The Chromatic Number of Kneser Graphs

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1 Introduction

The following is an adaptation of a section of *Proofs from THE BOOK* [1], a collection of some of the most beautiful proofs in mathematics. We will be exploring the chromatic number of Kneser graphs. This result from discrete math surprisingly relies on a fundamental theorem from the world of topology, the Borsuk-Ulam Theorem.

In order to understand this problem, we will start by defining some basic concepts.

Definition 1.1 (Graph)**.** A *graph* is a pair $G = (V, E)$ where V is a set of vertices, and E is a set of edges (u, v) where $u, v \in V$.

More intuitively, a graph is made up of a set of vertices and a set of edges between those vertices, which are defined by their endpoints. The study of graphs is a relatively new field of mathematics but has important applications. Much of graph theory is concerned with studying the properties of certain types of graphs. For the purposes of this problem, we will consider a special type of graph called a Kneser graph.

Definition 1.2 (Kneser graph)**.** A *Kneser graph* $K(n, k) = (V(n, k), E(n, k))$ is a graph whose vertices $V(n, k)$ are all possible subsets of $\{1, 2, ..., n\}$ of size k. There is an edge between two such subsets A and B if $A \cap B = \emptyset$.

We will refer to these subsets of size k as k-sets. Based on this construction, $|V(n, k)| = \binom{n}{k}$. A famous example of a Kneser graph is the Petersen graph, $K(5, 2)$, which has 10 vertices and 15 edges.

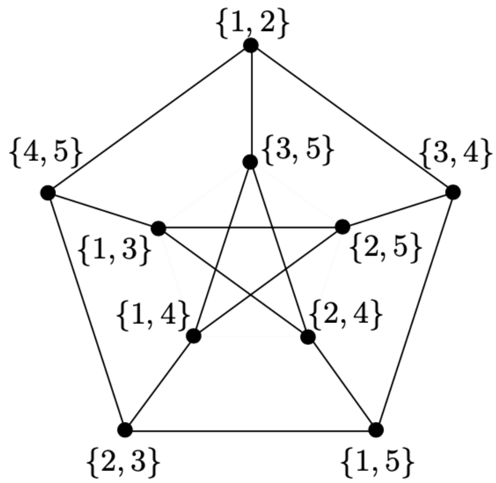

Figure 1: The Petersen graph, $K(5, 2)$ [1].

One property that we can use to describe the characteristics of a graph is its chromatic number. The chromatic number of graph G is the minimum number of colors that we can use to color the vertices of G such that no two vertices connected by an edge share the same color. To define this more formally, we will first define a vertex coloring.

Definition 1.3 (Vertex coloring)**.** A *vertex coloring* is a map $c : V(n, k) \to \{1, ..., m\}$ for some $m \geq 1$, defined as follows. Let $V_i = \{v \in V(n, k) \mid c(v) = i\}$ for all $i \in \{1, ..., m\}$. If vertices $v, w \in V_i$, then $(v, w) \notin E(n, k)$.

In other words, a valid vertex coloring defines a partition of $V(n, k)$ into m color classes, V_i for $1 \leq i \leq m$, such that each V_i is an independent set of vertices (i.e., there is no edge between any two vertices in the same color class). Then the chromatic number is defined as follows:

Definition 1.4 (Chromatic number)**.** The *chromatic number* of a graph G, written $\chi(G)$, is the minimum number of colors that we can use to create a valid vertex coloring of G.

Consider a color class for some vertex coloring of Kneser graph $K(n, k)$. Since there is no edge between any two vertices v and w in the color class, the k-sets corresponding to v and w must have a least one common element. However, notice that although all k-sets in a color class must be pairwise intersecting, all the k-sets in a color class need not share any single element. For example, if we were to begin to define a vertex coloring of the Petersen graph, a possible valid color class would be $V_1 = \{\{4,5\},\{3,5\},\{3,4\}\}$. Each pair of k-sets in V_1 shares an element and thus does not have an edge, but no single element is shared by all three k-sets.

Also notice that the chromatic number of a graph simply defines a lower bound for the number of colors we can use to create a valid vertex coloring. Given a vertex coloring using $\chi(G)$ colors, we can easily create a vertex coloring with $\chi(G)+1$ colors by arbitrarily splitting one of the color classes in two, as long as $\chi(G) < |V|$.

1.1 The Problem

Our problem aims to find the chromatic number for any Kneser graph. For the purposes of this proof, we will consider only Kneser graphs for which $n \geq 2k$, as our observations about the chromatic number of Kneser graphs will only hold under these conditions. If $n < 2k$, then by the Pigeonhole Principle there will always be at least one shared element between any two k-sets. Therefore, no two vertices will have an edge between them and the graph will be totally disconnected. So the chromatic number for any such Kneser graph will be 1, because no edges exist in the graph so we can color all vertices the same color.

Kneser's conjecture about the chromatic number of Kneser graphs was the following:

Theorem 1.5 (Kneser Conjecture)**.** *For $n \geq 2k$, $\chi(K(n,k)) = n - 2k + 2$.*

Because of the special construction of Kneser graphs, it is relatively straightforward to show that $\chi(K(n,k)) \leq n - 2k + 2$. We can prove this by constructing a proper vertex coloring with $n - 2k + 2$ colors.

Lemma 1.6. $\chi(K(n,k)) \leq n - 2k + 2$.

Proof. For $i \in \{1, ..., n - 2k + 1\}$, let V_i be the set of all k-sets where i is the smallest element. Since all elements of V_i share the element i, they are not adjacent in the graph. Therefore, we can color all vertices in V_i with the same color, and now we have used $n-2k+1$ colors. Consider the remaining uncolored k-sets. These k-sets cannot contain any elements of $\{1, ..., n - 2k + 1\}$. Suppose one such k-set A did contain some $j \in \{1, ..., n - 2k + 1\}$. If j is the minimum element of A, then A should have already been colored in color class V_j.

If not, then there is some $k < j$ in A, and thus A should have already been colored in color class V_k. Either way, this cannot happen.

So the remaining uncolored k-sets are all subsets of $\{n-2k+2, n\}$, which contains $2k-1$ elements. Then every pair of uncolored k-sets must share at least one element, because there is no way to divide $2k-1$ elements into two disjoint sets of size k. Therefore the remaining k-sets form an independent set, and so we can color them all the same color. In total, this vertex coloring uses $n-2k+2$ colors. □

Thus we have shown that the vertices of $K(n, k)$ can be properly colored with $n-2k+2$ colors. However, to prove the Kneser Conjecture, we still must show that we cannot find a vertex coloring with fewer than $n-2k+2$ colors. In order to show that $\chi(K(n, k)) \geq n-2k+2$, we must rely on a fundamental theorem from topology, the Borsuk-Ulam Theorem.

2 The Borsuk-Ulam Theorem in 2d

We are focusing at proving Borsuk-Ulam theorem in 2 dimensions with help of materials in chapter 9.2 and 9.6 regarding to circle functions and lifting.

Here are some definitions and theorems mentioned in the textbook that we will be using in our proof.

Definition 2.1 (Circle functions)**.** The continuous functions $f : S^1 \to S^1$ are called *circle functions*. By representing points on the circle with a variable θ, where θ is the usual angular measure taken from the positive x-axis in the plane, we can visualize the graph of a circle function $f : S^1 \to S^1$ by:

$$\{(x, y) \in S^1 \times S^1 | \, y = f(x)\} \tag{1}$$

Definition 2.2 (Degree of f)**.** For each circle function, $f : S^1 \to S^1$, there exists a unique $n \in \mathbb{Z}$ such that f is homotopic to $c_n(\theta) = n\theta$. The unique n associated to the circle function f is defined to be the *degree* of f and is denoted $\mathbf{deg}(f)$.

Then we need to define the essential idea used in our proof of the Borsuk-Ulam theorem, lifting. Before defining the related definitions and theorems that would be used later in our proof, we would represent the point $(cos(\theta), sin(\theta))$ in the circle as $e^{i\theta}$ in the imaginary plane in order to avoid confusing the two interpretations.

Definition 2.3 (Lifting)**.** Consider the function $p : \mathbb{R} \to S^1$ defined by $p(\theta) = e^{i\theta}$. This function p maps each interval $[r, r + 2\pi)$ bijectively around the circle and wraps the real line around S^1 infinite number of times. In figure2 below, it shows one of the covering map p.

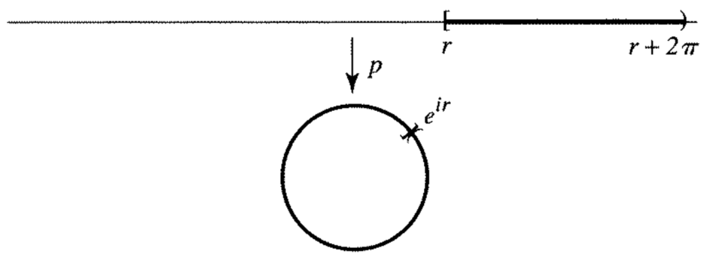

Figure 2: The function p wraps $[r, r + 2\pi)$ around S^1 [2].

Let $f : X \to S^1$ be continuous. A continuous function $f^* : X \to \mathbb{R}$ is called a *lifting* of f if $p \circ f^* = f$. Here is Figure 3 representing the lifting of f according to p:

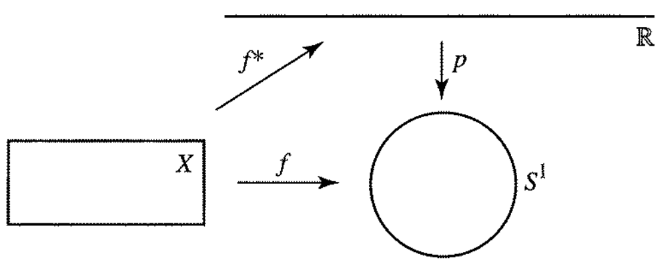

Figure 3: The lifting f^* of f satisfies $p \circ f^* = f$ [2].

Definition 2.4 (Antipode-preserving function)**.** A circle function $f : S^1 \to S^1$ is said to be antipode-preserving if f maps antipodal points to antipodal points. In other words, if z and $-z$ are the opposite two points in a circle, then $f(z)$ and $f(-z)$ are two opposite points in the image of f, i.e. $f(-z) = -f(z)$ shown in the figure 4 below.

Before actually proving the Borsuk-Ulam theorem when $n = 2$, we need to prove two lemmas and one theorem first.

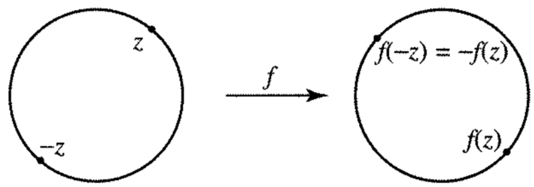

Figure 4: Antipode-preserving functions map antipodal points to antipodal points [2].

Lemma 2.5. *Assume that $f : [0, 2\pi] \to S^1$ is an antipode-preserving circle function, and let $f^* : [0, 2\pi] \to \mathbb{R}$ be a lifting of f, then there exists $n \in \mathbb{Z}$ such that $f^*(\theta+\pi) = f^*(\theta)+(2n+1)\pi$ for all $\theta \in [0, \pi]$.*

Proof. Define $p : \mathbb{R} \to S^1$ be $p(\theta) = e^{i\theta}$. Let $z \in S^1$, and $t \in P^{-1}(z)$, $\theta \in P^{-1}(-z)$.
Then $x = t - \theta \iff ix = i(t - \theta) \iff e^{ix} = e^{i(t-\theta)}$:

$$e^{ix} = e^{i(t-\theta)} = \frac{e^{it}}{e^{i\theta}} = \frac{p(t)}{p(\theta)} = \frac{z}{-z} = -1 \tag{2}$$

Then take the natural logarithm of both side in equation(2), we can get

$$\begin{aligned} ln(e^{ix}) &= ln(-1) \\ xi &= ln(-1) \\ ln(-1) &= i\pi(2n+1), \text{ for some } n \in \mathbb{Z} \end{aligned}$$

Then we can deduce the following equation for x:

$$xi = i\pi(2n + 1) \tag{3}$$

$$x = (2n + 1)\pi \tag{4}$$

Substitute equation(3) back into $x = t - \theta$, we get the formula:

$$(2n + 1)\pi = t - \theta \iff t = \theta + (2n + 1)\pi \tag{5}$$

Thus, we prove that for any pair of points $t, \theta \in \mathbb{R}$ in the preimage of antipodes in S^1

under p, they are always differed by $(2n+1)\pi$.

Then let f^* be the lifting of f defined in the lemma 2.5, then we have $p \circ f^* = f$ and

$$f^*(\theta+\pi) \in p^{-1}(f(\theta+\pi)),\ f^*(\theta) \in p^{-1}(f(\theta)$$

Since f is antipode-preserving circle function, we have $f(\theta+\pi) = -f(\pi)$ by definition, so they are antipodes in S^1. By the result proved earlier, we have

$$f^*(\theta+\pi) = f^*(\theta) + (2n+1)\pi \tag{6}$$

□

With help of lemma 2.5 and the following theorem 2.6 in the textbook [1], we can prove the following lemma 2.7 regarding to the nonzero degree of antipode-preserving functions.

Theorem 2.6. *Let $f : [0, 2\pi] \to S^1$ be a circle function, and let $f^* : [0, 2\pi] \to \mathbb{R}$ be a lifting of f. Then $f^*(2\pi) - f^*(0) = 2\pi \cdot deg(f)$.*

Lemma 2.7. *Assume that $f : [0, 2\pi] \to S^1$ is an antipode-preserving circle function, and let $f^* : [0, 2\pi] \to \mathbb{R}$ be a lifting of f, then $f^*(2\pi) - f^*(0) \neq 0$ and antipode-preserving circle functions have nonzero degree.*

Proof. By lemma 2.5, we have the following equations:

$$\begin{aligned} f^*(2\pi) - f^*(0) &= f^*(\pi) + (2n+1)\pi - f^*(0) \\ &= f^*(0) + (2n+1)\pi + (2n+1)\pi - (2n+1)\pi(0) \\ &= (4n+2)\pi, \text{ where n is an integer.} \end{aligned}$$

According to Theorem 2.6, we also know that $f^*(2\pi) - f^*(0) = 2\pi \cdot deg(f)$, so we can conclude that $2\pi \cdot deg(f) = (4n+2)\pi \iff deg(f) = 2n+1$. However, there does not exist any $n \in \mathbb{Z}$ such that $2n+1 = 0$, so $deg(f) \neq 0$. □

Theorem 2.8. *A circle function $f : S^1 \to S^1$ has degree 0 if and only if f extends to a continuous fucntion on the disk D. (that is , if and only if there exists a continuous functionF $: D \to S^1$ such that $F(x) = f(x)$ for all $x \in S^1$.*

Proof. $\Longrightarrow$ Assume that f has degree 0. Then there exists a homotopy $G : S^1 \times I \to S^1$ with $(\theta, 0) \to c_0(\theta)$ and $(\theta, 1) \to f(\theta)$, where we define the constant function $c_0 : S^1 \to S^1$ by $\theta \in S^1 \to 0 \in S^1$. We define F by $(r, \theta) \to (\theta, r)$. F inherits continuity from G. Since at $r = 0$, $F(0, 0) = F(0, \theta) = G(\theta, 0)$ is constant, F is well-defined on D. Meanwhile, $F(1, \theta) = G(\theta) = G(\theta, 1) = f(\theta)$ for all $\theta \in S^1$ as desired.

$\Longleftarrow$ Assume f extends to a continuous function $F : D \to S^1$ by $(\theta, r) = F(r, \theta)$. G is continuous because F is continuous. $F(0, 0)$ is a constant function with degree 0. To show f has degree 0, we first observe

$$G|_{S^1\times\{0\}}(\theta) = G(\theta, 0) = F(0, \theta) = F(0, 0) \text{ is a constant function}$$

$$G|_{S^1\times\{1\}}(\theta) = f(\theta).$$

Since G is continuous, it is a homotopy between $G|_{S^1\times\{0\}}$ and $G|_{S^1\times\{1\}}$. Thus they have the same degree and f has degree 0 [2].

Define the function $G : S^1 \times I \to S^1$ by $(\theta, t) \mapsto F(t, \theta)$. G inherits continuity from F.

□

Theorem 2.9. *There is no continuous antipode-preserving function $f : S^2 \to S^1$.*

Proof. We assume that there is an antipode -preserving function $f : S^2 \to S^1$ for a contradiction. Let f_0 be the circle function obtained by restricting f to the equator. Observe that $f_0 : S^1$ is defined as $f_0(\theta) = f(\theta, 0)$ is antipode-preserving and continuous. We let $f^* : S^1 \to S^1$ be a lifting of f. By Theorem 2.6, we conclude that $f^*(2\pi) - f^*(0) \neq 0$ and f_0 has nonzero degree. On the other hand, the restricted function of f, $f|_{\text{upperhalf and the equator}}$ is homotopic to a continuous function on the disk D, $F : D \to S^1$. f_0 extends to F because $F(x) = f_0(x)$ for all $x \in S^1$. By Theorem 2.8 from textbook [2], we conclude that the circle function $f_0 : S^1 \to S^1$ has degree 0. But f_0 must have nonzero degree. We arrive at a logical contradiction. Therefore, there is no such continuous antipode-preserving function. □

Theorem 2.10 (The Borsuk-Ulam Theorem in 2d)**.** *Let $f : S^2 \to \mathbb{R}^2$ be a continuous function. Then there exists a pair of antipodal points $x^*, -x^* \in S^2$ that are mapped to the same point $f(x^*) = f(-x^*)$.*

Proof. We assume no such x exists for a contradiction. We define a map $h : S^2 \to S^1$ by $x \mapsto \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$. Since f is continuous, h is also continuous. Furthermore, by construction,

$$h(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|} = -\frac{f(-x) - f(x)}{|f(x) - f(-x)|} = -h(x)$$

h is antipode-preserving. Theorem 2.9 says such function is impossible. This leads to a contradiction. We then conclude the existence of $x \in S^2$ such that $f(x) = f(-x)$. $\square$

An interesting real-world application of this theorem is the statement: "There are always two points on opposite sides of the earth with the same temperature and humidity." If we assume that temperature and humidity vary continuously and view earth as S^2, then we can apply Borsuk-Ulam to the function that sends a point on earth to the point in $\mathbb{R}^2$ with its temperature and humidity as coordinates.

In order to prove the Kneser conjecture, we will actually need a more general form of the theorem that applies to any dimension. Hopefully, the proof in two dimensions will provide some intuition for the result in higher dimensions. For the sake of brevity, we will omit the proof for dimensions greater than two.

Theorem 2.11 (Borsuk-Ulam Theorem)**.** *Let $f : S^d \to \mathbb{R}^d$ be a continuous function. Then there exists a pair of antipodal points $x^*, -x^* \in S^d$ that are mapped to the same point $f(x^*) = f(-x^*)$.*

3 The Lyusternik–Shnirel'man Theorem

The Borsuk-Ulam Theorem will allow us to derive the following result, which will prove to be surprisingly useful. Here, the author of [1] notes that a stronger version of the theorem is also true; if $U_1, \ldots, U_d$ are each either open or closed, the result of the theorem still holds. However, we will only need to use Theorem 3.1 as stated.

Theorem 3.1 (Lyusternik-Shnirel'man Theorem)**.** *If the d-sphere S^d is covered by $d+1$ sets,*

$$S^d = U_1 \cup \cdots \cup U_d \cup U_{d+1},$$

such that $U_1, \ldots, U_d$ are open, then at least one of the $d+1$ sets contains a pair of antipodal points $x^*, -x^*$.

Before we present the proof of this theorem, let us give the following definition.

Definition 3.2 (Distance)**.** The *distance* between a point x and a set A, denoted $\delta(x, A)$, is defined as $\inf\{\delta(x,a) \mid a \in A\}$, where $\delta(x,a)$ is the distance between two points in the metric space.

Proof of Lyusternik-Shnirel'man Theorem. Let $U_1, \ldots, U_d, U_{d+1}$ be a covering of S^d so that $U_1, \ldots, U_d$ are open. Consider the map $f : S^d \to \mathbb{R}^d$, with

$$f(x) \coloneqq (\delta(x, U_1), \delta(x, U_2), \ldots, \delta(x, U_d)).$$

This map is continuous. Let $\varepsilon > 0$ be given. Choose $\sigma = \frac{\varepsilon}{\sqrt{d}} > 0$. Notice that for all $x, y \in S^d$, $|\delta(x, U_i) - \delta(y, U_i)| \leq \delta(x, y)$ for each U_i by the reverse triangle inequality. Thus, if $\delta(x,y) < \sigma$ we have that

$$\begin{aligned}
\delta(f(x), f(y)) &= \delta\Big(\big(\delta(x, U_1), \ldots, \delta(x, U_d)\big), \big(\delta(y, U_1), \ldots, \delta(y, U_d)\big)\Big) \\
&= \sqrt{|\delta(x, U_1) - \delta(y, U_1)|^2 + \cdots + |\delta(x, U_d) - \delta(y, U_d)|^2} \\
&\leq \sqrt{\underbrace{\delta(x,y)^2 + \cdots + \delta(x,y)^2}_{d}} \\
&= \sqrt{d \cdot \delta(x,y)^2} \\
&< \sqrt{d \cdot \sigma^2} \\
&= \sqrt{d \cdot \frac{\varepsilon^2}{d}} \\
&= \varepsilon.
\end{aligned}$$

This allows us to use the Borsuk-Ulam Theorem, which tells us that there exists some pair of antipodal points $x^*, -x^*$ such that $f(x^*) = f(-x^*)$. If $x^*, -x^*$ are both in U_{d+1}, we are done. Otherwise, U_{d+1} does not contain both points, so at least one of $x^*, -x^*$ are contained in the other U_i's, say it is U_k $(k \leq d)$. Without out loss of generality, let $x^* \in U_k$. This means

$\delta(x^*, U_k) = 0$ so

$$\begin{aligned}(\delta(-x^*, U_1), \ldots, \delta(-x^*, U_k), \ldots, \delta(-x^*, U_{d+1})) &= f(-x^*) \\ &= f(x^*) \\ &= (\delta(x^*, U_1), \ldots, 0, \ldots, \delta(x^*, U_{d+1})).\end{aligned}$$

Thus, $\delta(-x^*, U_k) = 0$ as well. We assumed that U_k was open, so $\delta(-x^*, U_k) = 0$ implies that $-x^* \in \overline{U_k}$, the closure of U_k. Let $-U_k$ be the set consisting of the antipodes of all the points in U_k. Then $-x^* \in -U_k$ because $x^* \in U_k$. This means $-x^* \in \overline{U_k} \cap -U_k$. U_k is open so $-U_k$ is as well. Thus, there exist some neighborhood V of $-x^*$ so that $-x^* \in V \subseteq -U_k$. By the definition of closure, $V \cap U_k \neq \emptyset$ because $-x^* \in \overline{U_k}$. Thus, there exists $y \in V \cap U_k$. Then y is in both $U_k, -U_k$. Since $y \in -U_k$, $-y \in U_k$. Thus, $y, -y \in U_k$ so U_k contains some pair of antipodal points. $\square$

We are now almost ready to attack the Kneser conjecture. However, we have to give one more lemma before we proceed. We say that an arrangement of points on S^d is in *general position* if no $d+1$ of the points lie on a hyperplane through the center of the sphere.

Lemma 3.3. *We can always arrange finitely many points in general position on $S^d, d \geq 1$.*

Sketch of proof. Since $S^d \subseteq \mathbb{R}^{d+1}$, it takes $d+1$ points to define a hyperplane. We are concerned with hyperplanes defined by d points on S^d and the origin. We want to avoid hyperplanes with an "extra point." This can be done by adding points one at a time. Define a measure based on the surface area of the d-sphere. The set of points in S^d that will lead to a hyperplane with an extra points will be the intersection of finitely many hyperplanes with S^d and thus will always have measure 0. Since S^d has positive measure, we will always be able to avoid selecting such points. $\square$

4 Proof of the Kneser Conjecture

While it may seem unclear how the preceding theorems and lemma relate to the original problem of whether $\chi(K(n,k))$ is $n - 2k + 2$, we have actually made significant progress already. We are finally ready to put everything together.

Proof of the Kneser Conjecture. We have already shown that $n-2k+2$ is an upper bound for $\chi(K(n,k))$ in Lemma 1.6. We will now show that this bound is strict.

Let $n \geq 2k$. Assume to the contrary that we can properly color $K(n,k)$ with $n-2k+1$ colors. Then arrange n points in general position on S^{n-2k+1} (possible by Lemma 3.3). Every k-set of this set of points will correspond to exactly one vertex of $K(n,k)$. Suppose we partition the set $V(n,k)$ of all k-subsets of these points into $n-2k+1$ classes, $V_1, \ldots, V_{n-2k+1}$, based on the color of the corresponding vertex. Our goal will be to show that this is could not have been a proper coloring of $K(n,k)$, that is, we want to find a pair of disjoint k-sets A, B that belong to the same V_i. For $i = 1, \ldots, n-2k+1$, let

$$O_i = \{x \mid \text{the open hemisphere } H_x \text{ with pole } x \text{ contains a } k\text{-set from } V_i\}.$$

Each O_i will be open. Let $x \in O_i$, then H_x contains some k-set from V_i, call it D. Let $\varepsilon = \min_{y \in D}\{\delta(y, {O_i}^c\}$. Since ${O_i}^c$ is closed, each of these distances must be positive so $\varepsilon > 0$. Then if $z \in B(x, \frac{\varepsilon}{2})$, H_z will still contain all the points in D so $z \in O_i$. Thus, O_i is open.

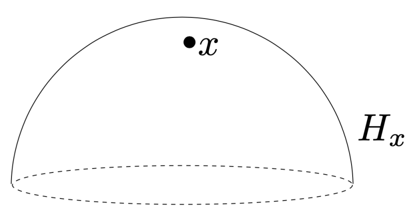

Figure 5: An open hemisphere in S^2 [1].

Let $C = S^{n-2k+1} \setminus (O_1 \cup \cdots \cup O_{n-2k+1})$. Then $O_1, \ldots, O_{n-2k+1}, C$ is a covering of S^{n-2k+1} that satisfies the Lyusternik-Shnirel'man Theorem, so we know that one of these sets contains a pair of antipodal points $x^*, -x^*$. Suppose the set is C, then H_{x^*}, H_{-x^*} must contain $k-1$ or fewer points. If they contained greater than or equal to k points, they would contain some k-set, but each k-set is in some V_i so then x^* or $-x^*$ would be in one of the O_i's. Thus, there are at least $n-2(k-1) = n-2k+2$ points in $S^{n-2k+1} \setminus (H_{x^*} \cup H_{-x^*})$. However, since $x^*, -x^*$ are antipodal, $S^{n-2k+1} \setminus (H_{x^*} \cup H_{-x^*})$ is just the equator with respect to x^*

as the north pole. The equator is on a hyperplane through the origin, so there are at least $n - 2k + 2$ points that share a hyperplane through the origin which contradicts the points being arranged in general position. Thus, it must be the case that $x^*, -x^*$ are in some O_i.

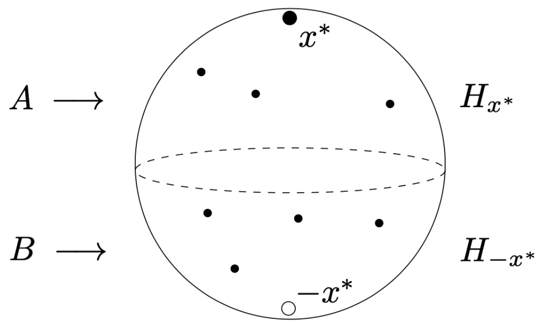

Figure 6: $A \subseteq H_{x^*}, B \subseteq H_{-x^*}$ [1].

This means there are k-sets A, B that are both in the class V_i, with $A \subseteq H_{x^*}$ and $B \subseteq H_{-x^*}$. Since H_{x^*}, H_{-x^*} are open hemispheres centered at opposite poles, they are disjoint so A, B are disjoint as well (see Figure 6). Then the vertices corresponding to A, B are adjacent but were both colored the same color which contradicts our assumption that we had a proper vertex coloring with $n - 2k + 1$ colors. Thus, $\chi(K(n, k)) \geq n - 2k + 2$ which completes the proof. $\square$

References

[1] M. Aigner, G.M. Ziegler, *Proofs from THE BOOK*, 2013, Springer-Verlag Berlin Heidelberg, pp. 251-55.

[2] Colin Adams, Robert Franzosa, *Introduction to Topology*, 2009, Pearson, pp. 295-302, 313-321.