FOURIER TRANSFORMS

s, z are complex no.'s

$e^{st} \longrightarrow H(s)e^{st}$

$z^n \longrightarrow H(z)z^n$

(complex exponentials are eigenfunctions of LTI systems.)

CONT. $\Rightarrow$ $x(t) = e^{st}$ $\quad y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,d\tau$ → converges (assume)

$$y(t) = e^{st}\left\{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau\right\} = H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} e^{-s\tau}h(\tau)\,d\tau$$

DISCRETE $\Rightarrow$ $x[n] = z^n$ $\quad y[n] = \sum_{-\infty}^{+\infty} h[K]z^{n-K} = z^n\sum_{-\infty}^{+\infty} h[K]z^{-K}$

$$\boxed{y[n] = \left(\sum_{-\infty}^{\infty} h[K]z^{-K}\right)\cdot z^n}$$ ie. $y[n] = H(z)\cdot z^n$

$\Rightarrow$ $\sum_K a_K e^{S_K t} \longrightarrow \boxed{\text{LTI system}} \longrightarrow \sum_K a_K H(S_K)\cdot e^{S_K t}$

in continuous time, $s = j\omega$ (Fourier) $\Rightarrow e^{j\omega t}$

in discrete time, $z = e^{j\omega}$ (unit mag.) $\Rightarrow e^{j\omega n}$

eg. $y(t) = x(t-3)$ $\quad e^{2jt} \rightarrow \boxed{} \rightarrow e^{2j(t-3)} = e^{-6j}\cdot e^{2jt}$

$$H(2j) = \int_{-\infty}^{\infty} h(\tau)e^{-2j\tau}\,d\tau = \int_{-\infty}^{\infty}\delta(\tau-3)\cdot e^{-2j\tau}\,d\tau = e^{-2j(3)} = e^{-6j}$$

so, $\boxed{y(t) = H(2j)\cdot e^{2jt}}$

PERIODIC SIGNALS

$x(t) = x(t+T)$ for all t

T: fundamental period ; $\omega_0 = 2\pi/T \Rightarrow$ fundamental frequency

e.g.- $\cos\omega_0 t$, $e^{j\omega_0 t}$

$$\phi_K = e^{jK\omega_0 t} = e^{jK\left(\frac{2\pi}{T}\right)t} \quad ; K = 0, \pm1, \pm2, \ldots$$

→ each of these $\phi_K(t)$ has a fundamental freq. as a multiple of ω_0 and each is periodic with period = T

$\boxed{x(t) = \sum a_K e^{jK\omega_0 t}}$ → periodic with period = T.

$\left.\begin{array}{l}K = +N\\K = -N\end{array}\right\}$ N^{th} - harmonic component

$\Rightarrow$ if $x(t)$ is real, $x(t) = x^*(t)$

$$\sum a_K e^{j\omega_0 Kt} = \sum_{-\infty}^{\infty} a_K^* e^{-j\omega_0 Kt} \xrightarrow{K\to(-K)} = \sum a_{-K}^* e^{jK\omega_0 t}$$

$\Rightarrow$ $\boxed{a_K = a_{-K}^*}$ for real $x(t)$

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right] \quad \text{and as} \quad a_{-k} = a_k^*$$

$$x(t) = a_0 + \sum_{1}^{\infty}\left(a_k e^{j\omega_0 tk} + a_k^* e^{-j\omega_0 tk}\right) = a_0 + \sum 2\,\mathrm{Re}\{a_k e^{j\omega_0 tk}\}$$

Rep. - 2

Rep. - 3

alt. represⁿ

$$\boxed{x(t) = a_0 + 2\sum_{1}^{\infty} a_k \cos(k\omega_0 t + \phi_k)} \quad ; \quad \boxed{a_k = B_k + jC_k}$$

$$x(t) = a_0 + 2\sum_{1}^{\infty} B_k \cos(\omega_0 t\,k) - C_k \sin(j\omega_0 t k)$$

Continuous-Time Periodic Signal

$$x(t)e^{-jn\omega_0 t} = \sum a_k e^{jk\omega_0 t} \cdot e^{-jn\omega_0 t}$$

Integrating both sides from 0 to $T\,(=2\pi/\omega_0)$

$$\int_0^T x(t)e^{-jn\omega_0 t} = \sum_{-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}\,dt$$

$$\Rightarrow \int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T \underbrace{\cos(k-n)\omega_0 t\,dt}_{0\ (T = n\times \text{period})} + j\int_0^T \underbrace{\sin(k-n)\omega_0 t\,dt}_{0}$$

$k \neq n$

$k = n \longrightarrow T$

$$\int_T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T \ ; & k=n \\ 0 \ ; & k \neq n \end{cases}$$

(over any length of T)

So,
$$\boxed{a_k = \frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\,dt}$$

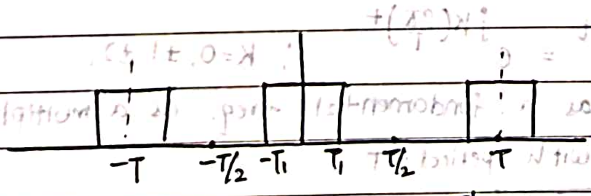
$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

e.g. $x(t) = \sin\omega_0 t \longrightarrow \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$

$$a_k = \frac{1}{T}\int_T \sin\omega_0 t\, e^{-kj\omega_0 t}\,dt = \frac{1}{T}\int_{(T)} \sin\omega_0 T\cos(-k\omega_0 t)\,dt + j\int_T \frac{\sin\omega_0 T\,dt}{\sin(-k\omega_0 T)}$$

$$a_1 = -j/2 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}, \quad a_k = 0,\ \text{otherwise}$$

eg. 2



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

$$a_k = \frac{1}{T}\int x(t)e^{-jk\omega_0 t} = \frac{1}{T}\int_{-T/2}^{T/2} e^{-jk\omega_0 t}\,dt = \frac{-1}{jk\omega_0 T}\left[e^{-j\omega k_0 T_1} - e^{jk\omega_0 T}\right]$$

$$a_k = \frac{2}{k\omega_0 T}\left\{\sin(k\omega_0 T_1)\right\} \Rightarrow \boxed{a_k = \frac{1}{k\pi}\sin(k\omega_0 T_1) \ ; \ k \neq 0}$$

Date
Page

if $T = 4T_1$ $\quad a_K = \dfrac{\sin\left(K\cdot\frac{2\pi}{T}\cdot\frac{T}{4}\right)}{K\pi} = \dfrac{\sin(K\pi/2)}{K\pi}$

$a_1 = a_{-1} = 1/\pi \quad a_3 = a_{-3} = 1/3\pi \quad a_5 = a_{-5} = 1/5\pi, \ldots$

convergence → $\int_T |x^2(t)| < \infty$

↳ Dirchlet conditions :- ① $\boxed{\int_T |x(t)|\,dt < \infty}$

$|a_K| \leq \frac{1}{T}\int |x(t)e^{-j\omega_0 tK}| = \frac{1}{T}\int |x(t)|\cdot dt \rightarrow a_K$ is finite

e.g. - $x(t) = \frac{1}{t}$ $\forall\ 0 < t \leq 1$ with a period $=1$ → (not integrable)

$\cos\left(\frac{2\pi}{t}\right)$ w.p ① ← ② Finite no. of maxima's & minima's in any single period

③ Finite no of discontinuities

PROPERTIES

$x(t) \xrightarrow{FS} a_K$ signal and its Fourier coefficients

1) linearity :- $x(t) \xrightarrow{FS} a_K$ $\quad$ $y(t) \xrightarrow{FS} b_K$ [same period T]

any L.C. is also periodic with period T

$z(t) = Ax(t) + By(t) \xrightarrow{FS} c_K = A\cdot a_K + B\cdot b_K$

2) Time Shifting :-

shift a periodic signal → period is retained (T)

$b_K\ (x(t-t_0)) \longrightarrow \frac{1}{T}\int_T x(t-t_0)e^{-jk\omega_0 t}dt$

$= \frac{1}{T}\int x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau = e^{-jk\omega_0 t_0}a_K$

$x(t) \xrightarrow{F.S.} a_K$ $\quad \boxed{x(t-t_0) \xrightarrow{F.S.} e^{-jk\omega_0 t_0}a_K}$

magnitude is unaltered, i.e. $|b_K| = |e^{-jk\omega_0 t_0}.a_K| = |a_K|$

3) Time reversal :- $y(t) = x(-t)$

$x(t) = \sum a_K e^{-jk\omega_0 T}$ $\quad$ $K = -m$

$x(-t) = \sum a_{-m}e^{jm\omega_0 T}$ $\quad a_{-m} = a_K$ i.e.

$\boxed{x(t) \xleftrightarrow{F.S.} a_K}$ Then $\boxed{x(-t) \longrightarrow a_{-K}}$

(Time) reversal of the corresponding co-efficients

→ if $x(t)$ is even, its fourier coeff. are also even

→ if $x(t)$ is odd, its F.C. are also odd.

Date
Page

④ Time scaling :- Changes the period of the signal.
F. coeff. don't change but the "series" changes
$$x(\alpha t) = \sum a_k e^{jk(\alpha \omega_0)t}$$

⑤ Multiplication :- $x(t), y(t)$ are periodic with time period T.
$$x(t)\,y(t) \xleftrightarrow{F.S.} h_k = \sum_{l=-\infty}^{+\infty} a_l b_{k-l}$$
convol^n of sequence representing $x(t)$ and $y(t)$

multiply Fourier series of $x(t)$ and $y(t)$ → k^{th} harmonic in the product ⇒ coeff. in sum of terms as $a_l b_{k-l}$

⑥ Conjugate & Conjugate symmetry :-
$$x(t) \longleftrightarrow a_k \qquad x^*(t) \xrightarrow{F.S.} a_{-k}^*$$
real $x(t) \Rightarrow a_k = a_{-k}^*$ (conjugate symmetric)
if $x(t)$ is real ⇒ 1) a_0 is real
⇒ $|a_k| = |a_{-k}|$

if $x(t)$ is real & even ⇒ $a_k = a_{-k}$ ⇒ $a_k = a_k^*$ ⇒ a_k is real
↳ F.coeff. are real
$x(t)$ is real & odd → F. coeff. are purely imaginary & odd.

⑦ Parseval's relation :-
$$\boxed{\frac{1}{T}\int |x(t)|^2 \cdot dt = \sum_{-\infty}^{+\infty} |a_k|^2}$$

⇒ Fourier series for sine & cosine
as $a_k = a_{-k}^*$ → $x(t) = a_0 + \sum_{1}^{\infty} a_k \cos(k\omega_0 t + \phi_k)$

$\cos\left(\frac{2\pi}{T} kt\right)$ and $\sin\left(\frac{2\pi}{T} kt\right)$ → form a basis

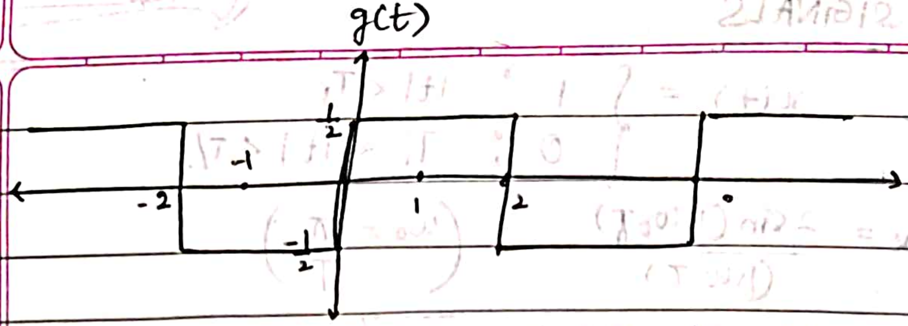
$$\langle x_1, x_2 \rangle = \int x_1 \overline{x_2}$$

$$p(t) = \sum_{1}^{\infty} A_k \cos(\omega_0 t k + \phi_k) + A_0 \qquad A_0 = \frac{1}{T}\int_{(T)} p(t) \cdot dt$$

$$A_k \cos\left(\frac{2\pi}{T} kt + \phi_k\right) = \left\{\frac{2}{T}\int_{(T)} p(\lambda) \cos\left(\frac{2\pi}{T} k\lambda\right) d\lambda\right\} \cos\left(\frac{2\pi}{T} \cdot kt\right)$$
$$+ \left\{\frac{2}{T}\int_{(T)} p(u) \sin\left(\frac{2\pi}{T} ku\right) du\right\} \sin\left(\frac{2\pi}{T} \cdot kt\right)$$

$$p(t) = A_0 + \frac{2}{T}\sum_{1}^{\infty}\left(\int p(\lambda)\cos\left(\frac{2\pi}{T} k\lambda\right) d\lambda\right) \cdot \cos\left(\frac{2\pi}{T} \cdot kt\right) + \frac{2}{T}\sum_{1}^{\infty}\left(\int p(\lambda) \sin\frac{2\pi}{T}k\lambda \, d\lambda\right) \sin\ldots$$

e.g.



with $T_0 = 4 \quad T_1 = 1 \qquad x(t-1) - \frac{1}{2} = g(t)$

$g(t) = x(t-1) - \frac{1}{2}$

$x(t) \longleftrightarrow a_K \qquad x(t-1) \longleftrightarrow e^{-jK\cdot\frac{2\pi}{4}(1)} \cdot a_K = a_K e^{-jK\pi/2}$

For DC offset, $-1/2$ $\qquad C_K = \begin{cases} 0 & K \neq 0 \\ -1/2 & K = 0 \end{cases}$

so, $d_K = \begin{cases} a_K e^{-jK\pi/2} & K \neq 0 \\ a_0 - 1/2 & K = 0 \end{cases}$ $\qquad a_K = \frac{\sin(K\pi/2)}{(K\pi)}$

$a_0 = \frac{1}{2}$

$d_K = \begin{cases} \frac{\sin(K\pi/2)}{K\pi} e^{-jK\pi/2} & ; K \neq 0 \\ 1/2 & ; K = 0 \end{cases}$

⑧ **Differenciation :-** $\quad x(t) \longleftrightarrow a_K$

$\frac{dx(t)}{dt} \longrightarrow j\omega_0 K a_K = j \cdot \frac{2\pi}{T} K a_K$

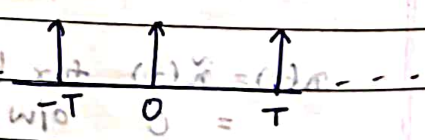
$x(t) = \sum a_K e^{+j\omega_0 K \cdot t} \qquad \frac{dx(t)}{dt} = \sum (j\omega_0 K a_K) e^{j\omega_0 K t}$

⑨ **Integration :-** $\int_{-\infty}^{t} x(t) \quad : \quad x(t) \longrightarrow a_K$

$\int_{-\infty}^{t} x(t) \rightarrow \left(\frac{1}{jK\omega_0}\right) a_K = \left(\frac{1}{jK(2\pi/T)}\right) a_K$

eg.- impulse-train ⇒ with period T

$x(t) = \sum_{K=-\infty}^{+\infty} \delta(t - KT)$



$a_K = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) \cdot e^{-jK\frac{2\pi}{T}t} \, dt = \frac{1}{T} \longrightarrow$ all fourier coeffⁿ are identical.

real $\qquad$ even(w.r.t. K)

For $s = j\omega$, $\quad H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega_0 t} dt \qquad e^{j\omega t} \longrightarrow \boxed{\delta} \longrightarrow H(j\omega) e^{j\omega t}$

For $x(t) = \sum_{-\infty}^{\infty} a_K e^{j\omega_0 t \cdot K} \longrightarrow y(t) = \sum_{-\infty}^{+\infty} a_K H(j \cdot K\omega_0) e^{jK\omega_0 t}$

Date ______
Page ______

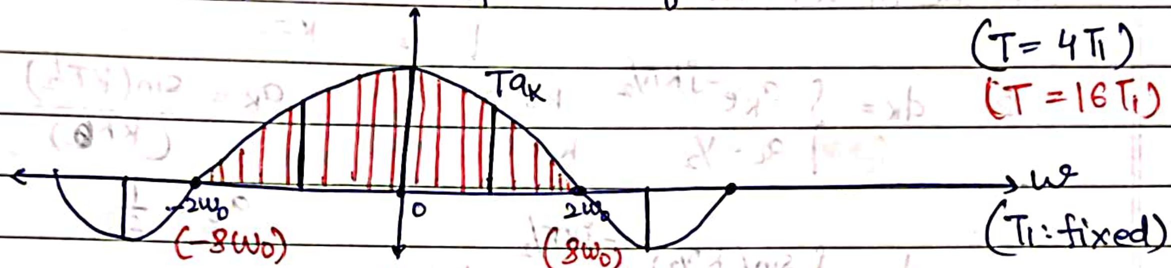
APERIODIC SIGNALS

For the signal :→ $x(t) = \begin{cases} 1 ; & |t| < T_1 \\ 0 ; & T_1 < |t| < T/2 \end{cases}$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{(k\omega_0 T)} \qquad \left(\omega_0 = \frac{2\pi}{T}\right)$$

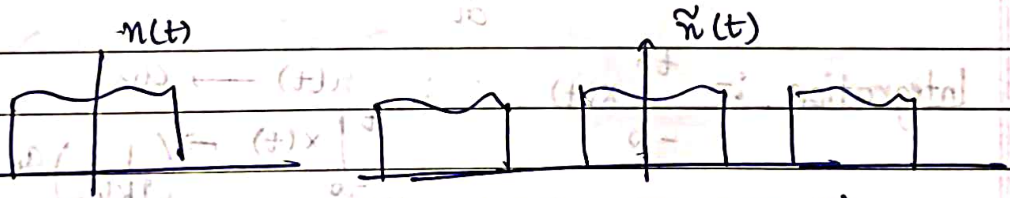
$$\boxed{Ta_k = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega = k\omega_0}}$$

→ with ω as a cont. variable, they are envelopes (samples) of Ta_k.
→ for fixed T_1, it is independent of 'T'.



→ as T increases (or $\omega\downarrow$) $[T = 2\pi/\omega_0]$ → envelope is sampled closer & closer.
→ as $T \Rightarrow \infty$, pulse → rectangular pulse : Fourier coeff. approach Ta_k as $T \to \infty$

i.e.
Let $x(t)$ be some finite (dur^n) signal ; i.e. $x(t) = 0 \ \forall \ |t| > t_1$
↳ we have a periodic signal $\tilde{x}(t)$ for which $x(t)$ is one period and then we choose its period $T \to \infty$



$$\tilde{x}(t) = \sum_{-\infty}^{+\infty} a_k e^{jk\omega_0 t} \ ; \quad a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t}\, dt$$

$x(t) = \tilde{x}(t)$ for $|t| < T/2$
$\quad = 0$ o/w

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

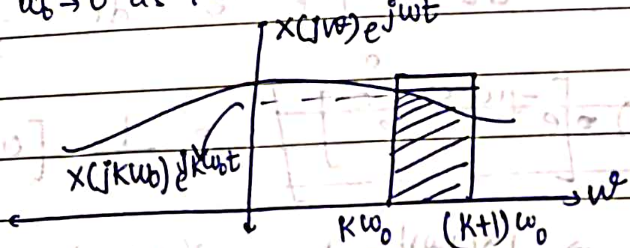
defining an envelope $X(j\omega)$ of Ta_k as $\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt}$

So, $\boxed{a_k = \frac{1}{T} X(jk\omega_0)}$ So, $\tilde{x}(t) = \sum \frac{1}{T} X(jk\omega_0) e^{+jk\omega_0 t}$

$$\tilde{x}(t) = \frac{1}{2\pi}\sum X(jk\omega_0) e^{jk\omega_0 t} (\omega_0)$$

as $T \to \infty$ $\tilde{x}(t) \to x(t)$ and the earlier eq^n becomes a rep^r of $x(t)$

$\omega_0 \to 0$, as $T \to \infty$ and so RHS $\longrightarrow$ integral (limit of sum)



as $\omega_0 \to 0$ (rect. $\longrightarrow$ pulse)

$\Sigma \to \int$

So, using $\tilde{x}(t) \to x(t)$ $T \to \infty$ $\omega_0 \to 0$

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega} \qquad \boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt}$$

(I.F.T. eq^n) (F.T. of $x(t)$)

$\Rightarrow$ For aperiodic signals, complex exponentials occur at a continuum of frequencies, and, have an amplitude $\left(X(j\omega)\,\frac{d\omega}{2\pi}\right)$

F.S. coefficients $\longrightarrow$ $X(j\omega)$ is the spectrum of $x(t)$

(#) a_k (F.coeff.) of some $\tilde{x}(t)$ (periodic, period = T)

$x(t) = \tilde{x}(t)$ for $s \le t \le s+T$ (over one period) for some s
(finite dur^n)

$x(t) = 0$ o/w

(4.37) $$a_k = \frac{1}{T}\int_s^{s+T} \tilde{x}(t)\, e^{-j\omega_0 t k} dt = \frac{1}{T}\int_s^{s+T} x(t)\, e^{-j\omega_0 t k} dt$$

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega_0 t k} dt \Rightarrow \boxed{a_k = \frac{1}{T} X(j\omega)\Big|_{\omega = k\omega_0}}$$

F. coeff. $\longrightarrow$ expressed in terms of equally-spaced samples of the Fourier Transform of one period of $\tilde{x}(t)$.

if $\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

$\hat{x}(t) = x(t)$ except for any 't' which is a discontinuity, where $\hat{x}$ is the average on either side

Dirichlet conditions :$\to$ 1) $x(t)$ is absolutely integrable :$\to$ $\int_{-\infty}^{+\infty} |x(t)| < \infty$

2) Finite no. of maxima & minima in any interval

3) Finite no. of discontinuities in any finite interval, each of these must also be finite

Date ______
Page ______

eg.- $x(t) = e^{-at}.u(t)$; $a>0$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt = \frac{-1}{a+j\omega}\left[e^{-(a+j\omega)t}\right]_0^{\infty}$$

$$= \cancel{\frac{-1}{a+j\omega}\left[(0-1)\cdot\left[e^{-j\omega(\infty)} - e^{0}\right]\right]} = \frac{-1}{a+j\omega}[0-1]$$

$$\boxed{X(j\omega) = \frac{1}{a+j\omega}} \Rightarrow |X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \quad \measuredangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

→ if a is complex, $x(t)$ is absolutely integrable if $Re(a) > 0$

(2) $x(t) = e^{-a|t|}$

$$X(j\omega) = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \boxed{\frac{2a}{a^2+\omega^2}}$$

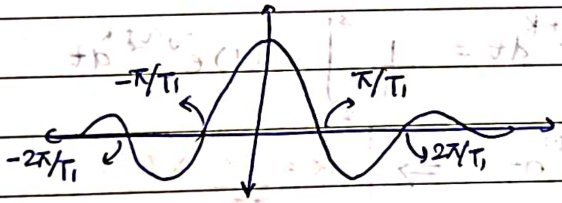
(3) $x(t) = \delta(t)$ (unit impulse)

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t}.dt = 1$$

(unit impulse has a F.T. consisting of equal contribⁿ at all frequencies)

(4) $x(t) = \begin{cases} 1 & ; |t| < T_1 \\ 0 & ; |t| > T_1 \end{cases}$ (rectangular pulse)

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\cdot dt = \frac{+1}{j\omega}\left[e^{j\omega T_1} - e^{-j\omega T_1}\right] = 2\left(\frac{\sin \omega T_1}{\omega}\right)$$



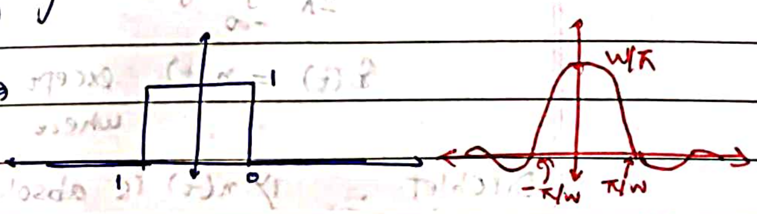
$\hat{x}(t) = x(t)$ ex. at the points of discontinuity at $(\pm T_1)$ where

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin\omega T_1}{\omega}\right) e^{j\omega t}\cdot d\omega$$

$\hat{x}(t) \to \frac{1}{2}$ [avg. on both sides of discontinuity]

$$= \lim_{W\to\infty} \frac{1}{2\pi}\int_{-W}^{W} 2\left(\frac{\sin\omega T_1}{\omega}\right) e^{j\omega t} d\omega$$

eg.(1) $X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$ ⇒

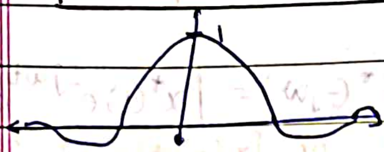


$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} 1\cdot e^{+j\omega t} d\omega$$

$$= \frac{1}{2\pi j t}\left[e^{jWt} - e^{-jWt}\right] = \frac{\sin Wt}{\pi t} = \frac{\sin Wt}{\pi t}$$

$$\boxed{\operatorname{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}}$$

$$\frac{2\sin \omega T_1}{\omega} = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)$$



$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wt}{\pi}\right)$$

as $W\uparrow$ in $\frac{\sin Wt}{\pi t}$ → $X(jW)$ becomes broader & peak of $x(t)$ becomes higher and width of I^{st} peak becomes narrower.

in limit as $W \to \infty$ $X(j\omega) = 1$ $\forall\ \omega$, and $x(t) \to$ impulse (δ)

inverse relⁿ b/w time and frequency domains.

FOURIER TRANSFORM OF PERIODIC SIGNALS

↳ train of impulses in freq. domain with areas ∝ F.S. coeff.

eg. $X(j\omega) = 2\pi\,\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi}\int 2\pi\,\delta(\omega-\omega_0)\,e^{+j\omega_0 t}\,d\omega = \boxed{e^{j\omega_0 t}}$$

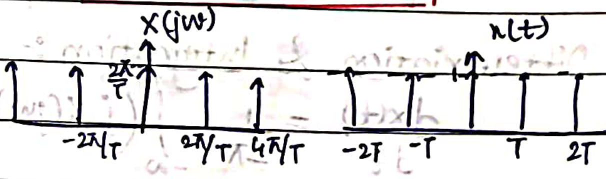
$$\boxed{\text{so, if } X(j\omega) = \sum_{K=-\infty}^{+\infty} 2\pi a_K\,\delta(\omega - K\omega_0) \Rightarrow x(t) = \sum_{-\infty}^{+\infty} a_K e^{jK\omega_0 t}}$$

e.g.- $X(j\omega) = \frac{2\pi}{T}\sum_{K=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi K}{T}\right) \Rightarrow x(t) = \frac{1}{2\pi}\int \frac{2\pi}{T}\,\delta\left(\omega - \frac{2\pi K}{T}\right) e^{j\omega t}\,d\omega$

$$\boxed{x(t) = \sum_{K=-\infty}^{+\infty} \delta(t - KT)}$$

periodic impulse train in time domain $(P = T)$ $\xrightarrow{F.T.}$ periodic impulse train in freq. domain ~~with~~ $(P = 2\pi/T)$

(inverse relⁿ)



PROPERTIES

$X(j\omega) = \mathcal{F}\{x(t)\}$ and $x(t) = \mathcal{F}^{-1}\{X(j\omega)\}$ $\quad x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$

① Linearity :- $x(t) \longleftrightarrow X(j\omega)$ $\quad y(t) \longleftrightarrow Y(j\omega)$

$$a\,x(t) + b\,y(t) \xleftrightarrow{\mathcal{F}} aX(j\omega) + bY(j\omega)$$

② Time Shifting :- $x(t - t_0) \longrightarrow \int_{-\infty}^{\infty} x(t-t_0)\,e^{-j\omega_0 t}\,dt$

$$x(t - t_0) \longrightarrow e^{-j\omega t_0} X(j\omega)$$

or $x(t) = \frac{1}{2\pi}\int X(j\omega)\,e^{j\omega t}\,d\omega$

$$x(t - t_0) = \frac{1}{2\pi}\int X(j\omega)\,e^{j\omega(t-t_0)}\,dt = e^{-j\omega t_0}\,\mathcal{F}(x(t))$$

Magnitude of F.T. → same
~~phase~~ → Shift by $-\omega t_0$

Date ______
Page ______

③ **Conjugation & conjugate symmetry :-**

$$x(t) \xrightarrow{F} X(j\omega) \Rightarrow \boxed{x^*(t) \xrightarrow{F} X^*(-j\omega)}$$

$$X(j\omega) = \int x(t) e^{-j\omega t} dt$$

$$X^*(j\omega) = \int x^*(t) e^{j\omega t} dt \quad \text{so,} \quad X^*(-j\omega) = \int x^*(t) e^{-j\omega t} dt = F(x^*(t))$$

if $x(t)$ is real $\Rightarrow \boxed{X(j\omega) = X^*(-j\omega)}$ or $\boxed{X(-j\omega) = X^*(j\omega)}$.

so, $Re\{X(j\omega)\} = Re\{X(-j\omega)\}$ → real part is even f^n^c of frequency.

and $Im\{X(j\omega)\} = -Im\{X(-j\omega)\}$ → im. part is an odd f^n^c of frequency.

$$X(j\omega) = |X(j\omega)| \cdot e^{j\angle X(j\omega)}$$

even f^n^c of ω ← $|X(j\omega)|$; $\angle X(j\omega)$ → odd f^n^c of ω

⇒ $x(t)$ is real and even, $X(-j\omega) = \int x(t) e^{j\omega t} \cdot dt \quad t \to -\tau$

if $x(t) = x(-t)$ $\quad = \int_{\infty}^{-\infty} x(-\tau) e^{-j\omega t} (-\tau) \cdot d\tau$

$\boxed{X(-j\omega) = X(j\omega)}$ $\quad \boxed{X(-j\omega) = F(x(t))}$

and $X(j\omega) = X^*(-j\omega) = X(-j\omega)$

⇒ $x(t)$ is real & even ⇒ $X(j\omega)$ is also real & even.

$x(t)$ is real & odd ⇒ $X(j\omega)$ is ~~also real~~ purely imaginary & odd.

so, (even)

$x(t) \xrightarrow{F} X(j\omega)$
$Ev\{x(t)\} \xrightarrow{F} Re\{X(j\omega)\}$
$Od\{x(t)\} \xrightarrow{F} j\,Im\{X(j\omega)\}$

eg. $e^{-a|t|} u(t) \rightarrow e^{-at} u(t) + e^{at} u(-t) = 2\{Ev(e^{-at} u(t))\}$

$$e^{-at} u(t) \rightarrow \frac{1}{a+j\omega}$$

$$2\,Ev\{e^{-at} \cdot u(t)\} \rightarrow 2 \cdot Re\left\{\frac{1}{a+j\omega}\right\} = \boxed{2a/(a^2+\omega^2)}$$

④ **Differenciation & Integration :-**

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (j X(j\omega) \cdot \omega) e^{j\omega t} d\omega$$

$$\frac{dx(t)}{dt} \xrightarrow{F} (j\omega) \cdot X(j\omega)$$

diff. in time domain → (multiply with $j\omega$ in freq. domain)

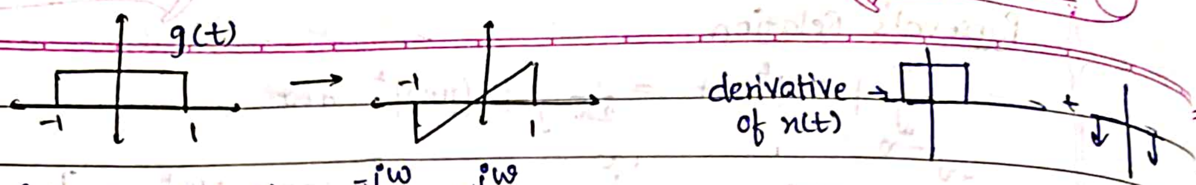
$$\int_{-\infty}^{t} x(\tau) d\tau \xrightarrow{F} \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega)$$

(DC/Average value)

eg. $u(t)$ $\qquad \delta(t) \rightarrow G(j\omega) = 1$

$$\int_{-\infty}^{t} \delta(t) \rightarrow \frac{1}{j\omega}(1) + \pi(1) \cdot \delta(\omega) = \boxed{\frac{1}{j\omega} + \pi\delta(\omega)}$$

eg. ②



$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{-j\omega} - e^{j\omega}$$

$1\cdot\delta(\omega+1) \rightarrow e^{j\omega}$ $\quad F(\delta) \Rightarrow e^{j\omega}$

(integr^n) $\qquad$ F.T.

as $G(0)=0 \longrightarrow X(j\omega) = \frac{G(j\omega)}{(j\omega)} \longrightarrow \boxed{\frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}}$

⑤ Time And Frequency Scaling :-

$x(t) \xleftrightarrow{F} X(j\omega) \Rightarrow \boxed{x(at) \xleftrightarrow{F} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)}$

$$F\{x(at)\} = \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau & ; a>0 \\ -\frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau & ; a<0 \end{cases}$$

For $a=-1 \Rightarrow \boxed{x(-t) \xrightarrow{F} X(-j\omega)}$

⑥ DUALITY :-

e.g.- $g(t) = \frac{2}{1+t^2}$ $\qquad x(t) = e^{-|t|} \xrightarrow{F} X(j\omega) = \frac{2}{1+\omega^2}$

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}d\omega \Rightarrow 2\pi e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}d\omega$$

replace$(t\to -t)$

So. $\qquad 2\pi e^{-|\omega|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-j\omega t}dt$ $\longleftarrow$ interchange names of t, ω

$$F\left\{\frac{2}{1+t^2}\right\} = 2\pi e^{-|\omega|}$$

So, $\boxed{\begin{array}{l} x(t) \xleftrightarrow{F} X(j\omega) \\ X(jt) \xleftrightarrow{F} 2\pi x(-\omega) \end{array}}$

$\int\left(\cancel{\int x(t)e^{-j\omega t}dt}\right)e^{j\omega t}d\omega$

$\cancel{\int X(jt)e^{j\omega t}d\omega = F(t)}$

$\cancel{2\pi F(-\omega)} = \cancel{\int X(-j\omega)e^{-j\omega t}}dt$ $\quad$ ~~replace names of t&w~~ $\cancel{\int X(-jt)e^{j\omega t}d\omega = F(-t)}\,2\pi$

~~again replace~~ $\cancel{2\pi x(-\omega)} = \int x(-jt)\,e^{-j\omega t}dt$

$\int X(jt)e^{-j\omega t}dt$

$\int X(j\omega)e^{-j\omega t}d\omega$ $\qquad f(t) = \int X(jt)e^{-j\omega t}d\omega \Rightarrow \frac{1}{2\pi}\int X(-jt)e^{j\omega t}d\omega = \frac{f(t)}{2\pi} = x(-t)$ $\quad t\to\omega$

$X(-j\omega)e^{j\omega t}d\omega$ $\quad 2\pi$

$2\pi x(-\omega)$ $\Rightarrow$ multiply by jt in time domain $\Rightarrow$ diff. in freq. domain

$$\boxed{-jt\,x(t) \xleftrightarrow{F} \frac{dX(j\omega)}{d\omega}} = \int_{-\infty}^{+\infty} -jt\,x(t)e^{-j\omega t}dt$$

$e^{j\omega_0 t}\,x(t) \xrightarrow{F} X(j(\omega-\omega_0))$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \xrightarrow{F} \int_{-\infty}^{\omega} X(\eta)d\eta$$

Parseval's Relation

$$\int_{-\infty}^{+\infty} |x(t)|^2 \cdot dt = \frac{1}{2\pi}\int |X(j\omega)|^2 \cdot d\omega$$

i.e. $$\int_{-\infty}^{+\infty} x(t)\,\overline{x(t)}\,dt = \int_{-\infty}^{+\infty} x(t)\cdot\frac{1}{2\pi}\left[\int_{-\infty}^{+\infty} \overline{X(+j\omega)e^{j\omega t}}\,d\omega\right]dt$$

$$= \int_{-\infty}^{+\infty} x(t)\cdot\frac{1}{2\pi}\left[\int_{-\infty}^{+\infty} \overline{X(j\omega)}\,e^{-j\omega t}\,d\omega\right]\cdot dt$$

reversing the order of integration

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{2\pi}\, x(t)\,\overline{X(j\omega)}\,e^{-j\omega t}\,d\omega\,dt$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi}\,\overline{X(j\omega)}\left[\int_{-\infty}^{+\infty} x(t)\,e^{-j\omega t}\,dt\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{X(j\omega)}\,X(j\omega)\cdot d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2\cdot d\omega$$

(4.45)

energy per unit time

$$\int_{-\infty}^{+\infty} |x(t)|^2\cdot dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2\,d\omega$$

Total energy

energy per unit frequency ⇒ energy - density spectrum

$X(j\omega) = \int x(t)e^{-j\omega t}dt$

$x(t) = \frac{1}{2\pi}\int X(j\omega)e^{j\omega t}d\omega$

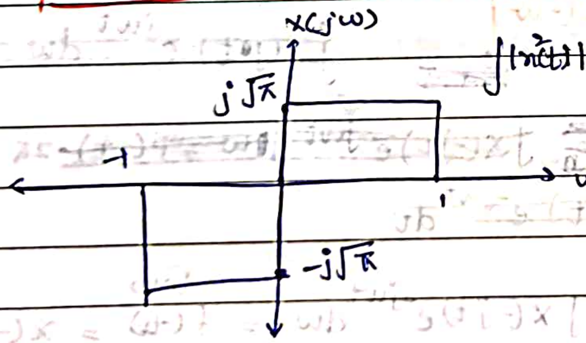
$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int X(j\omega)e^{j\omega t}\cdot d\omega\, j\omega$

$$x(t) \xrightarrow{\mathcal{F}} X(j\omega) \xrightarrow{\mathcal{F}^{-1}} x(t)$$

$$\frac{dx(t)}{dt} \xrightarrow{\mathcal{F}} j\omega X(j\omega) \xrightarrow{\mathcal{F}^{-1}} \frac{dx(t)}{dt}$$

$$\boxed{\left.\frac{dx(t)}{dt}\right|_{t=0} = \frac{1}{2\pi}\int X(j\omega)\cdot j\omega\, d\omega}$$

eg.-



$$\int |x(t)|^2 dt = \frac{1}{2\pi}\int |X(j\omega)|^2\cdot d\omega = \frac{1}{2\pi}\int_{-1}^{1} |j\sqrt{\pi}|^2\cdot d\omega$$

$$= \frac{\pi[2]}{2\pi} = \frac{2\pi}{2\pi} = 1$$

$$\left.\frac{dx(t)}{dt}\right|_{t=0} \cancel{= j\sqrt{\pi}(1-0) + (-j\sqrt{\pi})(-1)} \cancel{= 2j\sqrt{\pi}}$$

$$\left.\frac{dx(t)}{dt}\right|_{t=0} = \frac{1}{2\pi}\int_{-1}^{0}(-j\sqrt{\pi})\,j\omega\,d\omega + \frac{1}{2\pi}\int_{0}^{1}(j\sqrt{\pi})(j\omega)\,d\omega$$

$$= \frac{1}{2\pi}\left[\sqrt{\pi}\left[\frac{1}{2}(0-1)\right] + (-\sqrt{\pi})\left[\left(\frac{1}{2}\right)\right]\right] = \frac{-1}{2\sqrt{\pi}}$$

Date
Page

CONVOLUTION PROPERTY

complex exponentials are eigenfunctions of LTI systems.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega = \lim_{\omega_0\to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)\,e^{jk\omega_0 t}\cdot\omega_0$$

$$e^{jk\omega_0 t} \longrightarrow \boxed{\underset{\text{system}}{\overset{h(t)}{\text{LSI}}}} \longrightarrow H(jk\omega_0)\cdot e^{jk\omega_0} \qquad H(jk\omega_0) = \int_{-\infty}^{+\infty} h(t)e^{-jk\omega_0 t}\,dt$$

$\boxed{H(j\omega) = \mathcal{F}(h(t))}$ and so,
↳ frequency response

$$\tfrac{1}{2\pi}\sum X(jk\omega_0)e^{jk\omega_0 t}\cdot\omega_0 \longrightarrow \delta \longrightarrow \tfrac{1}{2\pi}\sum X(jk\omega_0)H(jk\omega_0)e^{jk\omega_0 t}\cdot\omega_0$$

so $$y(t) = \lim_{\omega_0\to 0}\left(\frac{1}{2\pi}\right)\sum_{k=-\infty}^{+\infty} X(jk\omega_0)\,H(jk\omega_0)\,e^{jk\omega_0 t}\cdot\omega_0 = \frac{1}{2\pi}\int X(j\omega)\,H(j\omega)e^{j\omega t}\,d\omega$$

so, $\boxed{Y(j\omega) = X(j\omega)\cdot H(j\omega)}$

$\Rightarrow$ $$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$\int_{-\infty}^{+\infty} y(t)\,e^{-j\omega t}\,dt = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau\right)e^{-j\omega t}\,dt$$ A.45

$$= \int_{-\infty}^{+\infty} x(\tau)\left(\int_{-\infty}^{+\infty} h(t-\tau)\,e^{-j\omega(t-\tau)}\,dt\right)e^{-j\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{+\infty}\left(x(\tau)e^{-j\omega\tau}\,d\tau\right)\int_{-\infty}^{+\infty} h(t)\,e^{-j\omega t}\,dt$$

$$\boxed{Y(j\omega) = X(j\omega)\cdot H(j\omega)}$$

$$h(t) * x(t) \xrightarrow{\ \mathcal{F}\ } H(j\omega)\cdot X(j\omega)$$
↳ completely characterizes a LTI system

$$x(t) \longrightarrow \boxed{H_1(j\omega)} \xrightarrow{x(t)*h_1(t)} \boxed{H_2(j\omega)} \xrightarrow{x(t)*h_1(t)*h_2(t)} y(t) \Rightarrow x(t) \longrightarrow \boxed{H_1(j\omega)\cdot H_2(j\omega)} \longrightarrow y(t)$$ eg.-

doesn't depend on order

$\Rightarrow$ LTI system is BIBO stable $\Rightarrow \int_{-\infty}^{+\infty} |h(t)| < \infty$ → absolutely integrable

① $$\mathcal{F}\{x(t) = C\} = C\int_{-\infty}^{+\infty} e^{-j\cdot 2\pi f t}\,dt \to 0 \text{ ex. when } n = 0$$

$$F[A] = \lim_{\gamma\to\infty} F\left[A\cdot \mathrm{rect}(t/\gamma)\right] = A\lim_{\gamma\to\infty} \gamma\,\mathrm{sinc}(\omega\gamma/2)$$

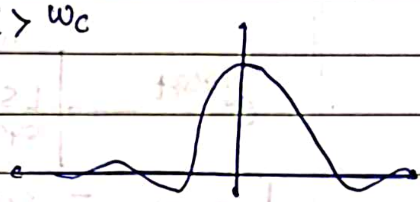
$$= A\lim_{\gamma\to\infty} \gamma\,\frac{\sin(\omega\gamma/2)}{(\omega\gamma/2)}$$

$F[A] \longrightarrow 2\pi A\,\delta(\omega)$

$F[e^{j\omega_0 t}] \longrightarrow 2\pi\,\delta(\omega+\omega_0)$

Frequency filter $\Rightarrow$ $H(j\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$

$\mathcal{F}^{-1}\{H(j\omega)\} = h(t) = \dfrac{\sin \omega_c t}{\pi t}$



i) $h(t)$ is NOT $=0$ for $t<0$ $\rightarrow$ NOT causal

e.g.- $h(t) = e^{-t}u(t)$
$H(j\omega) = \dfrac{1}{1+j\omega}$ } RC-Circuit

i) causal
ii) realizable

e.g. $h(t) = e^{-at}u(t)$ $x(t) = e^{-bt}u(t) \Rightarrow H(j\omega) = \dfrac{1}{a+j\omega}$ $X(j\omega) = \dfrac{1}{b+j\omega}$

$Y(j\omega) = \dfrac{1}{(a+j\omega)(b+j\omega)} = \dfrac{A}{a+j\omega} + \dfrac{B}{b+j\omega}$

$Y(j\omega) = \dfrac{1}{b-a}\left[\dfrac{1}{a+j\omega} - \dfrac{1}{b+j\omega}\right]$ so, $y(t) = \dfrac{1}{b-a}\left[e^{-at} - e^{-bt}\right]u(t)$

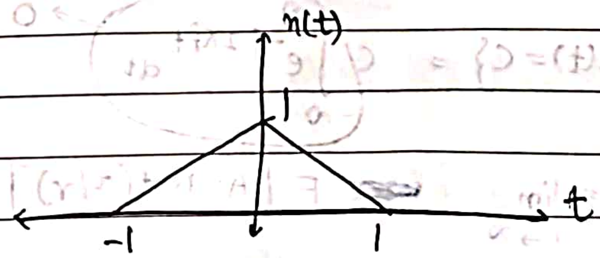
MULTIPLICATION PROPERTY

$s(t)p(t) \longrightarrow R(j\omega) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} S(j\theta)\,P(j(\omega-\theta))\,d\theta$

Some Properties

1. $Re\{X(j\omega)\} = Re\{X(-j\omega)\} = 0 \Rightarrow x(t) = -x(-t) \Rightarrow$ ODD & real
2. $Im\{X(j\omega)\} = -Im\{X(-j\omega)\} = 0 \Rightarrow x(t) = x(-t) \Rightarrow$ EVEN & real
3. $\exists$ a real α s.T. $e^{j\alpha\omega}X(j\omega)$ is real $\Rightarrow x(t+\alpha)$ is ~~odd~~ real & even
4. $\displaystyle\int_{-\infty}^{\infty} X(j\omega)\,d\omega = 0 \longrightarrow x(0) = 0$ 5. $\displaystyle\int X(j\omega)\cdot\omega\,d\omega = 0 \Rightarrow \left.\dfrac{dx}{dt}\right|_{t=0} = 0$
6. $X(j\omega)$ is periodic $\longrightarrow x(t)$ is periodic

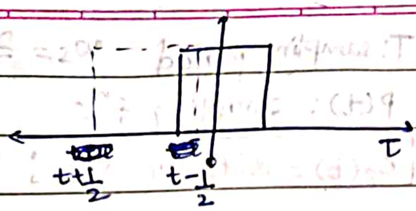
e.g. Find the fourier transform of



Ans) it is the convolution as $x_1(t) * x_1(t)$ where $x_1(t) = \begin{cases} 1 & |\omega| < \frac{1}{2} \\ 0 & \end{cases}$

$$\int_{-\infty}^{+\infty} x_1(\tau)\, x_1(t-\tau)\, d\tau$$

$|t-\tau| < \frac{1}{2}$



$-\frac{1}{2} < \tau < \frac{1}{2}$

$-\frac{1}{2} < t-\tau < \frac{1}{2}$

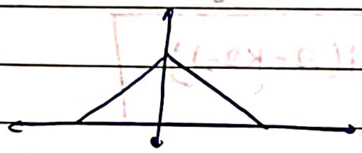
$\boxed{t-\frac{1}{2} < \tau < t+\frac{1}{2}}$ $\quad -\frac{1}{2} < \tau - t < \frac{1}{2}$

$\cancel{\int x_1(t-\tau)\, d\tau}$

for $t+\frac{1}{2} = -\frac{1}{2} \Rightarrow t=-1$ to $t-\frac{1}{2} = -\frac{1}{2} \Rightarrow t=0$

$$\int_{-\frac{1}{2}}^{t+\frac{1}{2}} 1 \cdot dt = \left(t+\frac{1}{2}\right) - \left(-\frac{1}{2}\right) = t+1 \quad \forall \; -1<t<0$$

and $$\int_{t-\frac{1}{2}}^{\frac{1}{2}} 1 \cdot dt = \frac{1}{2} - \left(t-\frac{1}{2}\right) = -t+1 \quad \forall \; 0<t<1$$



so, $F.T.(x(t)) = F.T.(x_1(t) * x_1(t)) = (X_1(j\omega))^2$

$$= \left(\frac{2\sin(\omega/2)}{\omega}\right)^2$$

\# $x(t) \xleftrightarrow{F.S.} a_k$

$x(t-t_0) \longleftrightarrow a_k e^{-j\omega_0 t_0 k}$ $\quad x(t+t_0) \longleftrightarrow a_k e^{j\omega_0 t_0 k}$

1) $x(t-t_0) + x(t+t_0) \longleftrightarrow a_k \cdot \{2\cos(\omega_0 t_0 \cdot k)\}$

2) $E_v(t) = \frac{x(t)+x(-t)}{2} \longleftrightarrow \left(\frac{a_k + a_{-k}}{2}\right)$

3) $Re\{x(t)\} = \frac{x(t) + \overline{x(t)}}{2} \longleftrightarrow \left(\frac{a_k + a_{-k}^*}{2}\right)$

4) $x(3t-1) \longleftrightarrow \sum a_k e^{jk\omega_0(3t-1)} = \sum a_k e^{-jk\omega_0} \cdot e^{3jk\omega_0 t}$ (ω')

(period $= T/3$) $= \boxed{a_k \cdot e^{-jk\omega_0}}$

\# $x(t) \longrightarrow X(j\omega)$, $x(-t) \longrightarrow X(-j\omega)$

$u(t) = \frac{1}{2}[1 + sgn(t)]$ $\quad sgn(t) = \lim_{a\to 0}\left[e^{-at}u(t) - e^{+at}u(-t)\right]$

$$F(sgn(t)) = \frac{2}{j\omega}$$