EE-229

SIGNAL PROCESSING - I

"Numbers don't have an existence of their own"

"If they shake hands, their numbers are multiplied"

"combined effect of all handshakes leads to new passengers"

"Quantization of 'δ' is in terms of the area it encapsulates."

"Those who look Inwards, can progress in a steady way."

"Do you like solving D.E. with me? I'll slap you".

"reasonably insane but sanely unreasonable."

"You can keep on having smoother and smoother signals"

"I had a niece & I pretended I didn't know the table of 13"

"being slow to learn isn't but being unwilling to learn is a disability"

"Is it easier to domesticate a lion or tiger? someone has tried?"

"You solve differential equations using your experience."

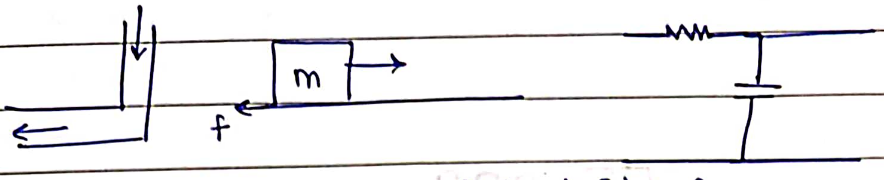
"Suppose you want to release the tiger, unleash him."

"move from शासन to अनुशासन, only then you are a successful IIT-ian"

"infinity is again an infinity itself, there are infinite ways to ∞"

⇒ Abstraction

⇒ expanding a context to several similar contexts.



(First-order dynamics)

Input signal ⟶ Output Signal

in control of experimenter.

(dynamics to relate them)

⟶ experimenter can observe as a consequence of an experiment.

abstraction of Order ⟶ expand it to several different contexts.

↓

common, important/relevant and helps build a 'common framework.'

↳ break the boundaries of context.

→ Signal :- physically relevant or meaningful functions of an independent variable.

⇒ it is a mapping from an independent variable to a dependent variable.

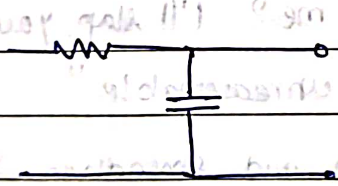
⇒ 'reasonable' functions of independent variable (e.g. 'unreasonable' $x(t) = \begin{cases} 1 \\ 0 \end{cases}$

→ system :- it is a mapping from signals to signals. i.e.

input signals $\xrightarrow{\text{map}}$ output signals.

(system is a mapping upon mappings.)

e.g.-



⇒ system description : simple/complex way to describe a system.

↳ explicit/implicit relⁿ b/w inputs and outputs.

Input signal : $x(t)$

↓

fⁿc of the independent variable

$x(t)$ → variable ; dependent variable

signal

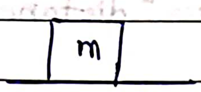
↓

independent variable & the depⁿ var. as a fⁿc of ind. var.

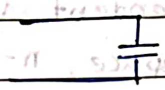
$x(t) \equiv$ System Description

examples:-

⟨1⟩

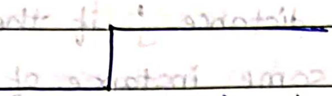


$F(t) = m\left(\frac{dv(t)}{dt}\right)$



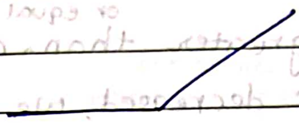
$i(t) = C\,\frac{dV_c(t)}{dt}$

Form : $x(t) = r\,\frac{dy(t)}{dt}$



unit step input

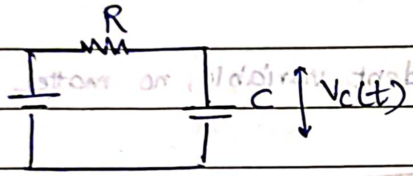
⟹



'ramp'

$\frac{dy(t)}{dt} = \frac{1}{r}$

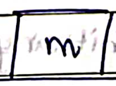
⟨2⟩ (composite systems)



$i(t) = C\,\frac{dV_c(t)}{dt}$

$V_c(t) + RC\,\frac{dV_c(t)}{dt} = V_{in}(t)$

as $V_c(0^+) = 0$ and $t \to \infty$ $V_c \to V_0$



friction $= K_0 V(t)$

$F(t) = m\,\frac{dV(t)}{dt} + K\,V(t)$

(same form) similarly,

$V(0^+) \to 0$

and as $t \to \infty$ $\boxed{V \to F(t)/K}$

⟹ Signal:- mapping from the sets ⟹

(real) $\mathbb{R} \longrightarrow \mathbb{C}$ (complex)

e.g. - $y(t) = 5\,\frac{dx(t)}{dt}$ (explicit)

$y(t)$ ↳ output signal

$x(t)$ ↳ input signal

$2\frac{d^2y(t)}{dt} + 3 + 6\frac{dx(t)}{dt} + 5x(t) = 0$ ⟨implicit⟩

$\frac{dy(t)}{dt} + 7 + 5x(t)y(t) = 0$ (implicit & inseperable)

e.g.- voltage signal ⟹ $V(t)$; dep. ; ↳ independent variable

; $E(x)$ ↳ space

; $E(t)$ (field intensity as a f^nc of time)

$E(x,t)$ ↳ hybrid independent variables (space + time)

$E(x,y)$ ↳ homogenous independent variables (time, space)

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Classification based on Nature

must satisfy Δ inequality

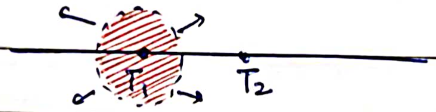
1) **Continuous :-** independent variables have a "distance" measure

e.g. – time, 1-D space, n-D space

→ we may/may not associate the notion of "ORDER".

time → earlier/before
1-D → farther/backward
} but, complex no.'s → NOT ordered.

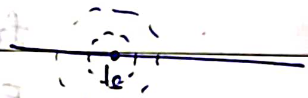
⇒ if we take all other independent variables which are at a distance greater than or equal to a particular distance ; if the particular distance is decreased, we still have some instance of independent variable which is less than that particular distance.



(Homogenous 2-D space)

continuity → arbitrary close independent variable, no matter how close w.g.

e.g. – continuous time



(complex no.'s also have continuity) → distance ✓ → continuity can exist.
→ order ✗

a time instance always exists inside the circle.

2) **Discrete :-** NOT continuous.

e.g. $t = \{1s, 2s, 3s, \ldots\}$

it can become smaller and smaller, but NOT zero

e.g. – instances when classes are conducted – discrete.

⇒ both dependent & independent variables can be discrete or continuous.

e.g. – a pressure-measurement

Dep. \ Indep.	CON. (t)	DIS. → (n)
CON.	✓ (A)	✓ (B)
DISC.		

(uniformly spaced)

(floor/ceil f^n^c)

(4 types of signals.)

SYSTEM

e.g. – Resistor : $v(t) = R\,i(t)$ or $i(t) = \frac{1}{R} v(t)$

mapping from signals to signals
(input) (output)

$i(t) = C\left(\frac{dV_c(t)}{dt}\right)$ → system description

e.g. – $y(t) = r\,x(t)$
output signal, input signal

(·) → continuous
[·] → discrete

$y[n] = R \cdot x[n]$
(explicit)

eg.- Implicit :- $x(t)y(t) + \frac{dx(t)}{dt}\cdot\frac{dy(t)}{dt} + 3 = 0$ → can't be converted to explicit easily.

Input - Output : C-C, D-D

System :-

Inp\Out	A	B
A	cont. systems variable ✓	
B		discrete variable systems ✓

System Property

⇒ The prescribed outcome of one or more prescribed experiments of the system.

<1> Additivity

Prescribed experiment :-

(t) independent variable
↳ assume it has an order.

$x_{1,2}(t) \rightarrow \boxed{\mathcal{S}} \rightarrow y_{1,2}(t)$ (multi-statement)
system

$x_1(t)+x_2(t) \rightarrow \boxed{\mathcal{S}} \rightarrow \boxed{y_1(t)+y_2(t)}$ (prescribed outcome)

$\forall\ x_1, x_2(t)$

→ Informally :- if we look at two outputs corresponding to x_1 and x_2, we get y_1 and y_2, irr. of x_1, x_2. [FOR ALL COMBINATIONS x_1, x_2]

⇒ One counter example is enough to prove the system is NOT additive.

eg. ① $y(t) = \text{Re}\{x(t)\}$ (inputs can be complex f^nc of the independent variable)

$\text{Re}\{x_1(t)\} = y_1(t)$; $\text{Re}\{x_2(t)\} = y_2(t)$ and

$\text{Re}\{x_1(t)+x_2(t)\} = y_1(t)+y_2(t)$ ⇒ Additive $\boxed{✓}$

② $y(t) = |x_1(t)|$

$|x_1(t)+x_2(t)| \neq |x_1(t)| + |x_2(t)|$ always

⇒ Additivity $\boxed{\times}$

Discrete systems :-

$x_{1,2}[n] \rightarrow \boxed{\mathcal{S}} \rightarrow y_{1,2}[n]$

⇓

$(x_1+x_2)[n] \rightarrow \boxed{\mathcal{S}} \rightarrow (y_1+y_2)[n]$ $\forall\ x_1, x_2$

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⟨2⟩ Homogenity / Scaling

$x(t) \rightarrow \boxed{S} \rightarrow y(t)$

$\Downarrow$

$c\,x(t) \rightarrow \boxed{S} \rightarrow c\,y(t)$ (prescribed outcome)

c is any COMPLEX constant, i.e. $c \in \mathbb{C}$

$\boxed{\forall x, \forall c}$

eg.- ① $y(t) = \mathrm{Re}\{x(t)\}$

"NOT" homogenous

$c\,x(t)$, if c is complex, $\mathrm{Re}\{c\,x(t)\} \neq c\,y(t)$

e.g.- $c = j$

② $y(t) = |x(t)|$

NOT homogenous $\quad y(t) = |c|\,|x(t)| \neq c\,y(t) \qquad |c| \neq c$

③ $y(t) = x(t-1)$

homogenous ; if input multiplied by c, $y(t) = c\,x(t-1) = c\,y(t)$

④ $y(t) = x(t) + 3x(t-1) \longrightarrow$ homogenous.

⑤ $y(t) = \begin{cases} \dfrac{x(t)\,x(t-1)}{x(t-2)} & ; x(t-2) \neq 0 \\ 0 & ; x(t-2) = 0 \end{cases}$ ($x(t)$ should be defined for all time 't')

c1⟩ $x(t-2) \neq 0$

$y(t) = \dfrac{c\,x(t) \cdot \cancel{c}\,x(t-1)}{\cancel{c} \cdot x(t-2)} \rightarrow c\,y(t) \qquad c \neq 0$

c2⟩ $x(t-2) = 0$

$y(t) = 0 = c \cdot 0 \rightarrow c\,y(t) \quad ; [c\,x(t-2) = 0]$

c3⟩ $c = 0$

$(c\,x(t-2) = 0 \Rightarrow y(t) = 0 = c \cdot 0 \rightarrow c\,y(t).)$

whole input is multiplied by '0' $\rightarrow$ output is identically zero

$0 \cdot x(t) \rightarrow \boxed{S} \longrightarrow 0 \cdot y(t) = 0.$

$x[n] \rightarrow \boxed{S} \longrightarrow y[n] \Rightarrow c\,x[n] \rightarrow \boxed{S} \rightarrow c\,y[n]$

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⟨3⟩ Linearity

Exp.1⟩ Is $\mathcal{S}$ additive ? Exp.2⟩ Is $\mathcal{S}$ homogenous ? (Prescribed outcome)

Yes ⟶ LINEAR ⟵ Yes

⟨4⟩ Shift-invariance (Time invariance)

continuous systems ⇒

$x(t) \longrightarrow \boxed{\mathcal{S}} \longrightarrow y(t)$ ⇒ $x(t-\tau) \longrightarrow \boxed{\mathcal{S}} \longrightarrow y(t-\tau)$

ANY real, constt. τ (+ve or −ve)   prescribed outcome $\boxed{\forall\ x, \tau}$

(Interpert^n :- system behavior does NOT change in time.)

eg. ① - $y(t) = x(t) + 3x(t-1)$

$x(t-\tau) + 3x(t-\tau-1) = y(t-\tau)$ ⟹ shift-invariant.

$x(t) \to y(t)$

$x(t-\tau) \to y(t-\tau)$ $\forall\ x, \tau$

② $y(t) = x(t) + 3x^2(t-1) \longrightarrow$ shift invariant

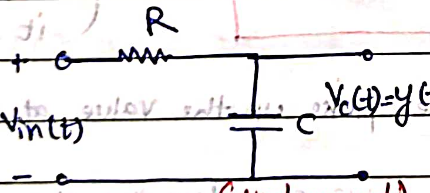
③ $y(t) = t\,x(t) \longrightarrow$ shift variant

④ $y(t) = |x(t)| \longrightarrow$ shift invariant

discrete systems ⇒ $x[n] \longrightarrow \boxed{\mathcal{S}} \longrightarrow y[n]$

$x[n-n_0] \longrightarrow \boxed{\mathcal{S}} \longrightarrow y[n-n_0]$; $n_0 \in \mathbb{Z}$ (integer shift)

$\forall x\ \forall n_0 \in \mathbb{Z}$



$x(t) = V_{in}(t)$ ⟶ $x_{1,2}(t) = y_{1,2}(t) + RC\dfrac{dy(t)}{dt}$

$\alpha x_1(t) + \beta x_2(t) = \alpha y_1(t) + RC\alpha\dfrac{dy_1(t)}{dt} + \beta y_2(t) + RC\beta\dfrac{dy_2(t)}{dt}$ $(\alpha, \beta \in \mathbb{C})$

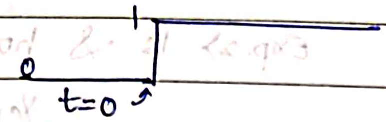
$= \alpha y_1(t) + \beta y_2(t) + RC\dfrac{d}{dt}\{\alpha y_1(t) + \beta y_2(t)\}$

⇒ as $\alpha y_1(t) + \beta y_2(t)$ satisfies the eq^n, hence the system is LINEAR.

⇒ if the capacitor has initially some voltage ⟶ SYSTEM IS NOT LINEAR (prove)

⇒ SHIFT-INVARIANT as $\dfrac{d(t-\tau)}{dt} = \dfrac{d(t-\tau)}{d(t-\tau)} \cdot \dfrac{d(t-\tau)}{dt} = (1)$

Unit step → input



$u(t)$

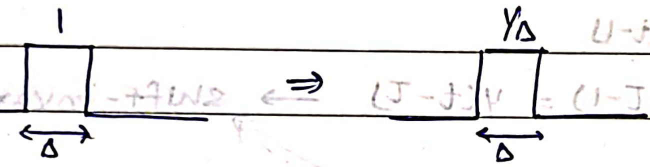
Unit step as input → $\boxed{\delta}$ → unit step response.

$u(t) \rightarrow \boxed{\begin{matrix}LSI\\ \delta\end{matrix}} \longrightarrow s(t)$ (unit step response)

linear, shift invariant

$u(t-\Delta) \rightarrow \boxed{\delta} \rightarrow s(t-\Delta)$ (shift invariant)

$(\Delta > 0)$ $u(t) - u(t-\Delta) \rightarrow \boxed{\delta} \rightarrow s(t) - s(t-\Delta)$ (linearity)



Unit impulse ← $\frac{1}{\Delta}\left(u(t)-u(t-\Delta)\right) \longrightarrow \boxed{\delta} \longrightarrow \frac{1}{\Delta}\left(s(t)-s(t-\Delta)\right)$

if Δ becomes smaller and smaller, height of step → ∞, width of step →
but the area = w×h remains ① const.

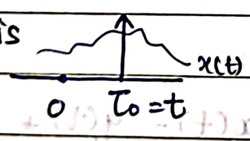
unit area starts to localize at a single point. (line)

$\delta(t) \Rightarrow$ unit impulse ; $\int_{-\infty}^{+\infty} \delta(t)\cdot dt = 1$

$x(t)\,\delta(t-T_0) \Rightarrow \boxed{x(T_0)\,\delta(t-T_0)}$ (SIFTING property)

(rest everywhere it is 0

assuming $x(t)$ is cont. at $t = T_0$

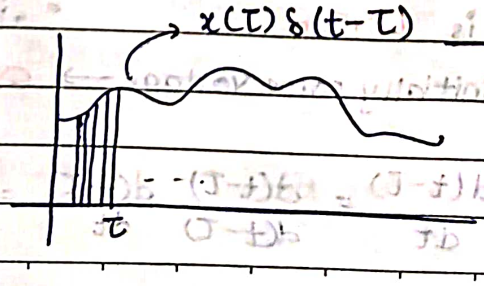


impulse picks up the value at the point

$\int_{-\infty}^{+\infty} x(T_0)\,\delta(t-T_0)\cdot dt = x(T_0)$ ⟶ we can describe the effect of an impulse under integral sign

↳ generalized f^n c.

→ Unit impulse's behavior can be understood only under an integrals



all such pulses stitched together forms the pulse back again.

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$x(\tau_1)\,\delta(t-\tau_1) \longrightarrow \boxed{\mathcal{S}} \longrightarrow x(\tau_1)\,h(t-\tau_1)$

$x(\tau_2)\,\delta(t-\tau_2) \longrightarrow \boxed{\mathcal{S}} \longrightarrow x(\tau_2)\,h(t-\tau_2)$

if the system is linear and S.I.

Stitch all together $\rightarrow \boxed{\mathcal{S}} \longrightarrow$ stitch all together.

→ if we have the impulse step response ⇒ we can solve for any $x(t)$.

Discrete linear shift invariant system

$x[n] \longrightarrow \underset{\text{L.S.I.}}{\boxed{\mathcal{S}}} \longrightarrow y[n]$

input, output are called 'SEQUENCES'

$$\delta[n] = \begin{cases} 1 & ; n=0 \\ 0 & ; \text{elsewhere} \end{cases}$$

→ unit impulse sequence / Kronecker-delta series.

$\delta[n] \longrightarrow \underset{\text{L.S.I.}}{\boxed{\mathcal{S}}} \longrightarrow h[n]$

Thm:- The unit impulse response of $\boxed{\mathcal{S}}$ completely characterizes $\mathcal{S}$ (L.S.I.).

→ Any input can be written as Kronecker delta.

Arrow notation:- $-1 \quad \underset{0}{\overset{3}{\uparrow}} \quad 2 \quad 4 \quad = x[n]$ ⇒ $-\delta[n+1] + 3\delta[n] + 2\delta[n-1] + 4\delta[n-2]$

(Finite length sequence)

$\left\{ \begin{array}{ll} \text{at } n=0 & x=3 \\ n=1 & x=2 \\ n=-1 & x=-1 \end{array} \right\}$

$$\delta[n-K] = \begin{cases} 1 & ; n=K \\ 0 & ; \text{elsewhere} \end{cases}$$

$\delta[n-K] \Downarrow \underset{K}{\overset{1}{\uparrow}}$

$\delta[n] \Rightarrow \underset{0}{\overset{1}{\uparrow}} \qquad \delta[n-1] \Rightarrow \underset{1}{\overset{1}{\uparrow}} \qquad \delta[n+1] \Rightarrow \underset{-1}{\overset{1}{\uparrow}}$

$x[n]$ is a L.C. of unit impulse and shift

$$\boxed{x[n] = \sum_{K\in\mathbb{Z}} x[K]\,\delta(n-K)}$$

(for a given K, $x[K]$ is a constant)

CONSTRUCTIVE Proof:- For any input sequence, $x[n] = \sum_{K\in\mathbb{Z}} x[K]\,\delta(n-K)$

For a fixed K, $\delta[n-K] \longrightarrow \boxed{\mathcal{S}} \longrightarrow h(n-K)$ [shift-invariant]

$x[K]\,\delta[n-K] \rightarrow \boxed{\mathcal{S}} \longrightarrow x[K]\,h(n-K)$ (homogenity)

$\sum_{K\in\mathbb{Z}} x[K]\,\delta[n-K] \rightarrow \boxed{\mathcal{S}} \rightarrow \sum_{K\in\mathbb{Z}} x[K]\,h(n-K) = y[n]$ [Additivity]

(output → input and impulse response)

$x[n]$ is a sequence ; $h[n]$ is also a sequence, so is $y[n]$.

sequences $\begin{cases} x[n] \\ h[n] \\ y[n] \end{cases}$ $\Rightarrow$ $y[n] = \underbrace{\sum_{k \in \mathbb{Z}} x[k]\, h[n-k]}$ $\forall\, n$

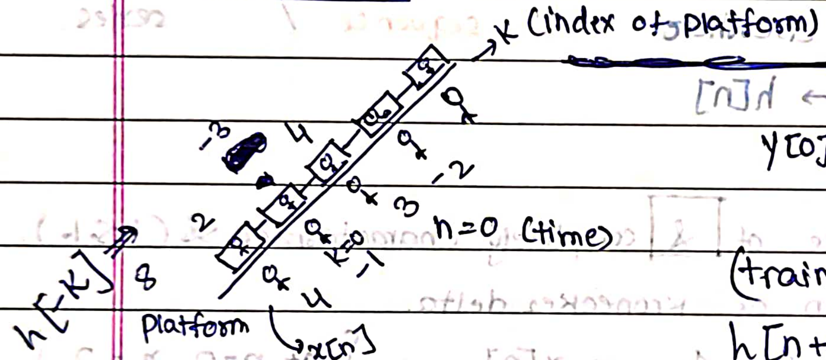
operation b/w two sequences $x[\cdot]$ and $h[\cdot]$

$\boxed{\text{CONVOLUTION}}$

eg- $x[n] = 4 \quad \underset{0}{\underset{\uparrow}{-1}} \quad 3 \quad -2$ $\qquad h[n] = \underset{-1}{\underset{\uparrow}{4}} \quad \underset{0}{\underset{\uparrow}{-3}} \quad 2 \quad 8$

$\Rightarrow$ Convolution of $x[\cdot]$ with $h[\cdot]$ $\Rightarrow y[n] = \sum_{k \in \mathbb{Z}} x[k]\, h[n-k]$

$\Rightarrow$ Platform-train



$y[0] = \sum_{k \in \mathbb{Z}} x[k]\, h[-k]$

(train moves step-by-step)

$h[n+k]$ $\longrightarrow$ move train by 'n' backwards.
(as the train moves)

$k+n=0 \Rightarrow (k=-n)$

$h[-k+n] \rightarrow$ flip the train w.r.t. the $(k=0)$ point after going backward by n.

$h[k]$ shifted backwards by 'n'. k is at $(-n)$ it takes value 0

$x[k] = 4 \quad \underset{0}{\underset{\uparrow}{-1}} \quad 3 \quad -2$

$h[-k] = 4 \quad -3 \quad 2 \quad 8 \Rightarrow h[n+k] = \underset{-n-1}{\underset{\uparrow}{4}} \quad \underset{-n}{\underset{\uparrow}{-3}} \quad \underset{-n+1}{\underset{\uparrow}{2}} \quad 8$

$h[n-k] = \underset{n-2}{\underset{\uparrow}{8}} \quad \underset{n-1}{\underset{\uparrow}{2}} \quad \underset{n}{\underset{\uparrow}{-3}} \quad \underset{n+1}{\underset{\uparrow}{4}}$ $\swarrow$ flip

$y[0] = \sum x[k] \cdot h[-k] = \begin{matrix} 4 & -1 & 3 & -2 \\ 8 & 2 & -3 & 4 \end{matrix}$

$y[0] = 23$

$y[1] \Rightarrow \begin{matrix} 4 & -1 & 3 & -2 \\ 8 & 2 & -3 & 4 \end{matrix} \Rightarrow 13$

$-8-9-2+32$

$y[2] \Rightarrow \begin{matrix} 4 & -1 & 3 & -2 \\ & 8 & 2 & -3 & 4 \end{matrix}$

$6+6-8 = 4$

Sampling property

$x[n] = \sum x[K] \cdot \delta[n-K]$ i.e. $n \to n-K$

$$\boxed{x[n] * \delta[n-C] = x[n-C]}$$

$= \sum x[K]\delta[n-C-K] = x[n-C]$

$y[3] \Rightarrow 4 \quad -1 \quad 3 \quad -2 \Rightarrow -4+24 \Rightarrow 20$

$\quad\quad\quad\quad\quad\quad\quad 8 \quad 2$

$y[4] \Rightarrow 4 \quad -1 \quad 3 \quad -2 \Rightarrow -16$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad 8$

$y[-1] \Rightarrow 4 \quad -1 \quad 3 \quad -2 \Rightarrow -16$

$\quad\quad -3, \; 4$

$y[-2] = 16$

16,	−16,	23,	13,	40,	20,	−16
↑	↑	↑	↑	↑	↑	↑
−2	−1	0	1	2	3	4

$$y[n] = \sum_{K \in \mathbb{Z}} x[K]\,h[n-K] \quad \forall n \Rightarrow \boxed{y = x * h}$$ convolution

→ Let x, h be finite length sequence

→ $x \neq 0$ for $N_1 \leq n \leq N_2$; $N_1, N_2 \in \mathbb{Z}$

→ $h \neq 0$ for $N_3 \leq n \leq N_4$; $N_3, N_4 \in \mathbb{Z}$

$K \in [N_1, N_2]$

$n-K \in [N_3, N_4] \Rightarrow n \in [N_3, N_4] + [N_1, N_2]$

so, $\boxed{y \neq 0 \text{ for } n \in [N_1+N_3, N_2+N_4]}$

eg.- x: $x \neq 0$ for $[0, \ldots, M-1]$ → M points $\Rightarrow y \neq 0$ for $[0, \ldots, M+L-2]$ → $M+L-1$ points

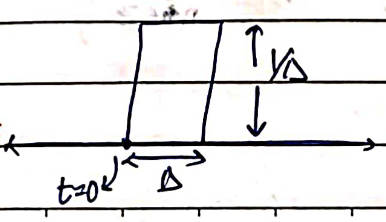
$h \neq 0$ for $[0, \ldots, L-1]$ → L points

CONTINUOUS-TIME SYSTEM

$x(t) \longrightarrow$ [$\mathcal{S}$ continuous time system] $\longrightarrow y(t)$

$\delta(t)$

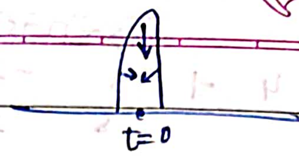
DIRAC DELTA



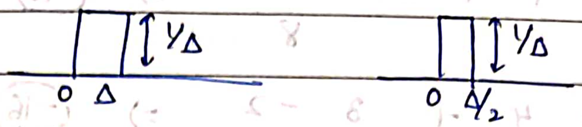
unit impulse compresses a non-zero area INTO a point

Properties Of Dirac-Delta

1) $\delta(t) = \delta(-t)$ (Time inversion plays no role)



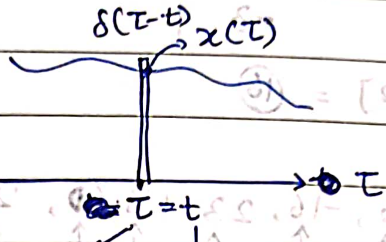
2) $\boxed{\delta(2t) = \frac{1}{2}\delta(t)}$



(encloses only half area)

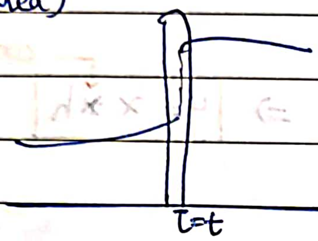
(all over the domain ex. a set of measure 0) ← (almost everywhere)

$$\boxed{x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau}$$



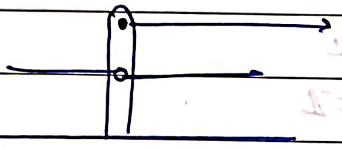
integration ↓ (bringing out the encapsulated area)

Assume x is CONTINUOUS AT $\tau = t$



If x is discontinuous, we get the average value of LHL & RHL.

⊛



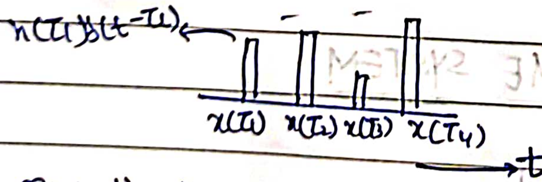
Blob uniformly takes a region around $\tau = t$, and it is converging FROM ALL THE SIDES (L & R) uniformly.

AVOID USING integral at a point of discontinuity.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(\tau - t)\cdot d\tau \longrightarrow \text{SIFTING Property}$$

(Stitch together an ∞ of impulses)

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \longrightarrow \text{generalization of a summation.}$$



⇒ Quantization of δ is in Terms of the area it encapsulates.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \xrightarrow{\text{discrete time}} x[n] = \sum_{n \in \mathbb{Z}} x[k]\,\delta[n-k]$$

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$\delta(t) \longrightarrow$ [cont. L.S.I. system] $\longrightarrow h(t)$ [unit impulse response]

$x(t) \longrightarrow \boxed{\mathcal{S}} \longrightarrow y(t)$

Thm: $\boxed{\mathcal{S}}$ is completely characterized by $h(t)$

$\delta(t) \xrightarrow{\mathcal{S}} h(t)$

$\delta(t-\tau) \xrightarrow{\mathcal{S}} h(t-\tau)$ (shift-invariance) (for $\tau \in \mathbb{R}$)

$x(\tau)\,\delta(t-\tau) \xrightarrow{\mathcal{S}} x(\tau)\,h(t-\tau)$ (homogeneity)

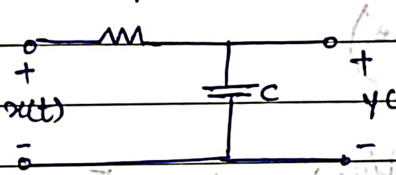
$x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau \xrightarrow{\mathcal{S}} \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau = y(t)$ (ignore identical points belonging to a set of measure zero)

↳ ADDITIVITY

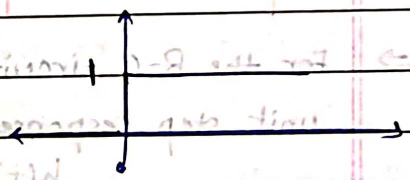
so, $\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau}$ (continuous-time convolution)

$y(t) = x(t) * h(t)$

eg. Impulse response of RC-circuit



$x(t) = \frac{1}{\Delta}\{u(t) - u(t-\Delta)\}$



$x(t) = y(t) + RC\frac{dy(t)}{dt}$

when $x(t) = u(t)$ →

$t<0 \quad y(t) = 0 \qquad t>0 \quad 1 = y + RC\frac{dy}{dt} \qquad 1-y = RC\frac{dy}{dt}$

~~$\int_0^y \frac{dy}{y - V_{in}} = \frac{-1}{RC}\int_0^t dt$~~

$\int_0^y \frac{dy}{1-y} = \int_0^t \frac{dt}{RC}$

~~$(\ln|y-V_{in}|)_0^y = \frac{-t}{RC} \Rightarrow \ln|y-1| = \frac{-t}{RC} \quad y-1 = e^{-t/RC}$~~

$-[\ln(1-y)]_0^y = t/RC$

~~$\ln\left|\frac{y-V_{in}}{V_{in}}\right| = -t/RC$~~

$\ln(1-y) = -t/RC$

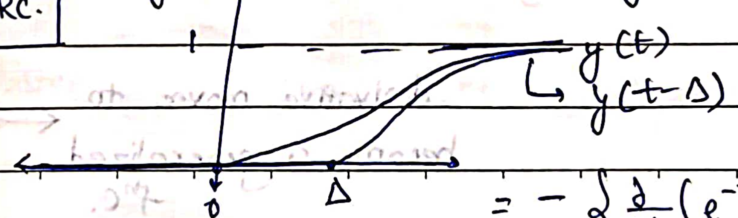
$1-y = e^{-t/RC}$

~~$y - V_{in} = V_{in}e^{-t/\tau} \quad y = V_{in}$~~

$\boxed{y = 1 - e^{-t/RC}}$

$\frac{1}{\Delta}\left\{(1-e^{-t/RC}) - (1-e^{-(t-\Delta)/RC})\right\}$

$= \frac{1}{\Delta}\left\{e^{-\frac{(t-\Delta)}{RC}} - e^{-t/RC}\right\}$



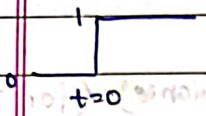
$\boxed{h(t) = \frac{1}{RC}\left(e^{-t/RC}\right)}$

$= -\left\{\frac{d}{dt}\left(e^{-t/RC}\right)\right\}$ $\quad \frac{f(x+(-\Delta)) - f(x)}{-\Delta}$

impulse response $\Rightarrow$ $\boxed{h(t) = \frac{1}{RC}\left(e^{-(t/RC)}\right).u(t)}$

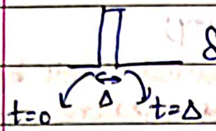
(for a R-C circuit)

$\Rightarrow$ unit step response :-



$u(t) \longrightarrow \boxed{S} \longrightarrow s(t)$

L.S.I.



$S_\Delta(t) = u(t) - u(t-\Delta) \longrightarrow \boxed{S} \longrightarrow s(t) - s(t-\Delta)$.

$\frac{1}{\Delta}\left(u(t) - u(t-\Delta)\right) \longrightarrow \boxed{S} \longrightarrow \frac{1}{\Delta}\left(s(t) - s(t-\Delta)\right)$

$\delta(t) = \lim_{\Delta\to 0} \frac{1}{\Delta}\left(u(t) - u(t-\Delta)\right) \longrightarrow \boxed{S} \longrightarrow \lim_{\Delta\to 0}\frac{1}{\Delta}\left(s(t) - s(t-\Delta)\right) = h(t)$

$\boxed{h(t) = \lim_{\Delta\to 0}\frac{1}{\Delta}\left(s(t) - s(t-\Delta)\right)}$ where $s(t) \longleftarrow \boxed{S} \longleftarrow u(t)$

derivative

$\boxed{h(t) = \frac{d\,s(t)}{dt}}$ $\Rightarrow$ (unit impulse response)

$\Rightarrow$ For the R-C circuit,

unit step response $\Rightarrow$ $s(t) = \left(1 - e^{-t/RC}\right).u(t)$ [holds for $t \geq 0$]

$$h(t) = \frac{d\,s(t)}{dt} = \frac{1}{RC}\left(e^{-t/RC}\right).u(t) + \left(1 - e^{-t/RC}\right)\left(\frac{d\,u(t)}{dt}\right)$$

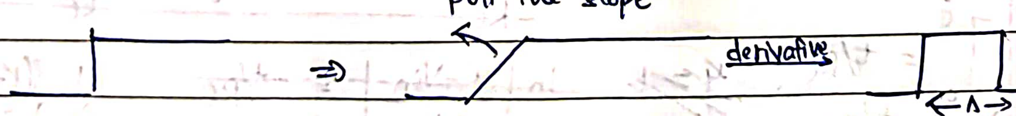
derivative of discontinuous functions

$h(t) \to \infty$ for $t \to 0$ as $\left(\frac{du(t)}{dt} \to \infty\right)$

and $\boxed{h(t) = \frac{1}{RC}\left(e^{-t/RC}\right).u(t)}$ $+ \cancel{(1-e^{-t/RC})}\,\delta(t)$ $\to 0$ at $t=0$ $= (1-e^{-0/RC})\delta(t) = \boxed{0}$.

($f^n c$ is CONT.)

pull the slope



derivative

continuous functions DRIVEN TO THEIR extremes

$\hookrightarrow$ change was so sharp

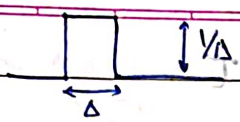
Take limit as $\Delta \to 0$

derivative of impulse $f^n c$

derivative moves to become a generalized $f^n c$.

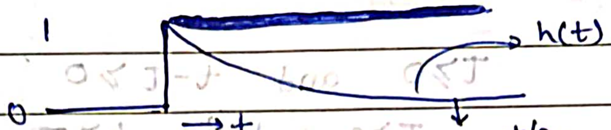
generalized $f^n c$.

RC circuit



can be driven to a limit

$\boxed{\dfrac{d(u(t))}{dt} = \delta(t)}$



$\frac{1}{RC} e^{-t/RC} \cdot u(t)$ ← (unit impulse response)

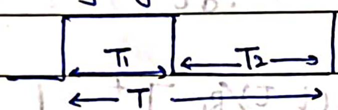
Coefficient associated with an impulse f^nc → Area associated with the impulse f^nc

→ how to apply unit impulse in Oscilloscope

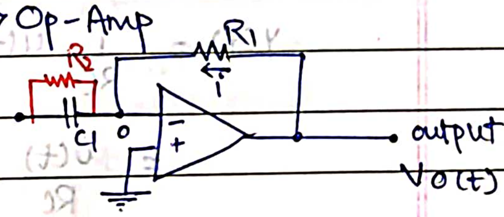
1) duty cycle control

$\Delta = \frac{T_1}{T}$

↳ make Δ small



2) Op-Amp



$\left(\begin{array}{c} R_2 \to \infty \\ \text{very large} \end{array}\right)$ $\frac{V_o(t)}{R} = +C \frac{dV_{in}(t)}{dt}$

'OR'

put a small resistance in series

$\boxed{V_o(t) = +RC\left(\frac{dV_{in}(t)}{dt}\right)}$

RC differentiator

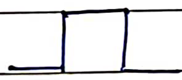
$\omega \uparrow$ $Z_c \downarrow$ → comparable to resistance

ratio $\not\to (\infty)$

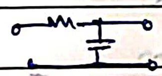
(diff. at low f, not at high f)

↓

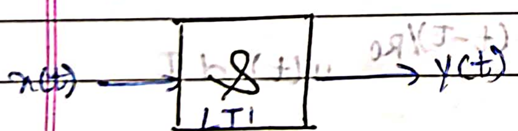
NOISE also differentiates



differentiator → R-C circuit → output



Unit step response $\xrightarrow{\text{diff.}}$ impulse response



y ⇒ continuous convolution of x with h,

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau}$$

RC circuit → $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$ $x(t) = u(t)$ [unit step]

$$y(t) = \int_{-\infty}^{\infty} u(\tau) \cdot \frac{1}{RC} e^{-(t-\tau)/RC} \, u(t-\tau) \cdot d\tau$$

$\tau \geq 0$ and $t-\tau \geq 0$

$\tau \geq 0$ and i.e. $t \geq \tau$ [if $t<0 \rightarrow$ never true so multiply with $u(t)$]

$$y(t) = \int_0^t \left(\frac{1}{RC} e^{\frac{-(t-\tau)}{RC}} d\tau \right) \cdot u(t)$$

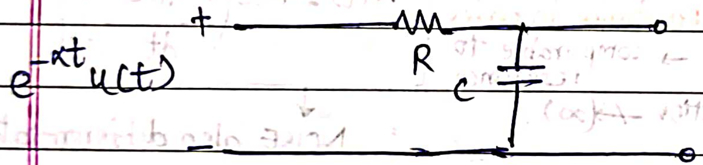
$$+ \int_t^{\infty} \frac{1}{RC} e^{-(t-\tau)/RC} d\tau \, (0) \rightarrow t-\tau < 0.$$

so,

$$y(t) = \frac{1}{RC} u(t) \int_0^t e^{-(t-\tau)/RC} \cdot d\tau$$

$$= \frac{u(t)}{RC} \left[RC \left[e^{-(t-\tau)/RC} \right]_0^t \right]$$

$$\boxed{y(t) = \left(1 - e^{-t/RC}\right) \cdot u(t)}$$



$\alpha \neq \frac{1}{RC}$ $\quad \alpha = \frac{1}{RC}$

$$\text{so, } y(t) = \int_{-\infty}^{\infty} e^{-\alpha\tau} u(\tau) \left(\frac{1}{RC} e^{-(t-\tau)/RC} u(t-\tau) \right) d\tau$$

$$= \int_0^t e^{-\alpha\tau} \cdot \frac{1}{RC} e^{-(t-\tau)/RC} u(t) \, d\tau$$

$$= \cancel{\frac{u(t)}{RC} \int_0^t e^{-(t+(1+\alpha)\tau)/RC} d\tau}$$

$$\cancel{\frac{u(t)}{RC} \left[\frac{e^{-(t+(\alpha-1)\tau)/RC}}{1/RC \, (\alpha-1)} \right]_0^t}$$

when $\alpha = \frac{1}{RC} \rightarrow \frac{u(t)}{RC} \int_0^t e^{-(t-\tau)/RC} \cdot e^{-\tau/RC} \cdot d\tau$

$$= \cancel{\frac{u(t)}{RC} \left(e^{\frac{-1}{RC}} t\right)}$$

$$\cancel{\alpha = \frac{1}{RC} \longrightarrow}$$

$$\cancel{\alpha \neq \frac{1}{RC} \longrightarrow \frac{u(t)}{\alpha-1}\left[e^{-(t+(\alpha-1)t)/RC} - e^{-(t)/RC}\right]}$$

$$\cancel{= \frac{u(t)}{\alpha-1}\left[e^{-\alpha t/RC} - e^{-t/RC}\right]}$$

$$\cancel{= \frac{u(t)}{\alpha-1}\, e^{-\alpha t/RC}\left[1 - e^{(-t+\alpha t)/RC}\right]}$$

$$Y(t) = \frac{1}{RC}\int_0^t e^{-\alpha\tau} \cdot e^{\tau/RC} \cdot e^{-t/RC}\, u(t)\, d\tau$$

$$= \frac{e^{-t/RC}\, u(t)}{RC} \int_0^t e^{\tau\left(\frac{1}{RC}-\alpha\right)} \cdot d\tau$$

$$= \frac{e^{-t/RC}}{RC}\, u(t) \left[\frac{e^{\tau\left(\frac{1}{RC}-\alpha\right)}}{\frac{1}{RC}-\alpha}\right]_0^t$$

$$= \frac{e^{-t/RC}}{1-RC\alpha}\, u(t) \left[e^{t\left(\frac{1}{RC}-\alpha\right)} - 1\right]$$

$$\boxed{Y(t) = \frac{u(t)}{1-RC\alpha}\left[e^{-\alpha t} - e^{-t/RC}\right]}$$

input ← $e^{-\alpha t}$; $e^{-t/RC}$ → impulse response

if $\boxed{\alpha = \frac{1}{RC}}$

$$y(t) = \frac{e^{-t/RC}}{RC}\, u(t) \int_0^t e^{\tau(0)}\, d\tau$$

$$\boxed{y(t) = \frac{t\, e^{-t/RC}}{RC}\, u(t)}$$ (reconant response)

when input resembles the impulse responce

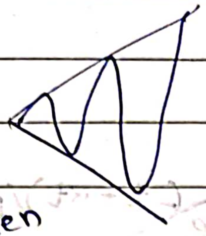
overall it does not grow as decay is faster

Output → sum of 2 components → input → FORCED response
→ impulse response → NATURAL response

natural response $\longrightarrow$ impulse response.

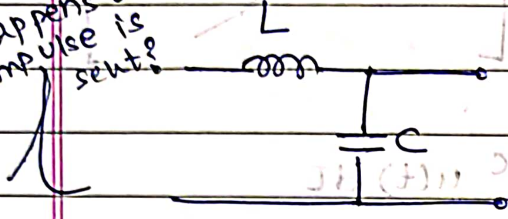
↓

jerk



sinusoidal output
gets multiplied by
't'.

what happens when an impulse is sent?



$f_{resonant} = \frac{1}{2\pi RC}$

CONVOLUTION

Commutative $\Rightarrow f * g = g * f$

Associative $\Rightarrow f*(g*h) = (f*g)*h$

Identity $\Rightarrow f * \delta = f$ $\quad [L(\delta) = 1]$

↓ identity element

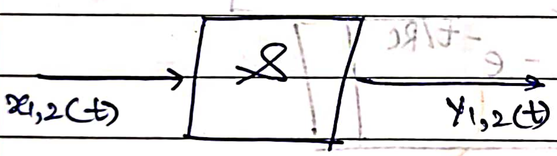
$\boxed{\text{Inverse}}$ $\quad \boxed{y = x * h}$

└ deconvolution operator

CAUSALITY (cause-effect)

prescribed experiment

$x_1(t) = x_2(t) \quad \forall \; t \leq t_0$



prescribed outcome :→ $\boxed{y_1(t) = y_2(t) \;\; \forall \;\; t \leq t_0 \;\; \forall \, t_0 \;\; \forall \, x_{1,2}}$

↓

difference in outputs only after inputs differ

independent of additivity, homogenity, linearity, & Inv.

e.g. - $\boxed{y(t) = x(t+2)}$

↓ non-causal

(#) output should be same as long as the inputs are same

e.g. 2) $y[n] = x[n+4]$

(#) A system is causal as long as the outputs depend only on the present inputs.

⇒ (Change in output CAN NOT preceed change in inputs)

⇒ For a discrete-time system to be causal, y[n] must NOT depend on x[K] for K>n.

↳ $\sum x[K]\, h[n-K] = y[n]$

so, the coeff. of $h[n-K]$ that multiply $x[K]$ must be 0 for all $K>n$.

~~→ so, if we have the unit impulse as input,~~

→

$$y[n] = \sum x[K]\, h[n-K]$$

$$= x[0]\,h[n] + x[1]\,h[n-1] + \dots + x[n]\,h[0] + x[n+1]\,h[-1] + \dots$$

0 0

⇒ $\boxed{h[n] = 0 \quad \forall\ n<0}$ $\boxed{\therefore \text{CAUSAL}}$

impulse responce must be zero before the impulse occurs.

$$y[n] = \sum_{0}^{\infty} x[K]\, h[n-K]$$

$\boxed{h(t)=0 \quad \forall\ t<0}$ $\boxed{\text{CAUSAL}}$

For a cont.-time system ⇒ and, $y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau = \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

⇒ Only one experiment is adequate to characterize the LTI system entirely.

$\delta(t) \longrightarrow \boxed{\mathcal{S}} \longrightarrow h(t).$

(unit impulse) (unit impulse response)

⇒ effect of the sum of 2 causes, is the sum of the effects → (Additivity)

⇒ scalability of causes & effects → (Homogenity)

⇒ system behavior doesn't change with time → (shift-invariance)

over all time / over reasonable time

⇒ LSI system → causal or NOT ?

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CAUSALITY OF LSI - Systems

$x(t) \longrightarrow \boxed{\mathcal{S} \text{ LSI}} \longrightarrow y(t)$

$h(t)$
↓
characterizes LSI system entirely.

$y(t) = x(t) * h(t)$ (commutative)

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau) \cdot d\tau$$

Theorem:- CAUSALITY $:\Leftrightarrow h(t) = 0 \quad \forall\, t \leq 0$
(for a LSI system)

Proof:- $(\Leftarrow)$ Let $h(t) = 0 \quad \forall\, t < 0$

$$y(t) = \int_{0}^{\infty} h(\tau) \cdot x(t-\tau)\, d\tau$$

Let $x_1(t) = x_2(t) \quad \forall\, t \leq t_0$

$$y_{1,2}(t) = \int_{0}^{\infty} h(\tau)\, x_{1,2}(t-\tau) \cdot d\tau$$

~~as $\tau \geq 0$~~ $\cancel{x_1(t-\tau) = x_2(t-\tau)}$

consider any $t \leq t_0 \Rightarrow \boxed{x_1(t-\tau) = x_2(t-\tau)}$ $(\forall\, t \leq t_0\ \forall\, \tau \geq 0)$ by assumption

$\Rightarrow y_1(t) = y_2(t) \quad \forall\, t \leq t_0$

Hence, the system is causal.

converse:- $(\Rightarrow)$

Corollary:- All zero-inputs result in an all-zero output for linear/LSI system

we know $y(t) = 0$ for $x(t) = 0$

Also for $x_2(t) = \delta(t) \rightarrow y(t) = h(t)$

as $x_1(t) = x_2(t)$ for $t < 0$

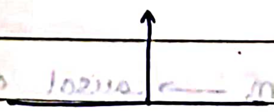
given the system is causal, $\delta(t)$ and 0 are inputs

$\boxed{h(t) = 0 \text{ for } t < 0}$ $\Leftarrow$ causality

$\Rightarrow$ if we have an impulse in the impulse response $\rightarrow$

e.g. $\Rightarrow y(t) = x(t)$

$\delta(t) \rightarrow \boxed{\mathcal{S}} \rightarrow \delta(t) = h(t)$.



it is A CAUSAL SYSTEM although $h(t) \neq 0$ at $t = 0$.

↳ $t = 0$ to be looked carefully for impulses.

$x_1[n] = x_2[n] \ \forall\ n \le n_0$
$\Rightarrow y_1[n] = y_2[n]\ \forall\ n \le n_0,\ \forall\ x_1, x_2$

$x[n] \longrightarrow$ [discrete, $h[n]$] $\longrightarrow y[n]$

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DISCRETE SYSTEMS (LSI) : $\Leftrightarrow$ causality $\Leftrightarrow h[n] = 0\ \forall\ n < 0$

(it can be non-zero for $n=0$)

$\Rightarrow$ causal system $\Rightarrow$

$$y[n] = \sum_{K \in \mathbb{Z}} x[K]\, h[n-K] = \ldots + x[n]\,h[0] + x[n+1]\,\cancel{h[-1]} + \ldots$$

So, $h[n] = 0 \ \forall\ n < 0$

$\Rightarrow h[n] = 0 \ \forall\ n<0$ & $x_1[K] = x_2[K]\ \forall\ K \le n_0$

Then, $\sum x_{1,2}[K]\, h[n-K] = y_{1,2}[n]$

i.e. if $x_1[K] = x_2[K]$

Then, $y_1[n] = y_2[n]$

$\Rightarrow$ The system is causal.

BOUNDED-INPUT BOUNDED-OUTPUT STABILITY (BIBO)

boundedness (property of the signal)

$x(t)$ (resp. $x[n]$) is bounded iff

$\exists\ 0 \le M_x < \infty$ (M_x is finite)

such that. $|x(t)| \le M_x\ \forall\ t$

(resp.) $|x[n]| \le M_x\ \forall\ n$. (input lies below the constant in magnitude)

eg- 1) $x(t) = \sin 2t \rightarrow$ bounded by "1".

2) $x[n] = 3\cos 4n \rightarrow$ bounded by "3".

3) $x(t) = e^{-t} \longrightarrow$ Not bounded ; $x(t) = e^{-t}.u(t) \rightarrow$ bounded by 1.

4) $x(t) = e^{-t^2} \longrightarrow$ bounded by "1".

STABILITY :-

Prescribed experiment :$\rightarrow$ give a bounded input to the system ((*) independent of the other 4 properties)

prescribed outcome :$\rightarrow$ The output is also bounded.

eg. 1) $y(t) = e^{x(t)} \rightarrow$ if $|x(t)| \le M_x < +\infty$

Then,

$$|y(t)| = |e^{x(t)}| = |e^{Re(x(t)) + j(Im(x(t)))}|$$

$$= |e^{Re(x(t))}| \cdot |e^{j(Im(x(t)))}|$$

$$= |e^{Re(x(t))}| \le \boxed{e^{M_x}}$$

Bounded!

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a) $\boxed{y(t) = \frac{dx(t)}{dt}}$ $\quad \delta(t) \rightarrow \boxed{\mathcal{S}} \rightarrow h(t) = \frac{d^2 u(t)}{dt^2}$

(unit doublet)

L.S.I. $\qquad$ (derivative of an impulse)

NOT BIBO STABLE

$x(t) = u(t) \rightarrow y(t) = \delta(t)$ $\qquad$ (unit doublet)

not bounded $\qquad$ BIBO unstable

$\Rightarrow$ $x(t) = \sin t^2 \qquad y(t) = \cos t^2 (2t) \Rightarrow$ NOT bounded

analytic & diff. everywhere $\rightarrow$ bounded

$\Rightarrow$ CAUSALITY $\longrightarrow$ differentiator is NOT CAUSAL $\longrightarrow$ need to look at a Left & Right neighbourho

$\Rightarrow$ WHEN IS A LSI system BIBO stable ?

$x(t) \longrightarrow \boxed{\mathcal{S},\ h(t)} \longrightarrow y(t)$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$ (CONVOL^n only for LSI)

Let $x(t)$ be bounded by $(0 \le M_x < \infty)$

$$|y(t)| = \left| \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau \right| \le \int_{-\infty}^{+\infty} \left| h(\tau)\, x(t-\tau) \right| d\tau$$

$$= \int_{-\infty}^{\infty} |h(\tau)|\, |x(t-\tau)|\, d\tau \le \int_{-\infty}^{+\infty} |h(\tau)|\, M_x\, d\tau$$

$$= M_x \int_{-\infty}^{+\infty} |h(\tau)|\, d\tau$$

Absolute integral of the impulse response $\rightarrow$ $\boxed{\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau}$

SUFF. $\rightarrow$ if the absolute integral of the impulse response is bounded, so is $y(t)$ i.e.

Sufficient Condition : $\rightarrow$ The impulse response is absolutely integrable $\rightarrow$ $\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau$ as finite.

→ unbounded input ⟶ nothing can be said

bounded input ⟶ BIBO stable ⟶ bounded

bounded input ⟶ BIBO unstable ⟶ Nothing can be said

→ In particular, consider $t=0$

$$y(0) = \int_{-\infty}^{+\infty} h(\tau)\, x(-\tau)\, d\tau$$

bounded input $x(t)$ which produces $y(0) = \int_{-\infty}^{\infty} |h(\tau)| \cdot d\tau$

Let

$$x(-\tau) = \begin{cases} \dfrac{\overline{h(\tau)}}{|h(\tau)|} & h(\tau) \neq 0 \\ 0 & h(\tau) = 0 \end{cases}$$

⟶ corresponding impulse response at the -ve point

(removing the angle)

$$y(0) = \int_{-\infty}^{+\infty} h(\tau) \cdot \frac{\overline{h(\tau)}}{|h(\tau)|} \cdot d\tau$$

$$= \boxed{\int_{-\infty}^{+\infty} |h(\tau)| \cdot d\tau}$$

input ⟶ bounded as $|x(\tau)| = 1$
$= 0$

if the system is BIBO stable ⇒ $y(0)$ must be FINITE.

FINITE ⟶ $\boxed{\text{Necessary condition}}$

⇒ A continuous variable LSI system is BIBO stable if and only if its impulse response is absolutely integrable.

Discrete LSI systems :-

A discrete variable LSI system is BIBO stable if and only if its impulse response is absolutely summable.

$\sum_{n \in \mathbb{Z}} |h[n]|$ is finite

Proof :- sufficiency : $|y[n]| \leq \sum |h[n]| \, |x[n-k]|$

assuming a bounded input, $|x[n-k]| \leq M_x$

so, $|y[n]| \leq \sum M_x (|h[n]|)$ ⟶ finite

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necessity :-

$$y[0] = \sum_{K=-\infty}^{\infty} h[K]\, x[-K]$$

Let $x[-K] = \begin{cases} \dfrac{\overline{h[K]}}{|h[K]|} & \text{if } h[K] \neq 0 \\ 0 & \text{else} \end{cases}$

$\Rightarrow y[0] = \sum_{K \in \mathbb{Z}} |h[K]|$ → if the system is BIBO stable, the SIGMA must be finite.

⇒ BIBO Stability → independent of Additivity, Homogenity, S.I., Causality.

Property :→ MEMORY

Prescribed experiment : → Apply- $x_1(t) = x_2(t) \ \forall \ t \neq t_0$

Prescribed outcome : → $y_1(t) = y_2(t) \ \forall \ t \neq t_0$

→ system is memoryless → output depends on the input ONLY at that time.

e.g.- 1) $y(t) = (x(t))^2$ → memoryless

2) $y[n] = \dfrac{x[n+1]}{5}$ → memory, non-causal

3) $y(t) = \dfrac{dx(t)}{dt}$ → Memory

discrete systems : → $x_1[n] = x_2[n]$, $\forall \ n \neq n_0$ → $y_1[n] = y_2[n]$ $\forall \ n \neq n_0$

Memoryless system → CAUSAL. memory → can't say about causality

eg.- $y(t) = (x(t-1))^3$ → with memory, causal.

REALIZABLE LINEAR - SHIFT - INVARIANCE - SYSTEMS

↓

finite resources required to implement the system.

Discrete :- $y[n] = \sum_{K=-\infty}^{+\infty} h[K]\, x[n-K]$ (convolution)

($h[n]$: impulse response)

Continuous :- $y(t) = \int h(\tau)\, x(t-\tau)\, d\tau$ ($h(t)$: impulse response)

∞ - summations
∞ - integrals ⇒ NOT realizable

eg.- $h[n] = u[n] \Rightarrow y[n] = \sum_{K=0}^{\infty} x[n-K]$

$$y[n] - y[n-1] = \sum_{K=0}^{\infty} x[n] - x[n-1-K]$$

$$x[n] - \cancel{x[n-1]}$$

$$\cancel{x[n-1]} - \cancel{x[n-2]} \;\ldots\; \cancel{x[n-L]} - \cancel{x[n-1-L]}$$

$$\boxed{y[n] - y[n-1] = x[n]}$$

or

$$\boxed{y[n] = x[n] + y[n-1]} \Rightarrow$$

Linear Constant Coefficient Difference Equation (LCCDE)

(Recursive LCCDE)

Realizable (finite length) systems

$$\boxed{\text{Realizable Discrete L.S.I. system} \longrightarrow \text{LCCDE (finite length)}}$$

eg.- $h(t) = A e^{-t/\tau_0} u(t)$

$$\cancel{y(t) = \int_0^{\infty} A e^{-\lambda/\tau_0} x(t-\lambda)\, d\lambda = \int_0^{\infty} x(t) A e^{-(t-\lambda)/\tau_0} d\lambda}$$

$$\cancel{\frac{dy(t)}{dt} = \int_0^{\infty} A e^{-\lambda/\tau_0} d}$$

$$\cancel{\frac{dy(t)}{dt} = \int_0^{\infty} \frac{dx(t)}{dt} A e^{-(t-\lambda)/\tau_0} d\lambda = A \frac{dx(t)}{dt} \left[\frac{e^{-(t-\lambda)/\tau_0}}{-(t-\lambda)/\tau_0}\right]}$$

$$\cancel{e^{-t/\tau_0} \int_0^{\infty} e^{\lambda/\tau_0} d\lambda}$$

$$\cancel{\frac{dy(t)}{dt} = A e^{-t/\tau_0} \frac{dx(t)}{dt}}$$

$$t - \lambda = \lambda_1 \Rightarrow -d\lambda = d\lambda_1$$

$$y(t) = \int_{-\infty}^{t} x(\lambda_1) A e^{-(t-\lambda_1)/\tau_0} d\lambda_1 = A e^{-t/\tau_0} \cdot \int_{-\infty}^{t} x(\lambda_1) e^{\lambda_1/\tau_0} d\lambda_1$$

$$\frac{dy(t)}{dt} = A\left(\frac{-1}{\tau_0}\right) \cdot y(t) + A \cancel{e^{-t/\tau_0}} x(t) \cancel{e^{t/\tau_0}}$$

$\boxed{\text{Realizable Continuous LSI systems} \Rightarrow \text{described as LCCDE}}$
↳ Differential

R-C circuit : order → No. of independent energy storage ele