Mobile Robots

Summer Semester 2024
Assignment 9

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Due: 02.07.2024, 10:15

Exercise 1 Kalman Filter (12 Points)

(a) Describe the at least two assumptions that apply to the Kalman Filter. (1 Point)

(b) Here, we will implement a basic Kalman Filter. Assume the following basic linear system:

$$x_t = 1.1x_{t-1} + 1.0u_t + \mathcal{N}(0, 0.8^2) \tag{1}$$
$$z_t = 2.1x_t + \mathcal{N}(0, 0.8^2) \tag{2}$$

Use the template $kf.py$ provided in moodle. Implement the functions $kalman\_update$, $kalman\_correction$, and $kalman$. The latter called the former two in the correct order and runs a loop over the given control commands and measurements.
The first few control inputs and measurements are $u_1 = 2, u_2 = -4, z_1 = 3, z_2 = 1$. Starting with an initial estimate of $\hat{x}_0 = \mathcal{N}(1, 3.6^2)$, compute two iterations of the Kalman Filter algorithm. Please provide all intermediate results for $\overline{\mu_i}$, $\overline{\Sigma_i}$, K_i, μ_i, and Σ_i. (6 Points)

(c) Now, read the data given in the .csv files in moodle. It contains a sequence of control commands and measurements. If a measurement is equal to zero it means, that there is no measurement available at this time step. Adapt your code, so that it can run with fewer measurements than control commands, performing only an update step at the time steps where there is no measurement.
Now implement the function $plot\_with\_uncertainty$. It takes two arrays of equal length, the first being the means, the second the standard deviations. The function should produce a plot similar to the one in the Kalman Filter script, slide 22, plotting the means as a graph and the standard deviations as vertical bars at each time step.
What can you observe? (5 Points)

Exercise 2 Kalman Filter Derivation (8 Points)

In this exercise we are going to derive the correction step of the Kalman Filter from the general Bayesian Filter assuming normally distributed noise and a linear system. For simplicity, we assume the state x_t, control input u_t and measurements z_t all to be in one dimension, i.e. $x_t, u_t, z_t \in \mathbb{R}$. The linear system is given by:

$$x_t = A_t x_{t-1} + B_t u_t + \epsilon_t, \qquad \epsilon_t \sim \mathcal{N}(0, R_t) \tag{3}$$
$$z_t = C_t x_t + \delta_t, \qquad \delta_t \sim \mathcal{N}(0, Q_t) \tag{4}$$

(Explain to yourself, why Equation (7) looks different to the script.) The Kalman Filter expressions are:

$$\mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \tag{5}$$

$$\Sigma_t = (1 - K_t C_t)\,\bar{\Sigma}_t \tag{6}$$

where

$$K_t = \frac{C_t\bar{\Sigma}_t}{C_t^2\bar{\Sigma}_t + Q_t}. \tag{7}$$

The correction step of the general Bayes filter is:

$$p(x_t|z_{t,\ldots 1}, u_{t,\ldots 1}) = \eta \cdot p(z_t|x_t) \cdot p(x_t|z_{t-1,\ldots 1}, u_{t,\ldots 1}) \tag{8}$$

The first distribution $p(x_t|z_{t,\ldots 1}, u_{t,\ldots 1})$ is the result of the Kalman filter iteration at time step t. We know from the lecture that we assume the result to be the normal distribution μ_t, Σ_t over the robot pose x_t,

$$p(x_t|z_{t,\ldots 1}, u_{t,\ldots 1}) = \mathcal{N}(x = x_t, \mu = \mu_t, \sigma^2 = \Sigma_t)$$

The second distribution $p(z_t|x_t)$ is the distribution of measurements according to equation 4,

$$p(z_t|x_t) = \mathcal{N}(x = z_t, \mu = C_t x_t, \sigma^2 = Q_t)$$

The third distribution $p(x_t|z_{t-1,\ldots 1}, u_{t,\ldots 1})$ is the result of a prediction according to the motion model: the robot pose x_t is assumed to be normally distributed around $\bar{\mu}_t$ with variance $\bar{\Sigma}_t$,

$$p(x_t|z_{t-1,\ldots 1}, u_{t,\ldots 1}) = \mathcal{N}(x = x_t, \mu = \bar{\mu}_t, \sigma^2 = \bar{\Sigma}_t)$$

(a) Based on this information, re-formulate Equation (8) with the probability density function for normal distributions. Simplify your equation such that there is only one exponential expression on each side.
Hint: Drop all normalization factors and replace these by constants η_i. Your solution should look like the following, where $f(\cdot)$ and $g(\cdot)$ are the functions you need to specify, (2 Points)

$$\eta_1 \exp\left(f(x_t, \mu_t, \Sigma_t)\right) = \eta_2 \exp\left(g(x_t, z_t, C_t, Q_t, \bar{\mu}_t, \bar{\Sigma}_t)\right)$$

(b) The equation from (a) needs to be valid for *any arbitrary* value of x_t. Factorize the two exponents $f(\cdot)$ and $g(\cdot)$ on both sides of the equation according to different powers of x_t, such that you obtain quadratic expressions of the form $x_t^2 \cdot a + x_t \cdot b + c$.
By comparing the coefficients a, b, c on both sides of the equation, you can derive one equation for μ_t and one for Σ_t, such that $\mu_t = h_1(Q_t, C_t, \bar{\Sigma}_t)$ and $\Sigma_t = h_2(z_t, C_t, Q_t, \bar{\Sigma}_t, \bar{\mu}_t factors)$.
Hint: You only need two pairs of coefficients. (3 Points)

(c) Use your results from (b) to derive the equations 5 and 6 for the 1D Kalman correction step. (3 Points)

Note that $C_t^T = C_t$, since we merely consider the 1D case throughout this exercise.
Hint: First, write μ_t as $\mu_t = z_t \cdot K + \bar{\mu}_t \cdot d$ to obtain the Kalman gain K, which should be identical to equation 7. Then, simplify d and your equation for Σ_t from (b) using K. Be patient and don't be scared if you have some strange intermediate formulas.