MODULE-(II)

FOURIER TRANSFORMS

properties $\longrightarrow$ belong to the systems.

Transform :- mapping from one paradigm (world-view) to another.

RC-circuit $\rightarrow$ natural domain $\rightarrow$ Time

image $\longrightarrow$ space ; video $\rightarrow$ both Time & space

$\rightarrow$ mapping on top of the 'signals/system' mappings.

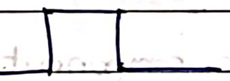
FOURIER :- Sinusoidal components.

$\rightarrow$ pseudo-periodicity is the way of nature. cyclic

$\Rightarrow$ Analytical functions :- differentiable under all orders.

$\rightarrow$ sinusoids are replicable under diff., integr., add^n, shift [CLOSED]

$\rightarrow$ RLC circuits $\longrightarrow$ sinusoid will have the same frequency all over it.



$\longrightarrow$ sinusoidal components $\longrightarrow$ $\boxed{\&}$ $\longrightarrow$ Effect

any periodic signal (L.S.I. systems) (L. combination of the components)

Periodic Signal

$$x(t+T) = x(t) \quad \forall t \quad (\text{period} = T)$$

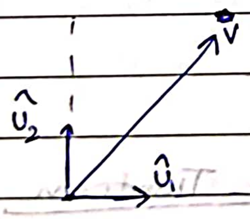
a L.C. of sinusoids which have the same period. $\rightarrow$ 0 frequency sine wave

Fourier series representation $\Rightarrow$ A_0, $A_1 \cos\left(\frac{2\pi}{T}t + \phi_1\right)$, $A_2 \cos\left(2\times \frac{2\pi}{T}t + \phi_2\right)$, $A_3 \cos\left(3\times\frac{2\pi}{T}t + \phi_3\right)$, ...

(fundamental)

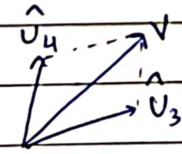
$A_1, \phi_1 \rightarrow$ determine

(periodic with submultiple of the period T)



$$\vec{V} = (\vec{V}\cdot\hat{u}_1)\cdot\hat{u}_1 + (\vec{V}\cdot\hat{u}_2)\cdot\hat{u}_2 \longrightarrow \text{decomposition.}$$

periodic signal $\Longrightarrow$ vector



complete $\|^{gm}$ $\longrightarrow$ we need to find the components along U_3 and U_4 in a "COUPLED" way.

$\perp^{ar}$ vectors $\longrightarrow$ dot product $= 0$.

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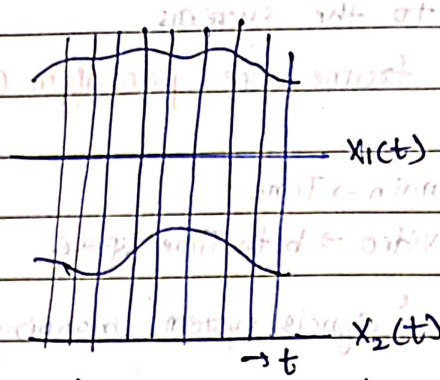
$\vec{V_1} = V_{11}\hat{U}_1 + V_{12}\hat{U}_2$ $\vec{V_2} = V_{21}\hat{U}_1 + V_{22}\hat{U}_2$

Inner Product (generalization of dot-product of

(signal is CONTINUOUS in this domain)



concatenate : SIGNAL

interval → small → signal is almost constant → associate a small pulse

Signal → vector with components as PULSES.

Inner product of x_1, x_2 ⇒ $\int x_1(t)\, x_2(t) \cdot dt$ (Real signals)

$\boxed{\langle x_1, x_2 \rangle = \int x_1(t)\, x_2(t) \cdot dt}$ (on the domain of the signals)

$\sum$ (Product of heights of pulses) × (pulse widths)

each points on the signal as a different component.

→ $\langle \alpha x_1 + \beta x_2, x_3 \rangle = \alpha \langle x_1, x_3 \rangle + \beta \langle x_2, x_3 \rangle$ [α, β: complex]

→ $\langle x_1, x_1 \rangle \geq 0$ and $\langle x_1, x_1 \rangle = 0$ iff $x_1 = 0$

$\boxed{\langle x_1, x_2 \rangle = \int x_1(t)\, \bar{x}_2(t)\, dt}$ (complex signals)

→ The V.S. of all signals which are periodic with period T → ∞-dimen (countable)

→ $\int_{(T)} A_m \cos\left(m \cdot \frac{2\pi}{T} t + \phi_m\right) \cdot A_n \cos\left(n \cdot \frac{2\pi}{T} t + \phi_n\right) \cdot dt$

(T) → over all continuous intervals of time

Transform → change of paradigm ⇒ Fourier Transform

$A_0 \cos(\Omega_0 t + \phi_0)$ → | Stable LSI system ($h(t)$) | →

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$h(t)$ is REAL

$$A_0 \cos(\Omega_0 t + \phi_0) = \frac{1}{2} A_0 e^{j(\Omega_0 t+\phi_0)} + \frac{1}{2} A_0 e^{-j(\Omega_0 t+\phi_0)}$$

$$= \frac{1}{2} A_0 e^{j(\Omega_0 t+\phi_0)} + \frac{1}{2} A_0 e^{-j(\Omega_0 t+\phi_0)}$$

$$\int_{-\infty}^{+\infty} h(\lambda) \cdot \frac{1}{2} A_0 e^{j(\Omega_0 (t-\lambda) + \phi_0)} d\lambda$$

$$= \frac{1}{2} A_0 e^{j(\Omega_0 t+\phi_0)} \int_{-\infty}^{\infty} h(\lambda) e^{-j(\Omega_0 \lambda)} d\lambda$$

Also, as

$$\left| \int_{-\infty}^{+\infty} h(\lambda) e^{-j\Omega_0 \lambda} d\lambda \right| \leq \int_{-\infty}^{\infty} \left| h(\lambda) e^{-j\Omega_0 \lambda} \right| d\lambda = \underbrace{\int_{-\infty}^{\infty} |h(\lambda)| \cdot d\lambda}_{\text{FINITE}}$$

$$\frac{1}{2} A_0 e^{j(\Omega_0 t+\phi_0)} \longrightarrow \boxed{\mathcal{S}} \longrightarrow H(\Omega_0) \cdot \frac{1}{2} A_0 e^{j(\Omega_0 t+\phi_0)}$$

where, $H(\Omega_0) = \int_{-\infty}^{+\infty} h(\lambda) e^{-j\Omega_0 \lambda} d\lambda$

with $\frac{1}{2} A_0 e^{-j(\Omega_0 t+\phi_0)} \longrightarrow \boxed{\mathcal{S}} \longrightarrow H(-\Omega_0) \cdot \frac{1}{2} A_0 e^{-j(\Omega_0 t+\phi_0)}$

$$H(-\Omega_0) = \int_{-\infty}^{\infty} h(\lambda) e^{j\Omega_0 \lambda} d\lambda$$

(complex conjugates) $\boxed{H(\Omega_0) = \overline{H(-\Omega_0)}}$

so, when the whole input is applied, output becomes →

$$\underbrace{\frac{1}{2} A_0 e^{j(\Omega_0 t+\phi_0)} H(\Omega_0)}_{z} + \underbrace{\frac{1}{2} A_0 e^{-j(\Omega_0 t+\phi_0)} \overline{H(\Omega_0)}}_{\bar{z}}$$

$$\frac{z+\bar{z}}{2} = \frac{2\,Re(z)}{2} = Re(z)$$

$$= Re\left\{ A_0 e^{j(\Omega_0 t+\phi_0)} \cdot |H(\Omega_0)| \cdot e^{j\measuredangle H(\Omega_0)} \right\}$$

$$= A_0 |H(\Omega_0)| \cos\{\Omega_0 t + \phi_0 + \measuredangle H(\Omega_0)\}$$ angle ($\tan^{-1}$)

sinusoid $\longrightarrow$ $\boxed{S}$ $\longrightarrow$ $\Rightarrow$ sinusoid with same freq
stable, real impulse response
$\Rightarrow$ Amplitude $\times |H(\Omega_0)|$
$\Rightarrow$ phase of $H(\Omega_0)$ is added

$$H(\Omega_0) = \int_{-\infty}^{+\infty} h(\lambda)\, e^{-j\Omega_0 \lambda}\, d\lambda$$

periodic signal : $p(t) = p(t+T) \;\; \forall t \rightarrow$ periodic signal of T.

all sinusoids with $\omega = K \cdot \left(\frac{2\pi}{T}\right)$ have the same period.

BASIS :→ $\cos\left(\frac{2\pi}{T} \cdot Kt\right)$ $\quad K = 1, 2, \ldots \rightarrow$ (harmonics)

Are the components $\perp^{ar}$?

$$\frac{1}{2} \int_{(T)} 2\cos\left(\frac{2\pi}{T} Kt\right) \cos\left(\frac{2\pi}{T} lt\right) \cdot dt$$

over any continuous interval T

$$= \frac{1}{2} \left[\int_{(T)} \left[\cos\left(\frac{2\pi}{T}(K+l)t\right) + \cos\left(\frac{2\pi}{T}(K-l)t\right) \right] \cdot dt \right]$$

$$= \frac{1}{2} \left[\frac{\left(\sin\left(\frac{2\pi}{T}(K+l)t\right)\right)_0^T}{\cdots} + \frac{\left(\sin\left(\frac{2\pi}{T}(K-l)t\right)\right)_0^T}{\cdots} \right]$$

orthogonal ← $\boxed{K \neq l \rightarrow 0}$

$$K = l \Rightarrow \frac{1}{2}\left[+ \cancel{\sin\left(\frac{2\pi}{T}(K+l)t\right)} \; (t)_0^T \right]$$

(modulus) ← $\boxed{K = l \Rightarrow +\frac{T}{2}}$

$\boxed{K = l = 0 \Rightarrow T}$

Fourier series representation :→ For $p(t)$, assuming $p(t)$ is real

$$p(t) = \sum_{K=1}^{\infty} A_K \cos\left(\frac{2\pi}{T} \cdot Kt + \phi_K\right) + A_0$$

$$A_0 = \frac{1}{T} \int_{(T)} p(t) \cdot dt \quad \text{[scribbled out]}$$

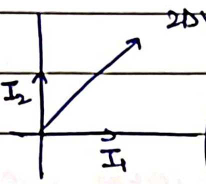
⇒ At any particular frequency, we have two components → f^n ⇒ sine/cosine.

→ We also have, $\sin\left(\frac{2\pi}{T}Kt\right)$ is $\perp^{ar}$ to itself for $K \neq \lambda$ and also to cosines of that frequency.

→

$$A_K \cos\left(\frac{2\pi}{T}Kt + \phi_K\right) = \left\{\frac{2}{T}\int_{(T)} p(\lambda)\cos\left(\frac{2\pi}{T}K\lambda\right)d\lambda\right\}\cos\left(\frac{2\pi}{T}\right)Kt$$
$$+ \left\{\frac{2}{T}\int_{(T)} p(\lambda)\sin\left(\frac{2\pi}{T}K\lambda\right)d\lambda\right\}\sin\left(\frac{2\pi}{T}\right)Kt$$

$\left|\frac{2}{T}\right|$



(LHS → U.V. → divide by $\sqrt{T/2}$
also on RHS → I_1 & I_2 are U.V. → divide by $\sqrt{T/2}$)

Periodic ⟶ non-periodic ; $(T \to \infty)$

as $T \to \infty$ discrete frequency axis ⟶ continuous frequency axis.

$$p(t) = A_0 + \frac{2}{T}\sum_{K=1}^{\infty}\left\{\int_{(T)} p(t)\cos\left(\frac{2\pi}{T}Kt\right)\cdot dt\right\}\cos\left(\frac{2\pi Kt}{T}\right)$$
$$+ \frac{2}{T}\sum_{K=1}^{\infty}\left\{\int_{(T)} p(t)\sin\left(\frac{2\pi}{T}Kt\right)\cdot dt\right\}\sin\left(\frac{2\pi}{T}\cdot Kt\right)$$

$\int_{(T)} p(t)\cdot dt$

⇒ (components of $p(t)$ along diff. directions) × (unit vector along that direction)

PERIODIC ⟶ NON-PERIODIC

$T \to \Omega$

instead of discrete points come closer & closer ⇒ occupy continuum of the axis.

Also, we assumed,

$p(t) = p(t+T)$ ⟶ Periodic, $\boxed{\text{REAL}}$
$\forall t$

$$A_0 + \sum_{k=1}^{\infty} A_K \cos\left(\frac{2\pi}{T}Kt + \phi_K\right)$$

$$\longrightarrow \frac{1}{2}A_K e^{j\left(\frac{2\pi}{T}Kt + \phi_K\right)} + \frac{1}{2}A_K e^{-j\left(\frac{2\pi}{T}Kt+\phi_K\right)}$$

(Two complex $f^n c$ of T)

as we change T $\longrightarrow$ the complex vector rotates with an angular frequency. $\left(\frac{2\pi}{T}\cdot K\right)$

Term1 > $t=0 \rightarrow \phi_K \longrightarrow$ rotate A.C.W.
Term2 > $t=0 \rightarrow -\phi_K \longrightarrow$ rotate CW

Two oppositely rotating complex no. of same amplitude, with equal & opposite starting phases

if we have complex $p(t)$, we need the start phases to not be equal & opposite.

NEW BASIS

$$e^{j\left(\frac{2\pi}{T}\cdot Kt\right)}, \; e^{-j\left(\frac{2\pi}{T}\cdot Kt\right)} \longrightarrow \text{both are periodic with period } T.$$

(inner product) $\langle x_1, x_2 \rangle = \int x_1 \cdot \overline{x_2}\, d\lambda$

For the basis $f^n c$ $\Rightarrow$

$$\int_{(T)} e^{j\left(\frac{2\pi}{T}Kt\right)} \cdot \overline{e^{-j\left(\frac{2\pi}{T}Kt\right)}}\, dt = \int_{(T)} e^{j\left(\frac{2\pi}{T}Kt + \frac{2\pi}{T}Kt\right)}$$

$$= \int e^{j\left(\frac{4\pi}{T}\cdot Kt\right)} = \frac{\left[e^{j\left(\frac{4\pi}{T}Kt\right)}\right]_0^T}{j\left(\frac{4\pi K}{T}\right)}$$

$$= \frac{e^{j(4\pi K)} - e^{j(0)}}{j\left(\frac{4\pi}{T}\cdot K\right)} = \frac{(1)-(1)}{j\left(\frac{4\pi}{T}\cdot K\right)} = \boxed{0}$$

The bases are $\perp^{ar}$

So $e^{j\Omega t}$, $e^{-j\Omega t}$ are $\perp^{ar}$ w.r.t. one another over an interval of $(2\pi/\Omega)$ $\Longrightarrow$ $\perp^{ar}$-ity over an interval of $K(2\pi/\Omega)$

as we start taking $T \to \infty$ $(\Omega \to 0)$

↳ 1 all components when we take the entire time-axis.

$$\cdots \quad -\frac{2\pi}{T} \quad 0 \quad \frac{2\pi}{T} \quad 2\cdot\frac{2\pi}{T} \quad \cdots \longrightarrow \Omega$$

$-\frac{2\pi}{T}$ ↓ place a complex no. rotating in CW dirⁿ

$\frac{2\pi}{T}$ ↓ place a complex no. rotating in ACW dirⁿ

⇓

PHASOR representation of signals (as $T \to \infty$ entire Ω-axis is occupied.)

$$A_0 \cos(\Omega_0 t + \phi_0) \longrightarrow \boxed{\text{LSI system}} \longrightarrow |H(\Omega_0)|\, A_0 \{\cos(\Omega_0 t + \phi_0 + \measuredangle H(\Omega_0))\}$$

$h(t)$: real

$$A_0 e^{j(\Omega_0 t + \phi_0)} \longrightarrow \boxed{\text{LSI system}} \longrightarrow \boxed{H(j\Omega_0) \cdot A_0 e^{j(\Omega_0 t + \phi_0)}}$$

$(A_0 > 0)$ $h(t)$: complex

multiplied by complex constants

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, A_0\, e^{j(\Omega_0 (t-\tau) + \phi_0)} \cdot d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, A_0\, e^{j\Omega_0 t} \cdot e^{-j\Omega_0 \tau} \cdot e^{j\phi_0}\, d\tau$$

$$= A_0 e^{j(\Omega_0 t + \phi_0)} \underbrace{\int_{-\infty}^{+\infty} h(\tau)\, e^{-j\Omega_0 \tau}\, d\tau}_{H(j\Omega_0)}$$

⇒ $A_0 e^{j(\Omega_0 t + \phi_0)}$ ⟶ comes out unchanged in form, multiplied by a complex constt.

$$H(j\Omega_0) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\Omega_0 \tau}\, d\tau = \langle h(t), e^{j\Omega_0 t} \rangle$$

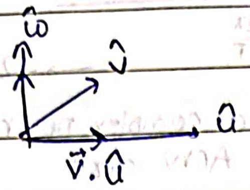
inner product of impulse response with $(e^{j\Omega_0 t})$. ⟶ assume $h(t)$ is aperiodic & take $T \to \infty$

Fourier Transform of $h(t)$ $\quad h(t) \xrightarrow{\text{F.T.}} H(j\Omega) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\Omega_0 \tau} \cdot d\tau$

angular frequency

any signal : $h(t) \xrightarrow{F.T.} H(j\Omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\Omega\tau} d\tau$

To reconstruct $h(t)$ from $H(j\Omega)$, we need all Ω over all Re



$\vec{V} = (\vec{V}\cdot\hat{u}) + (\vec{V}\cdot\hat{\omega})$

normalization

$K_0 \int_{-\infty}^{+\infty} H(j\Omega) \{ e^{j\Omega t} \} d\Omega$

(constant) → Turns out to be independent of Ω.

$K_0 = \frac{1}{2\pi}$

So, $\boxed{h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(j\Omega) e^{j\Omega t} d\Omega}$ (Inverse F.T.)

reconstruct $h(t)$ from each of the rotating comple

Fourier Transform

For any signal $x(t)$

(both of them have diff. independent variables)

$x(t) \xrightarrow{\mathcal{F}} X(j\Omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\Omega t} dt$

$X(j\Omega) \xrightarrow[\mathcal{F}^{-1}]{\text{inverse F.T.}} x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega) e^{j\Omega t} d\Omega$

⇒ signal & its F.T. are a pair.
 └ I.D. variable ⟶ dependent variable

System : signal ⟶ signal on same kind of I.D. variable

Transform : change of paradigm.

PROPERTIES OF FOURIER-TRANSFORMS → prescribed conseq transform when c oper^n are done on signal/ signals.

Transform property

oper^n on signal/ combin^n of signal ⟶ consequence on transform = ?

① Linear Transform

$ax_1(t) + bx_2(t) \longrightarrow aX_1(j\Omega) + b(X_2(j\Omega))$

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(2) **Shifting**

$$x(t-T_0) \xrightarrow{F} \int_{-\infty}^{\infty} x(t-T_0)\, e^{-j\Omega (t)}\, dt$$

$t - T_0 = m$

$$= \int_{-\infty}^{\infty} x(m)\, e^{-j\Omega(m+T_0)}\, dm$$

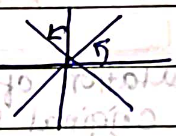
$$\boxed{x(t-t_0) \longrightarrow e^{-j\Omega t_0}\, X(j\Omega)}$$

change in phase | change the phase by $e^{-j\Omega t_0}$ | shows the change in phase from the starting phase.

rotating complex no. $\longrightarrow$ change in amplitude & phase
$(|H(j\Omega)|)$ $(+\angle H(j\Omega))$

$e^{j\Omega t}$ $(X(j\Omega))$ $\cdot e^{-j\Omega t_0}$ $\longrightarrow$ linear phase change $\Rightarrow$ all rotating complex no.'s are shifted by the same time.

(If not linear $\Rightarrow$ dispersion)



(3) **DUALITY**

$$x(t) \xrightarrow{F} X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$X(jt) \xrightarrow{F} \int_{-\infty}^{\infty} X(jt)\, e^{-j\Omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} X(j\Omega)\, e^{+j(-t)\Omega}\, d\Omega$$

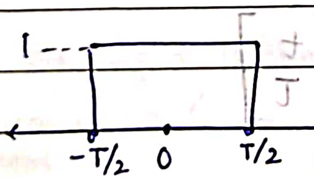
(rearrange the integral)

$$= 2\pi\, x(-\Omega)$$

$$\boxed{X(jt) \xrightarrow{F} 2\pi\, x(-\Omega)}$$

eg.

$x(t) =$

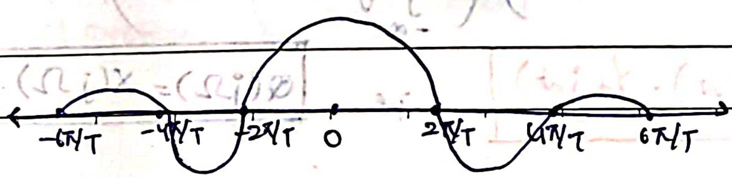


$$X(j\Omega) = \int_{-T/2}^{T/2} 1 \cdot e^{-j\Omega t}\, dt = \left[\frac{e^{-j\Omega t}}{(-j\Omega)}\right]_{-T/2}^{T/2}$$

$$= \frac{-e^{-j\Omega T/2} + e^{j\Omega T/2}}{(j\Omega)}$$

$$= \frac{2\cancel{j} \sin(\Omega T/2)}{\cancel{j}\Omega} = \frac{T \sin(\Omega T/2)}{(\Omega T/2)}$$

$$x(t) \xrightarrow{F} \boxed{\frac{T \sin(\Omega T/2)}{(\Omega T/2)}}$$



(sinc function)

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$\Omega = 2\pi f$

$\text{sinc}\,\lambda = \frac{\sin \pi\lambda}{\pi\lambda}$ so, $T \frac{\sin(\pi f T/2)}{(\pi f T/2)}$

$$\boxed{\mathcal{F}(x(t)) = T\,\text{sinc}(fT)}$$

$\lim_{\lambda \to 0} \text{sinc}\,\lambda = 1$ and $\text{sinc}\,\lambda = 0$ for all integers λ

$$\mathcal{F}(\text{rect. pulse}) \longrightarrow T\,\text{sinc}(fT)$$

$$\mathcal{F}(T\,\text{sinc}(fT)) \longrightarrow \text{rect. pulse}$$

Here, $x(t) = x(-t) \longrightarrow$ doesn't matter in duality

④ Duality of Time shift property

$$\boxed{e^{j\Omega_0 t} x(t)} \xrightarrow{\mathcal{F}} X(j(\Omega - \Omega_0))$$

(modulation of the original signal) ⟶ (shift on the Fourier axis)

⑤ Convolution

$x(t) * y(t) = q(t)$ (assume $\mathcal{F}(x(t))$ and $\mathcal{F}(y(t))$ and $\mathcal{F}(x*y)$ e

$$Q(j\omega) = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} x(\tau) y(t-\tau)\, d\tau \right) . e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} x(\tau) y(t-\tau) d\tau \right) e^{-j\omega t} . dt$$

Let $\lambda = t - \tau$

$\tau = \tau$

$$\begin{bmatrix} \lambda \\ \tau \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ \tau \end{bmatrix}$$

$|J| = 1$

so, $d\lambda\, d\tau = dt\, d\tau$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(\tau)\, y(\lambda)\, e^{-j\omega(\tau+\lambda)}\, d\tau\, d\lambda$$

$$Q(j\omega) = \left(\int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau}\, d\tau \right) \left(\int_{-\infty}^{+\infty} y(\lambda) e^{-j\omega\lambda}\, d\lambda \right)$$

$$\boxed{Q(j\omega) = X(j\omega) . Y(j\omega)}$$ i.e. $$\boxed{Q(j\Omega) = X(j\Omega) . Y(j\Omega)}$$

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convolve two signals $\longrightarrow$ Fourier transforms are multiplied.

$$x(t) \rightarrow \boxed{\begin{matrix} \mathcal{S} \\ \text{LSI} \end{matrix}}_{h(t)} \longrightarrow y(t)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega \longrightarrow$$ (linear combination of many rotating unit vectors with diff. Ω)

$$x(t) = e^{j\Omega t} \longrightarrow \boxed{\mathcal{S}} \longrightarrow H(j\Omega)\, e^{j\Omega t} \quad ; \quad H(j\Omega) = \int_{-\infty}^{+\infty} h(t)\, e^{-j\Omega t}\, dt$$

$\downarrow$ Fourier transform of $h(t)$

(component of $h(t)$ along the dir^n of rotation vector)

Homogenity) $X(j\Omega)\, e^{j\Omega t} \longrightarrow \boxed{\mathcal{S}} \rightarrow X(j\Omega) \cdot H(j\Omega) e^{j\Omega t}$

Additivity) $\frac{1}{2\pi}\int X(j\Omega) e^{j\Omega t} d\Omega \longrightarrow \boxed{\mathcal{S}} \rightarrow y(t) \Rightarrow \frac{1}{2\pi}\int X(j\Omega) H(j\Omega) e^{j\Omega t}\, d\Omega$

$$y(t) = \frac{1}{2\pi}\int \underbrace{\left(X(j\Omega) \cdot H(j\Omega)\right)}_{\text{IFT of } X(j\Omega) \cdot H(j\Omega)} \cdot e^{j\Omega t}\, d\Omega$$

so, $\boxed{\mathcal{F}\{y(t)\} = X(j\Omega) H(j\Omega)}$

eg.- $x(t) = e^{2t} \cdot u(t) \longrightarrow$ F.T. does NOT exist

$$X(j\Omega) = \int_0^{\infty} e^{2t} \cdot e^{-j\Omega t}\, dt = \int_0^{\infty} e^{(2-j\Omega)t} \cdot dt \longrightarrow \underline{\text{DOES NOT converge}}$$

$\Rightarrow$ $e^{j\Omega t} \rightarrow \boxed{\begin{matrix}\mathcal{S} \\ \text{LSI} \\ h(t)\end{matrix}} \longrightarrow [H(j\Omega)] e^{j\Omega t} \quad ; \quad H(j\Omega) = \int_{-\infty}^{+\infty} h(t)\, e^{-j\Omega t}\, dt$

→ Form of the signal is unchanged, only its parameters are changed.

→ rotating complex no. / PHASOR $\longrightarrow$ Eigenfunction of the LSI system $\longrightarrow$ ∞-dimensional V.S.

$x(t) \to$ vector
$e^{j\Omega t} \to$ E.fcn
$H(j\Omega) \to$ E.Value

↳ $Av = \lambda v$

if we have $A = A^T$ (symmetric) $\longrightarrow$ eigenvalues are real.

⇒ "eigen" $\longrightarrow$ your own

⇒ eigenproperty of LSI systems $\rightarrow$ (phasor analysis of LSI systems)

⑥ **Differenciation**

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$\frac{d\,x(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (j\Omega)\, X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

[interchange of integral and differential is admissable.]

$$\left(\frac{dx(t)}{dt}\right) = \frac{1}{2\pi}\int \left(j\Omega \cdot X(j\Omega)\right) e^{j\Omega t}\, d\Omega$$

$$\frac{dx(t)}{dt} \xrightarrow{F} \boxed{(j\Omega)X(j\Omega)}$$

For $e^{j\Omega t} \longrightarrow \frac{dx(t)}{dt} = (j\Omega)\, e^{j\Omega t} \longrightarrow$ each phasor is multiplied by $(j\Omega)$ and it is a LSI system.

⇒ For a differentiator, $y(t) = \frac{dx(t)}{dt} \longrightarrow h(t) = (j\Omega)$

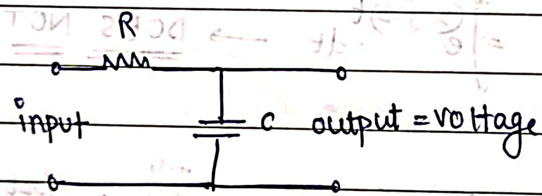
LSI
$h(t)$

$h(t)$ has a Fourier transform

$H(j\Omega) = F\{h(t)\}$ ⇒ Frequency response of the system

(how a system responds to a frequency of Ω ($e^{j\Omega t}$))

e.g.-



impulse response, $h(t) = \frac{1}{\tau_0} e^{-t/\tau_0} \cdot u(t)$ $\quad \tau_0 = RC$.

$$H(j\Omega) = \int_0^{\infty} \frac{1}{RC} e^{-t/RC} e^{-j\Omega t} \cdot dt$$

$$= \frac{1}{RC}\left[\int_0^{\infty} e^{\left(-\frac{1}{RC} - j\Omega\right)t} \cdot dt\right]$$

$$= \frac{1}{RC} \cdot \left[\frac{1}{\frac{1}{RC} + j\Omega}\right] = \frac{1/RC}{(1/RC) + j} = \frac{1}{1 + (jRC)\Omega}$$

$$\boxed{H(j\Omega) = \frac{1}{1 + j\Omega\tau_0}}$$

$V_{cap} = \dfrac{1/j\omega C}{R + 1/j\omega C}$ (complex impedance)

$V_{cap} = \dfrac{1}{1+j\omega CR} = \dfrac{1}{1+j\omega\tau_0}$ → same as earlier

⇒ If a system is LSI + BIBO stable ⇒ frequency response exists.
i.e. $h(t)$ has a Fourier transform

→ LSI, not BIBO stable ⟶ frequency response ?

⇒ If $h(t)$ is real

$A_0 \cos(\Omega_0 t + \phi_0) \longrightarrow \boxed{S}$ (LSI, $h(t)$) $\longrightarrow |H(j\Omega_0)|\, A_0 \cos(\Omega_0 t + \phi_0 + \measuredangle H(j\Omega_0))$

adds to phase

$A_0 e^{j(\Omega_0 t+\phi_0)} + A_0 e^{-j(\Omega_0 t+\phi_0)}$

two oppositely rotating complex vectors.

if $h(t)$ is not real: both components face diff. changes in amplitude/phase ⟶ $y(t)$ is no longer a sinusoid.

→ if $h(t)$ is real ⇒

$$H(j\Omega) = \int_{-\infty}^{+\infty} h(t)\, e^{-j\Omega t}\, dt \qquad \boxed{H(j\Omega) = \overline{H(-j\Omega)}}$$

$$H(-j\Omega) = \int_{-\infty}^{+\infty} h(t)\, e^{j\Omega t}\, dt$$

$$\overline{H(-j\Omega)} = \int_{-\infty}^{+\infty} \overline{h(t)}\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} h(t)\, e^{-j\Omega t} = H(j\Omega)$$

$\boxed{H(j\Omega) = \overline{H(-j\Omega)}}$ → magnitude at Ω and $(-\Omega)$ is same
→ phase is opposite (rotating complex no.)

Properties

① $x(t) \xrightarrow{F} X(j\Omega)$ $\qquad \displaystyle\int_{-\infty}^{+\infty} x(-t)\, e^{-j\Omega t}\, dt$

② $x(-t) \xrightarrow{F} X(-j\Omega)$

③ $(x(t))^* \xrightarrow{F} \overline{X(-j\Omega)}$ $\qquad \displaystyle\int x(\tau)\, e^{j\Omega t}\, dt$

time reversed $x(t)$ → each rotating complex no. is also time reversed → $X(-j\Omega)$.

$$\int x^*(\tau)\, e^{-j\Omega t}\, dt = \overline{\int x(t)\, e^{j\Omega t}} = \overline{X(-j\Omega)}$$

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(4) $x_{1,2}(t) \xrightarrow{\mathcal{F}} X_{1,2}(j\Omega)$

$x_1(t)\cdot x_2(t) \xrightarrow{\mathcal{F}}$

$$\int_{-\infty}^{+\infty} x_1(t)\, x_2(t)\, e^{-j\Omega t}\, dt \qquad x_2(t) = \frac{1}{2\pi}\int X_2(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$= \int_{-\infty}^{+\infty} x_1(t)\cdot \frac{1}{2\pi}\left(\int_{-\infty}^{+\infty} X_2(j\Omega_1)\, e^{j\Omega_1 t}\, d\Omega_1\right)\cdot e^{-j\Omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1(t)\, X_2(j\Omega_1)\, e^{j\Omega_1 t}\, e^{-j\Omega t}\, d\Omega_1\, dt$$

(interchange the two integrals)

$$\cancel{= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_2(j\Omega_1)\cdot\left(\int_{-\infty}^{+\infty} x_1(t)\, e^{j\Omega t}\, dt\right)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_2(j\Omega_1)\left\{\underbrace{\int_{-\infty}^{+\infty} x_1(t)\, e^{-j(\Omega-\Omega_1)t}\, dt}_{X_1(j(\Omega-\Omega_1))}\right\} d\Omega_1$$

So, $$\mathcal{F}(x_1(t)\cdot x_2(t)) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_2(j\Omega_1)\cdot X_1(j(\Omega-\Omega_1))\, d\Omega_1$$

if we use $\Omega_1 = 2\pi f_1$ → Hz frequency
angular frequency

$$\boxed{x_1(t)\, x_2(t) \xrightarrow{\mathcal{F}} \frac{1}{2\pi}\left[X_1(j\Omega) * X_2(j\Omega)\right]}$$

Parseval's Theorem

$x_1(t)$, $\overline{x_2(t)}$, $\Omega = 0$

$$\int_{-\infty}^{+\infty} x_1(t)\, \overline{x_2}(t)\, e^{-j\Omega t}\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_1(j\Omega_1)\, \overline{X_2(-(-j\Omega_1))}\cdot d\Omega_1$$

$\Omega = 0$ gives us $\langle x_1, x_2\rangle$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_1(j\Omega_1)\, \overline{X_2(j\Omega_1)}\cdot d\Omega_1$$

$$\frac{1}{2\pi}\langle X_1, X_2\rangle$$ (Inner product of the fourier transforms in Ω domain.)

$$\boxed{\text{inner product in time-domain} = \text{inner product in frequency domain}}$$

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⇒ signals have finite energy ⟶ the Fourier transform necessarily exists

↓

x_1, x_2 ⟶ parsevals theorem must be true

⇒ Energy of a signal = $\langle x_1, x_1 \rangle$ ⇒ inner product of x_1 with itself

$$\boxed{E = \int |x_1(t)|^2 \cdot dt}$$

eg.- $x(t) = V(t)$ across a 1 ohm resistance → $E = \int |x(t)|^2 \cdot dt$

$x(t) = i(t)$ across a 1 ohm resistance → $E = \int |x(t)|^2 \cdot dt$

$\boxed{\text{finite energy signals} \longrightarrow \text{F.Transform exists}}$

Parseval's Theorem :-
$$\boxed{\int_{-\infty}^{+\infty} |x(t)|^2 \cdot dt = \frac{1}{2\pi} \int |X(j\Omega)|^2 \cdot d\Omega}$$

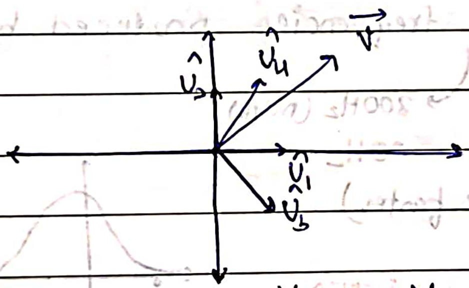
Energy in time and frequency-domain is the same (factor of 2π)

→ for a finite energy signal, $\int_{-\infty}^{+\infty} |X(j\Omega)|^2 \cdot d\Omega$ → energy spectral density

(spectrum = Fourier Transform)

$\int |x(t)|^2 \, dt$ → energy time density

$\boxed{V_i = \langle v, u_i \rangle}$



$\vec{V} = V_1 \hat{U}_1 + V_2 \hat{U}_2$

$\vec{V} = V_3 \hat{U}_3 + V_4 \hat{U}_4$

$\vec{W} =$ " "

$\langle v, w \rangle = V_1 W_1 + V_2 W_2 = V_3 W_3 + V_4 W_4$ (inner product is independent of basis)