University of Regensburg

Lecture XI: More K-Theory

Dec 14, 2024

Lecturer: Christoph Winges *Scribe: Shiyu Yan*

The goal of this talk is to prove the results we have assumed in the previous talk in order to develop the (complex) topological K-Theory.

1 Exact Sequences in K-Theory

Proposition 1.1. *If X is compact Hausdorff and $A \subset X$ is a closed subspace, then the inclusion and quotient maps $A \xrightarrow{i} X \xrightarrow{q} X/A$ induces homomorphisms $\tilde{K}(X/A) \xrightarrow{q^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(A)$ for which the kernel of i^* equals the image of q^*.*

Remark 1.2. Since A is a closed subspace of a compact Hausdorff space, it is also compact Hausdorff. X/A is compact since X is. It is Hausdorff because X is (by compact Hausdorffness) normal. So a point $x \in X - A$ outside of A and A have disjoint neighborhoods in X, projecting to disjoint neighborhoods of x and the point A/A in X/A. So $\tilde{K}(A), \tilde{K}(X/A), \tilde{K}(A/A)$ are well-defined.

Remark 1.3. Equivalently, we can say the sequence is exact, or the sequence is exact at $\tilde{K}(X)$.

Proof. We see that the homomorphism $i^*q^* = (qi)^* : \tilde{K}(X/A) \to$ is induced by the composition $qi : A \to X \hookrightarrow X/A$. Since the image of A in X/A is A/A, and the only vector bundle over A/A is the trivial one so $\tilde{K}(A/A)$ is trivial, so the pullback will also be trivial. This then implies that $\operatorname{Im} q^* \subset \ker i^*$.
For the opposite inclusion, $\ker i^* \subset \operatorname{Im} q^*$. Take $[p : E \to X] \in \ker i^* \subset \tilde{K}(X)$. This means that when $p : E \to X$ is restricted to A, it is stably trivial. In other words, when restricting to A, it is in the $\sim$-equivalent class of the trivial bundle. We might as well assume that E is trivial over A.[1] Choosing a trivialization $h : p^{-1}(A) \to A \times \mathbb{C}^n$, we define E/h to be the quotient space of E by identifying $h^{-1}(x, v) \sim h^{-1}(y, v)$ for all $x, y \in A$. It follows immediately that $E/h \to X/A$ is well defined by the universal property of quotient topology E/h. We would like to show that:

1. $E/h \to X/A$ is a vector bundle.
2. $E \approx q^*(E/h)$.

Then it is clear that the reverse inclusion holds. To address the first point, we observe that it is enough to show that around the point A/A it admits a trivialization[2].

Claim 1.4. *Since E is trivial over a closed subspace A, it is trivial over some neighborhood of A.*

Proof. In many cases[3], this holds because there is a neighborhood that deformation retracts to A. In this case, there exists an open neighborhood $A \subset U \subset X$ with $H : U \times [0, 1] \to U$ such that $H(-, 0) = id_U$, $r := H(-, 1) = A$, and that $H(a, t) = a$ for all $a \in A$ and t. We have the following pullback squares.

$$\begin{tikzcd} E_A \arrow[r, hook, "i^*"] \arrow[d] & r^*E_A \arrow[r, "r^*"] \arrow[d] & E_A \arrow[d] \\ A \arrow[r, hook, "i"] & U \arrow[r, "r"] & A \end{tikzcd}$$

[1]The independence of choice of representatives follows from the well-definedness of map $q^* : \tilde{K}(X/A) \to \tilde{K}(X)$.
[2]A is assumed to be a closed subspace of X, so A/A is a closed subspace of X/A. For a point $x \in X/A - A/A$, $x \in X - A$, and by normal condition x has a trivialization over a neighborhood in $X - A$ so also in $X/A - A/A$
[3]Such as when X is a CW-complex.

Call $E_A := p^{-1}(a)$. Notice that the $i^*r^*E_A = (ri)^*E_A = id_A^*E_A = E_A$. In particular, i^*r^* is an isomorphism, so is r^*. This shows that $r^*E_A \to U$ is trivial. Now observe that we also have the following pullback squares.

$$\begin{array}{ccccc} E_A & \longrightarrow & E_U = i'^*E & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ A & \xhookrightarrow{i} & U & \xhookrightarrow{i'} & X \end{array}$$

Since r, i are homotopy equivalences, ir is homotopic to the identity of U. By theorem 1.6[1], $id_U^* = (ir)^* = r^*i^*$, and $E_U = i'^*E = id_U^*i'^*E \approx r^*i^*i'^*E \approx r^*E_A$. Then we see that $r^*E_A \to U$ is indeed the restriction of $E \to X$. It is clear that U/A is the desired neighborhood of A/A. In the absence of such deformation retracts, prescribe the following argument.
Recall that an n-dimensional bundle $p : \mathbf{E}_A \to A$ is isomorphism to the trivial bundle iff it has n sections $s_1, \dots, s_n$ such that the vectors $s_1(a), \dots, s_n(a)$ are linearly independent in each fiber $p^{-1}(a)$. Choose a cover of A by open sets U_j in X trivializing E. Via a local trivialization, each section s_i can be regarded a map from $A \cap U_j$ to a single fiber (via a local trivialization s_i is $a \mapsto (a, v), v \in \mathbb{C}^n \approx \mathbb{R}^{2n}$ so we can regard it as $a \mapsto v$. We recall that a compact Hausdorff space is normal. So the Tietze extension theorem(4.1) applies, and we obtain a section $s_{ij} : U_j \to E$ extending s_i. Take $\{\varphi_j, \varphi\}$ a partition of unity to the cover $\{U_j, X - A\}$ of X[4] Now we consider the sum $\sum_j \varphi_j s_{ij}$. First we notice that it is defined on all of X[5]. At the same time we see that it extends s_i: for any $a \in A, \sum_j \varphi(a)s_{ij}(a) = \sum_{j'} \varphi(a)s_a = 1 * s_a = s_a$ since we can take a subcover $\{U_{j'}\}$ of $\{U_j\}$ where $\varphi_{j'}$ at a are non-zero – it is immediate that $s_{ij'}$ are equal to some s_a since $a \in U_{j'}$ for all j'. So it is a section extending s_i from $A \cap X_i$ on all of X. Since these new sections also form a basis in each fiber over A, they must form a basis in all nearby fibers. To spell this out, over U_j the extended s_i's can be regarded as a square matrix valued function $U_j \to M_n\mathbb{C} \approx \mathbb{C}^{n^2}$. Recall that the bundle is trivial over A is exactly if for each $a \in A$, $s_j(a)$'s are all linearly independent over the fiber of a, so the determinant of the matrix formed by $s_i(a)$'s is nonzero at all $a \in A$. Since the determinant function is continuous, the determinant must be nonzero also at a neighborhood of A contained in the cover $\{U_j\}$. □

The trivialization of h of E over a neighborhood U of A induces a trivialization of E/h over U/A, so E/h is a trivial bundle.

It remains to show the second point that that $E \approx q^*(E/h)$. In the commutative diagram below: since the quotient map $E \to E/h$ is an isomorphism on fibers over each $e \in E$ and fibers over $q(e)$, this defines a pullback square by the very construction of pullback of E by f. So $E \approx q * E/h$.

$$\begin{array}{ccc} E & \longrightarrow & E/h \\ {\scriptstyle p}\downarrow & & \downarrow \\ X & \xrightarrow[q]{} & X/A \end{array}$$

□

Next, we will see that there is a natural way to extend this exact sequence to a long exact sequence of $\tilde{K}$ groups. For this, we need this following lemma:

Lemma 1.5. *If A is contractible, the quotient map $q : X \to X/A$ induces a bijection $q^* : \mathit{Vect}^n(X/A) \to \mathit{Vect}^n(X)$ for all n.*

[4]Compact Hausdorff spaces are paracompact. By paracompactness of X, an open cover $\{U_i\}_{i \in I} \cup \{X - A\}$ of X admits a partition of unity subordinate to the cover. Consider the partition of unity $\{\phi_j\} \cup \{\phi_k\}$ induced by the cover, ie. $\{U_j = \varphi_j^{-1}((0,1])\}$ refining $\{U_i\}_{i \in I}$ and $\{V_k = \phi_k^{-1}\}$ refining $X - A$. Notice that the former is still a cover of A and the latter is a cover of $X - A$ so we might as well take the cover of X to be $\{U_j\} \cup \{V_k\}$ instead. But to alleviate the notation, we take $\phi = \sum_k \phi_k$. Then we can just consider $\{\phi_j, \phi\}$ subordinate to the cover $\{U_j, X - A\}$ of X. Lastly, it is clear that if $\{U_i\}_{i \in I}$ is a trivializing cover then so is $\{U_j\}$.

[5]Notice each $\phi_j s_{ij}$ is defined on X: ϕ_j is continuous of all of X such that it is only supported in U_j then consider $\phi_j|_{U_j} s_{ij}$ which is compactly supported on U_j.

Remark 1.6. If the quotient map $q : X \to X/A$ collapsing a contractible subspace to a point is a homotopy equivalence (for instance if our pair (X, A) has the homotopy extension property), then an earlier result gives us this bijection. But it is not always the case that we have a homotopy equivalence: for instance a dunce hat is contractible but not collapsible[5]. But the bijection between the sets of n-th dimensional vector bundles holds in general.

Proof. We aim to construct an inverse to q^*. Since A is contractible, A is homotopy equivalent to $*$, and a vector bundle $E \to X$ must be trivial over A. We have also already seen that a trivialization h of A gives a vector bundle $E/h \to X/A$. We would like to say that mapping E to E/h gives the inverse mapping. But first, it needs to be a well-defined map $\text{Vect}^n(X) \to \text{Vect}^n(X/A)$. We observe that the isomorphism classes of E/h do not depend on h. Given two choices of trivializations h_0 and h_1, there is $h_1 = (h_1 h_0^{-1})h_0$. Over each $x \in A$, $h_1 h_0^{-1}$ is an automorphism $g_x \in \text{GL}_n(\mathbb{C})$.This results in map $A \to \text{GL}_n(\mathbb{C})$. Since A is contractible we have a homotopy $G : A \times I \to \text{GL}_n(\mathbb{C})$ such that $G|_0 = g, G|_1 = const_\alpha : x \mapsto \alpha \in \text{GL}_n(\mathbb{C})$. We write αh_0 to be the map by composing h_0 with α in each fiber. Then G gives a homotopy $H : E \to E$ such that $H|_0 = h_1$ and $H|_1 = \alpha h_0$. We can now construct a vector bundle $(E \times I)/H \to (X/A) \times I$ just like how we construct $E/h \to X/A$. The vector bundle $(E \times I)/H$ restricts to E/h_1 at 0 and $E/\alpha h_0$ at 1, hence $E/h_1 \approx E/\alpha h_0$. This is because $X \times I$ deformation retracts to $X \times \{0\}$ and also to $X \times \{1\}$, so we have homotopy equivalences.

$$X \times \{0\} \underset{i_0}{\overset{r_0}{\leftrightarrows}} X \times I$$

$$X \times \{1\} \underset{i_1}{\overset{r_1}{\leftrightarrows}} X \times I$$

By taking the product of two obvious pullback squares, we have the following pullback square[6]:

$$\begin{tikzcd} i^*F \times I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & i^*F \arrow[d] \\ X \times I \arrow[r, "r"] & X \end{tikzcd}$$

Recall from an earlier argument, we have seen $F \approx (ir)^*F \approx r^*i^*F$ so $F \approx i^*F \times I$. Apply this to our situation, $E/h_1 \approx E/\alpha h_0$ is now clear.

Now we observe that composing h_0 with α in each fiber doesn't change E/h: $h_0^{-1}(x, \alpha v) \sim h_0^{-1}(y, \alpha v) \Leftrightarrow h_0^{-1}(x, v) \sim h_0^{-1}(y, v)$. This says that $E/\alpha h_0 = E/h_0$. It is also clear that if $E \approx E'$ then $E/h \approx E'/h'$. Thus, this is a well defined map $\text{Vect}^n(X) \to \text{Vect}^n(X/A)$.

This is inverse to q^*: from the preceding proposition, we have already seen that $q^*(E/h) \approx E$. It remains to check that for a bundle $E \to X/A$ we have $q^*(E)/h \approx E$ for the evident trivialization h of $q^*(E)$ over A . This is because we have two following pullback squares

$$\begin{tikzcd} q^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ X \arrow[r, "q"] & X/A \end{tikzcd} \qquad \begin{tikzcd} q^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & q^*E/h \arrow[d] \\ X \arrow[r, "q"] & X/A \end{tikzcd}$$

The second square is a pullback because $q^*(q^*E/h) \approx q^*E$. Thus, $X \times_{X/A} E \approx X \times_{X/A} q^*E/h$. So under projection $q^*E/h \approx E$. ☐

Remark 1.7. Notice that we can write for a (compact Hausdorff) space X, $\text{Vect}(X) = \bigoplus_{C \in \pi_0(X)} \text{colim}_n \text{Vect}^n(C)/\sim$ where $\text{Vect}^n(C) \to \text{Vect}^{n+1}(C), E \mapsto E \oplus \epsilon$ defines the filtration, and $E_1 \sim E_2$ iff $E_1 \oplus \epsilon_1 \approx E_2 \oplus \epsilon_2$. We observe

[6] The product of pullbacks is a pullback. Here I use F in place of $(E \times I)/H$ for readability. (r, i) are (r_0, i_0) and (r_1, i_1) respectively.

the following diagram is commutative

$$\begin{array}{ccccccc}
\dots & \longrightarrow & \mathrm{Vect}^n(X) & \longrightarrow & \mathrm{Vect}^{n+1}(X) & \longrightarrow & \dots \\
 & & \Big\uparrow{\scriptstyle q^*} & & \Big\uparrow{\scriptstyle q^*} & & \\
\dots & \longrightarrow & \mathrm{Vect}^n(X/A) & \longrightarrow & \mathrm{Vect}^{n+1}(X/A) & \longrightarrow & \dots
\end{array}$$

So this is an isomorphism of sequences. Then there is a bijection of colimits and then of reduced K-theory. So we can conclude from the lemma that if A is contractible, $\tilde{K}(X/A) \to \tilde{K}(X)$ is a bijection. It is straightforward to check this is a group homomorphism. So this map is a group isomorphism. It is worth noting that lemma 1.5 essentially shows the exactness axiom of the generalized cohomology theory[3].

There is an easy way to extend the exact sequence $\tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{A}$ to the left, by an instance of long cofiber sequence, where C and S denote cone and suspension:

$$\begin{array}{ccccccccc}
A & \hookrightarrow & X & \hookrightarrow & X \cup CA & \hookrightarrow & (X \cup CA) \cup CX & \hookrightarrow & ((X \cup CA) \cup CX)C(X \cup CA) \\
 & & & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
 & & & & X/A & & SA & & SX
\end{array}$$

In the first row, each space formed by attaching to its predecessor a cone on the subspace two steps back in the sequence. The maps are obvious inclusions. The vertical maps are the quotient maps obtained by collapsing the most recently attached cone to a point. For instance, attaching CX to $X \cup CA$ along X, then we can collapse X to a point, this turns the cone $X \cup CA$ into the suspension SA. In particular $\tilde{K}(SA) \approx \tilde{K}((X \cup CA)/X)$[7]. Then it follows that $\tilde{K}(SX) \to \tilde{K}(X/A) \to \tilde{K}(X)$ is exact.
In general, the sequence $\dots \to \tilde{K}(SX) \to \tilde{K}(SA) \to \tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A)$ is exact everywhere.

2 Deducing Bott Periodicity

Take X to be the wedge sum $A \vee B =$ then $X/A = B$ then the sequence $\tilde{K}(X/A) \approx \tilde{K}(B) \to \tilde{K}(X) \to \tilde{K}(A)$ admits a retraction $\tilde{K}(X) \to \tilde{K}(B)$[8]. So the sequence is split, $\tilde{K}(X) \xrightarrow{\sim} \tilde{K}(A) \oplus \tilde{K}(B)$ obtained by restriction to A and B is an isomorphism.

From last talk, we have established that

$$\begin{array}{ccccccccc}
K(X) \otimes K(Y) & \approx & \tilde{K}(X) \otimes \tilde{K}(Y) & \oplus & \tilde{K}(X) & \oplus & \tilde{K}(Y) & \oplus & \mathbb{Z} \\
\Big\downarrow & & \Big\downarrow & & \Big\| & & \Big\| & & \Big\| \\
K(X \times Y) & \approx & \tilde{K}(X \wedge Y) & \oplus & \tilde{K}(Y) & \oplus & \tilde{K}(Y) & \oplus & \mathbb{Z}
\end{array}$$

Recall the product theorem:

Theorem 2.1. *The homomorphism $\mu : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X \times S^2)$ is an isomorphism of rings for all compact Hausdorff spaces X.*

Take X to be a point, we obtain:

Corollary 2.2. *The map $\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$ is an isomorphism.*

Thus if we regard $\tilde{K}(S^2)$ as the kernel of $K(S^2) \to K(x_0) \approx \mathbb{Z}$, then it is generated as abelian group by $H-1$. We then have the following consequence:

[7] This is lemma 1.5 + remark 1.7 .

[8] for chosen basepoints a_0 of A and b_0 of B.

Theorem 2.3 (Bott periodicity)**.** *The homomorphism $\beta : \tilde{K}(X) \to \tilde{K}(S^2X), \beta(a) = (H-1) * a$, is an isomorphism for all compact Hausdorff spaces X.*

Proof. The map β is the composition

$$\tilde{K}(X) \to \tilde{K}(S^2) \otimes \tilde{K}(X) \to \tilde{K}(S^2 \wedge X) \approx \tilde{K}(S^2X)$$

where the first map is the external product with $H-1$, i.e. $a \mapsto (H-1) \otimes a$. This is an isomorphism because that $H-1$ is the generator in $\tilde{K}(S^2)$. ☐

Corollary 2.4. *$\tilde{K}(S^{2n+1}) = 0$ and $\tilde{K}(S^{2n}) \approx \mathbb{Z}$, generated by the n-fold reduced external product $(H-1) * \cdots * (H-1)$.*

Proof. The even case is clear in the light of the previous theorem. All complex bundles over S^1 are trivial, hence $K(S^1) = \{\mathbb{C}^m - \mathbb{C}^n\} \approx \mathbb{Z}$ and $\tilde{K}(S^1) = 0$. The odd case then follows[2]. ☐

3 Extending to a Cohomology Theory

Recall from the last talk, using Bott periodicity above, we have established the six-term periodic exact sequence from the long exact sequence of $\tilde{K}$ groups of a pair (X, A) of compact Hausdorff spaces:

$$\begin{tikzcd}
\tilde{K}^0(X,A) \arrow[r] & \tilde{K}^0(X) \arrow[r] & \tilde{K}^0(A) \arrow[d] \\
\tilde{K}^1(A) \arrow[u] & \tilde{K}^1(X) \arrow[l] & \tilde{K}^1(X,A) \arrow[l]
\end{tikzcd}$$

Recall that the reduced K theory of X can be viewed as the ideal of K theory, so multiplication is defined, and is commutative. But this is not the case in the cohomology ring $\tilde{K}^*(X)$ – this is only true up to a sign:

Proposition 3.1. *$\alpha\beta = (-1)^{ij}\beta\alpha$ for $\alpha \in \tilde{K}^i(X)$ and $\beta \in \tilde{K}^i(X)$.*

Proof. Recall the definition and an observation in negative degrees yield $\tilde{K}^i(X) = \tilde{K}(S^iX)$. Recall that $S^n \wedge X$ is (homeomorphic to) the n-th iterated reduced suspension $\Sigma^n X$. By an inductive argument, we see that S^n can be identified with an n-fold smash product $S^1 \wedge \cdots \wedge S^1$. We fix such a specific pre-chosen homeomorphism for S^i and S^j[9]. As a consequence, $S^i \wedge S^j = (S^1 \wedge \ldots_i \wedge S^1) \wedge (S^1 \wedge \ldots_j \wedge S^1) = S^1 \wedge \ldots_{i+j} \wedge S^1 = S^{i+j}$.[10] Recall that a product

$$\tilde{K}^i(X) \otimes \tilde{K}^j(X) \to \tilde{K}^{i+j}(X \wedge X) \to \tilde{K}^{i+j}(X)$$

is obtained from the external product

$$\tilde{K}(S^i \wedge X) \otimes \tilde{K}(S^j \wedge X) \to \tilde{K}((S^i \wedge X) \wedge (S^j \wedge X)) = \tilde{K}(S^i \wedge S^j \wedge (X \wedge X)) \to \tilde{K}(S^i \wedge S^j \wedge X)$$

where the first map is external product and the second map is induced by the diagonal map on X. We observe that swapping $\alpha\beta$ amounts to switching the two factors in the first map, which corresponds to switching the $S^i \wedge X$ and $S^j \wedge X$ in the second map. This amounts to swapping S^i with S^j and the second X, and then swapping first X with X^j and second X. We see that since the second map is pulling back along the diagonal, which is invariant under swapping X's. It reduces to checking swapping S^i and S^j. By our earlier identification, we view $S^i \wedge S^j$ as the smash product of $i+j$ copies of S^1. Switching S^i and S^j is a product of ij transpositions of adjacent factors. Transposing two factors of $S^1 \wedge S^1$ is equivalent to reflection of S^2 across a equator in the pictorial identification here:

[9] We don't allow re-identifying spheres. For example, an orientation-reversing automorphism of S^n will change signs in $\tilde{K}(S^n)$.

[10] One should be more careful that smash product is only associative up to homeomorphism. We can safely ignore this subtlety for our purpose – all in all this doesn't get detected once it is passed through the K-theory.

We can view S^2 as a suspension, thus it suffices to see that switching the two ends of a suspension SY induces a multiplication by -1 in $\tilde{K}(SY)$. We use the fact $[\Sigma X, BU]_* \xrightarrow{\approx} [\Sigma X, BU] \approx \tilde{K}(\Sigma X)$, where $BU = \mathrm{colim}_k \mathrm{Gr}_k(\mathbb{C}^\infty)$, and operation in $\tilde{K}(\Sigma X)$ corresponds to the Whitney sum $BU \times BU \xrightarrow{\oplus} BU$[11] . On the other hand, the pinch map $\Sigma X \to \Sigma X \vee \Sigma X$ also gives a multiplication on $[\Sigma X, BU]_*$ so by an Ackmann-Hilton argument, two operations are the same. Notice that now taking the reflection of ΣX across the equator via the pinch map corresponds to taking the inverse in $\tilde{K}(\Sigma X)$ via the Whitney sum. □

Remark 3.2. $\tilde{K}^*(X)$ has graded commutativity.

Rings $\tilde{K}^*(A)$ and $\tilde{K}^*(X,A)$ as abelian groups have a $\tilde{K}^*(X)$-module structure. On $\tilde{K}^*(A)$ we have the product $\tilde{K}^*(X) \otimes \tilde{K}^*(A) \to \tilde{K}^*(A)$ by $\xi \cdot \alpha = i^*(\xi)\alpha$ where $i : A \hookrightarrow X$. To define the module structure of $\tilde{K}^*(X,A)$, observe that the diagonal map $X \to X \wedge X$ induces a well-defined quotient map $X/A \to X \wedge X/A$, which gives a product $\tilde{K}^*(X) \otimes \tilde{K}^*(X,A) \to \tilde{K}^*(X \wedge X/A) \to \tilde{K}^*(X,A)$.

Proposition 3.3. *The exact sequence below is an exact sequence of $\tilde{K}^*(X)$-modules, with the maps homomorphism of $\tilde{K}^*(X)$-module.*

$$\begin{tikzcd}
\tilde{K}^*(X,A) \arrow[rr] & & \tilde{K}^*(X) \arrow[dl] \\
& \tilde{K}^*(A) \arrow[ul] &
\end{tikzcd}$$

Proof. This boils down to showing the following diagram is commutative where we allow j to vary.

$$\begin{tikzcd}
\tilde{K}(S^jSA) \arrow[r] \arrow[d] & \tilde{K}(S^j(X/A)) \arrow[r] \arrow[d] & \tilde{K}(S^jX) \arrow[r] \arrow[d] & \tilde{K}(S^jA) \arrow[d] \\
\tilde{K}(S^iX \wedge S^jSA) \arrow[r] \arrow[d] & \tilde{K}(S^iX \wedge S^j(X/A)) \arrow[r] \arrow[d] & \tilde{K}(S^iX \wedge S^jX) \arrow[r] \arrow[d] & \tilde{K}(S^iX \wedge S^jA) \arrow[d] \\
\tilde{K}(S^{i+j}SA) \arrow[r] & \tilde{K}(S^{i+j}(X/A)) \arrow[r] & \tilde{K}(S^{i+j}X) \arrow[r] & \tilde{K}(S^{i+j}A)
\end{tikzcd}$$

where the vertical maps between the first two rows are external product $\alpha \mapsto \xi \cdot \alpha$ with a fixed element $\xi \in \tilde{K}(S^iX)$, and the vertical maps between the second and third rows are induced by diagonal maps. We wish to show that the diagram commutes. For the upper two rows, notice that the horizonal maps are induced by maps between spaces, so commutativity follows from naturality of external product. The lower two rows are induced by suspension of maps between spaces

$$\begin{tikzcd}
X \wedge SA & X \wedge X/A \arrow[l] & X \wedge X \arrow[l] & X \wedge A \arrow[l] \\
SA \arrow[u] & X/A \arrow[l] \arrow[u] & X \arrow[l] \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

So it suffices to show this diagram commutes up to homotopy. The middle and right squares commute on the nose. For the left square, it is enough to consider a related square from the reduced long exact sequence (as the resulting K-theory is indistinguishable):

$$\begin{tikzcd}
S^1 \wedge X \wedge A \approx X \wedge S^1 \wedge A & X \wedge (X \cup \tilde{C}A) \arrow[l, "id_X \wedge p"'] \\
S^1 \wedge A \arrow[u, "id_{S^1} \wedge \Delta"] & X \cup \tilde{C}A \arrow[l, "p"] \arrow[u]
\end{tikzcd}$$

[11]This is by argument of example 1.13[1], where we see $E_{fg} \oplus \epsilon_n \approx E_f \oplus E_g$

Observe that the bottom map is just the boundary map from the long exact sequence:

$$\begin{tikzcd}
A \arrow[r] & X \arrow[r] \arrow[dr] & X \cup \widetilde{C}A \arrow[r] \arrow[d] \arrow[dr, "p"] & (X \cup \widetilde{C}A) \cup \widetilde{C}X \arrow[d] \\
& & X/A & \Sigma A
\end{tikzcd}$$

The right vertical map sends $x \in X$ to $(x, x, 0)$ and $(a, t) \in \widetilde{C}A$ to (a, a, t). The left vertical map is $id_{S^1} \wedge \Delta : S^1 \wedge A \to S^1 \wedge X \wedge A$ and the top map is $id_X \wedge p : X \wedge (X \cup \widetilde{C}A) \to X \wedge S^1 \wedge A$. It is straightforward to check that this is a commutative square, when we identify $S^1 \wedge X \wedge A \approx X \wedge S^1 \wedge A$.
All the little squares are now commutative squares. This finish the proof that the diagram is commutative everywhere.
□

4 Appendix

Theorem 4.1 (Tietze extension theorem for continuous functions [4]). *Let X be a normal topological space, and let $A \subset X$ be a closed subset. Suppose $f : A \to \mathbb{R}$ is a continuous function. Then there exists a continuous extension $\tilde{f} : X \to \mathbb{R}$ such that*

$$\tilde{f}(x) = f(x) \quad \textit{for all } x \in A.$$

References

[1] Allen Hatcher. Vector bundles and k-theory. https://pi.math.cornell.edu/ hatcher/VBKT/VBpage.html, 2009. Version 2.2, December 14, 2009.

[2] Lorena Martins. Reu paper. https://math.uchicago.edu/ may/REU2021/REUPapers/Martins.pdf, 2021. University of Chicago REU, 2021.

[3] nLab authors. Generalized (eilenberg-steenrod) cohomology – reduced cohomology. https://ncatlab.org/nlab/show/generalized+ Accessed: January 31, 2025.

[4] nLab authors. Tietze extension theorem. https://ncatlab.org/nlab/show/Tietze+extension+theoremForContinuousFunctions. Accessed: January 31, 2025.

[5] Emery Thomas. On a class of cohomology theories. *Topology*, 2(2):113–132, 1963.