EE 621

Markov Chains And Queueing Systems

- Discrete Time Markov Chains (DTMC)
- Continuous Time Markov Chains (CTMC's)
- Renewal processes
- Queueing Systems

DISCRETE TIME MARKOV CHAINS (DTMC's)

⇒ STOCHASTIC PROCESS / RANDOM PROCESS

↓

collection of Random variables

→ Random variable :-

↳ probability space ⇒ $(\Omega, \mathcal{F}, \mathbb{P})$

- Ω: sample-space (set of all possible outcomes)
- $\mathcal{F}$: (sigma-algebra) / Event space (collection of events of interest)
- $\mathbb{P}$: probability measure : $\mathbb{P}: \mathcal{F} \to [0,1]$ (function)

Event :- A subset of Ω (collection of 'some' possible outcomes)

$P(\Omega) \longrightarrow$ all possible events one can have.

$\mathcal{F} \longrightarrow$ collection of some events of interest. ⟶ we put a restriction on events in $\mathcal{F}$

↳ needs to be a $\boxed{\sigma\text{-field}}$

- ⇒ $\Omega \in \mathcal{F}$
- ⇒ if $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
- ⇒ if $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_n \in \mathcal{F}$

⇒ σ-field

There can be many σ-fields on a set

$\mathbb{P}: \mathcal{F} \longrightarrow [0,1]$ ⟹ each 'event' is associated a number.

⟹ assigned to EVENTS and not OUTCOMES.

↳ $\mathbb{P}(\Omega) = 1$

['Event A has occured' when outcome corresponds to the event being occured.]

↳ If $A_1, A_2, \dots \in \mathcal{F}$ and

$A_i \cap A_j = \phi \;\; \forall \; i \neq j$, Then,

$$\mathbb{P}(\cup A_n) = \sum \mathbb{P}(A_n)$$

⇒ Experiment with outcome uncertainity ⇒ $(\Omega, \mathcal{F}, \mathbb{P})$: probability space.

ω

Probability - Concepts Revision

⇒ Ω: Sample Space - set of all possible outcomes.

⇒ Event :- a collection of outcomes. $A \subseteq \Omega$.

⇒ event A occurs if outcome $\omega \in A$.

⇒ $\mathcal{F}$: collection of all events of interest. (It need not be $P(\Omega)$)

⇒ $\mathcal{F}$ has to be a σ-algebra (or σ-field)

- 1> $\Omega \in \mathcal{F}$ [certain event is certainly of interest]
- 2> If $A \in \mathcal{F}$, A^c MUST $\in \mathcal{F}$ [$\Omega^c = \phi$ also $\in \mathcal{F}$]
- 3> If $A_1, A_2, \ldots \in \mathcal{F}$, $\left(\bigcup_{i=1}^{\infty} A_n\right) \in \mathcal{F}$

⇒ $(\Omega, \mathcal{F})$ - Measurable space ⟶ $\mathbb{P}$ probability measure used to measure the same.

⇒ $\mathbb{P}$ $\mathcal{F} \to [0,1]$, interpreted as the probability that an event occurs.

(function) [can't be assigned to events not in $\mathcal{F}$.]

- ① $P(\Omega) = 1$
- ~~[illegible]~~
- ② If $A_1, A_2, \ldots \in \mathcal{F}$ s.t. $A_i \cap A_j = \phi \;\; \forall\, i \neq j$, Then

$$\boxed{P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)}$$ [σ-additivity]

⇒ $|\mathcal{F}| = 2^{|\Omega|}$ i.e. $|P(\Omega)| > |\Omega|$; if $\Omega = \{1, 2, \ldots\}$ $|\Omega| \Rightarrow$ countable ∞

if $\mathcal{F}$ is $P(\Omega)$

($|P(\Omega)| \equiv |\mathbb{R}| \to$ uncount. ∞) one-one with int.

if $\Omega = \mathbb{R}$, $|P(\Omega)| > |\mathbb{R}|$ → uncountably ∞-set ⇒ $\boxed{\text{problems}}$

⇒ Take a std. object → develop theory → MAPS

your object $\xrightarrow{s_1}$ std. object ⟹ apply developed theory, obtain results

(map the results back) $\xleftarrow{s_2}$

$(\mathbb{R}, \mathcal{B})$

set of real no's ; Borel σ-algebra

set of all possible outcomes ⇒ Ω
set of events of our interest ⇒ $\mathcal{B}$

Random Variable :- MAPS :- $\boxed{\Omega \to \mathbb{R}}$

⇒ A R.V. is a function, $f : \Omega \to \mathbb{R}$ [assign a real no. for every outcome in your experiment]

(randomness quantification needs to be replicated)

$(\Omega, \mathcal{F}) \rightleftarrows (\mathbb{R}, \mathcal{B})$ (std. object)

$(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow{\times} (\mathbb{R}, \mathcal{B}, \mathbb{P}_x)$

$\mathbb{P}_x(B) = \mathbb{P}(\omega \mid X(\omega) \in B)$

$\in \mathcal{B}$ (borel σ-field)

$$\boxed{\mathbb{P}_x(B) = \mathbb{P}(\omega : X(\omega) \in B)}$$

⇒ X is called Borel-measurable if $\forall\ B \in \mathcal{B}$, $\{w : X(w) \in B\} \in \mathcal{F}$

⇒ every event in std. probability space must be back-convertible to our own probability space.

(#) A R.V. X is a $\boxed{\text{Borel-Measurable function}}$ from $\boxed{\Omega \to \mathbb{R}}$.

→ Result :- To completely specify P_X, it is enough to specify $\boxed{P_X((-\infty, x]) \ \forall\ x \in \mathbb{R}}$

⇒ Define $\boxed{F_X(x) = P_X((-\infty, x])}$ ⟶ Cumulative Distribution Function (CDF)

↓ it is a parametrized functional form. → eg.- $1-e^{-\lambda x}$, $\sum \frac{e^{\lambda x}}{\lambda !}$

⇒ 2 or more RV's ⟶ Joint Distributions $\xrightarrow[\text{find}]{\text{can}}$ Marginals (only if independent)

⇒ (n RV's ⟶ n^{th}-order joint distrib^n)

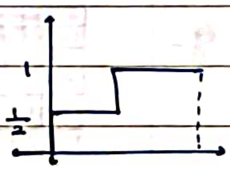
→ $X_1, X_2, \ldots$ or $\{X_n\}_{n \geq 1}$; e.g.- tossing a coin countably ∞-no. of times

→ $w = (w_1, w_2, \ldots)$ $w_n \in \{H, T\}$ $\forall n$ outcome- random sequence of H and T.

→ $X_n(w) = 1$ if $w_n = H$ and 0 o/w ⟶ X_n depends on n^{th}-coin toss only.

⇒ $P(X_1=1, X_2=1, \ldots) = 0$; $\underbrace{P(X_n=1) = 1/2}$; $P(X_n=0) = 1/2$

↓ doesn't give ANY INFO. — distrib^n f^{n}c is easy to calculate



Collection of ∞- RV's

→ To specify probability law for ∞-countable/uncountable collection of RV's ; we must specify all finite-dimensional joint distributions.

⇒ $P(X_{n_1} \leq x_{n_1}, X_{n_2} \leq x_{n_2}, \ldots X_{n_k} \leq x_{n_k})$ $\forall$ $(n_1, \ldots, n_k)$ $\forall$ $k = 1, 2 \ldots$ $\forall$ $(x_{n_1}, x_{n_2}, \ldots)$

(1D/2D/3D/4D. . . .)

↳ i.i.d. RV's ⟶ independent & identically distributed.

↳ Now, if we know marginals, we know all finite-dimensional distrib^n

STOCHASTIC/RANDOM PROCESSES

→ collection of RV's defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$

→ $\{X_t\}_{t \in I}$ — I is countably ∞ (discrete time stoc. Proc.)
— I is uncountable (continuous time stoc. proc.)

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Discrete Time Markov Chains (DTMC's)

① [STOCHASTIC PROCESSES]

→ Discrete-time :- countably-∞ RV's i.e. $\{X_n\}_{n\geq 0}$

→ Markov, Chains ⟹ Let S denote some countable set. Then, $\{X_n\}_{n\geq 0}$ is a CHAIN if $\boxed{P(X_n \in S) = 1 \ \forall n}$

→ If we have an instrument with a fixed precision to measure X_n, it can take values in a countable set only, & then X_n is called a chain.

⇒ ONLY countable values are possible for X_n.

→ We say that $\{X_n\}_{n\geq 0}$ satisfies Markov property if:

(S: state-space)

$$P(X_{n+k} = j \mid X_n = i,\ X_u, u<n) = P(X_{n+k} = j \mid X_n = i) \quad \forall\ k \geq 1,\ \forall\ i,j \in S$$

$(n+k)^{th}$ - time instance, probability that I'll be in state j given at X_n we were in i & given the entire past before n.

Past & Future are Independent, given present.

future is conditionally independent of past

eg.-

state is in which box the particle is at an instant of time.

⇒ $(n+k)^{th}$ minute in j^{th} box, given it was in i^{th} block at n and entire trajectory before it is given

⇓

HOW IT REACHES n^{th} - time instant DOES NOT MATTER

Future Trajectory is only a f^n c of its current place & how it reaches over there doesn't matter ←

$$P(X_{n+k} = j \mid X_n = i,\ X_{n-2} = i_2,\ X_{n-5} = i_3) = P(X_{n+k} = j \mid X_n = i)$$

→ before n : Past   n: current   after n: future.

⇒ Define :

$$P_{ij}^{(k)}(n) = P(X_{n+k} = j \mid X_n = i)$$

(from i to j in k-steps)

Probability of visiting state j starting from i (at time 'n') in k-steps

$y = \{y_i\}_{i \in E}$ and $y^T = x^T A$ is defined by $\boxed{y_i = \sum_{k \in E} x_k a_{ki}}$

$\boxed{z = \{z_i\}_{i \in E} \text{ given by } z = Ax \text{ is defined by } z_i = \sum_{k \in E} a_{ik} z_k}$

We only consider Time-Homogenous Markov Chains

$$P_{ij}^{(k)}(n) = P_{ij}^{(k)}(n+1) \quad \forall \; n \geq 0$$

i.e. Time has NO role in the transition from state i to state j

$\Rightarrow$ $P(X_{n+k} = j \mid X_n = i) = P(X_k = j \mid X_0 = i) = P_{ij}^{(k)}$

ie. $P_{ij}^{(k)}(n) = P_{ij}^{(k)}(0) = \boxed{P_{ij}^{(k)} = \mathbb{P}(X_k = j \mid X_0 = i)}$

(k-step transition probability)

$\Rightarrow$ Define :- $P_{ij} = P_{ij}^{(1)} \longrightarrow$ next-state transition

(ONE-STEP transition probability)

$\Rightarrow$ Define :-

$$P = \begin{array}{c|c} \text{from} \backslash \text{to} & j \\ \hline i & \begin{bmatrix} & & \\ & P_{ij} & \\ & & \end{bmatrix}_{|S| \times |S|} \end{array}$$

rows: from
columns: to

$\Rightarrow$ TPM :- Transition-Probability Matrix

and

$$P^{(k)} = \begin{array}{c|c} & j \\ \hline i & \begin{bmatrix} & & \\ & P_{ij}^{(k)} & \\ & & \end{bmatrix}_{|S| \times |S|} \end{array} \Rightarrow \text{k-state TPM}$$

$\rightarrow$ To specify the probability laws, we need all finite-order distributions.

$\Rightarrow$ $\boxed{P(X_n = j) = \sum_{i \in S} \mathbb{P}(X_n = j, X_0 = i)}$ $\qquad \boxed{P(B) = \sum_n P(B \cap A_n)}$

$$= \sum_{i \in S} P(X_n = j \mid X_0 = i) \cdot \mathbb{P}(X_0 = i) \qquad [P(X_0 = i) > 0]$$

Let α_i denote $P(X_0 = i)$; $\alpha = [\alpha_i]_{i \in S}$ [PMF defined on S]

$\alpha \to$ (PMF on S) $\to$ initial state distribution

$$\boxed{P(X_n = j) = \sum_{i \in S} \alpha_i P_{ij}^{(n)}}$$

$P(X_n = j) \to$ Marginal of n^{th}-RV

$\alpha_i \to$ initial states

$P_{ij}^{(n)} \to$ n-step TPM

(we need to know the TPM for every n)

eg.- $p_{ij}^{(2)} = \mathbb{P}(X_2=j \mid X_0=i) = \sum_{u \in S} P(X_2=j, X_1=u \mid X_0=i)$ (partition w.r.t. all diff. values possible)

$= \sum_{u \in S} P(X_2=j \mid \underset{\text{present}}{X_1=u}, \underset{\text{past}}{X_0=i}) \cdot P(X_1=u \mid X_0=i)$

future | present past

(Markov property)

$= \sum_{u \in S} P(X_2=j \mid X_1=u) \cdot P(X_1=u \mid X_0=i)$

$= \sum_{u \in S} (P_{uj}) \cdot (P_{iu})$

i.e. $\boxed{p_{ij}^{(2)} = \sum_{u \in S} P_{iu} \cdot P_{uj}}$ ⇒ Land somewhere in the I^{st} step, look at all steps possible & sum over them

[MATRIX MULTIPLICATION]

⇒ so, in general: $\boxed{P^{(n)} = P^n}$ (INDUCTION)

[n-step TPM] ↓

No additional info. is required

⇒ Initial state probability :- $\alpha = [\alpha_i]_{i \in S}$ where $\alpha_i = \mathbb{P}(X_0=i)$

where α is the PMF on S.

→ To define a stochastic process → need to find all finite order distributions

⇒ Marginals :- $\boxed{P(X_n=j) = \sum_{i \in S} p_{ij}^{(n)} \alpha_i}$

$\boxed{V_n^T = V_0^T P^n}$

we need α [I.S.P.] and P [TPM] $\qquad p_{ij}^{(n)} \Rightarrow (i,j)^{th}$ entry of $P^{(n)} = P^n$

⇒ 2-D Joint Distribution

$\mathbb{P}(X_{n_1}=j_1, X_{n_2}=j_2)$ WLOG $n_1 < n_2$; $j_1, j_2 \in S$

$= \sum_{i \in S} \mathbb{P}(X_{n_1}=j_1, X_{n_2}=j_2, X_0=i)$

$A \cap B \mid C = \mathbb{P}(B \mid A \cap C) \cdot \mathbb{P}(C \mid A)$

$= \sum_{i \in S} \mathbb{P}(X_{n_1}=j_1, X_{n_2}=j_2 \mid X_0=i) \cdot \underbrace{\mathbb{P}(X_0=i)}_{\alpha_i}$

$(n_2 > n_1)$

$= \sum_{i \in S} \alpha_i \, P(X_{n_2}=j_2 \mid X_0=i, X_{n_1}=j_1) \cdot P(X_{n_1}=j_1 \mid X_0=i)$

← latest time instance before n_1

$= \sum_{i \in S} \alpha_i \, p_{ij_1}^{(n_1)} \cdot p_{j_1 j_2}^{(n_2-n_1)}$

no impact as it is just the past

$P(X_{n_2}=j_2 \mid X_{n_1}=j_1, X_0=i) = P_{j_1 j_2}^{(n_2-n_1)}$

$$\boxed{P(X_{n_1}=j_1, X_{n_2}=j_2) = \sum_{i \in S} \alpha_i \, P_{ij_1}^{(n_1)} \cdot P_{j_1 j_2}^{(n_2-n_1)}}$$

(#) α and P are completely sufficient to describe probability laws of DTMC's. ⟶ initial state probability
⟶ TPM for the I^{st} transition

⇒ DTMC parameters moreover depends on P; impact of initial state on X_n for large n is less.

Properties of P (T.P.M.) ⇒ STOCHASTIC MATRIX

(1) $P_{ij} \geq 0 \;\; \forall \; i, j$ (+ve matrix)

(The dimension might be finite or ∞-ly countable)

(2) $\sum_{j \in S} P_{ij} = 1$ [starting from i, what is the probability that you end up in some state (j) in S?]

Each row is a probability MASS function on S

(3) The matrix has 1 as a right eigen-value

VISUAL REPRESENTATION :- Weighted Directed Graph

$G = (V, A)$ — V: vertices, A: arcs; $w: A \to \mathbb{R}$ (arc)

→ $V = S$ (each state is a node in the graph)

→ $w(i,j) = P_{ij}$; we only draw arcs that have +ve weights

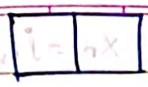
eg.- Given a sequence of fair coin tosses, find probability that HH appears before TH.

eg. - a sample path : T T H T H H H T H T T H

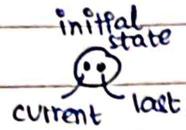
→ first two tosses must be HH otherwise we can never have this situation

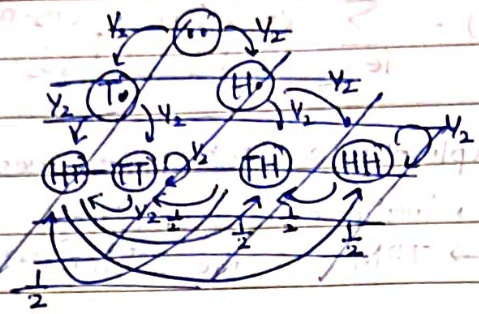
⇒ HH before HT. ⟶ T T T T T . . . T H H

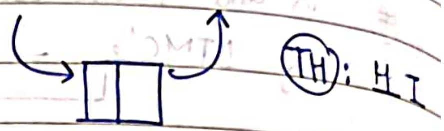
Formulation As a Markov Chain :-



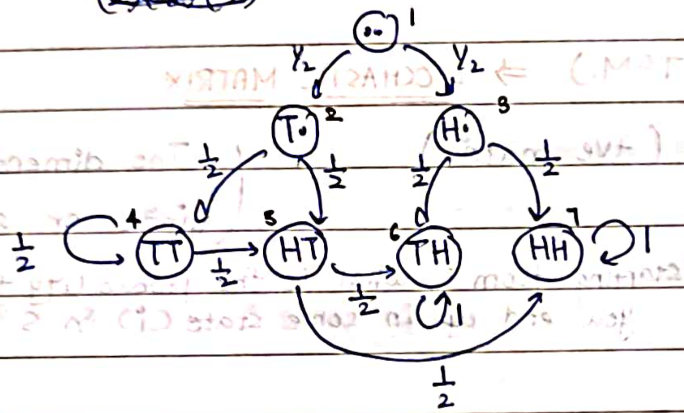
move this window one by one







(HH), (TH) is a stopping condition



(pictorial representation of a DTMC)

⇒ Once you reach HH or TH, you never leave from there
↓
I^{st} visit HH before TH.

⇒ Let P_k denote probability of visiting state 7 (HH) starting from k

$P_7 = 1$ $\quad P_6 = 0$ $\quad$ ~~$P_2 = \left(\frac{1}{2}\right)P_5 + \left(\frac{1}{2}\right)P_4$~~ $\quad P_1 = \frac{1}{2}(P_2 + P_3)$

$P_3 = 1/2$ $\quad$ ~~$= \frac{1}{8} + \frac{1}{4} = 3/8$~~ $\quad P_2 = \frac{1}{2}(P_4 + P_5)$

$P_5 = \frac{1}{2}$ $\quad$ ~~$P_1 = \left(\frac{1}{2}\right)P_2 + \left(\frac{1}{2}\right)P_3 = \frac{1}{4} + \frac{3}{16} = \frac{7}{16}$~~ $\quad P_3 = \frac{1}{2}(P_6 + P_7)$

$P_4 = \frac{1}{2}$

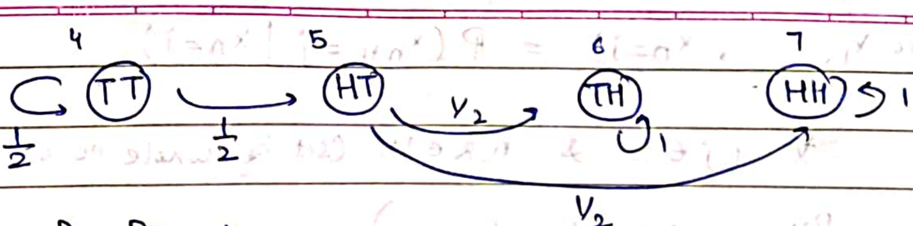
$$P_4 = \frac{1}{2}P_4 + \frac{1}{2}P_5 \Rightarrow P_4 = P_5 = \frac{1}{2}$$

$$P_3 = \frac{1}{2} \qquad P_2 = \frac{1}{2}P_5 + \frac{1}{2}P_4 = \frac{1}{2}$$

$$P_1 = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) = 1/2 \qquad \sum 1 \cdot P_1$$

as here $\alpha_1 = 1$ others 0

first two coin tosses do NOT play a role ; define $\alpha_i = 1/4$ for each state & call it the initial probability matrix



(Markov chain)

$P_4 = P_5 = \frac{1}{2}$

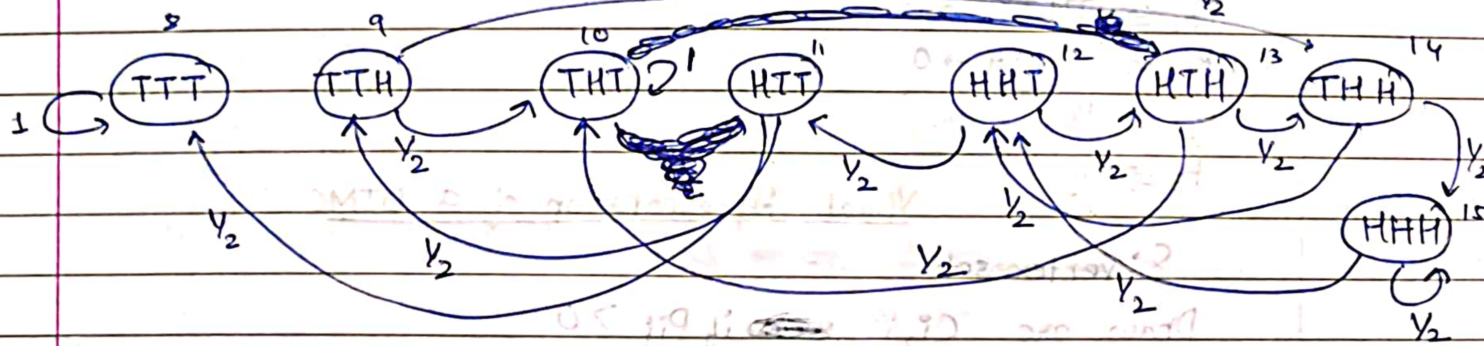
$P_6 = 0 \quad P_7 = 1$

Probability of visiting 7 $= \alpha_4 P_4 + \alpha_5 P_5 + \alpha_6 P_6 + \alpha_7 P_7$

$\left(= \sum_{i \in S} \alpha_i P_{ij} \right.$ $\quad j \to 7$, $i \to (4|5|6|7)$

$= \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2} + 1\right) = \frac{1}{2}$

Que:) Find probability of visiting TTT before THT.



$P_8 = 1 \qquad P_{10} = 0$

$P_9 = \frac{1}{2} P_{11} \qquad P_{11} = \frac{1}{2} P_{12} \qquad P_{12} = \frac{1}{2} P_{15} + \frac{1}{2} P_{14} \qquad P_{13} = \frac{1}{2} P_{12}$

$\Rightarrow \quad P_9 = \frac{1}{2} \quad P_{11} = \frac{1}{2} \quad P_{12} = 1 \quad P_{13} = \frac{1}{2} \quad P_{14} = 1 \quad P_{15} = 1$

so, $P = \frac{1}{8}\left(\frac{1}{2} + \frac{1}{2} + 1 + \frac{1}{2} + 1 + 1 + 1\right) = \frac{1}{8}\left(5 + \frac{1}{2}\right) = \frac{1}{8}\left(\frac{11}{2}\right) = \frac{11}{16}$

---x---

$P_8 = 1 \quad P_9 = 3/5 \quad P_{10} = 0 \quad P_{11} = 1/5 \quad P_{12} = 2/5 \quad P_{13} = 2/5 \quad P_{14} = 1/5 \quad P_{15} = 2/5$

(TTT → TTH → THT → HTT → HHH → HHT → HTH → THH)

Seq. of coin tosses :- HTTH . . . . . .

Let $\{C_n\}_{n \geq 1}$ denote outcome of the coin toss experiment

$C_n \in \{H, T\}$ is the outcome of n^{th}-toss $\forall n$

Define :-

$X_0 = (C_1, C_2) \quad X_1 = (C_2, C_3) \quad X_2 = (C_3, C_4) \; \ldots \; X_n = (C_{n+1}, C_{n+2})$

$\{X_n\}_{n \geq 0}$

$X_0 = \begin{cases} 0 & \text{if } (T,T) \\ 1 & \text{if } (T,H) \\ 2 & \text{if } (H,T) \\ 3 & \text{if } (H,H) \end{cases}$

$S = \{0, 1, 2, 3\}$ (state-space)

Note :- $P(X_n \in S) = 1 \;\; \forall n$ if S is countable

$$P(X_{n+k}=j \mid X_0, X_1, \ldots, X_n=i) = P(X_{n+k}=j \mid X_n=i)$$

$\forall\ i,j \in S$ & $n,k \in W$ (set of whole no.'s)

$\left(P(X_1=j \mid X_0=i) = P_{ij} \longrightarrow \text{can be found} \right)$

$P(X_1=0 \mid X_0=0) \Rightarrow TTT \Rightarrow 1/8$

$P(X_1=1 \mid X_0=0) \Rightarrow TTH \Rightarrow 1/8$

$P(X_1=2 \mid X_0=0) \Rightarrow TTx \rightarrow 0$

$P(X_1=3 \mid X_0=0) \Rightarrow TTx \rightarrow 0$

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

$P_{01} \Rightarrow THTT\ x \rightarrow 0$

$P_{11} \Rightarrow THTH \rightarrow 0$

$P_{21} \Rightarrow 1/8$

$P_{31} \Rightarrow 1/8$

Visual Representation of a DTMC

S: vertex set

Draw arc (i,j) if $P_{ij} > 0$

$w((i,j)) = P_{ij}$

Gambler's Ruin

A B

1Re 1Re $P(A \text{ wins?})$

winner takes 2Re

$P(A \text{ wins}) = p \qquad P(B \text{ wins}) = 1-p$

X_n = amount of money A has after n bets (B will have $(2c-n)$ rupees)

$S = \{0, 1, \ldots, 2c\}$

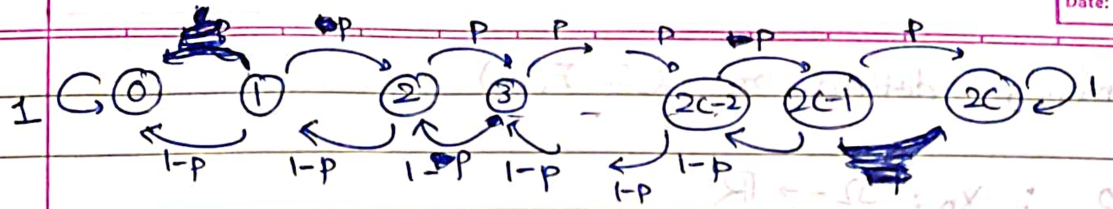
$$X_{n+1} = \begin{cases} X_n + 1 & \text{w.p. } p \\ X_n - 1 & \text{w.p. } (1-p) \end{cases}$$

when $X_n \in \{1, \ldots, 2c-1\}$

$X_{n+1} = X_n$ (stay in that state forever) w.p. 1

when $X_n = \{0, 2c\}$

alt. ⇒ recursion-based expection values



P(A wins all)

P_K : probability of A winning, from state K.

$P_0 = 0 \qquad P_{2C} = 1 \qquad P_C = ? = P(A \text{ wins})$

$P_1 = p \cdot P_2 + (1-p) \cdot P_0 = p \cdot P_2$

$P_2 = p \cdot P_3 + (1-p) \cdot P_1 = p \cdot P_3 + (1-p) \cdot p \, P_2$

$(1 - (p - p^2)) P_2 = p \cdot P_3$

$(1 - p + p^2) P_1 = p^2 P_3 \qquad P_3 = \left(\frac{1-p+p^2}{p^2}\right) P_1$ and so on

$P_3 = p \cdot P_4 + (1-p) \cdot P_2$ and so on ⟶ can be solved.

⇒ let every bet take a fixed amount of time ⟶ Expected time ?

⟶ T_K : Expected no. of bets until game ends, starting from state K.

$T_0 = 0 \qquad T_{2C} = 0$

$$\underset{K \neq \{0, 2C\}}{T_K} = 1 + (p \cdot T_{K+1}) + (1-p) T_{K-1}$$

One more coin toss to go to some other state

Tosses after you reach the next state

STRONG - MARKOV PROPERTY

⇒ Stopping Time :-

- Let T be a random variable s.t. $T : \Omega \longrightarrow \{0, 1, \ldots\}$ (sample/state space)
- A R.V. X_T ⟶ observation made at time T, where T itself is Random.

$X_T(\omega)$

$\{X_n\}_{n\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$

⇒ $\forall\, n \geq 0 \;:\; X_n : \Omega \to \mathbb{R}$

⇒ Thus, for every $\omega \in \Omega$, $\{X_n(\omega)\}_{n\geq 0}$ is simply a fixed sequence of real numbers.
(sample Path)

$X_T(\omega) = X_{T(\omega)}(\omega)$ (once we specify ω, everything is SPECIFIED)

K^{th} RV ⟺ $X_K(\omega)$

T_1 = first time A's worth is $\geq C+1$
T_2 = time of A's first loss which is followed by 3 consecutive wins.

ω: sequence of H and T.

Ⓗ T T H T T Ⓣ H H H T $\geq C+1$: no of H = no. of T + 1

T_1 T_2 ⟶ we need to go 3 steps in future to know if T_2 is here or not.

Stopping Times ⟶ think about outcomes important to us.

e.g.- $T_1 = n$ $\xrightarrow{\text{stopping Times}}$ check upto n the No. of H & T. → Need only the info. of tosses 'before' and 'AT' n

$T_2 = n$ $\xrightarrow{\text{not a stopping time}}$ Need info. of tosses of 'past' + 'current' + 'future' (n+3)

T is a stopping time if $I_{\{T>n\}} = g(X_1, \ldots, X_n)$ (PAST & current ONLY)

i.e.

$$= \begin{cases} 1 & ; T>n \\ 0 & ; o/w \end{cases}$$

→ a random variable $T: \Omega \to \{0,1,2,\ldots\}$

→ a random variable X_T S.T. $X_T(\omega) = X_{T(\omega)}(\omega)$

⇒ So, we say that T is a stopping time if $I_{\{t>n\}} = g(X_1, X_2, \ldots X_n)$

$$I_A^{(\omega)} = \begin{cases} 1 & ; \text{if } \omega \in A \\ 0 & ; o/w \end{cases}$$ ⇒ (Indicating occurrence of event A)

$$I_{\{t>n\}}(\omega) = \begin{cases} 1 & ; \text{ if } T(\omega) > n \\ 0 & ; \text{ o/w} \end{cases}$$

$$I_{\{t>n\}} = g(x_0, x_2, \ldots, x_n)$$

condition where we need to stop

a $f^n c$ of previous inputs ONLY

$$I_{\{t\le n\}} = 1 - I_{\{t>n\}} = 1 - g(x_0, \ldots x_n)$$
$$= g'(x_0, \ldots, x_n)$$

$$I_{\{t\le n\}} = g'(x_0, \ldots, x_n)$$

$$\boxed{I_{\{t\le n\}} = g'(x_0, \ldots, x_n)}$$
$$\boxed{I_{\{t=n\}} = g''(x_0, \ldots, x_n)}$$

$$I_{\{t=n\}} = I_{\{t\le n\}} - I_{\{t\le (n-1)\}}$$
$$I_{\{t=n\}} = h(x_0, x_1, \ldots x_n)$$

(independent of what happens in the future)

STRONG MARKOV PROPERTY

For DTMC $\{X_n\}_{n\ge0}$, let T be any proper $(P(T<\infty)=1)$ stopping time s.t. $P(T<\infty) = 1$, then,

$$\boxed{P(X_{T+k} = j \mid X_0, X_1, \ldots, X_T = i) = P_{ij}^{(k)} \quad \forall\, i,j \in S \;\; \forall\, k = \{0, 1, \ldots\}}$$

eg.- $x_0, x_1, \ldots, x_n, x_{n+1}, \ldots \qquad c, d \in S$

T: denotes first time of occurence of c followed by d

$$\ldots\ldots c \ldots\ldots \underset{\uparrow T}{c}d \ldots\ldots \Rightarrow (\text{NOT a stopping time})$$

$$P(X_{T+1} = i \mid X_0, \ldots, X_T = c) = \begin{cases} 1 & \text{if } i = d \\ 0 & \text{O/w} \end{cases}$$

probability (collection of all ω's where $X_{T+k}(\omega) = j$ and $x_0, x_1, \ldots, x_T(\omega) = i$) $= 1$

$$P(X_{T+k} = j \mid X_T = i) \qquad\qquad P(X_{n+k} = j \mid X_n = i)$$

$$\Rightarrow P\left(\{\omega : X_{T(\omega)+k}^{(\omega)} = j \mid X_{T(\omega)}^{(\omega)}\}\right) \qquad\qquad \Downarrow$$

$$P(\{\omega : X_{n+k}(\omega) = j \mid X_n(\omega) = i\})$$

(break the numerator w.r.t. T and solve, then reconstruct the same)

$$P(X_{T+k}=j \mid X_0, X_1, \ldots, X_T=i)$$

$$= \frac{P(X_{T+k}=j, X_0, \ldots, X_T=i)}{P(X_0, \ldots, X_T=i)} \quad \text{— (1)}$$

Numerator $= P(X_{T+k}=j, X_0, \ldots, X_T=i) = \sum_{t=0}^{\infty} P(T=t, X_{T+k}=j, X_0, \ldots)$

$$= \sum_{t=0}^{\infty} P(\underbrace{T=t, X_{t+k}=j, X_0, \ldots, X_t=i})$$

Consider $\Rightarrow$

$$= P(T=t \mid X_{t+k}=j, X_0, \ldots, X_t=i) \cdot P(X_{t+k}=j, X_0, \ldots, X_t=i)$$

$$= P(T=t \mid X_{t+k}=j, \ldots, X_t=i) \cdot P(X_{t+k}=j \mid X_0, \ldots, X_t=i) \cdot P(X_0, \ldots, X_t=i)$$

stopping time ($T=t$)   Future (NOT Needed) ($X_{t+k}=j$)

$$= P(T=t \mid X_0, \ldots, X_t=i) \cdot P_{ij}^{(k)} \cdot P(X_0, \ldots, X_t=i)$$

So,

$$P(X_{T+k}=j, \ldots, X_T=i) = \sum_{t=0}^{\infty} P_{ij}^{(k)} \cdot P(T=t, X_0, \ldots, X_t=i)$$

So,

$$= \sum_{t=0}^{\infty} P(T=t, X_0, X_1, \ldots, X_t=i) \cdot P_{ij}^{(k)}$$

$$= P_{ij}^{(k)} \sum_{t=0}^{\infty} P(T=t, X_0, X_1, \ldots, X_t=i)$$

reverse

$$= P_{ij}^{(k)} P(X_0, X_1, \ldots, X_T=i) \quad \text{— (2)}$$

Finally using (2) in (1), we get,

$$\boxed{P(X_{T+k}=j \mid X_0, X_1, \ldots, X_T=i) = P_{ij}^{(k)}}$$

Important stopping Times

⇒ $T = c$, where $c \in W$

⇒ time of first visit to state $j \in \mathcal{S}$ (Takes value 1 onwards 1, 2, 3, . . .)

- consider $X_0 = j$
- Let T denote time of first visit to j after time 0

$$\mathbb{P}(X_n = i \mid X_0 = j) = P_{ji}^{(n)}$$

So, we can say that $\boxed{\mathbb{P}(X_{T+n} = i \mid X_T = j) = P_{ji}^{(n)}}$ Strong-Markov

Markov :- when you are at a given point in space, the path does NOT matter

⇒ Process renews itself every time T arrives.

FIRST PASSAGE TIME

→ Time taken to visit a state for the first time

→ eg.- For state $j \in S$, let T_{ij} denote the first passage time given $X_0 = i$

$$\mathbb{P}(T_{ij} = n) = \mathbb{P}(X_1 \neq j, X_2 \neq j, \ldots, X_n = j \mid X_0 = i)$$
$$= f_{ij}^{(n)}$$

$[f_{ij}^{(n)}]_{n \geq 1}$ ⇒ PMF of T_{ij}

$$f_{ij}^{(1)} = \mathbb{P}(X_1 = j \mid X_0 = i) = P_{ij}$$

$$f_{ij}^{(2)} = \mathbb{P}(X_1 \neq j, X_2 = j \mid X_0 = i)$$
$$= \sum_{k \in S \setminus \{j\}} \mathbb{P}(X_1 = k, X_2 = j \mid X_0 = i)$$
$$= \sum_{k \in S \setminus \{j\}} P_{ik} \cdot P_{kj}$$

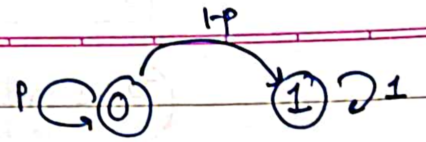
$$\boxed{f_{ij}^{(2)} = \sum_{k \in S \setminus \{j\}} P_{ik} \cdot P_{kj}}$$

Define :-

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} = \mathbb{P}(\text{Process visits state } j \mid X_0 = i)$$

$$\equiv \lim_{N \to \infty} \sum_{n=1}^{N} f_{ij}^{(n)}$$ ⇒ ($\exists$ some N s.t. you go from i to j.)

eg.- ①



$f_{00}^{(1)} = p$ $\quad$ $f_{00}^{(2)} = \cancel{p}$ ~~NOT DEFINED~~

0

$$= \mathbb{P}(X_1 \neq 0,\ X_2 = 0 \mid X_0 = 0)$$
$$= \mathbb{P}(X_2 = 0 \mid X_1 \neq 0) \cdot P(X_1 \neq 0 \mid X_0 = 0)$$
$$= \mathbb{P}(X_2 = 0 \mid X_1 = 1) \cdot P(X_1 = 1 \mid X_0 = 0)$$
$$0 \cdot (1-p) = \boxed{0}$$

$p_{00}^{(1)} = p$ $\quad$ $p_{00}^{(2)} = p^2$ and so on $p_{00}^{(n)} = p^n \ \forall\ n \geq 1$

$f_{00}^{(1)} = p$ $\quad$ $f_{00}^{(n)} = 0 \ \forall\ n \geq 2$

$\tau_{00} \Rightarrow$ improper R.V.

$P(\tau_{00} < \infty) = P(\tau_{00} = 1) = p$

$P(\tau_{00} = 1) = p$

$P(\tau_{00} = \infty) = 1 - p$

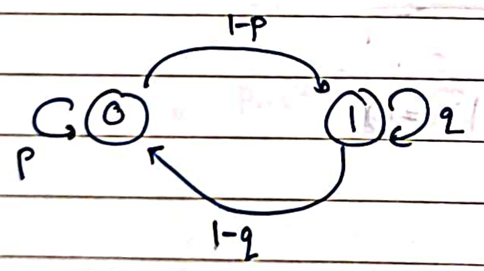
$\sum_{n=1}^{\infty} f_{ij}^{(n)} = f_{ij}^{(1)} + f_{ij}^{(\infty)} = 1$ X

$$\boxed{\begin{array}{l} \sum_{n=1}^{N} f_{ij}^{(n)} = f_{ij}^{(1)} = p \ \forall\ N \\ a_N = p \ \forall\ N \\ \text{so, } \lim_{N \to \infty} a_N = p \end{array}}$$

$P(\tau_{00} < \infty) = p$

For mathematical convenience, all the other mass is on ∞ now.
$P(\tau_{00} = \infty) = (1-p)$

eg.: ②



$f_{00}^{(1)} = p$ $\quad$ $f_{00}^{(2)} = (1-p)(1-q)$ $\quad$ $f_{00}^{(3)} = (1-p)\,q\,(1-q)$

$$f_{00} = \sum_{n=0}^{\infty} (1-p)(1-q)\,q^n + p$$

$$= p + (1-p)(1-q)\left(\frac{1}{1-q}\right)$$

$$= p + 1 - p = 1$$

It is certain that if we start in 0, we will end in 0.

$\Rightarrow P(RV) =$ finite w.p. 1 but has ∞ expectation $\Rightarrow$ INTEPRET

STATES CLASSIFICATION

State j

- TRANSIENT: $f_{jj} < 1$
- RECURRENT: $f_{jj} = 1$ $\Rightarrow$ (revisiting j ~~has~~ is ~~sure~~ certain, in a finite no. of steps)

CLAIM :- If $f_{jj} < 1$, then $P(\# \text{ of visits to state } j) < \infty) = 1$

proof using $\boxed{\text{Borel-Cantelli Lemma}}$

Define :- $\nu_{jj} = \sum_{n=1}^{\infty} n \cdot f_{jj}^{(n)} = E[\tau_{jj}]$ (proper RV only)

for recurrent j $(f_{jj}^{\circ} = 1)$

eg.- $f_{jj}^{(n)} \rightarrow \frac{1}{n^2} \cdot \frac{6}{\pi^2}$ $\qquad \sum f_{jj}^{(n)} = 1$ but $E[X]$ is ∞.

ν_{jj}
- $< \infty$: FINITE
- $= \infty$: INFINITE

State j

- (do not visit the state i.o.) Transient: $f_{jj} < 1$
  - $\boxed{\sum P_{jj}^{(n)} = 0}$ $\qquad \lim_{n\to\infty} P_{jj}^{(n)} = 0$ ✗
- Recurrent (guaranteed to come back to the state): $f_{jj} = 1$
  - Positive: $\mu_{jj} < \infty$ $\qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} P_{jj}^{(n)} > 0$
  - Null: $\mu_{jj} = \infty$ $\qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} P_{jj}^{(n)} = 0$

(μ_{jj} : expected time to come back to state j from state j)

Let M_j denote the no. of visits to state j

$E[M_j | X_0 = j]$

$$M_j = \sum_{n=1}^{\infty} I_{\{X_n = j\}} \qquad I_{\{X_n=j\}} = \begin{cases} 1 ; & X_n = j \\ 0 ; & o/w \end{cases}$$

$$E\left[\sum_{n=1}^{\infty} I_{\{X_n=j\}} \,\middle|\, X_0 = j\right]$$

$$= \sum_{n=1}^{\infty} E\left[I_{\{X_n=j\}} \,|\, X_0 = j\right] = \sum_{n=1}^{\infty} P(X_n = j | X_0 = j)$$

$$= \sum_{n=1}^{\infty} P_{jj}^{(n)}$$

$$\boxed{E[\underbrace{M_j \mid X_0 = j}_{\text{has to be } <\infty}] = \sum_{n=1}^{\infty} P_{jj}^{(n)}}$$

So, for transient states j, $\sum_{n=1}^{\infty} P_{jj}^{(n)} < \infty \rightarrow$ Transient

$= \infty \rightarrow$ recurrent

Monotone convergence theorem :- $E[\lim_{n\to\infty} A_n]$

limit and expectations can be exchanged

The sequence of RV's is MONOTONIC (inc./dec./non-inc./non-dec.)

$\lim_{n\to\infty} E[A_n]$

For our example, $A_{n+1} = \begin{cases} A_n \\ A_n + 1 \end{cases}$ (The summation is

Lemma:- ① If j is transient, then $\sum_{n=1}^{\infty} P_{jj}^{(n)} < \infty$

$$\Rightarrow \boxed{\lim_{n\to\infty} P_{jj}^{(n)} = 0}$$

② If j is recurrent, then $\sum_{n=1}^{\infty} P_{jj}^{(n)} = \infty$

Lemma 2:- For recurrent j, $\boxed{\lim_{N\to\infty} \frac{1}{N} \underbrace{\sum_{n=1}^{N} P_{jj}^{(n)}}_{\Downarrow\ \Theta[N]} = \underbrace{\gamma_j}_{\text{constant}}}$ $= \left(\frac{1}{\mu_{yy}}\right)$

$\gamma_j = 0$ if j is null

$\gamma_j > 0$ if j is positive

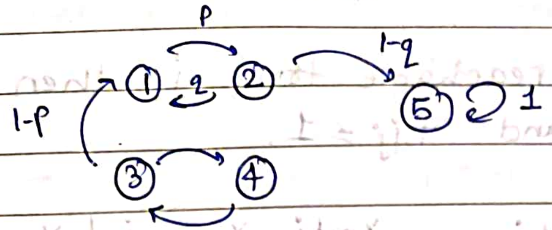
COMMUNICATING CLASSES

$\Rightarrow$ The entire state space S can be partitioned into communicating classes.

$\Rightarrow$ i communicates with j is an equivalence relⁿ & so the equivalence classes.

Types of communicating classes

OPEN ⟵ ⟶ CLOSED



Communicating classes :- $\{1,2\}$, $\{3,4\}$, $\{5\}$

open (for $\{1,2\}$); closed (for $\{3,4\}$, $\{5\}$)

⇒ A communicating class (C.C.) $\mathcal{C}$ is open if $\exists\, i \in \mathcal{C}$ & $j \notin \mathcal{C}$ S.T. $P_{ij} > 0$

($\exists$ a +ve probability to transitition to some other C.C.)

⇒ Any communicating class (C.C.) i.e. not open is called closed.
i.e. $\nexists\, i \in \mathcal{C}$ and $j \notin \mathcal{C}$ S.T. $P_{ij} > 0$.

Theorem :- $\boxed{\text{All the states in a communicating class are of the same type}}$

i.e. (Type is a class property)

Proof :- Consider a C.C. $\mathcal{C}$ S.T. $i \in \mathcal{C}$ and i is transient.

$$\Rightarrow \lim_{n\to\infty} P_{ii}^{(n)} = 0$$

Also, let $j \in \mathcal{C}$ ⇒ i and j communicate

⇒ i is reachable from j and j is reachable from i

$\exists\, r > 0$, S.T. $P_{ji}^{(r)} > 0$ and $\exists\, s > 0$, S.T. $P_{ij}^{(s)} > 0$

CLAIM :- $$\sum_n \left(P_{ii}^{(r+n+s)}\right) \geq \sum_n \left(P_{ij}^{(s)} \cdot P_{jj}^{(n)} \cdot P_{ji}^{(r)}\right)$$

Since this is a pointwise inequality, limit preserves it

$$\sum_n P_{ii}^{(n)} \geq \sum P_{ii}^{(r+n+s)} \geq \left(\sum P_{jj}^{(n)}\right) P_{ij}^{(r)} P_{ji}^{(s)}$$

⇒ H.P.

$$\lim_{n\to\infty} \sum_{n\geq 1}^{\infty} P_{ii}^{(r+n+s)} \geq \left(\lim_{n\to\infty} P_{jj}^{(n)}\right) P_{ij}^{(s)} \cdot P_{ji}^{(r)}$$

$$0 \geq \left(\lim_{n\to\infty,\, n\geq 1} \sum^{\infty} P_{jj}^{(n)}\right) \cdot c_1 c_2 = 0 \quad \text{[as all of them are +ve]}$$

$$\Rightarrow \lim_{n\to\infty} P_{jj}^{(n)} = 0 \Longrightarrow \text{TRANSIENT}$$

$\Rightarrow$ if $i,j \in \mathcal{C}_0$, then, if i is null-recurrent then j is also null.

Lemma:- If j is recurrent and i is reachable from j, then j is also reachable from i and $f_{ij} = 1$.

Proof:- **Define :-** $a_{ji}(n) = \mathbb{P}(X_1 \neq j, X_2 \neq j, \dots X_{n} \neq j, X_n = i \mid X_0 = j)$

$\exists\, s > 0$ s.T. $a_{ji}^{(s)} > 0$ as i is reachable from j

$f_{ij} \Rightarrow P(\text{reaching } j \text{ from } i)$

$a_{ji}^{(s)}(1 - f_{ij}) \Rightarrow P($reaching from j to i and not ever going back to $j)$

Now, $P(\text{not come back to } j) \geq P($reaching from j to i and not coming back to $j)$

$$1 - f_{jj} \geq a_{ji}^{(s)}(1 - f_{ij})$$

as j is recurrent $\Rightarrow f_{jj} = 1$

so,
$$a_{ji}^{(s)}(1 - f_{ij}) = 0$$

so, as $a_{ji}^{(s)} > 0$ (given reachability)

$\Rightarrow \boxed{f_{ij} = 1} \longrightarrow j$ is reachable from i (& i.e. $f_{ij} = 1$)

Corollary:- If i and j belong to recurrent class $\mathcal{C}$, then $f_{ij} = f_{ji} = 1$

$\downarrow$

j is reachable from i and i is recurrent $\Rightarrow$ i is reachable from j & $\boxed{f_{ji} = 1}$

and similarly, $\boxed{f_{ij} = 1}$

$\Rightarrow$ Let $x, y \in \mathcal{X}$ be in the same c.c. and y is recurrent. Then, $\boxed{\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P_{xy}^{(k)} = \frac{1}{\mu_{yy}}}$, and if the state is aperiodic, then, $\boxed{\lim_{n\to\infty} P_{xy}^{(n)} = \frac{1}{\mu_{yy}}}$

Lemma:- open communicating classes are transient

Proof:- Let C be an open communicating class, so,
$\exists\ j \in C$ and $i \notin C$ s.t. $P_{ji} > 0$

To show :- $f_{jj} < 1$ (Transient)

$$\boxed{f_{jj} = P_{jj} + \sum_{\substack{k \in S \\ k \neq j}} P_{jk} \cdot f_{kj}}$$ (S: state space)

$$= P_{jj} + \sum_{\substack{k \in S \\ k \neq i, j}} P_{jk} f_{kj} + \underbrace{P_{ji}}_{>0} \cdot \cancel{f_{ij}} \longrightarrow P(j \text{ is reachable from } i)$$

0

$\boxed{\text{NOT true}}$

$$f_{jj} = \sum_{\substack{k \in S \\ k \neq i, j}} P_{jk} f_{kj} + P_{jj} \le P_{jj} + \sum_{k \in S \setminus \{i,j\}} P_{jk} \quad \text{as } (f_{kj} \le 1)$$

$$\le \sum_{k \neq i} P_{jk}$$

also $\sum_{k \in S} P_{jk} = 1 \qquad \sum_{k \neq i} P_{jk} = 1 - P_{ji}$

so, $f_{jj} \le (1 - P_{ji}) < 1 \Rightarrow f_{jj} \le 1^{-} \Rightarrow \boxed{f_{jj} < 1}$

(open communicating classes are transient.)

Lemma:- Finite, closed communicating classes are +ve recurrent

Proof: Let C be a finite, closed communicating class.

$\forall\ i \in C$ and n, → Finite closed class.

$$\sum_{j \in C} P_{ij}^{(n)} = 1$$ → after every 'n'-steps, we remain in c only.

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} P_{ij}^{(n)} \ \begin{matrix} = 0 \\ > 0 \end{matrix} \leftrightarrow \text{+ve recurrent}$$

Here, $\sum_{j\in C}\left(\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}P_{ij}^{(n)}\right) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\Big(\underbrace{\sum_{j\in C}P_{ij}^{(n)}}_{1}\Big)$

$= 1$

i.e. $\boxed{>0}$ and $\neq 0$

so, j is +ve-recurrent.

Thm:- An irreducible DTMC is positive recurrent if and only if $\exists$ a probability mass function, π on S such that $\pi = \pi P$, & $\pi_i > 0$ $\forall\ i \in S$. such π is unique. (iff)

(A linear eqn) $\Rightarrow$ $\pi = \pi P$ — a fnc that assigns a positive probability (π_i) for all states (i) $\in$ state space(S)

irreducible DTMC $\Rightarrow$ if it has a single communicating class.

$\pi = \pi P$ (A Matrix Equation)

vector of dimension $|S|$ — Matrix of dimension $|S| \times |S|$

$$\boxed{\pi_i = \sum_{j \in S} \pi_j P_{ji}}$$ $\Rightarrow$ set of all these equations

also, $$\boxed{\sum_{i \in S} \pi_i = 1}$$

($\pi_i > 0\ \forall\ i \in S$)

suppose $\alpha = \pi$ (initial state distribn), i.e.

$P(X_0 = i) = \pi_i$

$P(X_1 = i) = \sum_{j \in S} P(X_0 = j) \cdot P(X_1 = i \mid X_0 = j) = \sum_{j \in S} \pi_j P_{ji}$

look at F.D. joint distribn

$= \pi_i$

So, essentially, $\boxed{P(X_n = i) = \pi_i} \longrightarrow \boxed{SSS}$

$\Rightarrow$ If the initial state distribution is π, then, all marginals are also π.

Invariant measure

Stationary stochastic process :- All finite-dimensional distributions are time-invariant.

$P(Y_{t_1} \le y_1, Y_{t_2} \le y_2, Y_{t_3} \le y_3, \dots, Y_{t_n} \le y_n)$

$=$

$P(Y_{t_1+\tau} \le y_1, Y_{t_2+\tau} \le y_2, \dots Y_{t_n+\tau} \le y_n) \quad \forall\ \tau \in \mathbb{R}$

$\Rightarrow$ If $\alpha = \pi$, then, the DTMC is a stationary stochastic process. Hence, π is also called the, stationary measure.

With identity as TPM, any probability distribution on the state space is a stationary distribution.

Proof :-

Part I) Let $\{X_n\}_{n\geq 0}$ be positive recurrent

- $\forall j \in S \Rightarrow$ ① $f_{jj} = 1$ (recurrence)

② $\frac{1}{n}\sum_{k=1}^{n} P_{ij}^{(k)} \xrightarrow[(n\to\infty)]{} x_j > 0$ if $f_{ij}=1$

Let $\alpha_i^{(n)} = P(X_n = i)$ [marginal for X_n]

$$c_i^{(n)} = \frac{1}{n}\sum_{k=1}^{n} \alpha_i^{(k)} \quad \forall\ i \in S$$

$$b_i^{(n)} = \frac{1}{n}\sum_{k=1}^{n} \alpha_i^{(k-1)} \quad \forall\ i \in S$$

$$\alpha_i^{(n)} = P(X_n = i) = \sum_{j\in S} \alpha_j P_{ji}^{(n)}$$

so $c_i^{(n)} = \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{j\in S} \alpha_j P_{ji}^{(k)}\right)$

$$\boxed{c_i^{(n)} = \sum_{j\in S} \alpha_j \frac{1}{n}\sum_{k=1}^{n} P_{ji}^{(k)}}$$

$P^0 = I$ i.e. $P_{ij}^{(0)} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{o/w} \end{cases}$

$$b_i^{(n)} = \frac{1}{n}\sum_{k=1}^{n} \alpha_i^{(k-1)} = \boxed{\sum_{j\in S} \alpha_j \frac{1}{n}\sum_{k=1}^{n} P_{ji}^{(k-1)}}$$

To show :- $c^{(n)} = b^{(n)} \cdot P$

$$c_i^{(n)} = \frac{1}{n}\sum_{k=1}^{n} P(X_k = i) = \frac{1}{n}\sum_{k=1}^{n}\sum_{j\in S} P(X_k = i \mid X_{k-1} = j)\cdot P(X_{k-1}=j)$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{j\in S} P_{ji}\cdot \alpha_j^{(k-1)}\right)$$

$$= \sum_{j\in S} P_{ji} \cdot \underbrace{\frac{1}{n}\sum_{k=1}^{n} \alpha_j^{(k-1)}}_{b_j^{(n)}}$$

So, $\boxed{c_i^{(n)} = \sum_{j\in S} P_{ji} \cdot b_j^{(n)}}$ $\forall\ i \in S$

$$c_i^{(n)} = \sum_{j\in S} P_{ji}\, b_j^{(n)} \quad \text{or,} \quad c^{(n)} = P \cdot b^{(n)}$$

Taking limit as $n \to \infty$ on both sides,

$$\lim_{n\uparrow\infty} c_i^{(n)} = \lim_{n\uparrow\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \alpha_i^{(k)} \right)$$

$$= \lim_{n\uparrow\infty} \left(\sum_{j\in S} \alpha_j \left[\frac{1}{n} \sum_{k=1}^{n} P_{ji}^{(k)} \right] \right) \quad \leq 1$$

(limit & sum can be exchanged $\Rightarrow$ monotone convergence theorem)

If $\{X_n\}$ s.t. $X_n \leq X_{n+1}$, then
$$\lim_{n\uparrow\infty} E[X_n] = E[\lim_{n\to\infty} X_n]$$
[can be exchanged]

BCT :- Bounded convergence Theorem
$\{X_n\}_{n\geq 1}$ s.t. $X_n \leq Y$ $\forall n$ and $E[|Y|] < \infty$, then
$$\lim_{n\uparrow\infty} E[X_n] = E[\lim_{n\uparrow\infty} X_n]$$

$$\boxed{E[X] = \sum_{i=1}^{\infty} a_i \cdot P(X = a_i)}$$

PMF on state space

$$\sum_{j\in S} \alpha_j \left(\frac{1}{n} \sum P_{ji}^{(k)} \right) \to \leq 1$$

$E[|1|] < \infty$

using the BCT,

$$\lim_{n\uparrow\infty} c_i^{(n)} = \sum_{j\in S} \alpha_j \lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} P_{ji}^{(k)} \right)$$

$$= \gamma_i \sum_{j\in S} \alpha_j \Rightarrow \boxed{\lim_{n\to\infty} c_i^{(n)} = \gamma_i}$$

now, $c^{(n)} = b^{(n)} \cdot P$

$\to r \quad \to r$

$\Rightarrow \boxed{r = rP}$ $\longrightarrow$ DTMC is positive recurrent

$r_i > 0 \;\forall\, i \in S$

$\to$ If we choose $\pi = r$, then π staisfies $\pi = \pi P$

$\hookrightarrow$ we need to argue π is a PMF on S.

$\lim_{n\to\infty} c^{(n)} = r$ (know) and $\lim_{n\to\infty} c_i^{(n)} = r_i \quad \forall\, i \in S$

Fix any (n) and observe,

$$\sum_{i\in S} c_i^{(n)} = \sum_{i\in S} \frac{1}{n} \sum_{k=1}^{n} \alpha_i^{(k)}$$

$$= \frac{1}{n} \sum_{k=1}^{n} \underbrace{\sum_{i\in S} \underbrace{\alpha_i^{(k)}}_{P(X_i = k)}}_{1}$$

$$\boxed{\sum_{i\in S} c_i^{(n)} = 1}$$

Now,

$$\sum_{i\in S} r_i = \sum_{i\in S} \lim_{n\to\infty} c_i^{(n)}$$

if the state space is finite, we can swap Σ and lim.

NOT 100% clear why we can do this for ∞. state space

eg.- $\lim_{n\uparrow\infty} \lim_{m\uparrow\infty} 2^{n-m} = 0$

$\lim_{m\to\infty} \lim_{n\to\infty} 2^{n-m} \Rightarrow \infty$

swapping the limits changes the value

$\Leftarrow$ $f^n c$ is monotone in both m, n

$\to$ ~~$\sum_{i\in S} \lim$~~ Taking a finite sum (limiting-case)

$$\sum_{i=1}^{m} r_i = \sum_{i=1}^{m} \lim_{n\to\infty} c_i^{(n)}$$

Page No.:
Date:

i.e. $\sum_{i=1}^{m} \gamma_i = \lim_{n\to\infty} \underbrace{\left(\sum_{i=1}^{m} c_i^{(n)} \right)}_{\leq 1 \ \forall n} \leq 1 \ \forall n$

i.e.

$\Rightarrow$ For every $m \in S$, $0 \leq \sum_{i=1}^{m} \gamma_i \leq 1$ $(\gamma_i > 0)$

Taking limit as $m \uparrow \infty$. (monotonic increasing in m)

$$\Rightarrow \boxed{0 < \lim_{m\uparrow\infty} \sum_{i=1}^{m} \gamma_i \leq 1}$$

i.e.

$$\boxed{0 < \sum_{i\in S} \gamma_i \leq 1}$$

$\sum_{i \in S} \gamma_i = \Delta$ — it is a finite value

Chose $\pi = \frac{\gamma}{\Delta}$, then π is a PMF on S and satisfies

$\pi = \pi P$, with $\boxed{\pi_i = \frac{\gamma_i}{\Delta}}$ $\boxed{\Delta : \sum_{i\in S} \gamma_i}$ $\pi_i > 0 \ \forall i \in S$

(if DTMC is P.R.)

$\Rightarrow$ If $\exists \ \pi > 0$ satisfying $\pi = \pi P$, then, the DTMC is (+ve) recurrent.
(π is a PMF on S)

$\rightarrow$ If $\Pi = \pi P$, then, $\Pi = \Pi P^{(n)}$

$$\frac{1}{n}\sum_{k=1}^{n} \pi = \frac{1}{n} \sum_{k=1}^{n} \pi P^{(k)} = \pi \left(\frac{1}{n} \sum_{k=1}^{n} P^{(k)} \right)$$

$$\pi = \pi \left(\frac{1}{n} \sum_{k=1}^{n} P^{(k)} \right)$$

each element is ≤ 1 → BCT

as $\lim n\to\infty$ $\Rightarrow$ $\lim_{n\uparrow\infty} \pi = \lim_{n\uparrow\infty} \left(\pi \cdot \underbrace{\frac{1}{n} \sum_{k=1}^{n} P^{(k)}}_{\text{vector}} \right)$

π: PMF

• expanding as matrix product → ~~expectat~~^{n} w.r.t. PMF Π

limit & expectation can be interchanged here (BCT)

$$\lim_{n\to\infty} \pi = \pi \underbrace{\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P^{(k)}}$$

$$\boxed{\lim_{n\to\infty} \pi = \gamma} \quad \boxed{\text{UNIQUE}} \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P_{ij}^{(k)} = \gamma_j$$

$\Rightarrow \quad \pi_i = \gamma_i \quad \forall \; i \in S$

$0 < \pi_i \leq 1 \quad \Rightarrow \quad 0 < \gamma_i \leq 1 \quad \forall \; i \in S$

So, the limits $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P^{(k)}$, > 0 and finite $\Rightarrow$ states are +ve recurrent.

$\Rightarrow$ say π_1, π_2 both satisfy this eq^n

$\Rightarrow \pi_1 = \gamma \Rightarrow \pi_2 = \gamma$ ($\Delta = 1$ here (turns out))

$\Rightarrow \pi_1 = \pi_2$

Lemma: For an irreducible positive recurrent DTMC with stationary measure π,

Fraction of time-spent in state-i $= \pi_i$

and, $\boxed{\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} I_{\{X_k = i\}} = \pi_i \text{ w.p. } 1}$

$\frac{1}{n}\sum_{k=1}^{n} Y_k \longrightarrow$ similar to LLN

$\rightarrow$ where the empirical mean converges? $\longrightarrow$ To the expectation for i.i.d.

$Y_k \rightarrow$ NOT IID's

$$\underbrace{\frac{1}{n}\underbrace{\sum I_{\{X_k = i\}}}_{\text{no. of times you visited state } i}}_{\text{Average time spent in state } i}$$

π_j : frac^n of time you spent in state-i on average.

PROPERTIES OF π :-

① π is called stationary measure if $\alpha = \pi$, then $\alpha^{(n)} = \pi \ \forall n$

(The DTMC is a strong-sense stationary process.)

→ π is also called steady-state prob. vector as → (irreducible DTMC)

$$\boxed{\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j} \quad \forall \ i \in S$$

→ aperiodic MC — self-loop in MC

i.e. the $\mathbb{P}(X_n = j) \longrightarrow \pi_j$

$$P^{(n)} = P^n = \begin{bmatrix} \pi \\ \pi \\ \vdots \end{bmatrix}$$

⇒ π is called ergodic measure. $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{I}\{X_k = j \mid X_0 = i\} = \pi_j$

$\forall \ i \in S$

→ we have already shown $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^{(k)} = \pi_j \quad \forall \ i \in S$

⇓

Frequency of Visits to state-j = π_j

Fraction of time spent in state-j = π_j

TRANSIENCE CRITERIA

⇒ Consider a DTMC, $\{X_n\}_{n\geq 0}$, on S with a TPM, P. Let the DTMC be irreducible. Consider, $\boxed{\tilde{S} \subsetneq S}$ (STRICT SUBSET)

Let $i \in \tilde{S}$ and define $y_i^{(n)} = \mathbb{P}(X_1 \in \tilde{S}, X_2 \in \tilde{S}, \ldots X_n \in \tilde{S} \mid X_0 = i)$

(start in state-i and never leave $\tilde{S}$ in n-steps)

$y_i^{(n)} \in [0,1] \quad \forall n, \; i \in \tilde{S}$

also,

$$y_i^{(n-1)} \geq y_i^{(n)} \longrightarrow \text{it is a } \boxed{\text{decreasing sequence}}$$

$\Rightarrow$ $y_i^{(n)} = \mathbb{P}\left(\bigcap_{k=1}^{n} \{ X_k \in \tilde{S} \mid X_0 = i \} \right)$

$\Rightarrow$ Let $\boxed{y_i = \lim_{n \to \infty} y_i^{(n)}}$ (bounded monotone sequence's limit converges)

and $y = [y_i]_{i \in \tilde{S}} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{|\tilde{S}|} \end{bmatrix}$

WLOG, let $\tilde{S} = \{1, 2, \ldots |\tilde{S}|\}$

$\Rightarrow$ Let Q, now, be a $\boxed{\text{restriction of } P \text{ to } \tilde{S}}$

$$P = \begin{array}{cc} & |\tilde{S}| \\ |\tilde{S}| & \begin{bmatrix} \begin{bmatrix} Q \end{bmatrix} & \\ & \end{bmatrix} \end{array} \longrightarrow \text{(Non-negative matrix)}$$

Theorem :- y is a MAXIMAL solution of $\boxed{x = Qx}$, $0 \leq x \leq 1$ (element-wise)

$$\underbrace{x}_{|\tilde{S} \times 1|} = \underbrace{Q}_{|\tilde{S} \times \tilde{S}|} \underbrace{x}_{|\tilde{S} \times 1|}$$

$\Rightarrow$ Additionally, either $\boxed{y = 0}$ or $\boxed{\sup_{i \in \tilde{S}} y_i = 1}$

(anywhere in $\tilde{S}$ you start, you'll eventually leave $\tilde{S}$)

(there exists one state in $\tilde{S}$ from where if you start, you never leave $\tilde{S}$)

PROOF :- $y_i^{(n)} = \mathbb{P}\left(X_1 \in \tilde{S}, X_2 \in \tilde{S}, \ldots, X_n \in \tilde{S} \mid X_0 = i \right)$

$$= \sum_{j \in \tilde{S}} \left(X_1 = j, X_2 \in \tilde{S}, \ldots \mid X_0 = i \right)$$

$$= \sum_{j \in \tilde{S}} \underbrace{\mathbb{P}(X_1 = j \mid X_0 = i)}_{P_{ij}} \cdot \mathbb{P}\left(X_2 \in \tilde{S}, \ldots \mid X_1 = j, X_0 = i \right)$$

(P_{ij} but $i, j \in \tilde{S}$)

$$\boxed{y_i^{(n)} = \sum_{j \in \tilde{S}} q_{ij} \cdot y_j^{(n-1)}} \Rightarrow \boxed{y^{(n)} = Q\, y^{(n-1)}}$$

Let $y(0) = I$ (similar to a 0-step matrix)
from $\tilde{S}$ in 0 steps, you'll be at $\tilde{S}$

$y^{(1)} = QI$

$y^{(2)} = Q^2 I \; . \; . \; . \; y^{(n)} = Q^n I$

$$y^{(0)} = P(X_0 \in \tilde{S} \mid X_0 \in \tilde{S}) = 1$$

$$y^{(n)} = Q\, y^{(n-1)}$$

$$\boxed{y_i^{(n)} = \sum_{j \in \tilde{S}} q_{ij}\, y_j^{(n-1)}}$$ Taking $\lim_{n\to\infty}$ on both sides,

$$\lim_{n\to\infty} y_i^{(n)} = \lim_{n\to\infty} \sum_{j \in \tilde{S}} q_{ij}\, y_j^{(n-1)}$$

$$y_i = \sum_{j \in \tilde{S}} q_{ij} \lim_{n\to\infty} y_j^{(n-1)}$$

$$y_i = \sum_{j \in \tilde{S}} q_{ij}\, y_j \qquad\qquad \sum_{j \in S} P_{ij} = 1$$

Now, $\sum_{j \in \tilde{S}} P_{ij} \le 1$

$$r = 1 - \sum_{j \in \tilde{S}} P_{ij} = \sum_{j \in S \setminus \tilde{S}} P_{ij}$$

Define :- A random variable Z, S.T.

$$Z_n^{(i)} = \begin{cases} 0 & \text{w.p. } r^{(1)} \\ y_j^{(n-1)} & \text{w.p. } P_{ij} \end{cases}$$

The

$$E[Z_n] = \sum_{j \in \tilde{S}} P_{ij}\, y_j^{(n-1)} \qquad\qquad E[Z_n] \in [0,1] \;\; \forall n$$

Using the Bounded Convergence Theorem (BCT),

$$\lim_{n\to\infty} E[Z_n] = E\left[\lim_{n\to\infty} Z_n\right] \implies y = Qy$$

$$\lim_{n\to\infty} Z_n = Z \implies \lim_{n\to\infty} Z_n(\omega) = Z(\omega)$$

P

\# The way we have defined Z_n, gives us,

$$\boxed{Z(\omega) = \lim_{n\to\infty} y_j^{(n-1)}}$$

Hence, y is a soln of $x = Qx$, $0 \le x \le 1$.

→ Let, x be any other solⁿ. Then, $x = Qx$

$x = Qx \le Q.(\underbrace{1}_{\text{vector}-1})$

$x = Q^n x \le Q^n . 1 = y^{(n)}$ $\forall n$

i.e. $x \le \lim_{n\to\infty} y^{(n)} \Rightarrow \boxed{x \le y} \Rightarrow$ y is a maximal solⁿ of the eqⁿ $x = Qx$. (element-wise)

→ Suppose :- $y \neq 0$

Define :- $c = \sup_{i \in \tilde{S}} y_i \in [0,1]$. Now, $y \le c \cdot 1$

Now, we know $y = Q^{(n)} \cdot y$, $\forall n$

$\Rightarrow y \le Q^n (c \cdot 1)$

$\Rightarrow y \le c(Q^n \cdot 1) = c(y^{(n)})$ $\forall n$

i.e. $y \le c\, y^{(n)}$ $\forall n$

$\Rightarrow y \le c\, y$ (Taking limit as $n \to \infty$.)

if $c < 1$, ⟶ Contradiction ($y < y$)

So, $\boxed{c = 1}$ or $\boxed{y = 0}$ Hence, $\boxed{\sup_{i \in \tilde{S}} y_i = 1}$ or $\boxed{y = 0}$

Theorem :- Consider an irreducible DTMC, $\{X_n\}_{n \ge 0}$ on $S = \{0, 1, 2, \ldots\}$ with TPM P, $\tilde{S} = S \setminus \{0\}$, Q is a restricting P to $\tilde{S}$ and let y is the maximal solⁿ to $y = Qy$. Then, DTMC is recurrent iff ~~[illegible]~~ $y = \bar{0}$

Intuition :- If you can't stay in $\tilde{S}$ forever ⟶ need to return to S_0

S_0 is recurrent

⟶ irreducible : DTMC is recurrent

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Page No.:

Date:

Proof :- $f_{00} = P_{00} + \boxed{\sum_{j \in \tilde{S}} P_{0j}(1 - y_j)}$

if $y = \bar{0}$ $\Rightarrow f_{00} = P_{00} + \sum_{j \in \tilde{S}} P_{0j} = \boxed{1}$

$\Rightarrow$ 0 is recurrent