EE621: Markov Chains and Queueing Systems

Quiz # 4 (20 Marks)

Time:- 9:30 - 11 am Date:- 16/03/2024

$\boxed{\text{Q.1}}$ Consider a machine that can run continuously for X units of time. Consider a strategy of performing maintainance when machine finishes T units of operation. If the machine fails before maintainance, then it need to be replaced. Let C_M and C_R denote down time for maintainance and replacement, respectively. Machine becomes new after maintainance. Find T that minimizes avg machine down time when: $\boxed{\text{8 Marks}}$

$X \sim \exp(1)$, $C_M = 10$ units and $C_R = 20$ units.

$\boxed{\text{Q.2}}$ Let $\{M(t)\}_{t \geq 0}$ be a renewal process with $X_2 \sim F$ and usual assumptions. Let X_e be any random variable having distribution given by:

$$F_{X_e}(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t P(Y(u) \leq x)\, du.$$

(a) Find $E[X_e]$ for $F = \text{Uniform}([0, 10])$. $\boxed{\text{4+3 Marks}}$

(b) Also, find $\lim_{t \uparrow \infty} \frac{1}{t} \int_0^t X(u)\, du$, $X(t) = Z_{M(t)+1} - Z_{M(t)}$.

$\boxed{\text{Q.3}}$ Let $\{M_1(t)\}_{t \geq 0}$ and $\{M_2(t)\}_{t \geq 0}$ be two independent renewal processes. Prove or disprove: $\{M_1(t) + M_2(t)\}_{t \geq 0}$ is a renewal process. $\boxed{\text{5 Marks}}$

Solutions

$\boxed{Q.1}$ Let $\{X_n\}_{n\geq 1}$ denote potential life-time of n^{th} machine, (a replacement or after maintenance

Define, $Y_n = T + C_m$ if $X_n > T$,

$= X_n + C_R$ if $X_n \leq T$.

Note that since $\{X_n\}_{n\geq 1}$ is iid, $\{Y_n\}_{n\geq 1}$ is also iid seq. Consider a renewal process with life times $\{Y_n\}_{n\geq 1}$.

Define reward $R_n = C_m$ if $X_n > T$,

$= C_R$ if $X_n \leq T$.

By RRT, avg m/c down time

$$= \frac{E[R_n]}{E[Y_n]} \quad \text{w.p. } 1.$$

Let us calculate the expectations.

$$E[R_n] = C_m P(X_n > T) + C_R P(X_n \leq T)$$

$$= C_m e^{-T} + C_R (1 - e^{-T})$$

$$= C_R - (C_R - C_m) e^{-T}.$$

Now, $E[Y_n] = \int_0^\infty P(Y_n > y)\, dy$

Note that

$$P(Y_n > y) = P(Y_n > y, X_n \leq T) + P(Y_n > y, X_n > T).$$

$$= \underbrace{P(X_n + C_R > y, X_n \leq T)}_{1} + \underbrace{P(T + C_m > y, X_n > T)}_{2}.$$

consider ②

$$P(T+C_m>y, X_n>T) = 0 \quad \text{if } y \geq T+C_m$$

$$= P(X_n>T) = e^{-T} \quad \text{if } y < T+C_m. \quad -ⓐ$$

consider ①

$$P(X_n > y - C_R, X_n \leq T)$$

$$= P(y-C_R < X_n \leq T)$$

$$= P(0 < X_n \leq T) \quad \text{if } y - C_R < 0 \quad (y < C_R)$$

$$= (1-e^{-T}). \quad -ⓑ$$

$$= P(y - C_R < X_n \leq T) \quad \text{if } y - C_R > 0 \ \& \ y - C_R \leq T$$

$$\Rightarrow C_R < y \leq T + C_R$$

$$= e^{-(y-C_R)} - e^{-T} \quad -ⓒ$$

$$= 0 \quad \text{if } y - C_R > T \Rightarrow y > T + C_R. \quad -ⓓ$$

Thus, $E[Y_n]$

$$= \underbrace{(\cancel{T}+C_m)e^{-T}}_{\text{from } ⓐ} + \underbrace{C_R(1-e^{-T})}_{\text{from } ⓑ} + \underbrace{[1-e^{-T}] - T\cancel{e^{-T}}}_{\text{from } ⓒ}$$

$$= (C_R+1) - (C_R+1-C_m)e^{-T}.$$

$$\frac{E[R_n]}{E[X_n]} = \frac{C_R - (C_R - C_m)e^{-T}}{(C_R+1) - (C_R+1-C_m)e^{-T}}.$$

decreases monotonically as $T \uparrow \infty$. (Take $\frac{d}{dT}$)

$\Rightarrow$ avg down time decreases with T

$\Rightarrow$ optimal $T = +\infty$.

$\boxed{Q.2}$ As shown in the class:

$$F_e^c(x) = \frac{1}{EX_2}\int_x^\infty F^c(u)\,du$$

$$\Rightarrow E[X_e] = \int_0^\infty F_e^c(x)\,dx$$

$$= \int_0^\infty \int_x^\infty F^c(u)\frac{du}{EX_2}$$

$$= \frac{1}{EX_2}\int_0^\infty \int_0^u dx\, F^c(u)\,du$$

$$= \frac{1}{EX_2}\int_0^\infty u F^c(u)\,du = \frac{EX_2^2}{2EX_2}.$$

$$EX^2 = \int_0^\infty x^2 f_X(x)\,dx$$

$$= \int_0^\infty \int_0^x 2u\,du\, f_X(x)\,dx$$

$$= 2\int_0^\infty u \int_u^\infty f_X(x)\,dx\,du$$

$$= 2\int_0^\infty u F_X^c(u)\,du$$

$$\Rightarrow \frac{EX^2}{2} = \int_0^\infty u F_X^c(u)\,du$$

Also as proved in the assignment:

$$\lim_{t\uparrow\infty}\frac{1}{t}\int_0^t X(u)\,du = \frac{EX^2}{EX_2}.$$

Thus, spread observed by random observer equals twice the waiting time of random observer.

$\boxed{Q.3}$ $M(t)$ is not necessarily a renewal process.

Let $\{M_1(t)\}$ & $\{M_2(t)\}$ be two r.p. with deterministic life times of length D.

Note that odd life-times of $\{M(t)\}$ have length D while even life-times have length 0.