EE621: Markov Chains & Queueing System

Quiz # 6 (20 Marks)

Date: 29/04/2024 Time: 6-7:30pm

Q.1 Let $\{M(t)\}_{t\geq 0}$ be an ordinary renewal process with non lattice, square integrable life-time distr. $F(\cdot)$. Find $\lim_{t\uparrow\infty} E[Y(t)]$, where

$$Y(t) = Z_{M(t)+1} - t.$$

7 Marks

Q.2 Let $\{M(t)\}_{t\geq 0}$ be an ordinary renewal process with non-lattice life-time distr. Uniform([0,1]). Show that $\forall t \in [0,1]$,

$$m(t) = e^{t} - 1.$$

Hint: You may want to write down renewal equation for $m(t)$.

6 Marks

Q.3 Consider a single queue with Poison(λ) arrivals. The service times are $\exp(\mu)$. However, after finishing service, the customer starts another independent $\exp(\mu)$ service w.p. p and leaves w.p. $(1-p)$. The decision to stay or leave is independent of the past. Find avg. waiting time in the system.

7 Marks

EE621: Markov Chains & Queueing Systems

Quiz #7 (20 Marks)

Date: 29/04/2024 Time: 7:30-9 pm

Q.1 Give an example of a positive recurrent CTMC whose EMC is Null recurrent.

6 Marks

Q.2 Consider M/M/c/c system with Poison(λ) arrivals and iid exp(μ) service times for all the servers. Let $A_c(t)$ denote the number of customers that are blocked until time t.

(a) Find $\lim_{t \uparrow \infty} \frac{A_c(t)}{t}$ using Renewal theory.

(b) Find fraction of arrivals blocked in the system.

5 + 2 Marks

Q.3 Consider an expert who starts his session at time 0. People come to seek his advice as per Poisson(λ) process. Each question needs exp(μ) time to address it. The expert leaves when there are no more people waiting. Find the expected time for which the expert answers the questions.

Solutions Quiz #6

We know that $Y(t) \to X_e$ in distribution as $t \uparrow \infty$, where X_e is any r.v. with distribution F_e. But convergence in distribution does not imply convergence in expection, i.e.

$$Y(t) \to X_e \text{ in distribution}$$

$$\not\Rightarrow E[Y(t)] \to E[X_e].$$

Recall $F_e(y) = \frac{1}{EX_2} \int_0^y F^c(u)\, du$

So, $E[Y(t)] = \int_0^\infty P(Y(t) > y)\, dy$

$$\lim_{t \uparrow \infty} E[Y(t)] = \lim_{t \uparrow \infty} \int_0^\infty P(Y(t) > y)\, dy$$

$$= \int_0^\infty \underbrace{\lim_{t \uparrow \infty} P(Y(t) > y)}_{F_e^c(y)}\, dy \quad 1$$

$$= \int_0^\infty \left(1 - \frac{1}{EX_2} \int_0^y F^c(u)\, du\right) dy$$

$$= \int_0^\infty \left(\frac{1}{EX_2} \int_0^\infty F^c(u)\, du - \frac{1}{EX_2} \int_0^y F^c(u)\, du \right) dy$$

$$= \int_0^\infty \frac{1}{EX_2} \int_y^\infty F^c(u)\, du \; dy$$

$$= \frac{1}{EX_2} \int_0^\infty \int_y^\infty F^c(u)\, du$$

$$= \frac{1}{EX_2} \underbrace{\int_0^\infty u F^c(u)\, du}_{E[X_2^2]/2}.$$

This approach fails unless exchange of limit & integral is properly justified in 1.

Right way: Note that

$$E[Y(t)] = \int_0^\infty P(Y(t)>y)\, dy. \quad -(*)$$

Consider

$$P(Y(t)>y) = \underbrace{P(Y(t)>y, X_1>t)}_{1} + \underbrace{P(Y(t)>y, X_1\le t)}_{2}$$

$$① = P(X_1 > t+y, X_1 > t) = P(X_1 > t+y). = F^c(t+y)$$

$$② = \int_0^t P(Y(t-u) > y)\, dF(u)$$

Substituting ① & ② in (*)

$$E[Y(t)] = \int_{y=0}^{\infty} F^c(t+y)\, dy + \int_{y=0}^{\infty} \int_{u=0}^{t} P(Y(t-u) > y)\, dF(u)\, dy.$$

$$= \int_t^{\infty} F^c(v)\, dv + \int_{u=0}^{t} \left[\int_{y=0}^{\infty} P(Y(t-u) > y)\, dy \right] dF(u)$$

$$= \int_t^{\infty} F^c(v)\, dv + \int_{u=0}^{t} E[Y(t-u)]\, dF(u).$$

Let $H(t) = E[Y(t)]$ & $a(t) = \int_t^{\infty} F^c(v)\, dv$

Thus, we get a renewal equation:

$$H = a + H * F$$

Sol^n for this is: $H = a + a * m$

Thus,

$$E[Y(t)] = \int_t^{\infty} F^c(v)\, dv + \int_0^t \int_{t-u}^{\infty} F^c(v)\, dv\, dm(u)$$

Note that $a(t)$ is monotone decreasing and non-negative. Thus, $a(t)$ is DRI. So by KRT

$$\lim_{t \uparrow \infty} E[Y(t)] = \underbrace{\lim_{t \uparrow \infty} \int_t^{\infty} F^c(v)\, dv}_{=0} + \frac{1}{EX_2} \int_0^{\infty} \int_u^{\infty} F^c(v)\, dv\, du$$

as $E[X_2] < \infty$.

$$= \frac{1}{E X_2} \int_{u=0}^{\infty} \int_{v=u}^{\infty} F^c(v)\,dv\,du$$

$$= \frac{1}{E X_2} \int_{v=0}^{\infty} \int_{u=0}^{v} du\; F^c(v)\,dv$$

$< \infty$ as X_2 is square integrable.

$$= \frac{1}{E X_2} \int_{v=0}^{\infty} v F^c(v)\,dv = \frac{E[X_2^2]}{2E[X_2]}.$$

$\boxed{Q.2}$ Consider $m(t) = E[M(t)]$

$$\Rightarrow \quad m(t) = \sum_{k=1}^{\infty} P(M(t) \geq k)$$

Now, $P(M(t) \geq k) = \underbrace{P(M(t) \geq k, X_1 > t)}_{=0 \;\forall k \geq 1.} + P(M(t) \geq k, X_1 \leq t)$

$$= \int_0^t P(M(t-u) \geq k-1)\,dF(u) \quad \forall k \geq 1.$$

$$\Rightarrow \quad m(t) = \sum_{k=1}^{\infty} \int_0^t P(M(t-u) \geq k-1)\,dF(u)$$

$$= \int_0^t \left[\sum_{k=1}^{\infty} P(M(t-u) \geq k-1)\right] dF(u)$$

$$= \int_0^t (1 + m(t-u))\,dF(u)$$

$$= F(t) + \int_0^t m(t-u)\,dF(u) \quad \forall t.$$

For $t \in [0,1]$

$$m(t) = t + \int_0^t m(t-u)\,du.$$

To verify $m(t) = e^t - 1$, substitute and check.

Consider
$$t + \int_0^t (e^{t-u} - 1)\, du$$
$$= \cancel{t} + e^t \int_0^t e^{-u} du - \cancel{t}$$
$$= e^t \left[-e^{-u} \Big]_0^t \right] = e^t \left[1 - e^{-t} \right] = e^t - 1.$$

This proves the required.

———— x ———— x ———— x ————

$\boxed{Q.3}$ Note that the service time for a customer can be written as $\sum_{n=1}^{N} S_n$, where $\{S_1, S_2, \ldots\}$ are iid $\exp(\mu)$ and $N \sim \text{Geometric}(p)$.

Moreover, N and $\{S_1, S_2, \ldots\}$ are independent.

We had shown that $\sum_{n=1}^{N} S_n \sim \exp((1-p)\mu)$

Thus, this is same as $M/M/1$ system with Poisson(λ) arrivals and service times iid $\exp((1-p)\mu)$.

Define, $\rho = \frac{\lambda}{\mu(1-p)}$ & as shown in the class the steady state distribution is

$$\pi_n = (1-\rho)\rho^n \quad \forall\, n \geq 0 \quad (\text{if } \rho < 1)$$

And avg. # of customers $= \sum_{n=1}^{\infty} n(1-\rho)\rho^n$

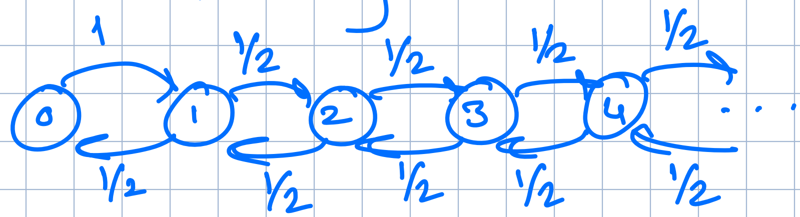
$$= \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$$

By Little's law: avg delay $= \frac{1}{\mu - \lambda}$.

—— x —— x —— x —— x ——

Solutions Quiz #7

Consider the following DTMC:



Use Foster's criteria to show that this is a Null recurrent DTMC.

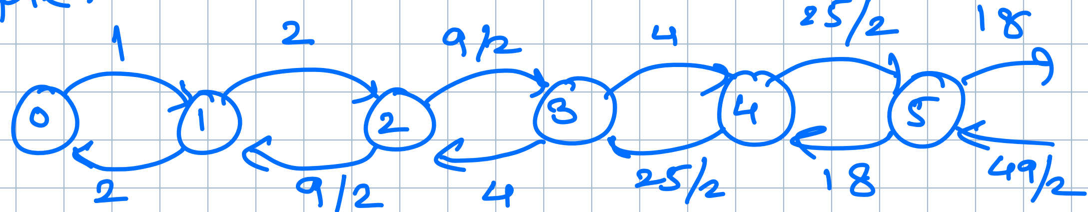
Now, lets solve for $\bar{u} = \bar{u}P$.

Note that $\bar{u} = [\frac{1}{2} \; 1 \; 1 \; 1 \; \ldots]$ is a sol^n.

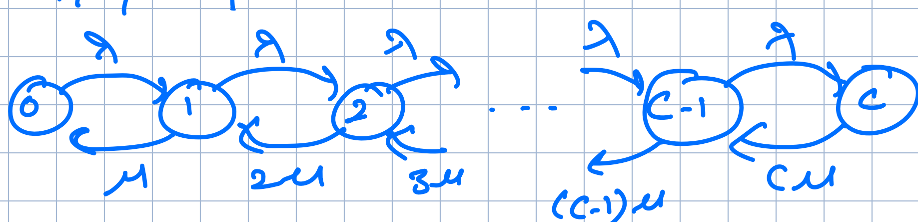
We need $\sum_{i \in S} \frac{u_i}{a_i} < \infty$ for positive recurrent.

choose $a_i = (i+1)^2$, $\forall i \in S$ satisfies required cond.

Thus, the CTMC (shown below) provides an example.



$\boxed{Q.2}$ M/M/1/C/C queue



Sojourn time in state C is $\exp(C\mu)$.

Consider renewal process with consecutive visits to state C. The expected life-times is $\frac{1}{\pi_C a_C} = \frac{1}{C\mu\pi_C}$

The expected reward in a life time

$= E[\#$ of Poisson arrivals in duration $\exp(C\mu)]$

$= \frac{\lambda}{C\mu}$

By RRT, $\lim_{t\uparrow\infty} \frac{\Lambda_C(t)}{t} = \lambda\pi_C$ w.p.1.

Now, $\lim_{t\uparrow\infty} \frac{\Lambda_C(t)}{\Lambda(t)} = \pi_C$ w.p.1

(verifying PASTA)

Recall that we have shown in the class:

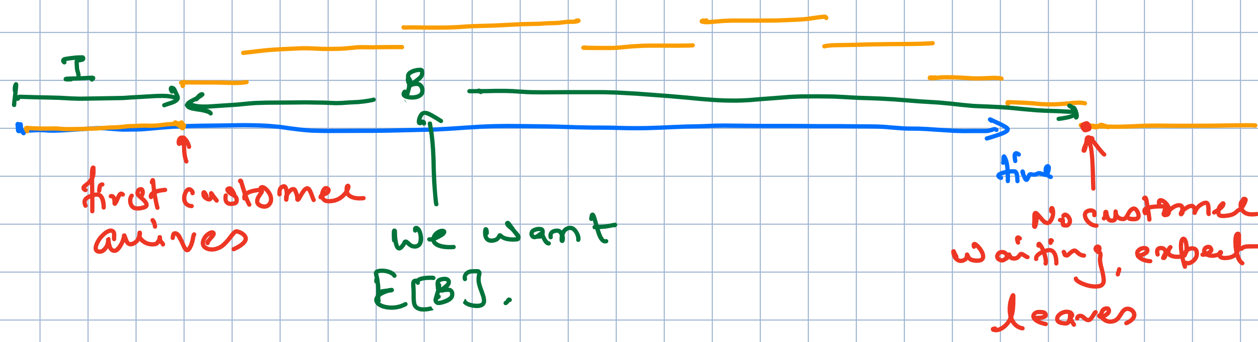
$$\pi_i = \frac{\rho^i}{i!}\pi_0 \quad \forall\ i = 0, 1, \dots, C.$$

$$\pi_0 \sum_{i=0}^{C} \frac{\rho^i}{i!} = 1 \Rightarrow \pi_0 = \frac{1}{\sum_{i=0}^{C} \rho^i/i!}$$

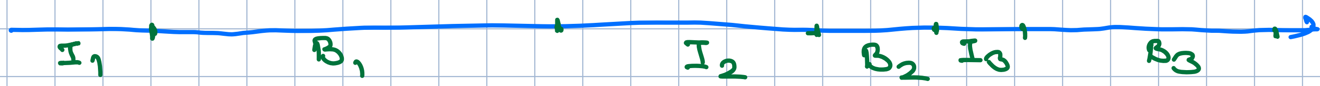
$$\Rightarrow \pi_i = \frac{\rho^i}{\sum_{j=0}^{C} \rho^j/j!}, \text{ where } \rho = \frac{\lambda}{\mu}.$$

——— x ——— x ——— x ———

Q.3 A sample path looks something like



Consider M/M/1 queue. Note that sample path of M/M/1 queue looks like:



Note that this is an alternating renewal process with life times $X_k = I_k + B_k$.

Let $R_k = B_k \quad \forall\, k = 1, 2, \ldots$

Thus, by RRT,

$$\underbrace{\text{fraction of time } \geq 1 \text{ customer in the system}}_{= (1-\pi_0) \text{ as shown in the class (if } \lambda < \mu)} = \frac{E[B]}{E[I] + E[B]} \quad \text{w.p. } 1$$

$E[I] = \frac{1}{\lambda}$.

$\rightarrow \pi_0 = 1 - \rho$, where $\rho = \frac{\lambda}{\mu}$.

$\Rightarrow$ If $\lambda < \mu$, then

$$\frac{\lambda}{\mu} = \frac{E[B]}{1/\lambda + E[B]}$$

$$\Rightarrow \quad \frac{\lambda}{\mu}\left[\frac{1}{\lambda} + E[B]\right] = E[B]$$

$$\Rightarrow \quad \frac{1}{\mu} + \frac{\lambda}{\mu} E[B] = E[B]$$

$$E[B] = \frac{1}{\mu} \div \left[1 - \frac{\lambda}{\mu}\right]$$

$$= \frac{1}{\cancel{\mu}(\mu-\lambda)} = \frac{1}{\mu-\lambda}.$$

Also, when $\lambda \geq \mu$, the CTMC is not positive $\Rightarrow$ the expected return time to 0 is ∞, and since $E[I] = \frac{1}{\lambda}$, $E[B] = \infty$.

— x — x — x —