DIFFERENTIAL COHOMOLOGY SEMINAR 8

TALK BY HANNES BERKENHAGEN

The aim of this talk is to review twisted cohomology theory with the aim of later discussing twisted differential cohomology theories [**?**]. For these talks the main source is [**?**, **?**].

1. Twisted Cohomology

Let R be a ring spectrum, meaning a monoid object in the ∞-category of spectra $\mathcal{S}\mathrm{p}$. From this we get a presentable stable ∞-category $\mathcal{M}\mathrm{od}_R$ of left R-module spectra. Objects therein are morphisms of the form $R \wedge M \to M$ satisfying the usual associativity and unit conditions up to coherent homotopies.

Note we have an adjunction diagram

$$\mathcal{S}\mathrm{p} \xrightarrow[\substack{\longleftarrow \\ \mathrm{Hom}_R(R,-)}]{\substack{R \wedge - \\ \perp}} \mathcal{M}\mathrm{od}_R \ ,$$

where the right adjoint is in fact the forgetful functor. This in particular means $\mathcal{M}\mathrm{od}_R$ has a distinguished object R i.e. the free R-module of rank 1. We now refine these constructions.

Definition 1.1. Let R be a ring spectrum. An R-*line* is an R-module L such that $L \simeq R$.

Definition 1.2. Let $\mathcal{L}\mathrm{ine}_R$ be the full sub-∞-groupoid of $\mathcal{M}\mathrm{od}_R$ spanned by the R-lines.

By construction, $\mathcal{L}\mathrm{ine}_R$ is equivalent to the category with a single object R and hom-space GL_1R, the ∞-group of R-linear automorphisms of R. Notice that $GL_1(R) \subseteq \mathrm{Hom}_{\mathcal{M}\mathrm{od}_R}(R, R) \simeq \mathrm{Hom}(\mathbb{S}, R) = \Omega^\infty R$.

Lemma 1.3. $GL_1(R)$ *fits into the pullback square*

$$\begin{array}{ccc}
GL_1(R) & \longrightarrow & \Omega^\infty R \\
\downarrow & & \downarrow \\
\pi_0(R)^\times & \longrightarrow & \pi_0(R)
\end{array}$$

Proof. $\pi_0(R) \simeq \mathrm{Hom}_{\mathrm{ho}\mathcal{M}\mathrm{od}_R}(R, R)$, where $\mathrm{ho}\mathcal{M}\mathrm{od}_R$ is the homotopy category of R-modules, the right-vertical arrow corresponds to $\mathrm{Hom}_{\mathcal{M}\mathrm{od}_R}(R, R) \to \mathrm{Hom}_{\mathrm{ho}\mathcal{M}\mathrm{od}_R}(R, R)$ mapping a morphism to its homotopy class, and $\pi_0(R)^\times \subseteq \mathrm{Hom}_{\mathrm{ho}\mathcal{M}\mathrm{od}_R}(R, R)$ is the set of isomorphisms. Finally, a morphism in a ∞-category $\mathcal{C}$ is an equivalence if and only if its homotopy class is an isomorphism in $\mathrm{ho}\mathcal{C}$. $\square$

In particular, $GL_1(R)$ consists of the connected components of equivalences in Ω^∞.

Definition 1.4. Let X be a space. Denote by $\mathcal{M}\mathrm{od}_R(X)$ the ∞-category of R-module spectra parametrized over X, i.e. the functor category $\mathrm{Fun}(X^{op}, \mathcal{M}\mathrm{od}_R)$.

Definition 1.5. Let X be a space. Denote by $\mathcal{L}\mathrm{ine}_R(X)$ the ∞-groupoid of R-line spectra parametrized over X, i.e. the $\mathrm{Fun}(X^{op}, \mathcal{L}\mathrm{ine}_R)$.

Since $\mathcal{L}\mathrm{ine}_R \simeq BGL_1(R)$, functors $X^{op} \to \mathcal{L}\mathrm{ine}_R$ are generalization of local systems.

Example 1.6. Let $R_X \colon X^{op} \to * \to \mathcal{L}\mathrm{ine}_R$ be the constant functor with value R.

We now proceed to the Thom construction.

Definition 1.7 (Thom spectrum). The *Thom R-module spectrum* is the functor

$$M \colon \mathcal{G}\mathrm{rpd}_\infty^{op}/\mathcal{L}\mathrm{ine}_R \to \mathcal{M}\mathrm{od}_R,$$

which sends $f \colon X^{op} \to \mathcal{L}\mathrm{ine}_R$ to the R-module spectrum $\mathrm{colim}(X^{op} \xrightarrow{f} \mathcal{L}\mathrm{ine}_R \xrightarrow{i} \mathcal{M}\mathrm{od}_R)$.

Remark 1.8. Notice that the definition of Thom R-module spectrum make sense for any functor $X^{op} \to R\mathrm{Mod}$.

Let us note an alternative characterization that will be important later.

Remark 1.9. For a given map $f: X \to Y$, we get a map $f^*: \mathrm{Mod}_R(Y) \to \mathrm{Mod}_R(X)$ by precomposition with f^{op}. Since f^* preserves both limits and colimits, we construct a left adjoint $f_!: \mathrm{Mod}_R(X) \to \mathrm{Mod}_R(Y)$ and a right adjoint $f_*: \mathrm{Mod}_R(X) \to \mathrm{Mod}_R(Y)$, using left and right Kan extension along f^{op}. Let $f = p: X \to *$ be the terminal functor, then left Kan extension along p^{op} is exactly taking colimit, therefore

$$Mf \simeq p_!(i \circ f).$$

Remark 1.10 ([**?**, 3.6]). Let Triv_R be the slice groupoid $\mathcal{L}\mathrm{ine}_R/R$, together with the canonical projection $\pi: \mathrm{Triv}_R \to \mathcal{L}\mathrm{ine}_R$. An object of Triv_R is a pair (L, ϕ) of a R-line L and an equivalence $\phi: L \to R$. Consider then the following:

(1) Triv_R is a slice ∞-groupoid, hence contractible, and π is a Kan fibration.
(2) $GL_1(R)$ is equivalent to the fiber of π over $R \in \mathcal{L}\mathrm{ine}_R$ and acts freely on the fibers of π.

These observations imply $\mathcal{L}\mathrm{ine}_R$ is the classifying space for $GL_1(R)$-bundles. Here we use the term $GL_1(R)$-bundle to mean a parametrized family of $GL_1(R)$-spaces with a free and transitive action.

We now proceed to define twisted cohomology theories.

Definition 1.11 (Twisted cohomology). Let R be a ring spectrum, X be a space, $p: X \to *$ the terminal functor, and $f: X^{op} \to \mathcal{L}\mathrm{ine}_R$ a R-line bundle over X. The *f-twisted R-cohomology of X* is defined as the mapping spectrum

$$R_f(X) := \mathrm{Map}_{\mathrm{Mod}_R}(Mf, R) \simeq \mathrm{Map}_{\mathrm{Mod}_R(X)}(f, p^*R) \simeq \mathrm{Map}_{\mathrm{Mod}_R(X)}(f, R_X).$$

Similarly, the *f-twisted R-homology of X* is defined as

$$R^f(X) := \mathrm{Map}_{\mathrm{Mod}_R}(R, Mf) \simeq Mf.$$

The f-twisted R-cohomology groups of X are defined as the homotopy groups of $R_f(X)$, i.e.

$$R_f^n(X) := \pi_0(\mathrm{Map}_{\mathrm{Mod}_R}(Mf, \Sigma^n R)) \cong \pi_{-n}(\mathrm{Map}_{\mathrm{Mod}_R(X)}(f, R_X)).$$

Similarly, the f-twisted R-homology groups of X are defined as the homotopy groups of $R^f(X)$, i.e.

$$R_n^f(X) := \pi_0(\mathrm{Map}_{\mathrm{Mod}_R}(\Sigma^n R, Mf)) \cong \pi_n(Mf).$$

Example 1.12 (Trivial twist). If $f: X \to \mathcal{L}\mathrm{ine}_R$ factors through $*$, then f factors as the $X^{op} \to * \to \mathcal{S}$, the constant factor with value $*$, and $R \wedge \Sigma_+^\infty(-): \mathcal{S} \to \mathrm{Mod}_R$. The latter functor commutes with colimits, being a left adjoint, while the colimit of the latter is X itself, then $Mf \simeq R \wedge \Sigma_+^\infty X$. In particular, f-twisted R-cohomology and R-homology of X reduce to ordinary (untwisted) R-cohomology and R-homology of X.

Definition 1.13. Given a vector bundle $\pi: E \to B$, define the *Thom space* of π, denoted $\mathrm{Th}(E)$, to be the homotopy cofiber of $E_0 \subseteq E$, where E_0 is the complement of the zero section.

Lemma 1.14. *Consider a space X and the ∞-categorical Yoneda's embedding $y: X \to \mathrm{Fun}(X^{op}, \mathcal{S})$. The colimit of y is the terminal pre-sheaf on X, i.e. the pre-sheaf with constant value the one-point space.*

Proof. Let S be a pre-sheaf on X, consider then the slice category $X_{/S}$ of pairs (x, ϕ), where x is an object of X and $\phi: y(x) \to S$. The density theorem for ∞-categories states that S is equivalent to the colimit of $X_{/S} \to X \xrightarrow{y} \mathrm{Fun}(X^{op}, \mathcal{S})$, the first map being the canonical projection. Take $S = *$, then $X_{/*} \to X$ is an equivalence, hence the claim. $\square$

Let G be a topological group and BG the ∞-groupoid with a single object 1 and hom-space G. The category $\mathcal{S}_G := \mathrm{Fun}(BG, \mathcal{S})$ is equivalent to the category of G-spaces.

Lemma 1.15. *Consider $X = BG$, a G-space $f: X \to \mathcal{S}$ and its left Kan extension $f_!: \mathcal{S}_G \to \mathcal{S}$, then $f_! \simeq (- \times E)/G$, where $E = f(1)$.*

Proof. Evaluate at 1, then $f_!(y(1)) = E$, by definition, and $y(1) \simeq G$, as G-spaces, hence $(y(1) \times E)/G \simeq (G \times E)/G \simeq E$. Since $f_!$ and $(- \times E)/G$ agree on representables and are colimit-preserving, they are equivalent. $\square$

Example 1.16. Take the space $BO(n)$ and $f_n : BO(n) \to \mathcal{S}_*$ the n-sphere S^n with $O(n)$-action coming from the one-point compactification of the regular action on $\mathbb{R}^n$. Let $\alpha_n = \Sigma^{\infty-n} f_n : BO(n) \to \mathcal{S}\mathrm{p}$, then $\alpha_n(1) = \Sigma^{\infty-n} f_n(1) = \Sigma^{\infty-n} S^n \simeq \mathbb{S}$, so α_n factors through $\mathcal{L}\mathrm{ine}_\mathbb{S}$. Let $X = BO(n)^{op}$ and $p : X \to *$ the terminal functor, then

$$M\alpha_n = p_! \Sigma^{\infty-n} f_n \simeq \Sigma^{\infty-n} p_!(f_n)_! y \simeq \Sigma^{\infty-n}(f_n)_! \underbrace{p_!(y)}_{\simeq *} \simeq \Sigma^{\infty-n}(* \times S^n)/O(n)$$

Let $P = EO(n)$ be the universal $O(n)$-bundle and M a $O(n)$-space, then $* \times_{O(n)} M$ is modelled by the *strict* quotient $(P \times M)/O(n)$, then

$$S^n/O(n) = \mathrm{cofib}(\mathbb{R}^n_0 \subseteq \mathbb{R}^n)/O(n) \simeq \mathrm{cofib}(\underbrace{* \times_{O(n)} \mathbb{R}^n_0}_{\simeq E^n_0} \subseteq \underbrace{* \times_{O(n)} \mathbb{R}^n}_{\simeq E^n}) = \mathrm{Th}(E^n)$$

where $E^n = P \times_{O(n)} \mathbb{R}^n \to BO(n)$ is the universal n-dimensional vector bundle, hence $M\alpha_n \simeq \Sigma^{\infty-n}\mathrm{Th}(E^n)$.

The functor $BO(n) \to \mathcal{L}\mathrm{ine}_\mathbb{S}$ induces a ∞-group homomorphism $j_n : O(n) \to GL_1(\mathbb{S})$, mapping ϕ to $\Sigma^{\infty-n}\mathrm{Th}(\phi)$. Consider the suspension morphism $s_n = \mathbb{R} \oplus - : O(n) \to O(1+n)$, then

$$j_n(\mathbb{R} \oplus \phi) = \Sigma^{\infty-n-1}\mathrm{Th}(\mathbb{R} \oplus \phi) \simeq \Sigma^{\infty-n-1} \underbrace{\mathrm{Th}(\mathbb{R})}_{\simeq S^1} \wedge \mathrm{Th}(\phi) \simeq \Sigma^{\infty-n}\mathrm{Th}(\phi) = j_n$$

Recall that the colimit over the suspension morphisms s_n is the stable orthogonal group O.

Definition 1.17. Denote by j the induced ∞-group homomorphism $O \to GL_1(\mathbb{S})$ induced by the sequence j_n, called the *J-homomorphism*.

Example 1.18. Let $X = O^{op}$ and take $Bj : BO \to \mathcal{L}\mathrm{ine}_\mathbb{S}$, then Mj is denoted MO and called the *real bordism spectrum*.

Denote by M the extended Thom spectrum functor $\mathcal{G}\mathrm{rp}_\infty^{op}/\mathrm{Mod}_R \to \mathrm{Mod}_R$, this is a left adjoint to the functor $\mathcal{O}$ sending a R-module to the functor $* \to \mathrm{Mod}_R$ picking out M. In particular, M preserves colimits and $Bj \simeq \mathrm{colim}_n Bj_n$, therefore we have the following:

Theorem 1.19. $MO \simeq \mathrm{colim}_n MO(n) = \mathrm{colim}_n \Sigma^{\infty-n}\mathrm{Th}(E^n)$.

Example 1.20. A group homomorphism $\xi : G \to O$ induces a functor $f : BG \to \mathcal{L}\mathrm{ine}_\mathbb{S}$. The Thom spectrum Mf is denoted MG or $M\xi$, and called *G-bordism spectrum*. For $G = U, SO, Spin$, and $String$, we obtain the *complex, oriented, spin*, and *string bordism spectra*.

Remark 1.21. In Equation (1.20) we might take $G = \{*\}$, the one-point group, then $MG \simeq \mathbb{S}$, which, if it didn't have a name, might be called *framed bordism spectrum*, following the naming convention in Equation (1.20) and in line with the theorem that $\pi_*(\mathbb{S}) \simeq \Omega_*^{\mathrm{fr}}$, the bordism ring of framed (trivialized tangent bundle) smooth manifolds.

Let R be a ring in sets, then R is a A_∞-ring spectrum (actually, E_∞), $\Omega^\infty R$ is equivalent to R with discrete topology ($\pi_0(\Omega^\infty R) \simeq R$, as sets, and every other homotopy group vanish). In particular, $GL_1(R)$ is simply $R^\times$ with discrete topology. Consider then the fiber sequence $SO \to O \to \mathbb{Z}^\times \simeq GL_1(\mathbb{Z})$.

Example 1.22. $X = BO^{op}$ and $\alpha = w_1 : BO \to \mathcal{L}\mathrm{ine}_\mathbb{Z}$, the 1st Stiefel-Whitney class (delooping of the determinant $O \to GL_1(\mathbb{Z})$), then Mw_1 is a $\mathbb{Z}$-module spectra. Let $i : SO \subseteq O$, then $w_1 i$ factors through the point, so $M(w_1 i) \simeq \mathbb{Z} \wedge \Sigma_+^\infty SO$.

Given $f : X^{op} \to \mathrm{Mod}_R$ and a sequence $F \xrightarrow{i} X \xrightarrow{\pi} Y$, there is an induced sequence of Thom R-module spectra $MF \to MX \to MY$. If πi factors through the point, $MF \simeq R \wedge \Sigma_+^\infty F$.

Lemma 1.23. *Let R be a ring spectrum and X a connected monoidal ∞-groupoid, then*

$$\mathrm{Hom}_{\mathrm{Mon}(\mathcal{S}\mathrm{p})}(\Sigma_+^\infty X, R) \simeq \mathrm{Hom}_{\mathrm{Mon}(\mathcal{S})}(X, GL_1(R))$$

Proof. Since X is connected, the space of homomorphisms $X \to GL_1(R)$ is equivalent to the space of homomorphisms $X \to \Omega^\infty R$, then use that $(\Sigma^\infty, \Omega^\infty)$ is a monoidal adjunction (The monoidal structure on spectra is such that Σ^∞ is strong monoidal). $\qquad\square$

Example 1.24. Let tmf be the E_∞-ring spectrum of topological modular forms and $\sigma : MString \to$ tmf the *String*-orientation of tmf. The group *String* is E_∞-monoidal, so $MString$ is a E_∞-ring spectrum and σ is a ring homomomorphism. In the fiber sequence $K(\mathbb{Z}, 3) \to BString \to BSO$, the fiber map $i : K(\mathbb{Z}, 3) \to BString$ is also a monoid homomorphism, so the composition

$$\Sigma_+^\infty K(\mathbb{Z}, 3) \xrightarrow{Mi} MString \xrightarrow{\sigma} \text{tmf}$$

is a homomorphism. Using Equation (1.23), we conclude that the induced homomorphism $K(\mathbb{Z}, 3) \to \Omega^\infty$tmf factors through $GL_1(\text{tmf})$, or equivalently, $K(\mathbb{Z}, 4) \to \mathcal{L}\text{ine}_{\text{tmf}}$, i.e. *2-bundle gerbes twist* tmf.

Let us go down one step in the chromatic ladder.

Example 1.25. Recall the fiber sequences for *Spin* and *Spin^c*:

$$\mathbb{Z}_2 \to Spin \to SO, \qquad S^1 \to Spin^c \to SO$$

All the spaces involved are E_∞-groups. Applying the Thom spectrum functor to the delooped sequences, we get

$$\Sigma_+^\infty K(\mathbb{Z}_2, 1) \to MSpin \to MSO, \qquad \Sigma_+^\infty K(\mathbb{Z}, 2) \to MSpin^c \to MSO$$

Let $\sigma : MSpin \to KO$ and $\sigma^c : MSpin^c \to KU$ be the Atiyah-Bott-Shapiro orientation of real and complex K-theory (see, [**?**]). Similar to Equation (1.24), we get homomorphisms

$$\Sigma_+^\infty K(\mathbb{Z}_2, 1) \longrightarrow MSpin \xrightarrow{\sigma} KO, \qquad \Sigma_+^\infty K(\mathbb{Z}, 2) \longrightarrow MSpin^c \xrightarrow{\sigma^c} KU$$

Using Equation (1.23) again and delooping, we obtain functors $K(\mathbb{Z}_2, 2) \to \mathcal{L}\text{ine}_{KO}$ and $K(\mathbb{Z}, 3) \to \mathcal{L}\text{ine}_{KU}$, i.e. *real, resp. complex, bundle gerbes twist real, resp. complex, K-theory.*

2. Twists via Picard Groupoids and Grading

This section requires some some further details. Up until now we defined everything via $\mathcal{L}\text{ine}_R$, however for many applications we need to work with $\mathcal{P}\text{ic}_R$ instead.

Definition 2.1. Given a monoidal ∞-category $(\mathcal{C}, \otimes, 1)$, an object M is *invertible* if there is an object D such that $D \otimes M \simeq M \otimes D \simeq 1$. The *Picard ∞-groupoid* of $\mathcal{C}$ is the sub-∞-groupoid generated by invertible modules.

Definition 2.2. If R is a ring spectrum, $\mathcal{M}\text{od}_R$ is monoidal. Denote by $\mathcal{P}\text{ic}_R$ the Picard groupoid of $\mathcal{M}\text{od}_R$.

Remark 2.3. It is a common fact that the ∞-groupoid $\mathcal{P}\text{ic}_R$ is equivalent to $\pi_0(R) \times \mathcal{L}\text{ine}_R$, meaning that the invertible R-modules are given by the R-lines together with a shift by an element in $\pi_0(R)$. Here it is important to note that this equivalence does not respect the monoidal structure is only an equivalence of underlying ∞-groupoids.

As mentioned in Equation (1.8), the Thom spectrum functor makes sense for every functor $f : X^{op} \to \mathcal{M}\text{od}_R$. However, all examples of twists encountered so far came from functors into $\mathcal{L}\text{ine}_R$. An example of twist that is not the result of a R-line bundle is the *degree shift*.

Definition 2.4. Denote by M the *Thom R-module spectrum* functor

$$\mathcal{G}\text{rp}_\infty^{op} / \mathcal{M}\text{od}_R \to \mathcal{M}\text{od}_R$$

sending a functor $f : X^{op} \to \mathcal{M}\text{od}_R$ to its colimit.

If L is a R-line, then $\Sigma^n L$ is invertible, with inverse $\Sigma^{-n} L$. Moreover, $\Sigma^n L$ is a line if and only if R is n-periodic.

Example 2.5. Let $f : X^{op} \to \mathcal{L}\text{ine}_R$ be a twist. Denote by $\Sigma^n f$ the composition of f with the shift functor $\Sigma^n : \mathcal{L}\text{ine}_R \to \mathcal{P}\text{ic}_R$. Since Σ^n is an equivalence, it commutes with colimits, so

$$M\Sigma^n f \simeq \Sigma^n M f$$

If $f = R_X$, then $M\Sigma^n f \simeq \Sigma^n R \wedge \Sigma_+^\infty X$, so $\Sigma^n f$-twisted R-cohomology and R-homology correspond to normal R-cohomology and R-homology with a degree shift by n.

References

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