TUTORIAL-1

Q1:) $E[n]$: expected no. of tosses before we get n-consecutive heads

Then,

(HHH.....) → T
$n-1$ H → H

$$E[n] = \begin{cases} E[n-1] + 1 & \text{if } H \\ (E[n-1]+1) + E[n] & \text{if } T \end{cases}$$

$$E[n] = \frac{1}{2}\left(E[n-1]+1\right) + \frac{1}{2}\left(E[n-1]+1+E[n]\right)$$

$$E[n] = 2\left(E[n-1]+1\right)$$

$$E[1] = \sum \left(1\cdot\frac{1}{2} + 2\cdot\frac{1}{2^2} + 3\cdot\frac{1}{2^3} + 4\cdot\frac{1}{2^4} + \cdots\right)$$

$$= 2 \quad \left[\text{it is the expected no. of tosses before getting a head}\right]$$

$$E[2] = 2(2+1) = 2^2+2$$

$$E[3] = 2(2^2+2+1) = 2^3+2^2+2$$

so,

$$E[n] = 2\left(2^{n-1}+2^{n-2}+\cdots+2^0\right)$$

$$= 2\left(\frac{1(1-2^n)}{1-2}\right) = 2(2^n-1)$$

so, $\boxed{E[n] = 2(2^n-1)}$ Ans.

If $A = \{X_{n+1} = j_1, \ldots, X_{n+k} = j_k\}$ and
$B = \{X_0 = i_0, \ldots, X_{n-1} = i_{n-1}\}$

we have

$$P(A \mid X_n = i, B) = P(A \mid X_n = i)$$

and so,

$$P(A \cap B \mid X_n = i) = P(A \mid X_n = i) \cdot P(B \mid X_n = i)$$

A and B are conditionally independent over $\{X_n = i\}$

$\Rightarrow$ Markov property is independent of the dir^n of times.

Theorem:- Let $\{Z_n\}_{n \geq 1}$ be an IID sequence of RVs with values in an arbitrary space F. Let E be a countable space, and $f: E \times F \longrightarrow E$ be some function. Let X_0 be a RV with values in E, independent of $\{Z_n\}_{n \geq 1}$. The recurrence rel^n

$$X_{n+1} = f(X_n, Z_{n+1})$$

then defines a HMC.

Proof:-

$$X_1 = f(X_0, Z_1)$$

$$X_2 = f(X_1, Z_2) = g_1(X_0, Z_1, Z_2) \quad \text{and so}$$

$$X_n = g_n(X_0, Z_1, Z_2, \ldots, Z_n)$$

$$P(X_{n+1} = j \mid X_n = i, \ldots)$$

$$\Rightarrow P(f(X_n, Z_{n+1}) = j \mid X_n = i, \ldots)$$

i.e.

$$P(f(i, Z_{n+1}) = j \mid \underbrace{X_n = i, \ldots}_{\Downarrow})$$

This event can be expressed in terms of $X_0, Z_1, Z_2, \ldots, Z_n$ and so is indep. of Z_{n+1}

$$\boxed{p_{ij} = P(f(i, Z_{n+1}) = j)} \Rightarrow \text{HMC}$$

Q1:) During day n, Z_{n+1} machines break down, and they enter a repair shop on day $(n+1)$, every day 1 machine is repaired.

X_n: no. of machines in shop on day 'n'.

$\Rightarrow$ $X_n = X_{n-1} + Z_n - 1$

$Z_n \Rightarrow$ IID and let $P(Z_1 = k) = a_k$ $(k \geq 0)$

Then, $\boxed{TPM = ?}$

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 1 & a_0 & a_1 & a_2 & \cdots \\ 2 & 0 & a_0 & a_1 & \cdots \\ 3 & 0 & 0 & a_0 & \cdots \end{array}$$

$X_1 = X_0 + Z_1 - 1$

$P(Z_1 = 0) = a_0$

$P(Z_1 = 1) = a_1$

$P(Z_1 = 2) = a_2$

and $P_{ij} = P(f(i, Z_{n+1}) = j)$

$X_{n+1} = (X_n + Z_{n+1} - 1)$

$f(X_n, Z_{n+1}) = X_n, Z_{n+1} - 1$

$= P(i + Z_{n+1} - 1 = j)$

$= P(Z_{n+1} = j - i + 1)$

$= P(Z_1 = (j-i) + 1)$ (except row 1)

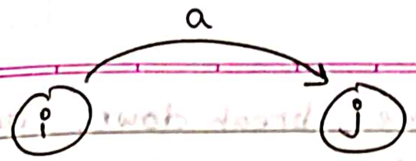
$= P(Z_1 = j - (i-1)^+)$

$= \boxed{a_{j-(i-1)^+}}$

$(i-1)^+ = \max(0, (i-1))$

STOCHASTIC AUTOMATA

→ A finite automation $(E; A, f)$ can read sequences of letters from a finite alphabet A written on some infinite tape. It can be in any state of a finite set. E, and its evolution is governed by a $f^n c$ $f: E \times A \to E$ as follows: when the automation is in state $i \in E$ and reads letters $a \in A$, it switches from state i to state $j = f(i, a)$ and then reads letters on the tape, from next letter to right.

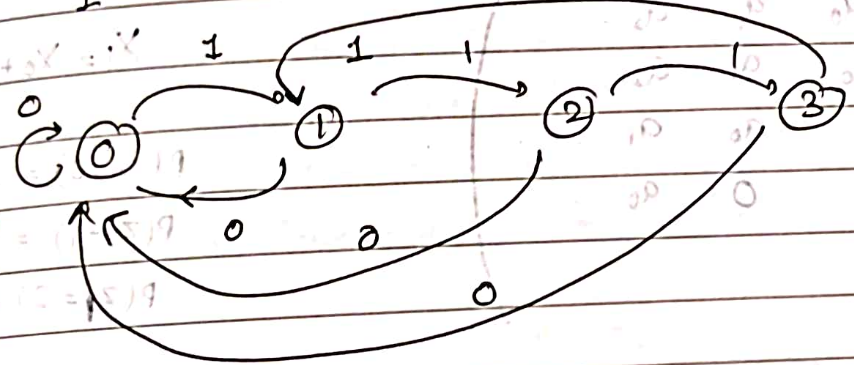


edge exists iff $\exists$ $a \in A$ and $j = f(i, a)$ and the edge receives a label a

if $j = f(i, a_1) = f(i, a_2)$ with $a_1 \neq a_2$, 2 edge from i to j

eg:- $A = \{0, 1\}$; the automation in state 0 reads the sequence from L to R, it passes successively through the states

1 0 0 1 1 1 1 0 0 1 1 1 1 1 1 0 1 0



states :- 0 1 0 0 1 2 3 1 0 0 1 2 3 1 2 3 0 1 0

$\Rightarrow$ If the sequence of letters read by automation is $\{Z_n\}_{n \geq 1}$, the sequence of states is $\{X_n\}_{n \geq 0}$, then,

$$X_n = \left(Z_n + Z_{n-1} + Z_{n-2}, \ 0 \text{ if } Z_n = 0 \right)$$

$$X_n = \begin{cases} 0 & \text{if } Z_n = 0 \\ Z_n & \text{if } Z_{n-1} = 0 \\ Z_n + Z_{n-1} & \text{if } Z_{n-2} = 0 \\ Z_n + Z_{n-1} + Z_{n-2} \end{cases} \qquad Z_n = \begin{cases} 0 & 1/2 \\ 1 & 1/2 \end{cases}$$

$$P = \begin{array}{c} i \backslash j \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ \frac{1}{2} & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & \frac{1}{2} & 0 & 0 \end{bmatrix} \end{array}$$

Closed set :- a set A is called closed if $\sum_{j \in A} P_{ij} = 1 \ \forall i \in A$

probability of absorption by a closed set — average times before absorption

FIRST-STEP ANALYSIS

Gambler's Ruin :- A: a, B: b (initial fortunes)

H → p
T ⇒ 1-p

outcomes are IID

X_n : fortune of A at time n, $X_{n+1} = X_n + Z_{n+1}$ $Z_n = +1 \ (p)$

$\{Z_n\}_{n \geq 1}$ is IID

$E = \{0, 1, 2, \dots a, \dots a+b = c\}$

dur^n is $T \Rightarrow$ first time where $X_n = 0$ or c

P(A wins) $\rightarrow$ $u(a) = P(X_T = c \mid X_0 = a)$

First step Analysis :- $u(i) = P(X_T = c \mid X_0 = i) \quad \forall \ (1 \leq i \leq c)$

generate a recurrence eq^n for $u(i)$'s → break down the event 'A wins' acc. to what can happen after the 1st step & using exclusive & exhaustive causes

if $X_0 = i \in [1, c-1]$, then $X_1 = i+1 \rightarrow p$

P(B's ruin starting with A having $i+1$) $= u(i+1)$

(" " $i-1$) $\Rightarrow u(i-1)$

so,

$$u(i) = p\,u(i+1) + q\,u(i-1) \longrightarrow \text{recurrence}$$

and also, $u(0) = 0 \quad u(c) = 1 \longrightarrow$ boundary

char eq^n : $p r^2 - r + q = 0 \rightarrow r_1 = 1$ and $r_2 = q/p$ $\quad p \neq q$

general sol^n : $u(i) = \lambda + \mu \left(\frac{q}{p}\right)^i \ (q \neq p)$ and $\lambda + \mu i \ (q = p)$

i.e. $u(i) = \dfrac{1 - (q/p)^i}{1 - (q/p)^c}$ $\quad = \dfrac{i}{c} \ \left(p = q = \frac{1}{2}\right)$

$v(i) = P(B \text{ wins with initial fortune } c-i,\ p = q = \tfrac{1}{2}) = \dfrac{c-i}{c}$ $= 1 - \frac{i}{c}$

i.e. $\boxed{u(i) + v(i) = 1}$ $\boxed{P(\text{game lasts forever}) = 0}$

So, $u(a) = \dfrac{a}{a+b}$ / ...

Let $Y = \{Y_n\}_{n\ge0}$ denote the MC obtained by delaying $X = \{X_n\}_{n\ge0}$ by one time unit :

$$Y_n = X_{n+1}$$

when $1 \le X_0 \le C-1$, the event " X is absorbed by 0" and " Y is absorbed by 0" are identical & so,

$$P(X \text{ is absorbed by } 0,\ X_1 = i\pm1,\ X_0 = i)$$
$$= P(Y \text{ is absorbed by } 0,\ X_1 = i\pm1,\ X_0 = i)$$

$\{Y_n\}_{n\ge0}$ & X_n are independent given X_1

RHS → $P(X_0 = i, X_1 = i\pm1) \cdot P(Y \text{ is absorbed by } 0 \mid Y_0 = i\pm1)$

and so

$$P(Y \text{ is absorbed by } 0 \mid Y_0 = i\pm1) = P(X \text{ is absorbed by } 0 \mid X_0 = i\pm1)$$

First-step Analysis → one state before other

First-step Analysis → average times before absorption

Gambler's Ruin (cont.) :-

the avg. duration, $m(i) = E[T \mid X_0 = i]$ → avg. durn of the game

$$m(i) = 1 + p\,m(i+1) + q\,m(i-1)$$

$m(0) = 0 \qquad m(C) = 0 \qquad y(1) = m(1)$

$$-1 = p(m(i+1) - m(i)) - q(m(i) - m(i-1))$$

$$y_i = m(i) - m(i-1)$$

$-1 = p\,y_{i+1} - q\,y_i \qquad m(i) = y_1 + y_2 + \dots + y_i$

$p = q = \frac{1}{2} \qquad -1 = \frac{1}{2}y_2 - \frac{1}{2}y_1$

$\boxed{-(i-1) = \frac{1}{2}y_i - \frac{1}{2}y_1} \Leftarrow \qquad -1 = \frac{1}{2}y_3 - \frac{1}{2}y_2$

$\vdots$

$y_i = y_1 - 2(i-1) \qquad -1 = \frac{1}{2}y_i - \frac{1}{2}y_{i-1}$

$$m(i) = i\,m(1) - 2(1+2+\cdots+(i-1)) = i\,m(1) - i(i-1)$$

$m(C) = 0 \rightarrow$

$$m(C) = i\,m(1) - i(i-1) \Rightarrow \cancel{C}\,m(1) = \cancel{C}(C-1)$$

$$m(1) = (C-1)$$

so, $m(i) = iC - \cancel{i} - i^2 + \cancel{i}$

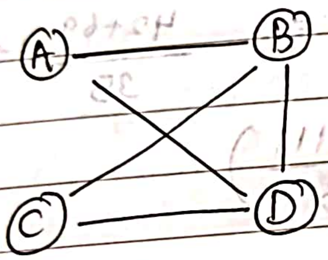
$$\boxed{m(i) = i(-i+C)}$$

$(c = a+b)$

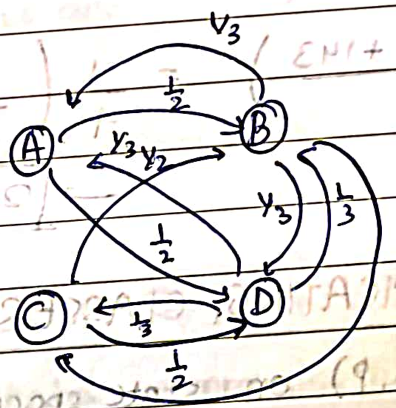
so, $\boxed{m(a) = a(C-a)}$ $\boxed{m(a) = ab}$!!

Tutorial-1

Q3)



DTMC model :-



$$P = \begin{array}{c|cccc} i\backslash j & A & B & C & D \\ \hline A & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ B & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ C & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ D & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{array}$$

$$\alpha = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \rightarrow \text{equally likely to be in any city.}$$

$$\nu_{AA} = E[\tau_{AA}] = \sum n\, f_{AA}^{(n)}$$

$$E[T \mid X_0 = A]$$

~~$f_{AA}^{(1)} = 0$ $f_{AA}^{(2)} =$~~

T: Time to reach state A.

$$m(i) = E(T \mid X_0 = i)$$

$m(A) = 0$ $\quad m(B) = 1 + \cancel{\frac{1}{3} m(A)}_{0} + \frac{1}{3} m(D) + \frac{1}{3} m(C)$

$m(C) = 1 + \frac{1}{2} m(A) + \frac{m(D)}{2}$ $\quad m(D) = 1 + \frac{1}{3} m(B) + \frac{1}{3} m(C)$

$m(D) = 1 + \frac{1}{3} m(B) + \frac{1}{3} + \frac{1}{6} m(B) + \frac{1}{6} m(D)$

$$\frac{5}{6} m(D) = \frac{4}{3} + \frac{1}{2} m(B) \Rightarrow m(D) = \frac{8}{5} + \frac{3}{5} m(B)$$

$$m(B) = 1 + \frac{8}{15} + \frac{1}{5} m(B) + \frac{1}{3} + \frac{1}{6} m(B) + \frac{8}{30} + \frac{1}{10} m(B)$$

$$\left(1 - \frac{2}{10} - \frac{1}{10}\right)(m(B)) = \left(1 + \frac{16+8}{30} + \frac{1}{2}\right)$$

$$\frac{7}{10} m(B) = \frac{3}{2} + \cancel{\frac{24}{30}}\ \frac{8}{10}\ \frac{4}{5} = \frac{15+8}{10} = \frac{23}{10}$$

$$\boxed{m(B) = \frac{23}{7}}$$

$$m(D) = \frac{8}{5} + \frac{69}{35} = \frac{42+69}{35} = \frac{111}{35}$$

$$m(C) = 1 + \frac{1}{2}\left(\frac{105+111}{35}\right)$$

$$m(C) = 1 + \frac{1}{\cancel{2}} \frac{\overset{108}{\cancel{216}}}{35} = \frac{108+35}{35} = \frac{143}{35}$$

$$\frac{1}{4}\left(\frac{105+111+143}{35}\right) = \frac{1}{4}\left(\frac{359}{35}\right)$$

$$\rightarrow \boxed{2.56}$$

COMMUNICATING CLASSES

Tut Q2 $\Rightarrow$ $\{X_n\}_{n \geq 0}$ with (α, P) on state space, S.

$\rightarrow$ state j is reachable from state i if $\exists$ $n \geq 0$ s.t. $P_{ij}^{(n)} > 0$ (note: $P_{ii}^{(0)} = 1 \ \forall \ i \in S$)

$\Rightarrow$ we say that i and j communicate if i is reachable from j and j is reachable from i.

(1) i and i always communicate, since, $P_{ii}^{(0)} = 1$ i.e. $\exists$ a $n=1$ s.t. $P_{ii}^{(n)} > 0$ $(1>0)$.

② if i communicates with j ⇒ i is reachable from j and j is reachable from i.

So, $\exists\ n_1 > 0$ s.t. $P_{ij}^{(n_1)} > 0$ and $\exists\ n_2 > 0$ s.t. $P_{ij}^{(n_2)} > 0$

Now,

j is reachable from i & i is reachable from j

⇒ j communicates with i

③ $i \boxed{C} K$ and $K \boxed{C} j$

⇒ $\exists\ n_1, n_2 > 0$ s.t. $P_{iK}^{(n_1)} > 0$, $P_{Ki}^{(n_2)} > 0$ and $\exists\ n_3, n_4$ s.t. $P_{Kj}^{(n_3)} > 0$, $P_{jK}^{(n_4)} > 0$

Now, $P_{ij}^{(n_1+n_3)} = \sum_{K \in S} P_{iK}^{(r)} \cdot P_{Kj}^{((n_1+n_3)-r)} > 0$ as we have at least one non-zero term

and similarly, $P_{ji}^{(n_2+n_4)} > 0$

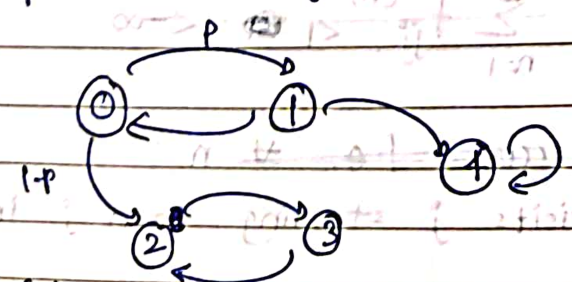
so, i and j also communicate.

④ $\tilde{S} \subseteq S$ is communicating class if $\forall\ i, j \in \tilde{S}$; i communicates with j.

→ If any state $k \in \tilde{S}_i$, $k \notin \tilde{S}_r$ where $r \neq i$ as then we have all the states communicating, i.e., a state can belong to only one class.

→ All the states communicate with themselves at least, and so are always a part of a class. Hence, $\tilde{S}$ form a partition.

⑤



$\{0, 1\}$

$\{4\}$ (0/1/2/3 not reachable from 4)

$\{2, 3\}$

⑥ $f_{00}^{(n)} = \begin{cases} 0 & n: \text{odd} \\ (p(1-p))^{n/2} & n: \text{even}, \end{cases}$

Borel–Cantelli Lemma's

① Let A_n be a sequence of events with an individual event occuring at time 'n' with $P(A_n)$. If the sum of probabilities of A_n is finite, i.e. $\sum_{n=1}^{\infty} P(A_n) < \infty$, then these events occur finitely often, P(infinitely many events occur) = 0,

or $P(\lim_{n\to\infty} \sup A_n) = 0$.

$\Rightarrow$ if $\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow P(A_n \text{ occurs i.o.}) = 0$

② If the sum of probabilities of an independent sequence of events A_n is infinite, s.t. $\sum_{n=1}^{\infty} P(A_n) > \infty$, then the probability that they occur ~~infinitely~~ infinitely often is 1, i.e. $P(\lim_{n\to\infty} \sup A_n) = 1$

$\Rightarrow$ if $\sum_{n=1}^{\infty} P(A_n) = \infty \Rightarrow P(A_n \text{ occurs i.o.}) = 1$

Claim :- If $f_{jj} < 1$, then, $P((\#\text{ visits to state } j) < \infty) = 1$

$f_{jj} = \sum_{n=1}^{\infty} f_{jj}^{(n)} \longrightarrow$ probability that it visits state j starting from j in n-steps.

~~A_n: it visits state j starting from state j in n steps. (for the 1st time)~~

$\cancel{\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} f_{jj}^{(n)} < 1}, < \infty$

~~so, A_n occurs f.o. $\forall n$~~

Also, let A_n: it visits j starting from j in the n^{th}-step

$P(A_n) = f_{jj}^{(n)} + f_{jj}^{(n-1)} P_{jj} + f_{jj}^{(n-2)} P_{jj}^{(2)} + \cdots$

and $\sum_{n=1}^{\infty} P(A_n) = \underbrace{(\sum f_{jj}^{(n)})}_{<1} + \underbrace{P_{jj} (\sum f_{jj}^{(n-1)})}_{< P_{jj}} + \underbrace{(P_{jj}^{(2)} \sum f_{jj}^{(n-2)})}_{< P_{jj}^{2}} + \cdots$

i.e. $\sum_{n=1}^{\infty} P(A_n) < 1 + P_{jj} + P_{jj}^2 + \cdots = \dfrac{1}{1 - P_{jj}}$

as $P_{jj} \neq 1$ because then $f_{jj} = 1$,

so, $\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow$ A_n's occur f.o.

$\Rightarrow$ there are finite visits of state j.

H.P.

Positive and Null Recurrences

$T_i = \min\{ n > 0 : X_n = i \mid X_0 = i \}$

we say that i is

(A) positive recurrent : $E[T_i] < \infty$

(B) null recurrent : $P(T_i < \infty) = 1$ but $E[T_i] = \infty$

(C) Transient : $P(T_i < \infty) < 1$

$\Rightarrow$ Finite-state MC's have no null-recurrent states

Def^n:- If there exists only one communication class, then the chain, its transition matrix and transition graph are said to be irreducible.

$$\Rightarrow \quad P_x(N_y = m) = \begin{cases} 1 - f_{xy} & ; \ m = 0 \\ f_{xy} f_{yy}^{m-1} (1 - f_{yy}) & ; \ m \in \mathbb{N} \end{cases}$$

starting from x

no. of visits to

state y

are m

$\Rightarrow P_x(N_y < \infty) = I_{\{f_{yy} < 1\}} + (1 - f_{xy}) I_{\{f_{yy} = 1\}}$

$\Rightarrow$ mean no. of visits starting from a state x is $E_x(N_y) = \dfrac{f_{xy}}{1 - f_{yy}} I_{\{f_{yy} < 1\}} + \infty \cdot I_{\{f_{xy} > 0,\, f_{yy} = 1\}}$

SYMMETRIC RANDOM WALK

A symmetric R.W. on $\mathbb{Z}$ can't have a stationary distrib^n.

balance eq^n :- $\pi(i) = \frac{1}{2}\pi(i-1) + \frac{1}{2}\pi(i+1)$ $\quad i \in \mathbb{R}^+$

$$\pi(i) = \pi(0) + (\pi(1) - \pi(0)) \cdot i \qquad (i \geq 0)$$

as $\pi(i) \in [0,1] \Rightarrow \pi(1) - \pi(0) = 0 \Rightarrow \pi(i) = \pi(0)$ (const)
and so, $\pi(i) = 0$ as total mass is finite

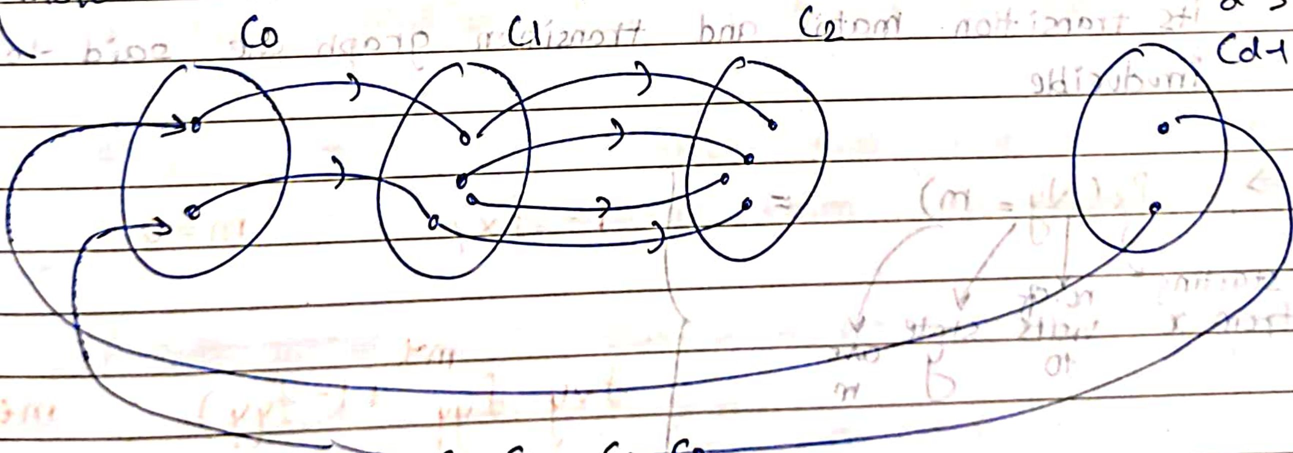
$\Rightarrow$ $\boxed{\pi(i) = 0 \ \forall i} \Longrightarrow$ NOT A P. Distrib^n !

PERIOD

Thm :- For any irreducible Markov chain, one can find an unique partition of E into d classes - $C_0, C_1, \ldots, C_{d-1}$ such that for all k and all $i \in C_k$,

$$\boxed{\sum_{j \in C_{k+1}} p_{ij} = 1}$$

and, by convention $C_d = C_0$, and where d is maximal, i.e. (there is no other such partition $C'_0, C'_1, \ldots, C'_{d'-1}$ with $d' > d$)



so, $P = \begin{array}{c|cccc} & C_0 & C_1 & C_2 & C_3 \\ \hline C_0 & 0 & A_0 & 0 & 0 \\ C_1 & 0 & 0 & A_1 & 0 \\ C_2 & 0 & 0 & 0 & A_2 \\ C_3 & A_3 & 0 & 0 & 0 \end{array}_{4\times4}$ $\qquad P^2 = \begin{pmatrix} 0 & 0 & B_0 & 0 \\ 0 & 0 & 0 & B_1 \\ B_2 & 0 & 0 & 0 \\ 0 & B_3 & 0 & 0 \end{pmatrix}_{4\times4}$

$P^d \longrightarrow$ BLOCK-diagonal form

Q:) Is there, in C_0, more than one P^d - communication class?

$$P^d = \begin{array}{c|c} & \begin{matrix} C_0 & C_1 & \cdots & C_{d-1} \end{matrix} \\ \begin{matrix} C_0 \\ C_1 \\ \vdots \\ C_{d-1} \end{matrix} & \begin{pmatrix} E_0 & & & \\ & E_1 & & \\ & & \ddots & \\ & & & E_{d-1} \end{pmatrix} \end{array} \rightarrow \text{stochastic matrix}$$

P-cyclic classes $C_0, C_1, \ldots$ are all in different P^d-communication classes

→ For an arbitrary TPM, not necessarily irreducible, the def^n of period is :-

The period d_i of state $i \in E$ is, by def^n,
$d_i = \gcd\{n \geq 1 : P_{ii}(n) > 0\}$

$d_i = +\infty$ if $\nexists\, n \geq 1$ with $P_{ii}(n) > 0$
$d_i = 1$ → the state i is aperiodic.

→ if i and j are communicating $d_i = d_j$

eg.- $E = \{0, 1, \ldots, N\}$

$$P = \begin{bmatrix} 0 & 1 & & & & \\ q_1 & r_1 & p_1 & & & \\ & q_2 & r_2 & p_2 & & \\ & & \ddots & & & \\ & & q_i & r_i & p_i & \\ & & & q_{N-1} & r_{N-1} & p_{N-1} \\ & & & & 1 & 0 \end{bmatrix}$$

$(p_i + q_i + r_i = 1)$ $\forall\, i$

→ Global balance eq^n ⇒
$\pi(i) = q_{i+1}\pi(i+1) + r_i\pi(i) + p_{i-1}\pi(i-1)$

and
$\pi(0) = \pi(1) q_1$ $\pi(N) = \pi(N-1) \cdot p_{N-1}$

also $\sum_{n=0}^{N} \pi(i) = 1$. $r_i = 1 - p_i - q_1$

$\pi(i+1) q_{i+1} - \pi(i) p_i = \pi(i) q_1 - \pi(i-1) p_{i-1}$

Page No.:
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$$\pi(1)\, q_1 - \pi(0) = 0$$

$$\pi(2)\, q_2 - \pi(1)\, p_1 = \pi(1)\, q_1 - \pi(0) = 0$$

so,

$$\boxed{\pi(i+1)\, q_{i+1} = \pi(i)\, p_i}$$

$$\pi(1) = \frac{\pi(0)}{q_1} \qquad \pi(2) = \pi(1)\,\frac{p_1}{q_2} = \frac{p_1}{q_2}\left(\frac{\pi(0)}{q_1}\right)$$

and so, $\pi(i) = \left(\dfrac{p_1\, p_2 \ldots p_{i-1}}{q_1\, q_2 \ldots q_i}\right) \pi(0) \qquad (2 \le i \le N)$.

and to find $\pi(0)$

$$\sum_{i=1}^{N} \pi(i) = 1$$