Lebesgue's Criterion for Riemann Integrability

James Y and Will F

March 2020

In this paper we look at a the Lebesgue criterion for Riemann integrability. The criterion states that for a bounded function, f, on [a,b], f is Riemann integrable if and only if the set of points in [a,b] at which f is not continuous has measure 0. We treat each direction of the criterion as a separate theorem, and the criterion will be a corollary of the results.

Lemma 1. *Let E be a measurable subset of $\mathbb{R}$. Let $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences of functions, each of which is integrable over E, such that $\{\varphi_n\}$ is increasing while $\{\psi_n\}$ is decreasing on E. Let the function f on E have the property that $\varphi_n \leq f \leq \psi_n$ on E for all $n \in \mathbb{N}$. If $\lim_{n\to+\infty} \int_E (\varphi_n - \psi_n) = 0$ then $\{\varphi_n\} \to f$ a.e., $\{\psi_n\} \to f$ a.e., f is integrable over E, and $\lim_{n\to+\infty} \int_E \varphi_n = \lim_{n\to+\infty} \int_E f = \lim_{n\to+\infty} \int_E \psi_n$.*

Proof. $\{\varphi_n\}$ and $\{\psi_n\}$ are sequences defined on extended real-valued numbers $\mathbb{R}\cup\{-\infty, +\infty\}$. The pointwise limit of the sequences are defined on the extended real valued numbers. Since the sequences are monotone, if they are bounded by a real number, then, for $x \in E$, $\{\varphi_n(x)\}$ and $\{\psi_n(x)\}$ are monotone sequences of real numbers and by Monotone Convergence Theorem, they converge pointwise. If $\exists x \in E$, they are not bounded by a real number, $\{\varphi_n(x)\}$ and $\{\psi_n(x)\}$ converge pointwise to $\pm\infty$. Therefore, we can properly define

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x) \text{ and } \psi(x) = \lim_{n\to\infty} \psi_n(x).$$

By Corollary 2.1.13, the pointwise limit of measurable functions are still measurable. Moreover, by construction,

$$_n \leq \varphi \leq f \leq \psi \leq \psi_n$$

on E for all n.

The linear combination of Lebesgue integrable functions $\psi_n - \varphi_n$ is again integrable. Now

$$0 \leq \psi - \varphi \leq \psi_n - \varphi_n \text{ for all n}$$

so $\psi - \varphi$ is integrable on E and

$$0 \leq \int_E (\psi - \varphi) \leq \int_E (\psi_n - \varphi_n) \text{ for all n}$$

But we also know that $\varphi \leq \psi$, so $\psi - \varphi$ is a nonnegative measurable function. Since $\int_E(\psi - \varphi) = 0$, by Lemma 2.2.14, we have $\psi - \varphi = 0$ a.e. or $\psi = \varphi$ a.e. on E. At the same time, $\varphi \leq f \leq \psi$ on E. Thus,

$$\{\varphi_n\} \to f \text{ and } \{\psi_n\} \to f \text{ pointwise a.e. on E}$$

and f is measurable. From $\varphi_1 \leq f \leq \psi_1$ we have $0 \leq f - \varphi_1 \leq \psi_1 - \varphi_1$, where ψ_1 and φ_1 are integrable so $\psi_1 - \varphi_1$ is also integrable. Thus we conclude $f - \varphi_1$ is integrable and f is integrable. It follows

$$0 \leq \int_E \psi_n - \int_E f = \int_E (\psi_n - f) \leq \int_E (\psi_n - \varphi_n)$$

and

$$0 \leq \int_E f - \int_E \varphi_n = \int_E (f - \varphi_n) \leq \int_E (\psi_n - \varphi_n)$$

Taking the limit of n to infinity

$$0 \leq \lim_{n\to\infty} \int_E \psi_n - \int_E f \leq \lim_{n\to\infty} \int_E (\psi_n - \varphi_n) = 0$$

and

$$0 \leq \int_E f - \lim_{n\to\infty} \int_E \varphi_n \leq \lim_{n\to\infty} \int_E (\psi_n - \varphi_n) = 0$$

Thus, we have

$$\lim_{n\to+\infty} \int_E \varphi_n = \lim_{n\to\infty} \int_E f = \lim_{n\to\infty} \int_E \psi_n.$$

□

Theorem 2. *Let f be a bounded function on $[a, b]$. If f is Riemann integrable over [a,b] then the set of points in [a,b] at which f is not continuous has measure 0.*

Proof. Assume f is Riemann integrable on $[a, b]$. Then for every $n \in \mathbb{N}$ we can find a partition P_n such that $U(f, P_n) - L(f, P_n) < \frac{1}{2^n}$. Assume that each P_{n+1} is a refinement of P_n since if it is not we can simply take it to be the common refinement of P_{n+1} and P_n. We define a lower and upper step function on the sequence of partition. Each lower and upper step function, denoted by φ_n and ψ_n, have the partition points of P_n and respectively takes the infimum and supremum of the bounded function on each open interval determined by the partition. We observe that by construction, we have

$$\int_a^b \varphi_n = \sum_{j=1}^{k} \inf_{x\in[x_{i-1},x_i]} (x_i - x_{i-1}) = L(f, P_n),$$

$$\int_a^b \psi_n = \sum_{j=1}^{k} \sup_{x\in[x_{i-1},x_i]} (x_i - x_{i-1}) = U(f, P_n)$$

for all n. Thus, the sequences of step functions are integrable. Also, by construction, $\varphi_n \leq f \leq \psi_n$ for all n. Because each P_{n+1} is a refinement of P_n , we have $\varphi_n \leq \varphi_{n+1} \leq f \leq \psi_{n+1} \leq \psi_n$ for all n on E. In other words, $\{\varphi_n\}$ is an increasing sequence and $\{\psi_n\}$ is a decreasing sequence. Lastly, we have for each $n \in \mathbb{N}$, $0 \leq U(f, P_n) - L(f, P_n) \leq \frac{1}{2^n}$.

Now

$$\lim_{n\to\infty} \int_a^b [\psi_n - \varphi_n] = \lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

By the previous lemma,

$$\{\varphi_n\} \to f \text{ and } \{\psi_n\} \to f \text{ pointwise a.e. on [a,b]}$$

Let E' be the points x where either $\{\psi_n(x)\}$or$\{\varphi_n(x)\}$ fail to converge to $f(x)$. Then the set has measure 0. Take $E = E' \cup \{$ the partition points of all $P_n\}$. Since each partition point has measure zero, a countable union of measure zero sets still has measure zero. Thus $m(E) = 0$. We wish to show that the function is continuous everywhere other than the measure zero set. Let $\epsilon > 0$ be given. By the pointwise convergence, at each $x_0 \in A = [a, b]\backslash E$, there exists some n such that, $f(x_0) - \varphi_n(x_0) < \epsilon$ and $\psi_n(x_0) - f(x_0) < \epsilon/2$. We have the inequality

$$f(x_0) - \epsilon/2 < \varphi_n(x_0) < f(x_0) < \psi_n(x_0) < f(x_0) + \epsilon/2.$$

Since x_0 is not a partition point of P_n, it is contained in some open interval I_n. If we take $(x_0 - \delta, x_0 + \delta) \subset I_n$ for $\exists\delta > 0$. We have $x \in (x_0 - \delta, x_0 + \delta)$, $\varphi_n(x) = \varphi_n(x_0)$ and $\psi_n(x) = \psi_n(x_0)$. Thus,

$$\varphi_n(x) \leq \varphi_n(x_0) \leq f(x) \leq \psi_n(x) \leq \psi_n(x_0) \text{ when } |x - x_0| < \delta$$

Thus, $f(x) - f(x_0) \leq f(x) - \varphi_n(x_0) \leq \psi_n(x_0) - \varphi_n(x_0) < \epsilon/2 + \epsilon/2 = \epsilon$. Thus we have f is continuous at x_0, so all discontinuous points are contained in E which has measure 0. Therefore, the set of discontinuous points in [a,b] has measure 0 as desired.

□

Theorem 3. *Let f be a bounded function on* $[a, b]$. *If the set of points on [a,b] at which f is not continuous has measure 0, then f is Riemann integrable over [a,b].*

Proof. For $n \in \mathbb{N}$, define the partition $P_n = \{a = x_0, a + (b-a)\frac{1}{2^n} = x_1, a + (b-a)\frac{2}{2^n} = x_2, \ldots, x_{2^n} = b\}$. For a partition, P, define gap P to be the length of the longest interval in the partition. Notice that $\lim_{n\to\infty}$ gap $P_n = \lim_{n\to\infty} \frac{b-a}{2^n} = 0$. For $n \in \mathbb{N}$, define the lower step function $\varphi_n(x) = \inf_{y\in[x_i,x_{i+1}]} f(y)$ where $x \in [x_i, x_{i+1}]$ in the partition P_n (if x is a partition point take the minimum of both intervals containing it). Similarly, define the upper step function $\psi_n(x) = \sup_{y\in[x_i,x_{i+1}]} f(y)$ where $x \in [x_i, x_{i+1}]$ in the partition P_n (if x is a partition point take the maximum of both intervals containing it). Define Z to be the set of points $x \in [a, b]$ such that f is discontinuous at x or $x \in P_n$ for any $n \in \mathbb{N}$.

Note that $m(Z) = 0$ since the set of discontinuous points has measure zero and there are countable partition points since each P_n has finite partition points. Let $x_0 \in [a,b] \setminus Z$. Let $\epsilon > 0$ be given. Then since f is continuous at x_0, there exists $\delta > 0$ such that if $|x_0 - x| < \delta$:

$$f(x_0) - \frac{\epsilon}{2} < f(x) < f(x_0) + \frac{\epsilon}{2}$$

Take $N \in \mathbb{N}$ such that if $n > N$, gap $P_n < \delta$. For $n > N$, let I_n be the open partition interval containing x_0 determined by P_n. Thus, $I_n \subseteq (x_0 - \delta, x_0 + \delta)$. Therefore, by the previous equation:

$$f(x_0) - \frac{\epsilon}{2} \le \varphi_n(x_0) \le f(x_0) \le \psi_n(x_0) \le f(x_0) + \frac{\epsilon}{2}$$

Thus, for $n \ge N$, $0 \le \psi_n(x_0) - f(x_0) \le \epsilon$ and $0 \le f(x_0) - \varphi_n(x_0) \le \epsilon$. Therefore, since $m(Z) = 0$, $\{\psi_n\} \to f$ pointwise a.e. and $\{\varphi_n\} \to f$ pointwise a.e.. Since $\{\psi_n\}$ and $\{\varphi_n\}$ are simple functions they are measurable. Since f is bounded, there exists $M > 0$ such that for all $x \in [a,b], -M \le f(x) \le M$. Therefore, for every $n \in \mathbb{N}$, $|\psi_n| \le M$ and $|\varphi_n| \le M$. Thus, by the Lebesgue Dominated Convergence Theorem, each $\{\psi_n\}$ and $\{\varphi_n\}$ and f are Lebesgue integrable over $[a,b]$ and

$$\lim_{n\to\infty} \int_a^b \psi_n = \int_a^b f = \lim_{n\to\infty} \int_a^b \varphi_n$$

so

$$\lim_{n\to\infty} \int_a^b [\psi_n - \varphi_n] = 0$$

. By construction, $U(f, P_n) = \int_a^b \psi_n$ and $L(f, P_n) = \int_a^b \varphi_n$. Thus,

$$\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0$$

so for any $\epsilon > 0$, we can find a partition, P (P_n for large enough n), such that $U(f, P_n) - L(f, P_n) < \epsilon$. Therefore, f is Riemann integrable over [a,b]. $\square$

Corollary 3.1 (Lebesgue's Criterion for Riemann Integrability)**.** *Let f be a bounded function on $[a,b]$. Then f is Riemann integrable over [a,b] if and only if the set of points in [a,b] at which f is not continuous has measure 0.*

References

[1] Gail S. Nelson. (2015). "A User-Friendly Introduction to Lebesgue Measure and Integration." AMS.

[2] Royden and Fitzpatrick, "Real Analysis" .