Z-TRANSFORM

$x[n] = z^{+n} \rightarrow$ $\boxed{\text{discrete LTI}}$ $\rightarrow y[n] = H(z)\, z^{n}$ where $H(z) = \sum_{n=-\infty}^{+\infty} h[n]\, z^{-n}$

h[n]

when $|z| = 1$ or $z = e^{j\omega} \rightarrow$ DTFT

↳ restriction is removed ⇒ $\boxed{X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n}}$

$z = re^{j\omega} \Rightarrow X(re^{j\omega}) = \sum \{x[n] r^{-n}\} e^{-j\omega n} \Rightarrow \boxed{X(re^{j\omega}) = \mathcal{F}\{x[n] r^{-n}\}}$

real exponential r^{-n}

z-transform reduces to DTFT on the circle $\boxed{|z| = 1}$

↳ ROC: if includes $|z|=1$, the F.T. is also converging

eg.- $x[n] = \alpha^{n} u[n]$ $\qquad X(z) = \sum_{-\infty}^{+\infty} \alpha^{n} u[n] z^{-n} = \sum_{0}^{+\infty} (\alpha/z)^{n}$

convergence : $|\alpha/z| < 1 \Rightarrow \boxed{|z| > |\alpha|}$ $\boxed{X(z) = \frac{z}{z-a}}$

⇒ For $a = 1$, $u[n] \longrightarrow X(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$; $|z| > 1$

$-n-1 \geq 0$

$-n \geq 1$

$n \leq -1$

$X(z) = \frac{z}{z-a}$ → $\alpha^{n} u[n]$ $\quad |z| > |a|$

→ $-\alpha^{n} u[-n-1] \longrightarrow X(z) = -\sum_{-\infty}^{-1} (\alpha/z)^{n}$ $\quad |z| < |a|$

$= -\sum_{1}^{\infty} (z/\alpha)^{n} = 1 - \frac{1}{1 - z/a}$

$= \frac{z}{z-a}$

<u>ROC: properties</u>

ROC →

- ring in z-plane centred about origin $\boxed{\sum |x[n]| r^{-n} < \infty}$ (A.1.)
- Two sided : ring
- ROC does not contain any poles
- $x[n]$ is right-sided / left-sided and if $|z| = r_0 \in$ ROC, then all finite values for which $|z| > r_0$ / $|z| < r_0$ also belong to ROC.
- $\boxed{x[n] \text{ is of finite dur}^{n}\text{, ROC is entire z-plane ex. possibly } z = 0 \text{ and/or } z = \infty}$

Let $X(z) = \sum_{N_1}^{N_2} x[n]\, z^{-n}$: $z \neq 0$ or ∞ ? finite terms

$N_1 < 0 \quad N_2 > 0 \Rightarrow$ as $|z| \to 0$: -ve : unbounded

$|z| \to \infty$: +ve : unbounded

eg. $\delta[n] \longrightarrow 1$

$\delta[n-1] \longrightarrow z^{-1}$: pole at $z = 0$

$\delta[n+1] \longrightarrow z$: pole at ∞

Date ____
Page ____

inverse Z-transform

$$X(re^{j\omega}) = F\{x[n]\, r^{-n}\}$$

so, $x[n]\, r^{-n} = F^{-1}\{X(re^{j\omega})\} \Rightarrow x[n] = r^{n} \cdot F^{-1}\{X(re^{j\omega})\}$

$$x[n] = \frac{1}{2\pi} r^n \int_{2\pi} X(re^{j\omega}) \cdot e^{j\omega n}\, d\omega \qquad \left(\begin{matrix} z = re^{j\omega} \\ z\, dz = d\omega \end{matrix} \right)$$

so, $$x[n] = \frac{1}{2\pi} \oint_C X(z)\, z^{n-1}\, dz$$

eg.- $X(z) = 4z^2 + 2 + 3z^{-1}$, $\infty > |z| > 0$

$$X(z) = \sum x[n]\, z^{-n}$$

$$x[n] = \begin{matrix} 4 & 0 & \underset{\substack{\uparrow \\ 0}}{2} & 3 \end{matrix} = 4[\delta(n+2)] + 2[\delta(n)] + 3\delta(n-1)$$

eg.- $X(z) = \log(1 + az^{-1})$

$\downarrow$

$$X(z) = az^{-1} - \frac{(a z^{-1})^2}{2} + \frac{(az^{-1})^3}{3} \ldots$$

$$x[n] = \underset{\substack{\uparrow \\ 0}}{a} \quad -a^2/2 \quad a^3/3 \ldots \qquad x[n] = \begin{cases} (-1)^{n+1} a^n/n & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

PROPERTIES

1) $a x_1[n] + b x_2[n] \xrightarrow{z} aX_1(z) + bX_2(z)$ ROC $\supset R_1 \cap R_2$

2) $x[n-n_0] \longrightarrow z^{-n_0} X(z)$ ROC $= R$ ex. $\{0\}$ or $\{\infty\}$ (possibly)

($n_0 < 0$: poles at ∞ may be cancelled)

3) $z_0^n\, x[n] \xleftrightarrow{z} X\left(\frac{z}{z_0}\right)$ ROC $= |z_0| R$

4) $x[-n] \xrightarrow{z} X\left(\frac{1}{z}\right)$ ROC $= \frac{1}{R}$

5) Let $x_{(k)}[n] = \begin{cases} x[n/k], & n = mk \\ 0; & n \neq mk \end{cases}$

$x_{(k)}[n] \xrightarrow{z} X(z^k)$ ROC: $R^{1/k}$

6) $n x[n] \xleftrightarrow{z} -z \frac{dX(z)}{dz}$