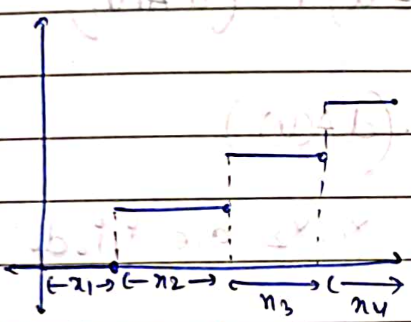
RENEWAL THEORY

⇒ Stochastic process $\{M(t)\}_{t \geq 0}$

→ Counting process, $M(0) = 0$

$M(t) \in \{0, 1, 2, \ldots\}$

$M(t) \leq M(t+\tau) \quad \forall\ \tau \geq 0$



→ each jump indicates an ⇑ in count.

⇒ $x_1, x_2, \ldots$ are called the lifetimes (eg. - turning on bulbs until they die & replacing them later.)

⇒ Life Times are INDEPENDENT.

⇒ $\{X_2, X_3, \ldots\}$ are iid variables (X_1 can be different because the time at which you start observing is almost unknown.)

$\underbrace{X_1 \sim A, \quad X_2 \sim F}_{\text{some distribution } f^n c}$

Define, $\underbrace{Z_K}_{\text{time of the } k^{th} \text{ increment}}$, $\boxed{Z_K = \sum_{i=1}^{k} X_i}$

$\Rightarrow$ Probability law for $\boxed{\{M(t)\}_{t\geq 0}}$ $\Longrightarrow$

$$P(M(t) \geq n) = \mathbb{P}(Z_n \leq t)$$

i.e., say, $P(M(t) \geq 1) = \mathbb{P}(Z_1 \leq t) = \mathbb{P}(X_1 < t) = A(t)$

(if at a time t, more than one bulbs are ON, it means that the 1^{st} transition must've occured before 't' itself.

$$P(M(t) \geq 2) = \mathbb{P}(Z_2 \leq t) = \mathbb{P}(X_1 + X_2 \leq t)$$

$$= \int_0^t \mathbb{P}(X_1 + X_2 \leq t \mid X_1 = u) \cdot (dA(u))$$

wow

$$= \int_0^t \mathbb{P}(X_2 \leq t-u \mid X_1 = u) \cdot (dA(u))$$

$$= \int_0^t \mathbb{P}(X_2 \leq t-u) \cdot (dA(u))$$

(since X_1, X_2 are i.i.d. RV's

$$P(X_1 + X_2 \leq t) = \int_0^t F(t-u)\, d(A(u)) = \boxed{F^* A(t)}$$

$$F^* A(t) = \int_0^t F(t-u) \cdot d(A(u))$$

$\Rightarrow$ $P(M(t) \geq 3) = P(X_1 + X_2 + X_3 \leq t)$

$$= \int_0^t (A^* F)(t-u) \cdot d(F(u))$$

⇒ since all $\{X_k\}_{k\geq 2}$ have same probability distribution, at every jump the process renews itself.
⇒ If we start observing from a jump point, we simply renew the process.

$$= \int_0^t P(X_2 + X_3 \leq t-u) \cdot d(A(u)).$$

$$= \int_0^t \underbrace{P(X_3 \leq t-u)}_{F*A(t-u)} \cdot d(F(u))$$

$$= \int_0^t A^*F(t-u) \cdot d(F(u)) \quad \Rightarrow P(M(t) \geq 3) = (A^* F^{(2)})(t).$$

So, $\boxed{P(M(t) \geq n) = (A^* F^{(n-1)})(t)}$

Renewal equation :- let $\boxed{m(t) = E[M(t)]}$ renewal f^nc $m: [0,\infty) \to [0,\infty)$ (deterministic; renewal f^nc)

→ At renewal points/epochs, you can start looking at the process beyond that point and it will look the same!

→ M(t) in itself is random, but has a fixed expectation.

$$m(t) = \sum_{k=0}^{\infty} k \cdot P(M(t) = k)$$

$$= \boxed{\sum_{k=1}^{\infty} P(M(t) \geq k)}$$ (alternate def^n of expectation value) [discrete, non-negative RV's]

$$\boxed{m(t) = \sum_{k=1}^{\infty} (A^* F^{(k-1)})(t)}$$

Lemma:- If $P(X_1 < \infty) = 1$ and $E[X_j] > 0$, then, $m(t) < \infty \;\; \forall t$
$\forall\, j \geq 2$

(all X_j have the same distrib^n for $j \geq 2$)

→ Here, we need to try to seperate out X_1 and the others.

If $E[X_j]=0$ $(\forall j\geq 2)$ $\Rightarrow$ $P(X_j=0)=1$ $\forall j\geq 2$

→ all events after x_1 happen at same time at x_1, and the process doesn't proceed further

Since, $E[X_j]>0$ $\Rightarrow$ Then, $\exists\ \delta>0$ s.t. $P(X_j>\delta)>0$ $\forall j\geq 2$

From the Borel-Cantelli Lemma,

$P(X_j\leq\delta \text{ i.o.})=1$

$B_j=\{X_j\leq\delta\}$ $\qquad$ $B_j^c=\{X_j>\delta\}$

$P(B_j)>0$ $\qquad$ $P(B_j^c)>0$

PROOF:- For any $n\geq 1$ and $0\leq m\leq n$, we have,

$$\boxed{F^{(n)}(t)\leq F^{(n-m)}(t)\cdot F^{(m)}(t)}$$

$P(X_1+X_2+\dots+X_{n+1}\leq t)$

↓ split in 'm'.

$(n-m)$ RV's $\qquad$ (m) RV's $\Rightarrow$ $\boxed{\text{iid sequence}}$

$P((n-m)\text{ variables}\leq t)$ and $P(m\text{ variables}\leq t)$

has more sol^n than $P(n\text{ variables}\leq t)$.

Mathematical Proof:-

$$F^{(n)}(t)=\int_0^t F^{(n-m)}(t-u)\,(dF^{(m)}(u))$$

$$\leq F^{(n-m)}(t)\int_0^t (dF^{(m)}(u))$$

$$\leq \boxed{F^{(n-m)}(t)\cdot F^{(m)}(t)}$$

hence proved

Note:- $F^{(nr+R)}(t)\leq F^{((n-1)r+R)}(t)\cdot F^{(r)}(t)$

applying again and again,

$$\boxed{F^{(nr+R)}(t)\leq\left(F^{(r)}(t)\right)^n\cdot F^{(R)}(t)}$$

$$\le \sum_{n=0}^{\infty}\sum_{k=0}^{r-1} F^{(k)}(t)\cdot\left(F^{(r)}(t)\right)^n \le \sum_{n=0}^{\infty}\sum_{k=0}^{r-1} 1\cdot\left(F^{(r)}(t)\right)^n$$

$$\le \sum_{n=0}^{\infty} r\left(F^{(r)}(t)\right)^n$$

$$m(t) = \sum_{n=1}^{\infty}\left(A^* F^{(n-1)}\right)(t)$$

$$\boxed{A^* F^{(n-1)}(t) \le (\le 1)\, A(t) F^{(n-1)}(t) \le F^{(n-1)}(t) \quad \forall t}$$

$$\le \sum_{n=0}^{\infty} F^{(n)}(t)$$

$$= \sum_{n=0}^{\infty} \underbrace{\sum_{k=0}^{r-1} F^{(nr+k)}(t)}_{\le r F^{(nr)}(t)} \quad : \text{true } \forall\, r \ge 1$$

$\le r F^{(nr)}(t)$ (true $\forall\, r\ge 1$)

SUM converges iff $\boxed{F^{(r)}(t) < 1}$

$$\le \sum_{n=0}^{\infty} r\left(F^{(r)}(t)\right)^n = \boxed{r\sum_{n=0}^{\infty}\left(F^{(r)}(t)\right)^n} \quad m(t)<\infty \;\; \forall\, t\in[0,\infty)$$

So, For any given t, $\exists\, r$ s.t. $F^{(r)}(t) \le 1$ (i.e. $P(X_2+\dots+X_{r+1}\le t) < 1$)

If $\nexists$ a r, $P(X_2+\dots+X_{r+1}\le t) = 1$ $\forall\, r\ge 1$ ⇐ we should have a ⊛.

~~$P(X_2\le t)=1$~~

$P(X_2+X_3\le t)=1 \rightarrow$ but $P(X_j>\delta)>0 \longrightarrow$ ~~$P(X_2+\dots+X_{r+1}>t)>0$~~ (For some r,δ) → and for this r, $P(X_2+\dots+X_{r+1}\le t)<1$

Elementary Renewal Theory :-

⇒ Also called the Strong Law of Large Numbers for Renewal Processes

Theorem :- Consider the following :-

1. $\forall\, k\ge 1$, $P(X_k<\infty)=1$ (all life times are proper RV's)
2. $0 \le E[X_1] \le \infty$
3. $\forall\, k\ge 2$, $E[X_k]<\infty$, let $E[X_2]=\frac{1}{\mu}$

Under these conditions, we have

a) $\boxed{\lim_{t\to\infty}\frac{m(t)}{t}=\mu}$ b) $\boxed{\lim_{t\to\infty}\frac{M(t)}{t}=\mu}$ almost surely (or w.p. 1)

For every ω, it is a fn of t ← $\left\{\frac{M(t)}{t}\right\}_{t\ge 0}$ ⟶ (deterministic process for some fixed ω)

⇓

$\lim_{t\to\infty}\frac{M(t)}{t}=\mu$ w.p. 1

⇓

$P\left(\left\{\omega : \lim_{t\to\infty}\frac{M(t,\omega)}{t}=\mu\right\}\right)=1$

⟶ there can be some ω for which the limit may not exist ⇒ w.p. 1 (AS)

$$P\left(\left\{\omega : \lim_{t\to\infty}\frac{M(t,\omega)}{t}=\mu\right\}\right)=1$$

$$\lim_{t\to\infty}\frac{m(t)}{t}=\frac{1}{E[X_2]}$$ ⟵ avg. life time ⇒ avg. period after which a renewal would occur

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$\boxed{E[X_2] = 1/\mu}$

$\boxed{m(t)/t}$

rate of renewal = actual rate at which renewals actually happen in the system.

as $t\to\infty$, the rate of renewals equals the expected renewals per unit time

(Strong LLN $\Rightarrow$ $Z_n = \sum_{k=1}^{n} X_k$, $\underbrace{\lim_{n\to\infty} \frac{Z_n}{n} = \frac{1}{\mu}}_{\text{strong law of large no's}}$ w.p. 1)

Can we say that $\lim_{t\to\infty} \frac{Z_{M(t)}}{M(t)} = \frac{1}{\mu}$ w.p. 1 ?

$Z_{M(t)}$ → Time of the last renewal before or at 't'

$Z_{M(t)+1}$ → Time of the first renewal after t

We need $M(t) \to \infty$ as $t\to\infty$.

(when we say $\lim_{t\to\infty} g(t) = \infty$ $\Rightarrow$ $\forall$ seq. $\{t_n\}_{n\geq 1}$, S.T. $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} g(t_n) = a$

$$\boxed{\lim_{n\to\infty} \frac{Z_{m(t_n)}}{m(t_n)}}$$

($Z_{M(t)}$: the time instance of the last renewal before/after t)

$m(t_1) = 1$

$m(t_2) = 2$

$m(t_3) = 4$

$Z_{m(t_1)} = Z_1$

$Z_{m(t_3)} = Z_4$

$Z_{m(t)+1}$ $\Rightarrow$ time instance of the first renewal after t.

it is a subset of $\left\{\frac{Z_n}{n}\right\}_{n\geq 1}$ ← $\left\{\frac{Z_{m(t_n)}}{M(t_n)}\right\}_{n\geq 1}$ → (it is an ω-wise convergence)

as $\left\{\frac{Z_n}{n}\right\}$ converges, so does $\left\{\frac{Z_{M(t)}}{M(t)}\right\}$

$$\boxed{\lim_{t\to\infty} M(t) = \infty \text{ almost surely}} \Rightarrow \text{CLAIM}$$

Suppose NOT,

let $P\left(\lim_{t\to\infty} m(t) = k\right) \Rightarrow P(X_1<\infty, \ldots, X_k<\infty, X_{k+1}=\infty)$

$\downarrow$ IID

$= P(X_{k+1}=\infty) = 0$

(as $P(X_n < \infty) = 1 \ \forall n \geq 1$)

$$\mathbb{P}\left(\lim_{t\to\infty} M(t) < \infty\right) = \mathbb{P}\left(\exists\, k<\infty \text{ s.t. } \lim_{t\to\infty} M(t) = k\right)$$

$$= \mathbb{P}\left(\bigcup_{k=1}^{\infty} \left\{\lim_{t\to\infty} M(t) = k\right\}\right)$$

(union-bound) $\leq \sum_{k=1}^{\infty} \mathbb{P}\left(\lim_{t\to\infty} M(t) = k\right) = 0$

(each case is mutually disjoint also)

as $\boxed{P\left(\lim_{t\to\infty} M(t) < \infty\right) = 0}$

and $\boxed{\lim_{t\to\infty} M(t) = \infty \quad \text{w.p. } 1}$ (almost surely)

$\lim_{t\to\infty} \frac{Z_{M(t)}}{M(t)} = \frac{1}{\mu}$ w.p. 1 $\left(\text{as } \lim_{n\to\infty} \frac{Z_n}{n} = \frac{1}{\mu},\ \lim_{t\to\infty} M(t) = \lim_{n\to\infty} n = \infty\right)$

Let $A = \left\{\omega : \frac{Z_n(\omega)}{n} \to \frac{1}{\mu}\right\}$ $(P(A)=1)$

$B = \left\{\omega : \lim_{t\to\infty} M(t) = \infty\right\}$ $(P(B)=1)$

Also, if $\omega \in A\cap B$, then, $\boxed{\frac{Z_{M(t)}(\omega)}{M_{(t)}(\omega)} \Rightarrow \frac{1}{\mu}}$ $\Rightarrow \left\{\omega : \frac{Z_{M(t)}}{M(t)} \to \frac{1}{\mu}\right\}$ is a superset of $A\cap B$

$\left(\frac{Z_{M(t)}}{M(t)} \to \frac{1}{\mu} \quad \text{w.p. } 1\right)$

$\Rightarrow P\left\{\omega : \frac{Z_{M(t)}}{M(t)} \to \frac{1}{\mu}\right\} \geq P(A\cap B) = 1$

also,

$$\frac{Z_{M(t)}}{M(t)} \leq \frac{t}{M(t)} \leq \frac{Z_{M(t)+1}}{(M(t)+1)} \cdot \frac{(M(t)+1)}{(M(t))}$$ (using sandwich theorem)

$$Z_{M(t)} \leq t \leq Z_{M(t)+1}$$

$\frac{1}{\mu} \longrightarrow \frac{1}{\mu} \longleftarrow \frac{1}{\mu}$

so, $\boxed{\frac{M(t)}{t} \longrightarrow \mu \quad \text{wp } 1}$ as $t\to\infty$.

We also must have $\lim_{n\to\infty} \frac{X_1}{n} \to 0$ w.p. 1

$$Z_{M(t)} \le t$$

$$E[Z_{M(t)}] \le t$$

$$E\left[\sum_{k=1}^{M(t)} X_k\right] \le t$$

$$Z_n = X_1 + \sum_{k=2}^{n} X_k$$

$$\frac{Z_n}{n} = \frac{X_1}{n} + \frac{1}{n}\sum_{k=2}^{n} X_k$$

$\rightarrow 0$

$$\frac{Z_n}{n} = \frac{X_1}{n} + \frac{(n-1)}{(n)} \cdot \frac{1}{(n-1)}\sum_{k=2}^{n} X_k \rightarrow 1/\mu$$

$\frac{1}{\mu} \leftarrow$ $\rightarrow 0$

Now,

$$E\left[\lim_{t\to\infty} \frac{M(t)}{t}\right] = \lim_{t\to\infty} E\left(\frac{M(t)}{t}\right)$$

For iid rv's

$$E\left[\sum_{k=1}^{n} X_k\right] = E[X_k] \times (n)$$

and

$$\boxed{E\left[\sum_{k=1}^{N} X_k\right] = E[N]\, E[X_k]}$$

$$E[Z_{M(t)}] = E\left[\sum_{k=1}^{M} X_k\right]$$

$$E[Z_{M(t)}] = E[M(t)] \cdot E[X_k]$$

$$\boxed{E[Z_{M(t)}] = m(t)\, E[X_k]}$$

Note :-

$$t \ge Z_{M(t)}$$

$$t \ge m(t) \cdot \frac{1}{\mu} \quad \forall\, t$$

$$\frac{m(t)}{t} \le \mu \quad \forall\, t$$

$$E[Z_{m(t)+1}] = \frac{(m(t)+1)}{\mu} \qquad t < \frac{m(t)+1}{\mu} \implies \mu < \frac{m(t)}{t} + \frac{1}{t} \quad \forall t$$

$$\boxed{\frac{m(t)}{t} \le \mu \le \frac{m(t)}{t} + \frac{1}{t}}$$

(The result now follows from SANDWICH THM.)

$\Rightarrow$ Now, we have $E\left[\sum_{K=1}^{N} X_K\right] = \sum_{n=1}^{\infty} E\left[\sum_{K=1}^{N} X_K \,\middle|\, N=n\right] \cdot P(N=n)$

$$= \sum_{n=1}^{\infty} E\left[\sum_{k=1}^{n} X_K\right] P(N=n)$$

$$= \sum_{n=1}^{\infty} n\,E[X_1]\,P(N=n) \qquad \left[\begin{array}{c} E[X_K] \text{ is the} \\ \text{same } \forall\, k \end{array}\right]$$

$$= E[X_1]\,E[N]$$

→ This can be done only if X_k's and N are independent.

$\Rightarrow$ But, in the previous proof ; $\boxed{M(t) \text{ and } X_K \text{ are NOT independent.}}$

WALD'S LEMMA :-

Let N be a stopping time for process $\{X_n\}_{n\geq 1}$ if

i) $E[N] < \infty$

ii) $E[|X_n|] \leq B$ (constant) $< \infty$

iii) $E[X_n] = E[X_1] \quad \forall\; n \geq 2$

Then, again, $E\left[\sum_{n=1}^{N} X_n\right] = E[N]\,E[X_1]$

Proof:- $E\left[\sum_{k=1}^{N} X_k\right] = E\left[\sum_{k=1}^{\infty} I_{\{K\leq N\}} \cdot X_K\right]$

1 if $K \leq N$, 0 o/w

as N is a stopping time, $I_{\{K\leq N\}}$ is a f^nc of $x_1, x_2, \ldots, x_{k-1}$

$\left(I_{\{k\leq N\}} = I_{\{k-1<N\}} = g(x_1, \ldots, x_{k-1}) \right)$

assuming that we can switch E and Σ,

$$\sum_{k=1}^{\infty} E[X_K]\,E[I_{\{N\geq k\}}] = E[X_1] \sum_{k=1}^{\infty} P(N\geq k)$$

$$= E[X_1]\,E[N]$$

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⇒ Proof for the swapping

Let $Y_n = \sum_{k=1}^{n} X_k I_{\{k \le N\}}$

$\{Y_n\}$ are monotones if X_n is positive,
However Wald's lemma doesn't say X_n should be positive.

DOMINATED CONVERGENCE THEOREM :-

$\{Y_n\}$ s.t. $|Y_n| \le Y$ for some y with $E[Y] < \infty$, then,

$$\lim E[Y_n] = E[\lim Y_n]$$

Now, lets define, $Y = \sum_{k=1}^{\infty} |X_k| I_{\{N \ge k\}}$

$$E[Y] = \sum_{k=1}^{\infty} E(|X_k|) P(N \ge k)$$

monotones

By BCT, $\le E[N] \cdot B. < \infty$ ⇒ Σ and E can be swapped

$$|Y_n| = \left|\sum_{k=1}^{n} X_k I_{\{N \ge k\}}\right| \le \sum_{k=1}^{n} |X_k| I_{\{N \ge k\}}$$

(Trivial Δ-inequality)

$|Y_n| \le Y$ and $E[Y] < \infty$ ⇒ E and Σ can be swapped for Y_n even if limit of Σ is ∞.

⇒ IS $M(t)$ A STOPPING TIME ?

To use Wald's lemma on $\sum_{k=1}^{M(t)} X_k$, it must be a stopping time.

$$\{M(t) > n\} = \{M(t) \ge n+1\} = \{Z_{n+1} \le t\}$$
$$= g(X_1, X_2, \ldots, X_{n+1})$$ as Z_{n+1} depends on them.

$\Rightarrow$ However, if $M(t)$ is a stopping time, it should depend only on $\{x_1, \ldots, x_n\}$, and not x_{n+1}.

$\Rightarrow$ Let's see $\{M(t)+1 > n\} = \{M(t) \geq n\}$

$$= \{Z_n \leq t\}$$

$$= g(x_1, \ldots, x_n)$$

$\boxed{M(t)+1 \text{ is a stopping time}}$

Then, $E\left[Z_{M(t)+1}\right] = E\left[\sum_{k=1}^{M(t)+1} X_k\right] = E[M(t)+1]\, E[X_k]$

$$= (m(t)+1) \cdot \frac{1}{\mu}$$

so' $t < Z_{M(t)+1}$

$t < E[Z_{M(t)+1}]$

$t < (m(t)+1) \cdot \frac{1}{\mu}$

$$\boxed{\mu < \frac{m(t)}{t} + \frac{1}{t} \quad \forall\, t}$$

(LIMIT MIGHT NOT HOLD somewhere)

Consider the sequence $\{a_n\}$, $\lim_{n\to\infty} \sup a_n$ exists.

$\Rightarrow$ $b_n = \sup_{k \geq n} a_k = \sup\{a_n, a_{n+1}, \ldots\}$

$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} b_n$ (exists because b_n decreases monotonically)

$\Rightarrow$ similarly, let $C_n = \inf_{n \geq k} a_k$

and, $\lim_{n\to\infty} \inf. a_n = \lim_{n\to\infty} C_n \rightarrow C_n$ is increasing

Now, if $\lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} \sup a_n = a$, then $\lim_{n\to\infty} a_n = a$

we have $\mu \le \frac{m(t)}{t} + \frac{1}{t} \quad \forall\, t$

$$\mu \le \lim_{t\to\infty} \inf \frac{m(t)}{t} + \lim_{t\to\infty} \inf \cdot \frac{1}{t}$$

$$\mu \le \lim_{t\to\infty} \inf \cdot \frac{m(t)}{t}$$

$\Rightarrow$ For other side of the inequality

$$t \ge Z_{M(t)}$$

$t \ge E[Z_{M(t)}]$ $\quad$ M(t) is not a stopping time

$\hookrightarrow$ wald's Lemma $\times$

$\rightarrow$ constructing another renewal process with lifetimes $\{X_n^c\}_{n\ge 1}$

$$t < Z_{M(t)+1}$$

Suppose $\exists\, c$ S.T. $Z_{M(t)+1} \le t + c$

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Renewal Processes

Review :-

$\{M(t)\}_{t\geq 0}$ is a counting process, with $M(0) = 0$

Z_k denote the k^{th}-jump epoch of $M(t)$, then,

$$Z_k = \sum_{j=1}^{k} X_j \,,$$ where $\{X_j\}_{j\geq 1}$ are called lifetimes.

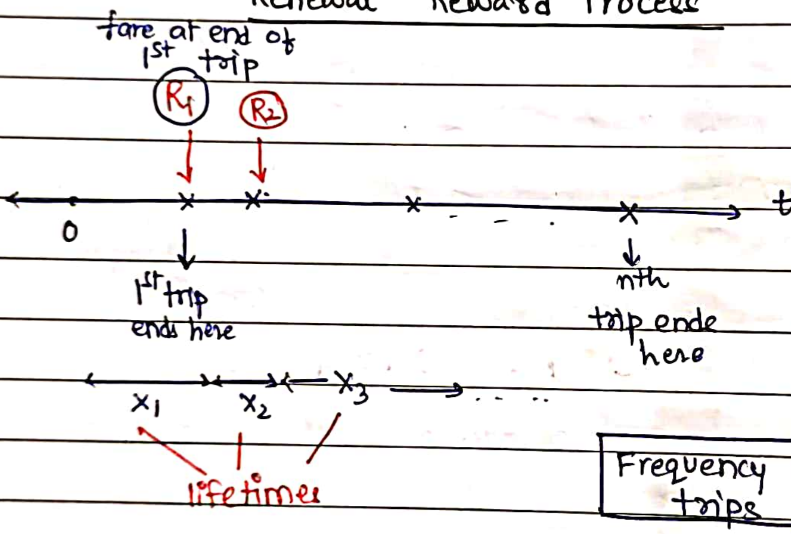
$P(X_j \geq 0) = 1$, X_j's are independent or $\{X_j\}_{j\geq 2}$ are i.i.d.

$\Rightarrow$ Elementary Renewal Theory :-

$$\lim_{t\to\infty} \frac{M(t)}{t} = \mu \quad \text{w.p. } 1 \quad \text{and}$$

$$\lim_{t\to\infty} \frac{m(t)}{t} = \mu \quad \text{where } E[X_2] = \frac{1}{\mu}$$

Renewal Reward Process



eg:- Uber-driver

$\boxed{\text{Frequency of trips} = ? \rightarrow 1/E[X_2]}$

Models for the fare:
- pre-paid
- $\boxed{\text{post-paid}}$
- Keep getting reward per unit time

Process :- $\{X_j, R_j\}_{j\geq 1}$

lifetimes — rewards

$\Rightarrow$ More the lifetime, more the rewards.

$\rightarrow$ R_j may depend on X_j $\Rightarrow$ $R_j = f(X_j)$

(not necessary linear)

NOT monotone rel^n necessary

(Joint distributions also must be considered)

Reward Process :- $\{R_j\}_{j\geq 1}$

⇒ R_j depends on X_j but not on X_i $\forall i \neq j$ (typically)

or,

$$\boxed{R_j \perp\!\!\!\perp X_i \quad \forall i \neq j}$$

as $X_i \perp\!\!\!\perp X_j \quad \forall i \neq j$ and if RV's are independent, their functions are also independent.

⇒ $\{R_j\}_{j\geq 1}$ is also a seq^n of independent RV's

→ Also as $\{X_j\}_{j\geq 2}$ are iid ⇒ $\{R_j\}_{j\geq 2}$ are also i.i.d. RV's

⇒ The joint distribution MUST be the same $(\{R_j, X_j\})$ can be a reward/a penalty (No restric^n to be non-negative)

⇒ Let, $C(t)$ denote the total reward earned until time t

$$\boxed{C(t) = \sum_{j=1}^{M(t)} R_j \quad : \text{POST-PAID}} \qquad \boxed{C(t) = \sum_{j=1}^{M(t)+1} R_j \quad : \text{PRE-PAID}}$$

$$C(t) = \sum_{j=1}^{M(t)} R_j + \underbrace{R(t)}_{\text{partial reward at } t} \quad : \text{UNIT-TIME reward model.}$$

RENEWAL REWARD THEOREM

If $\{(X_j, R_j)\}_{j\geq 1}$ is a renewal reward process, with $E[R_1] < \infty$ and $0 < E[X_1] < \infty$ and $E[X_2] > 0$ $(\mu < \infty)$, then,

a) $$\lim_{t\to\infty} \frac{C(t)}{t} = \frac{E[R_2]}{E[X_2]} \quad \text{w.p. } 1$$

b) $$\lim_{t\to\infty} \frac{c(t)}{t} = \frac{E[R_2]}{E[X_2]} \quad \text{w.p. } 1 \quad \text{where, } \boxed{c(t) = E[C(t)]}$$

say $R_j = 1$ w.p. 1 ⇒ $C(t) = M(t) \longrightarrow \lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{E[X_2]} \neq E\left[\frac{1}{X_2}\right]$

PROOF :- $C(t) = \sum_{k=1}^{M(t)} R_k$

$$\frac{C(t)}{t} = \frac{1}{t}\sum_{k=1}^{M(t)} R_k = \underbrace{\frac{M(t)}{t}}_{\text{as } t\to\infty} \cdot \underbrace{\frac{1}{M(t)}\sum_{k=1}^{M(t)} R_k}_{\left(\text{as } t\to\infty,\ M(t)\to\infty \text{ w.p. } 1\right)}$$

$\rightarrow \frac{1}{E[X_2]}$ w.p. 1 by ERT

$\longrightarrow E[R_2]$ w.p. 1 by Strong Law of Large No.'s (Strong LLN)

$$\boxed{\begin{array}{l} P(X_1<\infty)=1 \\ \infty > E[X_2] > 0 \ (\mu>0) \\ P(X_2=0)<1 \end{array}}$$

$\Rightarrow \exists\ A \subseteq \Omega$, $A\in\mathcal{F}$ and $P(A)=1$ and

$$\lim_{t\to\infty} \frac{M(t,\omega)}{t} = \frac{1}{E[X_2]}$$

and, $\exists\ B\subset\Omega$, $B\in\mathcal{F}$, S.T. $P(B)=1$ and

$$\lim_{t\to\infty} \frac{1}{M(t,\omega)} \sum_{k=1}^{M(t,\omega)} R_k(\omega) = E[R_2]$$

$\forall\ \omega\in(A\cap B)$.

$\longrightarrow$ since the probabilities are exactly 1, it does NOT matter.

(set of $P=0$ is independent of everything else
set of $P=1$ is " " " " ??)

EXAMPLE OF Renewal Reward Process

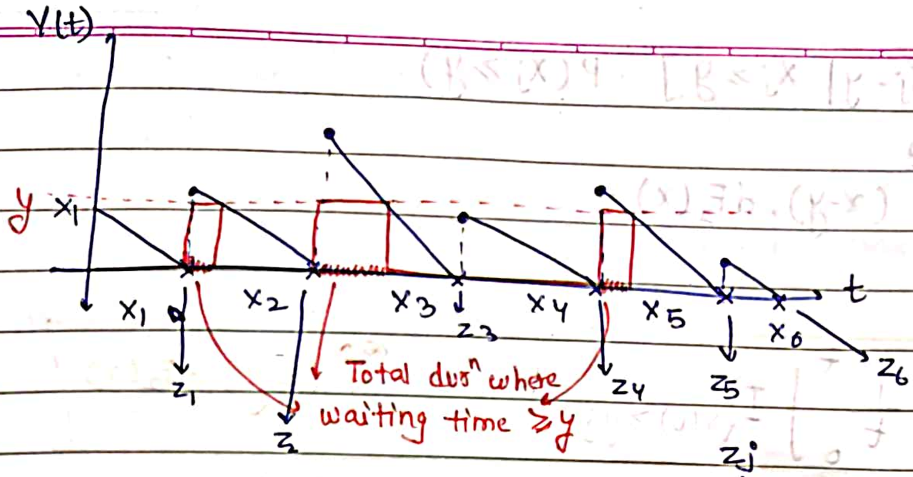
Define a residual time process

eg.:- $\{Y(t)\}_{t\geq 0}$ as $\boxed{Y(t) = Z_{M(t)+1} - t}$

(Time until the next renewal)

$$\lim_{t\to\infty} \underbrace{\frac{1}{t}\int_0^t I_{\{Y(u)\geq y\}}\, du}$$ [time to wait for the next bus]

frac^n of time for which waiting time $\geq y$ $\left[\frac{\text{Total duration for which residual time} \geq y \text{ in } [0,t]}{t}\right]$



We can define rewards as $R_j = \int_{z_{j-1}}^{z_j} I_{\{Y(u) \geq y\}} du$

(during the X_jth lifetime) (For how much time was the process Y, bigger than y)

$$C(t) = \sum_{j=1}^{M(t)} R_j + \int_{z_{M(t)}}^{t} I_{\{Y(u) \geq y\}} \cdot du$$

⇒ To show that $\{(X_j, R_j)\}_{j \geq 1}$ is a R.R.P., we know,

$$\int_{z_{j-1}}^{z_j} I_{\{Y(u) \geq y\}} \cdot du \equiv \int_0^{X_j} I_{\{Y(u - z_{j-1}) \geq y\}} du$$

$$= \int_0^{X_j} I_{\{Y(u) \geq y\}} \cdot du \Rightarrow \text{depends on } X_j \text{ only}$$

⇒ Here, $R_j = \begin{cases} 0 & \text{if } X_j \leq y \\ X_j - y & \text{o/w} \end{cases}$ ⇒ (R_j depends ONLY on X_j)

So, $E[R_j] = E[R_2] = \cancel{E[R_2 | X_j \leq y]} \cdot P(X_j \leq y)$ (conditional expectation)

$\rightarrow 0$

$+ \; E[R_2 | X_j \geq y] \cdot P(X_j \geq y)$

so,

$\cancel{E[R_2] = E[X_j - y] \; P(X_j \geq y)}$

$$\boxed{\cancel{E[R_2] = \left(\frac{1}{\mu} - y\right) F^c(y)}}$$

So, $\cancel{\lim_{t \to \infty} \frac{1}{t} \int_0^t I_{\{Y(u) \geq y\}} du = F^c(y) - \mu y F^c(y)}$

$\cancel{= \boxed{(1 - \mu y) F^c(y)}}$ w.p. 1

$$E[X_j - y \mid X_j \geq y] \cdot P(X_j \geq y)$$

$$= \int_0^\infty (x-y) \cdot dF_2(x)$$

so,

$$\boxed{\lim_{t\to\infty} \frac{1}{t} \int_0^t I_{\{Y(U) \geq y\}} \, du = \int_0^\infty (x-y) \, dF_2(x)}$$

eg.2) An observer visits a renewal process with lifetimes $\{X_j\}_{j\geq 1}$ at a random time in interval $[0,t]$, find the probability that the residual time he/she observes is $\leq y$, for some $y \geq 0$ and large t.

i.e.
$$\int_0^t P(Y(u) \leq y) \cdot \frac{1}{t} \, du$$

$\Rightarrow$ Let T: the time of visit of the observer.

$\rightarrow$ we need to find $P(Y(T) \leq y)$.

$$= \int_0^t P(Y(T) \leq y \mid T=u) \cdot f_T(u) \, du$$

$$= \int_0^t P(Y(u) \leq y) \, (1/t) \, du$$

uniform R.V. in $[0,t]$

$$= \int_0^t P(Y(u) \leq y) \cdot \frac{1}{t} \, du$$

So, we need to evaluate,

$$\lim_{t\to\infty} \underbrace{\int_0^t P(Y(u) \leq y)}_{g(t)} \cdot \frac{1}{t} \, du$$

★ Renewal Reward Theorem

$$E[I_A] = P(A)$$
$$P(Y(u) \leq y) = E\left[I_{\{Y(u) \leq y\}}\right]$$

Consider for any time t,

$$\frac{1}{t}\int_0^t P(Y(u)\le y)\,du = \frac{1}{t}\int_0^t E\left[I_{\{Y(u)\le y\}}\right] du$$

$$= E\left[\frac{1}{t}\int_0^t I_{\{Y(u)\le y\}}\,du\right]$$

Now,

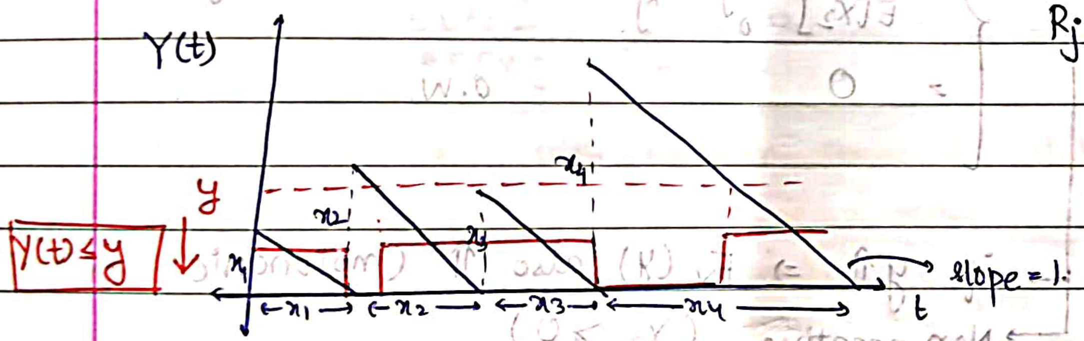
$$\lim_{t\to\infty} \frac{C(t)}{t} = \frac{E[R_2]}{E[X_2]}$$

i.e. $\boxed{\lim_{t\to\infty} \frac{E[C(t)]\,E[X_2]}{t} = E[R_2]}$

Let, $C(t) = \int_0^t I_{\{Y(u)\le y\}}\,du$

$$R_j = \int_{Z_{j-1}}^{Z_j} I_{\{Y(u)\le y\}}\,du$$

$$R_j = \begin{cases} x_j & \text{if } x_j \le y \\ y & \text{if } x_j \ge y \end{cases}$$



$\boxed{R_j = \min.\{X_j, y\}}$

$$E[R_j] = \int_0^\infty P(R_j > u)\,du$$

$$P(R_j > u) = P(\min\{X_j, y\} > u)$$

$$= P(X_j > u,\ y > u) \longrightarrow \text{intersection of two events}$$

null   certain (y, u are two constants)

$$= \begin{cases} 0 & \text{if } y \le u \\ \underbrace{P(X_j > u)}_{F^c_{X_j}(u)} & \text{if } y > u. \end{cases}$$

$$E[R_j] = \int_0^y P(R_j \geq u)\,du + \int_y^\infty \cancel{P(R_j > u)}\,du \quad \underbrace{}_{u \geq y} \; 0$$

$$\boxed{E[R_j] = \int_0^y F^c_{X_j}(u)\,.du}$$

Using the Renewal Reward Theorem,

$$\lim_{t\to\infty} \frac{1}{t} \underbrace{\int_0^t P(Y(U) \leq y)\,.du}$$

$$\frac{c(t)}{t} = \frac{E[R_2]}{E[X_2]}$$

i.e. $$\boxed{\lim_{t\to\infty} \frac{1}{t} \int_0^t P(Y(U) \leq y)\,.du = \frac{1}{E[X_2]} \int_0^y F^c_{X_j}(u)\,.du}$$

Let $F_Y(y) = \begin{cases} \dfrac{1}{E[X_2]} \displaystyle\int_0^y F^c_{X_j}(u)\,.du & \text{for } y \geq 0 \\ = 0 & \text{o.w.} \end{cases}$

→ if $y \Uparrow \Rightarrow F_Y(y)$ also $\Uparrow$ (motononic

→ Non-negative $(X_2 \geq 0)$

Also,

$$\lim_{y\to\infty} \frac{1}{E[X_2]} \int_0^y F^{(c)}_{X_j}(u)\,.du \quad E[X_2] \text{ as } y\to\infty$$

Note :- if $P(X_j > 0) = 1$, then,

$$E[X_j] = \int_0^\infty P(X_j > u)\,.du.$$

$$\boxed{\lim_{y\to\infty} F_Y(y) = 1}$$

eg. 3) Average waiting time for the observer, i.e.

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Y(u)\,du \qquad \left(\text{expected: } \frac{E[X_2]}{2}\right)$$

$\int_0^t Y(u)\,du$ → random variable ⇒ can be defined as $C(t)$ directly

$$C(t) = \int_0^t Y(u)\,du$$

$$R_j = \int_{Z_{j-1}}^{Z_j} Y(u)\,du \qquad \left(Y \text{ at a time } 'u'\right)$$

(integral in a lifetime)

$$R_j = \frac{X_j^2}{2} \qquad E[R_j] = \frac{1}{2}E[X_j^2]$$

So, $$\boxed{\lim_{t\to\infty} \frac{1}{t}\int_0^t Y(u)\,du = \frac{E[X_2^2]}{2E[X_2]} \quad \text{w.p. } 1}$$

$$\frac{E[X_2^2]}{2E[X_2]} > \frac{E[X_2]}{2}$$

avg. waiting time, with random times is always perceived to be higher than one with certain times.

↳ $\boxed{\text{BATCH BIASING}}$

DTMC as a Renewal Process

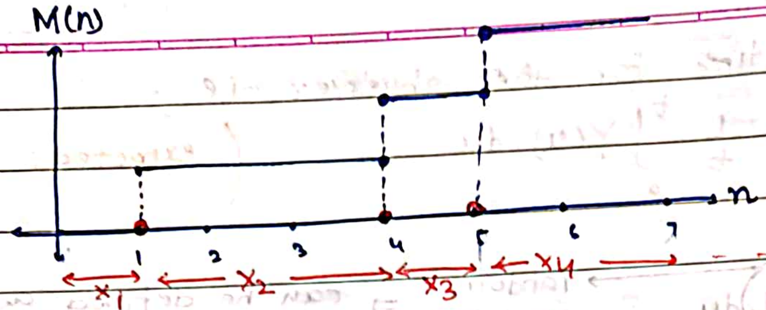
⇒ $\{D_n\}_{n\geq 0}$ defined by $\bar{\alpha}$ and P

⇒ Take any state $j \in S$, define

CLAIM :- $\{M_j(n)\}_{n\geq 1}$ is a renewal process

$$\boxed{M_j(n) = \sum_{k=1}^{n} 1_{\{X_k = j\}}}$$

(no. of visits to state j until n-steps)



$M_j(n)$ is a counting process

1) All the lifetimes (X_j's) are INDEPENDENT because of the STRONG MARKOV PROPERTY

2) so, X_j's, $j \geq 2$ i.e. $\{X_j\}_{j \geq 2}$ are identically distributed.

$\Rightarrow$ (first visit to state j, time, T is a stopping time)

Distribution of X_2 $\Rightarrow$ $\boxed{P(X_n = k) = f_{jj}^{(k)}}$ $\forall\ n \geq 2$

First Passage Time Distribution

$P(X_n = k) = P(D_1 \neq j, D_2 \neq j, \ldots, D_k = j \mid D_0 = j)$; $n \geq 2$

and

$$\boxed{P(X_1 = k) = \sum_{i \in S} \alpha_i f_{ij}^{(k)}}$$

probability of visiting the j^{th} state for the first time

$\longrightarrow$ X_2 : the steps we take from the I^{st} visit to the II^{nd} visit

SUPPOSITIONS OF ERT

1. $P(X_1 < \infty) = 1$
2. $E[X_2] \in [0, \infty]$

Propositions of ERT

1. $\lim_{n \uparrow \infty} \frac{M(n)}{n} = \frac{1}{E[X_2]}$ w.p. 1
2. $\lim_{n \uparrow \infty} \frac{m(n)}{n} = \frac{1}{E[X_2]}$

$$P(X_1 < \infty) = \sum_{i \in S} \alpha_i f_{ij}^{(k)}$$

Suppose DTMC is $\boxed{\text{irreducible}}$ and $\boxed{\text{recurrent}}$

irreducible, C.C. has recurrent states $\Rightarrow$ $f_{ij} = f_{ji} = 1$

and so, $P(X_1 < \infty) = 1$

$X_2 \geq 1$ i.e. $E[X_2] \geq 1$ and also the other condⁿ holds
$\Rightarrow E[X_2] \in [1, \infty]$

so, $\boxed{\text{irreducible \& recurrent DTMC's}}$

Thm :- If $\{D_n\}_{n\geq 0}$ is an irreducible, positive recurrent DTMC with a stationary measure π. then,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^{(k)} = \pi_j = \frac{1}{v_{jj}} \quad \forall\ i,j \in S$$

PROOF :- $P_{ij}^{(k)} = P(D_k = j \mid D_0 = i)$

Define $1\{D_k = j \mid D_0 = i\}$

$$M(n) = \sum_{k=1}^{n} 1\{D_k = j \mid D_0 = i\} \qquad E[M(n)] = \sum P(D_k = j \mid D_0 = i) = \boxed{\sum P_{ij}^{(k)}}$$

$\downarrow$

\# visits to state-j in first n-steps starting from state-i.

To show $\{M(n)\}_{n\geq 1}$ is a renewal process :-

$$P(X_1 < \infty) = f_{ij} = 1$$
$$E[X_2] \in (0, \infty]$$

so,
$$\boxed{\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1\{D_k = j \mid D_0 = i\} = \frac{1}{v_{jj}} \quad \text{w.p. } 1}$$

Also,
$$\lim_{n\uparrow\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^{(k)} = \frac{1}{v_{jj}} \qquad \left(\lim_{n\to\infty} \frac{m(n) = E[M(n)]}{n} \right)$$

Thm :- If $\{D_n\}_{n\geq 0}$ is an irreducible, positive recurrent DTMC with a stationary measure π. Then, find expected no. of visits to state i b/w two consecutive visits to state j. $E[V_{ji}]$

$\downarrow$

consider you earn a reward 1 every time state i is visited.

Total reward collected in the n^{th} lifetime $= \sum_{Z_{n-1}+1}^{Z_n - 1} \mathbb{1}_{\{D_K = i\}} = R_n$

$c(n) \Rightarrow$ # of times state-i is visited

$$\lim_{n\uparrow\infty} \frac{c(n)}{n} = \frac{E[V_{ji}]\ (v_{jj})}{E[X_2]} \quad \text{w.p. } 1$$

$\Rightarrow$ consider the R.R.P., $\{M_i(n)\}$ ⟶ total no. of visits to state i until 'n'.

$$\lim_{n\to\infty} \frac{M_i(n)}{n} = \frac{1}{v_{ii}} \quad \text{w.p. } 1$$

$\Rightarrow$ LHS is same

Finally, $\boxed{E[V_{ji}] = \frac{\pi_i}{\pi_j}}$

11/03 **RECAP :-**

Renewal Process (ERT)

Renewal Reward Process (RRT)

RRT
- Time-avg. distrib^n of residual time
- Time avg. of residual — batch biasing

Applications to DTMC :-

$$\lim_{n\uparrow\infty} \frac{1}{n} \sum_{k=1}^{n} p_{ij}^{(k)} = \pi_j \quad \forall\ i \in S$$

and

$$E[v_{ji}] = \frac{\pi_i}{\pi_j}$$

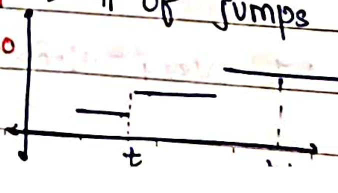
POISSON'S PROCESS

A poisson's process, $\{N(t)\}_{t\geq 0}$ is a COUNTING PROCESS with the following properties :-

① All jumps of $\{N(t)\}$ are of unit size w.p. 1.

$\Rightarrow$ ② $\boxed{N(t+s) - N(t)} \Rightarrow$ # of jumps in interval $(t, t+s]$

increments of $\{N(t)\}_{t\geq 0}$ in $(t, t+s]$



(no. of jumps = 1, $N(t+s) = 2$, $N(t) = 2$ ...)

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② $\{N(t)\}_{t\geq 0}$ has independent increments in NON-OVERLAPPING time intervals.

eg.- consider two time intervals $(t_1, t_1+s_1]$ and $(t_2, t_2+s_2]$, then,

$(N(t_2+s_1) - N(t_2)) \perp\!\!\!\perp (N(t_1+s_2) - N(t_1))$ as long as

$(t_1, t_1+s_1] \cap (t_2, t_2+s_2] = \{0\}$ (are DISJOINT)

③ $\{N(t)\}_{t\geq 0}$ has stationary increments

↓

Distribution of increments in I is same as no. of increments in interval $I+t$ $\forall$ $t \geq 0$

$$I = [t_1, t_2]$$

$$I+t = [t_1+t, t_2+t]$$

distrib^n of no. of intervals remains the same w.r.t. time.

⇒ Using the above 3, we can derive other properties of the Poisson's process :-

Lemma 1:- $\forall$ $0 \leq t_1 < t_2 < t_3 \dots < t_n < \infty$ and $0 \leq i_1 \leq i_2 \leq \dots i_n$ (whole numbers)

$$\boxed{(N(t_1) = i_1, N(t_2) = i_2, \dots, N(t_n) = i_n) = \prod_{k=1}^{n} P((N(t_k) - N(t_{k-1})) = i_k - i_{k+1}).)}$$

PROOF:- LHS ⇒ $(N(t_1) = i_1, N(t_2) - N(t_1) = i_2 - i_1, \dots, N(t_n) - N(t_{n-1}) = i_n - i_{n-1})$

↓

By the property of independent increments.

$$= \prod P(N(t_k) - N(t_{k-1}) = i_k - i_{k-1})$$

Lemma 2:- $\boxed{\exists \; \lambda \geq 0 \text{ s.T. } \forall \; t \geq 0, \; P(N(t) = 0) = e^{-\lambda}}$

PROOF:- Define $G(t) = P(N(t) = 0)$

$$G(t+s) = P(N(t+s) = 0)$$

⇒ $= P(N(t) = 0, N(t+s) - N(t) = 0) = P(N(t) = 0) \cdot P(N(t+s) - N(t) = 0)$

By independent increment property.

$G(t+s) = P(N(t)=0) \cdot P(N(s)=0)$

↳ stationary increment property.

so, $\boxed{G(t+s) = G(t) \cdot G(s)}$

so, $G(t) = e^{\alpha t}$ (α can be >0 or (<0))

since it is a probability

as $G(t) \in [0,1] \Rightarrow \alpha \in (-\infty, 0]$

$\rightarrow G(t) = e^{-\lambda t}$ for some $\lambda \geq 0$.

i.e., $\boxed{P(N(t)=0) = e^{-\lambda t} \text{ for some } \lambda \geq 0}$

Lemma 3:- $\boxed{\lim_{t \downarrow 0} \frac{1}{t} P(N(t) \geq 2) = 0}$

proved using property - 1.

Lemma 4:- $\boxed{\lim_{t \downarrow 0} \frac{1}{t} P(N(t)=1) = \lambda}$ $\Rightarrow$ (probability of $P(N(t+\tau) - N(t) = 1) \approx \lambda\tau$ for a small enough τ.)

if the interval is very small, the $P(N(t)=1)$ increases at

PROOF:- $P(N(t)=1) = 1 - P(N(t)=0) - P(N(t) \geq 2)$

$$\lim_{t \to 0} \frac{1}{t} (P(N(t)=1)) = \lim_{t \to 0} \left[\frac{1 - P(N(t)=0)}{t} \right] - \lim_{t \to 0} \frac{1}{t} P(N(t) \geq 2)$$

$$= \lim_{t \to 0} \left(\frac{1 - e^{-\lambda t}}{t} \right) - 0$$

$$\boxed{\lim_{t \to 0} \frac{1}{t} (P(N(t)=1)) = \lambda}$$

THEOREM:-

$$\boxed{P(N(t)=k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!} \quad \forall \ k = 0, 1, 2, \ldots}$$

POISSON PMF with parameter (λt).

PROOF:- Define $C(t) = E\left[Z^{N(t)} \right] = \sum_{k=0}^{\infty} Z^k \cdot P(N(t)=k)$

<u>Note that</u> :-

$$G(t+s) = E\left[z^{N(t+s)}\right]$$

$$= E\left[z^{(N(t+s)-N(t)+N(t))}\right]$$

$$= E\left[z^{(N(t+s)-N(t))} \cdot z^{N(t)}\right]$$

$$\cancel{= E\left[z^{N(s)} \cdot z^{N(t)}\right]}$$ (stationary-increment property)

$$= E\left[z^{(N(t+s)-N(t))}\right] \cdot E\left[z^{N(t)}\right]$$

$$= E\left[z^{N(s)}\right] \cdot E\left[z^{N(t)}\right]$$

$$= G(s)\, G(t)$$

So,

$$G(t) = e^{g(z) \cdot t}$$

$$g(z) = \lim_{t \to 0} \left[\frac{e^{g(z)t} - 1}{t}\right]$$

i.e.

$$g(z) = \lim_{t\to 0} \frac{1}{t}\left[e^{g(z)t} - 1\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[G(t) - 1\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[\sum_{k=0}^{\infty} z^k \cdot P(N(t)=k) - 1\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[P(N(t)=0) + z \cdot P(N(t)=1) + \sum_{k=2}^{\infty} z^k P(N(t)=k) - 1\right]$$

$$= \lim_{t\to 0}\left\{\left[\frac{e^{-\lambda t} - 1}{t}\right] + z\,\frac{1}{t} P(N(t)=1) + \frac{1}{t}\sum_{k=2}^{\infty} z^k P(N(t)=k)\right\}$$

$$\Downarrow \qquad\qquad z(\lambda) \qquad\qquad \Downarrow$$

$$-\lambda \qquad\qquad\qquad\qquad 0$$

$$\boxed{g(z) = -\lambda + \lambda z = (-1+z)\lambda}$$

So, we get,

$$G(t) = e^{(-1+z)\lambda t} = e^{-\lambda t}\, e^{\lambda z t}$$

$$= \left(e^{-\lambda t}\right) \cdot \sum_{k=0}^{\infty} \frac{(\lambda z t)^k}{k!}$$

so, $$G(t) = e^{-\lambda t} \sum_{k=0}^{\infty} z^k \frac{(\lambda t)^k}{k!}$$

comparing with $$G(t) = \sum_{k=0}^{\infty} z^k P(N(t)=k),$$

we get, $$\boxed{P(N(t)=k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}}$$

Hence Proved !!

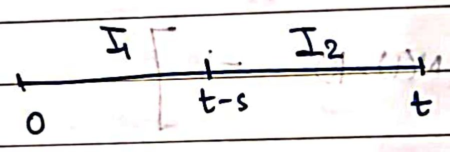
$\{N(t)=0\}$ ⇒ no jump till time 't'

⇒ Let x denote the time for the first jump, then,

$$\{N(t)=0\} \equiv \{x>t\}$$

⇒ $\boxed{P(x>t) = e^{-\lambda t}}$

⇒ $x \sim \exp.(\lambda)$ i.e. $f_x(x) = \lambda e^{-\lambda x}$ $\forall\ x>0$
$= 0$ o/w.



$N(t-s) = 0$ and $N(t) - N(t-s) = 0$

⇒ given an info. that say $x>(t-s)$, it gives us NO info. about as whatever happens in I_1, is ⫫ of I_2.

⇓

Memoryless Property $\boxed{P(x>t+s \mid x>s) = P(x>t)}$

so, the time until the first jump is exponential

(given the time elapsed, we have no info. about how much later will a jump actually occur)

general :- $P(x>t+s, x>s) = P(x>t)\, P(x>s)$ (Knowing that I've spent s-time, i spend (t+s) more time.)

$P(x>t+s) = P(x>t)\, P(x>s)$.

i.e. $G(t+s) = G(t)\, G(s)$ → exponential

Problem:- Show that :-

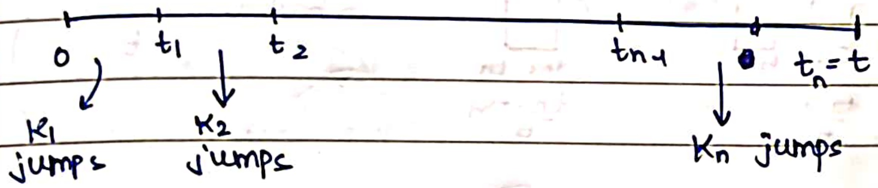
Let $N(t)$ be a poisson process with rate $\lambda > 0$, let $0 = t_0 < t_1 < t_2 < \dots < t_n = t$

Then, $\forall\ K, K_1, \dots, K_n$ s.T. $\sum_{i=1}^{n} K_i = K$,

$$P(N(t_1) = K_1, N(t_2) = K_1 + K_2, \dots N(t_n) = K \mid N(t) = K)$$

$$= \frac{K!}{\prod_{i=1}^{n}(K_i)!} \prod_{i=1}^{n} \left(\frac{t_i - t_{i-1}}{t}\right)^{K_i}$$

$\Rightarrow$



$$P(N(t_1) = K_1, N(t_2) = K_1 + K_2, \dots, N(t_n) = K \mid N(t) = K)$$

$$\Rightarrow P(N(t_1) = K_1, N(t_2) - N(t_1) = K_2, \dots, N(t_n) - N(t_{n-1}) = K_n \mid N(t) = K)$$

$$\Rightarrow \frac{P(N(t_1) = K_1, \dots, N(t_n) - N(t_{n-1}) = K_n, \cancel{N(t) = K})}{P(N(t) = K)}$$

if we pick any 'n' events from here, the $(n+1)^{th}$ HAS TO OCCUR.

$$\Rightarrow \frac{\prod_{i=1}^{n} P(N(t_i) - N(t_{i-1}) = K_i)}{P(N(t) = K)}$$

$$= \frac{\prod_{i=1}^{n} P(N(t_i - t_{i-1}) = K_i)}{P(N(t) = K)} = \frac{\prod_{i=1}^{n} \dfrac{e^{-\lambda(t_i - t_{i-1})}(\lambda(t_i - t_{i-1}))^{K_i}}{K_i!}}{\dfrac{e^{-\lambda t}(\lambda t)^K}{K!}}$$

$$= \frac{(K!)\, \cancel{e^{-\lambda((t_1 - t_0) + (t_2 - t_1) + \dots + (t - t_{n-1}))}} \cdot \cancel{\lambda^{K_1 + K_2 + \dots + K_n}}}{\cancel{e^{-\lambda t}}\, \cancel{\lambda^K}\, t^K \left(\prod_{i=1}^{n} K_i!\right)} \prod_{i=1}^{n} (t_i - t_{i-1})^{K_i}$$

$$= \boxed{\frac{K!}{\prod_{i=1}^{n} K_i!} \prod_{i=1}^{n} \left(\frac{t_i - t_{i-1}}{t}\right)^{K_i}} \qquad \left(\text{as } \prod_{i=1}^{n} t^{K_i} = t^K\right)$$

H.P.

→ Drop k points on the interval $[0,t]$. Let X_k denote the pos^n of the k^{th}-point, $\{X_k\}$ are i.i.d. Uniform $([0,t])$

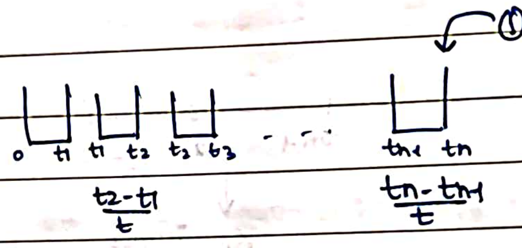
$$f_{X_k}(x) = \frac{1}{t} \quad \forall \; x \in [0,t]$$

$$= 0 \quad o/w$$

→ Let $Z(\tau, \tau+s)$ = # of points in interval $(\tau, \tau+s]$

⇒ Find $P(Z(0,t_1) = k_1,\ Z(t_1,t_2) = k_2, \ldots, Z(t_{n-1}, t_n) = k_n)$

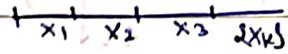
(Analogy :-



ball lands in bask-1 ⇒ $\frac{t_1}{t}$)

EQUIVALENT DEFINITIONS FOR A POISSON PROCESS

Thm:- A counting process $\{N(t)\}_{t\geq 0}$ is a poisson process with parameter λ, if and only if, successive jump times, $\{T_K\}_{K\geq 1}$ are renewal instances with i.i.d. exp.(λ) life times.



$\{X_K\}_{K\geq 1}$ are iid exp(λ)

$\Rightarrow$ consider a renewal process, $\{N(t)\}_{t\geq 0}$ with lifetimes $\{X_n\}_{n\geq 1}$. If X_n's are iid exp.(λ), then, $\{N(t)\}$ is a poisson process with a rate, λ.

$\Rightarrow$ A point process $\{N(t)\}_{t\geq 0}$ is a Poisson process iff

① $\forall$ $t_0 < t_1 < \cdots < t_n$; $\{N(t_K) - N(t_{K-1})\}$, $K=1,\ldots,n$ are independent R.V.'s

② $\exists$ $\lambda \geq 0$ s.t. $P((N(t) - N(t-s)) = K) = \boxed{e^{-\lambda s}\dfrac{(\lambda s)^K}{K!}}$ $k=0,1,2,\ldots$

Generalization of A Renewal Process-
REGENERATIVE PROCESSES

Def^n: A process $\{X(t)\}_{t\geq 0}$ is called a regenerative process if $\exists$ a stopping time, T_1, such that

① $\{X(t+T_1)\}_{t\geq 0}$ and $\{X(t)\}_{t\geq 0}$ have same probability law

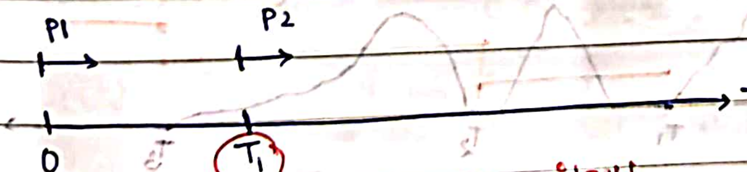
② $\{X(t+T_1)\}_{t\geq 0}$ $\perp\!\!\!\perp$ $\{T_1, X(u), u<T_1\}$.

$\rightarrow$ $\{X(t)\}$ can take any real value

$\Rightarrow$ T_1 is a stopping time $\Rightarrow$ $\{T_1 \geq t\}$ depends only on $\{X(u)\}_{u\leq t}$

(continue or not determined by the process before t ONLY.)

Property-1 :-



can't statistically differentiate b/w P_1 & P_2

process regenerates itself at point T_1

Property-2 :- Process after T_1 is independent of T_1, (whatever value T_1 took)

- T_1 won't change how process behaves after T_1
- also independent of the past.

Poisson process is a special type → process with difference of $(N(t)-N(t-s))$ is poisson, and all the epoch instants are also regenerative stopping times.

Lemma:- For a regenerative process, $\exists\ T_2 > T_1$, a stopping time satisfying properties ① & ②.

Corollary:- $\{T_1, T_2 - T_1, \ldots\}$ are i.i.d. random variables.

(T_2 ~~~~ plays the same role as that of T_1 for $\{X(t+T_1)\}_{t\geq 0}$)

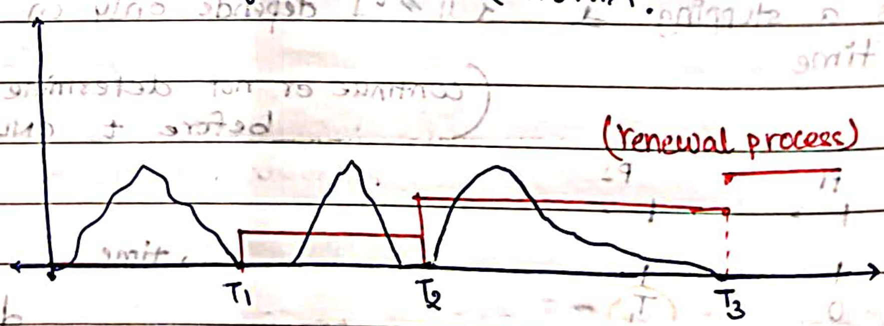
Corollary:- Define:-

$X_n = T_n - T_{n-1}$ $\quad (T_0 = 0)$

$\{X_n\}_{n\geq 1}$ are iid

↓

regenerative process has a renewal embedded in it with lifetimes as $\{X_n\}_{n\geq 1}$.



Find $\lim_{t\to\infty} \int_0^t X(u)\,du$

$\boxed{\text{Renewal Reward Theorem}}$

$R(u) = X(u) \longrightarrow$ reward

using RRT, it is $\dfrac{E\left[\int_0^{T_1} X(u)\,du\right]}{E[T_1]}$ w.p. 1

Note :-

renewal process → i.i.d. exp(λ) lifetimes

↙ Ordinary ↘ Delayed

$A = F$ $\qquad$ $A \neq F$

Regenerative process :- ∃ an embedded ordinary renewal process and the renewal epochs are $\{T_1, T_2, \ldots\}$

ERT, RRT } → limiting-time average behavior

POINT-WISE LIMITS

eg.- $a_n = \begin{cases} 1 & \text{if } n = \text{odd} \\ 0 & \text{o/w} \end{cases}$

$\lim_{n\to\infty} \frac{1}{n} \sum a_k = \frac{1}{2}$

but point-wise, $\lim_{n\to\infty} a_n \to$ DNE

$\{X_n\} \sim$ Bernoulli (p) [i.i.d]

$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X_k = p$ w.p. 1

$\lim_{n\to\infty} X_n$ (doesn't exist with prob = 1)

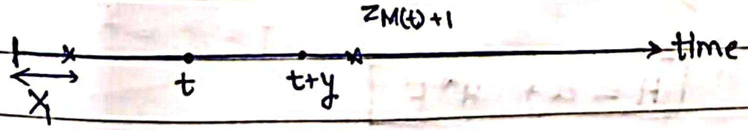
(Borel - Cantelli lemma)

⇒ If point-wise limit exists, the time-average ones should also exist.

Renewal Equation (Ordinary Renewal Processes)

The time avg. limit, $\frac{1}{t}\int_0^t P(Y(u) < y)\,du$ exists

point-wise limit ⇒ $P(Y(t) > y)$: If an observer arrives at time t, what is the probability that it sees a waiting time > y.



$\underbrace{Y(t) = Z_{M(t)+1} - t}_{\text{waiting time}}$

(The renewal after t occurs only after t+y.)

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⇒ Condition on the first renewal.

$$= P(Y(t) > y \mid X_1 < t) + P(Y(t) > y \mid t < X_1 < t+y) + P(Y(t) > y \mid X_1 > t+y)$$

A B C

0 — waiting time is less than y

1 (always true) $F^c(t+y)$

intersect with E is C itself.

C is a subset of E $\quad C \Rightarrow E(\,P(Y(t) > y)\,)$

$$P(Y(t) > y, X_1 < t) = \int_0^t P(Y(t) > y \mid X_1 = u) \cdot dF(u)$$

process after X_1 is independent of value of X_1, and so, once $Y(t-u)$ is being put in, P won't depend on X_1 anymore

$$= \int_0^t P(Y(t-u) > y)\, dF(u)$$

$$= \int_0^t G(t-u) \cdot dF(u)$$

Let $G(t) = P(Y(t) > y)$

so, $P(Y(t) > y, X_1 < t) = \int_0^t G(t-u)\, dF(u)$

$$= G * F(t)$$

So, we can write

$$\boxed{G(t) = G * F(t) + F^c(t)}$$

So, any eq^n of the form

$$\boxed{H(t) = a(t) + \int_0^t H(t-x)\, dF(x)}$$

where 'a' is any given f^nc and F is a $distrib^n$ f^nc is called a renewal eq^n.

consequently : $\boxed{H = a + H * F}$

Thm :- If distribution function, F, has a finite mean and function 'a' is a bounded f^nc, then the unique sol^n

$\boxed{H = a + H*F}$

that is bounded on finite intervals, is,

$$\boxed{H(t) = a(t) + \int_0^t a(t-u)\, dm(u)}, \quad \boxed{m(t) = \sum_{k=1}^{\infty} F^{(k)}(t)}$$

Given :- finite mean, i.e. $\int_0^x x \cdot dF(x) < \infty$ or $\int_0^x F^c(x)\, dx < \infty$

(expectation)

Bounded $\rightarrow \sup_{t \in R} |a(t)| < \infty$

Proof :- The ∞-sum, $m(t) = \sum_{k=1}^{\infty} F^{(k)}(t)$ is finite (proved earlier)

$F^{(k)}(t) = F * F^{(k-1)}(t)$

as it is an ordinary renewal process.

$P(M(t) \geq n) = P(Z_k \leq t) = \boxed{F^{(k)}(t)}$

$H = a + H*F$

If ① is a sol^n then,

$H = a + (a + a*m)*F$

$H = a + a*F + a*m*F$

$H = a + a*(F + m*F)$

$$m*F = \sum_{k=2}^{\infty} F^{(k)}$$

$$F + m*F = \sum_{k=1}^{\infty} F^{(k)} = m(t)$$

so, $H = a + a*m$ is indeed a solution.

$\Rightarrow$ Now, we show that H is bounded for every $T<\infty$

$$H(t) = a(t) + \int_0^t a(t-u)\, dm(u)$$

$$|H(t)| \le |a(t)| + \left| \int_0^t a(t-u)\, dm(u) \right|$$

$$\le |a(t)| + \int_0^t |a(t-u)|\, dm(u)$$

$$\sup_{0\le t\le T} |H(t)| \le \sup_{0\le t\le T} \left[|a(t)| + \int_0^t |a(t-u)|\, dm(u) \right]$$

$$\le \sup |a(t)| + \int_0^t \sup |a(t)|\, dm(u)$$

$$\left[\sup|a(t-w)| = \sup|a(t)| \text{ and } \sup(A) + \sup(B) \geq \sup(A+B) \right]$$

$$\le \sup|a(t)| + \sup|a(t)| \int_0^t dm(u)$$

$$= \sup|a(t)| + \sup|a(t)| \cdot m(t)$$

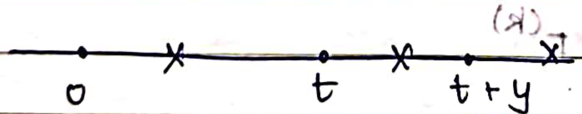
As a is bounded and $m(t)$ is finite,

$\sup|H(t)| < \infty \Rightarrow H(t)$ is bounded.

$\Rightarrow$ Uniqueness can be proved by contradiction.

DELAYED RENEWAL PROCESS

Let $K(t) = P(Y(t) > y)$



$$K(t) = P(Y(t)>y,\ X_1 \le t) + P(Y(t)>y,\ t \le X_1 \le t+y) + P(Y(t) \geq y,\ X_1 \geq t+y)$$

0

1

$$= P(X_1 \geq t+y) = \boxed{A^c(t+y)}$$

$$\boxed{P(Y(t) > y,\ X_1 \le t) = \int_0^t K_0(t-x)\, dA(x)}$$

(similar to the ordinary case)

where $K_0(t-x) = P(Y_0(t-x) > y)$ where Y_0 is the residual time of the ordinary renewal process.

Now,

$$\boxed{K(t) = A^c(t+y) + (K_0 * A)(t)}$$

renewal eq^n is of the form $\Rightarrow$ $\boxed{\begin{array}{l} H = a + H_0 * A \\ H_0 = a_0 + H_0 * F \end{array}}$

Theorem:- The solution of the general renewal eq^n is $H = a + a_0 * m$, where,

$$\boxed{m = \sum_{k=1}^{\infty} A * F^{(k-1)}(t)}$$ sol^n: $\boxed{H = a + a_0 * m}$

(renewal f^{n}c for a delayed renewal process)

Proof:- we have shown that the sol^n for

$H_0 = a_0 + H_0 * F$ is $H_0 = a_0 + a_0 * m_0$ (ordinary RP)

$H = a + H_0 * A$

$H = a + (a_0 + a_0 * m_0) * A = a + a_0 * A + a_0 * m_0 * A$

$H = a + a_0 * (A + m_0 * A)$

$= a + a_0 * (A + m_0 * A)$

(a_0, m_0 : ordinary RP, $m_0 = \sum f^{(k)}(t)$)

$\downarrow$

$\left(\sum F^{(k)}(t)\right) * A$

$$= a + a_0 * \left(A + \sum_{k=1}^{\infty} A * F^{(k)}(t)\right)$$

$$= a + a_0 * \left(\sum_{k=1}^{\infty} A * F^{(k-1)}(t)\right)$$

$$\boxed{H = a + a_0 * m}$$

APPLICATIONS :-

Poisson process :- Can be considered as an ordinary renewal process, with lifetimes $\sim \exp(\lambda)$

$$P(Y(t)>y) = F^c(t+y) + \int_0^t F^c(t+y-x)\,dm(x)$$

as $H_0 = a_0 + a_0 * m_0$ $\quad a_0(t) = F^c(t+y)$

$$F^c(t+y) = e^{-\lambda(t+y)}$$
$$F^c(t+y-x) = e^{-\lambda(t+y-x)}$$

$M(t)$ is a poisson process with parameter λt, if

$$m(t) = E[M(t)] = \lambda t$$
$$dm(x) = \lambda\,dx$$

Hence, $$P(Y(t)>y) = e^{-\lambda(t+y)} + \int_0^t e^{-\lambda(t+y-x)} \cdot \lambda\,dx$$

$$= e^{-\lambda(t+y)} + \left[e^{-\lambda y} - e^{-\lambda(t+y)}\right]$$

$$= e^{-\lambda y}$$

$$= P(X>y) \quad \text{where } X \text{ is a lifetime.}$$

Thus, $Y(t)$ has the same distribution as lifetimes, i.e., distribution of $Y(t)$ after time t is same as that of lifetime starting from zero.
i.e. from anywhere, the process looks the same and so it is a stationary process.

⇒ Stationary process :- A process in which from no matter where we start observing, the process remains the same.

Say we start from t, $Y(t)$ ⇒ renewal time after t
X_1 ⇒ renewal time after 0
They have the same distribution.

Note :- In general, the property of independent increments need not be true for a process to be stationary.

consider, $\lim_{t \to \infty} \frac{1}{t} \int_0^t P(Y(u) \leq y)\, du = \frac{1}{E[X_2]} \int_0^y F^c(x)\, dx$ w.p. 1

Define $F_e(y) = \frac{1}{E[X_2]} \int_0^y F^c(x)\, dx.$

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Examples :-

① Poisson process (λ) :-

$$P(Y(t) > y) = e^{-\lambda y} \quad \forall\ y \geq 0,\ \forall\ t \in \mathbb{R}$$

Alternatively, $Y(t) \sim F$
$\exp(\lambda) \quad \forall\ t \in \mathbb{R}.$

② $$A(y) = F_e(y) = \frac{1}{E[X_2]} \int_0^y F^c(x)\, dx \quad \forall\ y \geq 0$$

$$\hookrightarrow \lim_{t \to \infty} \frac{1}{t} \int_0^t P(Y(t) \leq y)\, dy$$

② Delayed renewal process

$$X \sim A = F_e$$

$$P(Y(t) > y) = F_e^c(t+y) + \int_0^t F^c(t+y-x)\, dm(x) \quad — ①$$

$$m(t) = \sum_{k=1}^{\infty} (A * F^{(k-1)})(t)$$

$\downarrow$ Laplace transform

$$\tilde{m}(s) = \sum_{k=1}^{\infty} \tilde{A}(s) (\tilde{F}(s))^{k-1}$$

$$\tilde{m}(s) = \tilde{A}(s) \sum_{k=0}^{\infty} (\tilde{F}(s))^k$$

$$\boxed{\tilde{m}(s) = \frac{\tilde{A}(s)}{1 - \tilde{F}(s)}}$$

$$A(t) = \frac{1}{E[X_2]} \int_0^t F^{(c)}(x)\, dx$$

$$\tilde{A}(s) = \frac{1}{E[X_2]} \cdot \frac{1}{s} \mathcal{L}(1 - F(t)) = \frac{1}{s E[X_2]} (1 - \tilde{F}(s))$$

$$\frac{\tilde{A}(s)}{1 - \tilde{F}(s)} = \frac{1}{s E[X_2]} = \tilde{m}(s)$$

Taking the l.L.T., $m(t) = \frac{t}{E(X_2)} = \mu t$

$\boxed{m(t) = \mu t}$

using ①,

$$P(Y(t) > y) = F_e^c(t+y) + \int_0^t F^c(t+y-x)\,\mu\,dx$$

$$= F_e^c(t+y) + \frac{1}{E[X_2]} \int_0^t F^c(t+y-x)\,dx$$

$$= F_e^c(t+y) + \frac{1}{E[X_2]} \int_y^{t+y} F^c(u)\,du$$

$$= F_e^c(t+y) + \underbrace{\frac{1}{EX_2} \int_0^{t+y} F^c(u)\,du}_{F_e(t+y)} - \underbrace{\frac{1}{EX_2} \int_0^{y} F^c(u)\,du}_{F_e(y)}$$

$$= \underbrace{F_e^c(t+y) + F_e(t+y)}_{1} - F_e(y)$$

$$= F_e^c(y)$$

$\boxed{P(Y(t) > y) = F_e^c(y)}$ $\Rightarrow Y(t) \sim F_e \quad \forall t$

$Y(t) \sim A$

For large t,

from ERT

$$\lim_{t\to\infty} \frac{m(t)}{t} = \mu \quad \Rightarrow \quad m(t) \sim \mu t$$

$$m(t+h) - m(t) = [\mu(t+h) + o(t+h)] - [\mu(t) + o(t)]$$

$$= \mu h + \underbrace{[o(t+h) - o(t)]}_{\to 0 \text{ assuming } h \ll t}$$

so, intuitively,

$\lim_{t\to\infty} m(t+h) - m(t) = \mu h$ $\Rightarrow$ The expected renewals are deterministic and ⇑ at a rate μ.

$dm(x) \longrightarrow$ propb. that there is a renewal in $[x, x+dx]$

$$m(x) = \underbrace{\sum_{k=1}^{\infty} \left(A^* F^{(k-1)}\right)(x)}$$

$\hookrightarrow$ distribution of R.V. Z_k, $\boxed{P(Z_k \le x)}$

$dm(x) \longrightarrow$ density of $Z_k \cdot x\, dx$

① Poisson Process :- $m(x) = \lambda x$

$$dm(x) = \lambda\, dx = \frac{1}{E[X_2]}\, dx$$

→ delayed poisson process with $A = F_e$:- (Time-averaged excess time distribution)

again, $dm(x) = \lambda\, dx = \dfrac{dx}{E[X_2]}$

ERT : $\displaystyle \lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{E[X_2]}$

for large enough t, $\boxed{m(t) \simeq \dfrac{t}{E[X_2]}}$ → $dm(t) = \dfrac{dt}{E[X_2]}$

Def^n:- A proper variable X is called lattice if $\exists\, d > 0$ s.t.

(a) $\displaystyle\sum_{n=-\infty}^{+\infty} P(X = nd) = 1$ o/w X is called non-latti

→ largest 'd' for which ⓐ holds is called the span of X.

Blackwell's Renewal Theorem (BRT)

→ For a delayed renewal process with arbitrary A and non-lattice F, and any $h > 0$

$$\lim_{t\to\infty} \left[m(t+h) - m(t) \right] = h\mu$$

→ For an ordinary renewal process with lattice F having a span 'd' a for $h = d, 2d, \ldots$

$$\lim_{t\to\infty} \left[m(t+h) - m(t) \right] = h\mu$$

$$H(t) = a(t) + \int_0^t a_0(t-x)\, dm(x)$$

$$\lim_{t\to\infty} H(t) = \lim_{t\to\infty} a(t) + \lim_{t\to\infty} \int_0^t a_0(t-x)\, dm(x)$$

↳ as $t\to\infty$ should converge to $\boxed{\mu \int_0^\infty a_0(x)dx}$

KEY RENEWAL THEOREM

Let A and F be proper distributions s.T. $\int_0^\infty F'(u)\, du = \frac{1}{\mu}$

① If F is non-lattice, then, (and a_0 is direct Reimann integrable.)

$$\lim_{t\to\infty} \int_0^t a_0(t-u)\, dm(u) = \mu \int_0^\infty a_0(u)\, du$$

② If A = F and F is lattice with span 'd', then, $\forall\ t>0$

$$\lim_{n\to\infty} \int_0^{t+nd} a_0(t+nd-u)\, dm(u) = \mu d \sum_{n=0}^{\infty} a_0(t+nd)$$

Lemma :- If a function $a_0 : R^+ \to R$ is non-negative, monotone non-increasing, reimann integrable, then it is Direct Reimann Integrable (DRI).

(sufficient but not necessary)

eg.- $$P(Y(t) > y) = A^c(t+y) + \int_0^t F^c(t+y-x)\, dm(x)$$

finding $\lim_{t\to\infty} P(Y(t)>y)$

Note :- $\lim_{t\to\infty} A^c(t+y) = 0$

and $\lim_{t\to\infty} \int_0^t \underbrace{F^c(t+y-x)}_{\text{DRI}}\, dm(x)$ → monotone, non-increasing, non-negative

using KRT $= \mu \cancel{\lim_{t\to\infty}} \int_0^\infty F^c(x-y)\, dx = \mu \int_y^\infty F^c(x)\, dx$

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So, $$\boxed{\lim_{t\to\infty} P(Y(t) > y) = \frac{1}{E[X_2]} \int_y^{\infty} F^c(x)\cdot dx}$$

(KRT)