DIFFERENTIAL COHOMOLOGY SEMINAR 8

TALK BY HANNES BERKENHAGEN

The aim of this talk is to review twisted cohomology theory with the aim of later discussing twisted differential cohomology theories [BG16]. For these talks the main source is [ABG+14, ABG18].

1. Twisted Cohomology

Let R be a ring spectrum, meaning a monoid object in the ∞-category of spectra $\mathcal{S}\mathrm{p}$. From this we get a presentable stable ∞-category $\mathcal{M}\mathrm{od}_R$ of left R-module spectra. Objects therein are morphisms of the form $R \wedge M \to M$ satisfying the usual associativity and unit conditions up to coherent homotopies.

Remark 1.1. Every stable category $\mathcal{C}$ is enriched over spectra. Given two objects C, D, we will denote by $\mathcal{H}\mathrm{om}_{\mathcal{C}}(C, D)$ the hom-spectrum and by $\mathrm{Hom}_{\mathcal{C}}(C, D) = \Omega^\infty \mathcal{H}\mathrm{om}_{\mathcal{C}}(C, D)$ the underlying hom-space.

Note we have an adjunction

$$\begin{tikzcd} \mathcal{S}\mathrm{p} \arrow[r, shift left, "R\wedge -"] & \mathcal{M}\mathrm{od}_R \arrow[l, shift left, "\mathcal{H}\mathrm{om}_{\mathcal{M}\mathrm{od}_R}(R,-)"] \end{tikzcd},$$

where the right adjoint is equivalent to the forgetful functor.

Definition 1.2. Let R be a ring spectrum. An *R-line* is an R-module L such that $L \simeq R$.

Definition 1.3. Let $\mathcal{L}\mathrm{ine}_R$ be the full sub-∞-groupoid of $\mathcal{M}\mathrm{od}_R$ spanned by the R-lines.

By construction, $\mathcal{L}\mathrm{ine}_R$ is equivalent to the category with a single object R and hom-space $GL_1 R$, the ∞-group of R-linear automorphisms of R. Notice that $GL_1(R) \subseteq \mathrm{Hom}_{\mathcal{M}\mathrm{od}_R}(R, R) \simeq \mathrm{Hom}_{\mathcal{S}\mathrm{p}}(\mathbb{S}, R) = \Omega^\infty R$.

Lemma 1.4. *$GL_1(R)$ fits into the pullback square*

$$\begin{tikzcd} GL_1(R) \arrow[r] \arrow[d] & \Omega^\infty R \arrow[d] \\ \pi_0(R)^\times \arrow[r] & \pi_0(R) \end{tikzcd}$$

In particular, the inclusion $GL_1(R) \to \Omega^\infty R$ induces an isomorphism on n-homotopy groups, for all $n \geq 1$.

Proof. $\pi_0(R) \simeq \mathrm{Hom}_{\mathrm{Ho}\mathcal{M}\mathrm{od}_R}(R, R)$, where $\mathrm{Ho}\mathcal{M}\mathrm{od}_R$ is the homotopy category of R-modules, the right-vertical arrow corresponds to $\mathrm{Hom}_{\mathcal{M}\mathrm{od}_R}(R, R) \to \mathrm{Hom}_{\mathrm{Ho}\mathcal{M}\mathrm{od}_R}(R, R)$ mapping a morphism to its homotopy class, and $\pi_0(R)^\times \subseteq \mathrm{Hom}_{\mathrm{Ho}\mathcal{M}\mathrm{od}_R}(R, R)$ is the set of isomorphisms. Finally, a morphism in a ∞-category $\mathcal{C}$ is an equivalence if and only if its homotopy class is an isomorphism in $\mathrm{Ho}\mathcal{C}$. □

Definition 1.5. Let X be a space. Denote by $\mathcal{M}\mathrm{od}_R(X)$ the ∞-category of R-module spectra parametrized over X, i.e. the functor category $\mathcal{F}\mathrm{un}(X^{\mathrm{op}}, \mathcal{M}\mathrm{od}_R)$.

Definition 1.6. Let X be a space. Denote by $\mathcal{L}\mathrm{ine}_R(X)$ the ∞-groupoid of R-line spectra parametrized over X, i.e. the functor category $\mathcal{F}\mathrm{un}(X^{\mathrm{op}}, \mathcal{L}\mathrm{ine}_R)$.

Since $\mathcal{L}\mathrm{ine}_R \simeq BGL_1(R)$, functors $X^{\mathrm{op}} \to \mathcal{L}\mathrm{ine}_R$ classify $GL_1(R)$-principal bundles over X. In this sense, elements of $\mathcal{L}\mathrm{ine}_R(X)$ are ∞-local systems.

Example 1.7. Let $R_X \colon X^{\mathrm{op}} \to * \to \mathcal{L}\mathrm{ine}_R$ be the constant functor with value R.

We now proceed to the Thom construction. We introduce it to full generality.

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Definition 1.8 (Thom spectrum)**.** The *Thom R-module spectrum* is the functor

$$R\colon \mathcal{S}/\mathcal{M}\mathrm{od}_R \to \mathcal{M}\mathrm{od}_R,$$

which sends $f\colon X^{\mathrm{op}} \to \mathcal{M}\mathrm{od}_R$ to the R-module spectrum R^f given by colimit of f.

Remark 1.9. The colimit of a functor $F : \mathcal{C} \to \mathcal{X}$ is the left Kan extension of F along the terminal functor $p : \mathcal{C} \to *$. If $\mathcal{X}$ is cocomplete, a functor $\pi : \mathcal{C} \to \mathcal{D}$ induces a pullback functor $\pi^* : \mathcal{F}\mathrm{un}(\mathcal{D}, \mathcal{X}) \to \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{X})$, which is cocontinuous, therefore using left Kan extension along π we can construct a left adjoint $\pi_!$.

Let us have a look at some examples of Thom spectra.

Example 1.10 (Trivial twist)**.** We begin by identifying the Thom spectrum of R_X. By the universal property of colimits, the Thom spectrum functor is left adjoint to the functor $\mathcal{M}\mathrm{od}_R \to \mathcal{S}/\mathcal{M}\mathrm{od}_R$ sending a module E to the corresponding functor $* \to \mathcal{M}\mathrm{od}_R$. In particular, the Thom spectrum functor is cocontinuous. Next, consider a R-module E and the functor $\mathcal{S} \to \mathcal{S}/\mathcal{M}\mathrm{od}_R$ sending a space X to the constant functor $E_X : X^{\mathrm{op}} \to \mathcal{M}\mathrm{od}_R$ with value E. This functor is also cocontinuous, by the way colimits are calculated in $\mathcal{S}/\mathcal{M}\mathrm{od}_R$. Therefore, the composition

$$\mathcal{S} \longrightarrow \mathcal{S}/\mathcal{M}\mathrm{od}_R \longrightarrow \mathcal{M}\mathrm{od}_R \tag{1.11}$$

is cocontinuous, and so completely determined by the value of the one-point space $*$, which is E. On the other hand, the following composition

$$\mathcal{S} \xrightarrow{\Sigma^\infty_+} \mathcal{S}\mathrm{p} \xrightarrow{E\wedge -} \mathcal{M}\mathrm{od}_R \tag{1.12}$$

is also cocontinuous and sends $*$ to E. We conclude that the colimit of E_X, for every space X, is equivalent to $E \wedge \Sigma^\infty_+ X$.

For the next example, we require some preliminary lemmas.

Definition 1.13. Given a vector bundle $V \to B$, the *Thom space* of V, denoted by S^V, is the homotopy cofiber of $V_0 \subseteq V$, where V_0 is the complement of the zero section.

Example 1.14. Let $B = *$ and $V = \mathbb{R}^n$, then S^V is homotopy equivalent to the n-sphere S^n.

Lemma 1.15. *Let $\mathcal{C}$ be a (small) category and $y : \mathcal{C} \to \mathcal{F}\mathrm{un}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ the ∞-categorical Yoneda embedding. The colimit of y is the terminal pre-sheaf, i.e. the pre-sheaf with constant value $*$.*

Proof. Apply the density theorem to the terminal pre-sheaf. □

We now proceed to define twisted cohomology theories.

Definition 1.16 (Twisted cohomology)**.** Let R be a ring spectrum, X be a space, $p\colon X \to *$ the terminal functor, and $f\colon X^{\mathrm{op}} \to \mathcal{L}\mathrm{ine}_R$ a R-line bundle over X. The *f-twisted R-cohomology of X* is defined as the mapping spectrum

$$R_f(X) \coloneqq \mathcal{H}\mathrm{om}_{\mathcal{M}\mathrm{od}_R}(Mf, R) \simeq \mathcal{H}\mathrm{om}_{\mathcal{M}\mathrm{od}_R(X)}(f, p^*R) \simeq \mathcal{H}\mathrm{om}_{\mathcal{M}\mathrm{od}_R(X)}(f, R_X).$$

Similarly, the *f-twisted R-homology of X* is defined as

$$R^f(X) \coloneqq \mathcal{H}\mathrm{om}_{\mathcal{M}\mathrm{od}_R}(R, Mf) \simeq Mf.$$

The f-twisted R-cohomology groups of X are defined as the homotopy groups of $R_f(X)$, i.e.

$$R^n_f(X) \coloneqq \pi_0(\mathcal{H}\mathrm{om}_{\mathcal{M}\mathrm{od}_R}(Mf, \Sigma^n R)) \cong \pi_{-n}(\mathcal{H}\mathrm{om}_{\mathcal{M}\mathrm{od}_R(X)}(f, R_X)).$$

Similarly, the f-twisted R-homology groups of X are defined as the homotopy groups of $R^f(X)$, i.e.

$$R^f_n(X) \coloneqq \pi_0(\mathcal{H}\mathrm{om}_{\mathcal{M}\mathrm{od}_R}(\Sigma^n R, Mf)) \cong \pi_n(Mf).$$

Example 1.17 (Trivial twist)**.** If $f : X \to \mathcal{L}\mathrm{ine}_R$ factors through $*$, then f factors as $X^{\mathrm{op}} \to * \to \mathcal{S}$, the constant factor with value $*$, and $R \wedge \Sigma^\infty_+(-) : \mathcal{S} \to \mathcal{M}\mathrm{od}_R$. The latter functor commutes with colimits, being a left adjoint, while the colimit of the latter is X itself, then $Mf \simeq R \wedge \Sigma^\infty_+ X$. In particular, f-twisted R-cohomology and R-homology of X reduce to ordinary (untwisted) R-cohomology and R-homology of X.

2. Examples of Twisted Cohomology

We now proceed to analyze several examples of twisted cohomology theories. This requires some preliminary lemmas.

Lemma 2.1. *Consider a space X and the ∞-categorical Yoneda's embedding $y : X \to \mathcal{F}\mathrm{un}(X^{\mathrm{op}}, \mathcal{S})$. The colimit of y is the terminal pre-sheaf on X, i.e. the pre-sheaf with constant value the one-point space.*

Proof. Let S be a pre-sheaf on X, consider then the slice category $X_{/S}$ of pairs (x, ϕ), where x is an object of X and $\phi : y(x) \to S$. The density theorem for ∞-categories states that S is equivalent to the colimit of $X_{/S} \to X \xrightarrow{y} \mathcal{F}\mathrm{un}(X^{\mathrm{op}}, \mathcal{S})$, the first map being the canonical projection. Take $S = *$, then $X_{/*} \to X$ is an equivalence, hence the claim. □

Let G be a topological group and BG the ∞-groupoid with a single object $*$ and hom-space G. The category $\mathcal{S}_G \coloneqq \mathcal{F}\mathrm{un}(BG, \mathcal{S})$ is equivalent to the category of G-spaces.

Lemma 2.2. *Consider $X = BG$, a G-space $f : X \to \mathcal{S}$ and its left Kan extension $f_! : \mathcal{S}_G \to \mathcal{S}$, then $f_! \simeq (- \times E)/G$, where $E = f(*)$.*

Proof. Evaluate at $*$, then $f_!(y(*)) = E$, by definition, and $y(*) \simeq G$, as G-spaces, hence $(y(*) \times E)/G \simeq (G \times E)/G \simeq E$. Since $f_!$ and $(- \times E)/G$ agree on representables and are colimit-preserving, they are equivalent. □

Example 2.3. Take the space $BO(n)$ and $f_n : BO(n) \to \mathcal{S}_*$ the n-sphere S^n with $O(n)$-action coming from the one-point compactification of the regular action on $\mathbb{R}^n$. Let $\alpha_n = \Sigma^{\infty-n} f_n : BO(n) \to \mathcal{S}\mathrm{p}$, then $\alpha_n(*) = \Sigma^{\infty-n} f_n(*) = \Sigma^{\infty-n} S^n \simeq \mathbb{S}$, so α_n factors through $\mathcal{L}\mathrm{ine}_{\mathbb{S}}$. Let $X = BO(n)^{\mathrm{op}}$ and $p : X \to *$ the terminal functor, then

$$M\alpha_n = p_! \Sigma^{\infty-n} f_n \simeq \Sigma^{\infty-n} p_! (f_n)_! y \simeq \Sigma^{\infty-n} (f_n)_! \underbrace{p_!(y)}_{\simeq *} \simeq \Sigma^{\infty-n} (* \times S^n)/O(n)$$

Let $P = EO(n)$ be the universal $O(n)$-bundle and M a $O(n)$-space, then $* \times_{O(n)} M$ is modelled by the *strict* quotient $(P \times M)/O(n)$, then

$$S^n/O(n) = \mathrm{cofib}(\mathbb{R}^n_0 \subseteq \mathbb{R}^n)/O(n) \simeq \mathrm{cofib}(\underbrace{* \times_{O(n)} \mathbb{R}^n_0}_{\simeq E^n_0} \subseteq \underbrace{* \times_{O(n)} \mathbb{R}^n}_{\simeq E^n}) = \mathrm{Th}(E^n)$$

where $E^n = P \times_{O(n)} \mathbb{R}^n \to BO(n)$ is the universal n-dimensional vector bundle, hence $M\alpha_n \simeq \Sigma^{\infty-n}\mathrm{Th}(E^n)$.

The functor $BO(n) \to \mathcal{L}\mathrm{ine}_{\mathbb{S}}$ induces a ∞-group homomorphism $j_n : O(n) \to GL_1(\mathbb{S})$, mapping ϕ to $\Sigma^{\infty-n}\mathrm{Th}(\phi)$. Consider the suspension morphism $s_n = \mathbb{R} \oplus - : O(n) \to O(1+n)$, then

$$j_n(\mathbb{R} \oplus \phi) = \Sigma^{\infty-n-1}\mathrm{Th}(\mathbb{R} \oplus \phi) \simeq \Sigma^{\infty-n-1} \underbrace{\mathrm{Th}(\mathbb{R})}_{\simeq S^1} \wedge \mathrm{Th}(\phi) \simeq \Sigma^{\infty-n}\mathrm{Th}(\phi) = j_n$$

Recall that the colimit over the suspension morphisms s_n is the stable orthogonal group O.

Definition 2.4. Denote by j the induced group homomorphism $O \to GL_1(\mathbb{S})$, called the *J-homomorphism.*

Example 2.5. Let $X = O^{\mathrm{op}}$ and take $Bj : BO \to \mathcal{L}\mathrm{ine}_{\mathbb{S}}$, then Mj is denoted MO and called the *real bordism spectrum.*

Denote by M the extended Thom spectrum functor $\mathcal{G}\mathrm{rp}^{\mathrm{op}}_{\infty}/\mathcal{M}\mathrm{od}_R \to \mathcal{M}\mathrm{od}_R$, this is a left adjoint to the functor sending a R-module to the corresponding functor $* \to \mathcal{M}\mathrm{od}_R$. In particular, M preserves colimits and $Bj \simeq \mathrm{colim}_n Bj_n$, therefore we have the following:

Theorem 2.6. $MO \simeq \mathrm{colim}_n MO(n) = \mathrm{colim}_n \Sigma^{\infty-n}\mathrm{Th}(E^n)$.

Example 2.7. A group homomorphism $\xi : G \to O$ induces a functor $f : BG \to \mathcal{L}\mathrm{ine}_{\mathbb{S}}$. The Thom spectrum Mf is denoted MG or $M\xi$, and called *G-bordism spectrum.* For $G = U, SO, Spin$, and $String$, we obtain the *complex, oriented, spin,* and *string bordism spectra.*

Remark 2.8. In Equation (2.7) we might take $G = \{*\}$, the one-point group, then $MG \simeq \mathbb{S}$, which, if it didn't have a name, might be called *framed bordism spectrum*, following the naming convention in Equation (2.7) and in line with the theorem that $\pi_*(\mathbb{S}) \simeq \Omega^{\mathrm{fr}}_*$, the bordism ring of framed (trivialized tangent bundle) smooth manifolds.

Let R be a ring in sets, then R is a A_∞-ring spectrum (actually, E_∞), $\Omega^\infty R$ is equivalent to R with discrete topology ($\pi_0(\Omega^\infty R) \simeq R$, as sets, and every other homotopy group vanish). In particular, $GL_1(R)$ is simply $R^\times$ with discrete topology. Consider then the fiber sequence $SO \to O \to \mathbb{Z}^\times \simeq GL_1(\mathbb{Z})$.

Example 2.9. $X = BO^{\mathrm{op}}$ and $\alpha = w_1 : BO \to \mathcal{L}\mathrm{ine}_\mathbb{Z}$, the 1st Stiefel-Whitney class (delooping of the determinant $O \to GL_1(\mathbb{Z})$), then Mw_1 is a $\mathbb{Z}$-module spectra. Let $i : SO \subseteq O$, then $w_1 i$ factors through the point, so $M(w_1 i) \simeq \mathbb{Z} \wedge \Sigma^\infty_+ SO$.

Given $f : X^{\mathrm{op}} \to \mathcal{L}\mathrm{ine}_R$ and a sequence $F \xrightarrow{i} Y \xrightarrow{\pi} X$, there is an induced sequence of Thom R-module spectra $MF \to MY \to MX$. If πi factors through the point, $MF \simeq R \wedge \Sigma^\infty_+ F$.

Lemma 2.10. *Let R be a ring spectrum and X a connected monoidal ∞-groupoid, then*

$$\mathrm{Hom}_{\mathcal{M}\mathrm{on}(\mathcal{S}\mathrm{p})}(\Sigma^\infty_+ X, R) \simeq \mathrm{Hom}_{\mathcal{M}\mathrm{on}(\mathcal{S})}(X, GL_1(R))$$

Proof. Since X is connected, the space of homomorphisms $X \to GL_1(R)$ is equivalent to the space of homomorphisms $X \to \Omega^\infty R$, then use that $(\Sigma^\infty, \Omega^\infty)$ is a monoidal adjunction (The monoidal structure on spectra is such that Σ^∞ is strong monoidal). □

Remark 2.11. Notice that we can weaken the result. Namely, if X is 1-connected (pointed and connected), then the space of functors (of ∞-groupoids) $X \to GL_1(R)$ is equivalent to the space of functors $X \to \Omega^\infty R$ such that $* \to X \to \Omega^\infty R$ is an equivalence. This last space is equivalent, via the $(\Sigma^\infty_+, \Omega^\infty)$ adjunction, to the space of morphisms of spectra $\Sigma^\infty_+ X \to R$, such that $\mathbb{S} \to \Sigma^\infty_+ X \to R$ represents a unit in $\pi_0(R)$.

Remark 2.12. Notice that we can also strengthen the result. Namely, if X is a connected, commutative monoid object and R is a commutative ring spectrum, then $\Omega^\infty R$ and $GL_1(R)$ are also commutative monoid objects. Using the same argument, together with the fact that $(\Sigma^\infty, \Omega^\infty)$ is actually a *symmetric* monoidal adjunction, we conclude that

$$\mathrm{Hom}_{\mathcal{CM}\mathrm{on}(\mathcal{S}\mathrm{p})}(\Sigma^\infty_+ X, R) \simeq \mathrm{Hom}_{\mathcal{CM}\mathrm{on}(\mathcal{S})}(X, GL_1(R))$$

Remark 2.13. Let $\mathcal{D}$ be a monoidal ∞-category. Consider $\mathcal{C}\mathrm{at}_\infty/\mathcal{D}$, the ∞-category of functors into $\mathcal{D}$, with monoidal structure given by

$$(F : \mathcal{A} \to \mathcal{D}, G : \mathcal{B} \to \mathcal{D}) \longmapsto (\mathcal{A} \times \mathcal{B} \xrightarrow{F \times G} \mathcal{D} \times \mathcal{D} \xrightarrow{\otimes} \mathcal{D})$$

The monoidal unit is the functor $* \to \mathcal{D}$ picking out the monoidal unit of $\mathcal{D}$. If $\mathcal{D}$ is symmetric monoidal, then so is $\mathcal{C}\mathrm{at}_\infty/\mathcal{D}$. A (commutative) monoid object in $\mathcal{C}\mathrm{at}_\infty/\mathcal{D}$ is given by a (symmetric) monoidal category $\mathcal{C}$ and a (symmetric) monoidal functor $F : \mathcal{C} \to \mathcal{D}$.

In view of Equation (2.13), let R be commutative ring spectrum, then $\mathcal{M}\mathrm{od}_R$ is a symmetric monoidal ∞-category and $\mathcal{L}\mathrm{ine}_R$ is a symmetric monoidal ∞-groupoid. The category $\mathcal{G}\mathrm{rpd}^{\mathrm{op}}_\infty/\mathcal{L}\mathrm{ine}_R$ is then symmetric monoidal and (commutative) monoid objects are given by (symmetric) monoidal ∞-groupoids X^{op} a (symmetric) monoidal functors $X^{\mathrm{op}} \to \mathcal{L}\mathrm{ine}_R$. One can then check that M is a symmetric monoidal functor, so that (commutative) monoid objects are sent to (commutative) monoid objects in $\mathcal{M}\mathrm{od}_R$, i.e. (commutative) R-algebras.

Example 2.14. Let Tmf be the commutative ring spectrum of topological modular forms (see Equation (2.15)) and $\sigma : MString \to \mathrm{tmf}$ the *String*-orientation of tmf. In the sequence

$$BString \longrightarrow BO \xrightarrow{Bj} \mathcal{L}\mathrm{ine}_\mathbb{S}$$

all functors are symmetric monoidal, so that $MString$ is a commutative $\mathbb{S}$-algebra, i.e. a commutative ring. The *String*-orientation of tmf is also a commutative ring homomorphism. In the fiber sequence $K(\mathbb{Z}, 3) \to BString \to BSO$, the fiber map $i : K(\mathbb{Z}, 3) \to BString$ is also a symmetric monoidal, so the composition

$$\Sigma^\infty_+ K(\mathbb{Z}, 3) \xrightarrow{Mi} MString \xrightarrow{\sigma} \mathrm{tmf}$$

is a commutative ring homomorphism. Using Equation (2.10), we conclude that the induced homomorphism $K(\mathbb{Z}, 3) \to \Omega^\infty \mathrm{tmf}$ (which is a homomorphism of commutative monoid objects, given Equation (2.12)) factors through $GL_1(\mathrm{tmf})$, and so it induces a (symmetric monoidal) functor $K(\mathbb{Z}, 4) \to \mathcal{L}\mathrm{ine}_{\mathrm{tmf}}$, i.e. *2-bundle gerbes twist* tmf.

Remark 2.15. The spectrum of topological modular forms comes in three main flavors, namely:

(1) TMF, i.e. the global sections of the spectral structure sheaf $\mathcal{O}^{top} : (\mathrm{Aff}/\mathcal{M}_{ell})^{\mathrm{op}} \to \mathcal{CM}\mathrm{on}(\mathcal{S}\mathrm{p})$ on the (étale site of the) moduli stack of elliptic curves.

(2) Tmf, i.e. the global sections of the spectral structure sheaf $\bar{\mathcal{O}}^{top} : (\mathrm{Aff}/\bar{\mathcal{M}}_{ell})^{\mathrm{op}} \to \mathcal{CM}\mathrm{on}(\mathcal{S}\mathrm{p})$ on the (étale site of the) *compactified* moduli stack of elliptic curves. The inclusion $\mathcal{M}_{ell} \hookrightarrow \bar{\mathcal{M}}_{ell}$ induces a commutative ring homomorphism Tmf $\to$ TMF.

(3) tmf, i.e. the connective cover of Tmf. By definition, there is a commutative ring homomorphism tmf $\to$ Tmf.

In [AHR10], tmf is used to denote our TMF, in [Goe09], tmf is used to denoted our Tmf, and [DFHH14] has the same notation as us. In Equation (2.14), we use tmf to mean the connective cover of Tmf.

Let us go down one step in the chromatic ladder.

Example 2.16. Recall the fiber sequences for *Spin* and *Spin*^{c}:

$$\mathbb{Z}_2 \to Spin \to SO, \qquad S^1 \to Spin^c \to SO$$

All the spaces involved are commutative groups. Applying the Thom spectrum functor to the delooped sequences, we get

$$\Sigma^\infty_+ K(\mathbb{Z}_2, 1) \to MSpin \to MSO, \qquad \Sigma^\infty_+ K(\mathbb{Z}, 2) \to MSpin^c \to MSO$$

Let $\sigma : MSpin \to KO$ and $\sigma^c : MSpin^c \to KU$ be the Atiyah-Bott-Shapiro orientation of real and complex K-theory (see [ABS64]). Similar to Equation (2.14), we get homomorphisms

$$\Sigma^\infty_+ K(\mathbb{Z}_2, 1) \longrightarrow MSpin \xrightarrow{\sigma} KO, \qquad \Sigma^\infty_+ K(\mathbb{Z}, 2) \longrightarrow MSpin^c \xrightarrow{\sigma^c} KU$$

Using Equation (2.10) again and delooping, we obtain functors $K(\mathbb{Z}_2, 2) \to \mathcal{L}\mathrm{ine}_{KO}$ and $K(\mathbb{Z}, 3) \to \mathcal{L}\mathrm{ine}_{KU}$, i.e. *real, resp. complex, bundle gerbes twist real, resp. complex, K-theory.*

3. Twists via Picard Groupoids and Grading

This section requires some some further details. Up until now we defined everything via $\mathcal{L}\mathrm{ine}_R$, however for many applications we need to work with $\mathcal{P}\mathrm{ic}_R$ instead.

Definition 3.1. Given a monoidal ∞-category $(\mathcal{C}, \otimes, 1)$, an object M is *invertible* if there is an object N such that $N \otimes M \simeq M \otimes N \simeq 1$. The *Picard ∞-groupoid* of $\mathcal{C}$ is the sub-∞-groupoid generated by invertible modules.

Remark 3.2. A monoidal category $(\mathcal{C}, \otimes, 1)$ is *closed* if, for every object M, the *left tensoring with M* functor $M^\otimes : \mathcal{C} \to \mathcal{C}$ admits a right adjoint $F(M, -)$. If M is invertible, with inverse N, then $M^\otimes$ is an equivalence with $N^\otimes$ as inverse. In particular, we can promote $(M^\otimes, N^\otimes)$ to an adjoint equivalence. By uniqueness of adjoint functors $F(M, -) \simeq N^\otimes$, and so

$$N \simeq N \otimes 1 = N^\otimes(1) \simeq F(M, 1) =: DM$$

Definition 3.3. Given a closed monoidal category $(\mathcal{C}, \otimes, 1)$, the functor $D := F(-, 1) : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ will be called *duality*. Given an object M, the object DM is called the *dual of M*.

Definition 3.4. If R is a ring spectrum, $\mathcal{M}\mathrm{od}_R$ is closed monoidal. Denote by $\mathcal{P}\mathrm{ic}_R$ the Picard ∞-groupoid of $\mathcal{M}\mathrm{od}_R$.

Remark 3.5. $\Sigma^n R$ is invertible, with inverse $\Sigma^{-n} R$. In particular, there is a map $\mathbb{Z} \times \mathcal{L}\mathrm{ine}_R \to \mathcal{P}\mathrm{ic}_R$. However, this map need not be neither injective (if R is n-periodic), nor surjective (see [HM17]).

As mentioned in, the Thom spectrum functor makes sense for every functor $f : X^{\mathrm{op}} \to \mathcal{M}\mathrm{od}_R$. However, all examples of twists encountered so far came from functors into $\mathcal{L}\mathrm{ine}_R$. An example of twist that is not the result of a R-line bundle is the *degree shift*.

Definition 3.6. Denote by M the *Thom R-module spectrum* functor

$$\mathcal{G}\mathrm{rpd}^{\mathrm{op}}_\infty / \mathcal{M}\mathrm{od}_R \to \mathcal{M}\mathrm{od}_R$$

sending a functor $f : X^{\mathrm{op}} \to \mathcal{M}\mathrm{od}_R$ to its colimit.

Example 3.7. Let $f : X^{\mathrm{op}} \to \mathcal{L}\mathrm{ine}_R$ be a twist. Denote by $\Sigma^n f$ the composition of f with the shift functor $\Sigma^n : \mathcal{L}\mathrm{ine}_R \to \mathcal{P}\mathrm{ic}_R$. Since Σ^n is an equivalence, it commutes with colimits, so

$$M\Sigma^n f \simeq \Sigma^n M f$$

If $f = R_X$, then $M\Sigma^n f \simeq \Sigma^n R \wedge \Sigma^\infty_+ X$, so $\Sigma^n f$-twisted R-cohomology and R-homology correspond to normal R-cohomology and R-homology with a degree shift by n.

4. Umkehr Map

We now proceed to the construction of the umkehr map in twisted cohomology theories. Here we follow [ABG18]. Let R be a ring spectrum, denote by D_R the duality of $\mathcal{M}\mathrm{od}_R$. Given a invertible R-twist $\alpha : X^{\mathrm{op}} \to \mathcal{P}\mathrm{ic}_R$, denote by $D_R\alpha$ the post-composition of α^{op} with D_R.

Remark 4.1. Depending on which cohomological degrees we want, we sometimes use the following alternative definition of twisted cohomology:

$$R^\alpha(X) = \pi_0 \mathrm{Hom}_{\mathcal{M}\mathrm{od}_R}(M(D_R\alpha), R)$$

meaning we use the dual/inverse of α. To justify the use of $D_R\alpha$, consider the following: Let $\beta = D_R\alpha$, then

$$\mathrm{Hom}_{\mathcal{M}\mathrm{od}_R}(M\beta, R) \simeq \mathrm{Hom}_{\mathcal{M}\mathrm{od}_R(X)}(D_R\alpha, R_X) \simeq \mathrm{Hom}_{\mathcal{M}\mathrm{od}_R(X)}(R_X, \alpha \otimes_{R_X} R_X) \simeq \mathrm{Hom}_{\mathcal{M}\mathrm{od}_R(X)}(R_X, \alpha)$$

The second equivalence is a consequence of the fact that, if M be an invertible R-module, left tensoring with M is left adjoint to tensoring with $D_R M$, and $D_R D_R M \simeq M$, see Equation (3.2). If we think of α as a bundle of invertible R-modules over X, then $\mathrm{Hom}_{\mathcal{M}\mathrm{od}_R(X)}(R_X, \alpha)$ is the spectrum of global sections of α, which aligns with the idea that twisted cohomology is the homotopy groups of the global sections of a bundle of spectra.

If $R = \mathbb{S}$, we denote the duality D_R by simply D.

Definition 4.2. The *Spanier-Whitehead dual* of a space X is the dual spectrum of $\Sigma^\infty_+ X$ with respect to the sphere spectrum.

Remark 4.3. Recall that $\mathcal{S}\mathrm{p}$ is a closed symmetric monoidal category, with $\mathcal{H}\mathrm{om}_{\mathcal{S}\mathrm{p}}(E, -)$ being the right adjoint to the functor given by smash product with E. Let $(\mathcal{C}, \otimes, 1)$ a closed monoidal category, where $F(X, -)$ is the right adjoint to the functor given by left tensoring with X, then

$$hom_{\mathcal{C}}(-, F(1, E)) \simeq \mathrm{Hom}_{\mathcal{C}}(1 \otimes -, E) \simeq \mathrm{Hom}_{\mathcal{C}}(-, E)$$

Using Yoneda's lemma, we conclude that $E \simeq F(1, E)$, for all E.

Example 4.4. By Equation (4.3), the Spanier-Whitehead dual to $*$ is the sphere spectrum, since

$$D* = \mathcal{H}\mathrm{om}(\Sigma^\infty_+ *, \mathbb{S}) = \mathcal{H}\mathrm{om}(\Sigma^\infty S^0, \mathbb{S}) = \mathcal{H}\mathrm{om}(\mathbb{S}, \mathbb{S}) \simeq \mathbb{S}$$

Definition 4.5. Denote by ϕ the functor $\mathcal{S}^{\mathrm{op}} \to \mathbb{S}/\mathcal{S}\mathrm{p}$ from spaces to the category of spectra under $\mathbb{S}$, mapping X to the map of spectra

$$\phi(X) : \mathbb{S} \simeq D* \xrightarrow{Dp} DX$$

where p is the terminal map $X \to *$.

Definition 4.6. Denote by $\widehat{\mathcal{M}\mathrm{fd}}_\infty$ the topological groupoid of closed smooth manifolds and diffeomorphisms. The set of diffeomorphisms are topologized by the weak C^∞ topology, see [Hir94]. Let $\mathcal{M}\mathrm{fd}_\infty$ be homotopy coherent nerve of $\widehat{\mathcal{M}\mathrm{fd}}_\infty$.

Let B be a connected compact space and $\pi\colon X \to B$ a continuous function such that, for every $b \in B$, the fiber $\pi^{-1}(b) =: X_b$ is equipped with the structure of a closed smooth manifold, which varies continuously in B, in the sense of being classified by a functor

$$f : B \to \mathcal{M}\mathrm{fd}_\infty$$

Given such a classifying map f, consider the following composition

$$B^{\mathrm{op}} \longrightarrow \mathcal{M}\mathrm{fd}_\infty^{\mathrm{op}} \longrightarrow \mathcal{S}^{\mathrm{op}} \xrightarrow{\phi} \mathbb{S}/\mathcal{S}\mathrm{p}$$

where the middle functor is the one forgetting the smooth structure. The above composition is then an object of

$$\mathcal{F}\mathrm{un}(B^{\mathrm{op}}, \mathbb{S}/\mathcal{S}\mathrm{p}) \simeq \mathbb{S}_B/\mathcal{F}\mathrm{un}(B^{\mathrm{op}}, \mathcal{S}\mathrm{p})$$

i.e. it is a natural transformation

$$\phi_{X/B} : \mathbb{S}_B \to D_B(f)$$

where D_B denotes the Spanier-Whithead duality applied to (the opposite of) $f : B \to \mathcal{S}$ point-wise. Let $p : B \to *$ be the terminal functor and $p_! : \mathcal{F}\mathrm{un}(B^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{p}$ the left Kan extension along p, i.e. the colimit functor. Applying $p_!$ to $\phi_{X/B}$, we obtain a morphism of spectra

$$\Sigma^\infty_+ B \simeq p_! p^* \mathbb{S} = p_! \mathbb{S}_B \xrightarrow{p_!(\phi_{X/B})} p_! D_B(f) \tag{4.7}$$

Let $T_{X/B} \to X$ be the vector bundle of fiber-wise tangent vectors, classified by $X^{\mathrm{op}} \to BO(n)$, where n is the manifold dimension of the fibers[1]. Denote by α the composition

$$X^{\mathrm{op}} \xrightarrow{T_{X/B}} BO(n) \longrightarrow \mathcal{S}_* \xrightarrow{\Sigma^\infty} \mathcal{S}\mathrm{p} \tag{4.8}$$

the middle arrow being the n-sphere S^n with $O(n)$-action coming from the one-point compactification of the regular action on $\mathbb{R}^n$. The above diagram takes values in the Picard ∞-groupoid of the sphere spectrum.

Theorem 4.9. $p_! D_B(f) \simeq M(-\alpha) =: X^{-T_{X/B}}$.

Definition 4.10. The morphism

$$\mathcal{PT}(f) : \Sigma^\infty_+ B \longrightarrow X^{-T_{X/B}}$$

is called *Pontryagin-Thom transfer map.*

Consider now a general situation. Let R be a ring spectrum and $\alpha : X^{\mathrm{op}} \to \mathcal{L}\mathrm{ine}_R$ a R-line bundle.

Definition 4.11. An *orientation* of $M\alpha$ is lift

$$\begin{tikzcd} & * \arrow[d] \\ X^{\mathrm{op}} \arrow[ur, dashed] \arrow[r, "\alpha"'] & \mathcal{L}\mathrm{ine}_R \end{tikzcd}$$

Explicitly, it is an equivalence of R_X-modules $\alpha \to R_X$.

Applying $p_!$, we see that an equivalence $t : \alpha \to R_X$ induces an equivalence

$$M\alpha \xrightarrow{p_!(t)} R \wedge \Sigma^\infty_+ X$$

In our case, α in Equation (4.8) is not valued in $\mathcal{L}\mathrm{ine}_{\mathbb{S}}$, but $\Sigma^{-n}\alpha$ is. An orientation of $\Sigma^{-n}\alpha$ induces an equivalence

$$\Sigma^{-n} M\alpha \xrightarrow{\simeq} R \wedge \Sigma^\infty_+ X \tag{4.12}$$

Definition 4.13. The isomorphism in cohomology induced by Equation (4.12) is called *Thom isomorphism.*

Finally, we have the following definition:

Definition 4.14. Assuming $\Sigma^{-n}\alpha : X^{\mathrm{op}} \to \mathcal{L}\mathrm{ine}_R$ is orientable, the *Umkehr map* is the map:

$$R^*(\Sigma^\infty_+ X) \xrightarrow{\text{Thom iso.}} R^{*-n}(X^{-T_{X/B}}) \xrightarrow{\mathcal{PT}(f)^{*-n}} R^{*-n}(\Sigma^\infty_+ B)$$

Remark 4.15. The name *Umkehr* comes from the map going in the opposite direction to $R^*(\Sigma^\infty_+ X) \to R^*(\Sigma^\infty_+ B)$, given by pulling back along π.

We now introduce the *twisted Umkehr map.* Given a twist $\beta \colon B^{\mathrm{op}} \to \mathcal{P}\mathrm{ic}_R$, we can smash to get a map

$$\beta \simeq \mathbb{S}_B \wedge_B \beta \longrightarrow D_B(f) \wedge_B \beta$$

Applying the functor $p_!$ once again, we obtain a morphism

$$M\beta \longrightarrow p_!(D_B(f) \wedge_B \beta) \simeq X^{-T_f + \beta\pi}$$

[1] B is connected, so the fibers have constant dimension.

where $\beta\pi : X^{\mathrm{op}} \to \mathcal{P}\mathrm{ic}_R$ is the composition of π and the twist β.

Definition 4.16. The *twisted Pontryagin-Thom transfer map* is the morphism:

$$\mathcal{PT}(f,\beta) : M\beta \longrightarrow X^{-T_f+\beta\pi}$$

The *twisted Umkehr map* is the map of twisted cohomology groups induced by $\mathcal{PT}(f,\beta)$.

Example 4.17. Assume that $X^{-T_{X/B}+\beta\pi}$ is orientable, i.e. the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{-T_f} & \mathcal{P}\mathrm{ic}_{\mathbb{S}} \\ \downarrow & & \downarrow \\ B & \xrightarrow{\alpha} & \mathcal{P}\mathrm{ic}_R \end{array}$$

then $X^{-T_{X/B}+\beta\pi} \simeq R \wedge \Sigma^\infty_+ X$ and the twisted Umkehr map becomes:

$$R^*(\Sigma^\infty_+ X) \longrightarrow R^{*-\beta}(B)$$

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