$i=1$

Tutorial- 2

Q5) irreducible DTMC, on S, P

$$D_i = \gcd\{n : f_{ii}^{(n)} > 0\}$$

$f_{ii}^{(n)} \Rightarrow$

(Start from i and reach i in n-steps.)

→ Here i and j are communicating

$\Rightarrow \forall\, m>0$ S.T. $f_{ij}^{(m)} > 0$ and

$\forall\, n>0$ ST $f_{ji}^{(n)} > 0$

let $D_i = a$ and $D_j = b$

$\Rightarrow m = K_1 a$ and $n = K_2 b$.

$\cancel{f_{ij} = \sum_{m=1}^{\infty} f_{ij}^{(m)} = 1}$ ~~(since we always visit i starting from j~~

$\Rightarrow$ If i and j communicate, they have the same period.

→ as they communicate, $\exists\, M, N$ and

$P_{ij}(M) > 0$ $\quad P_{ji}(N) > 0$.

Now,

$$P_{ii}^{(M+nK+N)} \geq P_{ij}^{(M)} \left(P_{jj}^{(K)}\right)^n \left(P_{ji}^{(N)}\right).$$

So, for any $R \geq 1$, s.t. $P_{jj}^{(R)} > 0$, we have $P_{ii}^{(M+nK+N)} > 0 \ \forall n \geq 1$

So, d_i divides $M+nK+N \ \forall n \geq 1$

particularly, d_i divides all R s.t. $P_{jj}^{(R)} > 0 \Rightarrow d_i \div d_j$

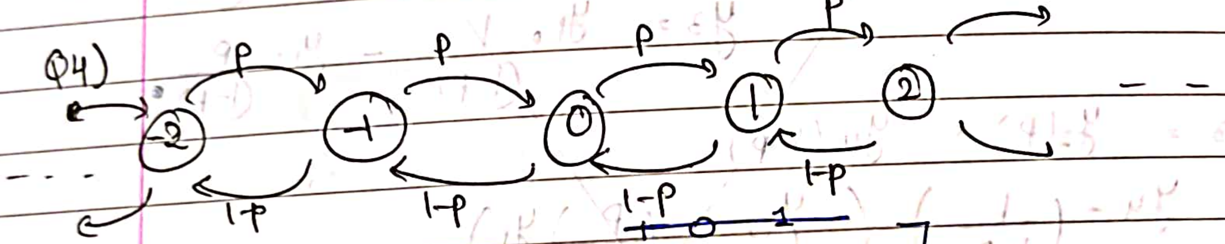
Symmetrically : $\underline{d_j \div d_i}$ H.P.

$\Rightarrow$ Here,

$$f_{ii}^{(M+nK+N)} \geq f_{ij}^{(M)} \left(f_{jj}^{(R)}\right) f_{ji}^{(N)}$$ one path only

and so, similarly, we can argue $\boxed{D_i = D_j}$

Q4)



b) ~~Here,~~

$$P = \begin{array}{c} \\ -1 \\ 0 \\ 1 \end{array} \left[\begin{array}{ccc} & & \\ & 1-p & p \\ & 1-p & p \\ & & \end{array}\right]_{n \times n}$$

Q = P without the 0^{th} row & column

0 is a soln

$$\begin{bmatrix} \vdots \\ y_{-2} \\ y_{-1} \\ y_1 \\ \vdots \end{bmatrix} = \begin{array}{c} -2 \quad -1 \quad 0 \quad 1 \quad 2 \\ \left[\begin{array}{ccc} 1-p & & \\ & 1-p & p \\ & & 1-p \end{array}\right]_{(n-1)\times(n-1)} \end{array} \begin{bmatrix} \vdots \\ y_{-1} \\ y_1 \\ \vdots \end{bmatrix}_{(n-1)\times 1}$$

$y_{-2} = 0$

$\cancel{y_{-1} = y_1} = y$

$y_2 = y/p$ $\quad y_{-2} = \frac{y}{1-p}$

~~if $y = 1$~~

~~$(1-p)\, y_{-1} + p = y_{-1}$~~

$p = p\, y_{-1}$

~~$y_{-1}(1-p) = y_{-1}$~~

~~$p y_{+1} = y_1$~~

~~$(1-p) y_2 = y_2$~~ $\Rightarrow y_2 = 0$

~~$(1-p)\, y_{-2} = y_{-1}$~~

~~$(1-p)$~~ $y_{-1} + p\, y_1 = y_{+1}$

$y_1 = p\, y_2$

$y_2 = y_1 p + y_3(1-p)$

$y_1 = 0$ $\quad$ ~~$y_{-1} = 0$~~ $\quad$ $y_{-2} = 0$ $\Rightarrow \boxed{Q}$

~~$y_2 = \ldots$~~

$y_2 p + y_4(1-p)$

$$\left(1-2(p-p^2)\right)\pi(0) = p^2\pi(-2) + (1-p)^2\pi(2)$$

WEIGHTED AM-GM

$$\frac{x_1+x_2+\dots+x_n}{n} \geq \sqrt[n]{x_1x_2\dots x_n} \geq \frac{n}{\frac{1}{x_1}+\dots+\frac{1}{x_n}}$$

and,

$$\frac{m_1x_1+\dots+m_nx_n}{n} \geq \left(x_1^{m_1}x_2^{m_2}+\dots+x_n^{m_n}\right)^{\frac{1}{m_1+m_2+\dots+m_n}}$$

Also,

$$\left(\frac{a_1^m+a_2^m+\dots+a_n^m}{n}\right) \leq \left(\frac{a_1+a_2+\dots+a_n}{n}\right)^m \quad \text{if } m\in[0,1]$$

$$> \quad m<0 \text{ or } m>1$$

$$\left(\frac{1}{n}\sum_{i=1}^{n}a_i^{k_1}\right)^{\frac{1}{k_1}} \geq \left(\frac{1}{n}\sum a_i^{k_2}\right)^{\frac{1}{k_2}} \quad \text{if } k_1\geq k_2>0$$

INVARIANT MEASURES

⇒ A non-trivial vector, $x = \{x_i\}_{i\in E}$ is called an invariant measure of the stochastic matrix $P = \{P_{ij}\}_{i,j\in E}$ if $\forall\, i \in E$

$$x_i \in [0,\infty)$$

and $$x_i = \sum_{j\in E} P_{ji}\, x_j \qquad \text{i.e.} \qquad \boxed{x^T = x^T P}$$

Thm. 1) Let P be the TPM of an irreducible, recurrent HMC, $\{X_n\}_{n\geq 0}$. Let 0 be an arbitrary state, and T_0 be the return time to 0. Let, for all $i \in E$

$$x_i = E_0\left[\sum_{n\geq 1} 1_{\{X_n=i\}} 1_{\{n \leq T_0\}}\right]$$

(x_i is the expected no. of visits to state-i before returning to 0.)

Then, $x_i \in (0,\infty)$ and x_i is an invariant measure of P.

⇒ if $1 \leq n \leq T_0$, then $X_n = 0$ iff $n = T_0$, and so

$x_0 = 1$

⇒ Also, $$\sum_{i\in E}\sum_{n\geq 1} 1_{\{X_n=i\}} 1_{\{n\leq T_0\}} = \sum_{n\geq 1}\left\{\sum_{i\in E} 1_{\{X_n=i\}}\right\} 1_{\{n\leq T_0\}}$$

$$= \sum_{n\geq 1} 1_{\{n\leq T_0\}} = T_0$$

and so, $$\boxed{\sum_{i\in E} x_i = E_0[T_0]}$$

Define :-

$${}_0P_{0i}(n) = E_0\left[1_{\{X_n=i\}} 1_{\{n\leq T_0\}}\right]$$
$$= P_0(X_1 \neq 0, \dots, X_{n-1}\neq 0, X_n = i)$$

(starting from state-0, of visiting i at a time n before returning back to 0.)

① ${}_0P_{0i}(1) = P_{0i}$

$$\boxed{x_i = \sum_{n\geq 1} {}_0P_{0i}(n)}$$

② For $n \geq 2$

$${}_0P_{0i}(n) = \sum_{j\neq 0} {}_0P_{0j}(n-1)\, P_{ji}$$

(summing up all these inequalities)

$$\boxed{x_i = P_{0i} + \sum_{j\neq 0} x_j P_{ji}}$$

$\Rightarrow$ If π is the unique stationary distribution of an irreducible positive recurrent chain, then,

$$\pi(i)\, E_i[T_i] = 1 \quad \text{or} \quad \boxed{\pi(i) = \frac{1}{E_i[T_i]}}$$

(T_i : time to return to state-i)

expected time to return to state-i $= \dfrac{1}{\pi(i)}$

As we know that, $\pi(i) = \dfrac{x_i}{\sum_{j\in E} x_j}$

and, particularly, for $i=0$, $\pi(0) = \dfrac{x_0}{\sum_{j\in E} x_j} = \dfrac{1}{E_0[T_0]}$

NOTE :- An irreducible HMC with finite state space is +ve recurrent.

$\downarrow$

if it were transient, $\sum_{n\geq 0} P_{ij}^{(n)} < \infty \quad \forall\, i,j \in E$

and since the state-space is finite,

$$\sum_{j\in E}\sum_{n\geq 0} P_{ij}^{(n)} < \infty$$

But, the latter sum is,

$$\sum_{n\geq 0}\sum_{j\in E} P_{ij}^{(n)} = \sum_{n\geq 0} 1 = \infty$$

which is a CONTRADICTION, and so the chain is recurrent.

$\Rightarrow$ let $\{X_n\}_{n\geq 0}$ be an irreducible, +ve recurrent HMC with stationary distribution π, and let $f : E \to \mathbb{R}$ be s.t.

$$\sum_{i\in E} |f(i)|\, \pi(i) < \infty$$

Then, for any initial distrib^n, μ. P_μ- as,

$$\boxed{\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(X_k) = \sum_{i\in E} f(i)\, \pi(i)}$$