Cohomology and Base Change

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Course: Abelian Varieties

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Contents

1 Preliminaries

In this section, we will fix notations that we will use in the report, and cite relevant notions and results from algebra and algebraic geometry.

In [Mumf. Abelian Varieties], it uses the terminology *prescheme*, but it is mostly outdated. In this report, I will simply call it a "scheme". The terminology *scheme* in the book is used for what I call a "separated scheme".
In the setting of the cohomology groups of complexes, I will use Z_n and $\ker \partial^n$ interchangeably, and resp. B_n and $\mathrm{Im}\partial^n$.

1.1 Algebraic Geometry

Theorem 1.1. *Let X be a scheme, and $\mathcal{F}$ an $\mathcal{O}_X$-module. The following are equivalent:*

1. *for all $U \subset X$ open affine we have $\mathcal{F}|_U \cong \widetilde{M}$, for some $\Gamma(U, \mathcal{O}_X)$-module M;*
2. *there exists an open cover $\{U_i\}$ of X with affine schemes such that for all i we have $\mathcal{F}|_{U_i} \cong \widetilde{M}$ for some $\Gamma(U_i, \mathcal{O}_X)$-module M_i;*
3. *for all $x \in X$ there exists an open neighborhood U of x in X, two sets I, J, and an exact sequence of $\mathcal{O}_X|_U$-modules*
$$(\mathcal{O}_X|_U)^{(I)} \to (\mathcal{O}_X|_U)^{(J)} \to \mathcal{F}|_U \to 0;$$
4. *for all inclusions $V \subset U$ of open affines in X, the natural map*
$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{F}(V)$$
is an isomorphism of $\mathcal{O}_X$-modules.

Here $\widetilde{M}$ denotes the tilde construction.

Definition 1.2. An $\mathcal{O}_X$-module satisfying the equivalent conditions from the theorem is called *quasi-coherent*.

Example: The invertible sheaf $\mathcal{O}(1)$ on projective space $\mathbb{P}^n$.

Definition 1.3. Let X be a noetherian scheme, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then we call $\mathcal{F}$ *coherent* if for all open affine subsets $U \subset X$ we have that $\Gamma(U, \mathcal{F})$ is a finitely generated $\Gamma(U, \mathcal{O}_X)$-module.

Definition 1.4. In the category of schemes, the fiber product always exists. For any morphism of schemes $X \to Z$ and $Y \to Z$, there is a scheme $X \times_Z Y$ with morphisms to X and Y, making the diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

commutative, and which is universal with this property.

Example(base change): If A and B are C-algebras, then $\mathrm{Spec}\,(A \otimes B)$, with the obvious arrows, is a (the) fiber product of $\mathrm{Spec}\,(A)$ and $\mathrm{Spec}\,(B)$ over $\mathrm{Spec}\,(C)$ in the category of schemes.

1.2 Homological Algebra

Lemma 1.5. *Given an abelian category $\mathcal{A}$, the category Comp($\mathcal{A}$) of complexes $M^\bullet$ of objects in $\mathcal{A}$ is an abelian category. If*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

is a short exact sequence in Comp($\mathcal{A}$) then there are natural maps $\delta^i : h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ giving rise to a Long Exact Sequence

$$\cdots \to h^i(A^\bullet) \to h^i(B^\bullet) \to h^i(C^\bullet) \to h^{i+1}(A^\bullet) \to \dots$$

in $\mathcal{A}$.

Definition 1.6. Assume that abelian category $\mathcal{A}$ has enough injectives, and let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Let $M \in \mathcal{A}$ be an object and let $M[0] \to I^\bullet$ be an injective resolution of M(such resolution always exists). We define R^iFM to be the object $h^i(F(I\bullet))$ of $\mathcal{B}$.
For X a topological space, and $\mathcal{F} \in \mathrm{Sh}(X)$ a sheaf we denote by $H^i(X, \mathcal{F})$ the right derived object $R^i\Gamma(X, \mathcal{F})$in Ab. We call the $H^i(X, \mathcal{F})$ the (sheaf) cohomology groups of $\mathcal{F}$. (The abelian category $\mathrm{Sh}(X)$ has enough injectives so this is well-defined.)

Definition 1.7. Let $f : X \to Y$ be a continuous map of topological spaces. Then we define the *higher direct image* functors $R^if_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ to be the right derived functors of the direct image functor f_*. (This makes sense because f_* is left exact.)

Definition 1.8. An R-module N is $flat$ if tensoring with N over R as a functor from RMod to itself is an exact functor (sends short exact sequences to short exact sequences)

Proposition 1.9. *A direct sum of R-modules is R-flat iff each module is R-flat.*

Corollary 1.10. *A free module is flat.*

Definition 1.11. For R a ring, a projective R-module is a projective object in $RMod$ the category of R-modules. In other words, for all diagrams of R-module homomorphisms of the form

$$\begin{tikzcd} & A \arrow[d] \\ N \arrow[r, "f"'] & B \end{tikzcd}$$

there exists a lift, hence a morphism $N \xrightarrow{\phi} A$ making a commuting diagram of the form

$$\begin{tikzcd} & A \arrow[d] \\ N \arrow[r, "f"'] \arrow[ur, dashed, "\phi"] & B \end{tikzcd}$$

Corollary 1.12. *Projective modules are flat. Over a noetherian ring, every finitely generated flat module is projective.*

Theorem 1.13. *Given a short exact sequence of A modules*

$$0 \to A \to B \to C \to 0$$

with B, C flat, then A is flat.

Lemma 1.14. *Given a finite long exact sequence of A-modules, $0 \to L^0 \xrightarrow{\partial^0} L^1 \xrightarrow{\partial^1} \cdots \to L^{n-1} \xrightarrow{\partial^{n-1}} L^n \xrightarrow{\partial^n} 0$ such that L^p is flat for $p > 0$, then all $\ker \partial^p$ are flat and L^0 is flat too.*

Proof. There is a short exact sequence $0 \to \mathrm{Im}\partial^{n-2} = \ker \partial^{n-1} \to L^{n-1} \to \mathrm{Im}\partial^n = L^n \to 0$. By theorem [1.13] $\ker \partial^{n-1}$ is also flat. Then we use the short exact sequences $0 \to \ker \partial^p \to K^p \to \mathrm{Im}\partial^p = \ker \partial^{p+1}$ inductively and conclude that all $\ker \partial^p$ are A-flat for $p \geq 0$ but $L^0 = \ker \partial^0$. $\square$

Lemma 1.15. *Let A be a ring, $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Suppose that M'' is flat, then the sequence $0 \to M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$ is exact for any A-module N.*

Proof. The tensor product is already right exact. We write N as a quotient of a free module, say $0 \to K \to A^N \to N \to 0$. Tensoring the exact sequence of Ms with the map $K \to A^N$ yields a diagram with all rows and columns exact:

$$\begin{tikzcd}
 & & & 0 \arrow[d, "b"] & \\
 & K \otimes M' \arrow[r] \arrow[d] & K \otimes M \arrow[r] \arrow[d] & K \otimes M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r, "a"] & (M')^{\oplus N} \arrow[r] & (M)^{\oplus N} \arrow[r] & (M'')^{\oplus N} \arrow[r] & 0
\end{tikzcd}$$

Here we use that free modules are flat to get the arrow (a) and that M'' is flat to get the arrow (b). The result follows from the snake lemma. $\square$

Theorem 1.16. *If R is a noetherian ring, then an R-module that is noetherian has to be finitely generated.*

Theorem 1.17. *If $f : X \to Y$ is a proper morphism of locally noetherian schemes and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$-modules on X, for all $p \geq 0$ the direct image sheaves $R^p f_*(\mathcal{F})$ are then coherent sheaves of $\mathcal{O}_Y$-modules.*

Definition 1.18. If $f : X \to Y$ is a morphism of schemes and $\mathcal{F}$ is a quasi-coherent sheaf on X, $\mathcal{F}$ is said to be *flat* over Y or f-flat if for each $x \in X$, $\mathcal{F}_x$, which has the natural structure of $\mathcal{O}_{Y,f(x)}$-module, is flat over $\mathcal{O}_{Y,f(x)}$.

Remark 1.19. *An equivalent definition of flatness is that there exists some affine cover $\{U\}$ of X and $\{V\}$ of Y, and $f(U) \subset V$, such that $\mathcal{F}(U)$ is a flat $\mathcal{O}(V)$-module.*

Definition 1.20.

Let X be a topological space. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X. Put a well-ordering on I. For $p \in \mathbb{Z}_{\geq 0}$ and for $i_0, \ldots, i_p \in I$, we set

$$U_{i_0 \ldots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}.$$

Let $\mathcal{F} \in \mathrm{Sh}(X)$ be a sheaf of abelian groups on X. For $V \subset U$ open and $s \in \mathcal{F}(U)$ we write $s|_V$ for the image of s in $\mathcal{F}(V)$ under the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$. We set

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0 \ldots i_p}), \quad (p \geq 0).$$

Moreover, we define maps

$$d = d^p : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$$

given by

$$(d\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\hat{i}_k,\ldots,i_{p+1}}|_{U_{i_0 \ldots, i_{p+1}}}.$$

The notation $\hat{}$ means "omit". A calculation shows that $d^{p+1} \circ d^p = 0$. We obtain a complex $C^\bullet(\mathcal{U}, \mathcal{F})$ in the category Ab of abelian groups. Up to isomorphism, this complex is independent of the choice of well-ordering. For all $p \geq 0$ we define the p-th Čech cohomology of $\mathcal{F}$ with respect to $\mathcal{U}$ to be the group $\check{H}^p(\mathcal{U}, \mathcal{F}) = h^p(C^\bullet(\mathcal{U}, \mathcal{F}))$.

Theorem 1.21. *Let X be a quasi-compact separated scheme, let $\mathcal{U}$ be an open affine cover of X, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for all $p \geq 0$, we have a natural isomorphism*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F}).$$

In the next sections, we will prove the following main result.

2 Main theorem

Theorem 2.1. *Let $f : X \to Y$ be a proper morphism of separated noetherian schemes with $Y = \mathrm{Spec}\, A$ affine, and $\mathcal{F}$ a coherent sheaf on X, flat over Y. There is a finite complex $K^\bullet : 0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$ of finitely generated projective A-modules and an isomorphism of functors*

$$H^p(X \times_Y \mathit{Spec}\, B, \mathcal{F} \otimes_{\mathcal{O}_{\mathit{Spec}\, A}} \mathcal{O}_{\mathit{Spec}\, B}) \cong H^p(K^\bullet \otimes_A B), \quad (p \geq 0)$$

on the category of A-algebras B.

First we build a Čech complex of X. Take a finite affine covering (possible because X is noetherian) $\mathfrak{U} = \{U_i\}_{i \in I}$ of open affine subsets. The Čech complex $C^\bullet(\mathfrak{U}, \mathcal{F}) = \oplus C^p(\mathfrak{U}, \mathcal{F})$. We note that this is a finite complex because the covering is finite (therefore the direct sum in place of direct product); since $\mathcal{F}(U_{i_0,\ldots,i_p})$ is a flat $\mathcal{O}_Y(V)$ module for the affine open $V = \mathrm{Spec}\, A = Y$, it is a finite complex of flat A-modules. Its cohomology groups are isomorphic to the cohomology groups $H^p(X, \mathcal{F})$ by theorem [1.21] (here we use the separatedness assumption). For any A-algebra B, it induces a map $\mathrm{Spec} B \to Y$. The following pullback square gives us $X \times_Y$ Spec B, which is also quasi-compact and has a finite affine covering by $\mathcal{U} := \{U_i \times_Y \operatorname{Spec} B\}$ (the covering is indeed affine by example [1.4]).

$$\begin{array}{ccc} X \times_Y \text{Spec } B & \longrightarrow & \text{Spec } B \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Since tensor product commutes with direct sums, we see that $C^p(\mathfrak{U}, \mathcal{F}) \otimes_A B$ is the module of Čech p-cochains of $\mathcal{F} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \mathcal{O}_{\operatorname{Spec} B}$ given our covering. Apply the same theorem again (as separatedness is stable under base change) we obtain

$$H^p(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \mathcal{O}_{\operatorname{Spec} B}) \cong \check{H}^p(\mathcal{U}, \mathcal{F} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \mathcal{O}_{\operatorname{Spec} B}) = H^p(C^\bullet \otimes_A B), \quad (p \geq 0)$$

for all B, and, functorial in B. We make an observation that $R^p f_*(\mathcal{F})$ is a sheaf of Y that sends an open $U \subset Y$ into $H^i(f^{-1}(U))$ so $R^p f_*(\mathcal{F})(Y) = H^p(X, \mathcal{F})$, which is isomorphic to $H^p(C^\bullet)$. By theorem [1.17], given $\mathcal{F}$ coherent and f proper, we have $R^p f_*(\mathcal{F})$ are coherent sheaves of $\mathcal{O}_Y$-modules. Thus $H^p(C^\bullet)$ are finitely generated A-modules. From here we can rephrase the theorem to be the following lemmas.

3 Proof of Main theorem

Lemma 3.1. *Let $C^\bullet$ be a complex of A-modules(A any noetherian ring) such that the $H^i(C^\bullet)$ are finitely generated A-modules for all i and such that $C^p \neq (0)$ only if $0 \leq p \leq n$ for some n. Then there exists a complex $K^\bullet$ of finitely generated A-modules such that $K^p \neq (0)$ only in $0 \leq p \leq n$ and K^p is free if $1 \leq p \leq n$ and a homomorphism of complexes $\phi : K^\bullet \to C^\bullet$ such that ϕ induces isomorphisms $H^i(K^\bullet) \xrightarrow{\sim} H^i(C^\bullet)$, for all i. Moreover, if the C^p are A-flat, then K^0 will be A-flat too.*

$$\begin{array}{ccccccccc} & & K^m & \xrightarrow{\partial'^m} & K^{m+1} & \xrightarrow{\partial'^{m+1}} & K^{m+2} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle \phi_m} & & \downarrow{\scriptstyle \phi_{m+1}} & & \downarrow{\scriptstyle \phi_{m+2}} & & \\ \dots \longrightarrow C^{m-1} & \longrightarrow & C^m & \xrightarrow[\partial^m]{} & C^{m+1} & \xrightarrow[\partial^{m+1}]{} & C^{m+2} & \longrightarrow & \dots \end{array}$$

Proof. (Note that the requirement of K^p to be A-projective in our theorem will be satisfied since it is A-flat and finitely generated over a noetherian A by corollary [1.12]) We will construct the complex $K^\bullet$ inductively in descending order. First of all, we put $K^p = 0$ for $p > n$.

Inductive hypothesis: Assume for $p > m$ we have constructed $(K^p, \phi_p : K^p \to C^p, \partial'^p : K^p \to K^{p+1})$ such that the following conditions hold ($p > m$):

1. $\partial'^{p+1} \circ \partial'^p = 0$
2. $\partial^p \phi^p = \phi_{p+1} \partial'^p$,
3. The ϕ^p induces isomorphisms in cohomology $H^q(K^\bullet) \xrightarrow{\sim} H^q(C^\bullet)$ for $q > m+1$, and a surjection $\ker \partial'^{m+1} \twoheadrightarrow H^{m+1}(C^\bullet)$,
4. The K^p are A-free and finitely generated,

We will show the conditions hold for $p = m$. Observe that item one ensures that we are building a complex (of A-modules), item two ensures that ϕ gives us a map between complexes. Notice that for $n = m$ the above conditions are trivially satisfied.

First we consider the case when $m \geq 0$. Take B^{m+1} to be the kernel of the homomorphism $\ker \partial'^{m+1} \twoheadrightarrow H^{m+1}(C^{\bullet})$. $B^{m+1} \subset \ker \partial'^{m+1} \subset K^{m+1}$ which is finitely generated so is B^{m+1} as A is noetherian (theorem [1.16]). Then B^{m+1} is some quotient of a finitely generated free module K'^m. We can find a surjection $\partial' : K'^m \twoheadrightarrow B^{m+1}$. At the same time, $H^m(C^{\bullet})$ is a finitely generated free A-module so we can find another surjection $\lambda : K''^m \twoheadrightarrow H^m(C^{\bullet})$ with K''^m a finitely generated free A-module. Take $\mu : K''^m \to Z^m(C^{\bullet}) = \ker \partial^m$ to be a lift of λ. Notice that we can define a map $\phi''_m : K''^m \xrightarrow{\mu} Z^m(C^{\bullet}) \hookrightarrow C^m$ as we have the latter natural inclusion. We construct $K^m := K'^m \oplus K''^m$ and define ∂'^m summandwise: on K''^m it becomes zero and on K'^m it is the same as ∂'. We observe that $\phi_{m+1} \circ \partial'^m(K^m) = \phi_{m+1} \circ \partial'(K'^m) \subset \phi_{m+1} \circ B^{m+1}$, and by definition of B^{m+1} it is a subset of $\partial^m C^m$. In other words, $\phi_{m+1} \circ \partial'(K'^m)$ is in the image of ∂^m so we can find a lift: $\phi'_m : K^m \to C^m$ such that $\partial^m \circ \phi'_m = \phi_{m+1} \circ \partial'$. We can define ϕ_m on K^m summandwise as ϕ'_m on K'^m and ϕ''_m on K''^m.
We check that the conditions are fulfilled with m. Firstly $\partial^{m+1} \circ \partial^m K^m = \partial^{m+1} \circ \partial' K'^m \subset \partial^{m+1} B^{m+1} \subset \partial^{m+1} \ker \partial^{m+1} = 0$. Next, recall that $\phi''^m K''^m \subset \ker \partial^m$, so $\partial^m \phi_m$ and $\phi_{m+1} \partial'^m$ on K''^m have the same effect, namely sending it to zero. Thus we have $\partial^m \phi_m K^m = \phi_{m+1} \partial'^m K^m$. So condition two is satisfied. For condition three, we recall that by definition we have the short exact sequence $0 \to B^{m+1} \to \ker \partial'^{m+1} \to H^{m+1}(C^{\bullet}) \to 0$ (the exactness at $H^{m+1}(C^{\bullet})$ is from the inductive hypothesis). Observe that now $\mathrm{Im}\partial'^m = \partial'^m K^m = \partial' K'^m = B^{m+1}$. At the same time, $K''^m \subset \ker \partial'^m$ and ϕ''_m takes K''^m to $Z^m(C^{\bullet})$ surjectively by definition. Thus the map $\ker \partial'^m \xrightarrow{\phi_m} \ker \partial^m = Z^m(C^{\bullet}) \to H^m(C^{\bullet})$ is surjective. It follows that condition three is satisfied. Lastly, condition four is obtained from the construction.

Now we isolate the case $m = -1$ because we only wish to construct finitely generated free modules down to K^1. Before we carry on, we make note that to construct K^m we only require conditions i-iii. So even though condition four no longer holds for K^0, we can assume conditions i-iii for $p \leq 0$. Up to this point, we have found $K^{\bullet}$ such that in positive degrees we have the desired properties. We remains to construct K^p with $p \leq 0$ such that each K^p is a finitely generated A-module and ϕ_p induces isomorphism of cohomology groups. Additionally, we ask for K^0 to be A-flat if C^p are. Notice that we need $H^0(K^{\bullet}) \cong H^0(C^{\bullet})$ but so far we only have a surjection $\ker \partial'^0 \to H^0(C^{\bullet})$. We modify our K^0 to be $K^0/(\ker \partial'^0 \cap \ker \phi_0)$ and take $\partial'^0 : K^0 \to K^1$ and $\phi_0 : K^0 \to C^0$ to be the induced maps. So $\ker \partial'^0 \xrightarrow{\sim} H^0(C^{\bullet})$ is now an isomorphism We also set $K^p = 0$ for all $p < 0$. It follows that $H^0(K^{\bullet}) \cong H^0(C^{\bullet})$. Immediately we can be assured that $\partial'^{p+1} \circ \partial'^p = 0$ and $\partial^p \phi^p = \phi_{p+1} \partial'^p$ for $p \leq 0$ still hold as our new K^0 is a quotient of the previous module.

Assuming all the C^p are A-flat, it still remains to be shown that K^0 is A-flat. Consider the mapping cylinder complex $L^{\bullet}$ where $L^p = K^p \oplus C^{p-1}$ and $\partial : L^p \to L^{p-1}$ is defined by $\partial(x, 0) = (\partial x, \phi(x)), \partial(0, y) = (0, -\partial y)$. We also define a complex $C'^{\bullet}$ by shifting the labels of $C^{\bullet}$ by 1, namely $C'^{p} = C^{p-1}$ and a shift in ∂ accordingly. Then by construction $0 \to C'^{\bullet} \to L^{\bullet} \to K^{\bullet} \to 0$ is a short exact sequence of complexes. Then we have the following long exact sequence of cohomology groups induced by the short exact sequence of complexes, where the connecting morphism of $H^p(K^{\bullet}) \to H^{p+1}(C'^{\bullet}) \cong H^p(C^{\bullet})$ is induced by $\phi : K^{\bullet} \to C^{\bullet}$.

$$\begin{tikzcd}
 & H^p(C^{\bullet}) \arrow[d, "\cong"] & & & H^{p+1}(C^{\bullet}) \arrow[d, "\cong"'] \\
H^p(K^{\bullet}) \arrow[r] & H^{p+1}(C'^{\bullet}) \arrow[r] & H^{p+1}(L^{\bullet}) \arrow[r] & H^{p+1}(K^{\bullet}) \arrow[r] & H^{p+2}(C'^{\bullet})
\end{tikzcd}$$

So all these connecting morphisms are isomorphisms, and all $H^p(L^{\bullet})$ must be (0). In other words, $0 \to K^0 = L^0 \to L^1 \to L^2 \to \cdots \to L^{n+1} \to 0$ is exact. As K^p is flat (by corollary 1.10) for $1 \le p \le n$ and C^p is flat for all p, lemma [1.13] gives that L^p are flat for $1 \le p \le n$. Then $K_0 = L_0$ flat follows from theorem [1.14]. □

Now, we sum up what we have accomplished: we have a complex $K^{\bullet}$ and a homomorphism $K^{\bullet} \to C^{\bullet}$ such that the induced maps on cohomology groups are isomorphisms. Namely, we have the following isomorphisms

$$H^p(K^{\bullet}) \xrightarrow{\sim} H^p(C^{\bullet}) \cong H^p(X, \mathcal{F}), \forall p$$

This is close to what we need, which is that for any A-algebra B, $H^p(K^{\bullet} \otimes_A B) \to H^p(C^{\bullet} \otimes_A B)$ is an isomorphism. For this, we formulate the following lemma regarding the base change.

Lemma 3.2. *Let $K^{\bullet}, C^{\bullet}$ be any finite complexes of flat A-modules, and let $K^{\bullet} \to C^{\bullet}$ be a homomorphism of complexes inducing isomorphisms $H^p(K^{\bullet}) \xrightarrow{\sim} H^p(C^{\bullet})$ for all p. Then for every A-algebra B, the maps $H^p(K^{\bullet} \otimes_A B) \xrightarrow{\sim} H^p(C^{\bullet} \otimes_A B)$ are isomorphisms.*

Proof. We construct the 'mapping cylinder' $L^{\bullet}$ where $L^p = K^p \oplus C^{p-1}$ exactly as before. We have seen that $L^{\bullet}$ is actually an exact finite sequence of flat A-modules. From an earlier lemma [1.14], all $\ker \partial^p$ are A-flat because L^p are. Thus we have the short exact sequence of flat modules $0 \to Z^p \to L^p \to B^p = Z^{p+1} \to 0$. By lemma [1.15],

$$0 \to Z^p \otimes_A B \to L^p \otimes_A B \to Z^{p+1} \otimes_A B \to 0$$

is exact. In other words, $L^{\bullet} \otimes B$ is exact. As before, we look at the long exact sequence of cohomology groups induced by $0 \to C'^{\bullet} \otimes_A B \to L^{\bullet} \otimes_A B \to K^{\bullet} \otimes_A B \to 0$ (the exactness here is given by the same lemma [1.15]).

$$\begin{tikzcd}
 & H^p(C^{\bullet} \otimes_A B) \arrow[d, "\cong"] & & & H^{p+1}(C^{\bullet} \otimes_A B) \arrow[d, "\cong"'] \\
H^p(K^{\bullet} \otimes_A B) \arrow[r] & H^{p+1}(C'^{\bullet} \otimes_A B) \arrow[r] & H^{p+1}(L^{\bullet} \otimes_A B) \arrow[r] & H^{p+1}(K^{\bullet} \otimes_A B) \arrow[r] & H^{p+2}(C'^{\bullet} \otimes_A B)
\end{tikzcd}$$

where $H^p(L^{\bullet} \otimes_A B) \cong 0$ implies that $H^p(K^{\bullet} \otimes_A B) \to H^p(C^{\bullet}_A B)$ are all isomorphisms. □

4 References

1 [RdBk] - *The Red Book of Varieties and Schemes*, by D. Mumford;

2 [HAG] - *Algebraic Geometry*, by R. Hartshorne;

3 [AM] - *Introduction to Commutative Algebra*, by M. F. Atiyah and I. G. MacDonald;

4 [MAV] - *Abelian Varieties*, by D. Mumford;

5 *The Stacks Project*, `https://stacks.math.columbia.edu`;

6 *The Rising Sea*, by R. Vakil `https://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf`.