What is an ∞-Category?

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∞-categories have become a standard tool in mathematics, especially in algebraic topology and algebraic geometry. However, ∞-category theory has primarily been developed using standard foundational assumptions. This leaves open how ∞-categories interact with different foundational assumptions. This talk aims to analyze some of these issues.

This talk is based on past work:

1. Filter quotients and non-presentable $(\infty,1)$-toposes

   DOI: j.jpaa.2021.106770 [Ras21]

2. Filter Quotient Model Categories

   arXiv:2508.07735 [Ras25a]

3. Non-Standard Models of Homotopy Type Theory

   arXiv: 2508.07736 [Ras25b]

4. Simplicial Homotopy Type Theory is not just Simplicial: What are ∞-Categories?

   arXiv:2508.07737 [Ras25c]

Here is a fun fact about these papers: There are three consecutive arXiv identifiers, I think that's a record!

1 What is an ∞-Category supposed to be?

What are the data and properties we expect from an ∞-category? Intuitively, it should be the following:

1. An $\mathbb{N}$-indexed collection of data (objects = 0-morphisms, 1-morphisms, 2-morphisms, ...)
2. Ways to relate these data (source and targets, ...)
3. Ways to compose these data
4. Ways to witness associativity, unitality, ... (higher coherences)

Fundamentally, we have two options to pursue such a definition:

1. **Synthetic:** We define our foundational object, such that it directly (maybe almost) encodes this data. We can then obtain higher morphisms and structure by unwinding definitions.
2. **Analytic:** We pick a foundation that explains what "collection" means and what "$\mathbb{N}$-indexed" means, and then work in that context.

Each approach has its benefits and drawbacks.

- **Analytic Approach Benefits:**
  1. Can pick a foundations that are well-studied and benefit from existing technology.
  2. Can construct examples coming from other areas of mathematics more directly.
- **Analytic Approach Drawbacks:**
  1. Definition and proofs can be unnecessarily involved and complicated due to foundation mismatch.
  2. We need to pick two foundations: one for the collection and one for the $\mathbb{N}$-indexing, which can diverge.
- **Synthetic Approach Benefits:**
  1. Foundation matches ∞-categorical intuition more closely (everything is equivalence invariant, functorial, ...)
  2. The aforementioned mismatch between the two foundations cannot occur.
- **Synthetic Approach Drawbacks:**
  1. Foundations are less well-studied and so less technology is available.
  2. Constructing examples from other areas of mathematics can be more complicated.

There are many different examples of ∞-categories all of which realize these conditions in some way.

- **Synthetic Examples:**
  1. **Simplicial Homotopy Type Theory:** Here ∞-categories are encoded via certain types, called *Rezk types*, and so the concrete data obtained by looking at maps out of certain test objects, which are called "shapes".
  2. **Regensburg ∞-Category Program:** Here ∞-categories are directly the foundational objects, similarly studied via test objects.
- **Analytic Examples:** These examples are often very point-set and have been used to construct many examples.
  1. **Quasi-categories** or **Complete Segal Spaces:**: Are simplicial objects, meaning the first two conditions are realized via the domain $\mathbb{\Delta}$, and the last two via lifting conditions. In these examples the foundation for "collection" are $\mathcal{S}$et or $\mathcal{A}$n, whereas the foundation for "$\mathbb{N}$-indexing" is $\mathbb{\Delta}$.
  2. **Enriched Categories:** Here again the first two conditions are realized via the objects and the enrichment, and the last two via the categorical structure. Again we have similar foundational assumptions.

We can summarize the situation as follows:

- The synthetic approach is in some sense "correct" as it will always match the situation appropriately.
- The analytic approach is nonetheless necessary, to connect to other areas of mathematics.

If we accept this, it leaves us with the following question: What is the correct way to do ∞-category theory analytically?

2 The Naive approach to Analytic ∞-Category Theory

The standard way people do analytic ∞-category theory in a general setup proceeds as follows:

1. Pick a foundation for "collection", e.g. an ∞-category $\mathcal{E}$ with finite limits.
2. Define ∞-categories as simplicial objects $\mathbb{\Delta}^{op} \to \mathcal{E}$ satisfying the Segal and completeness condition i.e. complete Segal objects or internal ∞-categories.

Remark 2.1. We are taking "external layer" also to be some previously agreed upon notion of ∞-category theory. This is does not make the mathematical arguments circular. As we are using an external layer to talk about an internal layer of mathematics.

This approach has been pursued in various places in the literature, for example in work of Martini-Wolf [MW21]. They develop significant amount of mathematics for ∞-categories internal to Grothendieck ∞-topoi. There are also alternative work by Ruit [Rui23] using a formal ∞-categorical approach, as well as other work [Ras22].

Let us summarize this approach from our perspective:

1. The "collection" are by definition objects in $\mathcal{E}$.
2. The $\mathbb{N}$-indexing is realized via the simplicial indexing category $\mathbb{\Delta}$.
3. Composition, unitality, associativity, ... are realized via the Segal and completeness conditions.

Why is this the naive approach? Crucially, $\mathbb{\Delta}$ does not depend on $\mathcal{E}$. Let's see an example that does not fit into this setup.

3 Filter Quotient Construction

We will now look at filter quotient ∞-categories.

Definition 3.1. A $(P, \leq)$ be a poset. A filter Φ is a subset that is:

1. **Non-empty:** Φ contains an element.
2. **Intersection closed:** Φ is co-filtered
3. **Upwards closed:** If $U \in \Phi$ and $V \in P$ and $U \leq V$, the $V \in \Phi$.

Definition 3.2. Let $\mathcal{E}$ be an ∞-category with finite products. A filter of subterminal objects is a filter of the (-1)-truncated objects.

By definition the terminal object 1 is in Φ.

Definition 3.3. Let $\mathcal{E}$ be an ∞-category with finite products and Φ a filter of subterminal objects. The *filter quotient ∞-category* $\mathcal{E}_\Phi$ is defined as

$$\mathcal{E}_\Phi = \mathrm{colim}_{U \in \Phi} \mathcal{E}_{/U}.$$

By construction there is a functor $P_\Phi \colon \mathcal{E} = \mathcal{E}_{/1} \to \mathcal{E}_\Phi$. It has the following cool properties.

Theorem 3.4 (R.)**.** *The functor P_Φ ...*

- *... preserves all finite properties (finite (co)limits, (local) Cartesian closure, ...)*
- *... does not preserve infinite properties (small (co)limits, presentability, ...)*

Let us look at one particular case of interest.

Example 3.5. Let I be a set. A filter Φ on the a set I, is a filter on the power-set PI. Such a filter induces a filter on the ∞-category $\prod_I \mathcal{E}$, which contains the following objects. For a given $U \in \Phi$, the subterminal object $(U_i)_{i\in I}$ with $U_i = *$ if $i \in U$ and $U_i = \emptyset$ if $i \notin U$ is in this filter. To simplify notation we also denote it by Φ.

Let $\mathcal{E}$ be an ∞-category and Φ a filter on the set I. The induced filter quotient is called *filter product* and is denoted $\prod_\Phi \mathcal{E} = (\prod_I \mathcal{E})_\Phi$. If the filter Φ is maximal, which is also called an ultrafilter, then the filter product is called an *ultraproduct*.

4 New ∞-Categories via Filter Quotients

We now look at an example of a filter quotient ∞-category whose objects are indeed (almost) ∞-categories, in the sense of above, but will not fit into the naive approach. This demonstrates there is a mismatch between "what an ∞-category is" in the analytic setting, and existing definitions via internal ∞-categories.

Let $\mathcal{A}\mathrm{n}^{\Delta^{op}}$ be the ∞-category of simplicial anima. Notice it includes as a reflective subcategory the ∞-category of ∞-categories $\mathcal{C}\mathrm{at}_\infty$, which are precisely our "internal ∞-categories". Take $\mathbb{N}$ with the Fréchet filter $\mathcal{F}$, meaning it contains the cofinite subsets. Then, the filter product $\mathcal{E} = \prod_{\mathcal{F}} \mathcal{A}\mathrm{n}^{\Delta^{op}}$ has objects and morphisms of the following form:

- Objects are sequences of simplicial anima $(\mathcal{C}_n)_{n\in\mathbb{N}}$.
- Morphisms from $(\mathcal{C}_n)_{n\in\mathbb{N}}$ to $(\mathcal{D}_n)_{n\in\mathbb{N}}$ are given by

$$\mathrm{Map}_{\prod_{\mathcal{F}} \mathcal{A}\mathrm{n}^{\Delta^{op}}}((\mathcal{C}_n)_{n\in\mathbb{N}}, (\mathcal{D}_n)_{n\in\mathbb{N}}) = \mathrm{colim}_{N\in\mathbb{N}} \prod_{n>N} \mathrm{Map}_{\mathcal{A}\mathrm{n}^{\Delta^{op}}}(\mathcal{C}_n, \mathcal{D}_n)$$

Let us understand the structure of the ∞-category $\mathcal{E}$:

1. There is an object of natural numbers of the form $(\mathbb{N})_{n\in\mathbb{N}}$
2. Hence, a natural number in $\mathcal{E}$ is a map $(*)_{n\in\mathbb{N}} \to (\mathbb{N})_{n\in\mathbb{N}}$, which corresponds to a sequence of natural numbers $(a_n)_{n\in\mathbb{N}}$, up to eventual equality.
3. The natural numbers include obvious examples, such as $(k)_{n\in\mathbb{N}}$ for a fixed natural number k. But it also includes non-trivial examples, such as $(n)_{n\in\mathbb{N}}$. Notice this has no non-trivial intersection with the (k). So, here we have uncountably many different natural numbers.
4. In particular the cocone $\{(k)_{n\in\mathbb{N}}\}_{k\in\mathbb{N}} \to (\mathbb{N})_{n\in\mathbb{N}}$ is not a coproduct cocone. Using some basic observations, this show that $\mathbb{N}$ is not a coproduct at all.
5. For each natural number $(a_n)_{n\in\mathbb{N}}$ there is a "free simplices" $(\Delta[a_n])_{n\in\mathbb{N}}$. This means $\mathcal{E}$ has "more" simplices.

The ∞-category $\mathcal{E}$ still has all finite limits and so we can impose Segal conditions and completeness conditions, to get a notion of ∞-category as a full subcategory of $\mathcal{E}$.

Using our newly gained understanding of simplices, we can unwind the structure of an object (our ∞-categories) $(\mathcal{C}_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ as follows:

1. 0-simplices (objects) are maps $(\Delta[0])_{n\in\mathbb{N}} \to (\mathcal{C}_n)_{n\in\mathbb{N}}$, which corresponds to sequences of objects $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \mathcal{C}_n$.
2. 1-simplices (1-morphisms) are maps $(\Delta[1])_{n\in\mathbb{N}} \to (\mathcal{C}_n)_{n\in\mathbb{N}}$, which corresponds to sequences of 1-morphisms $(f_n \colon x_n \to y_n)_{n\in\mathbb{N}}$ with f_n a 1-morphism in $\mathcal{C}_n$.

3. k-simplices (k-morphisms) are maps $(\Delta[k])_{n\in\mathbb{N}} \to (\mathcal{C}_n)_{n\in\mathbb{N}}$, which corresponds to sequences of k-morphisms $(\alpha_n)_{n\in\mathbb{N}}$ with α_n a k-morphism in $\mathcal{C}_n$.

4. For a given natural number $(a_n)_{n\in\mathbb{N}}$, (a_n)-simplices ((a_n)-morphisms) are maps $(\Delta[a_n])_{n\in\mathbb{N}} \to (\mathcal{C}_n)_{n\in\mathbb{N}}$, which corresponds to sequences of a_n-morphisms $(\alpha_n)_{n\in\mathbb{N}}$ with α_n an a_n-morphism in $\mathcal{C}_n$.

5. In particular the Segal condition implies equivalences $\mathcal{C}^{\Delta[a_\bullet+1]} \to \mathcal{C}^{\Delta[a_\bullet]} \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]}$. If a_n is a constant sequence, this unwinds to the usual Segal condition

$$\mathcal{C}^{\Delta[k]} \simeq \mathcal{C}^{\Delta[1]} \times_{\mathcal{C}} \ldots \times_{\mathcal{C}} \mathcal{C}^{\Delta[1]},$$

but in general this is not the case anymore.

We can see from this structure that our ∞-categories contain more "morphism-levels" than we are used to. This has some interesting implications.

1. On the one side, objects in $\mathcal{E}$ is a framework for some sort of ∞-category theory, which admits a good notion of internal ∞-categories. More precisely, it is a model for simplicial homotopy type theory, in a precise sense.

2. $\mathcal{E}$ is **not** equivalent to any ∞-category of simplicial objects, meaning for every ∞-category $\mathcal{C}$, there is no equivalence $\mathcal{C}^{\mathbb{\Delta}^{op}} \simeq \mathcal{E}$. So its full subcategory of local objects is **not** equivalent to any ∞-category of internal ∞-categories.

We will see more about the first item below. Regarding the second item the proof proceeds as follows:

Proof. The idea of the proof is to recognize $\mathcal{C}$ inside $\mathcal{E}$. Recall the following observation by Toën:

There is a unique ∞-category D with the following properties: The category $[1]$ is (up to equivalence) the only category with the following properties:

1. D is 1-category.

2. $\mathrm{Map}(1, D)$ has two elements.

3. The induced map $0 + 1 : 1 \sqcup 1 \to D$ is not an equivalence.

4. Every non-trivial subobject of D is also a subobject of $1 \sqcup 1$.

Just check $D = [1]$ does the trick and everything else is the same. These are equivalence invariant properties, so this means every equivalence of $\mathcal{C}\mathrm{at}_\infty$ needs to preserve $[1]$, up to equivalence. This is used by Toën to prove unicity [Toë05].

We want a slight generalization of this for $\mathcal{E}$ with the object $(\Delta[1])_{n\in\mathbb{N}}$. Using this we can see that any equivalence $\mathcal{C}^{\mathbb{\Delta}^{op}} \simeq \mathcal{E}$ restricts to an equivalence between $\mathcal{C} \simeq \mathcal{E}^{extdisc}$, the externally discrete objects, meaning objects that are local with respect to $(\Delta[k])_{n\in\mathbb{N}}$ for all natural numbers $k \geq 0$.

However, such an equivalence is not possible. Indeed the inclusion $\mathcal{C} \to \mathcal{C}^{\mathbb{\Delta}^{op}}$ admits a right adjoint (underlying discrete object), and so the analogous functor would also need to have a right adjoint, which it cannot. In some sense there are "too many" externally discrete objects and so we can pick an object that admits arbitrarily close externally discrete objects, and we cannot pick one. $\square$

5 Does this matter? Yes, completeness! (I)

Does this result matter beyond having a nice example? Yes, it matters for completeness! As of now we discussed the synthetic and analytic approaches to ∞-category theory. However, of course these two can be related in some way.

Let's try to be more precise. Recall that simplicial homotopy type theory (sHoTT) is a type theory due to Riehl and Shulman. Roughly speaking it is a foundation in which we have:

1. Types (objects)
2. Terms (morphisms)
3. Type constructors (universal properties)
4. Shapes (Test Objects)

Riehl–Shulman, and later on many others, show that if we give this foundation reasonable axioms, we can start proving ∞-categorical results, such as Yoneda lemma, Cartesian fibrations, ... Now, this leaves the question how this relates to more analytic approaches to ∞-category theory.

Here the idea is to find a technically rigorous way to interpret the types, terms and type constructors in suitable ∞-categories. Any such interpretation is known as a "model of sHoTT". This relation matters:

1. If we can prove something in sHoTT, then it will hold in any analytic model of sHoTT.
2. If something fails in a single analytic model of sHoTT, then it cannot be proven in sHoTT.

This means to better understand synthetic ∞-category theory, it is instructive to understand what we can and cannot prove in various analytic models. In particular, the eventual goal should be to have something like a "completeness theorem" establishing equivalences between type theories and models.

The standard way to mitigate between "models of homotopy type theory" and ∞-categories is to find strict models of an ∞-categories and show those strict models fit into various ways to turn categories into models of type theory. More concretely, one effective way is to show that the ∞-category is given as an underlying ∞-category of a certain model category.

In [RS17] Riehl–Shulman introduce the notion of a model category with $\mathfrak{T}$-shapes, as a model category that satisfies the minimal conditions to interpret the data mentioned above. They then prove the following result.

Theorem 5.1 (Riehl–Shulman)**.** *Let $\mathcal{M}$ be a right proper Cisinski model category. Then $\mathcal{M}^{\Delta^{op}}$ is a model category with $\mathfrak{T}$-shapes.*

Note this already applies to all "naive ∞-categories" internal to Grothendieck ∞-topoi. Our aim is to adjust this result to our setting. There is a more general framework, however, I will restrict here to the case of interest.

Theorem 5.2 (R.)**.** *Let $\mathcal{M}$ be a simplicial model category, I a set, and Φ a filter on I.*

- *The filter product 1-category $\prod_\Phi \mathcal{M}$ carries a simplicial model structure, such that $P_\Phi\colon \mathcal{M} \to \mathcal{M}_\Phi$ preserve weak equivalence and (co)fibrations.*
- *We have an equivalence $\mathrm{Ho}_\infty \prod_\Phi \mathcal{M} \simeq \prod_\Phi \mathrm{Ho}_\infty\mathcal{M}$.*
- *If $\mathcal{M}$ is a model category with $\mathfrak{T}$-shapes, then so is $\prod_\Phi \mathcal{M}$.*

Now applying this to the case $\mathcal{M} = s\mathcal{S}\mathrm{et}$ with the Kan model structure, we get the following result.

Corollary 5.3. *The ∞-category $\mathcal{E}$ is a model of sHoTT, that is not an ∞-category of simplicial objects. This means "naive internal ∞-categories" do not recover all possible models of sHoTT.*

So, if we hope to ever have a completeness theorem for sHoTT (and probably other synthetic approaches), we need to expand our understanding of analytic ∞-category theory beyond the naive approach.

6 Does this matter? Yes, completeness! (II)

Let us understand a second reason why this matters (and what got me interested in this). Let $\mathcal{G}$ be a Grothendieck ∞-topos (presentable ∞-category that satisfies descent). Then I can look at ∞-categories internal to $\mathcal{G}$ i.e. complete Segal objects in $\mathcal{G}^{\Delta^{op}}$.

In the case of ∞-topoi, we actually have interesting examples, obtained as follows: Fix some pullback stable class of morphisms S in $\mathcal{G}$. For a given object X in $\mathcal{G}$, denote $(\mathcal{G}_{/X})^S \hookrightarrow \mathcal{G}_{/X}$ as the full subcategory spanned by morphisms in S. This is pullback stable and hence assembles into a functor

$$(\mathcal{G}_{/-})^S \colon \mathcal{G}^{op} \to \mathcal{C}\mathrm{at}_\infty,$$

sending X to $(\mathcal{G}_{/X})^S$. Combining the fact that $\mathcal{G}$ satisfies a spacial version of descent and is locally Cartesian closed (or just satisfies descent), there exists a internal ∞-category $\underline{\mathcal{G}}$ that "represents this functor" in the sense that for a given object X

$$\mathrm{Map}_{\mathcal{G}^{\Delta^{op}}}(X, \underline{\mathcal{G}}) \simeq (\mathcal{G}_{/X})^S$$

This internal ∞-category is called the *internalization*. Notice we have to choose S to avoid some version of Russell's paradox.

Let us look at some prominent examples.

Example 6.1. Let κ be cardinal and S the class of relatively κ-compact morphisms in $\mathcal{G}$. Then $(\mathcal{G}_{/X})^S \simeq \mathcal{G}^\kappa_{/X}$, the full subcategory of κ-compact objects in $\mathcal{G}_{/X}$. The internalization $\underline{\mathcal{G}}^\kappa$ is an internal ∞-category, and the object of objects is called the *universe of κ-small objects* in $\mathcal{G}$.

Example 6.2. Let $S = \mathcal{M}\mathrm{ono}_\mathcal{G}$ denote the set of all monos in $\mathcal{G}$. Then $(\mathcal{G}_{/X})^S \simeq \mathrm{Sub}(X)$, the poset of subobjects of X. The internalization $\underline{\mathcal{G}}^{mono}$ is an internal 0-category, and the object of objects is called the *subobject classifier* Ω of $\mathcal{G}$.

Following work of Martini-Wolf [MW21], we have the following.

Theorem 6.3 (Martini–Wolf)**.** *Let $\mathcal{G}$ be a Grothendieck ∞-topos. Let S be a pullback stable class of morphisms in $\mathcal{G}$. Then $\underline{\mathcal{G}}$ is internally cocomplete.*

To summarize:

- Because the ∞-category is presentable, we can obtain representing internal ∞-categories.
- We can then prove these representing internal ∞-categories are internally cocomplete.

How about if we replace that first step by just assuming these internalizations exist? Can we still prove they are internally cocomplete? Here is a basic example.

Theorem 6.4. *Let $\mathcal{E}$ be a locally Cartesian closed 1-category with subobject classifier Ω. Then Ω lifts to an internal 0-category, which is internally cocomplete.*

Notice, we did not assume $\mathcal{E}$ has colimits! In fact, this results holds even in cases where $\mathcal{E}$ does not have all colimits. In light of this result and what Martini–Wolf did, one would expect some common generalization along the following lines.

Definition 6.5. Let $\mathcal{E}$ be a finitely complete ∞-category. An object $\mathcal{U}$ in $\mathcal{E}$ is a *univalent universe* if there is a map $\mathcal{U}_* \to \mathcal{U}$ in $\mathcal{E}$, such that the induced natural transformation

$$\mathrm{Map}(-, \mathcal{U}) \to (\mathcal{E}_{/-})^{\simeq}$$

is fully faithful.

The objects in the image of this maps are by definition *small*. These results suggest the following anticipated generalization.

Conjecture 6.6. *Let $\mathcal{E}$ be a locally Cartesian closed ∞-category. Let $\mathcal{U}$ be a univalent universe.*

1. *$\mathcal{U}$ lifts to an internalization $\underline{\mathcal{E}}$.*
2. *$\underline{\mathcal{E}}$ is internally complete.*

Now, at this step it really matters what I mean by "∞-category"! If I mean ∞-category in the naive analytic sense, then I have the following result.

Theorem 6.7 (R.)**.** *The statement* (1) *is true.*

Here $\underline{\mathcal{E}}_1 = [\mathcal{U}_* \times \mathcal{U}, \mathcal{U} \times \mathcal{U}_*]_{\mathcal{U}\times\mathcal{U}}$ and the higher ones are determined by the Segal conditions. The difficulty consists of stitching the objects into a simplicial object.

Theorem 6.8 (R.)**.** *The statement* (2) *is false. Meaning there exists a locally Cartesian closed ∞-category $\mathcal{E}$ with internalization $\underline{\mathcal{E}}$ that is not internally cocomplete.*

Let us analyze the situation further. Let $\mathcal{F}$ again denote the Fréchet filter on $\mathbb{N}$. Then we obtain the ∞-category $\mathcal{E} = \prod_{\mathcal{F}} \mathcal{A}\mathrm{n}$. It has a univalent universe given by $(\mathcal{U})_{n\in\mathbb{N}}$, where $\mathcal{U}$ is the univalent universe in $\mathcal{A}\mathrm{n}$. By the statement above this lifts to an internalization $\underline{\mathcal{E}}$.

Now completeness of this internalization in particular means, for every internal ∞-category I, the terminal map $I \to [0] \to \underline{\mathcal{E}}$ has a colimit, which after unwinding some definition corresponds to the geometric realization of I inside $\mathcal{E}$. However, we can construct an internal ∞-category I in $\mathcal{E}$ such that this geometric realization does not exist. This shows that the internalization $\underline{\mathcal{E}}$ is not internally cocomplete.

Let us discuss more explicitly how we can construct such an example.

Remark 6.9. Let D be a category with the following specifications:

1. It is weakly contractible.
2. It has arbitrarily long chains of compositions, meaning the nerve has arbitrarily high non-degenerate cells.
3. It has no isomorphisms.

Let $D_{\leq n}$ be the simplicial set given via embedding the n-skeleton. It comes with an embedding $D_{\leq n} \to D$, which is an isomorphism on k-simplices for $k \leq n$. Let S^n be the simplicial set given as $\Delta[n]/\partial\Delta[n]$. Then there is a unique map of simplicial sets $ND_{\leq n} \to S^n$, characterized by taking all non-degenerate elements in $(D_{\leq n})_n$ to the unique non-degenerate element in $(S^n)_n$. Observe that this map is homotopically non-trivial and in fact the induced map on π_n is surjective. We use the notation $\pi_n \colon D_{\leq n} \to S^n$ for the map described above.

Example 6.10. Here are two examples of categories satisfying the conditions of Remark 6.9.

1. The poset $\mathbb{N}$.
2. The poset J described in [Rez10]

Proposition 6.11. *The ∞-category $\prod_{\mathcal{F}} \mathcal{A}\mathrm{n}$ does not have geometric realization with respect to all internal ∞-categories.*

Proof. Let D be a category satisfying the conditions of Remark 6.9.

Let $\mathbb{D}\colon \Delta^{op} \to \prod_{\mathcal{F}} \mathcal{A}\mathrm{n}$ be given as $\mathbb{D} = P_{\mathcal{F}}\Delta_{\mathbb{N}}D = (D)_{n\in\mathbb{N}}$. Evidently this is an ∞-category internal to $\prod_{\mathcal{F}} \mathcal{A}\mathrm{n}$. We claim its geometric realization does not exist in $\prod_{\mathcal{F}} \mathcal{A}\mathrm{n}$.

Before that we more carefully analyze the data of a cocone under $\mathbb{D}$. By definition a cocone is a morphism in $(\prod_{\mathcal{F}} \mathcal{A}\mathrm{n})^{\Delta^{op}}$ from $\mathbb{D}$ into the constant diagram $\Delta(X_n)$. Unwinding this definition, such a morphism consists of the following data:

- An increasing sequence $(s_n)_{n\in\mathbb{N}}$. Here s_n is the natural number specifying from which point on a map out of D_n into ΔX always exists.
- A collection of morphisms $D_{\leq n} \to X_{s_n}$.

Pick an arbitrary object $(C_n)_{n\in\mathbb{N}}$ in $\prod_{\mathcal{F}} \mathcal{A}\mathrm{n}$ and assume it is the colimit of $\mathbb{D}$. We know there is a natural number $(a_n)_{n\in\mathbb{N}}$, such that $(C_n)_{n\in\mathbb{N}}$ is $(a_n)_{n\in\mathbb{N}}$-connected, but not $(a_n - 1)_{n\in\mathbb{N}}$-connected. First of all we see that $(a_n)_{n\in\mathbb{N}}$ is not a standard natural number. Indeed, it suffices to observe that the space of maps into every n-truncated object, where n is standard, is contractible. Pick an arbitrary k-truncated object $(X_n)_{n\in\mathbb{N}}$ and an arbitrary cocone into it with sequence $(s_n)_{n\in\mathbb{N}}$. By assumption there is an $N \in \mathbb{N}$, such that for all $n > N$, $s_n > k$. Then, the space of morphisms is contractible. Hence, $(a_n)_{n\in\mathbb{N}}$ is not a standard natural number.

We now finish the proof by observing it can also not be any other natural number. It suffices to witness that there is a non-trivial cocone into an object that is $(a_n - 1)_n$-truncated. Take the space $(K(\mathbb{Z}, a_n - 1))_{n\in\mathbb{N}}$. We now construct a non-trivial map as follows:

- We take $s_n = a_n - 1$.
- We take the cocone $ND_{\leq a_n - 1} \xrightarrow{\pi_{a_n-1}} S^{a_n-1} \to K(\mathbb{Z}, a_n - 1)$, where the first map π_n is defined in Remark 6.9 and the second map corresponds to the generator in $\pi_{a_n-1}(K(\mathbb{Z}, a_n - 1)) \cong \mathbb{Z}$.

This map is non-trivial for all n, hence a non-trivial cocone. $\square$

Now, on the other side, instead of looking at $\mathcal{E}^{\Delta^{op}} = (\prod_{\mathcal{F}} \mathcal{A}\mathrm{n})^{\Delta^{op}}$ we can look at $s\mathcal{E} = \prod_{\mathcal{F}}(\mathcal{A}\mathrm{n}^{\Delta^{op}})$. Here, we now have an object $(\underline{\mathcal{E}})_{n\in\mathbb{N}}$, and for any other ∞-category in $s\mathcal{E}$, we do actually have internal adjunction diagrams

$$\begin{tikzcd} {[(I_n)_{n\in\mathbb{N}}, (\underline{\mathcal{E}})]} \arrow[r, bend left] \arrow[r, bend right] & (\underline{\mathcal{E}}) \arrow[l] \end{tikzcd} .$$

Indeed, this just comes from a simple post-composition:

$$\mathcal{A}\mathrm{dj} \to \prod_{\mathbb{N}} \mathcal{A}\mathrm{n}^{\Delta^{op}} \xrightarrow{P_{\Phi}} \prod_{\mathcal{F}} \mathcal{A}\mathrm{n}^{\Delta^{op}}$$

7 Conclusion

Here is a summary:

1. ∞-category theory manifests in different ways.
2. The analytic approach is surprisingly tied to the foundation in a way that I did not anticipate.

3. Being able to “do” ∞-category theory properly in whichever context one wants requires a rethinking of its definition.

4. Hence, I end this talk with the following question: *What is an ∞-category?*

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