Final Project - Topics in Abstract Algebra Winter 2019

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1 Introduction

We begin with a motivating example. Imagine that we have a blank regular tetrahedron and we are given r colors and asked to color each face. Since the tetrahedron has 4 faces, it would seem like there would be r^4 colors. However, if we dropped some of the tetrahedra off our table it is possible that two different coloring would end up looking the same after being shuffled around. Therefore, we are interested in figuring out the number of *indistinguishable* colorings of a tetrahedron. More formally, we call two colorings of a tetrahedron indistinguishable if some orientation preserving symmetry of the tetrahedron makes one coloring look like the other.

In the specific instance in which we have two colors, red and blue, we can figure out how many indistinguishable colorings there are by counting them: there are:

- One using only red
- One using red on three faces on blue on one
- One using red on two faces and blue on two

- One using red on one face and blue on three
- One using only blue.

So we conclude that there are 5 indistinguishable colorings of the tetrahedron using two colors. We could try to repeat a similar process with different numbers of colors, but this will quickly become infeasible as there are 15 indistinguishable colorings with three colors, 36 with four colors, and 75 with 5 colors. With just 10 colors there are 925 indistinguishable colorings!

Moreover, this specific example hardly gives us any insights to coloring other geometric objects. We are interested in finding a general way to count the number of indistinguishable ways to color various objects that does not rely on enumerating cases. It turns out we will use some results from representation theory. By combining the group theory and combinatorial arguments, we can understanding the coloring problem in a general setting.

2 Burnside Counting Theorem

Let Ω be some finite point set and G be some group of permutations that acts on Ω. Then we can define a "natural" representation of G in $\mathbb{R}^n$. Think of the points of Ω as $1, 2, ..., n$, and then identify with them the natural basis vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$. Then for each $g \in G$, we can define $\rho(g)$ to be the invertible linear transformation represented by the permutation matrix $T(g)$ which has entries $t_{ij} = \begin{cases} 1 \text{ if } g(i)=j \\ 0 \text{ otherwise} \end{cases}$.

Let χ be the character of this representation, and let χ_1 be the trivial character. Then the number of times that the trivial character "appears in" χ will be $\langle \chi, \chi_1 \rangle = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} \chi(g)$.

Now we will digress to a seemingly unrelated issue, that of counting the number of orbits of Ω under G. Specifically, we will prove that the number of orbits is exactly the number of times that the trivial character appears in χ. More formally:

Theorem 1. Let G be a group acting on the point set Ω, and let χ be the character of the natural representation of G described above. Then

$$r = \frac{1}{|G|}\sum_{g\in G}\chi(g)$$

Where r is the number of orbits of Ω under G.

Proof. We denote the action of $g \in G$ on $x \in \Omega$ by $g(x)$. Note that $\chi(g)$ is the number of elements of Ω that are fixed by the action of g (that is the number of $x \in \Omega$ such that $g(x) = x$).

We will prove the theorem by counting the number N of pairs (g, x) such that $g(x) = x$ in two different ways.

First fix some $x \in \Omega$. We want to find the number of $g \in G$ such that $g(x) = x$. So we need $|G_x|$ where G_x is the stabilizer of x in G. So $N = \sum_{x\in\Omega} |G_x|$.

Before moving forward, we partition Ω into the union of orbits (where there are r orbits)

$$G(x_1) \cup G(x_2) \cup ... \cup G(x_r)$$

Then we can rewrite N as $\displaystyle\sum_{i=1}^{r}\sum_{x\in G(x_i)} |G_x|$. Since the size of the stabilizer is constant on orbits, $N = \displaystyle\sum_{i=1}^{r} |G(x_i)||G_{x_i}|$. Since the order of the stabilizer times the size of the orbit is just the order of G for any element in G, we get $N = \displaystyle\sum_{i=1}^{r} |G| = r|G|$.

Now we will find N by first holding $g \in G$ constant. Now we want to know the number of $x \in \Omega$ such that $g(x) = x$. As we noted above this is simply $\chi(g)$. So we see that $N = \displaystyle\sum_{g\in G}\chi(g)$

So $r|G| = N = \sum_{g \in G} \chi(g)$, and thus

$$r = \frac{1}{|G|} \sum_{g \in G} \chi(g)$$

$\square$

Later we will see a powerful application of this theorem that will allow us to figure out the number of ways to count the number of indistinguishable colorings of various objects.

3 Cycle Index of Permutation Groups

Let G be the permutation group acting on a point set $X = \{1, 2, ..., n\}$.

In general, the one cycles are omitted when writing g, but for our purposes we will include them. Each $g \in G$ is the composition of disjoint cycles. Furthermore, the sum of the lengths of these cycles will be n. So we can let α_i be the number of disjoint cycles of length i in g, and $\sum_{i=1}^{n} \alpha_i \cdot i = n$. We assign g an expression

$$\zeta_g(x_1, x_2, ..., x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

For this moment, we want to treat the x_i's as x in polynomial $\mathbb{N}[x]$. Let's run through a quick example. Let G be the group the symmetries of a regular hexagon including reflections and rotations. Label the corners clockwise as 1,2,3,4,5,6. Then we can write out the expression of ζ_g.

g	α_1	α_2	α_3	α_4	α_5	α_6	ζ_g
(1)(2)(3)(4)(5)(6)	6	0	0	0	0	0	x_1^6
(123456)	0	0	0	0	0	1	x_6
(135)(246)	0	0	2	0	0	0	x_3^2
(14)(25)(36)	0	3	0	0	0	0	x_2^3
(153)(264)	0	0	2	0	0	0	x_3^2
(165432)	0	0	0	0	0	1	x_6
(12)(36)(45)	0	3	0	0	0	0	x_2^3
(14)(23)(56)	0	3	0	0	0	0	x_2^3
(16)(25)(34)	0	3	0	0	0	0	x_2^3
(1)(4)(26)(35)	2	2	0	0	0	0	$x_1^2x_2^2$
(2)(5)(13)(46)	2	2	0	0	0	0	$x_1^2x_2^2$
(3)(6)(15)(24)	2	2	0	0	0	0	$x_1^2x_2^2$

If we take the formal sum of ζ_g for all $g \in G$ and divide by the order of the group, we obtain the **cycle index** of the group of permutations:

$$\zeta_G(x_1, x_2, ..., x_n) = \frac{1}{|G|}\sum_{g\in G}\zeta_g(x_1, x_2, ..., x_n).$$

Write the symmetries of the hexagon in our example in cycle index from the table:

$$\frac{1}{12}(x_1^6 + 4x_2^2 + 3x_1^2x_2^2 + 2x^3 + 2x_6)$$

Notice that the 'coefficients' sum up to $|G|$ because we have we have $|G|$ numbers of ζ_g and we divide $|G|$ out at the end.

Alternatively, we can write the cycle index of G as the sum of each type of permutation:

$$\zeta_G(x_1, x_2, ..., x_n) = \frac{1}{|G|} \sum c(\alpha_1, \alpha_2, ..., \alpha_n) x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

where $c(\alpha_1, \alpha_2, ..., \alpha_n)$ is the number of the cycle type in which there are α_1 one cycles, α_2 two cycles, ..., and α_n n cycles.

We now will briefly discuss how to interpret the cycle index of a group. A given term in the summation will be of the form $c \cdot \prod_{i=1}^{n} x_i^{\alpha_i}$. This term tells us that there are c elements of G that are the composition of α_1 1-cycles, α_2 2-cycles,..., up to α_n n-cycles. We know that the order of the composition of cycles is the least common multiple of the lengths of the cycles. This means that each of these c elements have order o=lcm$(\alpha_1, \alpha_2, ..., \alpha_n)$. Note that this does not mean that there are exactly c elements of this order because it is possible a different term in the cycle index also represents elements of order o.

4 Cyclic and Dihedral Symmetry

Recall that we are interested in finding the number of inequivalant colorings of objects. To do this, the most common type of symmetry that we would have to consider would be symmetries of some circular object. Before considering this we have to decide whether we are going to allow only circular rotations of the object or rotations and reflections of the object. For example, if we divide a disk into equal sectors (like a sliced pizza) and then color each sector (the author is hungry and imagines this as adding a single topping to each slice), then we would not allow reflections (as this would involve flipping over a perfectly good pizza). On the other hand, if we have a necklace with some equally spaced beads, then we would allow for both reflections and rotations (since it is very easy to put on a necklace "backwards").

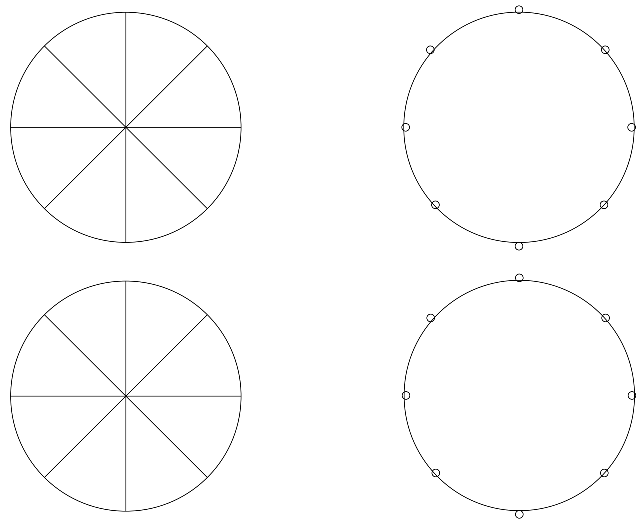

Figure 1: The same colorings applied to a disk and a necklace. The colorings of the disk are not equivalent since we do not allow for reflections, while the colorings of the necklace are equivalent.

First we will consider the situation in which reflections are not allowed. We can think of this as the group of orientation preserving symmetries of a planar n-gon whose vertices are labeled clockwise $1, 2, ..., n$. This will be the group of rotation generated by the rotation through $2\pi/n$ radians. We can think of this generating element as the permutation $\pi = (123...n)$. So the group of rotations is the cyclic group of order n generated by π ($C_n = \{e, \pi, \pi^2, ..., \pi^{n-1}\}$).

For example consider when $n = 8$. Then the permutations and the corresponding expressions for ζ_{π^i} are:

π^i	ζ_{π^i}
e	x_1^8
(12345678)	x_8
(1357)(2468)	x_4^2
(14725856)	x_8
(15)(26)(37)(48)	x_2^4
(16385274)	x_8
(1753)(2864)	x_4^2
(18765432)	x_8

So we get that the cycle index is $\zeta_{C_8}(x_1, ..., x_8) = \frac{1}{8}(x_1^8 + x_2^4 + 2x_4^2 + 4x_8)$

We can now prove the following theorem:

Theorem 2. Let C_n by the cyclic group of permutations generated by $(12...n)$. Then for each divisor d of n, there are $\phi(d)$ elements of C_n that are n/d disjoint cycles of length d. Thus, the cycle index is

$$\zeta_{C_n}(x_1, ..., x_n) = \frac{1}{n}\sum_{d|n}\phi(d)x_d^{n/d}$$

Proof. We know that there are $\phi(d)$ elements of order d in C_n. These elements will be of the form $\pi^{kn/d}$ where k is co-prime to d. Now we just need to show that these elements of actually just n/d d-cycles.

Let $i = kn/d$, and let m be the length of the shortest cycle in π^i. Let x be some element of a m-cycle in π^i. Then $(\pi^i)^m(x) = \pi^{im}(x) = x$.

Now let y be any element of $\{1, 2, ..., n\}$. Since y and x are in the same single n-cycle of π, we know that $y = \pi^r(x)$ for some r. This means that $\pi^{im}(y) = \pi^{im}\pi^r(x) = \pi^r\pi^{im}(x) =$

$\pi^r(x) = y$. So this means that y is in a cycle of length m' in π^i, where m' divides m. However, we assumed that m is the length of the smallest cycle in π^i, so $m' = m$. This means that every element of $\{1, 2, ..., n\}$ is in an m-cycle in π^i. It follows that there must be n/m of these m-cycles. Since the order of π^i is d, this means that all of these cycles are d-cycles ($m = d$). So $\pi^{kn/d}$ is n/d disjoint d-cycles.

Note that $\sum_{d|n} \phi(d) = n$. So when calculating the cycle index, if we sum over all d that divide n and multiply by $\phi(d)$ we will effectively be summing over all π^i for $1 \leq i \leq n$ (since ζ_g is constant for element of the same order). So we conclude that

$$\zeta_{C_n}(x_1, ..., x_n) = \frac{1}{n} \sum_{d|n} \phi(d) x_d^{n/d}$$

□

Now consider the case we also allow reflections on regular n-gons. The group of symmetries of a regular n-gon when reflections are allowed is denoted D_{2n}. It is this way because (as we will see later) there are $2n$ elements of this group.

Even n For an even n, there are essentially two different kinds of reflections. First, we can choose an axis which goes through the mid-point of an edge and the mid-point of the opposite edge. Let's look at the reflection when the axis crosses the mid-point of edge $\overline{1 \quad n}$ and edge $\overline{n' \quad n'+1}$ with $n' = n/2$. Clearly, this reflection, which we call σ, is

$$(1 \quad n)(2 \quad n{-}1) \cdots (n' \quad n'{+}1)$$

There are $n' = n/2$ 2-cycles in total in σ.

On the other hand, if we reflect around an axis that crosses a corner, say n, and the

Figure 2: The first type of reflection (in this case σ) is reflection over the depicted line

corner n'. The reflection σ' is:

$$(n)(n')(2 \quad n)(3 \quad n-1-1)\cdots(n' \quad n'+1).$$

Figure 3: The second type of reflection (in this case σ') is reflection over the depicted line

If we compose σ with π,

$$\sigma\pi = (1 \quad n)(2 \quad n–1)\cdots(n' \quad n'+1)(12\cdots n) = (n)(n')(1 \quad n–1)(2 \quad n–2)\cdots(n'–1 \quad n'+1) = \sigma'$$

and in turn

$$\sigma'\pi = (1 \quad 2)(n \quad 3)(n-1 \quad 4)\cdots(n'+1 \quad n'+2) = \sigma\pi^2$$

We see that we can have another reflection when we compose one reflection with rotations. Without repetition, we have n reflections in total, and the alternating pattern of two kinds of reflections tells us the numbers of each kind. In general, we have $n/2$ numbers of reflections which has an axis on the midpoints of the opposing edges and has $n/2$ cycles, they are:

$$\sigma, \sigma\pi^2, \sigma\pi^4, \sigma\pi^6, ..., \sigma\pi^{n-2}.$$

Similarly, we have $n/2$ numbers of reflections which has an axis on the opposing corners and has $n/2-1$ cycles:

$$\sigma\pi, \sigma\pi^3, \sigma\pi^5, \sigma\pi^7, ..., \sigma\pi^{n-1}$$

So finally, we see that the group of symmetries of a regular n-gon when we allow for reflections is the group $D_{2n} =< \pi, \sigma >$.

Odd n For odd n there is only one kind of reflection, the reflection over a line that connects a vertex to the midpoint of the opposite edge. In this case we can call the reflection about the line connecting the midpoint of the edge $\overline{n \quad 1}$ to the vertex $(n+1)/2$ σ. If we take $\sigma\pi^i$ then we get another reflection of this type. So there are n reflections of this type $(\sigma, \sigma\pi, ..., \sigma\pi^{n-1})$, and again $D_{2n} =< \pi, \sigma >$.

Figure 4: The one type of reflection (in this case σ) for odd n

Cycle Index for C_n and D_{2n}

Theorem 3. The cycle index of D_{2n} is

$$\frac{1}{2}\zeta_{C_n}(x_1, ..., x_n) + \begin{cases} \frac{1}{4}(x_2^{n/2}+x_1^{n/2-1}) \text{ if } n \text{ is even} \\ \frac{1}{2}x_1x_2^{(n-1)/2} \text{ if } n \text{ is even} \end{cases}$$

Proof. In the case that n is even, the n elements of D_{2n} fall into three categories. First, there are the n elements of C_n. Next, there are those $n/2$ reflections about a line connecting the midpoints of opposite sides, which are of the form $\sigma\pi^{2i}$. Recall that these are $n/2$ 2-cycles, so we associate with them $x_2^{n/2}$. Finally there are those $n/2$ refections that connect opposite vertices, which are of the form $\sigma\pi^{2i-1}$. These fix two points and swap the rest of the points in pairs so with them we associate $x_1^2x_2^{n/2-1}$. Putting this all together, we get that

$$\zeta_{D_{2n}}(x_1, ..., x_n) = \frac{1}{2n}\left(\sum_{d|n}\phi(d)x_d^{n/d} + \frac{n}{2}(x_2^{n/2} + x_1^{n/2-1})\right) = \frac{1}{2}\zeta_{C_n} + \frac{1}{4}(x_2^{n/2} + x_1^2 + x_2^{n/2-1})$$

In the case that n is odd there are only two types of elements. There are those elements of C_n and there are those n reflections about a line connecting a vertex and its opposite face. These reflections fix one point and the rest of the n-1 points are swapped in pairs, so we associate with these $x_1x_2^{(n-1)/2}$. Just like above, we put this together to get that

$$\zeta_{D_{2n}}(x_1, ..., x_n) = \frac{1}{2}\zeta_{C_n}(x_1, ..., x_2) + \frac{1}{2}x_1x_2^{(n-1)/2}$$

□

5 The Number of Inequivalent Colorings

After developing the tools, we are finally ready to attack the coloring problem in a general sense. Again, we are interested in the *distinguishable* coloring. For example, if we rotate or flip a circular string of evenly spread out black and white beads, we do not consider a new color pattern is produced. As usual, we define G to be a group of permutations which acts on the point set X of n elements. Let K be a set of r colors. A coloring is a function $X \to K$ that assigns each point in X a color in K. Let Ω be the set of colorings of X. Since each point of X has r choices of color, we have r^n colorings in Ω.

In the case of the colored beads, the permutations include rotating and flipping. Notice that with a specific coloring of the circular string, we can see that rotating or flipping the *coloring* gives as a new coloring. But the new coloring can also be treated as the previous coloring after a rotation or a flip of the *beads*.

In general, given a coloring $\omega \in \Omega$, we define the "action of g on ω", $\hat{g}(\omega)$, as:

$$(\hat{g}(\omega))(x) = \omega(g(x))$$

for every $x \in X$ We should give a check that such set $\hat{G} = \{\hat{g}\}$ forms a group.

Identity: Take e the permutation that fixes all x. Then there is $\hat{e} \in \hat{G}$ such that

$$(\hat{e}(\omega))(x) = \omega(e(x)) = \omega(x).$$

$\hat{e}$ fixes the coloring for all ω.

Closure: For $\hat{g}_1, \hat{g}_2 \in \hat{G}$, let

$$(\hat{g}_1(\omega))(x) = \omega(g_1(x))$$

$$(\hat{g}_2(\omega))(x) = \omega(g_2(x))$$

we act the composition on x and ω

$$(\hat{g}_1\hat{g}_2(\omega))(x) = (\hat{g}_1(\omega))(g_2(x)) = \omega(g_1g_2(x))$$

Since $g_1 g_2 \in G$, we conclude $\hat{g}_1 \hat{g}_2 \in \hat{G}$.

Inverse: Take any $\hat{g}$ such that

$$(\hat{g}(\omega))(x) = \omega(g(x)).$$

There is an inverse $\hat{g}^{-1}$

$$(\hat{g}^{-1}(\omega))(x) = \omega(g^{-1}(x)).$$

Use what we discovered previously

$$((\hat{g}^{-1}\hat{g})(\omega))(x) = \omega(g^{-1}g(x)) = \omega(x) = \omega(gg^{-1}(x)) = ((\hat{g}\hat{g}^{-1})(\omega))(x)$$

Associativity: Take $\hat{g}_1, \hat{g}_2, \hat{g}_3 \in \hat{G}$,

$$(\hat{g}_1(\omega))(x) = \omega(g_1(x))$$

$$(\hat{g}_2(\omega))(x) = \omega(g_2(x))$$

$$(\hat{g}_3(\omega))(x) = \omega(g_3(x))$$

We have

$$(((\hat{g}_1\hat{g}_2)\hat{g}_3)(\omega))(x) = \omega((g_1g_2)g_3(x)) = \omega(g_1(g_2g_3(x))) = ((\hat{g}_1(\hat{g}_2\hat{g}_3)(\omega))(x)$$

using the associativity of G.

Therefore, $\hat{G}$ is a group of permutations of Ω.

Furthermore, there is an isomorphism $\psi : G \to \hat{G}$ by $\psi(g) = \hat{g}$. Assume $\psi(g_1) = \hat{g}_1 = \hat{g}_2 = \psi(g_2)$, then

$$(\hat{g}_1(\omega))(x) = (\hat{g}_2(\omega))(x) \Rightarrow \omega(g_1(x)) = \omega(g_2(x))$$

is true for any ω and x. When it is not the trivial case, consider ω' which colors some x' one color and others a different color. The coloring forces $g_1(x') = g_2(x')$. Since we can find such coloring ω' for each x, $g_1 = g_2$ is in fact true for any x. Meanwhile, the injectivity is satisfied by the construction.

Homormorpism is also obvious. For $g_1, g_2 \in G$ and $\hat{g}_1, \hat{g}_2 \in \hat{G}$,

$$(\phi(g_1)\phi(g_2)w)(x) = (\hat{g}_1\hat{g}_2(\omega))(x) = (\hat{g}_1(\omega))(g_2(x)) = \omega(g_1g_2(x)) = (\phi(g_1g_2)w)(x).$$

Now, observe that if for some $\hat{g} \in \hat{G}$, $\hat{g}\omega = \omega'$, the two colorings ω, ω' are indistinguishable. This means that two colorings are indistinguishable if they are in the same orbit of Ω under $\hat{G}$. Next theorem follows from this fact.

Theorem 4. If G is a group of permutations of X, and $\zeta_G(x_1, ..., x_n)$ is its cycle index, then the number of inequivalent colorings of X with r colors available is

$$\zeta_G(r, r, ..., r)$$

Proof. Let $F(\hat{g})$ is the set of colorings $\omega \in \Omega$ such that $\hat{g}\omega = \omega$. Since Ω is a finite set, we can identity it with the set $\{1, 2, ..., r^n\}$ Recall from our discussion of Burnside's Counting Theorem that $|F(\hat{G})| = \chi(\hat{g})$. Suppose that $\hat{g}(\omega) = \omega, \omega \in F(\hat{g})$. Pick $(...xyz...)$ some cycle of g.

$$\omega(y) = \omega(g(x)) = \hat{g}(\omega)(x) = \omega(x).$$

x and y are assigned with the same coloring. Repeat the argument for $\omega(z)$ and so on. Also, we can apply the argument to any cycle. We can then conclude that all the elements in the same cycle are assigned with the same color. Say g is the composition of k disjoing cycles and we want to find the number of indistinguishable colorings of X under g. Since all elements of the same cycle must be the same color, there are r choice of coloring for each cycle. The number of colorings is r^k. $k = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, so we have the relation

$$|F(\hat{g})| = r^k = r^{\alpha_1+\alpha_2+\cdots+\alpha_n}$$

Thus

$$F(\hat{g})| = \zeta_g(r, r, ..., r)$$

by the definition of ζ.

The useful isomorphism tells us that $|\hat{G}| = |G|$. By Burnside's Counting Theorem, we have the number of orbits is

$$\frac{1}{|\hat{G}|}\sum_{\hat{g}\in\hat{G}}\chi(\hat{g}) = \frac{1}{|\hat{G}|}\sum_{\hat{g}\in\hat{G}}[F(\hat{g})| = \frac{1}{|G|}\sum_{g\in G}\zeta_g(r, r, ..., r) = \zeta_G(r, r.., r)$$

$\square$

5.1 Inequivalent Colorings of a Tetrahedron

We now return to the original motivating problem: to find a formula for the number of inequivalent colorings of a tetrahedron with r colors. But first we will show that coloring the faces of a tetrahedron is equivalent to coloring the vertices of a tetrahedron.

To understand this imagine making a "new" tetrahedron by placing a vertex at the center of every face of the "old" tetrahedron. If we color the faces of the old tetrahedron, then we are coloring the vertices of the new one. So every coloring of the faces of a tetrahedron corresponds to (bijectively) to a coloring of the verities of a tetrahedron. Thus, the number of inequivalent ways to color the vertices of a tetrahedron is exactly the number of inequivalent ways to color the faces of a tetrahedron.

Now we must find the group of permutations G that represents the orientation preserving symmetries of the vertices of a tetrahedron. We can label the vertices 1,2,3,4 in any way and now we are looking for a subgroup of S_4. Since the order of a subgroup must divide the order of the group, $|G| = 1, 2, 3, 4, 6, 12,$ or 24. There are 8 "obvious" symmetries of the tetrahedron: rotation by 120 or 240 degrees about the axes that connect a vertex to the center of the opposite face. These are all of the eight 3-cycles in S_4. Composing these three cycles will give all of the 3 elements of S_4 that are two disjoint two cycles. Along with the identity, we have now found 12 elements of G, so $|G| = 12$ or 24. Since there are many permutations that are not in G (for example (12)), we know that $|G| \neq 24$ and thus the 12 elements that we have found are all of G.

Now we can note that this group is actually A_4 (which is the only order 12 subgroup of S_4). Now we are nearly done. The cycle index of A_4 (as calculated in an earlier exercise) is $\zeta_{A_4}(x_1, x_2, x_3, x_4) = \frac{1}{12}(x_1^4 + 3x_2^2 + 8x_1x_3)$

By Theorem 3, the formula for the number of inequivalent colorings of a tetrahedron with r colors is:

$$\frac{1}{12}(r^4 + 11r^2)$$

References

[1] Norman L.Biggs. *Discrete Mathematics*, pages 441–455. Oxford University Press, New York, 1985.

[2] Victor E. Hill IV. *Groups and Characters*, pages 165–171. Chapman Hall, Boca Raton, Florida, 2000.

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