Department of Electrical Engineering, IIT Bombay

AUTUMN SEMESTER: JUL-NOV 2023

Mid-Semester Examination: EE 229 – Signal Processing I – (B. Tech.)

Maximum Marks: 50 (25 percent weight)

Date: Saturday 19 Sept. 2023 **Time: 13:30 to 15:30 hours**

Instructions:

1. Please begin the answer to each **main question** on a **fresh page** of the answer booklet.
2. This is a **closed book, closed notes** examination.
3. Show your reasoning and important steps clearly.
4. Unless otherwise stated, treat x as the input to the system and y, the output.

Q1. (4 + 6 = 10 marks):

In each of the parts (a) and (b) below, there is a given system description and two following statements, 'A' and 'R', pertaining to the system described. State and explain:

(i) whether the statement A is correct
(ii) whether the statement R is correct

(a) Continuous time system description: $y(t) = \dfrac{dx(t)}{dt}$

A: The response of this system to the bounded input $x(t) = 10\cos(Bt)$, for any finite angular frequency B, is bounded by $10B$.
R: This system is stable in a 'Bounded input, bounded output' (BIBO) sense.

(b) Discrete time system description:

$$y[n] = (x[n] + x[n-1])^{-1} \text{ when } (x[n] + x[n-1]) \neq 0,$$
$$y[n] = 0 \text{ when } x[n] + x[n-1] = 0$$

A: A periodic, bounded input to this system always results in a periodic, bounded output of the same period.
R: The system is causal, stable in a 'Bounded input, bounded output' (BIBO) sense and it is shift invariant as well.

Q2. (15 marks):

The impulse response of a discrete time, linear shift invariant system is: $h[n] = 1$ for $n = 0, 1, 2$ and 0 for other values of n.

Do **not** employ Fourier Transforms in this question.

(a) When a particular input, $x_1[n]$, which is non-zero for $n = 0, 1, 2, 3, 4$ and zero for other values of n is applied to this system, the output is $y_1[n] = 3, 10, 1, 2, 3$ respectively, for $n = 0, 1, 2, 3, 4$. Obtain $x_1[n]$ for all n and $y_1[n],\ n \geq 5$.

(b) When the input $x_2\ [n] = cos(Bn)$, B a positive angular frequency is applied to the system, show that the output is also of the form $y_2\ [n] = A\ cos(Bn + D)$ for appropriate $A, D; A > 0$ and D real, where A and D depend only on B. Obtain the possible values of the angular frequency B for which $y_2\ [n] = 0$ for all n.

Q3. (15 marks):

A periodic square wave $x(t)$, with a period of 1 second, is described by

$$x(t) = u(t) - \ u(t - r)\ for\ 0 < t < 1$$

and a chosen value of $r;\ 0 < r < 1$, with the time t in seconds.

(a) Show that $x(t)$ can be written in the form of a Fourier series expansion:
$x(t) = \ B_0 + \sum_{m,\ \text{for all positive integers}\ m} B_m \cos(2\,\pi\,m\,t\ +\ \alpha_m),$
where B_0 is a constant; B_m is the amplitude and α_m is the phase, of the m^{th} component. by obtaining B_0 , B_m and α_m in terms of r and m.

(b) Let $x(t)$ be the input voltage in Volts, applied in the R-C circuit shown in Fig.Q3. Obtain a Fourier series expansion for the output in a form similar to Q3(a).

(c) Obtain the value of $r,\ 0 < r < 1$ so that B_m is zero for all non-zero, **even** integers m, in Q3(a).

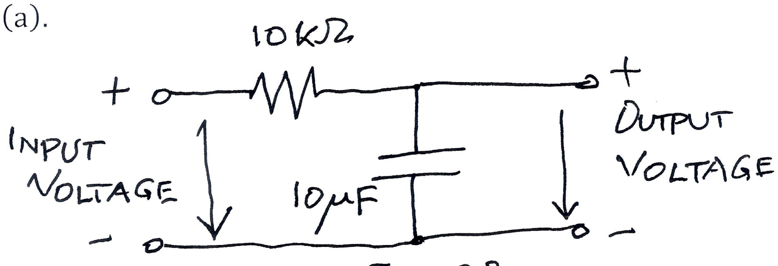


FIG. Q3

Q4. (10 marks)

Consider the continuous time signals $x_1(t) = \ e^{-2t}\,u(t)$ and $x_2(t) = \ e^{-5t}\,u(t)$ with respective Fourier Transforms $X_1\ (j\Omega)$ and $X_2\ (j\Omega)$. Evaluate:

(a) $\int_{-\,infinity}^{+\,infinity} |X_1\ (j\Omega)\ |^2\ d\Omega$

(b) $\int_{-\,infinity}^{+\,infinity} X_1\ (j\Omega)\ X_2\ (j\Omega)\ d\Omega$

(c) The inverse Fourier Transform of $|X_1\ (j\Omega)\ |^2$.

End of question paper – with best wishes.

Q1 -

(a)

(i) $x(t) = 10 \cos Bt$.

$$y(t) = \frac{dx(t)}{dt} = -10B \sin Bt$$

$|y(t)| \leq 10B$. Hence correct

(ii) Consider, for example $x(t) = 10 \cos Bt^2$ which is again bounded by 10.

Then the response $= -10(2Bt) \sin Bt^2$ which is unbounded. Hence incorrect.

(Any other correct counterexample to disprove the BIBO stability of the differentiator system, will do).

(b) The system is shift-invariant. Hence a periodic input to the system would result in a periodic output with the same period.

The system is clearly causal.

However, the system is NOT BIBO stable.

Consider the bounded input $x[n] = \left(\frac{1}{2}\right)^n u[n]$

The output would be

$$y[n] = 1, \quad n = 0$$

$$= \left(\left(\tfrac{1}{2}\right)^n + \left(\tfrac{1}{2}\right)^{n-1}\right)^{-1}, \quad n > 0$$

$$= \left(\left(\tfrac{1}{2}\right)^n (1+2)\right)^{-1} = \frac{1}{3} \cdot 2^n, \quad n > 0$$

which is unbounded.

Thus, (A) is correct but (R) is incorrect.

Q2 -

(a) $$y_1[n] = \sum_{k=0}^{2} h[k]x_1[n-k]$$

$$= x_1[n] + x_1[n-1] + x_1[n-2]$$

$n \rightarrow$	0	1	2	3	4
$y_1[n] \rightarrow$	3	10	1	2	3
=	$x_1[0]$	$x_1[0] + x_1[1]$	$x_1[0] + x_1[1] + x_1[2]$	$x_1[1] + x_1[2] + x_1[3]$	$x_1[2] + x_1[3] + x_1[4]$

Thus $x_1[0] = 3$

$x_1[1] = 10 - 3 = 7$

$x_1[2] = 1 - (3+7) = -9$

$x_1[3] = 2 - (7-9) = 2+2 = 4$

$x_1[4] = 3 - (-9+4) = 8$

$x_1[n] = 0 \;\; \forall n > 4.$

$y_1[5] = x_1[4] + x_1[3] = 8 + 4 = 12$

$y_1[6] = x_1[4] = 8 \quad y_1[n] = 0 \;\; \forall n > 6.$

(b) $y_2[n] = x_2[n] + x_2[n-1] + x_2[n-2]$

$= \cos Bn + \cos B(n-1) + \cos B(n-2)$

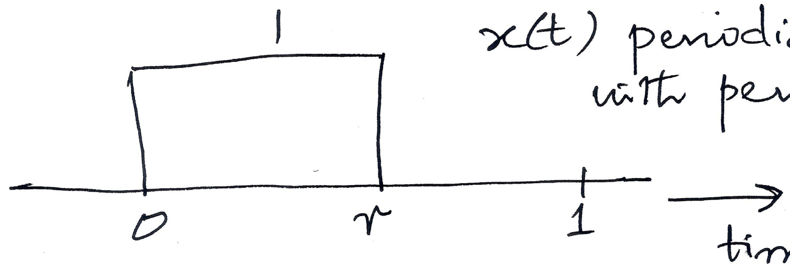
$= 2\cos \dfrac{Bn + B(n-2)}{2} \cos \dfrac{Bn - B(n-2)}{2} + \cos B(n-1)$

$= 2\cos B \cos B(n-1) + \cos B(n-1)$

$= (1 + 2\cos B)\cos B(n-1)$ of required form

Identically zero for $1 + 2\cos B = 0$

Q3 -



$x(t)$ periodic with period 1 sec

time t in sec

(a) $B_0 = \frac{1}{1}\int_0^1 x(t)\,dt = \tau.$

To obtain the m^{th} harmonic, we need to evaluate the integrals, $m \geq 1$

$$\frac{1}{1/2}\int_0^1 x(t)\cos 2\pi m t\,dt = 2\int_0^{\tau} \cos(2\pi m t)\,dt$$

$$= 2\,\frac{\sin 2\pi m t}{2\pi m}\Bigg|_0^{\tau} = \frac{2}{2\pi m}\sin(2\pi m \tau).$$

$$= \frac{1}{\pi m}\sin(2\pi m \tau).$$

$$\frac{1}{1/2}\int_0^1 x(t)\sin 2\pi m t\,dt = 2\int_0^{\tau} \sin(2\pi m t)\,dt$$

$$= -2\,\frac{\cos 2\pi m t}{2\pi m}\Bigg|_0^{\tau} = \frac{1}{\pi m}\cdot(1-\cos 2\pi m \tau).$$

Thus for $m \geq 1$

$$B_m \cos(2\pi m t + \alpha_m) = B_m\left(\cos\alpha_m \cos 2\pi m t - \sin\alpha_m \sin 2\pi m t\right)$$

$$= \frac{1}{\pi m}\sin 2\pi m \tau.\ \cos 2\pi m t + \frac{1}{\pi m}(1-\cos 2\pi m \tau)\sin 2\pi m t$$

leading us to $B_m \cos\alpha_m = \frac{1}{\pi m}\sin 2\pi m \tau$

$$-B_m \sin\alpha_m = \frac{1}{\pi m}(1-\cos 2\pi m \tau).$$

Squaring and adding,

$$B_m^2 = \left(\frac{1}{\pi m}\right)^2 \left\{ \sin^2 2\pi m\tau + (1-\cos 2\pi m\tau)^2 \right\}$$

$$= \left(\frac{1}{\pi m}\right)^2 \left\{ \sin^2 2\pi m\tau + 1 + \cos^2 2\pi m\tau - 2\cos 2\pi m\tau \right\}$$

$$= \left(\frac{1}{\pi m}\right)^2 2\left(1 - \cos 2\pi m\tau\right)$$

$$= \left(\frac{1}{\pi m}\right)^2 . 2 . 2 \sin^2 \frac{2\pi m\tau}{2}$$

$$= \left(\frac{1}{\pi m}\right)^2 . 2^2 \sin^2 \frac{2\pi m\tau}{2} .$$

Or $B_m = 2 . \frac{1}{\pi m} . |\sin \pi m\tau| .$

Dividing, $-\tan \alpha_m = \dfrac{1-\cos 2\pi m\tau}{\sin 2\pi m\tau}$

$$= \frac{2\sin^2 \pi m\tau}{2\sin \pi m\tau \cos \pi m\tau}$$

$$= \frac{\sin \pi m\tau}{\cos \pi m\tau} = \tan \pi m\tau$$

Or $\alpha_m = -\pi m\tau$.

(b) Frequency response of the LSI system shown in Fig. Q3

$$= \frac{\frac{1}{j\Omega C}}{R + \frac{1}{j\Omega C}}$$

$R = 10^4 \Omega$

$C = 10 \times 10^{-6}$ F

$RC = 0.1$ sec.

$$= \frac{1}{1 + j\Omega RC} = \frac{1}{1 + j\,0.1\,\Omega}$$

We need to evaluate this frequency response at $\Omega = 2\pi m$, $m \in \mathbb{Z}^+$

$$= \frac{1}{1 + j\,0.1 \times 2\pi m} = \frac{1}{1 + j\,0.2\pi m}$$

$$\text{magnitude} = +\sqrt{\frac{1}{1 + (0.2\pi m)^2}}$$

$$\text{phase (angle)} = -\tan^{-1} 0.2\pi m$$

Accordingly the output Fourier series would be

$$r + \sum_{m=1}^{\infty} \frac{2\,|\sin \pi m r|}{\pi m \sqrt{1 + (0.2\pi m)^2}} \cos\left(2\pi m t - \pi m r - \tan^{-1} 0.2\pi m\right)$$

Since the magnitude multiplies B_m and the phase angle adds to α_m.

(c) We desire $|\sin \pi m r| = 0$
for all $m = 2p$, $p \in \mathbb{Z}^+$, $p \neq 0$

or $|\sin 2p\pi r| = 0$

Thus $2pr$ is an integer

for $0 < r < 1$, in particular

$2r$ is integral $\Rightarrow$ $r = \frac{1}{2}$.

Q4 -

(a) Employ Parseval's theorem:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X_1(j\Omega)|^2 d\Omega = \int_{-\infty}^{+\infty} |x_1(t)|^2 dt$$

whereupon $$\int_{-\infty}^{+\infty} |X_1(j\Omega)|^2 d\Omega = 2\pi\int_{-\infty}^{+\infty} |x_1(t)|^2 dt$$

$$= 2\pi\int_0^{\infty} (e^{-2t})^2 dt = 2\pi\int_0^{\infty} e^{-4t} dt = \frac{2\pi e^{-4t}}{-4}\Big|_0^{\infty}$$

$$= \pi/2$$

(b) $X_1(j\Omega)X_2(j\Omega)$ is the Fourier Transform of $x_1(t) * x_2(t)$.

$$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} e^{-2\tau} u(\tau) e^{-5(t-\tau)} u(t-\tau) d\tau$$

$$= \int_0^t e^{-2\tau} e^{-5(t-\tau)} d\tau = e^{-5t}\int_0^t e^{-2\tau} e^{5\tau} d\tau$$

Now $$\Rightarrow \int_{-\infty}^{+\infty} X_1(j\Omega)X_2(j\Omega) d\Omega = 2\pi\left\{e^{-5t}\int_0^t e^{3\tau} d\tau\right\}_{t=0}$$

[Actually the instructor wanted to ask $\int_{-\infty}^{+\infty} X_1 \bar{X}_2 d\Omega$:-)]

$$= 0.$$

(c) $$|X_1(j\Omega)|^2 = X_1(j\Omega)\overline{X_1(j\Omega)}$$

$$\overline{\frac{1}{2\pi}\int_{-\infty}^{+\infty} X_1(j\Omega) e^{j\Omega t} d\Omega} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_1(j\Omega) e^{j\Omega(-t)} d\Omega$$

$$= x_1(-t)$$

Thus what is required is to convolve $x_1(t)$ and $\overline{x_1(-t)}$.

$= x_1(t)$ and $x_1(-t)$ since $x_1(t)$ is real.

The convolution of $x_1(t)$ and $x_1(-t)$

$$= \int_{-\infty}^{+\infty} e^{-2\tau} u(\tau) \cdot e^{+2(t-\tau)} u(\tau - t)\, d\tau$$

We consider 2 cases

$$t \geq 0 : \int_{-\infty}^{+\infty} e^{-2\tau} u(\tau) e^{2(t-\tau)} u(\tau - t) d\tau$$

$$= \int_{t}^{+\infty} e^{-2\tau} \cdot e^{2(t-\tau)} d\tau$$

$$= e^{2t} \int_{t}^{\infty} e^{-4\tau} d\tau = \left. \frac{e^{2t} \cdot e^{-4\tau}}{-4} \right|_{t}^{\infty}$$

$$= \frac{e^{2t} e^{-4t}}{4} = \frac{e^{-2t}}{4}.$$

$$t < 0 \int_{-\infty}^{+\infty} e^{-2\tau} u(\tau) e^{2(t-\tau)} u(\tau - t) d\tau$$

$$= \int_{0}^{\infty} e^{-2\tau} e^{2(t-\tau)} d\tau = e^{2t} \int_{0}^{\infty} e^{-4\tau} d\tau$$

$$= \left. \frac{e^{2t} e^{-4\tau}}{-4} \right|_{0}^{\infty} = \frac{e^{2t}}{4}.$$

The required convolution can be written together as $\frac{1}{4} e^{-2|t|}$. $\forall t \in \mathbb{R}$