1. Let $\Omega = \{1, 2, \cdots, 6\}$. Provide 3 distinct σ-fields on Ω.

2. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two σ-fields on Ω. Then, prove or disprove:

   a. $\mathcal{F}_1 \cup \mathcal{F}_2$ is a σ-field on Ω.

   b. $\mathcal{F}_1 \cap \mathcal{F}_2$ is a σ-field on Ω.

set of reals

3. Let $\mathcal{B}$ denote the σ-field (on $\mathcal{R}$) generated by a collection $\{(-\infty, x] : x \in \mathcal{R}\}$. Show that following type of sets belong to $\mathcal{B}$:

   (a) $(-\infty, x)$ (b) $(x, +\infty)$ (c) $[x, +\infty)$ (d) $[x_1, x_2]$ (e) (x_1, x_2) (f) $[x_1, x_2)$ (e) $(x_1, x_2]$.

4. Let $\{x_n\}_{n \geq 1}$ be a sequence of real numbers. Define, $y_n = \inf\limits_{k \geq n} x_k$ & $z_n = \sup\limits_{k \geq n} x_k$.

   a. Prove that $\{y_n\}_{n \geq 1}$ and $\{z_n\}_{n \geq 1}$ converge or diverge, but never oscillate. Hence, $\lim\limits_{n \uparrow \infty} y_n$ and $\lim\limits_{n \uparrow \infty} z_n$ is well define.

   b. Show that $\{x_n\}_{n \geq 1}$ converges to x if and only if $x = \lim\limits_{n \uparrow \infty} y_n = \lim\limits_{n \uparrow \infty} z_n$.

$\lim\limits_{n \uparrow \infty} y_n$ is called $\liminf\limits_{n \uparrow \infty} x_n$ and $\lim z_n$ is called $\limsup x_n$.

5. For a sequence of sets $\{A_n\}_{n\geq 1}$, define

$$\limsup_{n\uparrow\infty} A_n = \{\omega : \limsup_{n\uparrow\infty} 1_{A_n}(\omega) = 1\} \;\&$$

$$\liminf_{n\uparrow\infty} A_n = \{\omega : \liminf_{n\uparrow\infty} 1_{A_n}(\omega) = 1\}.$$

Show the following:

1. $\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$ (denote by $\bar{A}$)
2. $\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$ (denote by $\underline{A}$)
3. If $\{A_n\}_{n\geq 1} \in \mathcal{F}$, then $\bar{A} \in \mathcal{F}$ & $\underline{A} \in \mathcal{F}$.
4. $\bar{A} \supseteq \underline{A}$.
5. show that $P(\underline{A}) \leq \liminf_{n\uparrow\infty} P(A_n)$.
6. show that $P(\bar{A}) \geq \limsup_{n\uparrow\infty} P(A_n)$
7. show that if $\underline{A} = \bar{A}$, then define $\lim_{n\uparrow\infty} A_n = \underline{A} = \bar{A}$ and show that $P(\bar{A}) = \lim_{n\uparrow\infty} P(A_n)$.