

DIFFERENTIAL COHOMOLOGY SEMINAR 8

TALK BY HANNES BERKENHAGEN

The aim of this talk is to review twisted cohomology theory with the aim of later discussing twisted differential cohomology theories [BG16]. For these talks the main source is [ABG⁺14, ABG18].

1. TWISTED COHOMOLOGY

Let R be a ring spectrum, meaning a monoid object in the ∞ -category of spectra $\mathcal{S}p$. From this we get a presentable stable ∞ -category $\mathcal{M}od_R$ of right R -module spectra. Objects therein are morphisms of the form $R \wedge M \rightarrow M$ satisfying the usual associativity and unit conditions up to coherent homotopies.

Note we have an adjunction diagram

$$\mathcal{S}p \begin{array}{c} \xrightarrow{R \wedge -} \\ \xleftarrow[\text{Hom}_R(R, -)]{\perp} \end{array} \mathcal{M}od_R ,$$

where the right adjoint is in fact the forgetful functor. This in particular means $\mathcal{M}od_R$ has a distinguished object R i.e. the free R -module of rank 1. We now refine these constructions.

Definition 1.1. Let R be a ring spectrum. An R -line is an R -module L such that $L \simeq R$.

Definition 1.2. Let $\mathcal{L}ine_R$ be the full sub- ∞ -groupoid of $\mathcal{M}od_R$ spanned by the R -lines.

Note by construction this is equivalent to $BGL_1(R)$.

Definition 1.3. Let X be a space. Then we denote by $\mathcal{M}od_R(X)$ the ∞ -category of R -module spectra parametrized over X , i.e. the functor category $\text{Fun}(X^{op}, \mathcal{M}od_R)$.

Definition 1.4. Let X be a space. Then we denote by $\mathcal{L}ine_R(X)$ the full subcategory of $\mathcal{M}od_R(X)$ spanned by those functors $L: X^{op} \rightarrow \mathcal{L}ine_R$, meaning it is $\text{Fun}(X^{op}, \mathcal{L}ine_R)$.

Recall this generalization of local systems of abelian groups, which map out of groupoids instead of ∞ -groupoids.

Example 1.5. Let $R_X: X^{op} \rightarrow \mathcal{L}ine_R$ be the constant functor, then this is an object in $\mathcal{L}ine_R(X)$.

We now proceed to the Thom construction.

Definition 1.6 (Thom spectrum). The *Thom R -module spectrum* is the functor

$$M: \mathcal{G}rpd_{\infty}^{op} / \mathcal{L}ine_R \rightarrow \mathcal{M}od_R,$$

which sends $f: X^{op} \rightarrow \mathcal{L}ine_R$ to the R -module spectrum $\text{colim}(X^{op} \xrightarrow{f} \mathcal{L}ine_R \rightarrow \mathcal{M}od_R)$.

Let us note an alternative characterization that will be important later.

Remark 1.7. For a given map $f: X \rightarrow Y$, we get a map $f^*: \mathcal{M}od_R(Y) \rightarrow \mathcal{M}od_R(X)$ by precomposition with f^{op} . By the adjoint functor theorem, this functor admits a left adjoint $f_!: \mathcal{M}od_R(X) \rightarrow \mathcal{M}od_R(Y)$ and a right adjoint $f_*: \mathcal{M}od_R(X) \rightarrow \mathcal{M}od_R(Y)$. Now for $p: X \rightarrow *$, we have

$$Mf \simeq p_!(i \circ f).$$

We now proceed to define twisted cohomology theories.

Definition 1.8 (Twisted cohomology). Let R be a ring spectrum, X be a space and let $p: X \rightarrow *$ be the projection map, and let $f: X^{op} \rightarrow \mathcal{L}ine_R$ be an R -line bundle over X . The f -twisted R -cohomology of X is defined as the mapping spectrum

$$R_f(X) := \text{Map}_{\text{Mod}_R}(Mf, R) \simeq \text{Map}_{\text{Mod}_R(X)}(f, p^*R) \simeq \text{Map}_{\text{Mod}_R(X)}(f, R_x).$$

Similarly, we have f -twisted R -homology of X defined as

$$R^f(X) := \text{Map}_{\text{Mod}_R}(R, Mf) \simeq Mf.$$

In particular we can define the twisted cohomology groups of a space X with twist given by a map $f: X \rightarrow \mathcal{L}ine_R$, by

$$R_f^n(X) := \pi_0(\text{Map}_{\text{Mod}_R}(Mf, \Sigma^n R)) \cong \pi_n(\text{Map}_{\text{Mod}_R}(f, R)).$$

Similarly we define the twisted homology groups by

$$R_n^f(X) := \pi_0(\text{Map}_{\text{Mod}_R}(\Sigma^n R, Mf)) \cong \pi_n(Mf).$$

Remark 1.9. Note that if f is the constant map with value R , then we recover ordinary R -cohomology and R -homology. Concretely, we have

$$Mf \simeq R \wedge \Sigma_+^\infty X$$

and so it follows that

$$R_f(X) \simeq \text{Map}_{\text{Mod}_R}(R \wedge \Sigma_+^\infty X, R) \simeq \text{Map}_{\text{Sp}}(\Sigma_+^\infty X, R)$$

Applying π_n this recovers the regular R -cohomology groups of X , and analogous for homology.

Let us look at some examples of twisted cohomology theories.

Example 1.10. Recall that we have a diagram characterizing $BString$

$$K(\mathbb{Z}, 3) \rightarrow BString \rightarrow BSpin \rightarrow BGL_1(\mathbb{S}) = \mathcal{L}ine_{\mathbb{S}}$$

then applying the Thom construction to this diagram we get

$$\Sigma_+^\infty K(\mathbb{Z}, 3) \rightarrow MString \rightarrow MSpin.$$

Here we use the fact that the map $K(\mathbb{Z}, 3) \rightarrow \mathcal{L}ine_{\mathbb{S}}$ factors through the point, and hence its Thom spectrum is $\Sigma_+^\infty K(\mathbb{Z}, 3)$.

As a next step we have a string orientation map $MString \rightarrow \text{tmf}$, where tmf is the spectrum of topological modular forms, following the computation in [AHR10]. Using the fact that Σ_+^∞ is monoidal, we know that the units of $K(\mathbb{Z}, 3)$ include $K(\mathbb{Z}, 3)$ itself, meaning we get a map $K(\mathbb{Z}, 3) \rightarrow GL_1(\text{tmf})$. Applying B gives us a map $f: K(\mathbb{Z}, 4) \rightarrow BGL_1(\text{tmf}) = \mathcal{L}ine_{\text{tmf}}$, classifying the twist of tmf -cohomology. Further details can be found in [ABG10].

Let us try a more feasible example.

Example 1.11. Let

$$K(1, \mathbb{Z}/2\mathbb{Z}) \rightarrow BSpin \rightarrow BSO$$

be the fiber sequence and let

$$K(\mathbb{Z}, 2) \rightarrow BSpin^c \rightarrow BSO \rightarrow BGL_1\mathbb{S} = \mathcal{L}ine_{\mathbb{S}}.$$

Applying the Thom construction to the second sequence and again using the fact that $K(\mathbb{Z}, 2) \rightarrow \mathcal{L}ine_{\mathbb{S}}$ factors through the point, we get the sequence

$$\Sigma_+^\infty K(\mathbb{Z}, 2) \rightarrow MSpin^c \rightarrow MSO.$$

Now, using Atiyah–Bott–Shapiro orientation $MSpin^c \rightarrow KU$ [ABS64], we get a map $K(\mathbb{Z}, 2) \rightarrow GL_1 KU$, which induces a map $K(\mathbb{Z}, 3) \rightarrow \mathcal{L}ine_{KU}$. This gives us twisted KU -cohomology of $K(\mathbb{Z}, 3)$. As $K(\mathbb{Z}, 3)$ classifies bundle gerbes, this is the twist of KU -theory by bundle gerbes.

2. TWISTS VIA PICARD GROUPOIDS AND GRADING

This section requires some further details. Up until now we defined everything via $\mathcal{L}ine_R$, however for many applications we need to work with $\mathcal{P}ic_R$ instead.

Definition 2.1. Let R be a ring spectrum. The *Picard ∞ -groupoid* $\mathcal{P}ic_R$ is the full sub- ∞ -groupoid of $\mathcal{M}od_R$ spanned by the invertible R -modules, i.e. those R -modules M such that there exists an R -module N with $M \wedge_R N \simeq R$.

Remark 2.2. It is a common fact that the ∞ -groupoid $\mathcal{P}ic_R$ is equivalent to $\pi_0(R) \times \mathcal{L}ine_R$, meaning that the invertible R -modules are given by the R -lines together with a shift by an element in $\pi_0(R)$. Here it is important to note that this equivalence does not respect the monoidal structure is only an equivalence of underlying ∞ -groupoids.

We can now generalize [Definition 1.6](#) to $\mathcal{P}ic_R$.

Definition 2.3 (Thom spectrum). The *Thom R -module spectrum* is the functor

$$M: \mathcal{G}rpd_{\infty}^{op}/\mathcal{P}ic_R \rightarrow \mathcal{M}od_R,$$

which sends $f: X^{op} \rightarrow \mathcal{P}ic_R$ to the R -module spectrum $\text{colim}(X^{op} \xrightarrow{f} \mathcal{P}ic_R \rightarrow \mathcal{M}od_R)$.

We can now generalize [Definition 1.8](#) to $\mathcal{P}ic_R$.

Definition 2.4 (Twisted cohomology). Let R be a ring spectrum, X be a space and let $p: X \rightarrow *$ be the projection map, and let $f: X^{op} \rightarrow \mathcal{P}ic_R$. The *f -twisted R -cohomology of X* is defined as the mapping spectrum

$$R_f(X) := \text{Map}_{\mathcal{M}od_R}(Mf, R) \simeq \text{Map}_{\mathcal{M}od_R(X)}(f, p^*R) \simeq \text{Map}_{\mathcal{M}od_R(X)}(f, R_x).$$

Similarly, we have *f -twisted R -homology of X* defined as

$$R^f(X) := \text{Map}_{\mathcal{M}od_R}(R, Mf) \simeq Mf.$$

Let us see how this additional generality helps us. Let $f: X \rightarrow \mathcal{L}ine_R$, then we get a map $\Sigma^n f: \Sigma^n X \rightarrow \mathcal{P}ic_R$, which induces a cohomology $R^{*+n}(X)$. Now we have

$$R_{\Sigma^n f}(X) \simeq \text{Map}(M\Sigma^n f, R) \simeq \text{Map}(\Sigma^n f, R_x) \simeq \text{Map}(f, \Omega^n R_x) \simeq \Omega^n \text{Map}(f, R_x) \simeq \Omega^n R_f(X)$$

This equivalence can only work via $\mathcal{P}ic_R$. Indeed even though R is in $\mathcal{L}ine_R$, $\Sigma^n R$ is not in $\mathcal{L}ine_R$. However, it is in $\mathcal{P}ic_R$, with inverse $\Sigma^{-n}R$.

This show that gradings can be recovered via twisted cohomology theories using $\mathcal{P}ic_R$ as twists.

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