Q.1 Let $\{N(t)\}_{t\geq 0}$ be a Poisson(λ) process. Also, let $\{Y_n\}_{n\geq 1}$ be a sequence of i.i.d. Bernoulli(p) random variables for $p\in[0,1]$. Also, $\{N(t)\}_{t\geq 0}$ and $\{Y_n\}_{n\geq 1}$ are independent processes.

Define,
$$N_1(t) = \sum_{k=1}^{N(t)} 1_{\{Y_k=1\}}, \text{ and}$$
$$N_2(t) = \sum_{k=1}^{N(t)} 1_{\{Y_k=0\}}.$$

(a) Show that $\{N_1(t)\}_{t\geq 0}$ and $\{N_2(t)\}_{t\geq 0}$ are Poisson processes. Find parameters.

(b) Are the two processes independent?

—— x —— x ——

Q.2 Let $\{N_1(t)\} \sim$ Poisson(λ_1) and $\{N_2(t)\} \sim$ Poisson(λ_2). Define, $N(t) = N_1(t) + N_2(t)$.

(a) Show that $\{N(t)\}_{t\geq 0}$ is an ordinary Renewal process if $\{N_1(t)\} \perp\!\!\!\perp \{N_2(t)\}$.

(b) Find life-time distribution.

—— x —— x ——

Q.3 Let $F_e(y) = \lim_{t\uparrow\infty} \frac{1}{t}\int_0^t P(Y(u)\leq y)\,du$ as considered in the class for a renewal process s.t. $X_2 \sim F$. Let X & X_e be any two random variables satisfying

$X \sim F$ and $X_e \sim F_e$. Show that the following two statements are equivalent:

(i) $E[X-t \mid X>t] \leq E[X] \;\; \forall t \geq 0$,

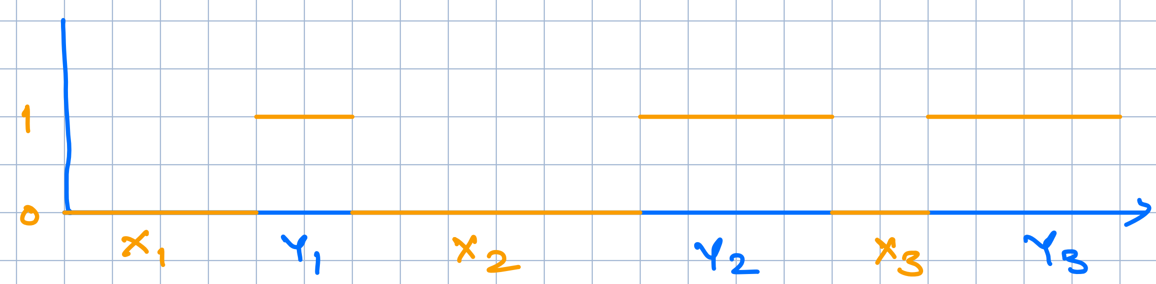
(ii) $\bar{F}_e(t) \geq \bar{F}(t) \;\; \forall t \geq 0$.

Also, show that $F_e(t) = F(t) \;\; \forall t \geq 0$, then F is exponential distribution.

Additional reading: Read about stochastic dominance. What does it mean if we say a random variable X is stochastically smaller or dominated by a random variable Y?

——— x ——— x ——— x ———

$\boxed{Q.4}$ Consider an alternating renewal process whose sample path is shown below:



Here $\{X_n\}_{n\geq 1}$ are iid F_0 and $\{Y_n\}_{n\geq 1}$ are iid F_1. Is this a renewal process? Let $\{Z_n\}_{n\geq 1}$ denote the life times of the renewal process. Show that $Z_n = X_n + Y_n$. If the process is in state i at time t, then define $Y_i(t)$

to be the time process spends in state i, i.e. the process jumps to state $1-i$ at time $t+Y_i(t)$.

Show that

$$\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t P(Y_i(u)>y)\,du = \frac{E[\tilde{X}_i]}{E[Z_i]} F_{ie}^c(y),$$

where $\tilde{X}_i = X_1$ if $i=0$ & $\tilde{X}_i = Y_1$ if $i=1$, and $F_{ie}(\cdot)$ is equilibrium distribution for process with life-times having distribution of $\tilde{X}_i$.

$\boxed{Q.5}$ Find $E[M^2(t)]$.

$\boxed{Q.6}$ Find $\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t P(U(x)>u, Y(x)>u)\,dx$.

$\boxed{Q.7}$ Consider a park and let $N(t)$ denote the number of visitors until time t. An arriving customer spends random time inside the park, the distribution is $G(\cdot)$. If $\{N(t)\}_{t\geq 0}$ is Poisson(λ), then find the distribution of $M(t)$ that denotes number of customers inside the park at time t. The answer can be in terms of $G(\cdot)$.