EE621: Markov Chains and Queueing Systems
Quiz # 1 (20 Marks)

Time:- 9:30 - 11 am Date:- 03/02/2023

Q.1 An EE621 student visits lake everyday morning and makes the following observations about the bird population:

1. If on a given day no birds are seen, then next day exactly one bird is seen.
2. If i birds are observed on a day, then on the next day either all birds are gone with prob. $1-g(i)$ or one extra bird is seen with prob. $g(i)$.

Let X_n denote number of birds observed on n^{th} day.

(a) Prove that $\{X_n\}_{n \geq 0}$ is a DTMC. **[2 Marks]**

(b) Find TPM of the DTMC and draw state transition diagram. **[3 Marks]**

(c) In the following 3 cases find type (transient, null recurrent, or positive recurrent) for state 100.

(i) $g(i) = 1/2 \quad \forall i.$

(ii) $g(i) = 1/(i+1) \quad \forall i.$ **[5 Marks each]**

(iii) $g(i) = (1/2)^i \quad \forall i.$

Solutions

(a) x_n denotes population of birds observed on the n^{th} day.

Hence, $x_n \in \{0, 1, 2, \dots\}$ $\forall n$

Thus, $S = \{0, 1, 2, \dots\}$ is the state space for $\{X_n\}_{n \geq 0}$ $\Rightarrow$ $\{X_n\}_{n \geq 0}$ is a chain.

Now, we just need to establish the MP.

Consider, for any n

$$P(X_{n+1} = j \mid X_0, \dots, X_{n-1}, X_n = i)$$

$$= \begin{cases} 1 & \text{if } i = 0 \;\&\; j = 1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 1 - g(i) & \text{if } i > 0 \;\&\; j = 0 \\ g(i) & \text{if } i > 0 \;\&\; j > 0. \end{cases}$$

Note that the one step transition prob depends only on the current state and the next state, past is irrelevant.

Now, consider for any n

$$P(X_{n+2} = j \mid X_0, \dots, X_{n-1}, X_n = i)$$

$$= \sum_{k \in S} P(X_{n+2} = j, X_{n+1} = k \mid X_0, \dots, X_n = i)$$

$$= \sum_{k \in S} P(X_{n+2}=j \mid X_0, \dots, X_n = i, X_{n+1}=k) \times P(X_{n+1}=k \mid X_0, \dots, X_n = i)$$

$$= \underbrace{\sum_{k \in S} P_{kj} P_{ik}}_{\text{independent of } k \text{ and depends only on } i \,\&\, j \,\&\, \text{equals } P(X_{n+2}=j \mid X_n = i)} \quad \leftarrow \text{ as shown above}$$

Continueing like this, we can show

$$P(X_{n+k}=j \mid X_0, \dots, X_n = i) = P(X_{n+k}=j \mid X_n = i).$$

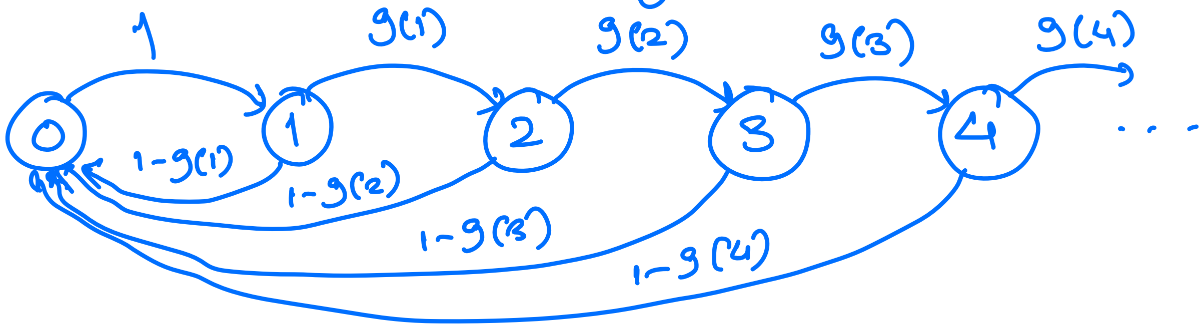
This proves the Markov property.

$\Rightarrow \{X_n\}_{n \geq 0}$ is a DTMC.

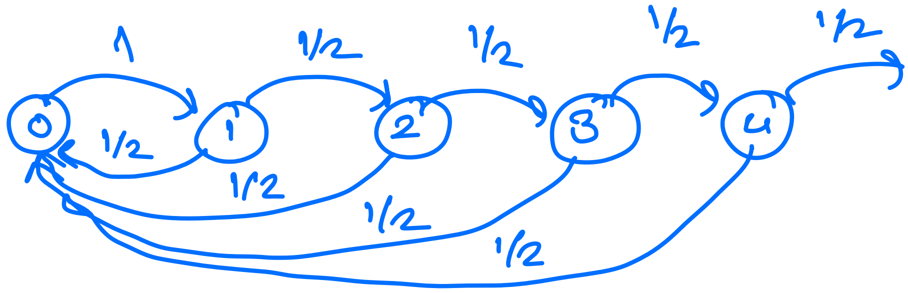
(b) Transition prob matrix (TPM)

$$
\begin{array}{c|ccccc}
\text{from} \backslash \text{to} & 0 & 1 & 2 & 3 & \\
\hline
0 & & & & & \\
1 & & & & & \\
2 & & & & & \\
3 & & & & & \\
4 & & & & & \\
\vdots & & & & &
\end{array}
\quad
P = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots \\
1-g(1) & 0 & g(1) & 0 & \cdots \\
1-g(2) & 0 & 0 & g(2) & \cdots \\
1-g(3) & 0 & 0 & 0 & g(3) \cdots \\
1-g(4) & 0 & 0 & 0 & 0 \\
\vdots & & & &
\end{bmatrix}
$$

State transition diagram



(c) (i)



Now consider $f_{00}^{(n)}$ and observe.

$$f_{00}^{(1)} = 0 \qquad f_{00}^{(3)} = \left(\tfrac{1}{2}\right)^3$$

$$f_{00}^{(2)} = \left(\tfrac{1}{2}\right)^2 \qquad f_{00}^{(4)} = \left(\tfrac{1}{2}\right)^4 \qquad \Rightarrow f_{00}^{(n)} = \left(\tfrac{1}{2}\right)^n.$$

Now, $$\nu_{00} = \sum_{n=1}^{\infty} n f_{00}^{(n)}$$

$$= \sum_{n=2}^{\infty} n \left(\tfrac{1}{2}\right)^n < \infty$$

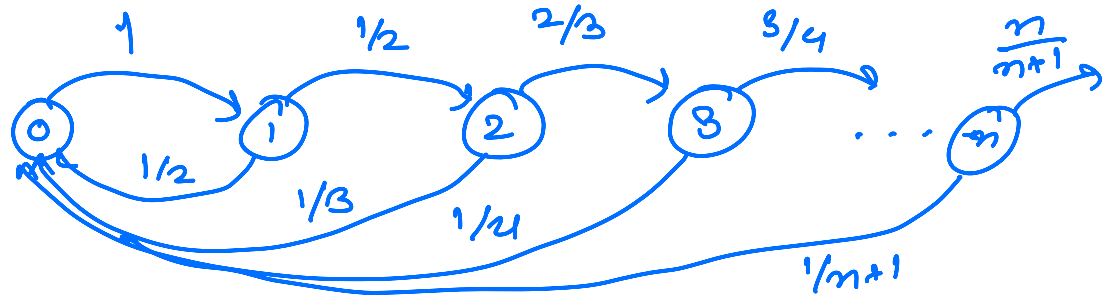
$\Rightarrow$ State 0 is positive recurrent

Also, for $g(i) = 1/2$ $\forall$ $i \geq 1$ and $P_{01} = 1$

$\Rightarrow$ DTMC is irreducible

$\Rightarrow$ State 100 is also positive recurrent.

(ii)



Now, $f_{00}^{(1)} = 0$

$f_{00}^{(2)} = \frac{1}{2}$, $f_{00}^{(3)} = \frac{1}{2} \cdot \frac{1}{3}$

In general only way to return to 0 in exactly n step starting from 0, is by making $(n-1)$ forward step and then return

$$\Rightarrow f_{00}^{(n)} = \underbrace{\frac{1}{n}}_{\text{prob. of returning from } (n-1)} \underbrace{\prod_{i=1}^{n-2} \frac{i}{i+1}}_{\text{prob of reaching state } (n-1) \text{ from } 0.}$$

Now, $\prod_{i=1}^{N} \frac{i}{i+1}$ is telescopic product.

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdots \frac{\cancel{N}}{N+1} = \frac{1}{N+1}.$$

$$\Rightarrow f_{00}^{(n)} = \frac{1}{n(n-1)}:$$

Now consider, $\sum_{n=1}^{\infty} f_{00}^{(n)} = \sum_{n=2}^{\infty} \frac{1}{n(n-1)}.$

$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right]$$

Again note that

$$\sum_{n=1}^{N}\left[\frac{1}{n}-\frac{1}{n+1}\right] = 1-\frac{1}{N+1}$$

Thus $$\sum_{n=1}^{\infty} f_{00}^{(n)} = \lim_{N\uparrow\infty}\left[1-\frac{1}{N+1}\right] = 1.$$

$\Rightarrow$ state 0 is recurrent. Now, consider

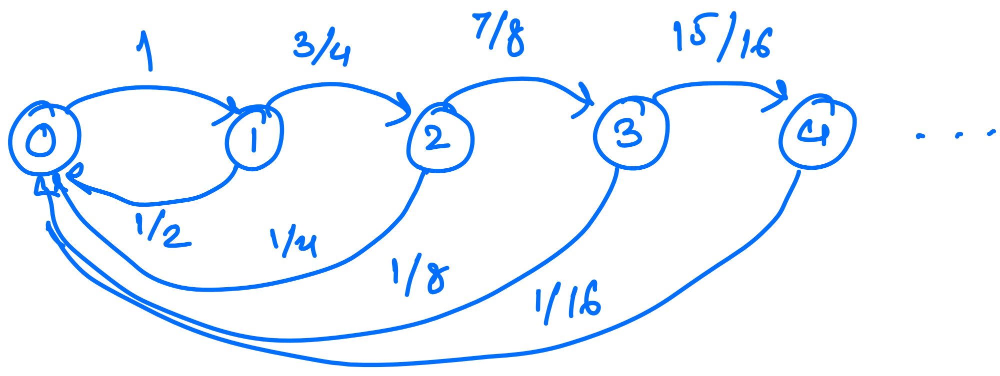
$$\nu_{00} = \sum_{n=2}^{\infty} n f_{00}^{(n)} = \sum_{n=2}^{\infty} \cancel{n}\cdot\frac{1}{\cancel{n}(n-1)}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n} = \infty .$$

$\Rightarrow$ 0 is null recurrent.

$\because$ the DTMC is irreducible for this choice of $g(\cdot)$, state 100 is also null recurrent.

(iii)



Note that $$f_{00}^{(n)} = \underbrace{\left(\tfrac{1}{2}\right)^{n-1}}_{\substack{\text{prob of} \\ \text{returning from} \\ n-1 \text{ to } 0}} \underbrace{\prod_{i=1}^{n-2}\left(1-g(i)\right)}_{\substack{\text{prob of reaching } (n-1) \\ \text{from } 0 \\ <1}}$$

Note that $f_{00}^{(n)} < \left(\frac{1}{2}\right)^{n-1}$

Thus, $\sum_{n=1}^{\infty} f_{00}^{(n)} < \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{n-1}$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n} = 1.$$

$\Rightarrow \sum_{n=1}^{\infty} f_{00}^{(n)} < 1 \Rightarrow 0$ is transient.

Again DTMC is irreducible, so 100 is transient.

—— x —— x —— x ——