DIFFERENTIAL COHOMOLOGY SEMINAR 4 (DRAFT)

TALK BY ALESSANDRO NANTO

In the last talk we learned the definition of a differential cohomology theory, as a sheaf valued in spectra on the site of manifolds. This talk continues our journey through differential cohomology theories, and focuses on the following three topics:

- (1) We want to learn how to construct non-trivial examples out of sheaves valued in chain complexes.
- (2) We want to understand how we can extend classical cohomology operations to the setting of differential cohomology theories.
- (3) We want to introduce suitable analogues of fiber-wise integration.

1. Abelian Groups, Spectra and the Heart

Let us start by reviewing the relation between abelian groups, rings and spectra.

Definition 1.1. Let $n \in \mathbb{Z}$ and X be a spectrum, define $\pi_n(X) := \pi_0(\Omega^{\infty + n}X) = \pi_0(X_{-n})$. We call π_n the n-th homotopy group of X.

Remark 1.2. Note that since $X_n \simeq \Omega^2 X_{n+2}$, for any n, the set $\pi_0(X_n)$ underlies the structure of an abelian group.

The category Sp underlies the structure of a symmetric monoidal ∞ -category ([Lur17, Corollary 4.8.2.19]). Following [Lur17], we denote by \otimes the tensor product on Sp.

Definition 1.3. A commutative algebra object in Sp is called an \mathbb{E}_{∞} -ring spectrum, see [Lur17, Definition 7.1.0.1]. Given an \mathbb{E}_{∞} -ring spectrum R, denote by Mod_R the corresponding category of left R-module spectra, see [Lur17, Definition 7.1.1.2].

Remark 1.4. The sphere spectrum \mathbb{S} acts as the monoidal unit of $\mathbb{S}p$, therefore it is a \mathbb{E}_{∞} -ring spectrum. The category $\mathrm{Mod}_{\mathbb{S}}$ is canonically equivalent to $\mathbb{S}p$.

Definition 1.5. Denote by $\operatorname{Sp}_{\geq 0} \subseteq \operatorname{Sp}$ the full sub-category generated by *connective spectra*, i.e. spectra X such that $\pi_n(X) \simeq 0$, for all n < 0. Denote by $\operatorname{Sp}^{\heartsuit} \subseteq \operatorname{Sp}_{\geq 0}$ the *heart of spectra*, i.e. the full sub-category generated by spectra X such that $\pi_n(X) \simeq 0$, for all n > 0.

We have the following result relating connective spectra and the heart, which follow immediately.

Lemma 1.6. Let X be a connective spectrum. The following are equivalent:

- (1) X is in the heart.
- (2) $\pi_n(\Omega^{\infty}X) = 0$, for all n > 0.
- (3) $\operatorname{Hom}_{S_{\alpha}}(S, \Omega^{\infty}X) \simeq 0$, for all connected, pointed spaces S.
- (4) X is local with respect to the class of maps $\Sigma^{\infty}S \to 0$, for every connected pointed space S.

The category $\mathrm{Sp}_{\geq 0}$ is presentable and π_0 induces an equivalence between the heart and Ab ([Lur17, Proposition 1.4.3.6]). The heart is a sub-category of local objects of connective spectra, therefore the inclusion $\mathrm{Ab} \simeq \mathrm{Sp}^{\heartsuit} \subseteq \mathrm{Sp}_{\geq 0}$ is a right adjoint. The category $\mathrm{Sp}_{\geq 0}$ is closed under \otimes and, given X, Y connective spectra,

(1.7)
$$\pi_0(X \otimes Y) \simeq \pi_0(X) \otimes \pi_0(Y)$$

see [Dav24, Theorem 2.3.28]

Definition 1.8. Given an abelian group A, denote by HA the (unique up to equivalence) spectrum of the heart such that $\pi_0(HA) \simeq A$. We call HA the Eilenberg-Mac Lane spectrum of A.

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Using Equation (1.7), one can prove H, viewed as a functor $Ab \to Sp$, is lax monoidal. In particular, if R is a commutative ring, then HR is a connective \mathbb{E}_{∞} -ring spectrum. On the other hand, if R is a connective \mathbb{E}_{∞} -ring spectrum and M a connective module, then $\pi_0(M)$ is a $\pi_0(R)$ -module.

Definition 1.9. Given a commutative ring R, denote by $Ch(R) = Ch(Mod_R)$ the ordinary category of unbounded chain complexes. Let $\mathcal{D}(R)$ be the ∞ -localization of Ch(R) at the class of quasi-isomorphisms.

Similar to the heart of spectra, given an \mathbb{E}_{∞} -ring spectrum R, denote by $\operatorname{Mod}_R^{\heartsuit} \subseteq \operatorname{Mod}_R$ the full subcategory generated by R-modules such that the underlying spectrum belongs to the heart of spectra.

Theorem 1.10 (Stable Dold-Kan Correspondence). Let R be a commutative ring.

- (1) $\operatorname{Mod}_R \simeq \operatorname{Mod}_{HR}^{\heartsuit}$ via taking Eilenberg-Mac Lane spectra.
- (2) The equivalence in (1) extends to an equivalence $H: \mathcal{D}(R) \simeq \mathrm{Mod}_{HR}$ of symmetric monoidal ∞ -categories.

Proof. (1) is [Lur17, Proposition 7.1.1.13], while (2) is [Lur17, Theorem 7.1.2.13].
$$\Box$$

An interesting consequence of Theorem 1.10 is the following:

Corollary 1.11. Given $F \in \mathcal{D}(R)$, then $\pi_n(HF) \simeq H_n(F)$, for all $n \in \mathbb{Z}$.

Proof.

$$\pi_n(HF) = \pi_0(\Omega^{\infty+n}HF)$$

$$\stackrel{\textcircled{1}}{\simeq} \pi_0(\operatorname{Hom}_{\operatorname{\mathbb{S}p}}(\Sigma^n\mathbb{S}, HF))$$

$$\stackrel{\textcircled{2}}{\simeq} \pi_0(\operatorname{Hom}_{\operatorname{Mod}_{HR}}(\Sigma^nHR, HF))$$

$$\stackrel{\textcircled{3}}{\simeq} \pi_0(\operatorname{Hom}_{\mathfrak{D}(R)}(R[n], F))$$

$$\stackrel{\textcircled{4}}{\simeq} H_n(F)$$

① The functor $\Omega^{\infty+n}$ is corepresented by the shifted sphere spectrum $\Sigma^n \mathbb{S}$. ② The forgetful functor $\operatorname{Mod}_{HR} \to \operatorname{Mod}_{\mathbb{S}} \simeq \operatorname{Sp}$ is right adjoint to tensoring by HR and $HR \otimes (\Sigma^n \mathbb{S}) \simeq \Sigma^n HR$. ③ Theorem 1.10 ④ π_0 of the mapping space $\operatorname{Hom}_{\mathcal{D}(R)}(R[n], F)$ is equivalent to the mapping space $R[n] \to F$ in the ordinary derived category of R, i.e. homotopy classes of maps $R[n] \to F$, which correspond exactly to classes in $H_n(F)$.

2. More ∞-categorical baggage

Let \mathcal{C} be a presentable ∞ -category. The ∞ -categorical background given in previous talks allows to conclude the existence of a number of functors. Here we give a (somewhat) explicit formula for one.

Remark 2.1. Recall &uc, the full sub-category of Mfd generated by Euclidean manifolds \mathbb{R}^n , for every $n \geq 0$. Denote by j the inclusion functor &uc \subseteq Mfd. Recall that the restriction along j induces an equivalence $\operatorname{Shv}(\operatorname{Mfd}, \mathfrak{C}) \simeq \operatorname{Shv}(\operatorname{Euc}, \mathfrak{C})$, see [ADH21, Corollary A.5.6].

Evaluation at $\{0\}$ induces an adjunction (Lconst, Γ): $\mathcal{C} \to \operatorname{Shv}(\mathcal{M}\mathrm{fd}, \mathcal{C})$, where the functor Γ is evaluation at $\{0\}$, while the left adjoint Lconst maps $C \in \mathcal{C}$ to the sheafification of the constant pre-sheaf with value C.

Remark 2.2. Every presentable ∞ -category \mathcal{C} is uniquely cotensored over \mathcal{S} , see [Lur09, Remark 5.5.2.6]. More explicitly, for every space S and object C, there is an object C^S together with a natural equivalence

$$\operatorname{Hom}_{\mathfrak{C}}(S, \operatorname{Hom}_{\mathfrak{C}}(-, C)) \simeq \operatorname{Hom}_{\mathfrak{C}}(-, C^S)$$

Definition 2.3. Denote by Sing the functor $\mathcal{M}fd \to \mathcal{S}$ mapping a manifold to its underlying space. Given a presentable ∞ -category \mathcal{C} , denote by \flat the composition $\mathcal{C} \to \operatorname{Fun}(\mathcal{S}^{op}, \mathcal{C}) \to \operatorname{Fun}(\mathcal{M}fd^{op}, \mathcal{C})$, the first functor coming from Remark 2.2, the second being pre-composition with Sing^{op}.

Explicitly, given an object $C \in \mathcal{C}$, the associated pre-sheaf bC maps a manifold M to $C^{\operatorname{Sing}(M)}$.

Lemma 2.4 ([BG21, Corollary 6.46]). \flat factors through $Shv(Mfd, \mathcal{C}) \subseteq Fun(Mfd^{op}, \mathcal{C})$.

Lemma 2.4 is the direct consequence of a weaker version of a generalized version of Seifert-van Kampen theorem, namely [Lur17, Proposition A.3.2], stating that, given a topological space X and a covering sieve \mathcal{O} , the space $\mathrm{Sing}(X)$ is the colimit of $\mathrm{Sing}(U)$ over $U \in \mathcal{O}$.

Theorem 2.5. $\flat : \mathcal{C} \to \operatorname{Shv}(\operatorname{Mfd}, \mathcal{C})$ is left adjoint to Γ .

Proof. The composition $\mathbb{C} \xrightarrow{\flat} \operatorname{Shv}(\operatorname{Mfd},\mathbb{C}) \xrightarrow{j_*} \operatorname{Shv}(\operatorname{\mathcal{E}uc},\mathbb{C})$ maps an object C to the sheaf $\flat C$ restricted to Euclidean spaces. Since \mathbb{R}^n is contractible, $(\flat C)(\mathbb{R}^n) = C^{\operatorname{Sing}(\mathbb{R}^n)} \simeq C$ and so \flat restricted to $\operatorname{\mathcal{E}uc}$ is equivalent to Const, the functor taking C to the the pre-sheaf with constant value C, which is left adjoint to Γ restricted to $\operatorname{\mathcal{E}uc}$.

Remark 2.6. Notice that the proof of Theorem 1.10 also shows that given an object $C \in \mathcal{C}$, the pre-sheaf on \mathcal{E} uc with constant value C is already a sheaf and C, viewed as a constant sheaf on \mathcal{E} uc, is equivalent to $j_* \flat C$.

3. Sheaves of complexes and spectra

The stable Dold-Kan correspondence allows us to move freely between sheaves of $H\mathbb{Z}$ -module spectras and sheaves valued in $\mathcal{D}(\mathbb{Z})$.

Remark 3.1. We identify the category of cochain complexes with Ch(R) by reversing grading. Namely, given a cochain V^* , we are implicitly identifying it with the chain complex $V_n = V^{-n}$.

Definition 3.2 ([BNV16, Definition 7.14]). Given $n \in \mathbb{Z}$, denote by $\tau^{\geq n}$, resp. $\tau^{\leq n}$, the naive truncation functors, mapping a cochain complex V^* to

$$\cdots \to 0 \to V^n \to V^{n+1} \to \cdots$$
, resp. $\cdots \to V^{n-1} \to V^n \to 0 \to \cdots$

Given $F: \mathcal{M}fd^{op} \to \mathrm{Ch}(\mathbb{Z})$, denote by $F^{\geq n}$ the composite $\mathcal{M}fd^{op} \xrightarrow{F} \mathrm{Ch}(\mathbb{Z}) \xrightarrow{\tau^{\geq n}} \mathrm{Ch}(\mathbb{Z})$, and similarly we define $F^{\leq n}$. If F is a sheaf, then so are its truncations.

Lemma 3.3 ([BNV16, Lemma 7.12]). Let $F : \mathcal{M}fd^{op} \to \mathrm{Ch}(\mathbb{Z})$ a sheaf of chain complexes of C^{∞} -modules, then $\mathcal{M}fd^{op} \xrightarrow{F} \mathrm{Ch}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$ is a sheaf.

Definition 3.4. Denote by Ω^* the sheaf $\mathcal{M}fd^{op} \to \mathrm{Ch}(\mathbb{Z})$ mapping a manifold to its de Rham complex.

Lemma 3.3 ensures that the sheaf in Definition 3.4 and the corresponding naive truncations remain sheaves after post-composition with the localization functor $Ch(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$.

Definition 3.5. Given a sheaf $F: \mathcal{M}fd^{op} \to \mathcal{D}(\mathbb{Z})$, denote by HF the Eilenberg-Mac Lane sheaf of $H\mathbb{Z}$ -module spectra obtained by applying point-wise the equivalence of Theorem 1.10.

Recall now the machinery set-up in Section 2.

Remark 3.6. Since $\mathcal{D}(\mathbb{Z})$ is presentable, we know that they is cotensored over S. Given a space S and a chain complex M_* , the cotensor M_*^S is the chain complex of graded linear maps $C_*(S,\mathbb{Z}) \to M_*$, from the (normalized) singular chain complex of S to M_* , see [Lur17, Definition 1.3.2.1]. In particular, let $M_* = M$ be concentrated in degree 0, then M_*^S is the singular cochain complex of S with values in M.

Definition 3.7. Consider the morphism $\Omega^*(M) \to (\flat \mathbb{R})(M) = C^*(M, \mathbb{R})$ taking a form $\omega \in \Omega^n(M)$ to the linear map $\int \omega : C_n(M, \mathbb{Z}) \to \mathbb{R}$. We call the induced transformation $dR : \Omega^* \to \flat \mathbb{R}$ the de Rham morphism.

Lemma 3.8 ([AS10, Theorem 3.25]). dR : $\Omega^* \to \flat \mathbb{R}$ is point-wise an \mathbb{A}_{∞} -quasi-isomorphism of DG algebras.

4. Deligne Cohomology

Finally, we have enough machinery to talk about Deligne cohomology.

Definition 4.1. Given $\ell \in \mathbb{N}$, define $\hat{\mathbb{Z}}(\ell) : \mathcal{M}fd^{op} \to \mathcal{D}(\mathbb{Z})$ as the pullback of

The vertical morphism being the composition $\Omega^{\geq \ell} \subseteq \Omega^* \xrightarrow{dR} \flat \mathbb{R}$. We call the corresponding Eilenberg-Mac Lane spectrum $H\hat{\mathbb{Z}}(\ell)$ the ℓ -th Deligne sheaf (see Definition 3.5).

Remark 4.2 (Model A, see [HS05, §3.2]). Let $\acute{C}(\ell)^n(M) \subseteq C^n(M,\mathbb{Z}) \oplus C^{n-1}(M,\mathbb{R}) \oplus \Omega^n(M)$ consist of triples (c,h,ω) for which $\omega=0$ if $n<\ell$, with differential $\delta(c,h,\omega)=(\delta c,\mathrm{dR}(\omega)-c-\delta h,d\omega)$. This complex $\acute{C}^*(\ell)(M)$ fits into a diagram

$$\dot{C}^*(\ell)(M) \longrightarrow \Omega^{\geq \ell}(M)
\downarrow \qquad \qquad \downarrow
C^*(M, \mathbb{Z}) \longrightarrow C^*(M, \mathbb{R})$$

which commutes up to homotopy given by the projections $\acute{C}^n(\ell)(M) \to C^{n-1}(M,\mathbb{R})$. The diagram above model the pullback of Definition 4.1, hence $\acute{C}^*(\ell)(M)$ is a model for $\mathring{\mathbb{Z}}(\ell)(M)$.

Remark 4.3 (Model B). Recall that \flat preserves cofiber sequences, since it is left adjoint, and that fiber sequences are the same a cofiber sequences in stable ∞ -categories. Consider the diagram

$$\hat{\mathbb{Z}}(\ell) \longrightarrow \Omega^{\geq \ell} \\
\downarrow \qquad \qquad \downarrow \\
\flat \mathbb{Z} \longrightarrow \flat \mathbb{R} \\
\downarrow \qquad \qquad \downarrow \\
0 \longrightarrow \flat (\mathbb{R}/\mathbb{Z})$$

Since the bottom square is an pullback, $\hat{\mathbb{Z}}(\ell)(M)$ is equivalent to the pullback of the diagram in red. Let $\check{C}^n(\ell)(M) \subseteq C^{n-1}(M,\mathbb{R}/\mathbb{Z}) \oplus \Omega^n(M)$ consist of pairs (χ,ω) for which $\omega=0$ if $n<\ell$, with differential $\delta(\chi,\omega)=(e^{2\pi i \mathrm{dR}(\omega)}-\delta\chi,d\omega)$. Similar to Remark 4.2, the complex $\check{C}^*(\ell)(M)$ fits into the above diagram so that the outer square is an pullback, hence it is equivalent to $\hat{\mathbb{Z}}(\ell)(M)$.

Take an n-cocycle $(n \ge \ell)$ in the model from Remark 4.3, i.e. $(\chi, \omega) \in C^{n-1}(M, \mathbb{R}/\mathbb{Z}) \oplus \Omega^n(M)$ such that $d\omega = 0$ and $\delta \chi = e^{2\pi i d\mathbb{R}(\omega)}$. Such a cocycle determines a differential character of degree n-1 for M, in the sense of the following definition:

Definition 4.4 ([HS05, Definition 3.4], see also [BB14, Chapter 5]). Consider a manifold M, a differential character of degree n consists of a character $\chi: Z_n^{\infty}(M,\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ on the group of smooth n-cycles of M together with a n-form $\omega \in \Omega^{n+1}(M)$, such that, for every smooth (n+1)-chain c,

$$\chi(\partial c) = e^{2\pi i \int_c \omega}$$

Remark 4.5 (Model C, see [ADH21, Lemma 7.3.4]). Consider the following diagram in the category of sheaves on $\mathcal{E}uc$

$$j_*\hat{\mathbb{Z}}(\ell) \longrightarrow j_*\Omega^{\geq \ell} \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\mathbb{Z} \longrightarrow j_*\Omega^* \longrightarrow j_*\Omega^{\leq \ell-1}$$

The left square is obtained by: ① Apply the restriction functor j_* to the pullback diagram of Definition 4.1 ② Substitute $j^*\flat\mathbb{R}$ with $j_*\Omega^*$ (see Lemma 3.8) and $j_*\flat\mathbb{Z}$ with \mathbb{Z} (see Remark 2.6). Since the right square is a pullback, $j_*\hat{\mathbb{Z}}(\ell)$ is equivalent to the pullback of the diagram in red. Let $\check{C}^*(\ell)$ be the sheaf of chain complexes $\mathbb{Z} \to \Omega^0 \to \cdots \to \Omega^{\ell-1} \to 0 \to \cdots$, where \mathbb{Z} is in degree 0 includes into Ω^0 as constant functions. The complex $\check{C}^*(\ell)$ fits into the above diagram, so that the outer square is an pullback, therefore $j_*\hat{\mathbb{Z}}(\ell) \simeq \check{\mathbb{C}}^*(\ell)$ and $j^*\check{C}^*(\ell) \simeq \hat{\mathbb{Z}}(\ell)$. Since $\hat{\mathbb{Z}}(\ell)$ is equivalent to $\check{C}^*(\ell)$ on Euclidean manifolds, consider a manifold M, let 0 be a good open cover and $\mathfrak{I}(0)$ the closure of 0 under finite intersections, then

$$\hat{\mathbb{Z}}(\ell)(M) \simeq \lim_{U \in \mathcal{I}(\mathcal{O})} \check{C}^*(\ell)(U) \simeq \lim_{n \in \Delta} \prod_{U_1, \dots, U_n \in \mathcal{O}} \check{C}^*(\ell)(U_1 \cap \dots \cap U_n)$$

We then apply [BNV16, Lemma 7.10] to calculate the last limit as a the total complex functor applied to the bicomplex

$$\check{C}^{m,n}(\ell)(\mathfrak{O}) := \prod_{U_1,\cdots,U_n \in \mathfrak{O}} \check{C}^m(\ell)(U_1 \cap \cdots \cap U_n)$$

5. Unfolding the fracture square of Deligne Cohomology

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