# EE621: Markov Chains and Queueing Systems

## Quiz #3 (20 Marks)

Time:- 11:30 - 1:30 pm                Date:- 27/02/2023

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**Q.1** Consider a DTMC $\{X_n\}_{n\geq 0}$ on $S = \{0,1,\ldots\}$ with TPM $P$.

Prove or disprove: If the DTMC is positive recurrent, then the steady state prob. of going from states in $\tilde{S}$ to $\tilde{S}^c (= S \setminus \tilde{S})$ equals that from $\tilde{S}^c$ to $\tilde{S}$ for every $\tilde{S} \subseteq S$, i.e. $\sum_{i\in\tilde{S}} \sum_{j\in\tilde{S}^c} \pi_i p_{ij} = \sum_{i\in\tilde{S}^c} \sum_{j\in\tilde{S}} \pi_i p_{ij}$. **[5 Marks]**

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**Q.2** Amol, an IITB student, commutes from H12 to LHC every morning and back every evening. He has a cycle, but he is forgetful. He takes his cycle if he has a cycle at the starting point, and either he rembers that or one of his friends reminds him. Prob. of him remembering is $p$ and his friends reminding is $q$, and both these events are independent. If the distance from the hostel to LHC is 3km, then find the avg. distance Amol walks every day. **[7 Marks]**

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**Q.3** Consider a renewal process $\{M(t)\}_{t\geq 0}$ with iid $\exp(\lambda)$ life times, i.e. $f_{X_k}(x) = \lambda e^{-\lambda x}$ $\forall x \geq 0$ & 0 otherwise.

(a) Find $P(M(t) = n)$. **[5 Marks]**

(b) Find $m(t)$. **[3 Marks]**

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# Solutions

$\boxed{Q.1}$

$$\text{LHS} = \sum_{i \in \tilde{S}} \sum_{j \in \tilde{S}^c} \pi_i p_{ij}$$

$$= \sum_{i \in \tilde{S}} \sum_{j \in \tilde{S}^c} \pi_i p_{ij} + \sum_{i \in \tilde{S}^c} \sum_{j \in \tilde{S}^c} \pi_i p_{ij} - \sum_{i \in \tilde{S}^c} \sum_{j \in \tilde{S}^c} \pi_i p_{ij}$$

$$= \sum_{i \in S} \sum_{j \in \tilde{S}^c} \pi_i p_{ij} - \sum_{i \in \tilde{S}^c} \sum_{j \in \tilde{S}^c} \pi_i p_{ij}$$

$$= \sum_{j \in \tilde{S}^c} \underbrace{\sum_{i \in S} \pi_i p_{ij}}_{\pi_j} - \sum_{i \in \tilde{S}^c} \sum_{j \in \tilde{S}^c} \pi_i p_{ij}$$

Exchange using MCT.

$$= \sum_{j \in \tilde{S}^c} \pi_j - \sum_{i \in \tilde{S}^c} \sum_{j \in \tilde{S}^c} \pi_i p_{ij}.$$

$$\text{RHS} :- \sum_{i \in \tilde{S}^c} \sum_{j \in \tilde{S}} \pi_i p_{ij}$$

$$= \sum_{i \in \tilde{S}^c} \pi_i \left[ 1 - \sum_{j \in \tilde{S}^c} p_{ij} \right]$$

$$= \sum_{i \in \tilde{S}^c} \pi_i - \sum_{i \in \tilde{S}^c} \sum_{j \in \tilde{S}} \pi_i p_{ij}.$$

Note that LHS = RHS, and this proves the required.

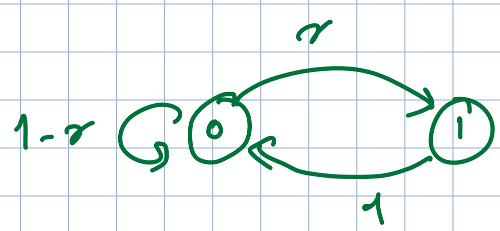
**Q.2** Two approaches are possible.

Approach 1:

Let $X_n$ denote the state just before Amol embarks on $n^{th}$ trip.

$$X_n = 0 \text{ if Amol \& cycle are at the same point}$$
$$= 1 \text{ otherwise.}$$

Then $\{X_n\}_{n\geq 0}$ is the DTMC as shown below:




$r = (1-p)(1-r)$
prob to forget

As long as $r > 0$, the DTMC is finite & irreducible $\Rightarrow$ positive recurrent

$$\pi_0 = (1-r)\pi_0 + \pi_1$$

$$\& \quad \pi_1 = r\pi_0.$$

also, $\pi_0 + \pi_1 = 1$

$$\Rightarrow \quad \pi_0 + r\pi_0 = 1$$

$$\Rightarrow \quad \pi_0 = \frac{1}{1+r} \quad \& \quad \pi_1 = 1 - \frac{1}{1+r} = \frac{r}{1+r}.$$

Thus, fraction of trips for which cycle is at the starting point is $\pi_0 = \frac{1}{1+r}$.

$\Rightarrow$ fraction of trips by walking

$$= r\pi_0 + \pi_1 = \frac{2r}{1+r}.$$

$\Rightarrow$ Thus, avg distance walked per trip

$$= \frac{6r}{1+r}$$

$\because$ he makes two trips a day,

avg distance walked per day

$$= \frac{12r}{1+r}.$$

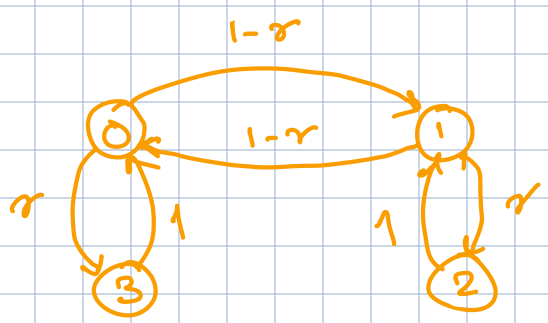
Approach 2 :

$X_n = 0$ if Amol & cycle, both at hostel

$= 1$ if Amol & cycle both at LHC

$= 2$ if Amol at hostel & cycle at LHC

$= 3$ if Amol at LHC & cycle at hostel.




DTMC

$$\pi_3 = r\pi_0\,, \quad \pi_2 = r\pi_1$$

$$\pi_0 = (1-r)\pi_1 + \pi_3$$

$$= (1-r)\pi_1 + r\pi_0$$

$\Rightarrow \quad \pi_0 = \pi_1 \quad \Rightarrow \quad \pi_2 = \pi_3 = r\pi_0.$

$$x_0 + x_1 + x_2 + x_3 = 1$$

$$2x_0 + 2r x_0 = 1 \implies x_0 = \frac{1}{2(1+r)}.$$

$$\implies \quad x_0 = x_1 = \frac{1}{2(1+r)} \quad \& \quad x_2 = x_3 = \frac{r}{2(1+r)}.$$

$$\Pr(\text{Amol walks}) = (x_2 + x_3) + r(x_0 + x_1)$$

$$= \frac{2r}{1+r}.$$

Avg distance per trip walked $= \dfrac{6r}{1+r}$

Avg distance per day walked $= \dfrac{12r}{1+r}.$

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$\boxed{Q.3}$ $\quad Z_n = \sum_{k=1}^{n} X_k$

$Z_n$ is the sum of $n$ i.i.d. $\exp(\lambda)$ random variables. Show that

$$Z_n \sim \text{Gamma}(n, \lambda), \text{ i.e.}$$

$$f_{Z_n}(z) = \frac{\lambda^n z^{n-1}}{(n-1)!} e^{-\lambda z} \quad \text{for } z \geq 0$$

$$= 0 \quad \text{otherwise.}$$

$$P(N(t) = n) = P(Z_n \leq t, Z_{n+1} > t).$$

$$= \int_0^t P(Z_n \le t, Z_{n+1} > t \mid Z_n = u)\, f_{Z_n}(u)\,du$$

$$= \int_0^t P(Z_{n+1} > t \mid Z_n = u)\, f_{Z_n}(u)\,du$$

$$= \int_0^t P(\overbrace{Z_n + X_{n+1}}^{\text{independent}} > t \mid Z_n = u)\, f_{Z_n}(u)\,du$$

$$= \int_0^t P(X_{n+1} > t-u)\, f_{Z_n}(u)\,du$$

$$= \int_0^t e^{-\lambda(t-\cancel{u})}\, \frac{\lambda^n u^{n-1}}{(n-1)!}\, e^{\cancel{-\lambda u}}\,du$$

$$= \frac{e^{-\lambda t}\lambda^n}{(n-1)!} \int_0^t u^{n-1}\,du = \frac{e^{-\lambda t}\lambda^n}{(n-1)!} \left.\frac{u^n}{n}\right|_0^t$$

$$= e^{-\lambda t}\,\frac{(\lambda t)^n}{n!}.$$

$N(t)$ is a Poisson random variable with parameter $\lambda t$.

(b) $m(t) = E[N(t)] = \lambda t$.