EE621: Markov Chains and Queueing Systems

Quiz # 1 (20 Marks)

Time:- 9:30 - 11 am Date:- 03/02/2023

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Q.1 An EE621 student visits lake everyday morning and makes the following observations about the bird population:

1. If on a given day no birds are seen, then next day exactly one bird is seen.
2. If $i$ birds are observed on a day, then on the next day either all birds are gone with prob. $1-g(i)$ or one extra bird is seen with prob. $g(i)$.

Let $X_n$ denote number of birds observed on $n^{th}$ day.

(a) Prove that $\{X_n\}_{n\geq 0}$ is a DTMC. [2 Marks]

(b) Find TPM of the DTMC and draw state transition diagram. [3 Marks]

(c) In the following 3 cases find type (transient, null recurrent, or positive recurrent) for state 100.

(i) $1-g(i) = 1/2 \quad \forall i.$

(ii) $1-g(i) = 1/(i+1) \quad \forall i.$ [5 Marks each]

(iii) $1-g(i) = (1/2)^i \quad \forall i.$

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Quiz # 2 (20 Marks)

Time:- 4:30 - 6 pm Date:- 18/02/2024

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**Q.1** For an irreducible DTMC $\{X_n\}_{n\geq 0}$ on $S = \{0,1,\ldots\}$ with TPM P, prove or disprove:

$$g.c.d.\{n : f_{jj}^{(n)} > 0\} = g.c.d.\{n : p_{jj}^{(n)} > 0\}.$$ 5 Marks

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**Q.2** Consider an irreducible $\{X_n\}_{n\geq 0}$ on $S = \{0,1,\ldots\}$ with TPM P. Prove or disprove: If $\exists$ a prob. mass function $\bar{\alpha} > 0$ on $S$ satisfying $\alpha_i p_{ij} = \alpha_j p_{ji}$ $\forall i,j \in S$, then the DTMC is positive recurrent. 5 Marks

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**Q.3** Consider the descrete time queueing system with two (instead of one) servers. In each slot, arrival occurs w.p. p in each slot. If a server is working, then prob. of it finishes service in a slot is q. The two servers are independent. The servers never idle as long as customers are waiting.

(a) Draw a state transition diagram. 5 Marks

(b) Find conditions under which the DTMC is positive recurrent. 5 Marks

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# EE621: Markov Chains and Queueing Systems

Quiz #3 (20 Marks)

Time:- 11:30 - 1:30 pm Date:- 27/02/2023

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**Q.1** Consider a DTMC $\{X_n\}_{n\geq 0}$ on $S=\{0,1,\ldots\}$ with TPM $P$. Prove or disprove: If the DTMC is positive recurrent, then the steady state prob. of going from states in $\tilde{S}$ to $\tilde{S}^c (= S \setminus \tilde{S})$ equals that from $\tilde{S}^c$ to $\tilde{S}$ for every $\tilde{S} \subseteq S$, i.e. $\sum_{i\in\tilde{S}}\sum_{j\in\tilde{S}^c} \pi_i p_{ij} = \sum_{i\in\tilde{S}^c}\sum_{j\in\tilde{S}} \pi_i p_{ij}$. **5 Marks**

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**Q.2** Amol, an IITB student, commutes from H12 to LHC every morning and back every evening. He has a cycle, but he is forgetful. He takes his cycle if he has a cycle at the starting point, and either he rembers that or one of his friends reminds him. Prob. of him remembering is $p$ and his friends reminding is $q$, and both these events are independent. If the distance from the hostel to LHC is 3km, then find the avg. distance Amol walks every day. Take $p=0.4$ and $q=0.6$. **7 Marks**

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**Q.3** Consider a renewal process $\{M(t)\}_{t\geq 0}$ with iid $\exp(\lambda)$ life times, i.e. $f_{X_k}(x) = \lambda e^{-\lambda x}$ $\forall x \geq 0$ & 0 otherwise.

(a) Find $P(M(t)=n)$. **5 Marks**

(b) Find $m(t)$. **3 Marks**

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Quiz # 4 (20 Marks)

Time: 9:30 - 11 am Date:- 16/03/2024

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$\boxed{Q.1}$ Consider a machine that can run continuously for $X$ units of time. Consider a strategy of performing maintainance when machine finishes $T$ units of operation. If the machine fails before maintainance, then it need to be replaced. Let $C_m$ and $C_R$ denote down time for machine maintainance and replacement, respectively. Machine becomes new after maintainance. Find $T$ that minimizes avg machine down time when: $\boxed{8 \text{ Marks}}$

$X \sim exp(1)$, $C_m = 10$ units and $C_R = 20$ units.

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$\boxed{Q.2}$ Let $\{m(t)\}_{t \geq 0}$ be a renewal process with $X_2 \sim F$ and usual assumptions. Let $X_e$ be any random variable having distribution given by:

$$F_{X_e}(x) = \lim_{t \uparrow \infty} \frac{1}{t} \int_0^t P(Y(u) \leq x) du.$$

(a) Find $E[X_e]$ for $F = \text{Uniform}([0, 10])$. $\boxed{4+3 \text{ Marks}}$

(b) Also, find $\lim_{t \uparrow \infty} \frac{1}{t} \int_0^t X(u) du$, $X(t) = Z_{m(t)+1} - Z_{m(t)}$.

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$\boxed{Q.3}$ Let $\{m_1(t)\}_{t \geq 0}$ and $\{m_2(t)\}_{t \geq 0}$ be two independent renewal processes. Prove or disprove: $\{m_1(t) + m_2(t)\}_{t \geq 0}$ is a renewal process. $\boxed{5 \text{ Marks}}$

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Quiz # 4 (20 Marks)

Time:- 2 - 3:30 pm Date:- 14/04/2024

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**Q.1** For a delayed renewal process $\{M(t)\}_{t \geq 0}$. Define, $U(t) = t - Z_{M(t)}$, $\forall t \geq 0$. Find $\lim_{t \uparrow \infty} P(U(t) \leq u)$ if $F$ is non-lattice. **[7 Marks]**

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**Q.2** Consider a delayed renewal process $\{M(t)\}_{t \geq 0}$ with $A = \text{Uniform}([0, C])$ and $F(x) = 0 \ \forall x < C$ and $= 1 \ \forall x \geq C$. Prove or disprove: the renewal process is stationary. **[6 Marks]**

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**Q.3** At a shopping mall, the number of customers that arrived until time $t$ is $N(t)$. Let $\{N(t)\}_{t \geq 0}$ be Poisson(2) process. Shopping times for the customers are iid Uniform([0, 10]). After shopping customers immediately leave the mall. Let $M(t)$ denote the number of customers in the mall at time $t$. Find the Prob. mass fn for $M(t)$. **[7 Marks]**

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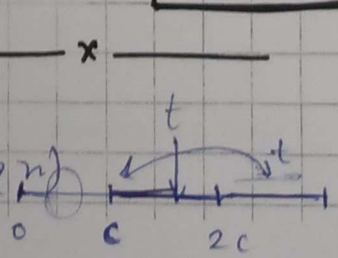
$P(X \geq y) = F(\ldots x)$



# EE621: Markov Chains & Queueing System

Quiz # 6 (20 Marks)

Date: 29/04/2024 Time: 6-7:30pm

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**Q.1** Let $\{M(t)\}_{t\geq 0}$ be an ordinary renewal process with non lattice, square integrable life-time distr. $F(\cdot)$. Find $\lim_{t\uparrow\infty} E[Y(t)]$, where

$$Y(t) = Z_{M(t)+1} - t.$$

**7 Marks**

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**Q.2** Let $\{M(t)\}_{t\geq 0}$ be an ordinary renewal process with non-lattice life-time distr. Uniform$([0,1])$. Show that $\forall t \in [0,1]$,

$$m(t) = e^{t} - 1.$$

Hint: You may want to write down renewal equation for $m(t)$.

**6 Marks**

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**Q.3** Consider a single queue with Poison$(\lambda)$ arrivals. The service times are exp$(\mu)$. However, after finishing service, the customer starts another independent exp$(\mu)$ service w.p. $p$ and leaves w.p. $(1-p)$. The decision to stay or leave is independent of the past. Find avg. waiting time in the system.

**7 Marks**

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# EE621: Markov Chains & Queueing Systems

## Quiz #7 (20 Marks)

Date: 29/04/2024 Time: 7:30-9 pm

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**Q.1** Give an example of a positive recurrent CTMC whose EMC is Null recurrent.

**6 Marks**

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**Q.2** Consider M/M/c/c system with Poison($\lambda$) arrivals and iid exp($\mu$) service times for all the servers. Let $\Lambda_c(t)$ denote the number of customers that are blocked until time $t$.

(a) Find $\lim_{t \uparrow \infty} \frac{\Lambda_c(t)}{t}$ using Renewal theory.

(b) Find fraction of arrivals blocked in the system.

**5 + 2 Marks**

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**Q.3** Consider an expert who starts his session at time 0. People come to seek his advice as per Poisson($\lambda$) process. Each question needs exp($\mu$) time to address it. The expert leaves when there are no more people waiting. Find the expected time for which the expert answers the questions.

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