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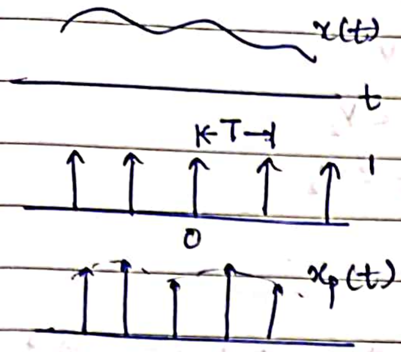
Module- 3

Impulse train sampling $\Rightarrow$ T: sampling period $\omega_s = \frac{2\pi}{T}$: sampling frequency



$P(t)$: sampling $f^n c$

$$\boxed{x_p(t) = x(t) \cdot p(t)} \; ; \; P(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

using $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\delta(t-nT)$$

$$X_p(j\Omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)\,P(j(\Omega-\theta))\,d\theta$$

$$P(j\Omega) = \frac{2\pi}{T}\sum_{K=-\infty}^{+\infty}\delta(\omega - K\omega_s)$$

convolution with an impulse simply shifts the signal,

$$X(j\Omega) * \delta(\omega - \omega_0) = X(j(\Omega - \Omega_0))$$

so, $$\boxed{X_p(j\Omega) = \frac{1}{T}\sum X(j(\Omega - K\Omega_s))}$$

Sampling Theorem :- $x(t)$ be a band-limited signal with $X(j\Omega) = 0 \;\forall\; |\Omega| > \omega_m$, then $x(t)$ is uniquely determined by its samples $(x(nT))$ for

$$\frac{2\pi}{T} = \omega_s > 2\omega_m$$

$x(t) \xrightarrow{\text{periodic impulse train}} x(t)p(t) \xrightarrow{F} X_p(j\Omega) \xrightarrow[\text{gain}=T]{\text{LPF}} x(t)$ $[F^{-1}]$

impulses with amplitudes as sample values

cutoff $f_s - f_m > f_c > f_m$

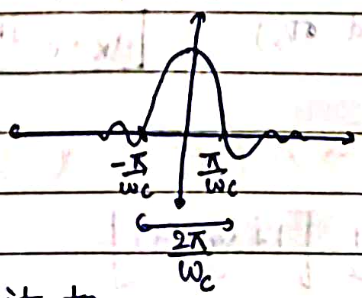
→ A signal is band-limited → there exists a max^m frequency component above which spectrum is entirely zero

there is a limit to how fast the signal changes. (PHYSICAL INTERPRET^n)

Ideal low-pass filters

$$H(j\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

$$h(t) = \underbrace{\frac{\sin \omega_c t}{\pi t}}_{\text{non-causal } (\infty - ly)}$$



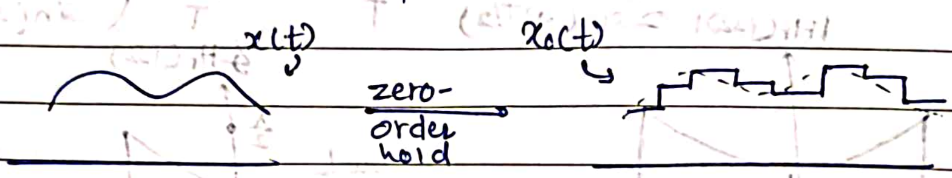
as we can't simply shift it to make it become causal.

→ causal filters exhibit some level of transition band b/w the passband and stopbands since it will take time for it to shift from allowing to attenuating frequencies, as it doesn't know about future inputs ('causal')

Sample & Hold

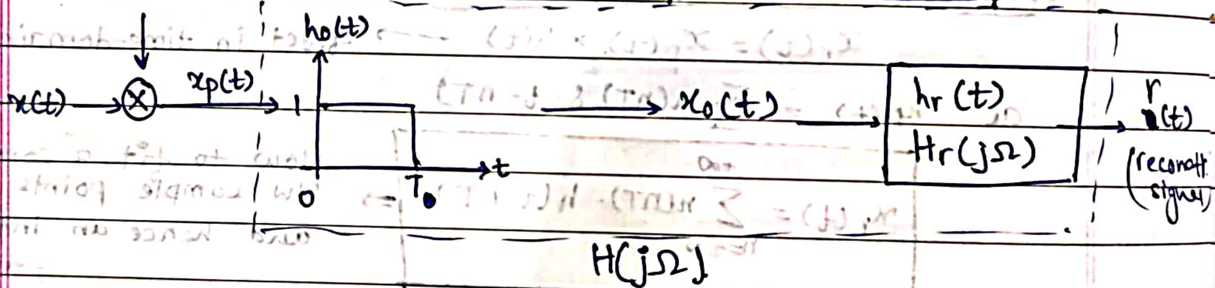
⇒ sampling theorem → band-limited signals UNIQUELY represented by its samples.

→ zero-order-hold → sample and hold till the next sample is taken.



reconstruction: filter → (no longer need constt. gain in the passband)

$x_0(t)$ ⟶ impulse-sampling + $\boxed{\text{LSI system with a rectL impulse response}}$



$H_r(j\Omega)$ such that $r(t) = x(t)$ ⟹ cascade of $h_0(t)$ & $h_r(t)$ is the ideal LPF $H(j\Omega)$ used.

$$h_0(t) \longrightarrow H_0(j\Omega) = \int_0^T 1 \cdot e^{-j\Omega t}\, dt$$

$$H_0(j\Omega) = \frac{1 - e^{-j\Omega T}}{j\Omega} = e^{-j\Omega T/2}\left[\frac{e^{j\Omega T/2} - e^{-j\Omega T/2}}{j\Omega}\right]$$

$$\boxed{H_0(j\Omega) = e^{-j\Omega T/2}\cdot\left[\frac{2\sin(\Omega T/2)}{\Omega}\right]}$$

and so, as $H(j\Omega) = H_0(j\Omega)\cdot H_r(j\Omega)$, we need,

$$\boxed{H_r(j\Omega) = \frac{e^{j\Omega T/2}\, H(j\Omega)}{\left(\frac{2\sin(\Omega T/2)}{\Omega}\right)}}$$

eg.- if the cutoff frequency of $H(j\Omega)$ is $\frac{\omega_s}{2} = \frac{1}{2T}$

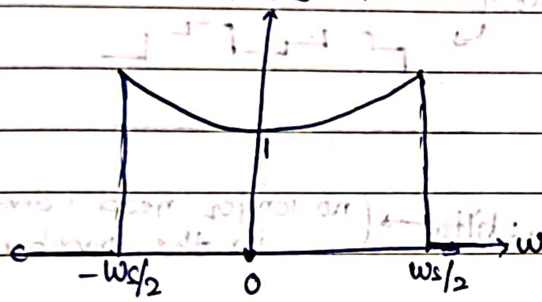
$$H(j\Omega) = \begin{cases} 1 & |\omega| < \omega_c = \omega_s/2 \\ 0 & o/w. \end{cases}$$

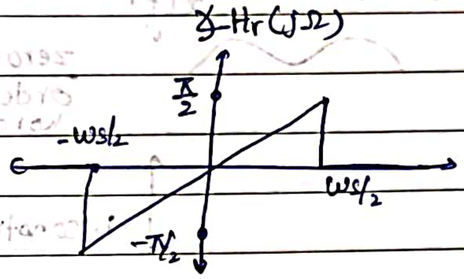
Then, $H_r(j\Omega) = \dfrac{\Omega\, e^{j\Omega T/2}}{2\sin(\Omega T/2)}$ $\quad \measuredangle H_r(j\Omega) = (\Omega T/2) \quad \Omega > 0$

$= (\Omega T/2) \quad \Omega < 0$

$$|H_r(j\Omega)| = \frac{(\Omega T/2)}{2\sin(\Omega T/2)} \cdot \frac{2}{T} = \frac{1}{T}\left(\frac{\Omega T/2}{\sin(\Omega T/2)}\right)$$

$|H_r(j\Omega)|$





output of a zero-order hold $\rightarrow$ adequate sample in itself.

RECONSTRUCTION / INTERPOLATION

$x_r(t) = x_p(t) * h(t)$ $\longrightarrow$ effect in time-domain of LPF

as $x_p(t) = \sum x(nT)\,\delta(t-nT)$

$$\boxed{x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \cdot h(t-nT)}$$ $\Rightarrow$ How to fit a cont. curve b/w sample points $x(nT)$ and hence an interpoln formulae.

For ideal LPF, $h(t) = \dfrac{\omega_c T}{\pi} \dfrac{\sin(\omega_c t)}{\omega_c(t)}$

and so,

$$\boxed{x_r(t) = \sum x(nT) \cdot \frac{\omega_c T}{\pi} \frac{\sin \omega_c(t-nT)}{\omega_c(t-nT)}}$$ (band-limited interpolation)

$\Rightarrow$ zero-order hold can also be used as an interpolation f^n^c, with the impulse response, $h_0(t)$ as the interpolating f^n^c.

$\hookrightarrow$ interpolating f^n^c, $h(t) = h_0(t)$ (impulse response of LSI system)

$\hookrightarrow$ $x_0(t) \xrightarrow{\text{approx}^m \text{ of}} x(t)$ and so system with $h_0(t) \xrightarrow{\text{approx}^m \text{ of}}$ Ideal LPF.

$$H_0(j\Omega) = \left(\frac{2\sin(\Omega T/2)}{\Omega}\right) e^{-j\Omega T/2} \qquad |H_0(j\Omega)| = \frac{\sin(\Omega T/2)}{(\Omega T/2)} \cdot T$$

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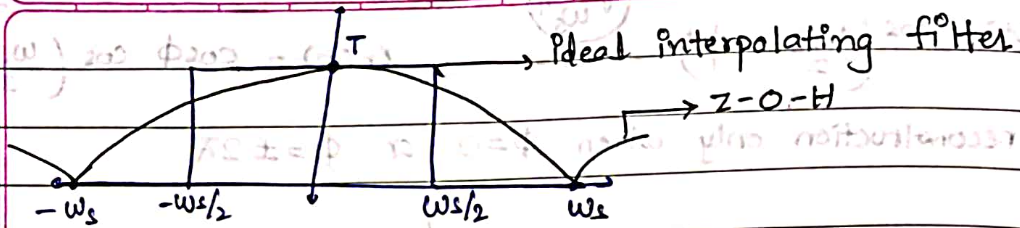
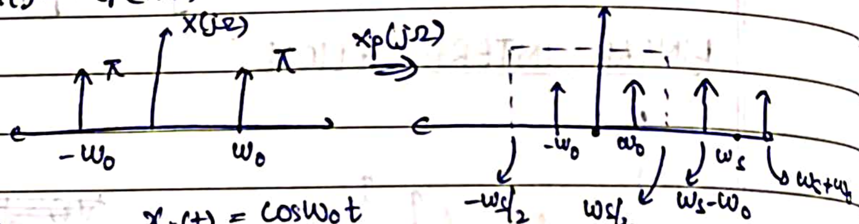


⇒ **NOTE :-** even if $\omega_s < 2\omega_m$, the reconstructed signal, $x_r(t)$ will be different from $x(t)$, but the value at the sampled points is same

i.e. $x(nT) = x_r(nT) \quad \forall n$

eg.- $x(t) = \cos \omega_0 t$

LPF : $\omega_c = \omega_s/2$



ⓐ $\omega_0 = \omega_s/6$; ⓑ $\omega_0 = 2\omega_s/6$ } no aliasing ⇒ $x_r(t) = \cos \omega_0 t$; $x_r(t) = \cos \omega_0 t$

ⓒ $\omega_0 = 4\omega_s/6 \longrightarrow x_r(t) = \cos((\omega_s - \omega_0)t) \neq x(t)$

ⓓ $\omega_0 = 5\omega_s/6 \longrightarrow x_r(t) = \cos((\omega_s - \omega_0)t) \neq x(t)$

when $\omega_0 = \omega_s \longrightarrow$ The output signal is a constant ⇒ 1

↳ when sampling one cycle, the samples are all equal and identical to those that would have been obtained by $\omega_0 = 0$ (const)

⇒ WHY NOT SAMPLE AT EXACTLY TWICE SAMPLING FREQUENCY.

eg.- $x(t) = \cos\left(\frac{\omega_s}{2} t + \phi\right) \longrightarrow$ impulse sampling at $2 \cdot (\omega_s/2) = \omega_s$

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\, \delta(t - nT) \qquad T = \frac{2\pi}{\omega_s}$$

$$x_p(t) = \sum_{n=-\infty}^{+\infty} \cos\left(\frac{\omega_s}{2} \cdot \frac{2\pi}{\omega_s} \cdot n + \phi\right) \delta(t - nT)$$

$$x_p(t) = \sum_{-\infty}^{+\infty} (\cos\phi)\, \delta(t - nT)$$

Here, $x(t) = \cos\left(\frac{\omega_s t}{2}\right)\cos\phi - \underbrace{\sin\left(\frac{\omega_s t}{2}\right)\sin\phi}_{g(t)}$

and $g(nT) = \sin\left(\frac{\omega_s}{2} \cdot \frac{2\pi}{\omega_c}\right)\sin\phi = 0$, so, $x(nT) = \cos\phi \cdot \cos\left(\frac{\omega_s nT}{2}\right)$

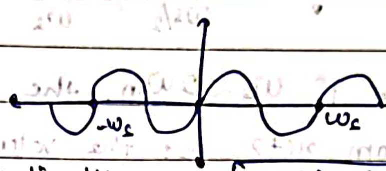
so, $X_p(t) \xrightarrow{LPF} Y_p(t) \qquad Y_p(j\Omega) = (\cos\phi) \cdot \{F^{-1}(\cos \omega_s T/2)\}$

so, $Y_p(t) = (\cos\phi) \cos\left(\frac{\omega_s T}{2}\right)$

$x(t) = \cos\left(\frac{\omega_s t}{2} + \phi\right) \xrightarrow{\omega_s^4} x_r(t) = \cos\phi \cos\left(\frac{\omega_s T}{2}\right)$

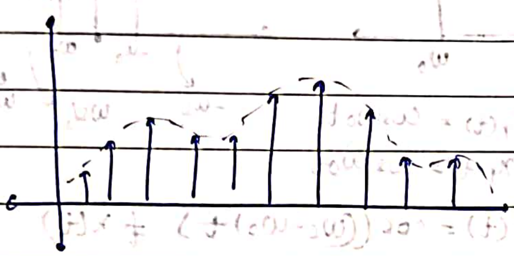
reconstruction only when $\phi = 0$ or $\phi = \pm 2\pi$

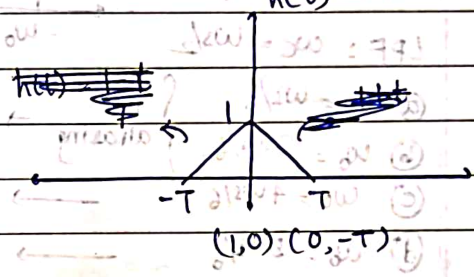
eg.2) $x_0(t) = \sin\left(\frac{\omega_s t}{2}\right)$

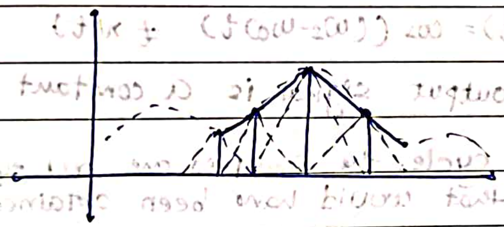


$\hookrightarrow x_p(t) = 0$ identically $\longrightarrow \boxed{x_r(t) = 0} \neq x(t)$

LINEAR INTERPOLATION







$H(j\Omega) \Rightarrow$

$h(t) = \begin{cases} 1 + t/T \\ 1 - t/T \end{cases}$

$H(j\Omega) = \int_{-T}^{0}\left(1+\frac{t}{T}\right) e^{-j\Omega t} + \int_{0}^{T}\left(1+\frac{t}{T}\right) e^{-j\Omega t}\, dt$

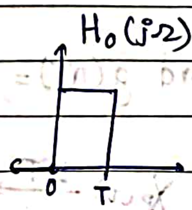
$t = -m$ $\quad = \int_0^T \left(\left(1-\frac{t}{T}\right) e^{j\Omega t} + \left(1+\frac{t}{T}\right) e^{-j\Omega t}\right) dt$ $\quad \Omega = \frac{\cancel{2}\pi}{\cancel{2}T}$

$= \int_0^T \left(2\cos\Omega t\right) - \frac{t}{T}(2j)\sin(\Omega t)\, dt$

$= \frac{-1}{T}\left[2\left(-t\frac{\cos\Omega t}{\Omega}\right)_0^T - 2\int_0^T \frac{1}{\Omega}\cos(\Omega t)\,dt\right]$

$= \frac{-1}{T}\left[\frac{0}{\Omega} + \frac{2}{\Omega^2}\left[\sin\Omega t\right]_0^T\right]$

$= \frac{1}{T}\left[\frac{\sin(\Omega T/2)}{(\Omega/2)^2}\right]$

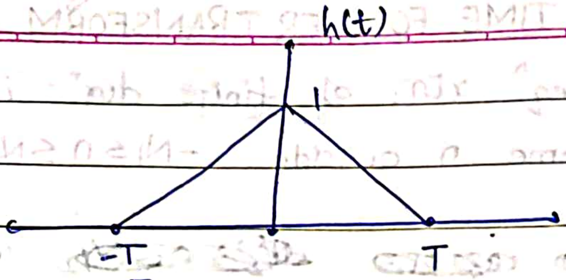


$h_0(t) * h_0(t)$

$\downarrow$

$h(t)$

$H(j\Omega) = H_0(j\Omega) * H_0(j\Omega)$



$$\cancel{H(j\Omega) = \int_{-T}^{T} h(t)\cdot e^{-j\Omega t}\,dt = \int_{-T}^{0}\left(1+\frac{t}{T}\right)e^{-j\Omega t}} + \int_{0}^{T}\left(1-\frac{t}{T}\right)e^{-j\Omega t}\,dt$$

$$\cancel{= \int_{-1}^{1}(2\cos\Omega t)\cdot dt + \frac{1}{T}\int t(}$$

$$t=-m$$

$$\int_{0}^{T}\left(1-\frac{t}{T}\right)e^{j\Omega t}\cdot dt$$

$$= \int_{0}^{T} 2\cos\Omega t + \frac{1}{T}\int\left(e^{j\Omega t} - e^{-j\Omega t}\right)\cdot t\,dt$$

$$= 2\left[\frac{\sin\Omega t}{\Omega}\right]_{0}^{T} + \frac{1}{T}\int_{0}^{T}(2\sin\Omega t)\cdot t\,dt$$

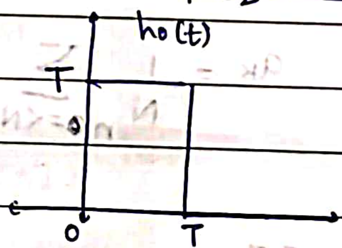
$$= \frac{2\sin\Omega T}{\Omega} + \frac{1}{T}\left[\left(2t\left(\frac{-\cos(\Omega t)}{\Omega}\right)\right)_{0}^{T} + \frac{1}{\Omega}\int_{0}^{T}\frac{\cos(\Omega t)}{\Omega}\right]$$

$$= \frac{2\sin\Omega T}{\Omega} + \frac{1}{\cancel{T}}\left[\frac{-2\cancel{T}\cos\Omega T}{\Omega}\right] + \frac{1}{T}\cdot\frac{1}{\Omega^2}\left[\frac{\sin(\Omega T)}{\Omega}\right]$$

$$\cancel{= \frac{2[\sin\Omega T - \cos\Omega T]}{\Omega} + \frac{1}{T}\cdot\frac{1}{\Omega^2}\frac{\sin(\Omega T)}{\Omega}}$$

$$h(t) = h_0^{(t)} * h_0^{(t)}$$

$$\mathcal{F}(h_0(t)) = \frac{\sin(\Omega T/2)}{\Omega/2}$$



so, $H(j\Omega) \longrightarrow$

$$H(j\Omega) = \frac{1}{T}\left[\frac{\sin(\Omega T/2)}{\Omega/2}\right]^2$$

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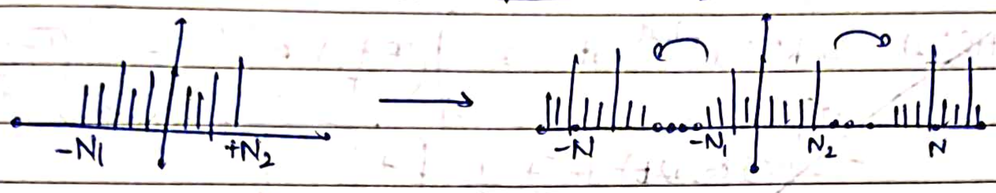
DISCRETE TIME FOURIER TRANSFORM

consider a general seq^n $x[n]$ of finite dur^n i.e.
$x[n] = 0$ for some n outside $-N_1 \le n \le N_2$.

$\tilde{x}[n] = x[n]$ with period N.



as $N \to \infty$ $\tilde{x}[n] \to x[n]$ for any finite n

$$\tilde{x}[n] = \sum_{K=\langle N\rangle} a_K e^{jK(\frac{2\pi}{N})n}$$

Note :- If $x[n] = x[n+N]$ $\quad \omega_0 = 2\pi/N$: fundamental frequency

$e^{j(\frac{2\pi}{N})n} \longrightarrow$ (periodic with period $= N$)

$\phi_K[n] = e^{jK\omega_0 n}$ $\quad K = 0, \pm 1, \pm 2, \dots \longrightarrow$ set of all discrete-time complex exponential signals periodic with period $= N$.

as

$\phi_0[n] = \phi_N[n]$

$\phi_1[n] = \phi_{N+1}[n]$ and so on, so, there are $\boxed{\text{ONLY N-distinct signals}}$

$$x[n] = \sum a_K \phi_K[n] = \sum_K a_K e^{jK\omega_0 n} = \sum_{K=\langle N\rangle} a_K e^{jK(\frac{2\pi}{N})n}$$

$\hookrightarrow$ N successive integers

and here, $$a_K = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jK(\frac{2\pi}{N})n}$$

Since $\tilde{x}[n] = x[n]$ over a period that includes $-N_1 \le n \le N_2$

then, $$a_K = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jK(2\pi/N)n} = \frac{1}{N} \sum_{-\infty}^{+\infty} x[n] e^{-jK(2\pi/N)n}$$

($x[n]$ is 0 o/w)

Let $$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}}$$, Then, $$\boxed{a_K = \frac{1}{N} X(e^{jK2\pi/N})}$$

proport^n to samples of $X(e^{j\omega})$

($\omega_0 = \frac{2\pi}{N}$ is spacing in the frequency-domain)

so, $$\boxed{\tilde{x}[n] = \sum_{K=\langle N\rangle} \frac{1}{N} X(e^{jK\omega_0}) \cdot e^{jK\omega_0}}$$

$\omega_0 = \frac{2\pi}{N}$ or $\frac{1}{N} = \frac{\omega_0}{2\pi}$

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) \cdot e^{jk\omega_0 n} \cdot \omega_0$$

as $N \to \infty$, $\omega_0 \to 0$ $\quad \Sigma \to \int$ $\quad$ Σ over N intervals of width $\omega_0 = \frac{2\pi}{N}$

$\int$ has total width 2π.

$$\boxed{x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega} \longrightarrow \text{inverse DTFT}$$

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}} \longrightarrow \text{discrete time FT (DTFT)}$$

→ Fourier co-eff. a_k of a periodic signal $\tilde{x}[n]$ can be expressed in terms of equally-spaced samples of the F.T. of a finite-duration aperiodic signal, $x[n] = \tilde{x}[n]$ over one period & zero o/w.

DTFT and Sampling

$$x_c(t) = x(t) \cdot \sum_{-\infty}^{+\infty} \delta(t-nT) = \sum_{n=-\infty}^{+\infty} x(t)\,\delta(t-nT) = \sum_{n=-\infty}^{+\infty} x(nT)\,\delta(t-nT)$$

(a C.T. Fourier T. can be defined)

Let $\boxed{x_d[n] \triangleq x(nT)}$

discrete time $\quad$ continuous time

$$X_c(j\Omega) = \int \left(\sum_{n=-\infty}^{+\infty} x(t)\,\delta(t-nT) \right) e^{-j\Omega t} \, dt$$

$$= \sum_{n=-\infty}^{+\infty} x(nT) \cdot e^{-j\Omega n T}$$

Two diff. representation of the same signal.

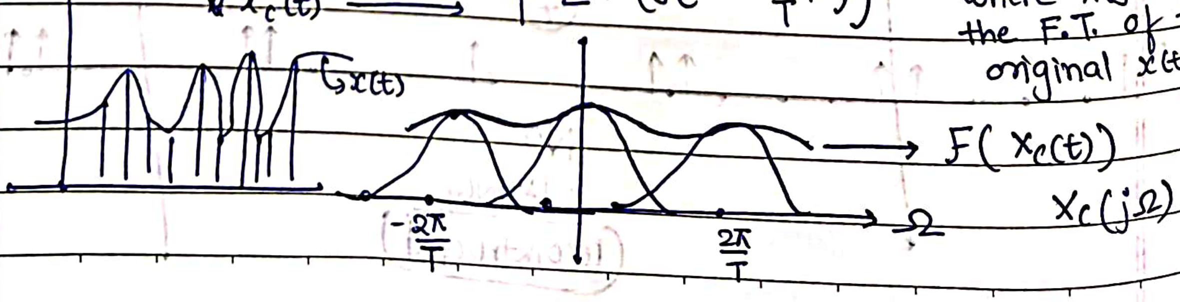
and $$X_d(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

So, $$\boxed{X_c\left(\frac{j\omega}{T}\right) = X_d(e^{j\omega})}$$

So, the DTFT is just the continuous Time Fourier Transform of a sampled signal.

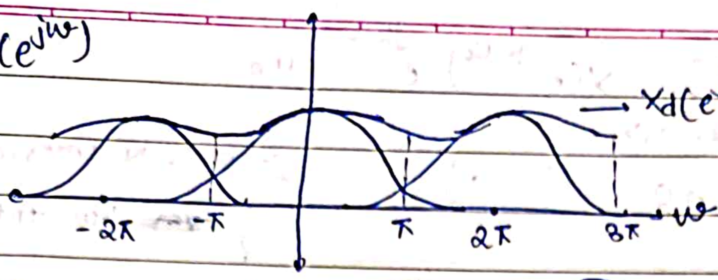
$$x_c(t) \xrightarrow{F} \frac{1}{T} \sum X\left(j\left(\Omega - \frac{2\pi}{T}k\right)\right)$$

where $X(j\Omega)$ is the F.T. of the original $x(t)$



$(\omega = \Omega T)$

$X_d(e^{j\omega})$

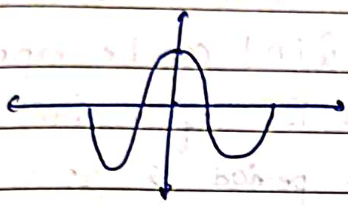


$\rightarrow X_d(e^{j\omega})$ (looks like the same.)

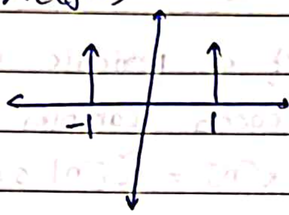
DTFT is periodic with period = 2π (in ω axis)
CTFT is periodic with period = $2\pi/T$ (in Ω) or 2π in (ω)

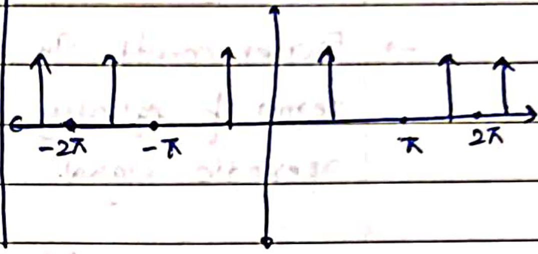
ALIASING

eg. $x_c = \cos(t)$ $\quad X_c(j\Omega)$ $\quad$ DTFT $= \sum X_c(j(\Omega - k\Omega_s))$



$\rightarrow$

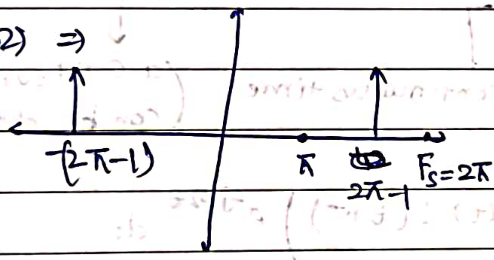


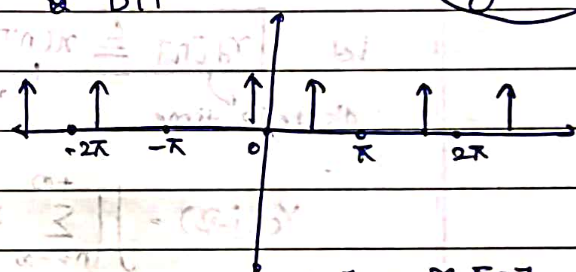


if the max. frequency, $f_m > \pi \Rightarrow$ aliasing in the signals

say $x_c(t) = \cos((2\pi-1)t)$

$X_{xc}(\Omega) \Rightarrow$ $\quad$ DTFT





DTFT of $x_2[n] \equiv x_1[n]$

$\rightarrow$ at a sampling rate of $T=1$, the frequencies 1 and $2\pi-1$ are indistinguishable $\longrightarrow$ $\boxed{\text{ALIASING}}$

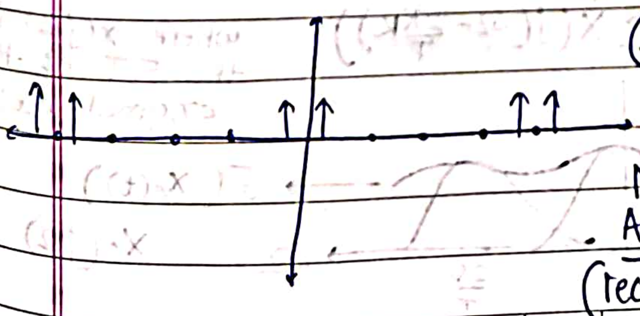
$f_s = 2\pi$ $\quad$ Nyquist criteria: $f_s - f_m > f_m$ or $f_s > 2f_m$

$2\pi > 2(1)$ but $\boxed{2\pi < 2(2\pi-1)}$

$\hookrightarrow$ thus, there is aliasing

if it was sampled as $f_s = 4\pi$ instead

$x_c \rightarrow$ DTFT



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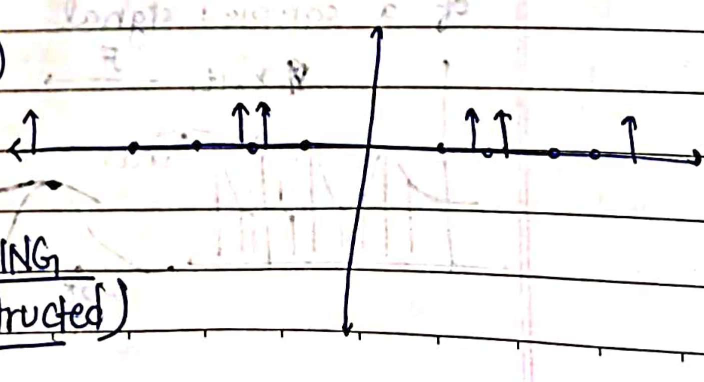

NO ALIASING (reconstructed)

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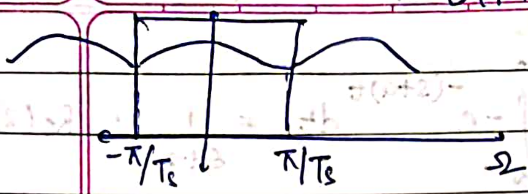
DTFT



$\longrightarrow$ restrict b/w $-\pi/T_s$ to π/T_s $\longrightarrow$ Let's say $x_1(t)$ has this as its F.T.

$\downarrow$

sample $x_1(t)$ at a $f = \frac{1}{T_s}$ $\Downarrow x[n]$ $\longrightarrow$ get the same spectrum as the DTFT

so, $x[n]$ can also be thought as the samples of $x_1(t)$ at $f = 1/T_s$. (NYQUIST CRITERIA)

$\rightarrow$ if we want each $\frown$ to contribute individually,
$(\omega)_{max.} = \pi \quad (\omega)_{min.} = -\pi$

MODULE-4

Laplace Transforms

$$e^{st} \longrightarrow \boxed{\text{LTI system}} \longrightarrow y(t) = H(s)\,e^{st}$$

$$y(t) = h(t) * e^{st}$$

$$= \int_{-\infty}^{+\infty} h(\tau)\,e^{s(t-\tau)}\,d\tau = e^{st} \underbrace{\int h(\tau)\,e^{-s\tau}\,d\tau}_{H(s)}$$

if $s = j\omega \longrightarrow$ F.T. ; general complex 's' $\longrightarrow$ Laplace Transform

$$\boxed{X(s) \triangleq \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt}$$ bilateral L.T.

$s = \sigma + j\omega$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)\,e^{-(\sigma + j\omega)t}\,dt$$

$$= \int_{-\infty}^{+\infty} \left(x(t)\,e^{-\sigma t}\right) e^{-j\omega t}\,dt$$

$$= \boxed{\mathcal{F}\left(x(t)\,e^{-\sigma t}\right)}$$

eg.- $x(t) = e^{-at}\,u(t)$ $\quad X(j\omega) = \int_{-\infty}^{+\infty} e^{-at}\,u(t)\,e^{-j\omega t}\,dt = \frac{1}{j\omega + a}$; $a > 0$
$(a > 0)$

$$X(s) = \int_{-\infty}^{+\infty} e^{-at}\,u(t)\,e^{-st}\,dt = \frac{1}{(\sigma + a) + j\omega} \quad ; \sigma + a > 0$$

$$\boxed{X(s) = \frac{1}{s + a} \quad ;} \quad (\mathrm{Re}(s) > -a)$$

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For $x(t) = -e^{-at}u(-t)$

$$X(s) = \int_{-\infty}^{+\infty} -e^{-at}u(-t)\cdot e^{-st}dt = \int_{-\infty}^{0} -e^{-(s+a)t}dt = \frac{1}{s+a} \; ; \; Re(s)+a<0$$

$$\boxed{X(s) = \frac{1}{s+a} \; ; \; Re(s) < -a}$$

→ Region of Convergence (R.O.C.)

⇒ ROC :- consists of those values of $s = \sigma + j\omega$ where the F.T. of $x(t)e^{-\sigma t}$ converges.

eg.- $x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)u(t)$

$$x(t) = \left[e^{-2t} + \frac{1}{2}e^{-(1+3j)t} + \frac{1}{2}e^{-(1-3j)t}\right]\cdot u(t)$$

$$\frac{1}{s+2} \qquad \frac{1}{s+(1+3j)} \qquad \frac{1}{s+(1-3j)} \longrightarrow \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)} \; ; \; Re(s) > -1.$$

$Re(s) > -2 \qquad Re(s) > -1 \qquad Re(s) > -1$

zeroes

poles

⇒ if order of deno. > num. ⇒ $X(s) \to 0$ as $s \to \infty$
num. > den. ⇒ $X(s)$ unbounded as $s \to \infty$ } zeroes / poles at ∞

eg.- $x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$

$$X(s) = \int_{-\infty}^{+\infty} \delta(t)e^{-st}\cdot dt = \int \delta(t)\cdot 1\, dt = 1 \quad \text{for any value 's'}$$

$$X(s) = 1 - \frac{4}{3}\cdot\frac{1}{s+1} + \frac{1}{3}\cdot\frac{1}{s-2}$$

$\boxed{Re(s) > 2}$

$Re(s) > -1$

$$= \frac{(s-1)^2}{(s+1)(s-2)}$$;- neither poles nor zeroes at infinity.

(#) If the ROC does NOT include the $j\omega$-axis, F.T. doesn't converge

↳ eg.- $e^{2t}u(t) \longrightarrow$ ~~F.T. DNC~~

$\frac{1}{s-2} \; ; \; Re(s) > 2$

(n^{th} order zero → repeated 'n' times)

Regions Of Convergence (R.O.C.)

Property 1) The ROC of $X(s)$ contains strips parallel to the $j\omega$-axis in the s-plane.

↓

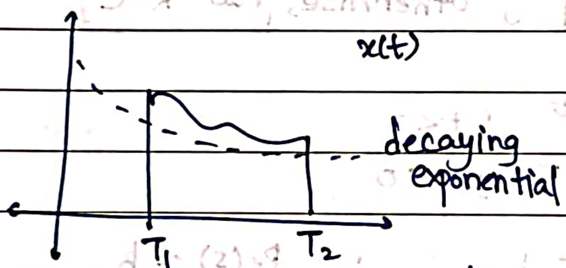
ROC contains those $s = \sigma + j\omega$ where F.T. of $x(t)e^{-\sigma t}$ converges
i.e. ROC has those values of s for which $x(t)e^{-\sigma t}$ is absolutely integrable

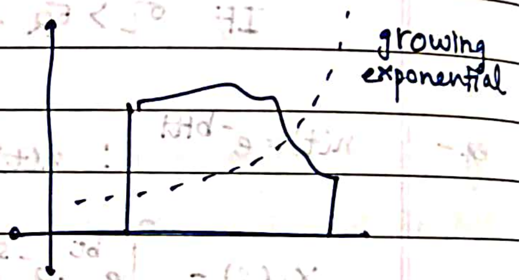
$$\text{ROC:} \rightarrow \boxed{\int_{-\infty}^{+\infty} |x(t)| e^{-\sigma t} \, dt < \infty}$$

Property 2) For rational Laplace transforms, the ROC does not contain ANY POLES.

↳ as $X(s) \rightarrow \infty$ at a pole, the integral doesn't converge at a pole.

Property 3) If $x(t)$ is of finite duration and absolutely integrable, then the ROC is the entire s-plane.





$\int |x(t)| e^{-\sigma t} \rightarrow$ can't be unbounded

Proof:- $\int_{T_1}^{T_2} |x(t)| \cdot dt < \infty$ — ①

$s = \sigma + j\omega$ to be in ROC, we need $\int_{T_1}^{T_2} |x(t)| e^{-\sigma t} \, dt < \infty$ — ②

① and ②: Verify that $s = 0$ is in the ROC

For $\sigma > 0$ $\quad \underbrace{\int |x(t)| e^{-\sigma t}}_{\text{also bounded}} < \underbrace{e^{-\sigma T_1} \int_{T_1}^{T_2} |x(t)| \, dt}_{\text{bounded}} \rightarrow$ finite

For $\sigma < 0$ $\quad \int |x(t)| e^{-\sigma t} < e^{-\sigma T_2} \int |x(t)| \, dt$ H.P.

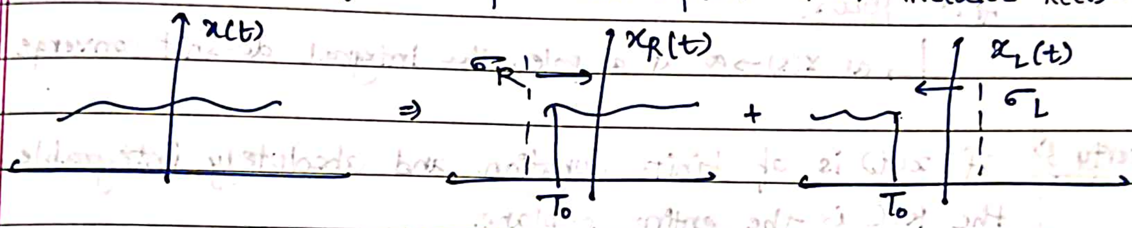
Property 4) If $x(t)$ is right sided i.e. $x(t) = 0 \; \forall \; t < T_1$, and if the line $Re(s) = \sigma_0$ is in ROC, then all s for which $Re(s) > \sigma_0$ are also in the ROC.

($x(t) = e^{t^2} \cdot u(t)$ ⟶ For no 's', this signal converges)

→ Here, $x(t) = e^{-\sigma_1 t}$ for $\sigma_1 > \sigma_0$ can't grow unbounded in the left-sided dir^n because $x(t)$ becomes 0 after a finite time.

Property 5) If $x(t)$ is left-sided, and if the line $Re(s) = \sigma_0 \in$ ROC, then all s for which $Re(s) < \sigma_0$ will also be in the R.O.C.

Property 6) If $x(t)$ is two sided and if the line $Re(s) = \sigma_0$ is in the ROC, then the ROC consists of a strip in the s-plane that includes $Re(s) = \sigma_0$.



IF $\sigma_L > \sigma_R$: $\boxed{\text{strip}}$; otherwise, $\boxed{\text{No ROC}}$

eg.- $x(t) = e^{-b|t|}$: $x(t) = \begin{cases} e^{bt} & ; t<0 \\ e^{-bt} & ; t>0 \end{cases}$

$$X_1(s) = \int_{-\infty}^{0} e^{bt} \cdot e^{-st} dt = \frac{-1}{s-b} \quad ; \quad Re(s) < b$$

$$X_2(s) = \frac{1}{s+b} \quad ; \quad Re(s) > -b$$

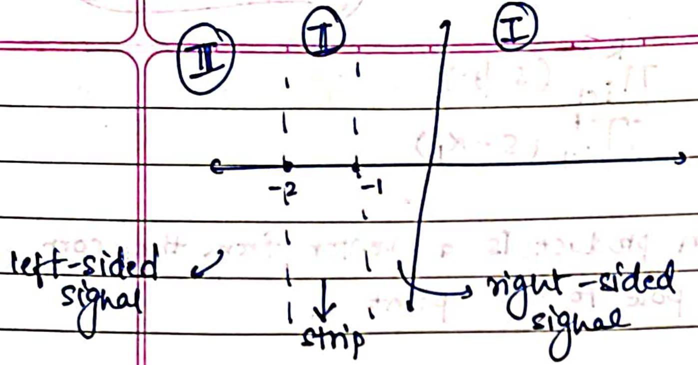
so, $X(s) = \frac{1}{s+b} - \frac{1}{s-b}$

$\boxed{\text{converges only if } b>0}$ ⇐ $\boxed{X(s) = \frac{-2b}{s^2-b^2} \quad \text{for} \quad -b < Re(s) < +b}$

Property 7) If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to ∞ and no poles of $X(s) \in$ ROC.

Property 8) $x(t)$ is right-sided ⇒ ROC is entire s-plane to right of rightmost pole.
$x(t)$ is left-sided ⇒ ROC is entire s-plane to left of leftmost pole.

eg.- $X(s) = \frac{1}{(s+1)(s+2)}$



$$X(s) = \frac{(s+2)-(s+1)}{(s+1)(s+2)}$$

$$= \frac{1}{s+1} - \frac{1}{s+2}$$

① $e^{-t}u(t) - e^{-2t}u(t)$

② $-e^{-t}u(-t) - e^{-2t}u(t)$

③ $-e^{-t}u(-t) + e^{-2t}u(-t)$

INVERSE LAPLACE TRANSFORM

$$X(\sigma+j\omega) = \mathcal{F}\{x(t)e^{-\sigma t}\} = X(s)$$

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}(X(s))$$

$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma+j\omega)\, e^{j\omega t}\, d\omega$$

so, $$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma+j\omega)\, e^{(\sigma+j\omega)t}\, d\omega}$$

$x(t)$ can be recovered from its Laplace transform, $X(\sigma+j\omega)$ by evaluating it for a set of values of $s=\sigma+j\omega$ in ROC with σ fixed and ω varying from $-\infty$ to $+\infty$.

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(s)\, e^{st}\, ds}$$

eg.- $$X(s) = \sum_{i=1}^{m} \frac{A_i}{s+a_i}$$

$X^{-1}(s) \longrightarrow A_i e^{-a_i t}u(t)$: if ROC is right of pole at $s=-a_i$

$\longrightarrow -A_i e^{-a_i t}u(-t)$: if ROC is left of pole at $s=-a_i$

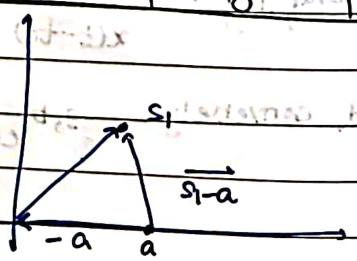
Geometric evaluation of F.T. from pole-zero plot

$X(s) = s-a$ (one zero)

$s = s_1$

$s_1 - a$ $\overrightarrow{s_1 - a}$: zero at $s=a$ to s_1

$s_1 + (-a)$



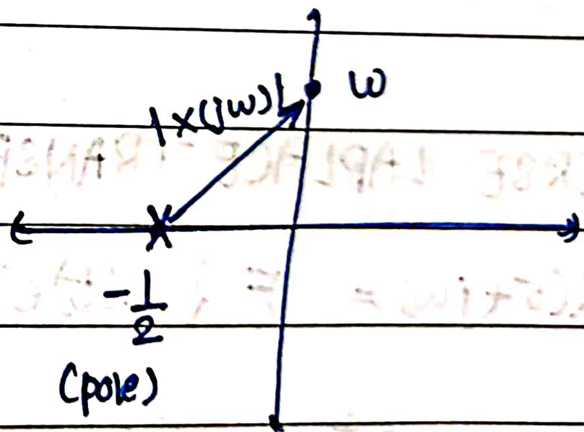
if $X(s) = \frac{1}{s-a} \longrightarrow$ deno. by the same vector sum of $s_1+(-a)$

rational L.T. : $X(s) = M \dfrac{\prod_{i=1}^{R} (s-\beta_i)}{\prod_{j=1}^{P} (s-\alpha_j)}$

$X(s)$ at $s = s_1$: each term in product is a vector from the co
zero or the pole to the point s_1.

eg.- $X(s) = \dfrac{1}{s+\frac{1}{2}}$, $Re(s) > -\frac{1}{2}$

$X(j\omega) = X(s)\big|_{s=j\omega} = \dfrac{1}{(j\omega)+(\frac{1}{2})}$



$|X(j\omega)|^2 = \dfrac{1}{\omega^2 + (\frac{1}{2})^2}$

$\measuredangle X(j\omega) = -\tan^{-1}(2\omega)$ ~~EX~~

PROPERTIES (list)