EE621: Markov Chains and Queueing Systems

Quiz # 2 (20 Marks)

Time:- 4:30 - 6 pm   Date:- 18/02/2023

Q.1 For an irreducible DTMC $\{X_n\}_{n\geq 0}$ on $S = \{0, 1, \ldots\}$ with TPM P, prove or disprove:

$$\text{g.c.d.}\,\{n : f_{jj}^{(n)} > 0\} = \text{g.c.d.}\,\{n : p_{jj}^{(n)} > 0\}.$$

5 Marks

Q.2 Consider an irreducible $\{X_n\}_{n\geq 0}$ on $S = \{0, 1, \ldots\}$ with TPM P. Prove or disprove: If $\exists$ a prob. mass function $\bar{\alpha} > 0$ on S satisfying $\alpha_i p_{ij} = \alpha_j p_{ji}$ $\forall\, i, j \in S$, then the DTMC is positive recurrent.

5 Marks

Q.3 Consider the descrete time queueing system with two (instead of one) servers. In each slot, arrival occurs w.p. p in each slot. If a server is working, then prob. of it finishes service in a slot is q. The two servers are independent. The servers never idle as long as customers are waiting.

(a) Draw a state transition diagram. 5 Marks

(b) Find conditions under which the DTMC is positive recurrent. 5 Marks

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Solution

Q.1 Note that $p_{jj}^{(n)} \geq f_{jj}^{(n)}$.

$$\Rightarrow \{n : f_{jj}^{(n)} > 0\} \subseteq \{n : p_{jj}^{(n)} > 0\}$$

$$\Rightarrow \gcd \{n : f_{jj}^{(n)} > 0\} \geq \gcd \{n : p_{jj}^{(n)} > 0\}. \quad (1)$$

Consider any n such that $p_{jj}^{(n)} > 0$ & $f_{jj}^{(n)} = 0$.

But note that

$$p_{jj}^{(n)} = \sum_{k=1}^{n} f_{jj}^{(k)} p_{jj}^{(n-k)}$$

$\because p_{jj}^{(n)} > 0$, then $\exists\ k \in \{1, 2, \ldots, n-1\}$ s.t.

$$f_{jj}^{(k)} p_{jj}^{(n-k)} > 0.$$

$$\Rightarrow f_{jj}^{(k)} > 0 \text{ and } p_{jj}^{(n-k)} > 0.$$

Applying same steps to $p_{jj}^{(n-k)}$, we see that any $n \in \{m : p_{jj}^{(m)} > 0\}$ is sum of elements of $\{m : f_{jj}^{(m)} > 0\}$.

Thus, g.c.d. $\{m : f_{jj}^{(m)} > 0\}$ also divides every element of $\{m : p_{jj}^{(m)} > 0\}$.

But this implies

$$\gcd \{m : p_{jj}^{(m)} > 0\} \geq \gcd \{m : f_{jj}^{(m)} > 0\} \quad (2)$$

The result follows from (1) & (2).

— x — x — x —

$\boxed{Q.2}$ We need to show that $\bar{\alpha}$ satisfies

$$\bar{\alpha} = \bar{\alpha} P.$$

Note that $\forall\, i \,\&\, j \in S$,

$$\alpha_i p_{ij} = \alpha_j p_{ji}$$

$$\Rightarrow \sum_{j \in S} \alpha_i p_{ij} = \sum_{j \in S} \alpha_j p_{ji}$$

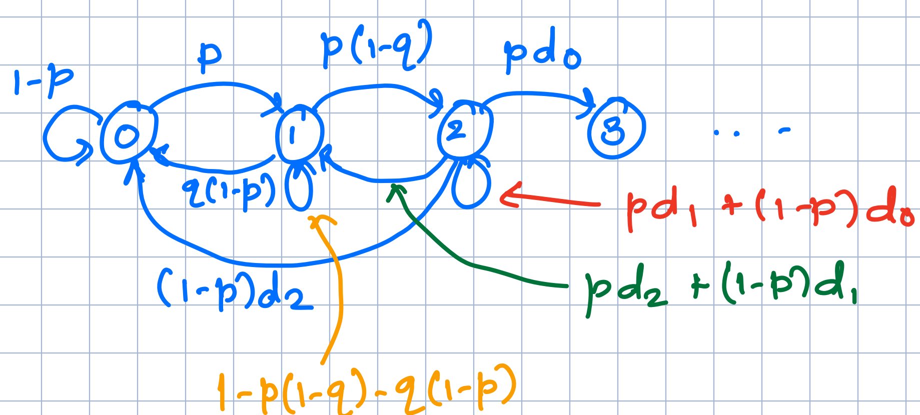
$$\Rightarrow \alpha_i = \sum_{j \in S} \alpha_j p_{ji}.$$

This proves the required.

—— x —— x —— x ——

$\boxed{Q.3}$ Let $d_i,\ i = 0, 1, 2$ denote the probability of i departures when both servers are working.

$$d_0 = (1-q)^2; \quad d_1 = q(1-q) \ \& \ d_2 = q^2$$



Note that $\forall\ k \geq 2$,

$$p_{k,k+1} = pd_0$$

$$p_{k,k} = pd_1 + (1-p)d_0$$

$$p_{k,k-1} = pd_2 + (1-p)d_1$$

$$p_{k,k-2} = (1-p)d_2$$

Now consider $A = \{0,1\}$ and $\varphi(k) = k$.

Now find the drift.

$$E[\varphi(X_{k+1}) - \varphi(X_k) \mid X_k = k]$$

$$= (k+1)pd_0 + k(pd_1 + (1-p)d_0) + (k-1)(pd_2 + (1-p)d_1) + (k-2)(1-p)d_2 - k$$

$$= k[\cancel{pd_0} + \cancel{pd_1} + (1-\cancel{p})d_0 + \cancel{pd_2} + (1-\cancel{p})d_1 + (1-\cancel{p})d_2 - 1] + pd_0 - pd_2 - (1-p)d_1 - 2(1-p)d_2$$

$$= k\underbrace{[d_0 + d_1 + d_2 - 1]}_{=0} + pd_0 - \cancel{pd_2} - d_1 + pd_1 - 2d_2 + \cancel{2}pd_2$$

$$= p[d_0 + d_1 + d_2] - d_1 - 2d_2$$

$$= p - d_1 - 2d_2.$$

Note that whenever $p < d_1 + 2d_2$, we can choose $\varepsilon = \frac{d_1 + 2d_2 - p}{2}$ and then note that $\text{Drift}(k) < -\varepsilon$ $\forall k \geq 2$.

Also show that $E[\varphi(X_{k+1}) \mid X_k = 0]$ & $E[\varphi(X_{k+1}) \mid X_k = 1]$ are $< \infty$.

Thus, Foster's criteria implies the DTMC is positive recurrent for $p < d_1 + 2d_2$ $(p < q(1-q))$.

Note the $d_1 + 2d_2$ is the expected number of departures in a slot when both servers are working.