Filter Quotient Models in Homotopy Type Theory

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The goal of this talk is to see how filter quotient methods can be used to construct new models of homotopy type theory. This talk is based on past work:

1. Filter quotients and non-presentable $(\infty,1)$-toposes

   DOI: j.jpaa.2021.106770 [Ras21]

2. Filter Quotient Model Categories

   arXiv:2508.07735 [Ras25a]

3. Non-Standard Models of Homotopy Type Theory

   arXiv: 2508.07736 [Ras25b]

4. Simplicial Homotopy Type Theory is not just Simplicial: What are ∞-Categories?

   arXiv:2508.07737 [Ras25c]

Here is a fun fact about these papers: There are three consecutive arXiv identifiers, I think that's a record!

1 From Type Theories to Categories and Back

Recall that type theory is one foundation for mathematics, consisting of types, terms, and rules for constructing new types and terms. This means that type theory has a very syntactic nature. This, of course, has many benefits, such as the formalization of mathematics. However, it does also come with drawbacks, such as challenges in understanding the meaning (i.e. semantics) of type theory.

Of course, these challenges have suitable solutions: For a given type theory (so chosen collection of types, terms, and rules), we define *models of type theory* as certain categories, where intuitively the objects correspond to types, the morphisms to terms, and the structure of the category to the rules of the type theory. Ideally, via these models, we can both obtain "soundness" result (i.e. there is a model) and "completeness" results (i.e. we have all models). In particular, for such categories, the type theory becomes the "internal language" of the category.

Broadly speaking, we would expect a diagram of correspondences like this:

$$\begin{tikzcd}
& \textbf{Type Theory} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & & \textbf{Category Theory} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\begin{matrix}\text{Extensional}\\ x=y \text{ simple}\end{matrix} \arrow[d, Rightarrow] & & \begin{matrix}\text{Intensional}\\ x=y \text{ complicated}\end{matrix} \arrow[r, leftrightsquigarrow] \arrow[d, Rightarrow] & \begin{matrix}1-\text{Category}\\ \text{Hom set}\end{matrix} \arrow[d, Rightarrow] & & \begin{matrix}\infty-\text{Category}\\ \text{Mapping space}\end{matrix} \arrow[d, Rightarrow] \\
\begin{matrix}\text{Constructor}\\ \text{(Product type)}\end{matrix} & & \begin{matrix}\text{Constructor}\\ \text{(Product type)}\end{matrix} & \begin{matrix}\text{Univ. Property}\\ \text{(Product in cats)}\end{matrix} & & \begin{matrix}\text{Univ. Property}\\ \text{(Product in } \infty\text{-cats)}\end{matrix}
\end{tikzcd}$$

Remark 1.1. Throughout this talk the term "∞-category" shall remain vague. We will take for granted:

1. An ∞-category has objects and between any two objects there is a space (∞-groupoid, homotopy type, anima, ...) of morphisms.
2. All standard categorical notions generalize to this setting.
3. There is an ∞-category of space $\mathcal{S}$, which is the analogue of sets in this setting.

2 Some Type Theory vs. Category Theory

Let us see one simple but prominent example of this correspondence. Let's say we have an extensional dependent type theory with Π-types.

- Dependent types should give us morphism between objects (projections), so closed types are objects
- Terms are given by section of those projections, so terms of closed types are just morphisms
- Extensionality gives us a set of morphisms between any two objects.
- Substitutions give us pullbacks.
- Π-types correspond to the object of sections, which is precisely the right adjoint to the pullback.

So, if we put these all together, naively speaking we should get a locally Cartesian closed category (LCCC), by definition a category with finite limits and where the pullback has a right adjoint.

Making this naive idea precise is actually much more challenging than one would expect. Indeed, a first try to prove this by Seely [See83, See84] contained a mistake, which was later fixed by Clairambault and Dybjer [CD14], using the technical tool of *categories with families* [Dyb96]. Their work in particular establishes an equivalence between the bicategory of such type theories and the bicategory of locally Cartesian closed categories. The original challenge arose from a translation of the pullback property from the very strict type theoretical side to the somewhat weaker categorical side, as pullbacks in categories are a universal property and hence determined up to isomorphism. Here we can already see what challenges lie ahead when we want to translate between type theory and category theory in more homotopical contexts.

How can we bring some homotopical data into our models of type theory? If we are given a higher category and want to interpret it as a model of type theory, we similarly need to strictify relevant categorical data and then interpret the type theory in that strictified setting. In very simple cases this can be done directly. One first prominent example is the groupoid model of Hofmann and Streicher [HS98], which gives us the first model of dependent type theory without UIP, still using categories with families.

Unfortunately these direct methods become more and more challenging when we increase the dimension of the category. How can we turn an ∞-category into a category with families? The trick is to leverage the historical literature on ∞-category theory. For many ∞-categories, we do have methods to strictify them into a category with additional data. The most prominent example are model categories, originally due to Quillen [Qui67] and with many application all throughout topology. So here the rough idea is that there every model category $\mathcal{M}$ can be turned into an ∞-category $\mathcal{H}\mathrm{o}_\infty(\mathcal{M})$.

Definition 2.1. An ∞-category $\mathcal{C}$ is *presented* by a model category $\mathcal{M}$ if $\mathcal{C}$ is equivalent to the underlying ∞-category $\mathcal{H}\mathrm{o}_\infty(\mathcal{M})$ of $\mathcal{M}$.

So, with such results at hand, we can summarize the general strategy as follows:

1. Show that a given ∞-category of interest can be represented by a model category.

2. Show that this data (in particular the fibrant objects and fibrations) gives us the structure to interpret the type theory.

First steps to implement this strategy were taken by Awodey–Warren [AW09], and Kapulkin–Lumsdaine–Voevodsky [KL21], resulting in the simplicial model, which shows that Kan complexes (i.e. ∞-groupoids) are a model of homotopy type theory, meaning the univalence axiom holds. Here we already knew, of course, that $\mathcal{S}$ can be presented by a model structure.

Given the success of this model categorical approach, there has been significant effort by a variety of figures trying to generalize to more and more intricate models of homotopy type theory. Prominent figures includes Arndt, Kapulkin, Lumsdaine, Shulman, and Warren, culminating in the following results [AK11, Shu15, LW15, LLS20, Shu19].[1]

Theorem 2.2 (Dugger–Rezk)**.** *For every Grothendieck ∞-topos $\mathcal{G}$, there exists a model category $\mathcal{M}$, called a model topos, that presents $\mathcal{G}$.*

Theorem 2.3 ([Shu19])**.** *For every Grothendieck ∞-topos $\mathcal{G}$, there exists a representing model category $\mathcal{M}$, called a type-theoretic model topos, that is a model of homotopy type theory with univalent universes.*

This theorem in particular applies to the ∞-category $\mathcal{S}$, but also many other examples, such as (pre)sheaf ∞-categories $\mathcal{S}\mathrm{hv}(\mathcal{C}, J)$. If we had expected that this approach recovers all possible models, we could have tried proving an equivalence. However, this is not what we expected. Indeed, it has long been anticipated that there should be many more models of homotopy type theory. So, how can we find them?

3 Further Models of Homotopy Type Theory

As we saw before, we have some models of homotopy type theory, but we do not have all of them. So, it remains to pursue further models using new techniques. How can we find such new models? Here it is helpful to take a more philosophical look.

Given an ∞-category that is an abstract model of homotopy type theory, we can also restrict to the 0-truncated objects, i.e. the "0"-types, meaning objects X such that $\mathrm{Map}(-, X)$ is a 0-truncated space. This gives us a 1-category that models "0-types in a homotopy type theory". It at least needs to satisfy the following conditions:

[1] These are of course not the only developments, and there are also advancements via, for example, fibration categories [Kap17, KS19]

1. Has finite limits
2. Is locally Cartesian closed
3. Has a Natural number object and even W-types
4. Satisfies Descent for monos and maybe even has subobject classifier

Interestingly enough these constitute the defining properties of what we call an *elementary topos with NNO* or at least a ΠW*-pretopos*. How about for the models we actually have? They satisfy a stronger condition, namely they are *Grothendieck topos*, which is an elementary topos with an additional local presentability property. This state of affairs suggests a natural way forward:

1. Look at ways to construct new elementary topoi out of Grothendieck topoi (in the 1-categorical context).
2. Lift those methods to the ∞-categorical setting.
3. Show that the resulting ∞-categories are models of homotopy type theory.

Here are some ways people have pursued with this mindset:

1. Given the category of sets, the sub-category of finite sets is an elementary topos, but not a Grothendieck topos. Can we lift this to the ∞-category setting? This was pursued by Lo Monaco [LM21] and only works with significant set-theoretic assumptions.
2. Given a suitable input one can construct a "effective topos". Higher categorical lifts are being pursued by Anel, Awodey and Barton [AAB25].
3. Given a Grothendieck topos and a suitable filter, one can construct a new elementary topos called the "filter quotient". Can we lift this to the ∞-category setting and build models of homotopy type theory? This is the topic of this talk.

4 The 1-Categorical Filter Quotient Construction

Let us explicitly recall the filter quotient construction in the 1-categorical setting. Of course there are many good references for this topic [Joh77, MLM94, Joh02a, Joh02b]. One interesting paper that studies these topics in detail is due to Adelman and Johnstone [AJ82]

Definition 4.1. A filter on a poset P is a non-empty subset Φ that is

- **Non-empty** There exists a U in Φ
- **Cofiltered:** For every U, V in Φ, there exists a $W \in \Phi$, such that $W \leq U, V$.
- **Upwards closed:** If $U \in \Phi$ and $U \leq V$, then $V \in \Phi$.

Recall that in a category $\mathcal{C}$ an object U is subterminal if $\mathrm{Hom}_{\mathcal{C}}(-, U)$ takes value in the empty or one element set.

Definition 4.2. Let $\mathcal{C}$ be a category. A *filter of subterminal objects* is a filter on the poset of isomorphism classes of subterminal objects.

Definition 4.3. Let $\mathcal{C}$ be a category with finite products and Φ a filter of subterminal objects. Define the filter quotient category $\mathcal{C}_\Phi$ as follows:

- Objects of $\mathcal{C}_\Phi$ are the objects of $\mathcal{C}$.

- Morphisms from X to Y are given by the set of morphisms of the form $f\colon X\times U \to Y$, for some U in Φ, subject to the following relation: Two maps $f\colon X\times U\to Y$, $g\colon X\times V\to Y$ are equivalent if there exists a $W\le U, V$ in Φ, such that the following diagram commutes:

$$\begin{tikzcd} X\times W \arrow[r, "X\times \mathrm{inc}_{WU}"] \arrow[d, "X\times \mathrm{inc}_{WV}"'] & X\times U \arrow[d, "f"] \\ X\times V \arrow[r, "g"] & Y \end{tikzcd}$$

Notice, by construction, we have a functor $P_\Phi\colon \mathcal{C}\to\mathcal{C}_\Phi$. This functor is very powerful.

Theorem 4.4. *Let $\mathcal{C}$ be a category with finite products and Φ a filter of subterminal objects.*

1. *P_Φ preserves finite (co)limits, Cartesian closure, natural number objects, ...*
2. *P_Φ does not preserve infinite (co)limits, local presentability, ...*

Let us look at a class of examples.

Example 4.5. Let I be a set and consider the category $\prod_I \mathcal{S}\mathrm{et}\cong \mathrm{Fun}(I,\mathcal{S}\mathrm{et})$. An object is just an I-indexed set $(X_i)_{i\in I}$. It is subterminal if X_i is empty or the point for all i.

Now, given any filter Φ on the poset PI (the power set), we can define a filter of subterminal objects on $\prod_I \mathcal{S}\mathrm{et}$. For an object U in Φ, choose the subterminal object $(U_i)_{i\in I}$ defined as follows:

$$U_i=\begin{cases} * & \text{if } i\in U \\ \emptyset & \text{else}\end{cases}.$$

The collection of these $(U_i)_{i\in I}$ for $U\in\Phi$ defines a filter of subterminal objects, also denoted Φ. The resulting filter quotient is often denoted $\prod_\Phi \mathcal{S}\mathrm{et}$ and called the *filter product*.

Let us look at two examples of this example, on the same category.

Example 4.6. Let $\mathbb{N}$ be the set of natural numbers and let $\mathcal{F}$ be the filter of cofinite subsets, also known as the *Fréchet Filter*. Unwinding the definition we have the following characterization of $(\prod_\Phi \mathcal{S}\mathrm{et})$:

- Objects are $(X_n)_{n\in\mathbb{N}}$.
- Morphisms are $\mathbb{N}$-indexed maps $f_n\colon \mathbb{N}\to\mathbb{N}$, where $(f_n)\sim(g_n)$ if there exists an $N\in\mathbb{N}$, such that for all $n>N$ we have $f_n=g_n$, i.e. the equivalence relation of eventual equality.

Notice this in particular means that $\prod_\Phi \mathcal{S}\mathrm{et}$ does not have infinite coproducts. Indeed, if it did, then the object $(\mathbb{N})_{n\in\mathbb{N}}$ would have to be the infinite coproduct of the point $(*)_\mathbb{N}$, which one can see it is not. Indeed we have the following pullback

$$\begin{tikzcd} (\emptyset)_{n\in\mathbb{N}} \arrow[r] \arrow[d] & (*)_{n\in\mathbb{N}} \arrow[d, "\Delta"] \\ (*)_{n\in\mathbb{N}} \arrow[r, "n"] & (\mathbb{N})_{n\in\mathbb{N}} \end{tikzcd},$$

so the natural number Δ is disjoint, meaning the $(n\colon (*)_{n\in\mathbb{N}}\to(\mathbb{N})_{n\in\mathbb{N}})$ are not jointly surjective and so not a coproduct cocone.

Example 4.7. Again, let $\mathbb{N}$ be the set of natural numbers and consider this time a non-principal ultrafilter $\mathcal{U}$, which is a filter that is maximal and not equal to one of the form $\{U : n\in U\}$, for some n. Then $\prod_\mathcal{U} \mathcal{S}\mathrm{et}$ is still not cocomplete and hence not equivalent to $\mathcal{S}\mathrm{et}$. However, the terminal object has no non-trivial subjects! So, it is "well-pointed" i.e. it only has global elements. We can think of this as a non-standard model of set theory.

Before we proceed, let us note the explicit filter quotient construction admits a characterization via universal property.

Proposition 4.8. *Let $\mathcal{C}$ be a category with finite products and Φ a filter of subterminal objects. Let $\mathcal{C}_{/-}\colon \Phi^{op} \to \mathcal{C}at$ be the functor that maps u to $\mathcal{C}_{/U}$ and $V \leq U$ to $- \times V\colon \mathcal{C}_{/U} \to \mathcal{C}_{/V}$. Then $\mathcal{C}_\Phi$ is the colimit of this diagram.*

5 The ∞-Categorical Filter Quotient Construction

Now can we lift this to the ∞-categorical setting? The answer is yes, and the construction is very similar, depending on universal properties. Here we follow [Ras21].

Definition 5.1. Let $\mathcal{C}$ be an ∞-category with finite products. A filter of subterminal objects of $\mathcal{C}$ is a filter on the poset of subterminal objects.

A filter induces a diagram of ∞-categories $\mathcal{C}_{/-}\colon \Phi^{op} \to \mathcal{C}at_\infty$.

Definition 5.2. Let $\mathcal{C}$ be an ∞-category with finite products and Φ a filter of subterminal objects. The filter quotient ∞-category $\mathcal{C}_\Phi$ is the colimit of $\mathcal{C}_{/-}$.

By definition of colimits, this again comes with a projection map $P_\Phi\colon \mathcal{C} \to \mathcal{C}_\Phi$.

Remark 5.3. As Φ^{op} is filtered, following the work of Rezk [Rez01], the colimit can actually be computed quite explicitly in contrast to general colimits of ∞-categories.

Remark 5.4. Working model dependently, using Kan enriched categories, we could repeat essentially the same steps as in the 1-categorical case to give an explicit construction of the filter quotient ∞-category.

Theorem 5.5. *Let $\mathcal{C}$ be an ∞-category with finite products and Φ a filter of subterminal objects.*

1. *P_Φ preserves finite (co)limits, Cartesian closure, natural number objects, ...*
2. *P_Φ does not preserve infinite (co)limits, presentability, ...*

We can summarize the examples from above as follows.

Example 5.6. Let $\mathcal{S}$ be the ∞-category of spaces, I a set and Φ a filter on PI, then we get a filter on $\prod_I \mathcal{S}$, whose filter quotient is the *filter product ∞-category* $\prod_\Phi \mathcal{S}$. Thus we recover the following examples:

- If $\Phi = \mathcal{F}$ we get the ∞-category $\prod_{\mathcal{F}} \mathcal{S}$, which is again not cocomplete.
- If $\Phi = \mathcal{U}$ we get the ∞-category $\prod_{\mathcal{U}} \mathcal{S}$, which again has a global point, and so is like a non-standard model of spaces.

So, we now have an ∞-category that has all the right properties. But is it a model of homotopy type theory? As we saw above, this requires bringing in some "strictification", and one prominent technique is to use model categories. So, how do these ∞-categorical results interact with model categories?

6 Filter Quotient Model Categories

We are now confronted with the following question: If an ∞-category is presented by a model category, can we present the filter quotient ∞-category via a model category? The answer is mostly yes! Here I will just state the main result immediately.

Definition 6.1 (Pseudo-definition)**.** Let $\mathcal{M}$ be a simplicial model category. A *simplicial model filter* on $\mathcal{M}$ is a filter on the poset of subterminal objects of the underlying category that interacts well with the model structure and with the simplicial enrichment.

Theorem 6.2. *Let $\mathcal{C}$ be an ∞-category presented by a simplicial model category $\mathcal{M}$. Assume the filter Φ of subterminal objects on $\mathcal{C}$ is presented by a simplicial model filter on $\mathcal{M}$.*

1. *The filter quotient* 1*-category $\mathcal{M}_\Phi$ admits a simplicial model structure induced from $\mathcal{M}$.*
2. *The underlying ∞-category of $\mathcal{M}_\Phi$ is equivalent to the filter quotient ∞-category $\mathcal{C}_\Phi$.*

Note we still have the functor $P_\Phi \colon \mathcal{M} \to \mathcal{M}_\Phi$ of model categories.

1. It preserves (co)fibrations and weak equivalences.
2. It preserves Cartesian closure, enrichment, ...
3. But it is not left or right Quillen in general!

Let us carry forward our examples.

Example 6.3. Let $s\mathcal{S}$et be the category of simplicial sets with the Kan-Quillen model structure, I a set and Φ a filter on PI, then we get a model filter on $\prod_I s\mathcal{S}\mathrm{et}$. The resulting filter product model category $\prod_\Phi s\mathcal{S}\mathrm{et}$ carries a model structure whose underlying ∞-category is $\prod_\Phi \mathcal{S}$.

- If $\Phi = \mathcal{F}$ we get a model structure on $\prod_\mathcal{F} s\mathcal{S}\mathrm{et}$, which is not combinatorial or cofibrantly generated.
- If $\Phi = \mathcal{U}$ we get a model structure on $\prod_\mathcal{U} s\mathcal{S}\mathrm{et}$, which again has a global point.

7 Filter Quotient Models of Homotopy Type Theory

Until now we have seen that the filter quotient construction can be lifted to the ∞-categorical setting, and that it will preserve model categorical presentations. The last step is to show that if the model category presents a model of homotopy type theory, then so does the filter quotient model category. we can summarize this via the following result.

Theorem 7.1. *The filter quotient construction preserves the following type constructors (for each row):*

1. *Unit Type, Σ-types, Π-types*
2. *identity types, function extensionality*
3. *empty type, Boolean type, coproduct types, pushout types, "cell complex" types: including spheres and tori*
4. *natural numbers type, W-types, propositional, truncations, James constructions, localizations*
5. *arbitrarily large univalent universes closed under Σ- and Π-types, identity types, binary sum types and containing "cell complex" types*

Remark 7.2. The first three rows are quite straightforward, but the last two cases are quite annoying. That is because all the original proofs involve many arguments, such as local presentability, small object argument, ... that do not directly translate anymore.

Let us look at some examples.

Example 7.3. Let $s\mathcal{S}$et be the category of simplicial sets with the Kan-Quillen model structure. Then, following the main result, both $\prod_\mathcal{F} s\mathcal{S}\mathrm{et}$ and $\prod_\mathcal{U} s\mathcal{S}\mathrm{et}$ are models of homotopy type theory with univalent universes. Following common analogies $\prod_\mathcal{U} s\mathcal{S}\mathrm{et}$ is what I would call a "non-standard model of spaces" or "non-standard model of HoTT".

8 Filter Quotient Models of Simplicial Homotopy Type Theory

We can add some further information to our homotopy type theory via simplicial homotopy type theory. which is homotopy type theory along with a directed non-contractible interval. It was designed by Riehl and Shulman as a place where "∞-category theory happens" [RS17]. As is shown in the appendix of that paper, the standard model for this type theory are simplicial spaces, i.e. $s\mathcal{S}\mathrm{et}^{\mathbb{\Delta}^{op}}$, with the Reedy (or injective) model structure. We now have the following summarized result and example.

Theorem 8.1. *Filter products preserve models of simplicial homotopy type theory.*

Remark 8.2. This result can be generalized to more general filter quotients, but that requires more work.

Example 8.3. $\prod_{\mathcal{F}} s\mathcal{S}\mathrm{et}^{\mathbb{\Delta}^{op}}$ and $\prod_{\mathcal{U}} s\mathcal{S}\mathrm{et}^{\mathbb{\Delta}^{op}}$ are models of simplicial homotopy type theory.

Note, these examples have the fun property that they have "too many " standard simplices, and so their underlying ∞-category cannot be equivalent to $\mathcal{C}^{\mathbb{\Delta}^{op}}$ for any $\mathcal{C}$.

9 What does this imply?

Here are some cool implications of this result. Fundamentally, to me these models show that

- *external infinity:* that comes from the ambient ∞-category theory
- *internal infinity:* that comes from the type theory

can significantly diverge. Here are some explicit implications:

1. Natural numbers may not be standard in a model of HoTT.
2. Relatedly, infinite coproducts are not type constructors
3. ∞-categories might not mean what we think they mean. More explicitly, up until now the consensus has been that an "∞-category in general foundations" can be obtained by looking at simplicial objects in that foundation. That has been pursued for quite a while by Martini–Wolf [MW21]. However, these examples demonstrate that there is a gap between the theory and the models that has remained unexplored. This is in stark contrast to categories, which are in fact foundation agnostic!

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