

# DIFFERENTIAL COHOMOLOGY SEMINAR 4 (DRAFT)

TALK BY ALESSANDRO NANTO

In the last talk we learned the definition of a differential cohomology theory, as a sheaf valued in spectra on the site of manifolds. This talk continues our journey through differential cohomology theories, and focuses on the following three topics:

- (1) We want to learn how to construct non-trivial examples out of sheaves valued in chain complexes.
- (2) We want to understand how we can extend classical cohomology operations to the setting of differential cohomology theories.
- (3) We want to introduce suitable analogues of fiber-wise integration.

## 1. ABELIAN GROUPS, SPECTRA AND THE HEART

Let us start by reviewing the relation between abelian groups, rings and spectra.

**Definition 1.1.** Let  $n \in \mathbb{Z}$  and  $X$  be a spectrum, define  $\pi_n(X) := \pi_0(\Omega^{\infty+n} X) = \pi_0(X_{-n})$ . We call  $\pi_n$  the  $n$ -th homotopy group of  $X$ .

*Remark 1.2.* Note that since  $X_n \simeq \Omega^2 X_{n+2}$ , for any  $n$ , the set  $\pi_0(X_n)$  underlies the structure of an abelian group.

The category  $\mathcal{S}p$  underlies the structure of a symmetric monoidal  $\infty$ -category ([Lur17, Corollary 4.8.2.19]). Following [Lur17], we denote by  $\otimes$  the tensor product on  $\mathcal{S}p$ .

**Definition 1.3.** A (commutative) algebra object in  $\mathcal{S}p$  is called an  $(\mathbb{E}_\infty)$ -ring spectrum, see [Lur17, Definition 7.1.0.1]. Given a ring spectrum  $R$ , denote by  $\text{Mod}_R$  the corresponding category of left  $R$ -module spectra, see [Lur17, Definition 7.1.1.2].

*Remark 1.4.* The sphere spectrum  $\mathbb{S}$  acts as the monoidal unit of  $\mathcal{S}p$ , therefore it is a  $\mathbb{E}_\infty$ -ring spectrum. The category  $\text{Mod}_{\mathbb{S}}$  is canonically equivalent to  $\mathcal{S}p$ .

**Definition 1.5.** Denote by  $\mathcal{S}p_{\geq 0} \subseteq \mathcal{S}p$  the full sub-category generated by *connective spectra*, i.e. spectra  $X$  such that  $\pi_n(X) \simeq 0$ , for all  $n < 0$ . Denote by  $\mathcal{S}p^\heartsuit \subseteq \mathcal{S}p_{\geq 0}$  the *heart of spectra*, i.e. the full sub-category generated by spectra  $X$  such that  $\pi_n(X) \simeq 0$ , for all  $n > 0$ .

We have the following result relating connective spectra and the heart, which follow immediately.

**Lemma 1.6.** *Let  $X$  be a connective spectrum. The following are equivalent:*

- (1)  $X$  is in the heart.
- (2)  $\pi_n(\Omega^\infty X) = 0$ , for all  $n > 0$ .
- (3)  $\text{Hom}_{\mathcal{S}_*}(S, \Omega^\infty X) \simeq 0$ , for all connected, pointed spaces  $S$ .
- (4)  $X$  is local with respect to the class of maps  $\Sigma^\infty S \rightarrow 0$ , for every connected pointed space  $S$ .

The category  $\mathcal{S}p_{\geq 0}$  is presentable and  $\pi_0$  induces an equivalence between the heart and  $\mathcal{A}b$  ([Lur17, Proposition 1.4.3.6]). The heart is a sub-category of local objects of connective spectra, therefore the inclusion  $\mathcal{A}b \simeq \mathcal{S}p^\heartsuit \subseteq \mathcal{S}p_{\geq 0}$  is a right adjoint. The category  $\mathcal{S}p_{\geq 0}$  is closed under  $\otimes$  and, given  $X, Y$  connective spectra,

$$(1.7) \quad \pi_0(X \otimes Y) \simeq \pi_0(X) \otimes \pi_0(Y)$$

see [Dav24, Theorem 2.3.28]

**Definition 1.8.** Given an abelian group  $A$ , denote by  $HA$  the (unique up to equivalence) spectrum of the heart such that  $\pi_0(HA) \simeq A$ . We call  $HA$  the *Eilenberg-Mac Lane spectrum* of  $A$ .

Using [Equation \(1.7\)](#) and the adjunction between  $H$  and  $\pi_0$ , one can prove  $H$ , viewed as a functor  $\mathcal{A}b \rightarrow \mathcal{S}p$ , is lax monoidal. In particular, if  $R$  is a commutative ring, then  $HR$  is a connective  $\mathbb{E}_\infty$ -ring spectrum. On the other hand, if  $R$  is a connective  $\mathbb{E}_\infty$ -ring spectrum and  $M$  a connective module, then  $\pi_0(M)$  is a  $\pi_0(R)$ -module.

**Definition 1.9.** Given a commutative ring  $R$ , denote by  $\mathrm{Ch}(R) = \mathrm{Ch}(\mathrm{Mod}_R)$  the ordinary category of unbounded chain complexes. Let  $\mathcal{D}(R)$  be the  $\infty$ -localization of  $\mathrm{Ch}(R)$  at the class of quasi-isomorphisms, called the *derived category*.

Similar to the heart of spectra, given an  $\mathbb{E}_\infty$ -ring spectrum  $R$ , denote by  $\mathrm{Mod}_R^\heartsuit \subseteq \mathrm{Mod}_R$  the full subcategory generated by  $R$ -modules such that the underlying spectrum belongs to the heart of spectra.

**Theorem 1.10** (Stable Dold-Kan Correspondence). *Let  $R$  be a commutative ring.*

- (1)  $\mathrm{Mod}_R \simeq \mathrm{Mod}_{HR}^\heartsuit$  via taking Eilenberg-Mac Lane spectra.
- (2) The above equivalence extends to an equivalence  $H : \mathcal{D}(R) \simeq \mathrm{Mod}_{HR}$  of symmetric monoidal  $\infty$ -categories.

*Proof.* (1) is [[Lur17](#), Proposition 7.1.1.13], while (2) is [[Lur17](#), Theorem 7.1.2.13]. □

An interesting consequence of [Equation \(1.10\)](#) is the following:

**Corollary 1.11.** *Given  $F \in \mathcal{D}(R)$ , then  $\pi_n(HF) \simeq H_n(F)$ , for all  $n \in \mathbb{Z}$ .*

*Proof.*

$$\begin{aligned}
 \pi_n(HF) &= \pi_0(\Omega^{\infty+n} HF) \\
 &\stackrel{\textcircled{1}}{\simeq} \pi_0(\mathrm{Hom}_{\mathcal{S}p}(\Sigma^n \mathbb{S}, HF)) \\
 &\stackrel{\textcircled{2}}{\simeq} \pi_0(\mathrm{Hom}_{\mathrm{Mod}_{HR}}(\Sigma^n HR, HF)) \\
 &\stackrel{\textcircled{3}}{\simeq} \pi_0(\mathrm{Hom}_{\mathcal{D}(R)}(R[n], F)) \\
 &\stackrel{\textcircled{4}}{\simeq} H_n(F)
 \end{aligned}$$

- ① The functor  $\Omega^{\infty+n}$  is corepresented by the shifted sphere spectrum  $\Sigma^n \mathbb{S}$ .
- ② The forgetful functor  $\mathrm{Mod}_{HR} \rightarrow \mathrm{Mod}_{\mathbb{S}} \simeq \mathcal{S}p$  is right adjoint to tensoring with  $HR$  and  $HR \otimes (\Sigma^n \mathbb{S}) \simeq \Sigma^n HR$ .
- ③ [Equation \(1.10\)](#)
- ④  $\pi_0$  of the mapping space  $\mathrm{Hom}_{\mathcal{D}(R)}(R[n], F)$  is equivalent to the mapping space  $R[n] \rightarrow F$  in the *ordinary* derived category of  $R$ , i.e. homotopy classes of maps  $R[n] \rightarrow F$ , which correspond exactly to classes in  $H_n(F)$ . □

## 2. LOCALLY CONSTANT SHEAVES ON MANIFOLDS

Let  $\mathcal{C}$  be a presentable  $\infty$ -category. The  $\infty$ -categorical background given in previous talks allows to conclude the existence of a number of functors. Here we give a (somewhat) explicit formula for one.

*Remark 2.1.* Let  $\mathcal{B}$  be a full, dense sub-site of  $\mathrm{Mfd}$ , recall then that, for every presentable category  $\mathcal{C}$ , the restriction functor induces an equivalence  $\mathrm{Shv}(\mathrm{Mfd}, \mathcal{C}) \simeq \mathrm{Shv}(\mathcal{B}, \mathcal{C})$ .

Evaluation at  $\{0\}$  induces an adjunction  $(L, \Gamma) : \mathcal{C} \rightarrow \mathrm{Shv}(\mathrm{Mfd}, \mathcal{C})$ , where the functor  $\Gamma$  is evaluation at  $\{0\}$ , while the left adjoint  $L$  maps  $C \in \mathcal{C}$  to the sheafification of the constant pre-sheaf with value  $C$ .

*Remark 2.2.* Every presentable  $\infty$ -category  $\mathcal{C}$  is uniquely *cotensored over*  $\mathcal{S}$ , see [[Lur09](#), Remark 5.5.2.6]. More explicitly, for every space  $S$  and object  $C$ , there is an object  $C^S$  together with a natural equivalence

$$\mathrm{Hom}_{\mathcal{S}}(S, \mathrm{Hom}_{\mathcal{C}}(-, C)) \simeq \mathrm{Hom}_{\mathcal{C}}(-, C^S)$$

**Definition 2.3.** Denote by  $\mathrm{Sing}$  the functor  $\mathrm{Mfd} \rightarrow \mathcal{S}$  mapping a manifold to its underlying space of *smooth* simplexes. Given a presentable  $\infty$ -category  $\mathcal{C}$ , denote by  $\flat$  the composition  $\mathcal{C} \rightarrow \mathrm{Fun}(\mathcal{S}^{\mathrm{op}}, \mathcal{C}) \rightarrow \mathrm{Fun}(\mathrm{Mfd}^{\mathrm{op}}, \mathcal{C})$ , the first functor coming from [Equation \(2.2\)](#), the second being pre-composition with  $\mathrm{Sing}^{\mathrm{op}}$ .

Explicitly, given an object  $C \in \mathcal{C}$ , the associated pre-sheaf  $\flat C$  maps a manifold  $M$  to  $C^{\mathrm{Sing}(M)}$ .

*Remark 2.4.* Given a topological space  $X$ , denote by  $\text{Sing}'(X)$  the corresponding space, i.e. its singular simplicial set. If  $X = M$  is a smooth manifold, then  $\text{Sing}(M) \subseteq \text{Sing}'(M)$  and the inclusion is a homotopy equivalence by Whitney's Approximation Theorem, see [Zuo21, Theorem 1.6].

**Lemma 2.5** ([BG16, Corollary 6.46]). *Sing is a cosheaf. In particular,  $\flat$  factors through  $\text{Shv}(\text{Mfd}, \mathcal{C}) \subseteq \text{Fun}(\text{Mfd}^{\text{op}}, \mathcal{C})$ .*

**Equation (2.5)** is essentially the consequence of a generalized version of Seifert-van Kampen theorem, namely [Lur17, Proposition A.3.2], stating that, given a topological space  $X$  and a covering sieve  $\mathcal{O}$ , the space  $\text{Sing}'(X)$  is the colimit of  $\text{Sing}'(U)$ , over  $U \in \mathcal{I}(\mathcal{O})$ .

**Theorem 2.6.**  *$\flat : \mathcal{C} \rightarrow \text{Shv}(\text{Mfd}, \mathcal{C})$  is left adjoint to  $\Gamma$ .*

*Proof.* The composition  $\mathcal{C} \xrightarrow{\flat} \text{Shv}(\text{Mfd}, \mathcal{C}) \xrightarrow{|\mathcal{E}\text{uc}} \text{Shv}(\mathcal{E}\text{uc}, \mathcal{C})$  maps an object  $C$  to the sheaf  $\flat C$  restricted to Euclidean spaces. Since  $\mathbb{R}^n$  is contractible,  $(\flat C)(\mathbb{R}^n) = C^{\text{Sing}(\mathbb{R}^n)} \simeq C$  and so  $\flat$  restricted to  $\mathcal{E}\text{uc}$  is equivalent to  $L$ , the functor taking  $C$  to the pre-sheaf with constant value  $C$ , which is left adjoint to  $\Gamma$  restricted to  $\mathcal{E}\text{uc}$ .  $\square$

*Remark 2.7.* The proof of **Equation (2.6)** shows that, given an object  $C \in \mathcal{C}$  and a dense sub-site  $\mathcal{B} \subseteq \text{Mfd}$  of *contractible* manifolds, the constant pre-sheaf with value  $C$  is equivalent to the sheaf  $\flat C$  restricted to  $\mathcal{B}$ , hence it is already a sheaf.

### 3. CHAIN COMPLEXES AND SHEAVES

Denote by  $\text{Ch}$  the category of chain complexes of abelian groups and  $\mathcal{D}\text{r}$  the associated derived category. Moreover, denote by  $\text{Mod}$  the category of  $H\mathbb{Z}$ -module spectra. The equivalence of **Equation (1.10)** applied point-wise induces an equivalence of sheaf categories

$$\text{Shv}(\text{Mfd}, \mathcal{D}\text{r}) \simeq \text{Shv}(\text{Mfd}, \text{Mod})$$

**Definition 3.1.** Given a sheaf  $V : \text{Mfd}^{\text{op}} \rightarrow \mathcal{D}\text{r}$ , denote by  $HV$  the corresponding sheaf of  $H\mathbb{Z}$ -module spectra, called *Eilenberg-Mac Lane sheaf*.

In this section, we introduce some technical lemmas regarding sheaves on manifolds valued in  $\mathcal{D}\text{r}$ . We implicitly identify (sheaves of) cochain complexes  $V^*$  with (sheaves of) chain complexes with reversed indexing, i.e.  $V_n := V^{-n}$ .

**Definition 3.2** ([BNV16, Definition 7.14]). Given  $n \in \mathbb{Z}$  and a (sheaf of) cochain complex  $V^*$ , denote by  $V^{\geq n}$ , resp.  $V^{\leq n}$ , the *naive truncations* of  $V^*$

$$\cdots \rightarrow 0 \rightarrow V^n \rightarrow V^{n+1} \rightarrow \cdots, \quad \text{resp.} \quad \cdots \rightarrow V^{n-1} \rightarrow V^n \rightarrow 0 \rightarrow \cdots$$

Denote by  $\widehat{\mathbb{R}}$  the sheaf of sets associated to  $\mathbb{R}$  by the Yoneda embedding, mapping a manifold  $M$  to the set of smooth maps  $M \rightarrow \mathbb{R}$ . The sheaf  $\widehat{\mathbb{R}}$  is a ring object in  $\text{Shv}(\text{Mfd}, \text{Set})$ , a  $\widehat{\mathbb{R}}$ -module is called a *smooth module*. Explicitly, a smooth module is a sheaf  $V$  together with a natural  $C^\infty(M)$ -module structure on  $V(M)$ , for every  $M$ .

**Lemma 3.3** ([BNV16, Lemma 7.12]). *Let  $V : \text{Mfd}^{\text{op}} \rightarrow \text{Ch}$  a sheaf of chain complexes of smooth modules, then the composition  $\text{Mfd}^{\text{op}} \xrightarrow{V} \text{Ch} \rightarrow \mathcal{D}\text{r}$  is a sheaf.*

**Definition 3.4.** Denote by  $\Omega^*$  the sheaf  $\text{Mfd}^{\text{op}} \rightarrow \text{Ch}$  mapping  $M$  to its de Rham complex  $\Omega^*(M)$ .

**Equation (3.3)** ensures that the sheaves in **Equation (3.4)** is a sheaf after post-composition with the localization functor  $\text{Ch} \rightarrow \mathcal{D}\text{r}$ .

### 4. HIGHER DE RHAM THEOREM

*Remark 4.1.* Given a smooth manifold  $M$ , denote by  $C_*^{\text{sm}}(M, \mathbb{Z})$  the *smooth* singular complex generated by  $\text{Sing}(M)$ . Given an abelian group  $A$ , denote by  $C_{\text{sm}}^*(M, A)$  the singular cochain complex valued in  $A$  associated to  $C_*^{\text{sm}}(M, \mathbb{Z})$ .

The de Rham homomorphism is a cochain homomorphism  $\Omega^*(M) \rightarrow C_{\text{sm}}^*(M, \mathbb{R})$ . Restriction to smooth cycles induces a quasi-isomorphism  $C^*(M, \mathbb{R}) \simeq C_{\text{sm}}^*(M, \mathbb{R})$ , the de Rham theorem states that the de Rham homomorphism induces an isomorphism of graded rings between de Rham cohomology and smooth singular cohomology with real coefficients, therefore  $H_{\text{dR}}^*(M) \simeq H^*(M, \mathbb{R})$ , as graded rings. In this section, we rephrase the de Rham homomorphism as a morphism of sheaves valued in  $\mathcal{D}\mathbf{r}$  and reference a result generalizing the de Rham theorem.

*Remark 4.2.* Since  $\mathcal{D}(\mathbb{Z})$  is presentable, we know that it is cotensored over  $\mathcal{S}$ . Given a space  $S$  and a chain complex  $V_*$ , the cotensor  $V_*^S$  is the cochain complex (viewed as a chain complex) of graded linear maps  $C_*(S, \mathbb{Z}) \rightarrow V_*$ , from the (normalized) singular chain complex of  $S$  to  $V_*$ , see [Lur17, Definition 1.3.2.1]. In particular, let  $V_* = V$  be concentrated in degree 0, then  $V_*^S$  is the singular cochain complex of  $S$  with values in  $V$ .

Recall the functor  $\flat$  defined in Equation (2.3). By definition of  $\text{Sing} : \mathbf{Mfd} \rightarrow \mathcal{S}$  and Equation (4.2), given  $V_*$  chain complex,  $\flat V$  maps a manifold  $M$  to the cochain complex of graded linear maps  $C_*^{\text{sm}}(M, \mathbb{Z}) = C_*(\text{Sing}(M), \mathbb{Z}) \rightarrow V_*$ .

**Definition 4.3.** Consider the natural morphism  $\Omega^*(M) \rightarrow (\flat \mathbb{R})(M) = C_{\text{sm}}^*(M, \mathbb{R})$  taking a form  $\omega \in \Omega^n(M)$  to the linear map  $\int \omega : C_n^{\text{sm}}(M, \mathbb{Z}) \rightarrow \mathbb{R}$ . We call the natural transformation  $\text{dR} : \Omega^* \rightarrow \flat \mathbb{R}$  the *de Rham morphism*.

*Remark 4.4.* [LV12, Proposition 10.2.12] spells out the data of a  $A_\infty$ -morphism between general  $A_\infty$ -algebras. An  $A_\infty$ -morphism is a  $A_\infty$ -quasi-isomorphism if the underlying map of chain complexes is an equivalence.

We can now state the main theorem.

**Theorem 4.5** ([Aa10, Theorem 3.25]).  $\text{dR} : \Omega^* \rightarrow \flat \mathbb{R}$  underlies a natural  $A_\infty$ -quasi-isomorphism of sheaves of differential graded algebras.

## 5. DELIGNE COHOMOLOGY

In this section, we give the definition of the  $k$ -th Deligne sheaf as the Eilenberg-Mac Lane spectrum (see Equation (3.1)) associated to a sheaf  $V : \mathbf{Mfd}^{\text{op}} \rightarrow \mathcal{D}\mathbf{r}$ . Following that, we give explicit cochain complexes that are quasi-isomorphic to the value of  $V$  at a manifold  $M$ .

**Definition 5.1.** Given  $k \in \mathbb{N}$ , define  $\hat{\mathbb{Z}}(k) : \mathbf{Mfd}^{\text{op}} \rightarrow \mathcal{D}\mathbf{r}$  as the pullback

$$\begin{array}{ccc} \hat{\mathbb{Z}}(k) & \longrightarrow & \Omega^{\geq k} \\ \downarrow & & \downarrow \\ \flat \mathbb{Z} & \longrightarrow & \flat \mathbb{R} \end{array}$$

The vertical morphism being the composition  $\Omega^{\geq k} \hookrightarrow \Omega^* \xrightarrow{\text{dR}} \flat \mathbb{R}$ . We call the corresponding Eilenberg-Mac Lane spectrum  $H\hat{\mathbb{Z}}(k)$  the  $k$ -th Deligne sheaf.

*Remark 5.2* (Model A, see [HS05, §3.2]). Let  $\acute{C}(k)^n(M) \subseteq C_{\text{sm}}^n(M, \mathbb{Z}) \oplus C_{\text{sm}}^{n-1}(M, \mathbb{R}) \oplus \Omega^n(M)$  consist of triples  $(c, h, \omega)$  for which  $\omega = 0$  if  $n < k$ , with differential  $\delta(c, h, \omega) = (\delta c, \text{dR}(\omega) - c - \delta h, d\omega)$ . This complex  $\acute{C}^*(k)(M)$  fits into a diagram

$$\begin{array}{ccc} \acute{C}^*(k)(M) & \longrightarrow & \Omega^{\geq k}(M) \\ \downarrow & & \downarrow \\ C_{\text{sm}}^*(M, \mathbb{Z}) & \longrightarrow & C_{\text{sm}}^*(M, \mathbb{R}) \end{array}$$

which commutes up to homotopy given by the projections  $\acute{C}^n(k)(M) \rightarrow C_{\text{sm}}^{n-1}(M, \mathbb{R})$ . The diagram above model the pullback of Equation (5.1), hence  $\acute{C}^*(k)(M) \simeq \hat{\mathbb{Z}}(k)(M)$ .

*Remark 5.3* (Model B). Recall that  $\flat$  preserves cofiber sequences, since it is left adjoint, and that fiber sequences are the same as cofiber sequences in stable  $\infty$ -categories. Consider the diagram

$$\begin{array}{ccc} \hat{\mathbb{Z}}(k) & \longrightarrow & \Omega^{\geq k} \\ \downarrow & & \downarrow \\ \flat\mathbb{Z} & \longrightarrow & \flat\mathbb{R} \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{\text{red}} & \flat(\mathbb{R}/\mathbb{Z}) \end{array}$$

Since the bottom square is a pullback,  $\hat{\mathbb{Z}}(k)(M)$  is equivalent to the pullback of the diagram in red. Let  $\check{C}^n(k)(M) \subseteq C_{\text{sm}}^{n-1}(M, \mathbb{R}/\mathbb{Z}) \oplus \Omega^n(M)$  consist of pairs  $(\chi, \omega)$  for which  $\omega = 0$  if  $n < k$ , with differential  $\delta(\chi, \omega) = (e^{2\pi i \text{id}_{\mathbb{R}}(\omega)} - \delta\chi, d\omega)$ . Similar to Equation (5.2), the complex  $\check{C}^*(k)(M)$  fits into the above diagram so that the outer square is a pullback, hence it is equivalent to  $\hat{\mathbb{Z}}(k)(M)$ .

Take an  $n$ -cocycle ( $n \geq k$ ) in the model from Equation (5.3), i.e.  $(\chi, \omega) \in C_{\text{sm}}^{n-1}(M, \mathbb{R}/\mathbb{Z}) \oplus \Omega^n(M)$  such that  $d\omega = 0$  and  $\delta\chi = e^{2\pi i \text{id}_{\mathbb{R}}(\omega)}$ . Such a cocycle determines a differential character of degree  $n-1$  for  $M$ , in the sense of the following definition:

**Definition 5.4** ([HS05, Definition 3.4], see also [BB14, Chapter 5]). Consider a manifold  $M$ , a *differential character* of degree  $n$  consists of a character  $\chi : Z_n^{\text{sm}}(M, \mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z}$  on the group of smooth  $n$ -cycles of  $M$  together with a  $n$ -form  $\omega \in \Omega^{n+1}(M)$ , such that, for every smooth  $(n+1)$ -chain  $c$ ,

$$\chi(\partial c) = e^{2\pi i \int_c \omega}$$

**Definition 5.5.** Given a full sub-site  $\mathcal{B} \subseteq \mathbf{Mfd}$  and a manifold  $M$ , a  $\mathcal{B}$ -good open cover of  $M$  consists of an open cover  $\mathcal{O}$  such that every finite intersection of elements in  $\mathcal{O}$  is either empty or diffeomorphic to an object of  $\mathcal{B}$ . If  $\mathcal{B} = \mathcal{Euc}$  the sub-site of Euclidean spaces, we call a  $\mathcal{B}$ -good open cover a *differentially good open cover*, see [FSS11, Definition 6.3.9].

If  $\mathcal{B}$  is the sub-site of contractible manifolds, we recover the notion of a *good open cover*. In our next model, we will construct a sheaf  $F$  equivalent to  $\hat{\mathbb{Z}}(k)$  on a dense sub-site  $\mathcal{B}$ . Given a generic manifold  $M$ , we can evaluate  $\hat{\mathbb{Z}}(k)(M)$  using  $F$ , if we assume the existence of a  $\mathcal{B}$ -good open cover. Here we care mainly about  $\mathcal{B} = \mathcal{Euc}$  and so we require the following result.

**Theorem 5.6** ([FSS11, Theorem A.1]). *Every paracompact smooth manifold admits a differentially good open cover.*

*Remark 5.7* (Model C, see [ADH21, Lemma 7.3.4]). Let  $\mathcal{B} \subseteq \mathbf{Mfd}$  be a dense sub-site of contractible manifolds, consider the following diagram in the category of sheaves on  $\mathcal{B}$

$$\begin{array}{ccccc} \hat{\mathbb{Z}}(k)|_{\mathcal{B}} & \longrightarrow & \Omega^{\geq k}|_{\mathcal{B}} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \text{red} \\ \mathbb{Z} & \longrightarrow & \Omega^*|_{\mathcal{B}} & \longrightarrow & \Omega^{\leq k-1}|_{\mathcal{B}} \end{array}$$

The left square is obtained by: ① Apply the restriction to  $\mathcal{B}$  functor to the pullback diagram of Equation (5.1) ② Substitute  $\flat\mathbb{R}|_{\mathcal{B}}$  with  $\Omega^*|_{\mathcal{B}}$  (see Equation (4.5)) and  $\flat\mathbb{Z}|_{\mathcal{B}}$  with  $\mathbb{Z}$  (see Equation (2.7)). Since the right square is a pullback,  $\hat{\mathbb{Z}}(k)|_{\mathcal{B}}$  is equivalent to the pullback of the diagram in red. Let  $\check{C}^*(k)$  be the sheaf of chain complexes  $\mathbb{Z} \rightarrow \Omega^0 \rightarrow \dots \rightarrow \Omega^{k-1} \rightarrow 0 \rightarrow \dots$ , where  $\mathbb{Z}$  in degree 0 includes into  $\Omega^0$  as constant functions. The complex  $\check{C}^*(k)$  fits into the above diagram, so that the outer square is a pullback, therefore  $\hat{\mathbb{Z}}(k)|_{\mathcal{B}} \simeq \check{C}^*(k)$ .

*Remark 5.8* (Model C, Continued). Since  $\hat{\mathbb{Z}}(k)$  is equivalent to  $\check{C}^*(k)$  restricted to  $\mathcal{B}$ , consider a manifold  $M$  having a  $\mathcal{B}$ -good open cover  $\mathcal{O}$  and let  $\mathcal{I}(\mathcal{O})$  the closure of  $\mathcal{O}$  under finite intersections, then recall

$$\hat{\mathbb{Z}}(k)(M) \simeq \lim_{U \in \mathcal{I}(\mathcal{O})} \hat{\mathbb{Z}}(k)(U) \simeq \lim_{U \in \mathcal{I}(\mathcal{O})} \check{C}^*(k)(U)$$

By rearranging the limit, we see it is equivalent to the limit of the cosimplicial object taking  $n$  to the product over all tuples  $(U_0, \dots, U_n) \in \mathcal{O}^{n+1}$  of  $\check{C}^*(k)(U_0 \cap \dots \cap U_n)$ . We can then apply [BNV16, Lemma 7.10] to calculate the limit of this cosimplicial cochain complex as the total cochain complex of

$$\check{C}^{n,m}(k) := \prod_{\mathcal{O}^{n+1}} \check{C}^m(k)(U_0 \cap \dots \cap U_n)$$

The differential in the  $m$ -direction simply comes from  $\check{C}^*(k)$ . The differential in the  $n$ -direction is the alternating sum of the morphisms  $\delta^i : \check{C}^{n,m}(k) \rightarrow \check{C}^{n+1,m}(k)$ , for  $0 \leq i \leq n+1$ , described in components as follows: Given  $\mu \in \check{C}^{n,m}(k)$ , the  $(U_0, \dots, U_{n+1})$ -component of  $\delta^i \mu$  is the  $(U_0, \dots, \widehat{U}_i, \dots, U_{n+1})$ -component of  $\mu$  restricted to  $U_0 \cap \dots \cap U_{n+1}$ , where the hat under  $U_i$  means we're removing it from the tuple.

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