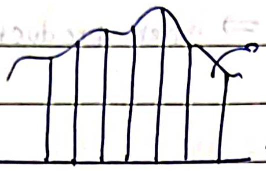
# MODULE - 3

## BRING TOGETHER CONTINUOUS & DISCRETE SYSTEMS

**Sampling :-**

waveform at discrete instants & reconstruct the signal




↳ something is lost in b/w

e.g. - $x(t) = a_0 e^{-\alpha t}$ $(\alpha > 0 \;\; a_0 \in \mathbb{R})$

↳ we only need $(a_0, \alpha)$ to reconstruct the waveform samples

[$(a_0, \alpha) \to$ unknown]

↳ need only two values $x(t_1) = a_0 e^{-\alpha t_1}$ $x(t_2) = a_0 e^{-\alpha t_2}$

$$\log\left(\frac{x(t_1)}{x(t_2)}\right) = \alpha(t_2 - t_1)$$

⇒ Representing a signal is a combination of A-priori knowledge (before hand) and measurement.

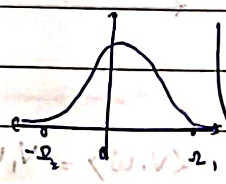
→ Audio signals ⟶ can't be changed at a rate which is very high

⇒ Spectrum of Audio :- limit to frequencies produced by human voice

(max.) 20KHz ⇓ (vocal cords need to vibrate faster.)

300Hz (min.) 20Hz

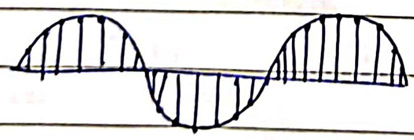
Apriori knowledge ⟶ BAND-LIMITED




all sinusoids upto a maximum frequency

(Fourier transform ⟶ 0 after a certain frequency)

(sinusoid)



linear system ⇒ reconstruct each component sinusoids
get the system back (ADD UP)

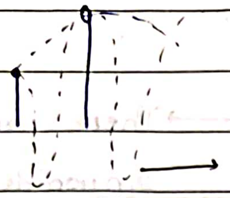
⇒ If a sine wave can be reconstructed ⇒ the linear system can be reconstructed.

$$\boxed{s(t) = A_0 \cos(\Omega_0 t + \phi_0)}$$

uniform sampling ⇒ $t = nT_0, \; n \in \mathbb{Z}$

rate of sampling $\to \frac{1}{T_0}$

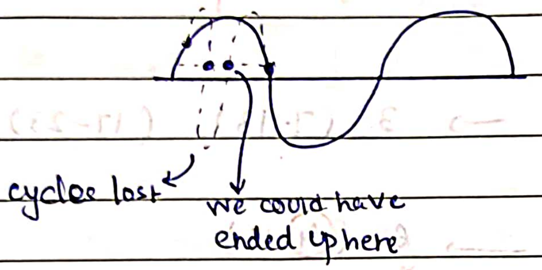
$\tilde{s}[n] = A_0 \cos(\Omega_0 n T_0 + \phi_0)$



b/w any two samples

we could have lost one / two / three / . . . . sinusoids

→ Also, any point occurs at two points




Pair of points on sinusoid which take the same value.

(① Lost cycles
② wrong edge)

$\tilde{s}[n] \longrightarrow$ Ambiguities :- 1) $\cos(\alpha) = \cos(-\alpha)$

2) $\cos(\alpha) = \cos(2K\pi + \alpha)$

i.e., $\cos(\alpha) = \cos(2K\pi \pm \alpha)$

$\hat{s}[n] = A_0 \cos(\Omega_0 n T_0 + \phi_0)$

(n: original integer)

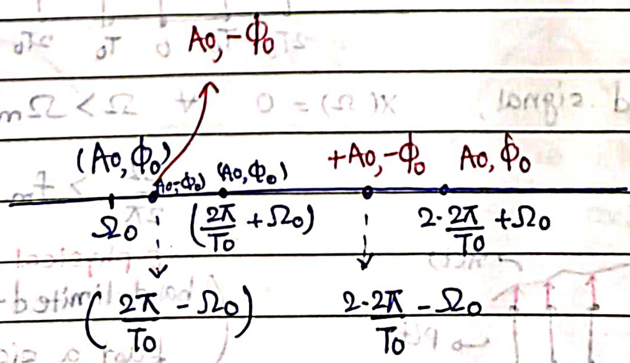
$$\hat{s}[n] = \boxed{A_0 \cos\left\{ 2\pi K n \pm (\Omega_0 n T_0 + \phi_0) \right\}}$$

cycle-loss index    Time index

This captures all sinusoids which have the same sample at the same points.

$$\tilde{s}[n] = A_0 \cos\left\{ \frac{2\pi}{T_0} K \pm \Omega_0 n T_0 \pm \phi_0 \right\}$$

(same samples at same points)




(frequency axis)

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e.g.- Suppose, $\frac{1}{T_0} = 100$ KHz $\quad \frac{\Omega_0}{2\pi} = 1$ KHz

($A_0, \phi_0$) of the original signal.

→ 1KHz, (100+1), (100−1), (201), (199) ⟶ There are too many sinusoids to choose from

$(100+1) \downarrow (A_0, \phi_0)$ $\quad (100-1) \downarrow (A_0, -\phi_0)$

↳ one criteria to resolve (BAND-LIMITED) → $\boxed{\text{Choose the lowest frequency}}$

e.g.- $\frac{1}{T_0} = 10$ KHz $\quad \frac{\Omega_0}{2\pi} = 3 \longrightarrow 3, (7-13), (17-23), \ldots$

$\frac{\Omega_0}{2\pi} = 6 \longrightarrow \cancel{6}, (4, 16)$

mistake if $\overbrace{\frac{\Omega_0}{2\pi}}^{f_0} > \frac{1}{2}\cdot\left(\frac{1}{T_0}\right)$.

Criteria :- $\frac{1}{T_0} - f_0 > f_0 \quad \forall f_0$ in the signal

$\Rightarrow \boxed{\frac{1}{T_0} > 2f_0} \quad \forall f_0$ present in the signal

constrains this relⁿ → Highest $f_0$

NYQUIST THEOREM OF SAMPLING

$\boxed{(\text{SAMPLING RATE}) > 2 \times (\text{maximum sinusoidal frequency in the signal})}$

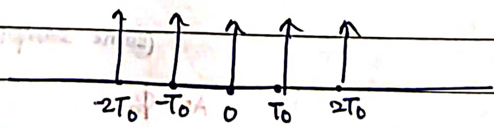
(SAMPLING THEOREM)

(Nyquist-Shannon-Whittaker)

Formal Proof of the Sampling Theorem

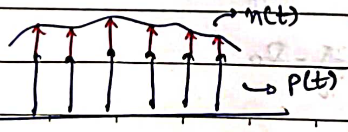
Idealized sample :- multiply the signal with a uniform impulse train (To: impulse interval)

$p(t) = \boxed{\text{impulse train} = \sum_{l=-\infty}^{+\infty} \delta(t - lT_0)}$




$x(t)$: Band-limited signal, $\quad X(\Omega) = 0 \quad \forall \ \Omega > \Omega_m$

$\frac{\Omega}{2\pi} > f_m$

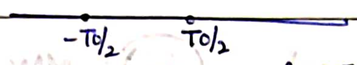



↳ physical significance ?
(band-limited ⟶ limit to how fast a signal changes.)

(shifting the spectrum by $\frac{2\pi}{T_0}\cdot K$)

$$x(t)p(t) = x(t)\sum_{l=-\infty}^{+\infty}\delta(t-lT_0) = \underbrace{\sum_{l=-\infty}^{+\infty} x(lT_0)\,\delta(t-lT_0)}_{\text{(Train of uniformly-spaced impulses)}} \qquad \text{(SIFTING)}$$

**Fourier Series :-**
(of p(t))




$$a_K(t) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\delta(t)\cdot e^{-jKt}\,dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\delta(t)\,dt$$

So, $p(t) = \sum_{K=-\infty}^{+\infty}\frac{1}{T_0}e^{jK\frac{2\pi t}{T_0}}$ $\qquad a_K(t) = \frac{1}{T_0} \quad \forall K.$

$$x(t)\cdot p(t) = x(t)\cdot\sum_{K=-\infty}^{+\infty}\frac{1}{T_0}e^{j\frac{2\pi}{T_0}Kt}$$

$$= \frac{1}{T_0}\sum_{K=-\infty}^{+\infty}x(t)e^{j\frac{2\pi}{T_0}Kt}$$

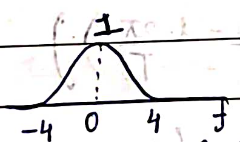
$$x(t) \longrightarrow X(j\Omega)$$

$$x(t)\cdot e^{j\frac{2\pi}{T_0}Kt} \longrightarrow X\left(j\left(\Omega-\frac{2\pi}{T_0}K\right)\right)$$

**OVERALL :-**

$$x(t)p(t) \xrightarrow{\mathcal{F}} \boxed{\frac{1}{T_0}\sum_{K=-\infty}^{+\infty}X\left(j\left(\Omega-\frac{2\pi}{T_0}\cdot K\right)\right)}$$

eg.- $x(t)$: Speech signal $\Rightarrow$ $f_m = 4$ KHz, let $\frac{1}{T_0} = 10$ KHz



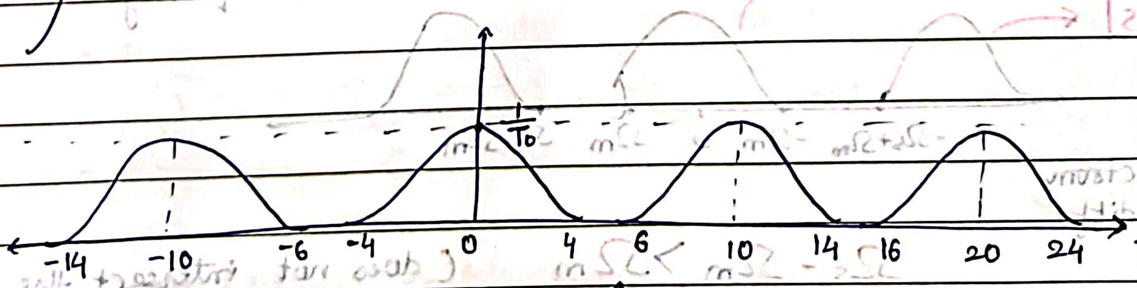

$\left(=\frac{\Omega}{2\pi}\right)$

(spectrum of speech signal)

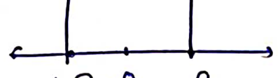
$\Omega = 2\pi f$

$\Omega-\frac{2\pi}{T_0}$ $\qquad \Omega+\frac{2\pi}{T_0}$

(shifting the spectrum by $\frac{2\pi}{T_0}\cdot K$)




(Complex axis: shift both FWD & BWD)

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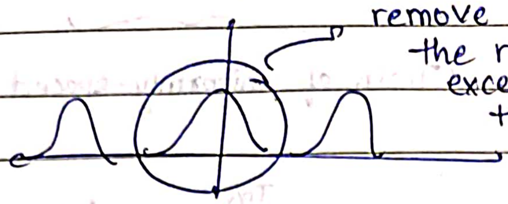
reconstruct the signal → ?

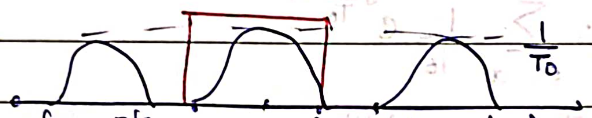


remove all the rest except this → get the Fourier spectrum back.

⇒ In general ⇒

$f_s = \frac{1}{T_0}$ (sampling frequency) $\quad f_m$ : MAX. frequency component in the signal

Suppose : $f_s > 2f_m$ →




$\left( \begin{aligned} f_s - f_m &> f_m \\ f_s &> 2f_m \end{aligned} \right)$

⇒ Reconstruct :- multiply with $H(j\Omega) = \begin{cases} 1 & ; |\Omega| \leq \Omega_1 \\ 0 & \text{o/w} \end{cases}$

$\boxed{2\pi f_m < \Omega_1 < 2\pi (f_s - f_m)}$

stopping all frequencies after $f_1 = \frac{\Omega_1}{2\pi}$ ← LOWPASS FILTERING

(CUT-OFF)

signal back ← Then, take the I.F.T.

$x(t) \longrightarrow X(j\Omega)$

↓ multiplied by $\sum_{l=-\infty}^{\infty} \delta(t - lT_0) \longrightarrow \frac{1}{T_0} \sum X\left(j\left(\Omega - l \cdot \frac{2\pi}{T_0}\right)\right)$

$\frac{2\pi}{T_0} = f_s$ (sampling (angular) frequency)

$\boxed{\text{Aliases}}$


Aliasing
↓
original spectrum assuming a diff. identity

$\Omega_s - \Omega_m > \Omega_m$ "(does not intersect the original spec

$\Omega_s > 2\Omega_m \quad \forall\, l \neq 0$ $\quad \boxed{\text{NYQUIST CRITERIA}}$

$X\left(j\left(\Omega - l \cdot \frac{2\pi}{T_0}\right)\right) \cap X(j\Omega) = \phi$

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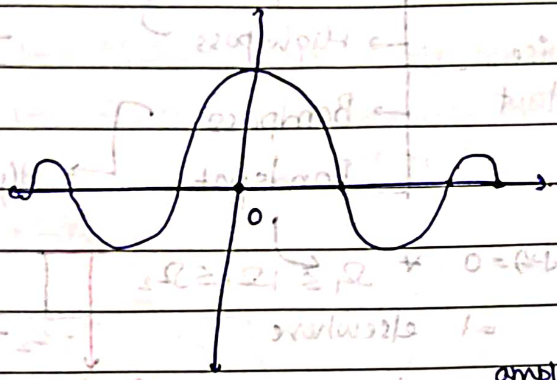
Ideal low-pass-filter with cut-off $\Omega_L$ such that

$$H(j\Omega) = 1 \quad 0 \le |\Omega| \le \Omega_L$$
$$= 0 \quad \text{else}$$

$$h(t) = \frac{1}{2\pi}\int H(j\Omega)e^{j\Omega t}\, d\Omega \Rightarrow h(t) = \frac{\sin(\Omega_L t)}{\pi t}$$



Frequency response, $|H(j\Omega)| \rightarrow$ amplitude of sinusoids in sequence

$\measuredangle H(j\Omega) \rightarrow$ phase of sinusoids is added.

↳ if we add a linear phase

CAUSALITY.

$y(t) = x(t-t_0) \Rightarrow h(t) = \delta(t-t_0)$

$\boxed{h_1(t)} \longrightarrow \boxed{y(t) = x(t-t_0)}_{h_2(t)} \longrightarrow$

$h'(t) = h_1 * h_2$
$= h(t) * \delta(t-t_0)$
$\boxed{h'(t) = h(t-t_0)}$

$h(t-t_0)$ → magnitude : same
→ phase : changed.

↳ even if $h(t)$ is non-causal, we can make it causal by shifting the impulse response, only if $h(t)$ is non-zero for ~~∞~~ finite time.

Here, $\boxed{h(t) :\longrightarrow \infty\text{-ly NON-CAUSAL}}$

STABILITY

$$\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau = \int_{-\infty}^{+\infty} \left|\frac{\sin \Omega_L t}{\pi t}\right| . dt \longrightarrow \text{NOT absolutely integrable}$$

Ideal LPF ⟶ 1> Infinitely Non-causal
2> Unstable
3> Irrational (unrealizable)

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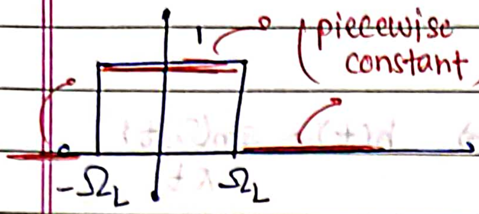
**Filter** → LSI system which has a frequency resp[onse]

(F.T. of its impulse response)

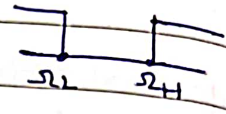



(piecewise constant)

ideal

$H(j\Omega) = 1 \quad \forall \ |\Omega| \geq$ [cut off]

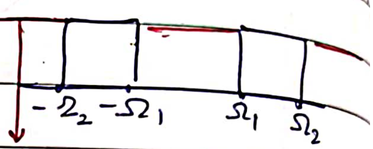
Piecewise constant
- → low-pass
- → High-pass
- → Bandpass
- → Bandspot




$H(j\Omega) = 1 \quad \forall \ \Omega_1 \leq |\Omega|$ [cut off]

$H(j\Omega) = 0 \quad \forall \ \Omega_1 \leq |\Omega| \leq \Omega_2$

$= 1$ elsewhere

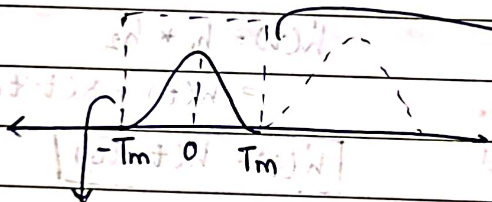



(Bandstop)

Passband
Stopband

1> Passband is flat
2> Stopband is flat
3> Passband & stopband are ~~touching~~ transitioning sharply

non-c[ausal] unsta[ble] realiz[able]




LSI system $h(t)$ → (LPF)

spectrum of sampled signal, $T_0$: sampling interval.

$$\sum_{-\infty}^{+\infty} x(nT_0)\,\delta(t-nT_0)$$

$$\delta(t) \longrightarrow h(t)$$

$$\delta(t-nT_0) \longrightarrow h(t-nT_0)$$

homogenity: $x(nT_0)\,\delta(t-nT_0) \longrightarrow x(nT_0)\,h(t-nT_0)$

additivity: $\sum_{-\infty}^{+\infty} x(nT_0)\,\delta(t-nT_0) \longrightarrow \sum_{-\infty}^{+\infty} x(nT_0)\,h(t-nT_0)$

Ideal LPF :- $h(t) = \dfrac{\sin w_c t}{\pi t}$ ; $w_c$: cutoff

at any point in time, we are adding an $\infty$ number of sinc f^n's

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Here, it is not necessary that sum of $\infty$-sinc's may diverge.
→ ① Reconstruction is NON-CAUSAL using ideal LPF
→ ② Sinc is NOT absolutely integrable.

\# strictly piecewise constant filters are unrealizable with finite resources

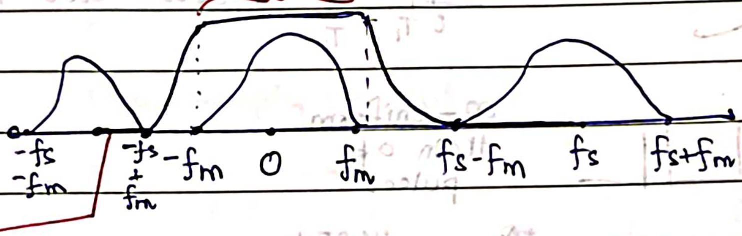
Alternative :→ MUST leave a gap b/w the aliases.

We can't use $f_s = 2f_m$ ⟶ 1> single frequency at that point can't "occur"
2> problem discussed.

$f_s > 2f_m$

as flat as possible

→ aliases are seperated from each other.




passband & stopband ~~still touch~~

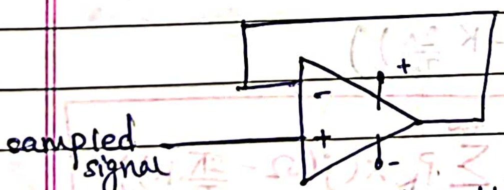
1> reach 0 at a point before $f_s - f_m$ 3> flat b/w $-f_m$ to $+f_m$.
2> remains 0 afterwards
→ stability ✓ ; non-causality & irrational (unrealizable)

sampling & reconstruction ⟶ some distortion.
→ $\infty$-non causality & $\infty$-non realizable always.

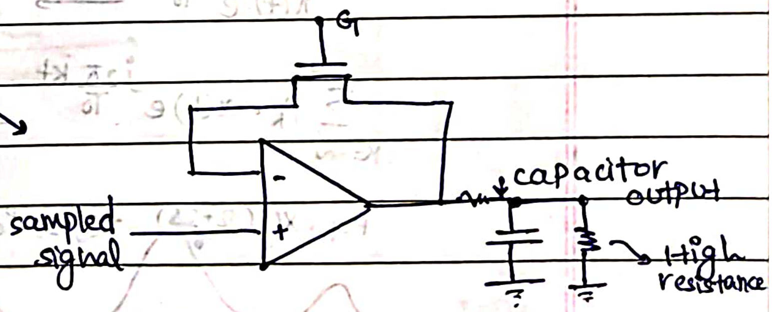
⇓

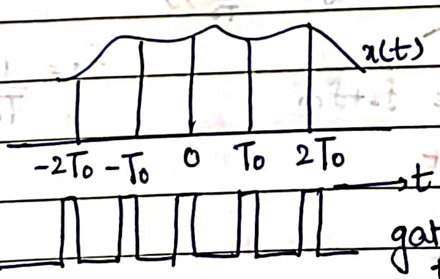
$\boxed{\text{Minimize Distortion}}$

voltage follower [NO CURRENT is drawn from the source]




Sample And Hold



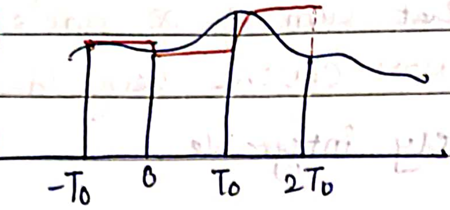




during the gate pulse ⟶ capacitor voltage follows the input

Band-limited signal → rate of change

**natural Sampling**

(Sample & Hold)

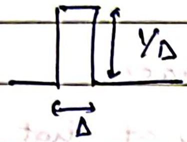



(capacitor voltage drops smoothly)

→ multiplying the signal with pulses

(sample & hold)

impulses are non-realizable



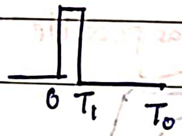

$E = \left(\frac{1}{\Delta}\right)^2 \cdot \Delta \longrightarrow$ diverges

$\boxed{\text{impulse has } \infty \text{ energy; un-realizable}}$

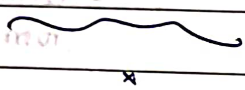
Practical pulses $\Rightarrow e^{-t/T_0}(u(t) - u(t-T_1))$

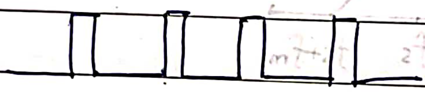
$x(t)$ (x) [Train of pulses]

$T_1 < T_0$




$\Rightarrow u(t) - u(t-T_1)$





$\infty$ - uniform train of pulses

Any periodic $f^n c$, $p(t) = \sum_{K=-\infty}^{+\infty} P_K \cdot e^{jK\frac{2\pi}{T_0}t}$

→ For a uniform impulse train, $P_K = \frac{1}{T_0} \;\forall\; K.$

→ real periodic $f^n c \Rightarrow P_K = \overline{P_{-K}}$.

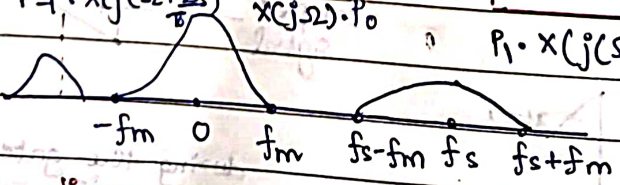
Band-limited

$x(t)p(t) \longrightarrow$ spectrum of $\sum_{-\infty}^{+\infty} P_K\, x(t) \cdot e^{j\frac{2\pi}{T_0}Kt}$

$x(t) \xrightarrow[\text{spectrum}]{\text{F.T. or}} X(j\Omega)$

$x(t) \cdot e^{j\frac{2\pi}{T_0}Kt} \longrightarrow X\left(j\left(\Omega - K \cdot \frac{2\pi}{T_0}\right)\right)$

$\sum_{K=-\infty}^{+\infty} P_K \cdot x(t) e^{j\frac{2\pi}{T_0}Kt} \longrightarrow \boxed{\sum_{K=-\infty}^{+\infty} P_K X\left(j\left(\Omega - \frac{2\pi}{T_0} \cdot K\right)\right)}$




$f_s = \frac{1}{T_0} > 2f_m$

All aliases are now different

Filter ⟶ remove aliases except the central one

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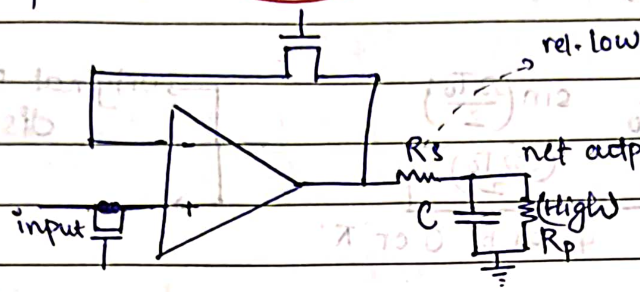
<u>Analog To Digital converstion</u> →

convert samples to a stream of bits | range → divide in $2^K$ intervals → which interval the sample falls → k bits

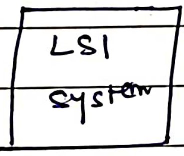
↳ retain the sample value → HOLDING

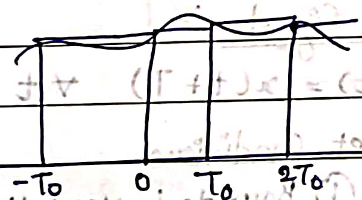



gate both MOSFETS → ip = op
↓
gate is opened (NOT conducting)
↓
decoupled i.p. and o.p.

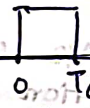
when gate pulce is removed, C discharges, but slow as Rp is ↑.

Ideal sampler → Hold




impulse response →



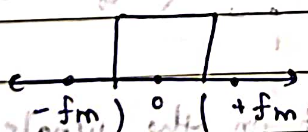

$$H(j\Omega) = \int_{-\infty}^{+\infty} h(t)\cdot e^{-j\Omega t}\, dt$$

$$= \int_{0}^{T_0} e^{-j\Omega t}\, dt = \boxed{\frac{1-e^{-j\Omega T_0}}{(j\Omega)}}$$

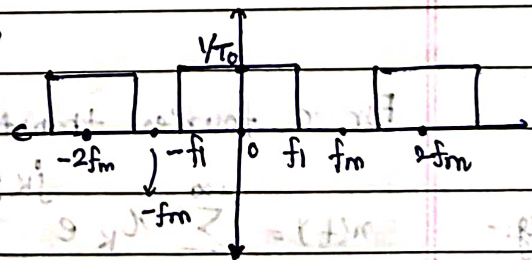
let's say X(jΩ)

$f_1 < f_m$

$\frac{1}{T_0} = 2f_m$

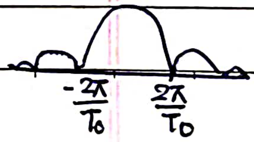


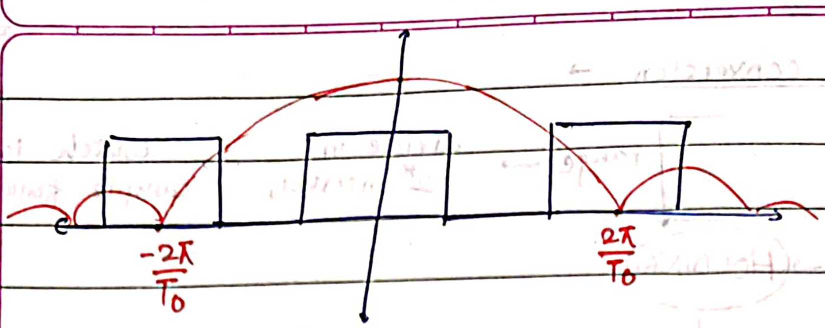

ideal sampler →




$$H(j\Omega) = \frac{j}{\Omega} e^{-\frac{j\Omega T_0}{2}} \left(e^{-\frac{j\Omega T_0}{2}} - e^{\frac{j\Omega T_0}{2}}\right) = \frac{2j\, e^{-\frac{j\Omega T_0}{2}} \sin(\Omega T_0/2)}{j\Omega}$$

$$H(j\Omega) = e^{-j\Omega T_0/2} \frac{\cdot T_0}{(\Omega T_0/2)} \sin\left(\frac{\Omega T_0}{2}\right) \Rightarrow |H(j\Omega)| \rightarrow$$






$$\underbrace{e^{-\frac{j\Omega T_0}{2}}}_{\phi = \left(-\frac{\Omega T_0}{2}\right)} \cdot T_0 \underbrace{\frac{\sin\left(\frac{\Omega T_0}{2}\right)}{\left(\frac{\Omega T_0}{2}\right)}}_{\phi \text{ can be `0' or `}\pi\text{'}}$$ → original spectrum is distorted

1) Fourier Transforms of periodic signals
2) Fourier Tranoform of a sequence

] → sampling

Periodic Signals :-

$x(t) = x(t+T) \quad \forall t$

Dirchlet Conditions

eg.- ($x(t) = \frac{1}{t}$, $0 \le t \le 1$ and $T = 1$) ←

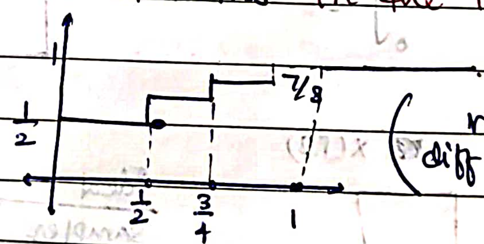
→ ① Bounded variation (finite range within which the function lies)

→ ② Finite extrema points in the interval T.

↳ eg.- $x(t) = \sin\left(\frac{1}{t}\right)$

→ ③ Finite no. of discontinuities in the interval T.

eg.-




(rises by half the diff. everytime we reach the MPT)

For a fourier transform → Dirchlet on the whole interval $\mathbb{R}$.

eg.- $$x(t) = \sum_{k=-\infty}^{+\infty} x_k e^{jk\frac{2\pi}{T}t} \qquad x_k = \frac{1}{T}\int_{(T)} x(t) e^{-j\frac{2\pi}{T}kt} dt$$

$$= \left\langle x(t), e^{j\frac{2\pi}{T}kt} \right\rangle$$

assuming $x(t)$ has a F.T.

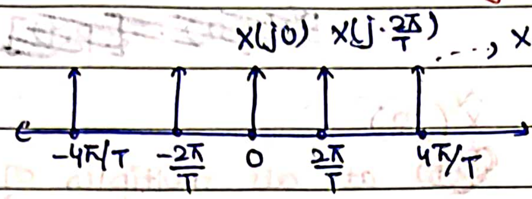
$$x(t) \xrightarrow{\mathcal{F}} X(j\Omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

impulses located at every multiple of $\frac{2\pi}{T}\cdot k$ (in the $\Sigma$)

$x_k$ $\longrightarrow$ values 'picked' up.

$\Rightarrow$ clearly, $X(j\Omega)$ is a ~~uniform~~ spaced train of impulses located at every multiple of $2\pi/T$.




UNIFORMLY-SPACED

$$x(t) = \frac{1}{2\pi}\int X(j\Omega)\, e^{j\Omega t}\, d\Omega = \frac{1}{2\pi}\sum X\left(j\cdot\frac{2\pi}{T}k\right) e^{j\frac{2\pi}{T}kt}$$

compare with $x(t) = \sum x_k e^{j\frac{2\pi}{T}kt}$

$$\boxed{x_k = \frac{1}{2\pi} X\left(j\cdot\frac{2\pi}{T}k\right)} \quad \text{or,} \quad \boxed{X\left(j\cdot\frac{2\pi}{T}k\right) = 2\pi x_k}$$

strength of impulse

$\boxed{\text{impulse in the Fourier domain} \Longrightarrow e^{j\Omega t} \text{ (PHASORS) in time-domain}}$ (phase changes but not amplitude) (coefficients are complex)

eg.- $x(t) = a_0 \cos(\Omega_0 t + \phi_0)$

$$= \frac{a_0}{2} e^{j\phi_0}\left[e^{j\Omega_0 t} + e^{-j\Omega_0 t - 2j\phi_0}\right]$$

$$X(j\Omega) = \frac{a_0}{2} e^{j\phi_0}\left[\delta(\Omega-\Omega_0) + e^{-2j\phi_0}\delta(\Omega+\Omega_0)\right]$$

$$X(j\Omega) = \frac{a_0}{2} e^{j\phi_0}\left[\delta(\Omega-\Omega_0) + e^{-2j\phi_0}\delta(\Omega+\Omega_0)\right]$$

$$= \frac{a_0 e^{j\phi_0}}{2}\delta(\Omega-\Omega_0) + \frac{a_0 e^{-j\phi_0}}{2}\delta(\Omega+\Omega_0)$$

F.T. of a periodic signal As a DUAL to Sampling

$x(t)$: periodic $\longrightarrow$ obeys the Dirchlet conditions

$\hookrightarrow$ restrict $x(t)$ to a period $T$, to produce $\tilde{x}(t)$.

$$\tilde{x}(t) \xrightarrow{\mathcal{F}} \tilde{X}(j\Omega) = \int \tilde{x}(t) e^{-j\Omega t}\, dt$$

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$\tilde{x}(t) = x(t)$ over $(T)$
$= 0$ elsewhere

$$\tilde{X}(\Omega) = \int_{-\infty}^{+\infty} \tilde{x}(t) e^{-j\Omega t} dt \Rightarrow \int_{(T)} x(t) e^{-j\Omega t} dt$$ (Fourier Series)

$$x_K = \frac{1}{T} \tilde{X}\left(j \frac{2\pi}{T} K\right)$$

↓

Sampling the F.T. of $\tilde{X}(j\Omega)$ at all multiples of $\frac{2\pi}{T}$

$$X_K = \frac{1}{T} \tilde{X}\left(j \cdot \frac{2\pi}{T} K\right)$$

↓

Sampling ⟶ Aliases in the time domain with $\frac{2\pi}{(2\pi/T)} = T$ as difference b/w two aliases

$\tilde{x}(t) \longrightarrow$ aliases $\longrightarrow x(t+T) \longrightarrow$ periodic signal.

$x(t)$ periodic $\longrightarrow$ restrict to a period $T$ $\longrightarrow$ sample in freq. domain $\longrightarrow$ shift in time axis by T.

Aliasing :- $f_s > 2f_m$

↓ DUAL

$\frac{1}{T_s} > \frac{2}{T}$

$T > 2T_s$ $\quad$ $T_s < T/2$

dual of aliasing (?)

F.T. of a sequence ⟶ Train of impulses

$x[n]$ can be thought as

$$x(t) = \sum_{n=-\infty}^{+\infty} x[n]\, \delta(t-n)$$ ⟶ (a trail of impulses)

$\xrightarrow{F.T.}$ Discrete-Time Fourier Transform (DTFT)

## Discrete Time Fourier Transform

$$x(t) = \sum_{n=-\infty}^{+\infty} x[n]\,\delta(t-n)$$

Then, $$\mathcal{F}(x(t)) = \int_{-\infty}^{+\infty}\left(\sum_{n=-\infty}^{+\infty} x[n]\,\delta(t-n)\right) e^{-j\Omega t}\,dt = \sum_{n=-\infty}^{+\infty} x[n] \int_{-\infty}^{+\infty} \delta(t-n) e^{-j\Omega t}\,dt$$

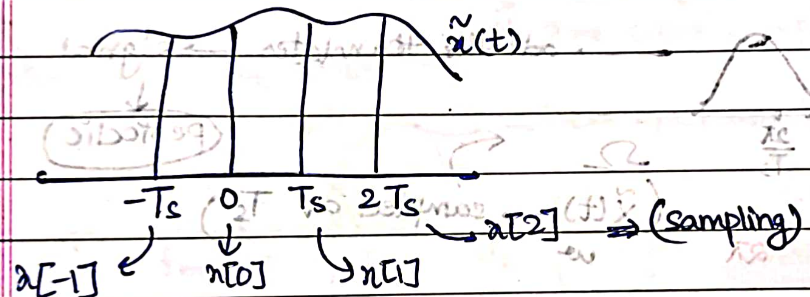
$$F(x(t)) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\Omega n}$$

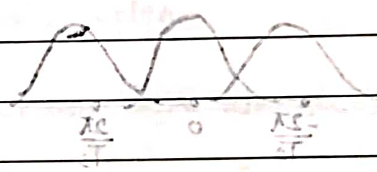
(Think of a sequence as a train of impulses in time and find its F.T.)

$$\boxed{\text{DTFT of } x[n] = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\Omega n}}$$

⇒ we could have as well got $x[n]$ by sampling a continuous time signal, $\tilde{x}(t)$ at a rate $\left(\frac{1}{T_s}\right)$. [at every multiple of $T_s$]

i.e. $\boxed{\tilde{x}(nT_s) = x[n]}$ ⟶ we wish to relate $\tilde{X}(j\Omega)$ and DTFT of $x[n]$.



DTFT ⟶ spacing of 1 unit, here ⟶ spacing of $\frac{2\pi}{T_s}$

$$\tilde{x}(t) \xrightarrow[\text{at } t=nT_s]{\text{ideal sampling}} \sum_{n\in\mathbb{Z}} \tilde{x}(nT_s)\,\delta(t-nT_s)$$

$\downarrow$ F.T.

$$\frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X\left(j\left(\Omega - \frac{2\pi}{T_s}k\right)\right)$$

$$\cancel{\int_{-\infty}^{+\infty}\left(\sum_{n\in\mathbb{Z}} \tilde{x}(nT_s)\,\delta(t-nT_s)\right) e^{-j\Omega t}\,dt}$$

$$\cancel{= \sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty}\left(\tilde{x}(nT_s)\,\delta(t-nT_s)\right) e^{-j\Omega t}\,.d\Omega}$$

$$\cancel{= \sum \tilde{x}(nT_s) \int e^{-j\Omega nT_s}\,.d\Omega = \sum \tilde{x}(nT_s)\,\frac{e^{-j\Omega nT_s}}{-jnT_s}}$$

$$\int \left(\sum_{n\in Z} \tilde{x}(nT_s)\,\delta(t-nT_s)\right) e^{-j\Omega t}\, dt$$

$$= \sum_{n\in Z} \underbrace{\tilde{x}(nT_s)}_{x[n]} \cdot e^{-j\Omega n T_s} \quad \text{equal} \quad = \sum_{n\in Z} x[n]\, e^{-j\Omega n T_s}$$

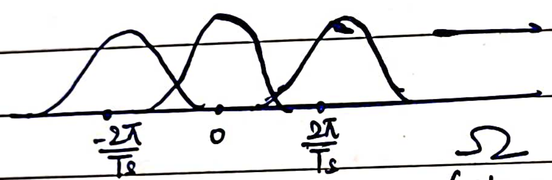
(equal)

$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \tilde{X}\left(j\left(\Omega - \frac{2\pi k}{T_s}\right)\right)$$

Put $\omega = \Omega T_s$ or $\omega = \Omega \cdot T_s = \frac{\Omega}{(1/T_s)} \rightarrow$ sampling rate

normalize $\Omega$ by the sampling

$\sum_{n\in Z} x[n]\, e^{-j\omega n} \longrightarrow x[n]$ is a sequence spaced by '1' sample.




add the translates → signal → periodic

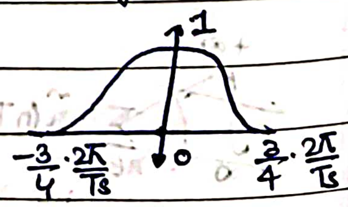
$(\tilde{x}(t) \rightarrow$ sampled at $T_s)$

→ in $\Omega$, period → $\frac{2\pi}{T_s}$, in $\omega$, period → $2\pi$.

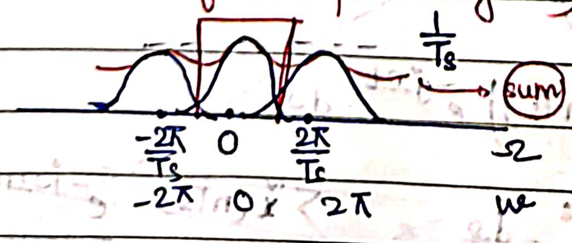
→ A factor of $(1/T_s)$ also needs to be included.

1) normalize ind. variable by $1/T_s$

2) multiply the dependent variable by




(Spectrum of sampled Signal)




$x[n] = \tilde{x}(nT_s) = \tilde{x}(t)\big|_{t=nT_s}$

DTFT of $x[n] = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$

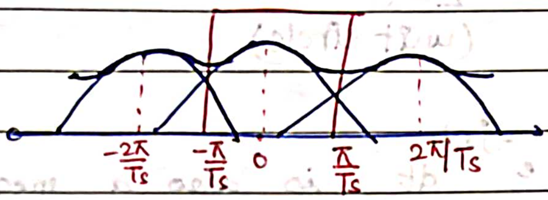
$\boxed{X(e^{j\omega})}$ [DTFT OF A SEQUENCE]

$\left(\omega = \frac{\Omega}{1/T_s}\right)$

(aliasing also ...)

DTFT is periodic with a period of $(2\pi)$ ⟶ $2\pi$ → normalized angular frequency

└→ $\pi = \omega$ : max$^m$ frequency present in the signal without any ALIASING.




shift and add ⎍ to get the spectrum back.

restrict b/w $-\pi/T_s$ to $+\pi/T_s$ ⟶ $x(t)$ has its F.T. as this

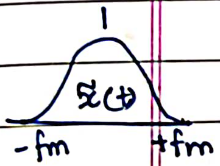
sample $x_1(t)$ at $f = 1/T_s$ ← obeys the Nyquist criteria

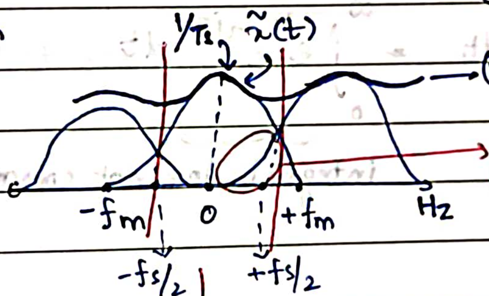
└→ same spectrum

$x[n]$ ⟶ could also be thought as samples of $x_1(t)$

└→ $x[n]$ = samples of $\tilde{x}(t)$ & samples of $x_1(t)$ (IFT and restricting b/w $-\pi/T_s, \pi/T_s$)

both give same samples.



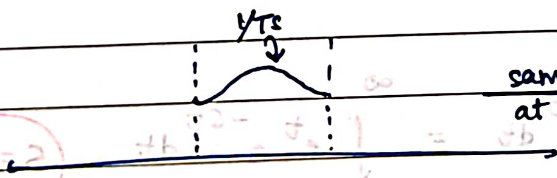




~~(new spectrum looks like this)~~

(it is these frequencies that cause false aliasing)

sampled at $f_s = 1/T_s$ ⟶ the spectrum of copies overlap with the original spectrum which add up.

sampled at $f_s$




$\xrightarrow{\text{sampled at } f_s}$ $\tilde{x}(nT_s) = x[n]$

$$\text{DTFT} = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

when $\omega = \pi$ and $\omega = -\pi$

max$^m$ frequency that could've been present in the signal without being aliased.

normalized angular frequency ← $\omega = \Omega T_s = \frac{\Omega}{1/T_s} = \frac{\Omega}{f_s}$

DTFT ⟶ samples of the signal b/w $\omega = \pm\pi$ after aliasing

original signal ⟶ aliased signal ⟶ restrict b/w $\pm f_s$ ⟶ sampled to give ~~us $x[n]$ back.~~

$\frac{1}{T_s} \rightarrow 1$

(IFF it has a DTFT.)

# Properties Of DTFT

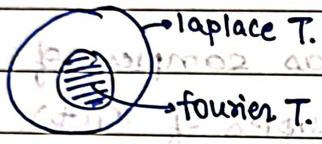
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n} \qquad \omega = \frac{\Omega}{1/T_s} = \Omega T_s = \frac{\Omega}{f_s}$$

→ DTFT is a special case of

(z-transform) $X(Z) = \sum_{n=-\infty}^{+\infty} x[n]\, Z^{-n}$ $\quad Z = e^{j\omega}$ (unit circle)

$x(t) \xrightarrow{F} X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$ is also a special case

(laplace transform) $\leftarrow X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$ for $\underbrace{s = j\Omega}_{\text{imaginary axis}}$




(MODULE 4)

Signals that do not have a F.T.:

Cont. Time → $x(t) = e^{t} u(t)$

$$X(j\Omega) = \int_{-\infty}^{+\infty} e^{t} u(t)\, e^{-j\Omega t}\, dt = \int_{0}^{\infty} e^{t} e^{-j\Omega t} dt$$

F.T. does not exist $\qquad$ integral does not converge

→ discrete time → $x[n] = e^{n} u[n]$

→ $e^{t} u(t) \longrightarrow e^{-\Sigma t} e^{t} u(t)$ : $\sigma > 0$

$e^{-t} u(-t) \longrightarrow e^{\Sigma t} e^{-t} u(-t)$ : $\sigma > 0$

$e^{-(\Sigma - 1)t} u(t) \rightarrow \Sigma > 1$

$$\int_{0}^{\infty} e^{-\Sigma t} e^{t} u(t)\, e^{-j\Omega t}\, dt = \int_{0}^{\infty} e^{t} e^{-st} dt \qquad s = \Sigma + j\Omega$$

$$= \int_{0}^{\infty} e^{(1-s)t} dt = \left(\frac{e^{(1-s)t}}{1-s}\right)_{0}^{\infty} \qquad \Sigma > 1$$

as long as $\Sigma > 1$,

$$\boxed{\text{F.T.} = \frac{1}{s-1}, \quad Re(s) > 1}$$

(region of convergence)

⇒ (Laplace Transform of the signal)

↓

(F.T. of a signal i.e. restricted)

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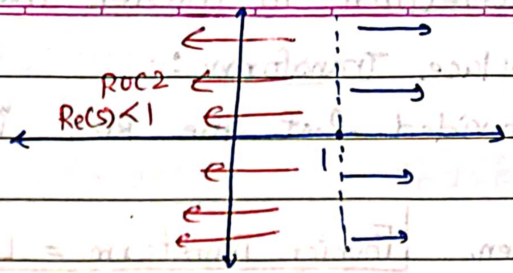
## Region of Convergence

Re(s) > 1 ⇒ s-plane




$$\mathcal{L}(e^{t}u(-t)) \rightarrow \int_{-\infty}^{+\infty} e^{t}u(-t)\, e^{-st}\, dt = \int_{-\infty}^{0} e^{(1-s)t}\, dt$$

$$= \left(\frac{e^{(1-s)t}}{1-s}\right)_{-\infty}^{0} = \frac{1-0}{1-s} = \frac{1}{1-s} \qquad \begin{matrix} 1-s>0 \\ (s<1) \end{matrix}$$

Re(s) < 1

$$\boxed{\mathcal{L}(-e^{t}u(-t)) = \frac{1}{s-1} \qquad Re(s)<1}$$

- $e^{t}u(t)$ and $e^{-t}u(-t) \longrightarrow \mathcal{L} \Rightarrow \frac{1}{s-1}$ ; ROC different

## Z-TRANSFORM

Z-transform of $x_1[n] = \sum_{n=-\infty}^{+\infty} x_1[n]\, z^{-n}$ in an appropriate ROC on z-plane.

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## Connection To Fourier Transform

**Laplace Transform :-**

⇒ provided that the ROC includes $s = j\omega$ (or $s = j\Omega$), [imaginary axis]

then $\boxed{\text{Fourier Transform} = \text{Laplace Transform}}$ evaluated on imaginary axis ($s = j\Omega$)

eg.- $x(t) = e^{-t}u(t)$

$$\mathcal{L}(x(t)) = \int_{-\infty}^{+\infty} e^{-t}u(t)\cdot e^{-st}\,dt$$

$$= \int_{0}^{\infty} e^{-(1+s)t}\cdot dt \qquad s = \sigma + j\omega$$

$\left(\begin{array}{l} -1-s<0 \\ s>-1 \end{array}\right)$

$$= \left(\frac{e^{-(1+s)t}}{-(1+s)}\right)_0^{\infty} = \frac{1}{1+s}\ ; \ Re(s)>-1$$

$$X(s) = \mathcal{L}(x(t)) = \frac{1}{1+s}\ ;\ Re(s) > -1$$

$$\text{So, } X(j\Omega) = \mathcal{F}(x(t)) = \frac{1}{1+j\Omega}$$

⇒ For a discrete sequence, $x[n]$,

Z-transform, $X(Z) = \sum_{n=-\infty}^{+\infty} x[n]\,z^{-n}$ (expression & R.O.C.)

→ if the R.O.C. includes $|z| = 1$, $r = 1$, then the DTFT = Z-transform on $Z = e^{j\omega}$

eg.- $x_1[n] = \begin{cases} 3^n ; & n \geq 0 \\ 0 & o/w \end{cases}$ $\qquad x_2[n] = \begin{cases} -3^n ; & n \leq -1 \\ 0 & o/w \end{cases}$

$$X_1(Z) = \sum_{n=0}^{\infty} 3^n z^{-n} = \frac{z}{z-3}\ ; \ |z|>3 \quad ; \quad X_2(Z) = \frac{Z}{Z-3}\ ;\ |z|<3$$

DTFT ✗ ← ROC doesn't include $|z| = 1$

Has a DTFT

DTFT of $x_2[n] = \boxed{\dfrac{e^{j\omega}}{e^{j\omega}-3}} = \dfrac{1}{1-3e^{-j\omega}}$

(#) R.O.C.'s of a Laplace Transform are always b/w two VERTICAL lines

↳ for the Z-transform → b/w two circles centred at the origin.
(modulus)

## PROPERTIES

① **Linearity :-** $x_1(t) \longrightarrow X_1(s)\ ;\ R_{x_1}$

$x_2(t) \longrightarrow X_2(s)\ ,\ R_{x_2}$

then, $\alpha x_1(t) + \beta x_2(t) \rightarrow \alpha X_1(s) + \beta X_2(s)\ ;\ \cancel{R_{x_1} \cap R_{x_2}}$ $\quad ROC = ?$

eg.- $x_1(t) = \left\{ 3^t + \left(\frac{1}{2}\right)^t \right\} u(t) \qquad x_2(t) = -3^t u(t)$

$X_1(s) = \int_0^\infty \left(3^t + \left(\frac{1}{2}\right)^t\right) e^{-st}\, dt \qquad 3^t = e^{\ln 3^t} = e^{t\ln 3}$

$= \int_0^\infty e^{t\ln 3 - st} \cdot dt + e^{-t\ln 2 - st} \cdot dt$

$X_1(s) = \dfrac{1}{s-\ln 3} + \dfrac{1}{s+\ln 2} \qquad X_2(s) = \dfrac{1}{\ln 3 - s} = \dfrac{-1}{s-\ln 3}$

$Re(s) > \ln 3 \qquad\qquad Re(s) > \ln 3$

so, for $x_1(t) + x_2(t) \xrightarrow{\mathcal{L}} \dfrac{1}{s+\ln 2}\ ;\ Re(s) > -\ln 2$

$R_{x_1} \cap R_{x_2} \neq ROC$ for $x_1 + x_2(t)$

$\Rightarrow$ ROC for $\alpha x_1(t) + \beta x_2(t)$ can expand beyond the intersection

$\quad \hookrightarrow$ ROC INCLUDES at least the intersection

$\rightarrow$ Analogously, for Z-transforms:-

$x_1[n] \xrightarrow{\mathcal{Z}} X_1(z),\ R_{x_1} \qquad x_2[n] \xrightarrow{\mathcal{Z}} X_2(z),\ R_{x_2}$

$\alpha x_1[n] + \beta x_2[n] \xrightarrow{\mathcal{Z}} \alpha X_1[z] + \beta X_2[z]\ ;\ ROC \supset R_{x_1} \cap R_{x_2}$

② **SHIFTING :-**

$x(t) \xrightarrow{\mathcal{L}} X(s),\ R_x$

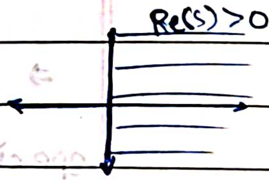
$T_0 \in \mathbb{R},\ x(t-T_0) \xrightarrow{\mathcal{L}} e^{-sT_0} X(s)\ ,\ ROC = ? \rightarrow$ possibly including or excluding the boundaries.

$X(s) = \int x(t) e^{-st} \cdot dt$

$\int_{-\infty}^{+\infty} x(t-T_0) e^{-st} \cdot dt \qquad t - T_0 = m \qquad \Rightarrow \int x(m) e^{-s(m+T_0)} \cdot dm$

$\qquad t = m + T_0 \qquad = e^{-sT_0} X(s)$

Re(s)>0



eg.- $x(t) = u(t) \Rightarrow X(s) = \int_0^\infty e^{-st} \cdot dt = \dfrac{1}{s}\ ;\ Re(s) > 0$

$x_1(t) = u(t+1) \Rightarrow X(s) = \int_{-1}^{\infty} e^{-st} \cdot dt = \dfrac{e^{+s}}{s}\ ;$

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$\frac{e^s}{s}$ will not converge as $s \to \infty$ ; so, ROC doesn't include the $\boxed{\infty\text{- contour here}}$

$x_2(t) = u(t-1) \Rightarrow X_2(s) = \int_1^{\infty} e^{-st}.dt = \frac{e^{-s}}{s}$ Re(s) > 0 (includes $\infty$-contour)

$\Rightarrow$ Re(s) $\to +\infty$ ($\infty$-contour) : if x(t) is the impulse-response of a LTI system, if the system is CAUSAL, then the $\infty$-contour is a part of the R.O.C.

~~x(t) is the impulse response of a causal system~~

$\Rightarrow$ Re(s) $\to +\infty$ : boundary

Z-transform :- $x[n] \xrightarrow{z} X(z)$, $R_x$

$x[n-n_0] \xrightarrow{z} z^{-n_0} X(z)$, $R_x$ $\to$ (check the boundaries)

(Re(s) and |z| are relevant to convergence)

③ CONVOLUTION

$x(t) \xrightarrow{L} X(s)$, $R_x$

$h(t) \xrightarrow{L} H(s)$, $R_H$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Now, $$L(y(t)) = \int_{-\infty}^{+\infty} y(t) e^{-st}.dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, e^{-st}\, d\tau\, dt$$

$$\begin{bmatrix} \tau \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} t \\ \tau \end{bmatrix}$$ ($t-\tau = \lambda$, $-d\tau = d\lambda$, $\tau = t+\lambda$)

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)\, h(\lambda)\, e^{-s(\tau+\lambda)}\, d\tau\, d\lambda = \int_{-\infty}^{+\infty} \left(x(\tau) e^{-s\tau} d\tau\right) \cdot \int_{-\infty}^{+\infty} h(\lambda) e^{-s\lambda}$$

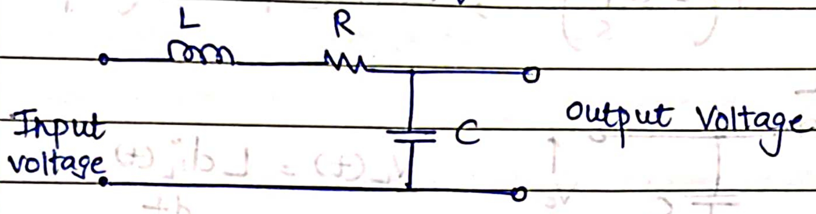
$\Rightarrow$ $\boxed{Y(s) = X(s) \cdot H(s)}$ $\boxed{ROC \supseteq R_x \cap R_H}$

special case: $s = j\Omega \to$ Fourier Transform

$x(t) \longrightarrow$ LSI system $h(t)$ $\longrightarrow y(t)$

Ratio, $\boxed{\dfrac{Y(s)}{X(s)} = \underbrace{H(s)}}$

(system function of the LSI system) $\Rightarrow \{H(s), R_H\}$




④ Formal inverse of the L.T.

$$x(t) \xrightarrow{\mathcal{L}} X(s),\ R_X \qquad \text{(I.L.T.)}$$

e.g.

$$\frac{1}{s+\alpha},\ Re(s) > -\alpha \xrightarrow{\text{I.L.T.}} e^{-\alpha t}u(t)$$

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, \underbrace{e^{-\Sigma t}}_{\text{cage it}} e^{-j\Omega t}\, dt$$

Take the F.T. $\longrightarrow$ I.F.T. and then multiply by $e^{\Sigma t}$

$$x(t) = \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(s)\, e^{j\Omega t}\, d\Omega\right) \cdot e^{\Sigma t}$$

($\Sigma$ must belong to the ROC)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(s)\, e^{(\Sigma + j\Omega)t} \cdot d\Omega \qquad s = \Sigma + j\Omega \text{ for any vertical line in ROC}$$

$(ds = j\, d\Omega)$

$$\boxed{x(t) = \frac{1}{2\pi j}\int_{\substack{\text{vertical line}\\ \text{in ROC}}} X(s)\, e^{st}\, ds}$$

FORMAL INVERSE

$\hookrightarrow$ choose imaginary axis for integrn $\longrightarrow$ I.F.T.

⑤ Differenciation

$$x(t) \xrightarrow{\mathcal{L}} X(s),\ R_X$$

$$x(t) = \frac{1}{2\pi j}\int_{\Sigma:\text{constant}} X(s)\, e^{st}\, ds \qquad \frac{dx(t)}{dt} \longrightarrow sX(s),\ R_X$$

$R_X \downarrow$ (except the boundaries) $\downarrow$ [CHECK]

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\Sigma:\text{constant}} X(s) \cdot s e^{st}\, ds$$

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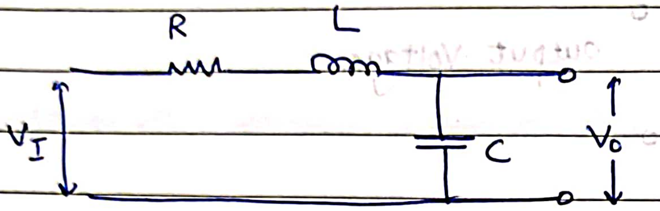
$x(t) \longrightarrow X(s)$ ; $R_x$

$\frac{dx(t)}{dt} \longrightarrow sX(s)$ ; $ROC \geq R_x$, but we need to check the boundaries

$u(t) \xrightarrow{\mathcal{L}} 1/s$ , $Re(s) > 0$

$\delta(t) \xrightarrow{\mathcal{L}} 1 \left(= s \cdot \frac{1}{s}\right)$ , entire plane (expanded ROC)




$V_L(t) = L\frac{di_L(t)}{dt}$

$V_L(s) = sLi_L(s)$

So, $\frac{V_L(s)}{i_L(s)} = \boxed{Z_L = sL}$ $\longrightarrow$ Laplace domain impedance of the inductor

similarly, $\frac{V_R(s)}{i_R(s)} = \boxed{R}$ $\frac{V_C(s)}{i_C(s)} = \boxed{\frac{1}{sC}}$

Impedance $\longrightarrow$ how the flow of current is obstructed.

Now, $\frac{V_o(s)}{V_i(s)} = \frac{1/sC}{1/sC + R + sL} = \frac{1}{s^2LC + sRC + 1}$

$H(s) = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$ $\frac{1}{LC} = \Omega_m^2 \longrightarrow$ natural resonant frequency

system function of the RLC circuit $\longrightarrow$ ratio of output & input of the LSI system, must be independent of input

$$H(s) = \frac{\Omega_m^2}{s^2 + 2\zeta\Omega_m s + \Omega_m^2}$$

$2\zeta\Omega_m = \frac{R}{L} \Rightarrow 2\zeta\frac{1}{\sqrt{LC}} = \frac{R}{L} \Rightarrow \boxed{\zeta = \frac{1}{2}R\sqrt{\frac{C}{L}}}$

$\boxed{\Omega_m = \frac{1}{\sqrt{LC}}}$

$\xi \rightarrow$ damping factor of the circuit $= \frac{R}{2}\sqrt{C/L}$

$0 < \xi < 1 \rightarrow$ underdamped  $\xi = 1 \rightarrow$ critically damped  $\xi > 1 \rightarrow$ overdamped

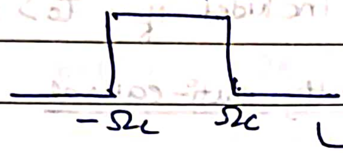
$\rightarrow$ Here the system $f^n$c is $\boxed{\text{RATIONAL}}$ $\rightarrow$ ratio of 2 finite length polynomials

$(P/Q) \longrightarrow$ polynomials in the base

1> a rational no. exists b/w any two rational no.'s

2> an irrational no. can be approximated to any degree using a rational no.

ideal LPF




1> $\cancel{\phantom{xx}}$ $\infty$ - non causal

2> unstable

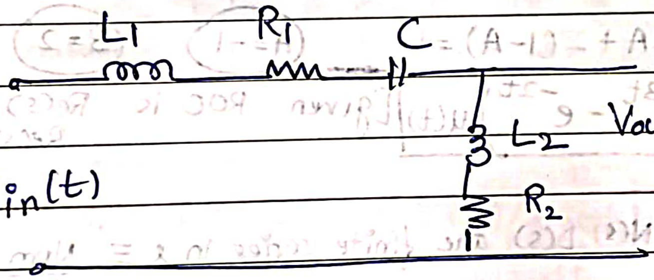
3> irrational → can't be expressed in terms of ratio of 2 polynomials

$\hookrightarrow$ FT exists, but L.T. is irrational

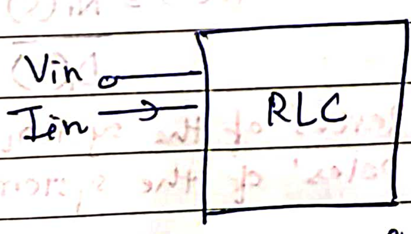
can't realize with a finite no. of resources

$$\sum \left(\frac{s}{2}\right)^n = \frac{1}{1 - s/2} \Rightarrow \left|\frac{s}{2}\right| < 1 \text{ (ROC)}$$

**RATIONAL SYSTEMS:** LSI system with a system function i.e. $\underbrace{\text{rational}}$ can be expressed as ratio of two polynomials




$$\frac{V_{in}(s)}{V_{out}(s)} = \frac{R_2 + sL_2}{R_1 + sL_1 + sL_2 + \frac{1}{sC}} = \underbrace{\frac{sC(R_2 + sL_2)}{1 + sC(R_1 + s(L_1 + L_2))}}_{\text{rational system}}$$




$\longleftarrow$ driving-point impedance $= \frac{V_{in}(s)}{I_{in}(s)}$

(positive, real)

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## RATIONAL SYSTEMS

→ A LSI system which has the laplace transform of its impulse response exists and is RATIONAL.

eg.- $y(t) = x(t - T_o)$

$Y(s) = e^{-sT_o} X(s) \Rightarrow \frac{Y(s)}{X(s)} = \boxed{e^{-sT_o} = H(s)}$ (irrational)

All of s-plane except possibly $Re(s) = \pm\infty$

→ Not a rational system function
→ system function depends on the input

$T_o \cdot Re(s) > 0$

R.O.C.

if $T_o > 0 \Rightarrow Re(s) > 0$

if $T_o < 0 \Rightarrow Re(s) < 0$

CHECK

extreme contours → $Re(s) \to \infty$ NOT included if $T_o < 0$

$Re(s) \to -\infty$ NOT included if $T_o > 0$

⇒ System can be strictly causal or strictly anti-causal

⇒ rational systems ⟶ H(s) : ratio of two polynomials.

H(s): L.T. of the impulse response.

eg.-

$$H(s) = \frac{s+1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3} = \frac{2}{s+3} - \frac{1}{s+2}$$

$A + B = 1 \qquad 3A + 2B = 1$

$3A + 2(1-A) = 1 \qquad A = -1 \qquad B = 2$

Then, $\boxed{h(t) = (2e^{-3t} - e^{-2t}).u(t)}$ [given ROC is $Re(s) > -2$, $Re(s) > -3$ ]

$$\boxed{H(s) = \frac{N(s)}{D(s)} \quad N(s), D(s) \text{ are finite series in } s \equiv \frac{\text{Num. polynomial}}{\text{Den. polynomial}}}$$

deg. (N(s)) ⩾ deg. (D(s)) ⇒ polynomial in s + proper fraction

$Q(s)$ $\qquad R(s) = \frac{N_r(s)}{D(s)}$

numerator polynomial, $N(s) = 0$ ⟶ 'zeroes' of the system.

$D(s) = 0$ ⟶ 'poles' of the system.

⇒ ROC can't pass through a pole, there is singularity (divergence) at it.

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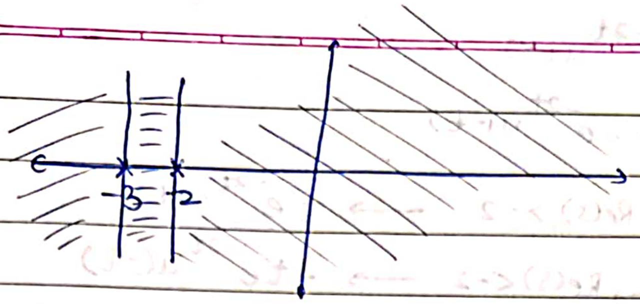



each of the regions have a different L.T.

if ROC is to right of the pole → RIGHT-SIDED func

→ if ROC is $-3 < Re(s) < -2$

$$\boxed{h(t) = 2e^{-3t}u(t) + e^{-2t}u(-t)}$$

$\left( \frac{1}{s+\alpha} \xrightarrow{\text{left of it}} -e^{-\alpha t}u(-t) \right)$

→ if ROC is $Re(s) < -3$ & $Re(s) < -2$

$$h(t) = -(2e^{-3t} - e^{-2t})u(-t) = \boxed{(e^{-2t} - 2e^{-3t})u(-t)}$$

IMP.
$$\boxed{\frac{1}{s+\alpha} \begin{cases} Re(s) > -Re(\alpha) \rightarrow e^{-\alpha t}u(t) \\ Re(s) < -Re(\alpha) \rightarrow -e^{-\alpha t}u(-t) \end{cases}}$$

ROC: depends on poles only

$H(s) = \frac{s+1}{(s+2)^2(s+3)}$ → (same 3 parts of ROC)

Steps to invert

- identify poles & look at ROC & L or R → Invert the term corresponding to the ~~~~ L.T.

$Q(s) = 5 \xrightarrow{\text{I.L.T.}} 5\delta(t)$

$Q(s) = 5+3s \xrightarrow{\text{I.L.T.}} 5\delta(t) + 3\left(\frac{d\,\delta(t)}{dt}\right)$ (unit doublet)

Polynomial in s with deg. $\geq 1$

Repeated Poles

⇒ Derivative of Laplace Transform

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt \Rightarrow \frac{dX(s)}{ds} = \int_{-\infty}^{+\infty} (-t)\,x(t)e^{-st}\,dt$$

$$\boxed{-\frac{dX(s)}{ds} = \mathcal{L}(t\,x(t))}$$

$$x(t) \xrightarrow{\mathcal{L}} X(s)$$
$$t\,x(t) \xrightarrow{\mathcal{L}} -dX(s)/ds$$

$$\frac{1}{s+2} \begin{cases} \xrightarrow{Re(s) > -2} e^{-2t} u(t) \\ \xrightarrow{Re(s) < -2} -e^{-2t} u(-t) \end{cases}$$

$$-\frac{d}{ds}\left(\frac{1}{s+2}\right) = \underbrace{\frac{1}{(s+2)^2}}_{\text{(repeated poles)}} \begin{cases} \xrightarrow{Re(s) > -2} t e^{-2t} u(t) \\ \xrightarrow{Re(s) < -2} -t e^{-2t} u(-t) \end{cases}$$

Rational system $\longrightarrow$ proper fractions ( deg.(N) < deg.(D))

$\hookrightarrow$ I.L.T. $h(t)$ $\longrightarrow$ Polyex $\begin{cases} \text{right-sided} \\ \text{left-sided} \end{cases}$

(polynomials in 't') × (exponentials in 't')

$$\left[\text{degree} = \left(\begin{matrix}\text{multiplicity} \\ \text{of the pole}\end{matrix}\right) - 1\right]$$

## Z-TRANSFORM : formal inverse

$$\underbrace{\sum_{n=-\infty}^{+\infty} x[n] r^{-n} e^{-j\omega n}}_{\text{DTFT}} = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \quad (z = re^{j\omega})$$

$= \text{Inner product of } \langle x[n] r^{-n}, e^{+j\omega}$

DTFT of $x[n] = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} = X(e^{j\omega})$

(component of $x[n]$ on $e^{j\omega n}$)

= Dot-Product (inner product) of $x[n]$ and $e^{j\omega n}$

$\omega$: normalized angular frequency , $\omega = \frac{\Omega}{1/T_s}$ ; $T_s$ : sampling period

$X(e^{j(\omega+2\pi)}) = X(e^{j\omega}) \;\forall\; \omega$ $\quad x[n] = K_0 \int X(e^{j\omega}) e^{j\omega n} d\omega$

Inverse DTFT :- $x[n] = K_0 \int_{2\pi} \underbrace{X(e^{j\omega}) e^{j\omega n}}_{\text{continuum of components}} d\omega$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[k] e^{-j\omega k}$$

$$x[n] = K_0 \int_{2\pi} \sum_{k=-\infty}^{+\infty} x[k] e^{j\omega(n-k)} d\omega$$

$$= K_0 \sum_{k=-\infty}^{+\infty} x[k] \int_{2\pi} e^{j\omega(n-k)} d\omega$$

$$= K_0 \sum_{K=-\infty}^{+\infty} x[K] \cdot I \qquad I = \left( \frac{e^{j\omega(n-K)}}{j(n-K)} \right)_0^{2\pi}$$

when $n \neq K$, $I = 0$ when $n = K$, $I = 2\pi$

$$x[n] = K_0 \sum_{K=-\infty}^{\infty} x[K] \cdot 2\pi\, \delta(n-K) = K_0 \cdot 2\pi\, x[n]$$

$$\Rightarrow K_0 = \frac{1}{2\pi} \qquad \boxed{x[n] = \frac{1}{2\pi} \int_{(2\pi)} X(e^{j\omega}) \cdot e^{j\omega n} d\omega}$$

Formal inverse of z-transform

z-transform of $x[n] \longrightarrow$ DTFT of $x[n]\, r^{-n}$

(r: chosen such that $z \in$ R.O.C.)

$$x[n]\, r^{-n} = \frac{1}{2\pi} \int_{(2\pi)} X(re^{j\omega}) \cdot e^{j\omega n} d\omega$$

$$\boxed{x[n] = \frac{1}{2\pi} \int_{(2\pi)} X(re^{j\omega}) \cdot r^n e^{j\omega n} d\omega}$$ r: $z \in$ ROC

(inverse z-transform)

(DTFT of $x[n] r^{-n}$)

$$= \frac{1}{2\pi} \int_{(2\pi)} X(z)\, z^n\, d\omega \qquad z = re^{j\omega}$$

$$dz = r(j)\, e^{j\omega} d\omega = j z\, d\omega$$

$$= \frac{1}{2\pi j} \oint_C X(z)\, z^n z^{-1} dz \qquad d\omega = dz/jz = \frac{1}{j} z^{-1} dz$$

C: circle centred at origin (O) in ROC

$$\boxed{x[n] = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1} \cdot dz}$$

$$x[n] \xrightarrow{\mathcal{Z}} X(z) \; ; \; R_x$$

$$h[n] \xrightarrow{\mathcal{Z}} H(z) \; ; \; R_H$$

$$y[n] = x[n] * h[n] \qquad y[n] \xrightarrow{\mathcal{Z}} \sum_{n=-\infty}^{+\infty} \sum_{K=-\infty}^{+\infty} x[K]\, h[n-K]\, z^{-n}$$

$=$

$n - K = l$ For a fixed K, when n runs over all integers, l also runs over all integers

$$\sum_{K=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} x[K]\,h[l]\,z^{-(K+l)}$$

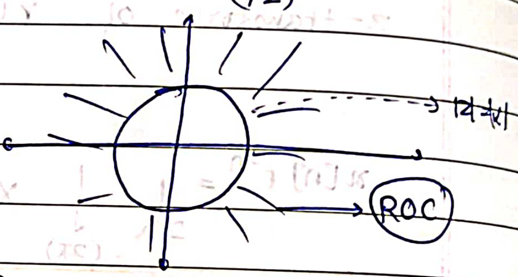
$$=\left(\sum_{K=-\infty}^{+\infty} x[K]z^{-K}\right)\left(\sum_{l=-\infty}^{+\infty} h[l]\,z^{-l}\right) \qquad R_Y = \text{ROC} \supset R_X \cap R_H$$

eg.- $x[n] = \alpha^n u[n]$

Z-transform of $x[n] = \sum_{n=-\infty}^{+\infty} \alpha^n u[n]\, z^{-n}$

$$= \sum \alpha^n z^{-n} = \frac{1}{1-(\alpha/z)}$$

$$\boxed{X(z) = \frac{1}{1-\alpha z^{-1}}\ ; \quad |z|>|\alpha|}$$




Let $h[n] = \underset{0}{\underset{\uparrow}{1}} \quad \underset{1}{\underset{\uparrow}{-\alpha}}$

~~$H(z) = \sum_{n=0} 1\cdot z^{-0} + \sum_{n=1}^{\infty} (-\alpha)\, z^{-n}$~~ ~~$|z|<1$~~

~~$= 1 + (-\alpha)\left(\frac{1/z}{1-1/z}\right)$~~ ~~$= 1-\alpha\left(\frac{1}{z-1}\right)$~~

~~$x[n] * h[n] \Rightarrow Y(z) = \frac{1}{1-\alpha/z} = \frac{z}{z-\alpha}$~~

$$H(z) = 1\cdot z^{-0} + (-\alpha)\cdot z^{-1} = 1-\alpha z^{-1} \qquad \underset{R_H}{|z|>0}$$

$$Y(z) = X(z)\,H(z) = \underbrace{1}_{\text{(entire z-plane)}} \Rightarrow \boxed{y[n] = \delta[n]}$$

M2> $y[n] = x[n] * h[n]$

$x[n] \Rightarrow 1 \quad \alpha \quad \alpha^2 \quad \alpha^3 \ \ldots$

$h[n] \Rightarrow 1 \quad -\alpha$

$y[0] = \sum x[K]\,h[n-K] = 1$

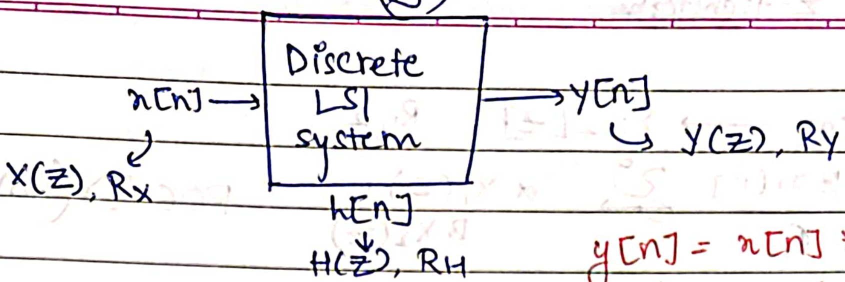
$y[1] = \sum x[K]\,h[1-K] = \alpha + (-\alpha) = 0 \qquad y[n] = \delta[n]$

$y[-1] = \sum x[K]\,h[-1-K] = 0$

$R_H: |z|>0 \qquad R_X: |z|>|\alpha| \qquad R_X \cap R_H: |z|>|\alpha|$

$R_Y: |z|>0$ expands

(8)




$y[n] = x[n] * h[n]$

$Y(Z) = X(Z)\,H(Z)$

$$\boxed{H(Z) = \frac{Y(Z)}{X(Z)}}$$

$R_Y \supset (R_X \cap R_H)$

→ For a LSI system, ratio of output & input z-transforms (→ provided they converge) is independent of the input.

→ $(H(Z), R_H) \longrightarrow$ system function of $\mathscr{S}$

eg. $e^{z^{-1}}$ ROC: $|z| > 0$ $\quad 0! = 1 \quad n! = n(n-1)! \;\forall\; n \geq 1$

$X(z) = \sum_{n=0}^{\infty} \frac{1}{n!} (z^{-n}) \Rightarrow x[n] = \frac{1}{n!} \cdot u[n]$

(Irrational z-transform) $\quad$ ↓ impulse response of a LSI system → irrational system as H(Z) is irrational

⇒ System function $= \frac{Y(Z)}{X(Z)} \longrightarrow$ in an LSI system (discrete)

↳ 'rational' system func. $= \frac{\text{finite length series in } Z}{\text{finite length series in } Z}$

eg.- $\frac{3z + 2 + 4z^{-1}}{7z + 5 + 2z^{-1}} \longrightarrow$ rational system

It can always be expressed as either

① $z^{D} \frac{P_1(Z)}{P_2(Z)}$ or $z^{K} \frac{P_1(z^{-1})}{P_2(z^{-1})}$

→ Inversion of Z-transform normally by inspection

↓

$\boxed{H(Z) \text{ is rational}} = z^{D} \frac{N(z^{-1})}{D(z^{-1})}$

$D(z^{-1}) = 0 \longrightarrow$ poles of $H(Z)$

$N(z^{-1}) = 0 \longrightarrow$ zeroes of $H(Z)$

# Properties of z-transform

① **Linearity :-**

$$x_{1,2}[n] \longrightarrow X_{1,2}[Z] \quad R_{1,2}$$

$$\alpha x_1[n] + \beta x_2[n] \xrightarrow{\mathcal{Z}} \alpha X_1(Z) + \beta X_2(Z) \qquad ROC \supseteq (R_1 \cap R_2)$$

$$\forall \alpha, \beta \in \mathbb{C}$$

②

$$x[n] \xrightarrow{\mathcal{Z}} X(Z), \; R_x$$

$$x[n-n_0] \xrightarrow{\mathcal{Z}} z^{-n_0} X(Z), \; \left(R_x \text{ but check for the boundaries}\right)$$

Indication of CAUSALITY

③

$$x[n] \longrightarrow X(Z), \; R_x$$

$$h[n] \longrightarrow H(Z), \; R_H$$

$$x * h[n] \longrightarrow X(Z)H(Z), \; ROC \supseteq (R_x \cap R_H)$$

④

$$X(Z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}, \; R_x$$

$$\frac{d(X(z))}{dz} = \sum_{n=-\infty}^{+\infty} (-n x[n]) z^{-n-1}, \; R_x \text{ but check boundaries}$$

$$\boxed{-z\frac{d(X(z))}{dz} = \sum_{n=-\infty}^{+\infty} (n x[n]) z^{-n}}$$

ROC: $R_x$, but check boundaries (0 radius, $\infty$ radius circles)

$$\boxed{n x[n] \xrightarrow{\mathcal{Z}} -z\left(\frac{dX(z)}{dz}\right)}$$

eg.- $x[n] = 2^n u[n]$

$$X(z) = \frac{1}{1-2z^{-1}}, \; |z| > 2$$

$$\frac{dX(z)}{dz} = +1\left(\frac{1}{1-2z^{-1}}\right)^2 \left(+2z^{-2}\right) = \frac{-2z^{-2}}{(1-2z^{-1})^2}$$

$$-z\frac{dX(z)}{dz} = \frac{+2z^{-1}}{(1-2z^{-1})^2} \quad (|z|>2) \rightarrow \text{(boundaries are preserved)}$$

so, $n \cdot 2^n u[n] \xrightarrow{\mathcal{Z}} \dfrac{2z^{-1}}{(1-2z^{-1})^2}$ → how to handle multiple poles

diff.

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## Inverting a Rational z-transform

shift forward by (-D)

$$z^{D}\,\frac{N(z^{-1})}{D(z^{-1})} \Rightarrow \frac{N(z^{-1})}{D(z^{-1})} \longrightarrow \text{long division} \Rightarrow Q(z^{-1}) + \frac{R(z^{-1})}{D(z^{-1})}$$

(polynomial in $z^{-1}$)

eg.- $Q(z^{-1}) = \frac{1}{2} + 3z^{-1} + 5z^{-2} \xrightarrow{\mathcal{Z}^{-1}} \frac{1}{2} \quad 3 \quad 5$

inverse z-transform of a finite-length polynomial in $z^{-1}$ → finite-length sequence.

$$\frac{R(z^{-1})}{D(z^{-1})} = \sum_{l=1}^{m} \frac{\text{polynomial in } z^{-1} \text{ of (deg.} < P_l)}{(1-\alpha_l z^{-1})^{P_l}}$$

($P_l$ : multiplicity of pole)

Partial fraction decomposition → seperate terms of each pole

each of the terms can be inverted using the properties.

→ look at multiplicity → differentiate that many times → multiply z each time → find the final expression.

each term gives you a polyex term in 'n'

exponential factors → poles

zero → manifest the contribution of the poles

zero → cancels the pole if it sits on it

zero far from pole ✓
zero near pole → suppress

$H(s) = \frac{N(s)}{D(s)}$ → plot $|H(s)|$ on the splane

tent is held up at the poles.

$Q(s) + \frac{R(s)}{D(s)} = \frac{N(s)}{D(s)}$

TENT ANALOGY

b/w the poles and zeroes of a rational fun$^c$, the complex f$^n$c must be varying everywhere

$$\frac{As+B}{(s+1)^2} \qquad \frac{1}{s+1} \xrightarrow{\text{derivative}} \frac{-1}{(s+1)^2}$$

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Bilateral Laplace Transform

<u>Causal system</u> :- $H(s) = \int_0^{\infty} e^{-st} \cdot h(t)\, dt$

~~H(s)~~ system function, $H(s) = \int_{-\infty}^{+\infty} e^{-st} h(t)\, dt$

causal system, $H(s) = \int_0^{\infty} h(t) e^{-st}\, dt$

$= \int_0^{+\infty} h(t) e^{-\Sigma t} \cdot e^{-j\Omega t}\, dt$

Integral must converge $\forall$ Re(s) >

if we had $\int_{-\Sigma_0}^{0} h(t) e^{-\Sigma t} e^{-j\Omega t}\, dt \longrightarrow$ never converge on $Re(s) \to +\infty$

$\boxed{Re(s) \to +\infty}$

indicative of causality or non-causality

$(z = re^{j\omega})$

$H(Z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n}$

Causal : $H(Z) = \sum_{n=0}^{\infty} h[n] r^{-n} e^{-j\omega n}$

Non-causal systems : we have $\sum_{-n_0}^{-1} h[n] z^{-n}$

does not converge AT ALL if $|z| \to \infty$.

$r \to +\infty$ always included (converges)

($|z| \to +\infty$ contour always included for causal systems.

$s = j\Omega$ (imag. axis) $\Rightarrow Re(s) = 0$

Causality $\longrightarrow$ check if ROC includes $+\infty$

stability ? $\longrightarrow$ if $\boxed{deg.(N) > deg.(D)}$

there is some differenciation involved, and

$\boxed{\text{DIFFERENTIATOR IS UNSTABLE}}$

$deg.(N) \le deg.(D) \longrightarrow$ polyex terms (L.o.l.)

one term can't cancel the other or force it to converge.

each of these exponentials MUST BE DECAYING i.e.

$\boxed{\text{location of poles :-}}$

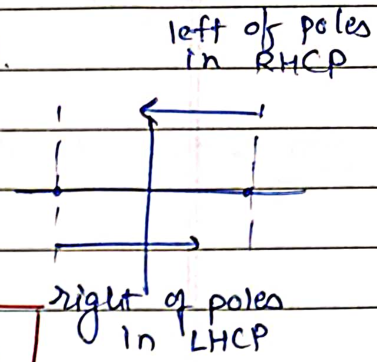
pole at $s_p$

RHCP $(Re(s_p) > 0)$

Sp on RHCP

$e^{2t}$ is decaying → left-sided exponential ($t<0$)

Sp on LHCP → $e^{-2t}$ → right-sided exponential.




pole on LHCP → contribute to RHS

pole on RHCP → contribute to LHS

↓

imaginary axis always included ⇒ If a pole on the IMAGINARY AXIS, ⇒ UNSTABLE

$$\boxed{\text{Stability} \iff \text{Imaginary Axis} \in \text{R.O.C}}$$

($|z|=1$ for discrete rationals)