$ *with values in* $\mathcal{D}$ is a limit preserving functor $F\colon \mathcal{S}\mathrm{hv}(\mathcal{C}, J)^{op} \to \mathcal{D}$, which corresponds to an object in $\mathcal{S}\mathrm{hv}(\mathcal{C}, J) \otimes \mathcal{D}$.

Of course, this description is very abstract and ideally we want a more explicit description that we can use in the future. That is the aim of the next sections.

## 5. An Explicit Description of Sheaves valued in Spectra

We have given a formal definition of sheaves valued in spectra, via the tensor product of presentable $\infty$-categories. We now want a more explicit description thereof. For this we make use of the following result.

**Proposition 26** ([Lur17], see also [GGN15])**.** *The inclusion $\mathrm{Pr}^L_{st} \to \mathrm{Pr}^L$ of stable presentable $\infty$-categories into the category of presentable $\infty$-categories admits a left adjoint, which is explicitly given by $- \otimes \mathcal{S}\mathrm{p}$.*

We can recall from our previous talk that the stabilization of a presentable $\infty$-category $\mathcal{C}$ is given by the $\infty$-category $\mathcal{S}\mathrm{p}(\mathcal{C})$ of spectrum objects in $\mathcal{C}$, therefore $\mathcal{C} \otimes \mathcal{S}\mathrm{p} \simeq \mathcal{S}\mathrm{p}(\mathcal{C})$, for every presentable $\infty$-category $\mathcal{C}$. In particular, the $\infty$-category $\mathcal{S}\mathrm{hv}(\mathcal{C}, J) \otimes \mathcal{S}\mathrm{p}$ is equivalent to the category of spectrum objects in $\mathcal{S}\mathrm{hv}(\mathcal{C}, J)$. Finally, since the inclusion $\mathcal{S}\mathrm{hv}(\mathcal{C}, J) \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ preserves and reflects limits, therefore a commutative square in $\mathcal{S}\mathrm{hv}(\mathcal{C}, J)$ is a pullback square if and only if it is a pullback square in $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$.

**Proposition 27.** *A spectrum object in sheaves on the Grothendieck site $(\mathcal{C}, J)$ is given by a spectrum object in presheaves on $(\mathcal{C}, J)$ that is point-wise a sheaf.*

This gives us the following explicit description of sheaves valued in spectra. Recall that $\Omega^{\infty-n}\colon \mathcal{S}\mathrm{p} \to \mathcal{S}$ is the functor mapping a spectrum object into its $n$-th component.

**Theorem 28.** *There is a canonical equivalence between $\mathcal{S}\mathrm{hv}(\mathcal{C}, J) \otimes \mathcal{S}\mathrm{p}$ and the category of functors $F\colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{p}$ such that $\Omega^{\infty-n}F\colon \mathcal{C}^{op} \to \mathcal{S}$ is a sheaf of spaces, for every $n$.*

## 6. An Explicit Description of Sheaves on the Site of Manifolds

We continue our analysis of sheaves with the aim of providing explicit descriptions. Now we focus on the case of sheaves on the site of manifolds. Recall that a sieve $S$ on a manifold $M$ is a covering sieve if an open cover $\mathcal{O}$ of $M$ exists, such that every morphism in $S$ factors through the inclusion $U \subseteq M$ of some element $U \in \mathcal{O}$. From the definition of the topology on $\mathcal{M}\mathrm{fd}$ we have the following: Given a covering sieve $S$ generated by an open cover $\mathcal{O}$. Denote by $\mathcal{I}(\mathcal{O})$ the poset (viewed as a category) consisting of finite intersections of elements of $\mathcal{O}$.

**Lemma 29.** *The functor $i : \mathcal{I}(\mathcal{O}) \to \mathcal{M}\mathrm{fd}_{/S}$ is cofinal.*

In particular, since $\mathrm{Map}(\mathcal{M}\mathrm{fd}_{/S}, F)$ is defined as the limit $(\mathcal{M}\mathrm{fd}_{/S})^{op} \to \mathcal{M}\mathrm{fd}^{op} \xrightarrow{F} \mathcal{S}$ and $i^{op}$ is *final*, there is a canonical equivalence

$$\mathrm{Map}(\mathcal{M}\mathrm{fd}_{/S}, F) \simeq \lim(\mathcal{I}(\mathcal{O})^{op} \to \mathcal{M}\mathrm{fd}^{op} \xrightarrow{F} \mathcal{S}) = \lim_{U \in \mathcal{I}(\mathcal{O})} F(U)$$

**Proposition 30.** *A presheaf $F\colon \mathcal{M}\mathrm{fd}^{op} \to \mathcal{S}\mathrm{p}$ is a sheaf if and only if for every manifold $M$ and every open cover $\mathcal{O}$ of $M$, the canonical morphism $F(M) \to \lim_{U \in \mathcal{I}(\mathcal{O})} F(U)$ is an equivalence.*

In fact we can check the sheaf condition for sheaves on the site of manifolds with a very specific type of covering families. For a proof of the following result, we refer to [ADH21, Proposition 3.6.6].

**Proposition 31.** *A presheaf $F\colon \mathcal{M}\mathrm{fd}^{op} \to \mathcal{S}\mathrm{p}$ is a sheaf if and only if*

*(1) $F(\emptyset)$ is terminal.*
*(2) For every $M = U \cup V$, the canonical square*

$$\begin{tikzcd} F(M) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ F(V) \arrow[r] & F(U \cap V) \end{tikzcd}$$

*is a pullback square.*

(3) *For a sequential family of opens $U_1 \subseteq U_2 \subseteq \cdots$ covering $M$, the canonical map $F(M) \to \lim_n F(U_n)$ is an equivalence.*

## 7. Equivalences of Sheaves

We now have etsablished a very solid understanding of sheaves on the site of manifolds with arbitrary values as presheaves with suitable limits conditions. We now want to use these explicit descriptions to better understand equivalences of sheaves on the site of manifolds.

**Lemma 32.** *Let $\varphi\colon G \to F$ be a morphism of sheaves on $\mathcal{M}\mathrm{fd}$. Then $\varphi$ is an equivalence if and only if $\varphi(\mathbb{R}^n)$ is an equivalence, for every $n \geq 0$.*

**Definition 33.** Denote by $\mathcal{E}\mathrm{uc}$ the full sub-category of $\mathcal{M}\mathrm{fd}$ generated by the Euclidean manifolds $\mathbb{R}^n$, for every $n \geq 0$. The category $\mathcal{E}\mathrm{uc}$ inherits a Grothendieck topology from $\mathcal{M}\mathrm{fd}$.

**Theorem 34.** *Denote by $j$ the inclusion functor $\mathcal{E}\mathrm{uc} \subseteq \mathcal{M}\mathrm{fd}$. The following hold:*

(1) *The adjoint functors $(j_*, j^*) : \mathrm{Fun}(\mathcal{E}\mathrm{uc}^{op}, \mathcal{S}\mathrm{p}) \to \mathrm{Fun}(\mathcal{M}\mathrm{fd}^{op}, \mathcal{S}\mathrm{p})$ both preserves sheaves.*

(2) *The restricted adjunction $\mathcal{S}\mathrm{hv}(\mathcal{E}\mathrm{uc}^{op}, \mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{hv}(\mathcal{M}\mathrm{fd}^{op}, \mathcal{S}\mathrm{p})$ is an equivalence.*

Let us see one more way to understand equivalences of sheaves via stalks.

**Definition 35.** Let $x \in M$ be a point, denote by $\mathcal{O}\mathrm{pen}_{M,x} \subseteq \mathcal{O}\mathrm{pen}_M$ the poset of open neighborhoods of $x$ in $M$. The *stalk* of a presheaf $F$ at $x$ is defined as the colimit

$$x^*(F) = \mathrm{colim}((\mathcal{O}\mathrm{pen}_{M,x})^{op} \subseteq (\mathcal{O}\mathrm{pen}_M)^{op} \xrightarrow{F} \mathcal{S}\mathrm{p})$$

Taking stalks at $x$ is a functor $x^* : \mathrm{Fun}((\mathcal{O}\mathrm{pen}_M)^{op}, \mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{p}$, which we extend to presheaves on manifolds by pre-composing with the inclusion $\mathcal{O}\mathrm{pen}_M \subseteq \mathcal{M}\mathrm{fd}$. We will also denote by $x^*$ the resulting functor $\mathrm{Fun}(\mathcal{M}\mathrm{fd}^{op}, \mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{p}$.

**Theorem 36.** *A morphism $\varphi\colon G \to F$ of sheaves on $\mathcal{M}\mathrm{fd}$ is an equivalence if and only if for every manifold $M$ and every point $x \in M$, the induced map $x^*(\varphi)\colon x^*(G) \to x^*(F)$ is an equivalence.*

Note, the general condition above is equivalent to the following simpler condition. Denote by $0_n$ the origin of $\mathbb{R}^n$.

**Corollary 37.** *A morphism $\varphi\colon G \to F$ of sheaves on $\mathcal{M}\mathrm{fd}$ is an equivalence if and only if for all $n \geq 0$, the induced map $(0_n)^*(f)\colon (0_n)^*(F) \to (0_n)^*(F')$ is an equivalence.*

## References

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