Q.1 Let X be a random variable from $(\Omega, \mathcal{F}, P)$ to $(\mathcal{X}, \mathcal{B}, P_X)$. Show the following:

(a) If $A = X^{-1}(B)$, then $A^c = X^{-1}(B^c)$.

(b) If $A_n = X^{-1}(B_n)$ $\forall n = 1, \dots$, then $\bigcup_{n=1}^{\infty} A_n = X^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right)$.

(c) If $A_1 = X^{-1}(B_1)$ and $A_2 = X^{-1}(B_2)$ and $B_1 \cap B_2 = \phi$, then $A_1 \cap A_2 = \phi$.

(d) Show that $\{A : \exists B \in \mathcal{B} \text{ s.t. } X^{-1}(B) = A\}$ is a σ-field on Ω and is a subset of $\mathcal{F}$.

Remark: $\{A : \exists B \in \mathcal{B} \text{ s.t. } X^{-1}(B) = A\}$ is called σ-field generated by X. Moreover, X can provide information only about this σ-field. This σ-field is denoted by $X^{-1}(\mathcal{B})$.

(e) Show that P_X is a prob. measure of $(\mathcal{X}, \mathcal{B})$.

Q.2 Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{F} = \mathcal{P}(\Omega)$. Let X be defined as follows:

$$
\begin{aligned}
X(\omega) &= 0 \quad \text{if } \omega = 1, 2, 3 \\
&= \omega \quad \text{o.w.}
\end{aligned}
$$

(a) Verify if X is a valid random variable.

(b) Find a set in $\mathcal{F}$, but not in $X^{-1}(\mathcal{B})$.
(Use your intuition to find a set and then prove).

(c) Find distribution $F_X(\cdot)$. How many points of discontinuity $F_X(\cdot)$ have?

Q.3 Show that a distribution fn can have at most countable (finite/infine) number of discontinuities.

Q.4 Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two collection of subsets of Ω. Also, let Σ_1 and Σ_2 denote the σ-fields generated by $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. If $\mathcal{A}_1 \subseteq \Sigma_2$ and $\mathcal{A}_2 \subseteq \Sigma_1$, then $\Sigma_1 = \Sigma_2$.

$\boxed{Q.5}$ Use Q.4 to show that the smallest σ-field containing $\{(x_1, x_2) : x_1 < x_2, \ x_1, x_2 \in \mathbb{R}\}$ is the Borel σ-field.