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Questions for additivity, homogenity & shift-invariance

e.g.① $y(t) = Re\{x(t)\}$

Additivity :-

$y(t_1) = Re\{x(t_1)\}$
$y(t_2) = Re\{x(t_2)\}$

$Re\{x(t_1) + x(t_2)\} = Re\{x(t_1)\} + Re\{x(t_2)\}$

$y(t_1) + y(t_2) = Re\{x(t_1) + x(t_2)\}$ $\Rightarrow$ Additivity $\boxed{\checkmark}$

Homogenity :-

$Re\{x(t)\} = y(t)$

ascuming 'c' is complex, $\Rightarrow$ Homogenity $\boxed{\times}$

$Re\{c\,x(t)\} \neq c\,y(t)$

Shiftinvariance :-

$y(t-t_0) = Re\{x(t-t_0)\}$ $\Rightarrow$ Shift-invariant $\boxed{\checkmark}$

To make them shift variant :-

add a time $f^n c$

eg.② - $y(t) = \begin{cases} \dfrac{x(t)\,x(t-1)}{x(t-2)} & ; x(t-2) \neq 0 \\ 0 & ; x(t-2) = 0 \end{cases}$

$y(t_1) = \begin{cases} \dfrac{x(t_1)\,x(t_1-1)}{x(t_1-2)} & ; x(t_1-2) \neq 0 \\ 0 & ; x(t_1-2) = 0 \end{cases}$

→ Not Additive

→ Homogenous $\dfrac{(c\,x(t))(\cancel{c}\cdot x(t-1))}{(\cancel{c}\cdot x(t-2))}$ $[c \neq 0]$

as $c\,x(t-2) = 0$

$\Rightarrow x(t-2) = 0$ if $c \neq 0$ $c=0 \rightarrow y(t) = 0 \Rightarrow$ Homogenity $\boxed{\checkmark}$

→ Shift invariancy:- $\boxed{Yes}$

Complex eponential & sinusoidal signals

$x(t) = Ce^{at}$

$\Rightarrow$ If a is constrained to be purely imaginary,

$x(t) = e^{jw_0 t}$

For periodicity, $e^{jw_0 T} = 1$

if $w_0 = 0 \Rightarrow x(t) = 1 \rightarrow$ Periodic ✓

$w_0 \neq 0 \Rightarrow w_0 T = 2\pi \Rightarrow T = \dfrac{2\pi}{|w_0|}$ (period)

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sinusoidal signal : $x(t) = A\cos(\omega_0 t + \phi)$; $\boxed{\omega_0 = 2\pi f_0}$ ↳ hertz (hz)

fundamental period, $T_0 = 2\pi/\omega_0$

⇒ Euler's relⁿ : $e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$

⇒ $A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi} e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi} e^{-j\omega_0 t}$ (complex amplitudes)

→ These signals have infinite total energy but finite average power.

$$E_{period} = \int_0^{T_0} |e^{j\omega_0 t}|^2 \cdot dt = \int_0^{T_0} 1 \cdot dt = T_0$$

$$P_{period} = \frac{E_{period}}{T_0} = 1 \quad \Rightarrow P_{avg.} = 1$$

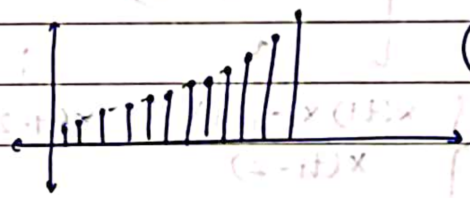
$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |e^{j\omega_0 t}|^2 \cdot dt = 1$$

Discrete-time

$\boxed{x[n] = C\alpha^n}$ C, α are complex no.'s.

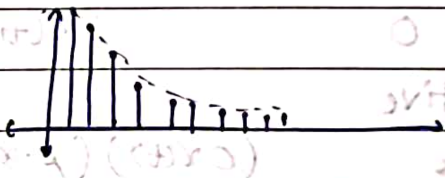
$x[n] = Ce^{\beta n}$, $\alpha = e^{\beta}$

$C, \alpha \in \mathbb{R}$ ⇒ $|\alpha| > 1$



(increases exponentially)

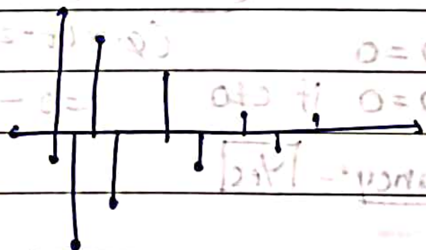
$|\alpha| < 1$ and $\alpha > 0$



(decreases exponentially)

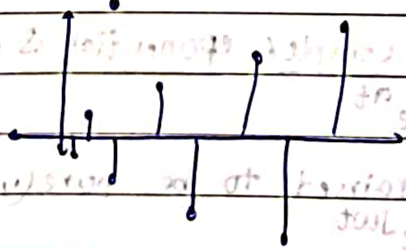
~~α>0~~ ~~cosca~~

$-1 < \alpha < 0$



(sign alternates)

$\alpha < -1$



(sign alternates)

$\alpha = 1$ → constant $\alpha = -1$ → $+C$ and $-C$

System Properties

⇒ If a system is either additive or homogenous, it has the property that if the input is identically zero, then the output is also identically zero.

⇒ $y(t) = (x(t))^2 \longrightarrow$ neither A nor H, but zero output for 0 input
$y(t) = \sin(x(t))$

e.g.1) – $y[n] = x[n]\,\{g[n] + g[n-1]\}$

i) $g[n] = 1$ for all n

Then, $g[n-1] = 1 \Rightarrow y[n] = x[n]\{2\} \Rightarrow y[n] = 2x[n]$

The system is T.I.

ii) $g[n] = n$

$g[n-1] = n-1 \qquad y[n] = (2n-1)[x[n]] \Rightarrow$ T.V.

iii) $g[n] = 1 + (-1)^n$

$1 + (-1)^n + 1 + (-1)^{n-1} = 2 \longrightarrow$ T.I.

eg.2) – $x[n] \longrightarrow \boxed{S\text{-}1} \longrightarrow \boxed{S\text{-}2} \longrightarrow \boxed{S\text{-}3}$

S-1: $y[n] = \begin{cases} x[n/2] & ; n = \text{even} \\ 0 & ; n = \text{odd} \end{cases}$ S-2: $y[n] = x[n] + \frac{1}{2}x[n-1] + \frac{1}{4}x[n-2]$ S-3: $y[n] = x[2n]$

is the net system linear, T.I. ?

$$y[n] = \begin{cases} x[n/2] + x[n] + \frac{1}{2}x[n-1] + \frac{1}{4}x[n-2] + x[2n] & ; n = 2m \\ x[n] + \frac{1}{2}x[n-1] + \frac{1}{4}x[n-2] + x[2n] & ; n = 2m+1 \end{cases}$$

additivity ⇒ ✓ homogenity ⇒ ✓ ~~linear~~

$y_1[n] = x_1[n/2]$ (n=even) $\qquad y_2[n] = 0$ (n=odd)

even + odd no. → odd ⇒ $y_1[n] + y_2[n]$ ✓

eg.3) $y(n) = \begin{cases} x(n) & ; n = 2m \\ x(n-1) & ; n = 2m+1 \end{cases}$

$y_1(n) = \begin{cases} x_1(n) \\ x_1(n-1) \end{cases} \qquad y_2(n) = \begin{cases} x_2(n) \\ x_2(n-1) \end{cases}$

$y_3(n) = \begin{cases} x_1(n) + x_2(n) \\ x_1(n-1) + x_2(n-1) \end{cases} = y_1(n) + y_2(n) \rightarrow$ linear ✓

homogenous ✓

T.I. ✓

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eg.4> $y(n) = \max.\{x(n), x(n+1)\}$

$y_1(n) = \max.\{x_1(n), x_1(n+1)\}$ $\quad y_2(n) = \max.\{x_2(n), x_2(n+1)\}$

$\max.\{x_1(n)+x_2(n),\ x_1(n+1)+x_2(n+1)\}$

$\neq \max.\{x_1(n), x_1(n+1)\} + \max.\{x_2(n), x_2(n+1)\}$

Additive × Homogenous ✓ T.I. $\boxed{✓}$ always (i.e. $\forall n$)

⇒ if the input to a linear system is 0 b/w t_1 and t_2 in continuous time or b/w n_1 and n_2 in discrete time, then its output must also be zero in that time?

$\boxed{\text{NO}}$

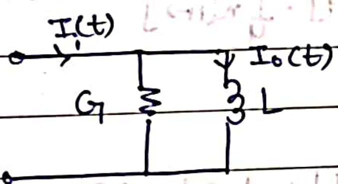
The case where output is a delayed version of the input

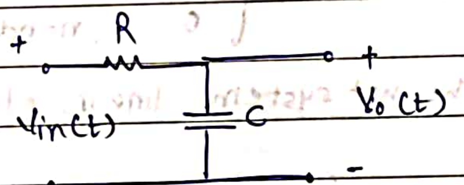
e.g. – $\boxed{y(t) = x(t-1)}$

if $x(t) = \begin{cases} 0 & 3<t<4 \\ 1 & \text{else} \end{cases}$ $\quad y(t) = \begin{cases} 0 & 4<t<5 \\ 1 & \text{else.} \end{cases}$

R–L circuits

parallel





$I_i(t) = I_o(t) + GL\dfrac{dI_o(t)}{dt}$ ←similarity→ $V_o(t) + RC\dfrac{dV_o(t)}{dt} = V_i(t)$

Discrete time convolution

e.g. $x[n] = a^n u[n]$ $\quad h[n] = \begin{cases} 1 & N_1 \le n \le N_2 \\ 0 & \text{otherwise} \end{cases}$; $x[n] * h[n]$.

⇒ $y[n] = x[n] * h[n] = \sum_{K\in\mathbb{Z}} x[K]\,h[n-K]$

For $n \in [0, N_1]$. $\quad n-K \in [0, N_1] \Rightarrow h[n-K] = 0$

$y[n] = \sum_{K\in\mathbb{Z}} a^n u[K]\,h[n-K] = 0.$ $\quad a^0 \quad a^1 \quad a^2 \quad a^3 \ldots a^{N_1} \ldots a^{N_2} \ldots$ $\quad$ N_2-N_1

$n \in [N_1; N_2]$ $\quad n=N_1 \to a^0$ $\quad n=N_1+1 \to a^1+a = 1+a$

$y[n] = \sum_{K\in\mathbb{Z}} a^n u[K]\,h[n-K] =$ $\boxed{\dfrac{(1-a^{n-N_1+1})}{1-a}}$ $\quad a^0+a^1+\ldots+a^{n-N_1}$

$n = N_2$

$a^0 \quad a^1 \ldots a^{N_2-N_1} \quad a^{N_1} \ldots a^{N_2}$

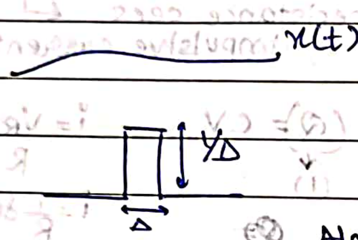
$1 \ldots \ldots 1 \ldots$

$$y[n] = \frac{a^{n-N_2}\left(1 - a^{N_2-N_1+1}\right)}{1-a} = \frac{a^{n-N_2} - a^{n-N_1+1}}{(1-a)} = a^{n-N_2}\left(\frac{1-a^{N_2-N_1+1}}{1-a}\right)$$

$$y[n] = \begin{cases} 0 & ; \; n < N_1 \\ \dfrac{1-a^{n-N_1+1}}{1-a} & ; \; N_1 \le n < N_2 \\ a^{n-N_2}\left(\dfrac{1-a^{N_2-N_1+1}}{1-a}\right) & ; \; n \geq N_2 \end{cases}$$

eg. 4)

CONVOLUTION



$$\int_{-\infty}^{+\infty} x(t)\,\delta_\Delta(t-t_0)\cdot dt = \int_{-\infty}^{+\infty} x(t_0)\,\delta_\Delta(t-t_0)\cdot dt$$

$$= x(t_0)$$

Also, $x(t)\underbrace{\delta_\Delta(t-t_0)}_{\text{it SIFTS out } x(t_0)} = x(t_0)\,\delta_\Delta(t-t_0)$

Also, $\delta_\Delta(t) = \delta_\Delta(-t)$;

$$\int_{-\infty}^{+\infty} \underset{\text{cont. pulse}}{x(t)}\, \underset{\text{Unit pulse}}{\delta_\Delta(t-t_0)}\, dt = \int_{-\infty}^{+\infty} x(t)\,\delta_\Delta(t_0-t)\cdot dt = x(t_0)$$

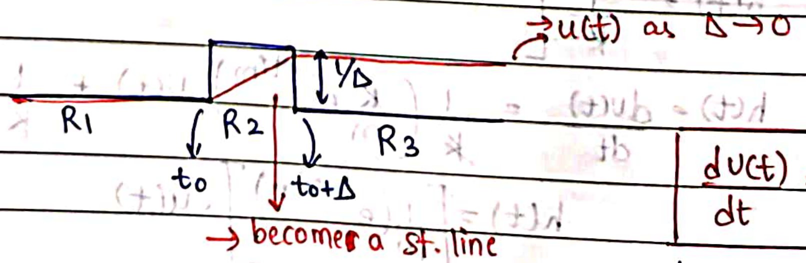
basic variable — variable w.r.t. which f^n is described (index)

index variable

$$\left(\int_{-\infty}^{\infty} x(\tau)\,\delta_\Delta(t-\tau)\,d\tau\right)$$

δ_Δ is centred at t_0, shifted by 't' each time we move to $x(t)$.

$\Rightarrow$ Unit step function, $u(t) = \int_{-\infty}^{t} \delta(t)\cdot dt$



$$\boxed{\frac{du(t)}{dt} = \delta(t)}$$

or $\dfrac{d\,u(t-t_0)}{dt} = \delta(t-t_0)$

(tolerated) ⟵ ...

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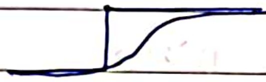
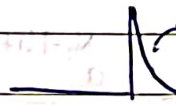
$\lim_{\Delta \to 0} \frac{u(t+\Delta)-u(t)}{\Delta}$ or $\lim_{\Delta \to 0} \frac{u(t)-u(t-\Delta)}{\Delta}$ or $\lim_{\Delta \to 0} \frac{u(t+\Delta/2)-u(t-\Delta/2)}{\Delta}$

$\downarrow$ FWD $\qquad \downarrow$ BKWD $\qquad \downarrow$ Balanced

$\frac{dx(t)}{dt} \longrightarrow \boxed{S} \longrightarrow \frac{dy(t)}{dt}$ (continuous/ piecewise cont. $x(t)$)



Unit impulse



(exponential discharge) → cap. suddenly undergoing a change in voltage
$\Downarrow$
sudden amt. of charge

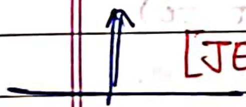
Unit impulse responce of a R-C circuit

$\left[\int \delta(t)\,dt\right]$ $\int i \cdot dt \to$ finite amount of charge ← goes to capacitor also ← resistance sees an impulsive current

↳ suddenly puts a voltage on a capacitor

$(Q) = CV$ $\quad i = \frac{V_R}{R}$ $\quad V = iR$
(1)

Voltage pulse



[JERK]

$i = \frac{1}{R}\delta(t)$ $\quad Q \Rightarrow \frac{1}{R}$

$V_{out} = \frac{1}{C}\cdot\frac{1}{R}\int\delta(t)$ $\quad V \to \frac{1}{RC}$ (at $t=0$)

and then it starts to discharge.

$\boxed{m} \to F$ $\qquad f = KV_0(t)$

f $\quad F(t) = m\frac{dV(t)}{dt} + KV(t)$
input $\qquad$ outputs

$1 = m\frac{dv(t)}{dt} + KV(t)$

$m\frac{dv}{dt} = 1 - KV$

$\int \frac{dv}{1-KV} = \int \frac{dt}{m} \Rightarrow \left[\frac{\ln(1-KV)}{-K}\right]_0^V = \left(\frac{t}{m}\right)_0^t$

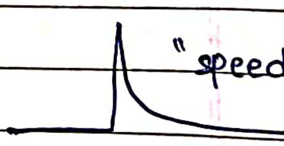
$1-KV = e^{-Kt/m}$ $\qquad V = \frac{1}{K}(1-e^{-Kt/m})$

$U(t) \longrightarrow \frac{1}{K}(1-e^{-Kt/m})\cdot u(t)$

~~So, $h(t) = \frac{1}{K}\left[(1-e^{-Kt/m}) - (1-e^{-K(t-\Delta)/m})\right]$~~

$h(t) = \frac{dU(t)}{dt} = \frac{1}{\cancel{K}}\left(\frac{\cancel{K}}{m}e^{-Kt/m}\right)U(t) + \cancel{\frac{1}{K}(1-e^{-Kt/m})}\,\delta(t-0)$ $\quad 0$

$h(t) = \left[\frac{1}{m}(e^{-Kt/m})\right]\cdot u(t)$



"speed"

on giving an impulse $\Rightarrow x\Uparrow \Rightarrow f\Uparrow \Rightarrow v\Downarrow$

Also, $u(t) = \int \frac{du(t)}{dt} \longrightarrow \boxed{\delta} \longrightarrow \int h(t)\cdot dt$

So, $\int_{-\infty}^{t} \frac{1}{RC}\left(e^{-t/RC}\right)\cdot u(t)\, dt$

$= 0 \ (t<0)$ and $\int_{0}^{t} \frac{1}{RC}\left(e^{-t/RC}\right)\cdot dt$

$= \frac{1}{RC}\left[\frac{e^{-t/RC}}{-1/RC}\right]_0^t = \frac{e^{-t/RC}(-\cancel{RC}) - 1(-\cancel{RC})}{\cancel{RC}}$

$= \begin{cases} 1-e^{-t/RC} ; & t \geq 0 \\ 0 ; & t<0 \end{cases} \Rightarrow \left(1-e^{-t/RC}\right).u(t)$

"unit step response"

CONVOLUTION OPERATOR

ASSOCIATIVE :→

$(f*g)*h\,(u) = \int_R f*g\,(x)\cdot h(u-x)\, dx$

$= \int_R \left[\int_R f(y)\, g(x-y)\, dy\right] h(u-x)\, dx$

$= \iint_{RR} f(y)\, g(x-y)\, h(u-x)\, dy\, dx$

Change of variables by Fubini's theorem

$= \int_R \left[\int_R g(x-y)\, h(u-x)\, dx\right] f(y)\, dy$

$x \to x+y$

$= \int_R \left[\int_R g(x)\, h(u-(x+y))\, dx\right] f(y)\cdot dy$

$= \int_R \left[\int_R g(x)\, h((u-y)-x)\cdot dx\right] f(y)\, dy$

$= \int_R (g*h)(u-y)\, f(y)\cdot dy$

$\boxed{(f*g)*h = f*(g*h)}$ Hence Proved...

⇒ Inverse of the convolution operator

$f * g = h$

if we are given f, h we need a g such that this holds as

$$\mathcal{L}(f*g) = \mathcal{L}(f)\cdot\mathcal{L}(g)$$

so, $\mathcal{L}(f)\cdot\mathcal{L}(g) = \mathcal{L}(h) \Rightarrow \mathcal{L}(g) = \dfrac{\mathcal{L}(h)}{\mathcal{L}(f)}$

$$\boxed{g = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}(h)}{\mathcal{L}(f)}\right\}}$$

(inverse convol^n operator)

So, if $\mathcal{L}(f)$ exists and is not 0 at any point, g is the inverse of convolution operator.

For discrete-time systems ⇒ Z-transform

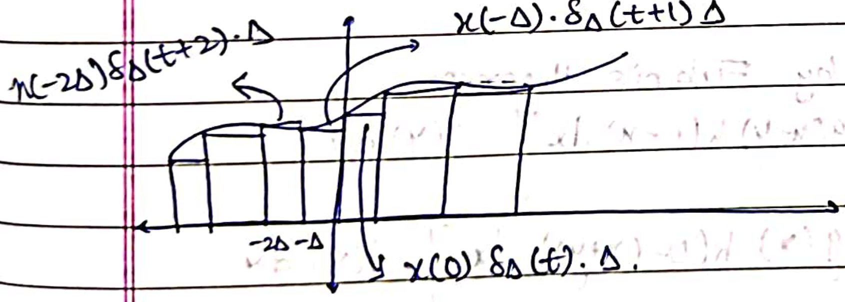
if we want to define the inverse of a f w.r.t. convol^n operator, i.e. $f * g = \delta$

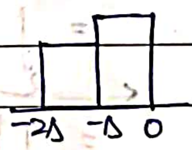
or $\mathcal{L}(g) = \mathcal{L}^{-1}\left\{\dfrac{1}{\mathcal{L}(f)}\right\}$

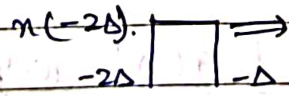
$$\boxed{g = \mathcal{L}^{-1}\left(\frac{1}{\mathcal{L}(f)}\right)}$$

~~CAUSALITY~~

⇒ Derivation of Convolution-Operator :→







$\Rightarrow x(-2\Delta)\cdot\delta_\Delta(t+2\Delta)\cdot\Delta$

Let $\delta_\Delta(t) = \begin{cases} 1/\Delta & ; 0 \le t \le \Delta \\ 0 & ; \text{otherwise} \end{cases}$

Since $\delta_\Delta(t)\cdot\Delta$ has unit amplitude,

$$\hat{x}(t) = \sum_{K=-\infty}^{\infty} x(K\Delta)\,\delta_\Delta(t-K\Delta)\cdot\Delta$$

For any value of t, only one term is NON-ZERO

$$x(t) = \lim_{\Delta\to 0} \sum_{K=-\infty}^{\infty} x(K\Delta)\,\delta(t-K\Delta)\cdot(\Delta\cdot K)$$

as $\Delta \to 0$ $\delta_\Delta \to$ unit Impulse f^{n}c and so

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau}$$

For $x(t) = u(t)$

$u(t) = \int_0^\infty \delta(t-\tau)\,d\tau \longrightarrow$ area enclosed by combinat^n of ∞-many impulses spaced ∞-close.

$u(\tau) = 0,\ \tau < 0$ $\quad u(\tau) = 1,\ \tau > 1$ $\quad \Downarrow u(t)$

$\delta(t-\tau) \longrightarrow$ f^{n}c of τ with t being fixed

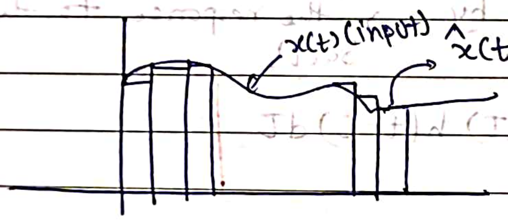
↳ impulse located at $\tau = t$.

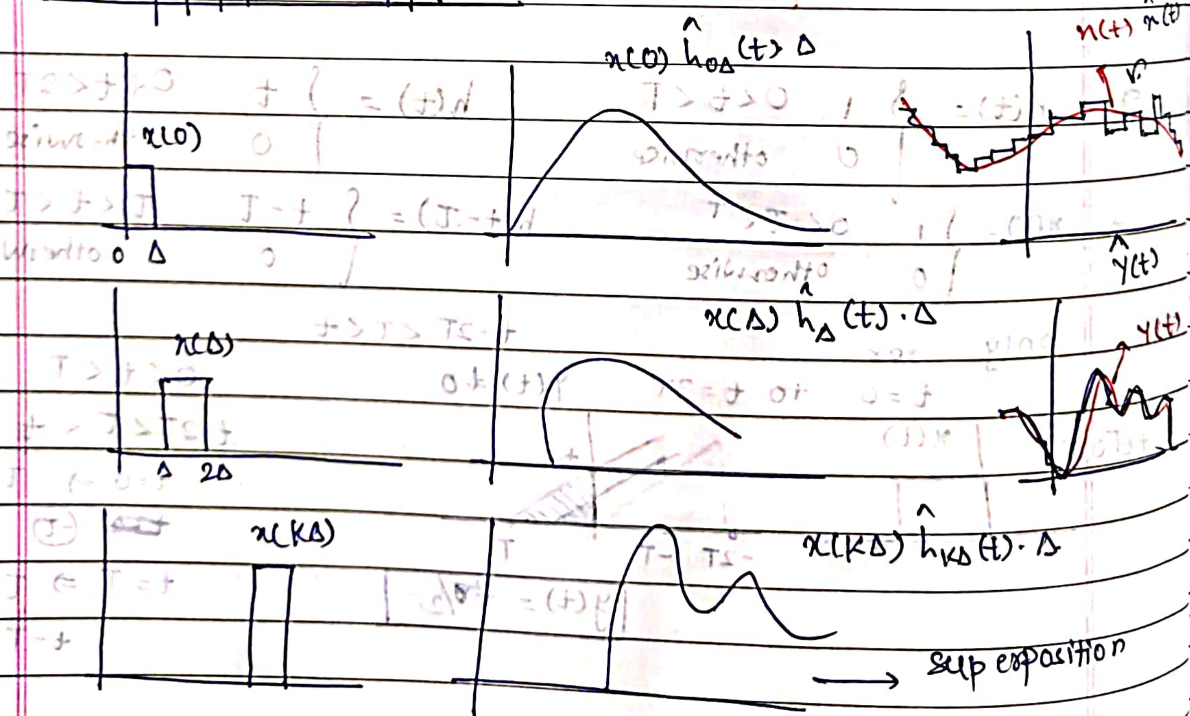
so, $x(\tau)\,\delta(t-\tau) = x(t)\,\delta(t-\tau)$.

⇒ The responce, $\hat{y}(t)$ of this system [Linear] to a signal will be a superposition of the responses to the scaled & shifted versions of $\delta_\Delta(t)$.

$\hat{h}_{k\Delta}(t) \longrightarrow$ responce to $\delta_\Delta(t-k\Delta)$

Then, $\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\hat{h}_{k\Delta}(t)\cdot\Delta$





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$\hat{x}(t) \rightarrow \boxed{\mathcal{S}} \rightarrow \hat{y}(t)$ $\hat{y}(t)$ is superposⁿ of all the responses

$\Rightarrow$ $\hat{x}(t) \rightarrow x(t)$ as $\Delta \rightarrow 0$ and so $\hat{y}(t) \rightarrow y(t)$

$\Rightarrow$ For Δ small enough $\delta_\Delta(t-k\Delta) \longrightarrow$ same response as unit impulse.

Let $h_\tau(t)$ denote the response at a time t to $\delta(t-\tau)$, located at time τ,

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\, \hat{h}_{k\Delta}(t)\cdot\Delta = \int_{-\infty}^{\infty} x(\tau)\, h_\tau(t)\cdot d\tau$$

as $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \longrightarrow$ (SUM OF WEIGHTED, SHIFTED impulses)

weight on the ~~impulse~~ response $h_\tau(t)$ to the shifted $\delta(t-\tau)$ is also $x(\tau)\cdot d\tau$ $\Leftarrow$ weight on impulse $\delta(t-\tau)$ is $x(\tau)d\tau$

$\rightarrow$ If the system is T.I. also,

$h_\tau(t) = h_0(t-\tau)$ i.e. response ~~of~~ an LTI system to the unit impulse $\delta(t-\tau)$.

$\Rightarrow$ Unit impulse response $\Rightarrow$ $\boxed{h(t) = h_0(t)}$

$\rightarrow$ response of an LTI system to the unit impulse $\delta(t-\tau)$, shifted by τ seconds from the origin $\rightarrow$ is also a shifted version of the response to the unit impulse $\delta_0(t)$.

AND THUS,
$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau}$$

eg. $x(t) = \begin{cases} 1 & 0<t<T \\ 0 & \text{otherwise} \end{cases}$ $\qquad h(t) = \begin{cases} t & 0<t<2T \\ 0 & \text{otherwise} \end{cases}$

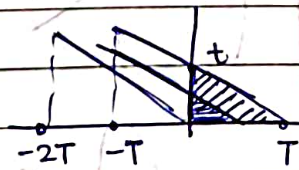
$\Rightarrow$ $x(\tau) = \begin{cases} 1 & 0<\tau<T \\ 0 & \text{otherwise} \end{cases}$ $\qquad h(t-\tau) = \begin{cases} t-\tau & \tau<t<\tau+2T \\ 0 & \text{otherwise} \end{cases}$

only for $t-2T < \tau < t$

$t=0$ to $t=3T$ $y(t) \neq 0$

$0<t<T$

(1) $t \in [0,T]$



$t-2T < \tau < t$

$t=0 \Rightarrow \tau \in [-2T, 0]$

~~t=0~~ $(-\tau)$

$\boxed{y(t) = t^2/2}$

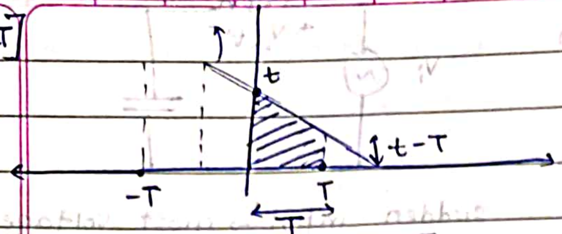
$t=T \Rightarrow \tau \in [-T, T]$

$t-T$ $\qquad t-\tau=0$

$y_0 = t - x$ $\quad$ $(x=t) \to \tau$

$Y = t - \tau$

c2> $t \in [T, 2T]$



$t - 2T < \tau < t$ $\qquad$ $T - \tau$

$-T < \tau < 2T$

$t = T$ $\qquad$ $-T < \tau < T$

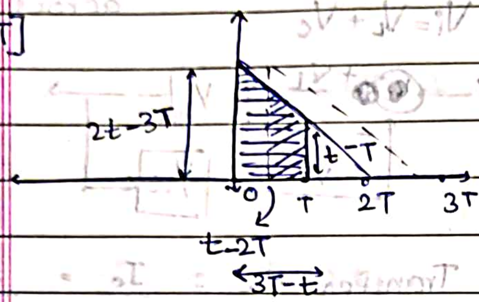
$$Y(t) = \frac{T}{2}\left(t + (t - T)\right)$$

$$\boxed{Y(t) = tT - \frac{T^2}{2}}$$

$y = t - (3T - t)$

$y = t - \tau$ $\qquad$ $2t - 3T$

c3> $t \in [2T, 3T]$



$T - (t - 2T)$

$3T - t$

$0 < \tau < 3T$

$\cancel{\frac{1}{2}(3T - t)\,[3t - 4T]}$

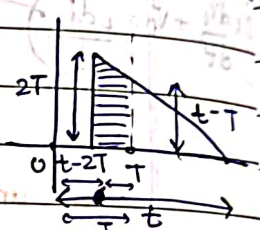
$\cancel{Y(t) = \frac{1}{2}(2T)(t - 2T)}(t - 3T)$ $\qquad$ $y = t - \tau$

$\cancel{= \frac{t^2}{2}}$

$y = t - (t - 2T)$

$2T = t - \tau$

$\tau = t - 2T$



$\cancel{T - (t - 2T)}$

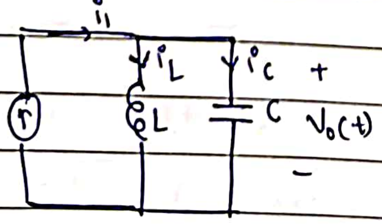
$\cancel{3T - t}$

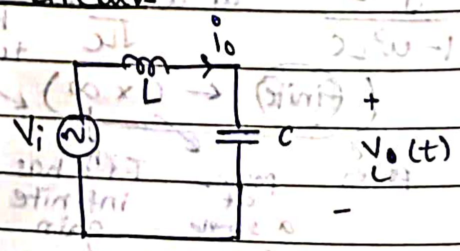
$$Y(t) = \frac{1}{2}(3T - t)(t + T) \qquad T - (t - 2T) = 3T - t$$

$$\boxed{Y(t) = \frac{3T^2}{2} + Tt - \frac{t^2}{2}}$$

$$Y(t) = \begin{cases} 0 & ; t \leq 0 \\ \frac{t^2}{2} & 0 \leq t \leq T \\ tT - \frac{T^2}{2} & T \leq t \leq 2T \\ \frac{3T^2}{2} + Tt - \frac{t^2}{2} & ; 2T \leq t \leq 3T \\ 0 & t \geq 3T \end{cases}$$

Parallel and Series LC circuits

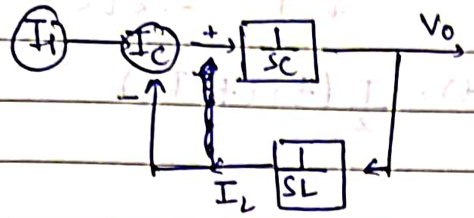


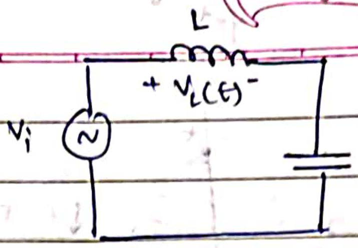


Parallel

$i_1 = i_L + i_C$ (feedback also)

Laplace: $I_1 = I_L + I_C$

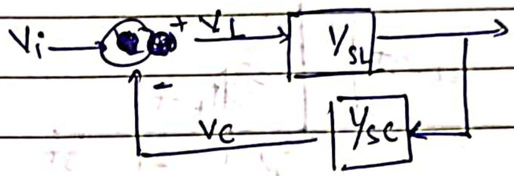




sudden input → most voltage on the inductor

Voltage across capacitor ← i ← change → reduce the voltage across inductor

$$V_i = V_L + V_C$$



$$L(s) = \frac{1}{s^2 LC} = \frac{\omega_0^2}{LC}$$

closed loop poles: $1 + L(s) = 0$

(purely imag^n) ← $s = \pm j\omega_0$

Transfer f^{n}c ⇒ $\frac{V_0}{I_1} = \frac{sL}{1+s^2LC}$

$$= \frac{1}{C}\left(\frac{s}{s^2+\omega_0^2}\right)$$

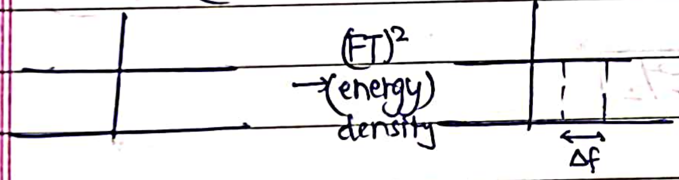
$$LC\frac{d^2V_0}{dt^2} + V_0 = L\frac{di_1}{dt}$$

Transfer f^{n}c : $\frac{I_0}{V_i} = \frac{sC}{1+s^2LC}$

$$I_0/V_i = \frac{1}{L}\left(\frac{s}{s^2+\omega_0^2}\right)$$

when an impulse of current is applied → flows through cap. → cap. discharges through ind. ⇒ OSC. (SINUSOID)

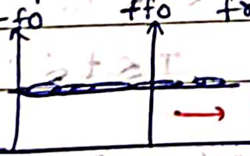
F.T. of δ



$$P = 1 \cdot \Delta f \to 0$$

$\delta(t)$ has No power at a single frequency

F.T. of a sine wave



→ ∞ power at this frequency

impulse at f_0 [Finite power]

$$\frac{j\omega L}{1-\omega^2 LC}$$

at $\omega = \frac{1}{\sqrt{LC}}$ → Tr. F^{n}c blows to ∞

(finite) ← $0 \times \infty$)

power

power at a single freq.

T.f^{n}c has infinite gain

MEMORY

→ a system is memoryless if its output at any time depends only on the input at that time.

⇒ If $x_1(t) = x_2(t) \ \forall \ t \neq t_0 \Rightarrow y_1(t) = y_2(t) \ \forall \ t \neq t_0$

⇒ $h[n] = 0 \ \forall \ n \neq 0 \Rightarrow h[n] = h[0] \cdot \delta[n]$

$$\boxed{h[n] = K\,\delta[n]}$$

and $y[n] = \sum h[K]\, x[n-K] = \sum_{K \in Z} x[K]\, h[n-K]$

$$= \sum_{K \in Z} x[K]\, K\,\delta[n-K] = K\,x[n]$$

$$\boxed{y[n] = K\,x[n]}$$

→ If $h[n] \neq 0 \ \forall \ n \neq 0$

e.g. – (memory): $h[n] = \begin{cases} 1 ; & n = 0, 1 \\ 0 ; & \text{otherwise} \end{cases}$

i.e. $y[n] = x[n] + x[n-1]$

if $x_1[n] = x_2[n] \ \forall \ n \neq n_0$

i.e. $y_1[n] = x_1[n] + x_1[n-1] \neq y_2[n]$ for $n = n_0$

and, $y_1[n] \neq y_2[n]$ for $n = n_0 + 1$

So, the system HAS MEMORY

⇒ Continuous-Time :-

$\boxed{y(t) = K\,x(t)}$ $\boxed{h(t) = K\,\delta(t)}$

INVERTIBILITY

→ an inverse system exists, if, connected in series with the original system, produces an output = input to the 1st system.

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \longrightarrow \boxed{h_1(t)} \longrightarrow x(t)$

$$\boxed{h(t) * h_1(t) = \delta(t)}$$

discrete time → $h[n] * h_1[n] = \delta[n]$

eg.1) $y(t) = x(t - t_0)$

$h(t) = \delta(t - t_0)$

$\delta(t) \longrightarrow \boxed{} \longrightarrow \delta(t - t_0)$

$\delta(t - t_0) \longrightarrow \boxed{h_1(t)} \longrightarrow \delta(t)$

$\delta(t-t_0) * h_1(t) = \delta(t)$

so, $\displaystyle\int_{-\infty}^{\infty} \delta(\tau - t_0)\, h_1(t-\tau)\, d\tau = \delta(t)$

$\downarrow$ sifting property

$h_1(t-t_0) = \delta(t) \Rightarrow \boxed{h_1(t) = \delta(t+t_0)}$

So, the system is invertible

$x(t) \rightarrow \boxed{\text{Delay}} \rightarrow \boxed{\text{Advance}} \rightarrow x(t).$

$h[n] = u[n]$

$\delta(t) \rightarrow \boxed{\text{int.}} \rightarrow u(t) \rightarrow \boxed{\text{diff}} \rightarrow \delta(t)$

$y_1[n] = \sum x(k)\, h(n-k) = \sum x(k)\, u(n-k)$

$y_1[n] = \sum_{k=0}^{n} x(k) \rightarrow$ input to a system

$\cancel{y_2[n] = x[n]}$

$\cancel{x[n] = \sum y_1[k]\, h_1[n-k] = \sum (x[0] + x[1] + x[2] + \cdots + x[k])\, h_1[n-k]}$

$\cancel{x[0] = \sum_{k \in Z} y_1[k]\, h_1[-k] = \sum \left(\sum_{r=0}^{k} x[r]\right) h_1[-k]}.$

$\cancel{\sum_{r=0}^{n}}$ $\quad y[n] - y[n-1] = x[n].$

(summer/accumulator)

$\rightarrow$ inverse $\Rightarrow$

$y[n] = x[n] - x[n-1]$ (First difference operator)

So, $h_1[n] = \delta[n] - \delta[n-1]$

$h[n] * h_1[n] = \sum u[k]\left[\delta[n-k] - \delta[n-1-k]\right]$

$= u[n] - u[n-1] = \delta[n]$

CAUSALITY

⇒ $y[n]$ must not depend on $x[K]$ for $K>n$.
└→ coeff. of $h[n-K]$ that multiply with $x[K]$ must be 0 for all $K>n$ i.e. $\boxed{h[n]=0 \;\forall\; n<0}$

e.g.- $y[n]=2x[n]+3$ → causal but does'nt satisfy initial rest condition.

STABILITY

sufficient condition ⇒ $\sum_{-\infty}^{-\infty} |h[K]| < +\infty$

or $\int_{-\infty}^{\infty} |h(\tau)| d\tau < +\infty$

eg.2>

integrator, $h(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = u(t)$

and $\int_{-\infty}^{\infty} |h(\tau)| \cdot d\tau = \int_{-\infty}^{\infty} |u(\tau)| \cdot d\tau \to \infty$ [NO bound]

so, the system is BIBO unstable.

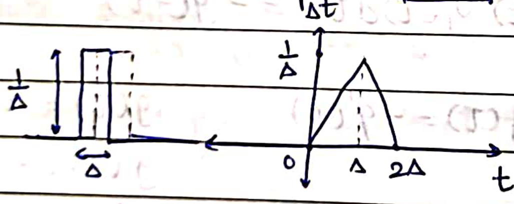
SINGULARITY FUNCTIONS

$x(t) * \delta(t) = x(t)$ $\qquad$ $x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$

if we take $x(t)=\delta(t)$, $\qquad = x(t)$.

$\delta(t) * \delta(t) = \delta(t)$

Let $r_\Delta(t) = \delta_\Delta(t) * \underbrace{\delta_\Delta(t)}_{\text{short, rectangular pulse}}$



$\int \delta_\Delta(\tau)\, \delta_\Delta(t-\tau) \cdot d\tau$

Then, $\lim_{\Delta \to 0} r_\Delta(t)$ must also be an unit impulse and so.

$\lim_{\Delta\to 0} r_\Delta(t)$, $r_\Delta(t) * r_\Delta(t)$, $r_\Delta(t) * \delta_\Delta(t)$ must also be unit impulses.

⇒ we define $\delta(t)$ as the signal for which $x(t) = x(t) * \delta(t)$, for any $x(t)$

1> If $x(t)=1 \implies \int_{-\infty}^{\infty} \delta(\tau) x(t-\tau) d\tau = 1$ $\qquad$ $(x(t) = x(t) * \delta(t) = \delta(t) * x(t))$

$\Rightarrow \int_{-\infty}^{\infty} \delta(\tau) = 1$

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$\Rightarrow$ $g(-t) = g(-t) * \delta(t) = \int_{-\infty}^{\infty} g(\tau - t)\, \delta(\tau) d\tau$

For $t = 0$

$\Rightarrow \int_{-\infty}^{\infty} g(\tau)\delta(\tau) d\tau = g(0)$ Let $g(\tau) = x(t-\tau)$

Then $g(0) = x(t)$

$x(t) = g(0) = \int_{-\infty}^{+\infty} g(\tau)\delta(\tau) d\tau = \int_{-\infty}^{\infty} x(t-\tau)\delta(\tau) d\tau$

UNIT DOUBLET

$y(t) = \frac{dx(t)}{dt}$ $\Rightarrow$ $x(t) * h(t) = \frac{dx(t)}{dt}$

if $x(t) \to \delta(t) \Rightarrow y(t) = ?$ $h(t) = u_1(t)$

$$\boxed{x(t) * u_1(t) = \frac{dx(t)}{dt}}$$

also $\frac{d^2 x(t)}{dt^2} = \frac{d}{dt}\left(\frac{dx(t)}{dt}\right)$

$= (x(t) * u_1(t)) * u_1(t)$

So, $\boxed{u_2(t) = u_1(t) * u_1(t)}$

So, the k^{th} - derivative of $\delta(t)$ has

$u_k(t) = \underbrace{u_1(t) * u_1(t) * u_1(t) \ldots}_{k\text{-times}}$

$\Rightarrow$ $x(t) = 1 \Rightarrow x(t) * u_1(t) = 0 \Rightarrow \int u_1(\tau) \cdot 1 \, d\tau = 0$

$\int_{-\infty}^{\infty} u_1(\tau) d\tau = 0$ $\to$ zero area

$\Rightarrow$ $g(-t) * u_1(t) = \int_{-\infty}^{\infty} g(\tau - t)\, u_1(\tau) d\tau = -g'(-t)$

For $t = 0$ $\int_{-\infty}^{\infty} g(\tau)\, u_1(\tau) = -g'(0)$

$g(\tau) = x(t-\tau)$

$g'(\tau) = -x'(t-\tau)$

$x'(t) = \int g(\tau)\, u_1(\tau) d\tau$ $g'(0) = -x'(t)$

i.e. $\boxed{x'(t) = \int_{-\infty}^{\infty} x(t-\tau)\, u_1(\tau) d\tau}$ (derivat^n)