Talk II: Zariski Descent I

Shiyu Yan
Department of Mathematics
University of Regensburg
Organizer: Marc Hoyois

July 25, 2025

Contents

1. Introduction

A foundational fact is that every *finitely generated projective* module is, Zariski-locally, nothing more exotic than a free module of finite rank. Equivalently,

$$\textit{finitely generated projective } R\textit{-modules} \iff \textit{locally free } R\textit{-modules of finite rank.}$$

The aim of this talk is to establish that equivalence and to develop the idea of "patching modules" that lies behind it. Concretely we proceed in two stages.

1. **Local picture.** We recall the basic properties of finitely generated / finitely presented / projective / flat modules, apply a key property that finitely generated projective modules are finitely presented, and show that over a *local* ring every finitely generated projective module is free (Proposition 2.13).

2. **Global picture via Zariski descent.** We then explain how to glue local data along overlaps: first for a two-element cover $D(f) \cup D(g) = \mathrm{Spec}R$, then for arbitrary finite covers. The fiber-product description of modules (§3.1) and an identification of two categories yield the key descent theorem (4.1), from which it follows that (finitely generated) projective $\Leftrightarrow$ locally free (Corollary 5.2).

Along the way we highlight tools, such as localization, exactness criteria, and functoriality of various constructions. Readers who are comfortable with the language of sheaves may recognize the argument as nothing but the sheaf condition that the presheaf $U \mapsto \mathrm{Proj}_{\mathrm{fg}}(\mathcal{O}(U))$ is a $\mathcal{O}_{\mathrm{Spec}R}$ sheaf for the Zariski topology.
In the end, Serre-Swan theorem (6.1) connects two notions of local triviality: that of complex vector bundles over a compact Hausdorrff space X and that of projective $C(X, \mathbb{C})$ modules.

2. Preliminaries

First we will recall some properties of finitely generated, finitely presented modules of a commutative ring.

Definition 2.1. *If R is a commutative ring, an R-module M is called:*

1. *finitely generated if there is an epimorphism $R^n \to M$,*
2. *finitely presented if M is the cokernel of a map $R^n \to R^m$ (equivalently, M is finitely generated, and for some surjection $\varphi : R^m \to M$, the kernel $\ker\varphi$ is finitely generated as well).*
3. *coherent if M is finitely generated and any finitely generated (not necessarily proper) submodule is itself finitely presented,*
4. *flat if $- \otimes_R M$ is an exact functor on Mod_R,*
5. *projective if $\hom_R(M, -)$ is an exact functor on Mod_R,*
6. *invertible if $- \otimes_R M$ is an auto-equivalence of Mod_R.*

If $f : R \to S$ is a homomorphism of commutative rings, then we say that f is a flat homomorphism iff S is a flat R-module.

We start with the elementary fact that localizations are flat homomorphisms.

Theorem 2.2. *If R is a commutative unital ring, $S \subset R$ a multiplicative subset, and M is an R-module, then*

1. *$M[S^{-1}] \cong M \otimes_R R[S^{-1}]$,*
2. *The assignment $M \mapsto M[S^{-1}]$ is an exact functor $Mod_R \to Mod_{R[S^{-1}]}$,*
3. *$R \to R[s^{-1}]$ is a flat ring homomorphism.*

Lemma 2.3. *Let R be a commutative unital ring. If P is an R-module, TFAE:*

1. *P is projective,*
2. *Any R-module epimorphism $M \to P$ is split,*
3. *P is a direct summand of a free R-module.*

Proof. ($1 \Rightarrow 2$) Given an exact sequence $A \xrightarrow{\varphi} P \to 0$, and exact functor $\hom_R(P, -)$, $\hom_R(P, A) \xrightarrow{\varphi_*} \hom_R(P, P) \to 0$ is also exact. Take $id_P \in \hom_R(P, P)$, a lift is exactly the section $\psi : P \to A$ such that $P \xrightarrow{\psi} A \xrightarrow{\varphi} P$ is the identity.
($2 \Rightarrow 3$) By choosing generators of P, we can always build an epimorphism $\varphi : F \to P$ such that F is free. Since it is split by assumption, $F \cong P \oplus \ker\varphi$.
($3 \Rightarrow 1$) Assume $P \oplus Q \cong F$ a free module. F is then projective, i.e. $\hom_R(F, -) \cong \hom_R(P \oplus Q, -) \cong \hom_R(P, -) \oplus \hom_R(Q, -)$ is an exact functor. In general, any summand of a projective module is projective, showing the claim.

□

Lemma 2.4. *R is a commutative unital ring, any finitely generated projective R-module is finitely presented.*

Proof. Since projective modules are summands of free modules, say P is finitely generated projective then there is $R^n \cong P \oplus Q$ so Q is the kernel of the epimorphism $R^n \to P$; Q is finitely generated. On the other hand, R^n also surjects onto Q, so

$$R^n \twoheadrightarrow Q \hookrightarrow R^n \twoheadrightarrow P$$

induces an exact sequence $R^n \to R^n \to P \to 0$. □

Lemma 2.5. *R is commutative unital ring, M_1, M_2 are R-modules then the following are true:*

1. *If M_1 and M_2 are flat, then $M_1 \otimes_R M_2$ is flat,*
2. *If M_1 and M_2 are projective, then $M_1 \otimes_R M_2$ is projective.*

Proof. (1) follows from the isomorphic associativity of tensor products. (2) follows from viewing each projective module as a direct summand of a free one. The tensor product of free modules is free, thus also projective, which direct summands are all projective. □

Lemma 2.6. *If $f : R \to S$ is any ring homomorphism, then "extension of scalars", i.e. sending $M \to M \otimes_R S$ determines a functor $Mod_R \to Mod_S$. Extension of scalars sends*

1. *flat R-modules to flat S-modules,*
2. *(finitely generated) projective R-modules to (finitely generated) projective S-modules.*

Proof. (1) Say M is a flat R module, then the functor $(M \otimes_R S) \otimes_S - \cong M \otimes_R (S \otimes_S -)$ which is exact.
(2) let P be a (finitely generated) projective R-module, then there is $P \oplus Q \cong F$ a (finitely generated) free module. It then follows from $(P \oplus Q) \otimes_R S \cong P \otimes_R S \oplus Q \otimes_R S$ which is a (finitely generated) free S-module, that it is projective and so are the direct summands. □

Theorem 2.7. *The category Mod_R for a commutative ring R equipped with the usual structure of direct sum and tensor product is abelian and symmetric monoidal.*

Lemma 2.8 (Atiyah–Macdonald §2, Ex. 3)**.** *Let R be a commutative unital ring and suppose*

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of finitely generated sequence of R-modules. The following statements hold:

1. *Any extension of finitely generated R-modules is finitely generated, i.e., if M' and M'' are finitely generated, then so is M.*
2. *Any extension of finitely presented R-modules is finitely presented, i.e., if M' and M'' are finitely presented, then so is M.*
3. *Any quotient of a finitely generated module is finite, i.e., if M is finitely generated, so is M''.*
4. *Any quotient of a finitely presented module by a finitely generated submodule is finitely presented, i.e., if M is a finitely presented R-module, and M' is finitely generated, then M'' is finitely presented as well.*
5. *If M'' is finitely presented, and M is finitely generated, then M' is finitely generated as well.*

One result we will apply numerously is the Nakayama lemma. We shall see that this is particularly useful in the situation of local ring R when the Jacobson radical $J(R)$ is just its unique maximal ideal, to analyze finitely generated R-modules.

Lemma 2.9 (Nakayama). *If M is a finitely generated R-module and $M/J(R)M = 0$, then $M = 0$.*

Remark 2.10. It is convenient to use the isomorphism $M \otimes_R R/J(R) \cong M/J(R)M$.

Corollary 2.11. *Let M be a finitely generated R-module, N a submodule of M. Then $M = J(R)M + N \Rightarrow M = N$.*

Using the corollary, we have the following proposition, which we state without proof.

Proposition 2.12 (Atiyah–Macdonald §2, 2.18)**.** *Let $(R, \mathfrak{m})$ be a local ring and $\kappa = A/\mathfrak{m}$ is the residue field, and let $e_i (1 \leq i \leq n)$ be elements of R-module M whose images in $M/\mathfrak{m}M$ form a basis of this κ-vector space. Then the e_i generate M.*

Proposition 2.13. *If R is a local ring, then every finitely generated projective R-module is free.*

Proof. Suppose $\mathfrak{m}$ is the maximal ideal of R and $\kappa := R/\mathfrak{m}$ is the residue field. Take a finitely generated projective R-module P. Here $P/\mathfrak{m}P = P \otimes_R R/\mathfrak{m} \cong P \otimes_R \kappa$ is a finitely generated κ-module, i.e. a finite dimensional κ-vector space κ^n.
Fix a basis $\overline{e_1}, \dots, \overline{e_n}$ for $P/\mathfrak{m}P$, take a lift the basis $e_1, \dots, e_n \in P$. Take a free R-module R^n and a morphism $\varphi : R^n \to P$ by $(a_1, \dots, a_n) \mapsto a_1 e_1 + \dots + a_n e_n$. We aim to show that this is an isomorphism. Tensoring with $R/\mathfrak{m}$, we have $\overline{\varphi} : R^n \otimes_R R/\mathfrak{m} \to P/\mathfrak{m}P$. But tensoring is right exact, i.e. $\mathrm{coker}(\varphi) \otimes_R R/\mathfrak{m} \cong \mathrm{coker}(\overline{\varphi})$. It follows that R^n and P are finitely generated that $\mathrm{coker}(\varphi)$ is. Since $\overline{\varphi}$ is epimorphism, $\mathrm{coker}(\overline{\varphi})$ is trivial. Nakayama's lemma says $\mathrm{coker}(\varphi)$ is trivial too.
Since tensoring is not left exact, this is where projectiveness is used. P is projective so $R^n \cong \ker(\varphi) \oplus P$. In particular, $\ker(\varphi)$ is finitely generated. After tensoring we have the isomorphism:

$$\kappa^n \cong R^n \otimes_R R/\mathfrak{m} \cong (P \oplus \ker(\varphi)) \otimes_R R/\mathfrak{m} \cong \kappa^n \oplus \ker(\varphi) \otimes_R R/\mathfrak{m}$$

so $\ker(\varphi)$ mod $\mathfrak{m}$ is trivial, and Nakayama's lemma concludes that $\ker(\varphi)$ is trivial and P is free. □

Though it is needed for the later results, there is a more refined version of this proposition.

Proposition 2.14. *Suppose R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field κ and P is finitely generated R-module. The following conditions are equivalent:*

1. *P is free;*

2. *P is projective; and*

3. *$Tor_1^R(R/\mathfrak{m}, P) = 0$.*

Proof. (1) $\Rightarrow$ (2) is clear.
To see (2) $\Rightarrow$ (3), recall that projective modules are flat, and tensoring P to the exact sequence

$$0 \to \mathfrak{m} \to R \to R/\mathfrak{m},$$

the following sequence is exact

$$0 \to \mathfrak{m} \otimes_R P \to P \to P/\mathfrak{m}P.$$

For showing (3) $\Rightarrow$ (1), we use the fact that the symmetry of Tor, namely, $\mathrm{Tor}_1^R(R/\mathfrak{m}, P) \cong \mathrm{Tor}_1^R(P, R/\mathfrak{m})$. Now, take $e_1, \ldots, e_n$ to be elements of P whose image in $P/\mathfrak{m}P$ form a basis of the κ-vector space. By 2.12, the e_i generate P over R. Consider R^n and morphism $\varphi : R^n \to P, (r_1, \ldots, r_n) \mapsto r_1e_1 + \cdots + r_ne_n$ is an epimorphism. We obtain a short exact sequence

$$0 \to \ker\varphi \to R^n \to P \to 0.$$

By assumption $\mathrm{Tor}_1^R(P, R/\mathfrak{m}) = 0$, after tensoring by $R/\mathfrak{m}$, the resulting sequence is exact

$$0 \to \ker\varphi \otimes_R R/\mathfrak{m} \to \kappa^n \xrightarrow{\overline{\varphi}} \kappa^n \to 0$$

where $\overline{\varphi}$ is an isomorphism. Since $\ker\varphi$ is finitely generated as a submodule of the Noetherian module R^n, we deduce that $\ker\varphi = 0$ by Nakayama's lemma. This shows that P is free and thereby completes the proof.

□

Remark 2.15. It is an easy observation that a projective module is flat by viewing it as a direct summand of a free one. Over Noetherian rings, every finitely generated flat module is projective(by the equational criterion of flatness). We have the following correspondence,

Over Noetherian ring R: finitely generated flat modules $\Leftrightarrow$ finitely generated projective modules,

Over Noetherian local ring R: finitely generated flat modules $\Leftrightarrow$ finitely generated projective modules $\Leftrightarrow$ finitely generated free modules.

If we expand the scope to the more general setting, where R is not necessarily local, the previous propositions combined with the fact that finitely generated projective modules are finitely presented has a key consequence that projective modules are locally free.

Proposition 2.16. *Assume R is a commutative unital ring, $\mathfrak{p}$ is a prime ideal in R and P is a finitely generated projective R-module.*

1. *The localization $P_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module of some finite rank n.*
2. *There exists an element $s \in R\backslash\mathfrak{p}$ such that the localization of P away from s is free, i.e., $P[\frac{1}{s}]$ is a free $R[\frac{1}{s}]$-module of rank n.*
3. *If $\mathfrak{p}'$ is any prime ideal not containing s, $P_{\mathfrak{p}'}$ is a free $R_{\mathfrak{p}'}$-module of rank n.*

Proof. (1) Since $R_\mathfrak{p}$ is local and $P_\mathfrak{p}$ is finitely generated projective $R_\mathfrak{p}$ module, it is free of some finite rank n by 2.13.
(2) $P_\mathfrak{p}$ is finitely generated projective so it is finitely presented. Choose $e_1, \ldots, e_n \in P$ whose image in $P_\mathfrak{p}$ gives an $R_\mathfrak{p}$ basis, they generate P by 2.12. Set $\varphi : R^n \to P$ be the canonical map. We have the exact sequence

$$0 \to \ker\varphi \to R^n \to P \to \mathrm{coker}\varphi \to 0.$$

Since $\varphi_\mathfrak{p}$ is an isomorphism, by the flatness of localization, $0 = \mathrm{coker}(\varphi_\mathfrak{p}) = (\mathrm{coker}\varphi)_\mathfrak{p}$. $\mathrm{coker}\varphi$ is finitely generated. Take $x_1, \ldots, x_r$ to be the generators. Then there is some $f_i \in R\backslash\mathfrak{p}$ such that $f_ix_i = 0$. Take $f = \prod f_i$, then f vanishes $\mathrm{coker}\varphi$, and $0 = (\mathrm{coker}\varphi)_f = \mathrm{coker}(\varphi_f)$. So

$$0 \to \ker\varphi_f \to R_f^n \to P_f \to 0$$

is exact.
P is finitely presented so is P_f. Thus, $\ker \varphi_f$ is (finitely presented so) finitely generated by 2.8. Notice that localization first at f then at $\mathfrak{p}$ is the same as localization at $\mathfrak{p}$. Again, since $0 = \ker(\varphi_{\mathfrak{p}}) = (\ker \varphi_f)_{\mathfrak{p}}$, and that $\ker \varphi_f$ is finitely generated, and there is some $g \in R \backslash \mathfrak{p}$ vanishing $\ker \varphi_f$. In other words, $s = fg \in R \backslash \mathfrak{p}$ vanishes $\ker \varphi$, thus $R_s^n \cong P_s$ is a free R_s-module of rank n.
(3). Notice that for $\mathfrak{p}' \in D(f)$, localization at $\mathfrak{p}'$ is a further localization of s, i.e. $P_{p'} \cong (P_s)_{p'}$. Again, by flatness of localization, $P_{\mathfrak{p}'}$ is free of rank n. $\square$

We establish a slightly stronger statement below.

Proposition 2.17. *If R is a commutative unital ring and P is a finitely generated projective R-module, then there is an integer r and finitely many elements $f_1, \ldots, f_r \in R$ such that the family $f_1, \ldots, f_r$ generate the unit ideal in R and such that $P[\frac{1}{f_i}]$ is a free $R[\frac{1}{f_i}]$-module of finite rank for each i.*

Remark 2.18. This is equivalent to ask P to be finitely generated, and for every prime $\mathfrak{p}$ the module $M_{\mathfrak{p}}$ is free, and the function

$$\rho_P : \mathrm{Spec}(R) \to \mathbb{Z}, \mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})} P \otimes_R \kappa(\mathfrak{p}).$$

In a sense, this is the right notion of "locally free" for Zariski topology of Spec$R$. Namely, this proposition says that a finitely generated projective R-module is locally free for the Zariski topology on Spec$R$ of finite rank.

Proof. For each $\mathfrak{p} \in \mathrm{Spec}R$. By appealing to 2.16, there is some $s_{\mathfrak{p}}$ such that $\mathfrak{p} \in D(s_{\mathfrak{p}})$ and that $P[\frac{1}{s_{\mathfrak{p}}}]$ is a free $R[\frac{1}{s_{\mathfrak{p}}}]$-module. Note that $\{s_{\mathfrak{p}}\}$ covers Spec$R$, i.e. $s_{\mathfrak{p}}$'s generate the unit ideal in R. Since Spec$R$ is quasi-compact, it follows that a finite number of these modules already generate the unit ideal, which we relabel as $f_1, \ldots, f_r$. This shows the claim. $\square$

Definition 2.19. *An R-module M is called (Zariski) locally free (of finite rank) if there exists a family of elements $\{f_i\}_{i \in I}$ that generate the unit ideal such that the R_{f_i}-module M_{f_i} is free (of finite rank).*

A natural question to ask is that when we have a locally free module, when can we conclude it is projective. For this to be fully addressed, we need to first introduce a way to building a module by "patching" up localizations.

3. Zariski Descent I

We will first deal with the simplest case: suppose we have an affine scheme $X = \text{Spec}R$, covered by a pair of basic open sets $U_1 = \text{Spec}R_f$ and $U_2 = \text{Spec}R_g$ for elements f and g generating the unit ideal. The coproducts of affine schemes correspond to products of rings, so the cover $U_1 \coprod U_2 \to X$ corresponds to a ring homomorphism $R \to R_f \times R_g$. Extension of scalars induces a functor $\text{Mod}_R \to \text{Mod}_{R_f \times R_g}$. Since $\text{Mod}_{R_f \times R_g}$ and $\text{Mod}_{R_f} \times \text{Mod}_{R_g}$ are an equivalence of categories, this functor is $M \mapsto M \otimes_R (R_f \times R_g) \cong (M_f, M_g)$, so the image of the pullback functor consists of pairs of modules that are localizations of a single module. Moreover, there is a distinguished identification to equate M_{fg} and M_{gf} by the functoriality of localizations. This is because M_{fg} is obtained by first considering M_f as an R_f-module and then $M_f \otimes_{R_f} R_{fg}$ as an R_{fg}-module. Similarly, M_{gf} is obtained by $M_g \otimes_{R_g} R_{gf}$. The associativity isomorphism for the tensor product of R-modules and the functoriality of localizations now yield the distinguished identification:

$$(M \otimes_R R_f) \otimes_{R_g} R_{fg} \cong M \otimes_R (R_f \otimes_{R_f} R_{fg}) \cong M \otimes_R (R_{fg}) \cong M \otimes_R (R_g \otimes_{R_g} R_{gf}) \cong (M \otimes_R R_g) \otimes_{R_g} R_{gf}.$$

Proposition 3.1. *Suppose R is a commutative unital ring and M is an R-module. Given an open cover of SpecR by two basic open sets $D(f)$ and $D(g)$ whose intersection is $D(fg)$, we obtain modules M_f, M_g and M_{fg} over R_f, R_g and R_{fg}. Note that we can view M_f, M_g and M_{fg} as R-modules, The module M is the pullback of the diagram of R-modules:*

$$\begin{tikzcd} M \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & M_g \arrow[d] \\ M_f \arrow[r] & M_{fg} \end{tikzcd}$$

Concretely, M is the fiber product $M_f \times_{M_{fg}} M_g$, or the submodule of $M_f \oplus M_g$ consisting of pairs (α, β) that localize to the same element of M_{fg}.

Proof. We first show that $M \to M_f \oplus M_g$ is injective. Since $D(f), D(g)$ cover Spec$R$, f, g generate the unit ideal. In other words, f and g are comaximal elements of R, i.e. $1 = af + bg$ for some $a, b \in R$. Note that then f^r and g^r are also comaximal as $(af + bg)^{2r-1} = 1^{2r-1} = 1$. If $m \in M$ localizes to 0 in M_f and M_g, then $f^r m = 0 = g^r m$ for some r sufficiently large. Then as $cf^r + dg^r = 1$ for some $c, d \in R$, $m = cf^r m + dg^r m = 0$.
The image of the map $M \mapsto M_f \oplus M_g$ consists of pairs of elements (α, β) that agree upon further localization to M_{fg}. To alleviate notation, set $S = \{f^r, r \geq 0\}, T = \{g^r, r \geq 0\}$. We can immediately observe that any elements of the two sets are comaximal. Now suppose $\frac{m}{s} \in M_f$ and $\frac{n}{t} \in M_g$ localize to the same element in M_{fg}, then there is $s' \in S, t' \in T$ such that $s't'(tm - sn) = 0$. Thus, $(tt')(s'm) = (ss')(t'n)$, and replacing $\frac{m}{s}$ by $\frac{s'm}{s's}$ and $\frac{n}{t}$ by $\frac{t'n}{t't}$, we may say $sn = tm$. Since s, t are comaximal, for some $x, y \in R$, we have $xs + yt = 1$. Take $q = xm + yn$, we can verify that q localizes to $\frac{m}{s}$ and $\frac{n}{t}$:

$$sq = s(xm) + s(yn) = (xs)m + y(sn) = (xs)m + y(tm) = (xs + yt)m = m$$

and similarly $tq = n$. □

Remark 3.2. That M is the pullback of the above square is the same as that the following sequence of R-modules is exact:

$$0 \to M \to M_f \oplus M_g \to M_{fg}$$

where the second map is induced by localization and the third map sends each (a, b) to $a - b$ in the localization.

In the special case where $M = R$, this is actually an exact sequence of flat R-algebras of the form:

$$0 \to R \to R_f \oplus R_g \to R_{fg}$$

So, M is the zeroth cohomology group of two-term complex $M \otimes (R_f \oplus R_g \to R_{fg})$.

We proceed to show that this construction is functorial:

Proposition 3.3. *Suppose R is a commutative unital ring and M and N are R-modules. Suppose f and g are comaximal elements of R and $\alpha_f : M_f \to N_f, \alpha_g : M_g \to N_g$ are a pair of homomorphism that localize to the same homomorphism $M_{fg} \to N_{fg}$.*

1. *There exists a unique R-module homomorphism $\alpha : M \to N$ that localizes to α_f and α_g.*
2. *The morphism α is an isomorphism (resp. monomorphism) if and only if α_f and α_g are isomorphisms (resp. monomorphisms).*[1]

Proof. Since pullback is functorial, there is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & M_f \oplus M_g & \longrightarrow & M_{fg} & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle \alpha_f \oplus \alpha_g} & & \Big\downarrow{\scriptstyle \alpha_{fg}} & & \\ 0 & \longrightarrow & N & \longrightarrow & N_f \oplus N_g & \longrightarrow & N_{fg} & \longrightarrow & 0 \end{array}$$

We will construct $\alpha : M \to N$ by diagram chase. For $m \in M$, $\alpha_f(m) \oplus \alpha_g(m)$ is contained in the kernel of the map $N_f \oplus N_g \to N_{fg}$ by commutativity. Thus, there is a lift to a unique $\alpha(m) \in N$. We check α is a well defined R-module map: $\alpha_f(am + bn) \oplus \alpha_g(am + bn) = (a\alpha_f(m) + b\alpha_f(n)) \oplus (a\alpha_g(m) + b\alpha_g(n)) = a\alpha_f(m) \oplus \alpha_g(m) + b\alpha_f(n) \oplus \alpha_g(n)$. Then $\alpha(am + bn)$ is its unique lift, which is $a\alpha(m) + b\alpha(n)$.
For point (2), if α_f, α_g and α_{fg} are injective or isomorphism, then an analogous diagram chase can be carried out to show that β is respectively an monomorphism or isomorphism. $\square$

[1]The statement is also true for epimorphism. It just doesn't follow from the diagram chase argument we are applying.

4. Zariski Descent II

We would like to upgrade above construction to a functor of the module categories, namely, given modules over localizations and "patching data": a choice of isomorphism on the intersection, we now can build the modules over the ring itself functorially.
But we must be careful: in the above case, we start with an R-module then pass it to its local data and ask if they can the patched to get this R-module back. But now we start with "local data"+ "patching data" instead. Moreover, *Cat* is a 2-category, where the above pullback diagram also encodes the data of a 2-isomorphism.

$$\alpha : (pr_f(-))_g \Rightarrow (pr_g(-))_f.$$

We will formalize this idea with the following construction: Consider the category of patching datum $\mathrm{Mod}_R(f,g)$ where objects are triples $(M_1, M_2, \alpha_{M_1 M_2})$, M_1 is an R_f module, M_2 is an R_g module, and $\alpha_{M_1 M_2} : (M_1)_g \xrightarrow{\sim} (M_2)_f$ is an isomorphism of R_{fg} modules. In particular, $\alpha_{M_f M_g} : (M_f)_g \xrightarrow{\sim} (M_g)_f$ denotes the canonical isomorphism in 3. A morphism triples consists of the following data:

1. $\beta_1 : M_1 \to M_1'$ and $\beta_2 : M_2 \to M_2'$, such that
2. $$\begin{tikzcd} (M_1)_g \arrow[r, "\alpha_{M_1 M_2}"] \arrow[d, "(\beta_1)_g"'] & (M_2)_f \arrow[d, "(\beta_2)_f"] \\ (M_1')_g \arrow[r, "\alpha_{M_1' M_2'}"'] & (M_2')_f \end{tikzcd}$$

   is a commutative diagram.

This construction is exactly the (homotopy) pullback $R_f \times_{R_{fg}} R_g$.

Theorem 4.1. *Suppose R is a commutative unital ring and f, g are comaximal elements in R. The functor $Mod_R \to Mod_R(f,g)$ sending an R-module M to (M_f, M_g, c) is an equivalence of categories.*

Proof. Given an R-module M, observe that $(M_f, M_g, \alpha_{M_1 M_2})$ is the same data as the localization diagram.

$$\begin{tikzcd} & M_g \arrow[d] \\ M_f \arrow[r] & M_{fg} \end{tikzcd}$$

The functoriality $\mathrm{Mod}_R \to \mathrm{Mod}_R(f,g)$ sending an R-module M to (M_f, M_g, c) essentially follows from that of extension of scalars and that of localization.

Step 1. Given a triple $(M_1, M_2, \alpha_{M_1 M_2})$, we construct a candidate quasi-inverse by taking the R-module as the pullback from the diagram

$$\begin{tikzcd} M \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & M_2 \arrow[d] \\ M_1 \arrow[r] & (M_1)_g \cong (M_2)_f \end{tikzcd}$$

Thus, functoriality of pullbacks makes this a functor $\mathrm{Mod}_R(f,g) \to \mathrm{Mod}_R$.

Step 2. We study the composite $\mathrm{Mod}_R \to \mathrm{Mod}_R(f,g) \to \mathrm{Mod}_R$. Starting with an R-module M, the first functor takes it to (M_f, M_g, c) and by 3.1, the second functor recovers M. On morphisms, let $\varphi : M \to N$ be a morphism of R-modules. the first functor takes it to $(\varphi_f : M_f \to N_f, \varphi_g : M_g \to N_g, \varphi_{fg} : M_{fg} \to N_{fg})$. By the functoriality of pullback, the second map recovers φ.

Step 3.1 Next, we look at the composite $\mathrm{Mod}_R(f,g) \to \mathrm{Mod}_R \to \mathrm{Mod}_R(f,g)$. If we start with $(M_1, M_2, \alpha_{M_1 M_2})$ then the first functor sends it to the fiber product M, and the second functor sends it to (M_f, M_g, c). By the lemma, observe that the map $i : M \to M_1$ factors through a unique morphism $i' : M_f \to M_1$ by the universal property of localization. Similarly, let $j : M \to M_2$ and $j' : M_g \to M_2$ be the corresponding morphism from localization. We will show i' is an isomorphism.
To show injectivity, suppose $m \in M$ and $i'(\frac{m}{f^n}) = 0 \in M_1$. In this case $i(m) = 0$, and it follows that $j(m) \in M_2$ localizes to zero in $(M_2)_f$ by the commutativity of the pullback diagram, i.e. $f^k j(m) = j(f^k m) = 0$ for some non-negative integer k. On the other hand, $i(f^k m) = f^k i(m) = 0$ so $f^k m = 0$, $\frac{m}{f^n} = 0$.
For surjectivity, suppose $m_1 \in M_1$ then $\frac{m_2}{f^r} \in (M_2)_f$ is the image of m_1 under localization and isomorphism α. We observe that $f^r m_1$ and m_2 localize to the same element of $(M_1)_g \cong (M_2)_f$. Therefore, there exists an element $m \in M$ such that $i(m) = f^r m_1, j(m) = m_2$. So $i'(\frac{m}{f^r}) = m_1$ as desired. By symmetry, j' is an isomorphism.
Lastly, we shall verify the commutativity of the following diagram

$$\begin{tikzcd}
(M_f)_g \arrow[r, "(i')_g"] \arrow[d, "c"'] & (M_1)_g \arrow[d, "\alpha"] \\
(M_g)_f \arrow[r, "(j')_f"'] & (M_2)_f
\end{tikzcd}$$

Adopting the notations from earlier, if we start with $\frac{m_1}{g^l} \in (M_1)_g$ then it is lifted to $\frac{m/f^r}{g^l} \in (M_f)_g$. At the same time, it is sent by α to $\frac{m_2/f^r}{g^l} = \frac{m_2/g^l}{f^r} \in (M_2)_f$, which is lifted to $\frac{m_2/g^l}{f^r} \in (M_g)_f$. This shows commutativity.

Step 3.2 Note that what we showed in step 3.1 already implies the composite is essentially surjective. We now check that for fully faithfulness. Take two triples $(M_1, M_2, \alpha_{M_1,M_2})$, $(M_1', M_2', \alpha_{M_1',M_2'})$ and morphism (φ_1, φ_2) between them. Define $k' : M_f' \to M_1', l' : M_g' \to M_2'$ similar to i' and j'. Then we can define φ_f by the following commutative diagram using that i', k' are isomorphisms:

$$\begin{tikzcd}
M_f \arrow[r, "\varphi_f"] \arrow[d, "i'"'] & M_f' \arrow[d, "k'"] \\
M_1 \arrow[r, "\varphi_1"'] & M_1'
\end{tikzcd}$$

Define φ_g accordingly, (φ_1, φ_2) are sent to (φ_f, φ_g). Recall that a morphism of triples has to satisfy certain commutative square (being compatible with the 2-isomorphism α). We have the following diagram

$$\begin{tikzcd}
& (M_f)_g \arrow[rr, "(\varphi_f)_g"] \arrow[dl, "c=\alpha_{M_f M_g}"'] \arrow[dd, "i'_g"] & & (M'_f)_g \arrow[dl, "c'=\alpha_{M'_f M'_g}"'] \arrow[dd, "k'_g"] \\
(M_g)_f \arrow[rr, "(\varphi_g)_f"] \arrow[dd, "j'_f"] & & (M'_g)_f \arrow[dd, "l'_f"] & \\
& (M_1)_g \arrow[rr, "(\varphi_1)_g"] \arrow[dl, "\alpha_{M_1 M_2}"] & & (M'_1)_g \arrow[dl, "\alpha_{M'_1 M'_2}"] \\
(M_2)_f \arrow[rr, "(\varphi_2)_f"] & & (M'_2)_f &
\end{tikzcd}$$

where the commutativity of the left and right faces are given by the first commutative diagram in step 3.1; the top face commute by the functoriality of pullbacks and localizations; the bottom face commutes by setup; the front and back faces commute because commutativity is preserved by localizations. Thus, it is a commutative cube. Finally, given any $\varphi_f : M_f \to M'_f$, the fullness is immediate as $\varphi_1 = k'^{-1}\varphi_f i'^{-1}$ is always defined, and the faithfulness is also clear by from injectivity of $M \to M_f \oplus M_g$ (3.1).

□

Remark 4.2. But since $((M_2)_g)_f \cong ((M_2)_f)_g \cong ((M_1)_g)_g \cong (M_1)_g \cong (M_2)_f$, we might actually take $\frac{m_1}{g^l} \mapsto \frac{m_2/g^l}{f^r}$.

We can easily generalize Zariski descent for any finite cover $D(f_1), \dots, D(f_n)$ of Spec$R$. If $f_1, \dots, f_n$ generate the unit ideal then then $f_1g_1 + \cdots + f_ng_n = 1$ for $g_i \in R$. Take $f = f_1, g = f_2g_2 + \cdots + f_ng_n$, then we have the same setup as earlier. By inductive hypothesis, Zariski descent applies to SpecR_g, with the cover $D(f_2), \dots, D(f_n)$, so it also applies to Spec$R$ and the cover $D(f_1), \dots, D(f_n)$.

5. Zariski Descent III

Definition 5.1. *A property P for R-modules that is stable by localization is called local for the Zariski topology on SpecR if an R-module M has property P if and only if for comaximal elements $f_1, \ldots, f_n$, $M_{f_1}, \ldots, M_{f_n}$ have property P.*

By the final remark from the previous section, it is sufficient to ask this for a pair of comaximal element (f, g) and R_f, R_g-modules M_f and M_g, as the general case follows from a similar inductive arguement.

Proposition 5.2. *The property that an R-module is finitely generated or finitely generated projective is local for the Zariski topology on SpecR.*[2]

Proof. It is sufficient to consider for a pair (f, g) comaximal from above comment, so we will follow the same setup as we have had in the previous section. Let $(M_1, M_2, \alpha_{M_1 M_2})$ be the patching datum, and M the module obtained by patching.
Finite generation.
Assuming M_i are finitely generated, we can pick elements $x_1, \ldots, x_n$ of M such that $i(x_1), \ldots, i(x_n)$ form M_f. Without loss of generality, we can assume $M_1 = M_f$, and $M_2 = M_g$ that are finitely generated projective, then so is M_{fg} . We would like to show M is finitely generated projective itself. Similarly pick $y_1, \ldots, y_m$ of M that form R_g -generators of M_g under j. Let $M' \subset M$ be the submodule generated by $x_1, \ldots, x_n, y_1, \ldots, y_m$, then M' localizes to the patching datum. It then follows from the universal property of fiber product that $M' \to M$ is an isomorphism.
Projectivity.
By 4.1, without loss of generality, we can assume $M_1 = M_f$ and $M_2 = M_g$. In this case M_{fg} is also finitely generated projective (M_f and R_{fg} are projective, so its projectivity follows from 2.5). We already saw that M is finitely generated from the previous part. Pick generators $x_1, \ldots, x_n$ of M, then localize the surjection $R^{\oplus n} \to M$ at f and g. This by 2.3 admits M_f as a summand of $R_f^{\oplus n}$ and M_g as a summand of $R_g^{\oplus n}$, i.e. after localizing at f and g, it becomes split surjection. We would like to conclude that $R^{\oplus n} \to M$ is then split. But we pinpoint the first obstacle that two splittings need not to coincide as maps $R_{fg}^{\oplus n} \to M_{fg}$. In other words, let $s_f : M_f \to R_f^n$ and $s_g : M_g \to R_g^n$ be the sections. After localizing, $s_f|_g \alpha_{M_1 M_2}$ isn't necessarily $s_g|_f$. But we shall show that up to changing bases, the splittings will coincide and patch to give the required split. To alleviate the readability this proof, we formulate this into three separate claims.
<u>Claim:</u> Set $R' = R_f \oplus R_g$, a sequence $M_1 \to M_2 \to M_3$ is exact if and only if it is exact after after tensoring R'.
<u>Proof:</u> R_f and R_g are both localizations, so R' a is flat R-module and the induced morphism $R \to R'$ is flat. This is the content of 2.2, and the "only if" direction follows.
For the "if" direction, Take an arbitrary sequence $M_1 \to M_2 \to M_3$ such that $M_1 \otimes_R R' \to M_2 \otimes_R R' \to M_3 \otimes_R R'$ is exact. Set $H = \ker(M_2 \to M_3)/\mathrm{im}(M_1 \to M_2)$. We will show that H is zero. Firstly,

$$0 \to \mathrm{im}(M_1 \to M_2) \to \ker(M_2 \to M_3) \to H \to 0$$

is exact. By the flatness of R',

$$\ker(M_2 \to M_3) \otimes_R R' \cong \ker(M_2 \times_R R' \to M_3 \otimes_R R'),$$

$$\mathrm{im}(M_1 \to M_2) \otimes_R R' \cong \mathrm{im}(M_1 \otimes_R R' \to M_2 \otimes_R R'),$$

[2]Being finitely presented is also a local property for the Zariski topology, but being injective or torsion-free are not local properties. Without a finiteness hypothesis, we lose finite presentation that comes from finite generative projectivity, and our proof fails right away. However, the local property still holds for projectivity. See the MathOverflow discussion.

and lastly

$$H \times_R R' \cong \ker(M_2 \times_R R' \to M_3 \otimes_R R')/\mathrm{im}(M_1 \otimes_R R' \to M_2 \otimes_R R') = 0.$$

Now take an element $x \in H$, consider the induced morphism $R \to H$ by multiplying x. Set $I = \{r \in R | rx = 0\}$, then this map factors through the injection $R/I \hookrightarrow X$. Now again by flatness of $R \to R'$. $R'/IR' \cong R/I \otimes_R R' \hookrightarrow H \otimes_R R' = 0$. If $I \neq R$, there is a maximal ideal $\mathfrak{m}$ containing I and $R'/\mathfrak{m}R' = 0$. However, we see this is impossible. Notice here $\mathrm{Spec}R' \cong \mathrm{Spec}R_f \sqcup \mathrm{Spec}R_g \to \mathrm{Spec}R$ is an open covering, and $\mathfrak{m}$ is in $D(f)$ or $D(g)$. It follows that $R/\mathfrak{m}R$ is non-zero after localizing at f or g. Writing $R'/\mathfrak{m}R' \cong R_f/\mathfrak{m}R_f \oplus R_g/\mathfrak{m}R_g$, the contradiction is clear. ■

We would like $- \otimes_R R'$ to be an exact functor. But in fact we can show that $\otimes_R S$ is exact for any flat R-module S.

Claim: If $\varphi : R \to R'$ is a flat ring homomorphism, and if M and N are R-modules with M finitely presented, then the map induced by the extension of scalars

$$\hom_R(M, N) \otimes_R S \to \hom_S(M \otimes_R S, N \otimes_R S)$$

is an isomorphism.

Proof: Choose a presentation $R^n \xrightarrow{A} R^m \to M \to 0$, where A is a $m \times n$ matrix over R. By left exactness of $\hom(-, N)$, we have the exact complex of R-modules

$$0 \to \hom_R(M, N) \to N^m \xrightarrow{A^*} N^n$$

where $A^* = (a^*_{ij})$ is the right multiplication by the transpose of A. Now S is a flat R-module, then

$$0 \to \hom_R(M, N) \otimes_R S \to N^m \otimes_R S \xrightarrow{A^* \otimes 1_S} N^n \otimes_R S$$

Rewrite the last morphism to $(N \otimes_R S)^m \xrightarrow{A^*_S} (N \otimes_R S)^n$ where $(A^*)_S$ has entries $(a^*_{ij} \otimes_R 1_S)$. On the other hand, first tensoring with S yields

$$S^n \xrightarrow{A_S} S^m \to M \otimes_R S \to 0$$

where A_S has entries $a_{ij} \otimes_R 1_S$. Applying the functor $\hom(-, N \otimes_R S)$ to the exact sequence gives

$$0 \to \hom_S(M \otimes_R S, N \otimes_R S) \to (N \times_R S)^m \xrightarrow{(A_S)^*} (N \otimes_R S)^n.$$

where $(A_S)^*$ is the right multiplication by the transpose of A_S, thus it is identical to $(A^*)_S$, and their kernels agree. ■

Lastly, recall that finitely generated projective modules are finitely presented by 2.4. It suffices to prove the following claim.

Claim: Let $\varphi : M \to N$ be a homomorphism of R-modules with N finitely presented over R. The homomorphism φ is a split surjection if and only if the induced map $\varphi' : R_f \otimes_R M \oplus R_g \otimes_R M \to R_f \otimes_R N \oplus R_g \otimes_R N$ is split surjection.

Proof: The second map is obtained by tensoring R'; writing φ into a three term exact sequence with its cokernel, the first claim says that φ is surjective if and only if φ' is. We reduces the claim to that of splitting. Consider the map $\hom_R(N, M) \xrightarrow{\varphi\circ} \hom_R(N, N)$; a splitting $N \to M$ provides a lift of any endomorphism of N to a homomorphism $N \to M$, i.e. the map is surjective. Conversely, if the map is surjective, then a lift of 1_N is a splitting. In other words, φ is split if and only if $\hom_R(N, \varphi)$ is surjective, which is if and only if $\hom_R(N, \varphi) \otimes_R R'$ is surjective by the first claim again. Given that N is finitely presented, $- \otimes_R R'$ is exact by the second claim i.e. $\hom_R(N, \varphi) \otimes_R R' \cong \hom_{R'}(N \otimes_R R', \varphi \otimes_R R)$. So the splitness of φ is the same as asking for

$$\hom_{R'}(N \otimes_R R', M \otimes_R R') \xrightarrow{\varphi \otimes_R R' \circ} \hom_{R'}(N \otimes_R R', N \otimes_R R')$$

to be surjective, but this is if and only if $\varphi \otimes_R R'$ is split, by the same reasoning as before. ■

This last claim concludes the proof of the proposition. □

Consider locally free R-modules (of finite rank) in the sense of 2.19, since free modules are projective, 5.2 thus says that every locally free R-module is finitely generated projective. This concludes to an answer to the question at the end of section 2, that is

Corollary 5.3. *Every finite rank locally free R-module is a finitely generated projective R-module. Moreover, the categories of finitely generated projective R-modules and locally free R-modules are equivalent.*

Proof. The first part is clear. Note that the homomorphism sets in each category are simply R-module homomorphisms. So the equivalence of the two categories is also immediate. □

6. Discussion

Suppose X is a topological space. A finite rank, continuous vector bundle on X is intuitively a continuous family of finite dimensional vector spaces parametrized by the points of X. Vector bundles have the key feature that they are locally trivial, i.e., for every $x \in X$, there is an open neighborhood U_x such that the restriction of the family to U_x is homeomorphic to a product of U_x with a finite dimensional vector space. Recall that a (global) section of a vector bundle $\pi : E \to X$ is a continuous map $s : X \to E$ such that $\pi \circ s = id_X$. A map $s : U \to E$ with $U \subset X$, is called a local section. Evaluating at each x, a section gives an element of the fiber $\pi^{-1}(x)$. Thus, identifying the fiber with $\mathbb{R}^n$ or $\mathbb{C}^n$ the set of all global section $\Gamma(\pi)$ has a vector space structure by

$$(a_1 s_1 + a_2 s_2)(x) = a_1 s_1(x) + a_2 s_2(x)$$

with $a_i \in \mathbb{R}^n$ or $\mathbb{C}^n$. Moreover, $\Gamma(\pi)$ assembles to a $C(X)$-module by defining $(fs)(x) = f(x)s(x)$ for $f \in C(X)$ and $s \in \Gamma(\pi)$. fs is again a section, and the it is straightforward to check the module structure. We will now call $\Gamma(\pi)$ the module of sections instead. Now if X is compact Hausdorff, then $\Gamma(\pi)$ is in fact a finitely generated projective $C(X)$-module. It's true the other way around: a finitely generated projective $C(X)$-module can be view as the module of sections of some vector bundle $\pi : E \to X$. This is the content of the following theorem.

Theorem 6.1 (Vaserstein's Serre-Swan Theorem)**.** *If X is a topological space, write $C(X, \mathbb{C})$ (resp. $C(X, \mathbb{R})$) for the set of complex (resp. real) valued continuous functions on X. The functor assigning to a complex (resp. real) vector bundle $\pi : E \to X$, the $C(X, \mathbb{C})$-module (resp. $C(X, \mathbb{R})$-module) of sections yields an equivalence between the category of complex (resp. real) vector bundles on X and the category of finitely generated projective $C(X)$-modules.*

We will outline the proof for the complex case below.

Lemma 6.2. *Let R be a ring and P an R-module. The following are equivalent.*

1. *P is finitely generated projective.*
2. *There exists $n \in \mathbb{N}$ and an idempotent $e \in M_n(R)$ such that $P \cong eR^n$.*

Proof. We give a sketch. If P is finitely generated, choose a surjection $R^n \twoheadrightarrow P$ with splitting because P is projective; represent the composition $R^n \to P \to R^n$ by e, an idempotent matrix. Conversely, eR^n is a direct summand of $R^n \cong eR^n \oplus (1-e)R^n$ for an idempotent e. □

If $\pi : E \to X$ is a complex vector bundle of finite rank over a compact Hausdorff space X, we will show $\Gamma(E)$ is a finitely generated projective $C(X)$–module.

Finite generation.

Choose a finite trivializing cover $\{U_i\}_{i=1}^r$ of X and a partition of unity $\{\rho_i\}$ subordinate to it (this is possible because being compact hausdorff $\Rightarrow$ paracompact). For each i pick a local frame $(s_{i1}, \dots, s_{in})$ over U_i. Extending by zero outside U_i yields global sections $\sigma_{ij} := \rho_i s_{ij} \in \Gamma(E)$. Any $t \in \Gamma(E)$ can be written as $t = \sum_{i,j} a_{ij}\sigma_{ij}$ with coefficients $a_{ij} \in C(X)$. Indeed, $t|_{U_i}$ corresponds to $x \mapsto (f_{i1}(x), \dots, f_{in}(x))$. By taking $a_{ij}(x) = \rho_i(x) f_{ij}(x)$, we assemble t back. Hence $\Gamma(E)$ is generated by the finite set $\{\sigma_{ij}\}$.

Projectivity.

Let $N = rn$. Using the same frames, define a bundle monomorphism $\iota : E \hookrightarrow X \times \mathbb{C}^N$ that is fiberwise linear: on U_i send $\sum_j z_j s_{ij}(x)$ to $(0, \dots, 0, z_1, \dots, z_n, 0, \dots, 0)$ where the block $(z_1, \dots, z_n)$ lies in the i–th slot. The cokernel bundle $E^\perp$ [3] exists because ι is fiberwise injective and X is compact. Then $X \times \mathbb{C}^N \cong E \oplus E^\perp$, and after taking sections, the LHS is $\Gamma(X \times \mathbb{C}^N) =$

[3] defined fiberwise as the orthogonal complement with respect to the standard Hermitian inner product on $\mathbb{C}^N$.

$\{s : X \to X \times \mathbb{C}^N, s(x) = (x, v(x))\}$ such that $pr_1 \circ s = id_X$. So the sections are exactly the coordinate maps $v : X \to \mathbb{C}^N$. But we may write $v(x) = (f_1(x), \ldots, f_N(x))$ as N-tuple of continuous complex valued functions; hence the LHS is the same as $C(X)^N$. With the observation that $\Gamma(E \oplus E^\perp) \cong \Gamma(E) \oplus \Gamma(E^\perp)$.

$$C(X)^N \;\cong\; \Gamma(E) \oplus \Gamma(E^\perp).$$

Thus $\Gamma(E)$ is a direct summand of the free module $C(X)^N$.

Next, let P be a finitely generated projective $C(X)$–module. We build a vector bundle E with $\Gamma(E) \cong P$. By Lemma 6.2 there exist n and an idempotent matrix $e = e^2 \in M_n\big(C(X)\big) \cong C\big(X, M_n(\mathbb{C})\big)$ such that $P \cong eC(X)^n$. For $x \in X$ let $e(x) \in M_n(\mathbb{C})$ denote the evaluation of the matrix of functions at x. Define

$$E := \big\{(x, v) \in X \times \mathbb{C}^n \mid v = e(x)v\big\} \subset X \times \mathbb{C}^n, \quad \pi : E \to X,\ \pi(x, v) = x.$$

For each x, the fiber $E_x = e(x)\mathbb{C}^n$ is a complex vector space of dimension $r(x)$. Equipping E with the subspace topology of $X \times \mathbb{C}^n$, we next show it is a complex vector bundle over X. For $x \in X$, define $r(x) = \operatorname{rank} e(x)$. It is locally constant on X and constant on the connected components[4] Now fix $x_0 \in X$, choose a neighbourhood U of x_0 on which r is constant. Use linear algebra to find a continuous map $u : U \to \mathrm{GL}_n(\mathbb{C})$ such that $u(x)^{-1}e(x)u(x)$ is the fixed projection $\operatorname{diag}(I_k, 0)$. Then

$$\phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^k, \quad (x, v) \mapsto \big(x, \pi_k\big(u(x)^{-1}v\big)\big)$$

where π_k projects to the first k coordinates, is a trivialization; its inverse sends (x, w) to $(x, u(x)\,(w, 0))$. Continuity of u ensures ϕ is a homeomorphism. ϕ restricts to a linear map on each fiber, and the collection of such charts gives local triviality.
Denote the constructed vector bundle $\pi : E \to X$. We have an isomorphism

$$\begin{aligned} P &\longrightarrow \Gamma(\pi), \\ e(f_1, \ldots, f_n) &\longmapsto \Big(x \longmapsto \big(x, e(x)(f_1(x), \ldots, f_n(x))\big)\Big). \end{aligned}$$

where $(f_1, \ldots, f_n) \in C(X)^n$. Observe this is a $C(X)$-module homomorphism. Injectivity follows from injectivity of pointwise evaluation; surjectivity is immediate because any section $s \in \Gamma(E)$ can be viewed as an n–tuple of continuous functions satisfying $e(x)s(x) = s(x)$.
Define the functor

$$\Gamma : \mathbf{Vect}(X) \longrightarrow \mathbf{Proj}_{\mathrm{fg}}(C(X)), \quad \pi : E \to X \longmapsto \Gamma(\pi).$$

On morphisms, Γ sends a bundle map $\alpha : E \to F$ covering id_X to the induced map on sections. Observe that the above already imply Γ is essentially surjective and faithful. Fullness follows from standard sheaf arguments: a $C(X)$–linear map $\Gamma(E) \to \Gamma(F)$ is determined fiberwise by evaluation and thus defines a bundle map.

[4]Each $e(x)$ is a projection, so its eigenvalues are 0 and 1. The rank equals the trace, $r(x) = \mathrm{tr}(e(x))$, which is continuous as a composite of continuous maps. Since r is integral valued and X is Hausdorff, r must be locally constant.

References

[1] Andrew Critch, "Is projectiveness a Zariski-local property of modules?", MathOverflow (Nov. 24, 2009), https://mathoverflow.net/questions/6719/is-projectiveness-a-zariski-local-property-of-modules-answered-yes, accessed Jul. 23, 2025.

[2] Aravind Asok, *Algebraic Geometry from an* $\mathbf{A}^1$*-Homotopic Viewpoint*, Course notes (MATH 614, Spring 2021, University of Southern California), May 3, 2021, available at https://dornsife.usc.edu/aravind-asok/wp-content/uploads/sites/213/2023/06/MATH614.S21.pdf, accessed 23 July 2025.

[3] D. Husemoller, *Fibre Bundles*. Springer GTM 20, 1994.

[4] L. N. Vaserstein, *Vector Bundles and Projective Modules*, Translations of Mathematical Monographs, vol. 82, American Mathematical Society, Providence, RI, 1988.

[5] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, Massachusetts, 1969.

[6] R. G. Swan, *Vector Bundles and Projective Modules*, Trans. Amer. Math. Soc. 105 (1962), 264–277.

[7] T. Y. Lam, *Lectures on Modules and Rings*. Springer GTM 189, 1999.