EE621: Markov Chains and Queueing Systems

Quiz #4 (20 Marks)

Time:- 2 - 3:30 pm Date:- 14/04/2024

Q.1 For a delayed renewal process $\{M(t)\}_{t\geq 0}$. Define, $U(t) = t - Z_{M(t)}, \forall t \geq 0$. Find $\lim_{t\uparrow\infty} P(U(t) \leq u)$ if F is non-lattice. **[7 Marks]**

Q.2 Consider a delayed renewal process $\{M(t)\}_{t\geq 0}$ with $A = \text{Uniform}([0, C])$ and $F(x) = 0 \;\forall x < C$ and $= 1 \;\forall x \geq C$. Prove or disprove: the renewal process is stationary. **[6 Marks]**

Q.3 At a shopping mall, the number of customers that arrived until time t is $N(t)$. Let $\{N(t)\}_{t\geq 0}$ be Poisson(2) process. Shopping times for the customers are iid Uniform([0, 10]). After shopping customers immediately leave the mall. Let $M(t)$ denote the number of customers in the mall at time t. Find the Prob. mass fn for $M(t)$. **[7 Marks]**

Solutions

Consider, $P(U(t)>u)$ and note that

$$P(U(t)>u) = P(U(t)>u, X_1>t) + P(U(t)>u, X_1\in[t-u,t]) + P(U(t)>u, X_1 < t-u)$$

Now, $P(U(t)>u, X_1>t) = P(X_1>t)$ if $t>u$.

$$= A^c(t)\, 1_{\{t>u\}}. \quad - \; (1)$$

$$P(U(t)>u, X_1\in[t-u,t]) = 0 \quad - \; (2)$$

$$P(U(t)>u, X_1 < t-u) = \int_0^{t-u} P_0(U(t-v)>u)\, dA(v)$$

$$= \int_0^t P_0(U(t-v)>u)\, 1_{\{t-u>v\}}\, dA(v). \; - (3)$$

Define, $H(t) \triangleq P(U(t)>u)$

$$= P(U(t)>u)\, 1_{\{t>u\}}$$

(If $t\leq u$, then $U(t)\leq u$ w.p.1)

from (1), (2) & (3), conclude

$$\left.\begin{aligned} H(t) &= a(t) + (H_0 * A)(t) \quad \& \\ H_0(t) &= a_0(t) + (H_0 * F)(t) \end{aligned}\right\} \quad (4)$$

where $a(t) = A^c(t)\, 1_{\{t>u\}}$ &

$$a_0(t) = F^c(t)\, 1_{\{t>u\}}.$$

Solution of renewal equation (4) is

$$H(t) = a(t) + (a_0 * m)(t)$$

$$= \underbrace{A^c(t)\, 1_{\{t>u\}}}_{\to 0 \text{ as } t \uparrow \infty} + \int_0^t \underbrace{F^c(t-v)\, 1_{\{t-u>v\}}}_{\substack{\text{monotone decreasing} \\ \text{fn of } t \text{ \& non-negative} \\ \Rightarrow \text{DRI.}}} dm(v)$$

Thus, by KRT

$$\lim_{t\uparrow\infty} H(t) = \frac{1}{EX_2} \int_0^\infty F^c(v)\, 1_{\{v>u\}}\, dv .$$

$$\lim_{t\uparrow\infty} P(U(t)>u) = \frac{1}{EX_2} \int_v^\infty F^c(v)\, dv .$$

Now, note that

$$\lim_{t\uparrow\infty} P(U(t)\le u) = 1 - \lim_{t\uparrow\infty} P(U(t)>u)$$

$$= 1 - \frac{1}{EX_2} \int_v^\infty F^c(v)\, dv .$$

$$= \frac{1}{EX_2} \int_0^\infty F^c(v)\, dv - \frac{1}{EX_2} \int_v^\infty F^c(v)\, dv$$

$$= \frac{1}{EX_2} \int_0^v F^c(v)\, dv .$$

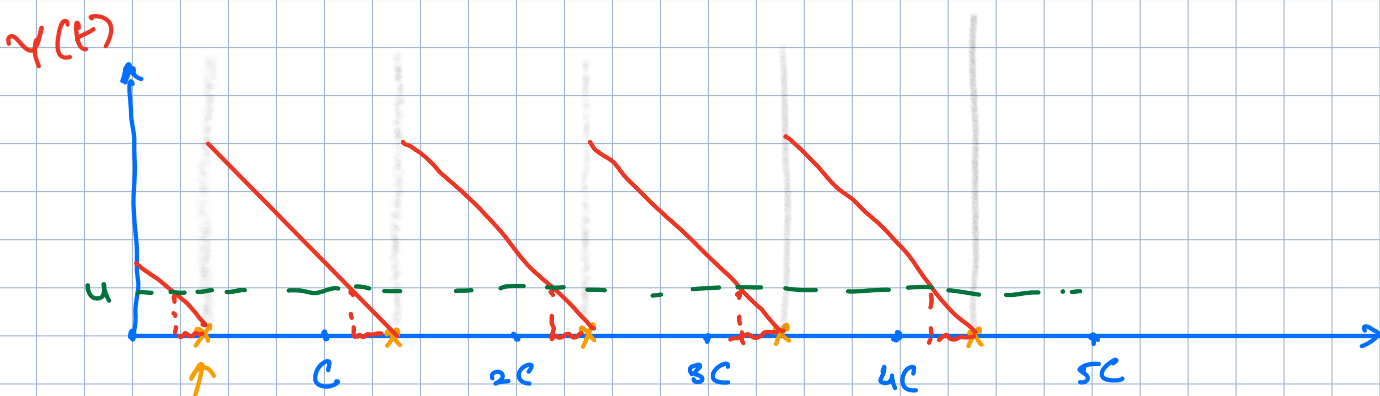
$\boxed{Q.2}$ To show that the renewal process is stationary, we need to show that

$$A = F_e .$$

Many ways to prove it. A straightforward way is to evaluate

$$\lim_{t \uparrow \infty} \frac{1}{t} \int_0^t 1_{\{Y(t) \le y\}} \, dy .$$

If you divide time into intervals of length c as shown below:



first renewal Uniform([0, c])

subsequent renewals are $X_1 + c$, $X_1 + 2c$, ...

Note that $\forall\, y \in [0, c]$

$$\int_{nc}^{(n+1)c} 1_{\{Y(t) \le y\}} \, dt = y$$

$$= 0 \quad \text{if } y < 0$$

$$= c \quad \text{if } y > c .$$

Thus,

$$\lim_{t \uparrow \infty} \frac{1}{t} \int_0^t 1_{\{Y(t) \le y\}} \, dt = \begin{cases} 0 & \text{if } y < 0 \\ y/c & \text{if } y \in [0, c] \\ 1 & \text{if } y > c \end{cases}$$

Thus $F_e(y) = \lim_{t \uparrow \infty} \frac{1}{t} \int_0^t 1_{\{Y(t) \le y\}} dt$ = Uniform([0,c])

$= A(y)$

$\Rightarrow$ The renewal process is stationary.

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$\boxed{Q.3}$ Fix $t > 0$. Now recall that given that there are n Poisson points in interval $[0,t]$, then unordered arrivals can be thought of as independent and uniform $(U[0,t])$ points in the interval.

Two cases. ① $t \le 10$.

1 Here, any point arriving at time $v \in [0,t]$ being in the system at time t $= 1 - \frac{v}{t}$. incorrect

2 Thus, prob of a point in system at time

$$t = \int_0^t \left(1 - \frac{v}{t}\right) \cdot \frac{1}{t} \, dv.$$

$$= 1 - \frac{1}{t^2} \int_0^t v \, dv = 1 - \frac{1}{t^2} \left. \frac{v^2}{2} \right|_0^t.$$

$$= 1/2.$$

Thus, if n points arrived in $[0,t]$, then

$$P(M(t) = k \mid \text{arrivals} = n) = \binom{n}{k} \left(\frac{1}{2}\right)^n$$

$$\Rightarrow P(M(t) = k) = \sum_{n=k}^{\infty} \binom{n}{k} \cancel{\left(\frac{1}{2}\right)^n} e^{-2} \frac{\cancel{2^n}}{n!}$$ 3

$$= \frac{e^{-2}}{k!}\sum_{n=k}^{\infty}\frac{1}{(n-k)!} = \frac{e^{-2}}{k!}\underbrace{\sum_{u=0}^{\infty}\frac{1}{u!}}_{e}$$

$$= \frac{1}{e\,k!}$$

Case ② : $t > 10$.

Consider interval $[t-10, t]$ and repeat above calculations. Use the fact that arrivals in $[t-10, t]$ are independent of arrivals in $[0, t-10)$.

Corrected sol^n :

① $= 1 - \frac{t-v}{10}$

Prob customer arriving at v leaves before t

② $$= \int_0^t \left(1 - \frac{t-v}{10}\right)\frac{1}{t}\,dv$$

$$= \int_0^t \frac{1}{t}\,dv - \int_0^t \frac{1}{10}\,dv + \int_0^t \frac{v}{10t}\,dv$$

$$= \frac{v}{t}\Big|_0^t - \frac{v}{10}\Big|_0^t + \frac{v^2}{20t}\Big|_0^t$$

$$= 1 - \frac{t}{10} + \frac{t}{20} = 1 - \frac{t}{20} = p(t)$$

③ $$= \sum_{n=k}^{\infty}\binom{n}{k}\left(p(t)\right)^k \left(1-p(t)\right)^{n-k} e^{-2}\frac{2^n}{n!}$$

$$= [2p(t)]^k \sum_{n=k}^{\infty} \frac{\cancel{n!}}{k!(n-k)!} [2(1-p(t))]^{n-k} \frac{e^{-2} 2^{\cancel{n-k}}}{\cancel{n!}}$$

$$= \frac{[2p(t)]^k e^{-2}}{k!} \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!} [2(1-p(t))]^n}_{e^{2(1-p(t))}}$$

$$= \frac{[2p(t)]^k}{k!} e^{2[\cancel{1}-p(t)\cancel{-1}]} = e^{-2p(t)} \frac{[2p(t)]^k}{k!}$$

$$\Rightarrow \boxed{\begin{array}{l} m(t) \sim \text{Poisson}(2p(t)), \text{ where} \\ \quad p(t) = 1 - \frac{t}{20}. \end{array}}$$

Case 2: We only need to consider $[t-10, t]$, in other words we need to consider interval of length 10 instead of t.

$\Rightarrow m(t) \sim \text{Poisson}(2p(10))$, where

$$p(10) = 1/2.$$