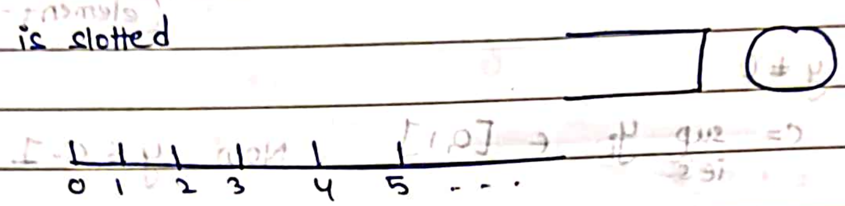
Proof:- $f_{00} = P_{00} + \boxed{\sum_{j \in \tilde{S}} P_{0j}(1 - y_j)}$

if $y = \bar{0}$ $\Rightarrow$ $f_{00} = P_{00} + \sum_{j \in \tilde{S}} P_{0j} = \boxed{1}$

$\Rightarrow$ 0 is recurrent

eg.- DISCRETE TIME QUEUE

Time is slotted



Observe the system at these points in time

$\Rightarrow$ Let $\{A_n\}_{n \geq 1}$ denote customer-arrival process

$\Rightarrow$ Specifically, A_n denotes no. of customers arriving in slot n.

C3 C2 C1 $\xrightarrow{\text{arrive}}$ service $\longrightarrow$ departs (FCFS)

queue; arrive → beginning; service instantly; departs → just before the next slot starts

$\Rightarrow$ Let X_n: denotes the no. of customers in the system, in slot n observed after potential arrivals.

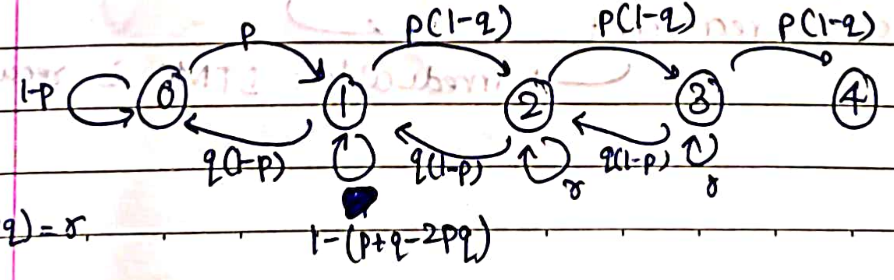
$$X_{n+1} = (X_n + A_{n+1} - D_n)^+ \qquad \boxed{(x)^+ = \max.(x, 0)}$$

$\{A_n\} \Rightarrow$ IID, Bernoulli (p) $\qquad \{D_n\} \Rightarrow$ IID, Bernoulli (q)

$$A_n = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{o/w} \end{cases} \qquad D_n = \begin{cases} 1 & \text{w.p. } q \\ 0 & \text{o/w} \end{cases}$$

$$\boxed{X_{n+1} = (X_n - D_n)^+ + A_{n+1}}$$

if there were some customers & not empty



$(1 - p - q + 2pq) = r$

Discrete Time Queue

from\to	0	1	2	3	...
0	$1-p$	p	0	0	
1	$q(1-p)$	w	$p(1-q)$	0	..
2	0	$q(1-p)$	w	$p(1-q)$	..
3	0	0		...	

$w = [pq + (1-pq)]$

$S = \{0, 1, 2, \ldots\}$

⇒ ⇑ q increases the P ⇒ customer goes back fast.

⇒ $\forall\ (p, q) \in (0, 1) \Rightarrow$ The DTMC is irreducible

→ say the chain is recurrent → if the DTMC is +ve recurrent, we must have, $\exists\ \pi$ satisfying

$$\boxed{\pi = \pi P}$$

or

$$\boxed{\pi_i = \sum_{j \in S} \pi_j P_{ji}} \quad \forall\ i \in S$$

π_i : steady-state probability of being in state-i

⇒ after the system has achieved a steady state, the distribution of any state is π_i (for the +ve recurrent DTMC).

Now, from graph,

$$\pi(i) = q(1-p)\,\pi(i+1) + p(1-q)\,\pi(i-1) + w\,\pi(i)$$

or,

$$\pi(0) = (1-p)\,\pi(0) + q(1-p)\,\pi(1)$$

$$\pi_0(p) = q(1-p)\,\pi(1) \Rightarrow \pi(0) = \frac{q}{p}(1-p)\,\pi(1)$$

$$\boxed{\pi(1) = \frac{p}{q(1-p)}\,\pi(0)} \longrightarrow (1)$$

Similarly, $\pi(1) = p\,\pi_0 + w\,\pi(1) + q(1-p)\,\pi_2$

and on solving,

$$\boxed{\pi(2) = \frac{p^2(1-q)}{q^2(1-p)^2}\,\pi(0)} \quad \text{——} \quad (2)$$

$$\boxed{\pi(3) = \frac{p^3(1-q)^2}{q^3(1-p)^3}\,\pi(0)} \quad \text{——} \quad (3)$$

and so, in general,

$$\boxed{\pi_n = \left(\frac{p}{q(1-p)}\right)^n (1-q)^{n-1}\,\pi_0} \quad \forall\ n \geq 1$$

Also,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \pi_i = 1$$

$$\Rightarrow \left(\frac{\pi_0}{1-q}\right)\left[\frac{1}{1-\frac{p(1-q)}{q(1-p)}}\right] = 1$$

$$\frac{\pi_0}{(1-q)}\left[\frac{q(1-p)}{q-\cancel{qp}-p+\cancel{pq}}\right] = 1$$

$$\frac{\pi_0(1-p)}{q(1-q)(q-p)} = 1 \Rightarrow \pi_0 = \frac{(q-p)(1-q)q}{(1-p)}$$

we can solve this to find π_0, given that

$$\rho \quad \frac{p(1-q)}{q(1-p)} < 1$$

$$p - pq < q - qp \Rightarrow \boxed{p<q}$$

If $\rho<1$,

$$\pi_0 = \left[\frac{1}{1+\frac{\rho}{(1-q)(1-p)}}\right] > 0\ !$$

$\rightarrow \pi$ is unique,

$\Rightarrow$ the recurrent DTM

Physical meaning :- q : service rate
p : arrival rate

when the service rate is greater than the arrival rate, we get a +ve - recurrent DTMC.

Average Queue Length

x_n = Queue length at any time in slot n

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_n \longrightarrow \text{Time avg. of queue length}$$

$$\boxed{\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_n = \underbrace{\sum_{i\in S} \pi_i \cdot i}_{\text{convergence of } \infty \text{ R.V. (almost sure).}} \longrightarrow \text{expectation w.r.t. } \pi_i}$$

π_i : fraction of time spent in state i

$$\pi_i = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} I_{\{x_n = i\}} \quad \text{w.p. } 1$$

→ For the prev. example,

$$\text{avg. queue length} = \sum_{i\in S} i \left(\frac{p}{(1-p)q}\right)^i (1-q)^{i-1} \pi_0.$$

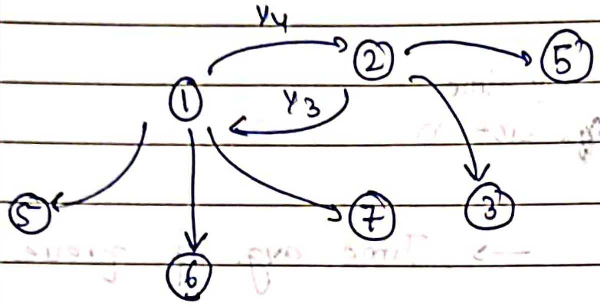
→ The DTMC is (+ve) recurrent if $\rho<1$ (i.e. $q>p$)

Let Q is restriction of P to $\tilde{S} = \{1,2,\ldots\}$

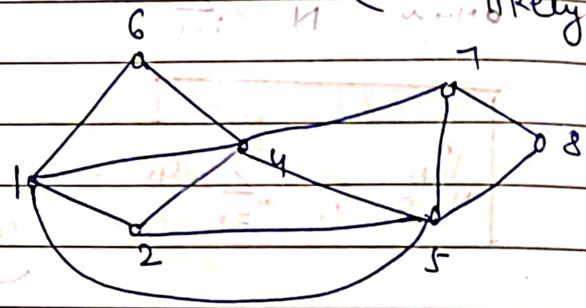
then, $\boxed{\bar{y} = Q\bar{y}}$

$p > q \longrightarrow$ transient
$p = q \longrightarrow$ null recurrent
$p < q \longrightarrow$ +ve recurrent

eg. - $\{X_n\}_{n \geq 0}$ $\quad X_n$: city index on n^{th} - day



(all equally likely)



(irreducible & finite DTMC)

$\pi_1 = P_{21}\pi_2 + P_{41}\pi_4 + P_{51}\pi_5 + P_{61}\pi_6 \longrightarrow$ not easy to solve

as we have assumed, +ve recurrent
$\pi = \pi P$ exists.
Let $\pi_i = c \cdot d_i$

(1) $c[4] = \left(\frac{1}{3}\right) c[3] + \left(\frac{1}{5}\right) c[5] + \left(\frac{1}{2}\right)(2c) + \left(\frac{1}{5}\right)$
(guess)

$= 4(c) \longrightarrow$ Verified

$\pi_i = c[d_i]$ (no. of edges).

as $\sum \pi_i = 1$

$\frac{1}{c} = \sum_{i \in S} d_i \Rightarrow 2 \times$ (no. of edges in graph)

Theorem:- An irreducible DTMC on $S \in \{0,1,2,\dots\}$ is recurrent if $\exists$ a +ve function $\phi : S \longrightarrow R^+$ s.T. $\phi(j) \longrightarrow \infty$ as $j \longrightarrow \infty$, and a finite set $A \subset S$ s.T.

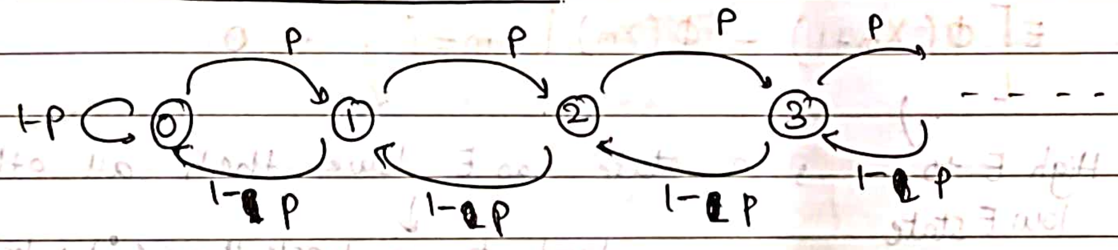
$$E\left[\phi(X_{n+1}) - \phi(X_n) \mid X_n = i\right] \leq 0 \quad \forall\ i \notin A$$

$E[\phi(X_{n+1}) - \phi(X_n) \mid X_n = i] \longrightarrow$ drift in state-i

$$= E[\phi(X_{n+1}) \mid X_n = i] - \phi(i)$$

$$= \underbrace{\sum_{j \in S} \phi(j) P_{ij}}_{\text{avg. energy in next state}} - \underbrace{\phi(i)}_{\text{energy in this state}} \leq 0$$

eg.- <u>A reflected random walk</u>



($p \longrightarrow 1-p$, we walk entirely reverse)

$A = \{0\}$, $\phi(i) = i \quad \forall\ i \in S$

$\rightarrow$ Let any i, $i \notin A$, then, the drift =

$$\text{drift} = \left[\phi(i+1)(1-p) + \phi(i-1)(p)\right] - \phi(i)$$

$$= (i+1)(1-p) + (i-1)(p) - i$$

$$= -1 + p(2) \Rightarrow 2p-1 \Rightarrow$$ so, for recurrence

$$2p - 1 \leq 0$$

$$\boxed{p \leq \frac{1}{2}}$$

⇒ This result just says that if $p < (1/2)$, the DTMC is the recurrent but it doesn't say if $p > (1/2)$ implies the DTMC is NOT the recurrent. It may or may not be true.

⇒ There needs to exist only ONE ϕ, A for this theorem to hold true.

Theorem :- An irreducible DTMC $\{X_n\}_{n \geq 0}$ on $S = \{0, 1, \ldots\}$ is transient if $\exists$ a $\phi : S \longrightarrow \mathbb{R}^+$ and a finite set $A \subset$ s.t.

① $\exists\ j \notin A$ and $\phi(j) < \min_{i \in A} \phi(i)$ and

[a state j has a lower PE than any other states in A

② $\forall\ i \in A$,

$$E\left[\phi(X_{m+1}) - \phi(X_m) \mid X_m = i\right] \leq 0$$

[High E to low E state → a state has E lower than all others → tend to go back there (j) ⇒ transient]

eg. → in earlier example, now, let $A = \{0\}$

$\phi(j) = \alpha^j, \quad 0 \leq \alpha < 1$

Drift ⇒ $(1-p)\alpha^{j-1} + p\alpha^{j+1} - \alpha^j$

$$= \alpha^i \left[\underbrace{\frac{1-p}{\alpha} + p\alpha - 1}_{<0} \right]$$

$$p\alpha^2 - \alpha + 1 - p \leq 0$$

$\frac{d}{d\alpha}(p\alpha^2 - \alpha + 1 - p) < 0 \Rightarrow 2p\alpha - 1 < 0 \Rightarrow \alpha < \frac{1}{2p}$

$$(\alpha - 1)\left(\alpha - \left(\frac{1}{p} - 1\right)\right) \leq 0$$

+ $\quad$ $\frac{1}{p}-1$ $\quad$ − $\quad$ 1 $\quad$ +

$f(\alpha) = \alpha^k < 1$

$\alpha^{\frac{1}{p}-1} < 1$

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Theorem :- <u>Foster's Criteria</u>

→ An irreducible DTMC $\{X_n\}_{n\geq 0}$ on $S = \{0, 1, 2, \dots\}$ is +ve recurrent if $\exists$

$\phi : S \longrightarrow \mathbb{R}^+$, $\varepsilon > 0$, and a finite set $A \subset S$, s.t.

① $\forall\, i \notin A$, drift $\leq -\varepsilon$ (negative & bounded energy from 0)

② $\forall\, i \in A$, $E[\phi(X_{m+1}) \mid X_m = i] < \infty$

eg.- $A = \{0\}$, $\phi(i) = i$, drift $= 2p - 1$

Let $p = \frac{1}{2} - \delta p$ (fixed) drift $= 1 - 2\delta p - 1 = -2\delta p$

so, we can set ε as δ, and for every $p < 1/2$, we have $a(\varepsilon, \delta)$.