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Digital Signal Processing

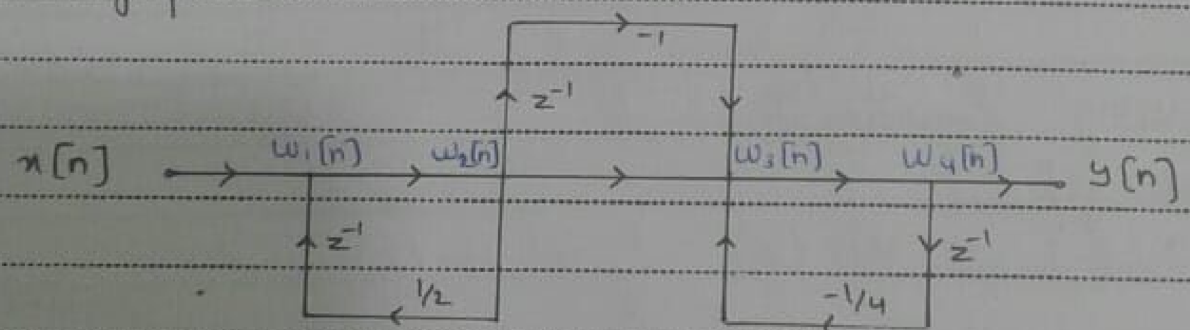
Assignment: 04

Questions:

Question: 1

Determine the system function of the following signal flow graphs.

a)



Solution:

$$w_1[n] = x[n] + \frac{1}{2} w_2[n-1]$$

$$w_2[n] = w_1[n]$$

$$w_3[n] = w_2[n] - w_2[n-1] - \frac{1}{4} w_4[n-1]$$

$$w_4[n] = w_3[n]$$

$$y[n] = w_4[n] = w_3[n]$$

In z-transform:

$W_1(z) = X(z) + \frac{1}{2} z^{-1} W_2(z) \longrightarrow (1)$

$W_2(z) = W_1(z) \longrightarrow (2)$

$W_3(z) = W_2(z) - z^{-1} W_2(z) - \frac{1}{4} z^{-1} W_4(z) \longrightarrow (3)$

$W_4(z) = W_3(z) - (4)$

$Y(z) = W_3(z) - (5)$

W Put eq (2) in eq (1)

$W_2(z) = X(z) + \frac{1}{2} z^{-1} W_2(z) \longrightarrow (A)$

put eq (4) in eq (3)

$W_3(z) = W_2(z) - z^{-1} W_2(z) - \frac{1}{4} z^{-1} W_3(z)$

$W_3(z) = W_2(z) \left\{1 - z^{-1}\right\} - \frac{1}{4} z^{-1} W_3(z) - (B)$

By using Equation (A)
we get

$X(z) = W_2(z) - \frac{1}{2} z^{-1} W_2(z)$

$X(z) = W_2(z) \left\{1 - \frac{1}{2} z^{-1}\right\}$

$$W_2(z) = \frac{X(z)}{\left(1 - \frac{1}{2} z^{-1}\right)}$$

Put $W_2(z)$ in eq (B)

$$W_3(z) = \frac{X(z)}{\left(1 - \frac{1}{2} z^{-1}\right)} \left(1 - z^{-1}\right) - \frac{1}{4} z^{-1} W_3(z)$$

As

$$W_3(z) = Y(z)$$

$$W_3(z) + \frac{1}{4} z^{-1} W_3(z) = X(z) \frac{(1-z^{-1})}{(1-\frac{1}{2}z^{-1})}$$

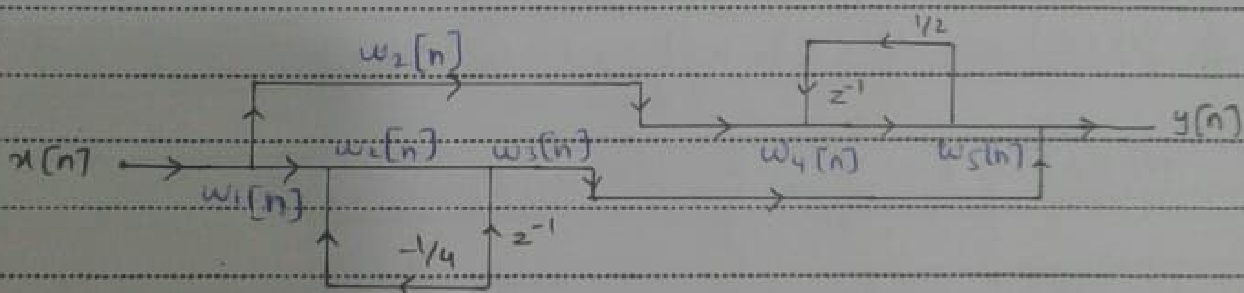
$$W_3(z) \left\{ 1 + \frac{1}{4} z^{-1} \right\} = \left(\frac{1-z^{-1}}{1-\frac{1}{2}z^{-1}} \right) X(z)$$

$$Y(z) \left\{ 1 + \frac{1}{4} z^{-1} \right\} = X(z) \left(\frac{1-z^{-1}}{1-\frac{1}{2}z^{-1}} \right)$$

$$\frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

$$H(z) = \frac{1-z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

b.



Solution:

$$w_1[n] = x[n]$$

$$w_2[n] = w_1[n] - \frac{1}{4} w_3[n-1]$$

$$w_3[n] = w_2[n]$$

$$w_4[n] = w_1[n] + \frac{1}{2} w_5[n-1]$$

$$w_5[n] = w_4[n]$$

$$y[n] = w_5[n] + w_3[n]$$

In z-transform:

$$W_1(z) = X(z)$$

$$W_2(z) = W_1(z) - \tfrac{1}{4} z^{-1} W_3(z) \quad — ①$$
$$W_3(z) = W_2(z) \quad — ②$$
$$W_4(z) = W_1(z) + \tfrac{1}{2} z^{-1} W_5(z) \quad — ③$$
$$W_5(z) = W_4(z) \quad — ④$$
$$Y(z) = W_5(z) + W_3(z) \quad — ⑤$$
$$W_2(z) = X(z) - \frac{1}{4} z^{-1} W_2(z)$$

$$W_2(z)\left\{1 + \frac{1}{4} z^{-1}\right\} = X(z)$$

$$W_2(z) = X(z) \Big/ \left(1 + \frac{1}{4} z^{-1}\right)$$

$$W_2(z) = W_3(z)$$
$$W_3(z) = X(z) \Big/ \left(1 + \frac{1}{4} z^{-1}\right) \quad — Ⓐ$$

$$W_5(z) = W_1(z) + \frac{1}{2} z^{-1} W_5(z)$$

$$W_5(z)\left\{1 - \frac{1}{2} z^{-1}\right\} = W_1(z)$$

$$W_5(z)\left\{1 - \frac{1}{2} z^{-1}\right\} = X(z) \qquad \because W_1(z) = X(z)$$

$$W_5(z) = X(z) \Big/ \left(1 - \tfrac{1}{2} z^{-1}\right) \quad — Ⓑ$$

put the value of Ⓐ and Ⓑ in eq ⑤

$$Y(z) = W_5(z) + W_3(z)$$
$$Y(z) = X(z)\left\{\frac{1}{1 - \frac{1}{2} z^{-1}}\right\} + X(z)\left\{\frac{1}{1 + \frac{1}{4} z^{-1}}\right\}$$
$$Y(z) = X(z)\left\{\frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{4} z^{-1}}\right\}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 - \frac{1}{4} z^{-1}}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 + \frac{1}{4} z^{-1}\right)}$$

Question: 2

Design IIR Filters using Impulse Invariance And Bilinear transformation based on Butterworth Approximation.

Solution:

IIR Filter Design Using Impulse Invariance based on Butterworth Approximation:

Given:

$$0.23306 \le |H(e^{j\omega})| \le 1 \qquad 0 \le |\omega| \le 0.4\pi$$
$$|H(e^{j\omega})| \le 0.17782 \qquad 0.5\pi \le |\omega| \le \pi$$

Discrete Time specification $\xrightarrow{\text{Convert into}}$ Continuous Time specification

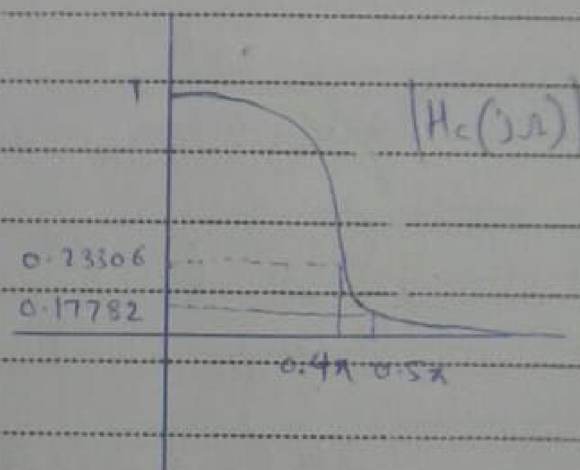
Role of Td is cancel out by the Impulse invariance method so we assume,

$$T_d = 1 \qquad \omega = \Omega T_d \qquad \omega = \Omega$$

$$0.23306 \le |H_c(j\Omega)| \le 1 \qquad 0 \le |\Omega| \le 0.4\pi$$
$$|H_c(j\Omega)| \le 0.17782 \qquad 0.5\pi \le |\Omega| \le \pi$$

By using Butterworth Approximation:

$$|H_c(j\Omega)|^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$



$$(0.23306)^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

$$1+\left(\frac{\Omega}{\Omega_c}\right)^{2N} = \frac{1}{(0.23306)^2}$$

$$\left(\frac{0.4\pi}{\Omega_c}\right)^{2N} = \frac{1}{(0.23306)^2} - 1$$

$$\left(\frac{0.4\pi}{\Omega_c}\right)^{2N} = 17.410 \longrightarrow (A)$$

$$(0.17782)^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

$$1+\left(\frac{\Omega}{\Omega_c}\right)^{2N} = \frac{1}{(0.17782)^2}$$

$$\left(\frac{0.5\pi}{\Omega_c}\right)^{2N} = \frac{1}{(0.17782)^2} - 1$$

$$\left(\frac{0.5\pi}{\Omega_c}\right)^{2N} = 30.6256 \longrightarrow (B)$$

Divide Equation (A) by Equation (B)

$$\frac{(0.4\pi)^{2N}/\cancel{(\Omega_c)^{2N}}}{(0.5\pi)^{2N}/\cancel{(\Omega_c)^{2N}}} = \frac{17.410}{30.6256}$$

$$\left(\frac{0.4\cancel{\pi}}{0.5\cancel{\pi}}\right)^{2N} = 0.5684$$

$$(0.8)^{2N} = 0.5684$$

Taking $\log_{10}$ on b/s

$$\log_{10}(0.8)^{2N} = \log_{10}(0.5684)$$

$$2N\log_{10}(0.8) = \log_{10}(0.5684)$$

$$2N(-0.0969) = -0.2453$$

$$2N = 2.5314$$

$$N = 1.2657$$

$$N \cong 2$$

When we put $N = 1.2657$ in eq Ⓐ and eq Ⓑ, we get the same answer of Ω_c.

In Equation A:

$$\left(\frac{0.4\pi}{\Omega_c}\right)^{2(1.2657)} = 17.410$$

$$\left(\frac{0.4\pi}{\Omega_c}\right)^{2.5314} = 17.410$$

$$\left(\frac{0.4\pi}{\Omega_c}\right) = (17.410)^{1/2.5314}$$

$$\frac{0.4\pi}{3.0914} = \Omega_c$$

$$\Omega_c = 0.4064$$

In Equation B:

$$\left(\frac{0.5\pi}{\Omega_c}\right)^{2(1.2657)} = 30.6256$$

$$\left(\frac{0.5\pi}{\Omega_c}\right)^{2.5314} = 30.6256$$

$$\left(\frac{0.5\pi}{\Omega_c}\right) = (30.6256)^{1/2.5314}$$

$$\frac{0.5\pi}{\Omega_c} = 3.864$$

$$\Omega_c = 0.4064.$$

When we put $N = 2$ in eq Ⓐ and eq Ⓑ, we get the different answer of Ω_c.

In Equation A:

$$\left(\frac{0.4\pi}{\Omega_c}\right)^{2\times 2} = 17.410$$

$$\left(\frac{0.4\pi}{\Omega_c}\right)^{4} = 17.410$$

$$\frac{0.4\pi}{\Omega_c} = (17.410)^{1/4}$$

$$\Omega_c = 0.61519$$

In Equation B:

$$\left(\frac{0.5\pi}{\Omega_c}\right)^{2\times 2} = 30.6256$$

$$\left(\frac{0.5\pi}{\Omega_c}\right)^{4} = 30.6256$$

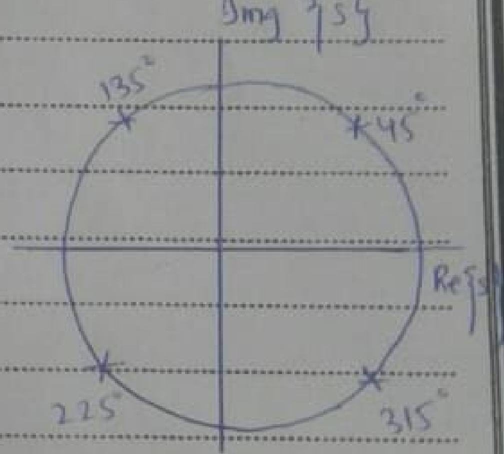
$$\left(\frac{0.5\pi}{\Omega_c}\right) = (30.6256)^{1/4}$$

$$\Omega_c = 0.6677$$

Now, we choose $\Omega_c = 0.61519$, $N=2$

Angular Distance b/w poles $= \frac{\pi}{N}$

$$= \frac{\pi}{2} = \frac{180^\circ}{2} = 90^\circ$$



$N=2$ = even = No pole on x-axis

Number of poles = $2N$

$= 2\times 2 = 4$

for casual and stable system choose those poles which lies on the left side of s-plane

. The resulting transfer function has the following poles

$$S_1 = 0.61519\, e^{j(135^\circ)}$$

$$= 0.61519\,\{\cos(135) + j\sin(135)\}$$

$$= 0.61519\,\{-0.9960 + j\,0.08836\}$$

$$= -0.6127 + j\,0.05435$$

$$S_2 = -0.6127 - j\,0.05435$$

Resulting in:

$$H(s) = \frac{(-\Omega_c)^N}{(s-s_1)(s-s_2)}$$

$$H(s) = \frac{(0.61519)^2}{(s-(-0.6127+j0.05435))(s-(-0.6127-j0.05435))}$$

$$H(s) = \frac{0.61519}{(s+0.6127-j0.05435)(s+0.6127+j0.05435)}$$

$$= \frac{0.3784}{s^2 + 1.2254s + 0.3774}$$

Mapping to z-domain:

poles $s = s_k$ in s-domain transform into pole at $z = e^{s_k T_d}$ in z-domain

$$H(z) = \frac{0.3784}{(1-e^{s_1}z^{-1})(1-e^{s_1}z^{-1})}$$

$$= \frac{0.3784}{(1-e^{-0.6127+j0.05435}z^{-1})(1-e^{-0.6127-j0.05435}z^{-1})}$$

$$= \frac{0.3784}{(1-0.5418e^{j0.05435}z^{-1})(1-0.5418e^{-j0.05435}z^{-1})}$$

$$H(z) = \frac{A}{(1-0.5418e^{j0.05435}z^{-1})} + \frac{B}{(1-0.5418e^{-j0.05435}z^{-1})} \quad \text{— } (x)$$

$$A = \left.\cancel{(1-0.5418e^{j0.05435}z^{-1})}\frac{0.3784}{\cancel{(1-0.5418e^{j0.05435}z^{-1})}(1-0.5418e^{-j0.05435}z^{-1})}\right|_{z=0.5418e^{j0.0543}}$$

$$= \frac{0.3784}{\left(1-0.5418e^{-j0.05435} \times \frac{1}{0.5418e^{j0.05435}}\right)}$$

$$A = \frac{0.3784}{1 - \dfrac{0.5418\left(\cos(-0.05435) + j\sin(-0.05435)\right)}{0.5418\left(\cos(0.05435) + j\sin(0.05435)\right)}}$$

$$A = \frac{0.3784}{1 - \left(\dfrac{0.99 - j9.48\times10^{-4}}{0.99 + j9.48\times10^{-4}}\right)}$$

$$= \frac{0.3784}{1\times10^{-3} + 1.9151i}$$

$$A = 1.0317\times10^{-4} - 0.1975j$$

$$B = \left[\left(1-0.5418e^{-j0.05435}z^{-1}\right)\frac{0.3784}{\left(1-0.5418e^{j0.05435}z^{-1}\right)\left(1-0.5418e^{-j0.05435}z^{-1}\right)}\right]_{z=0.5418e^{-j0.05435}}$$

$$B = \frac{0.3784}{1-0.5418e^{j0.05435} \times \dfrac{1}{0.5418e^{-j0.05435}}}$$

$$B = \frac{0.3784}{1 - \dfrac{0.5418\left(\cos(0.05435) + j\sin(0.05435)\right)}{0.5418\left(\cos(-0.05435) + j\sin(-0.05435)\right)}}$$

$$B = \frac{0.3784}{1 - (0.999 + 1.897j)}$$

$$B = 1.0515\times10^{-4} + 0.199j$$

Put the value of A and B in eq (x)

$$H(z) = \frac{1.0317\times10^{-4} - 0.1975j}{\left(1-0.5418e^{j0.05435}z^{-1}\right)} + \frac{1.0515\times10^{-4} + 0.199j}{\left(1-0.5418e^{-j0.05435}z^{-1}\right)}$$

IIR Filter Design using Bilinear Transformation based on Butterworth approximation:

Given:

$$0.23306 \le |H(e^{j\omega})| \le 1 \qquad 0 \le |\omega| \le 0.4\pi$$

$$|H(e^{j\omega})| \le 0.17782 \qquad 0.5\pi \le |\omega| \le \pi$$

Solution:

Discrete Time Specification $\xrightarrow{\text{Convert into}}$ Continous Time Specification

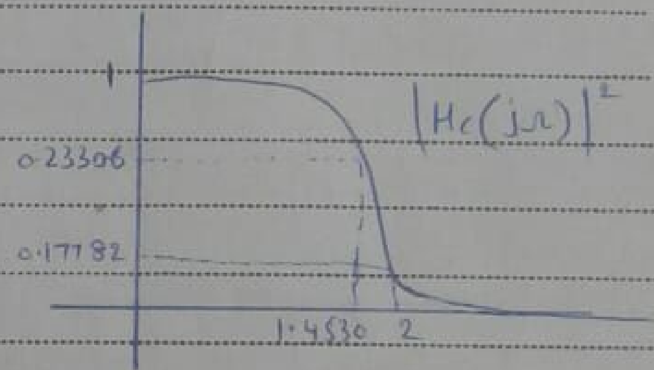
$$\Omega = \frac{2}{T_d} \tan\left(\frac{\omega}{2}\right)$$

When $\omega = 0.4\pi$

$$\Omega = \frac{2}{T_d} \tan\left(\frac{\omega}{2}\right) \qquad \because T_d = 1$$

$$= \frac{2}{1} \tan\left(\frac{0.4\pi}{2}\right)$$

$$= 1.4530$$



When $\omega = 0.5\pi$

$$\Omega = \frac{2}{T_d} \tan\left(\frac{\omega}{2}\right)$$

$$= \frac{2}{1} \tan\left(\frac{0.5\pi}{2}\right)$$

$$\Omega = 2$$

$$0.23306 \le |H(j\Omega)| \le 1 \qquad 0 \le |\Omega| \le 1.4530$$

$$|H(j\Omega)| \le 0.17782 \qquad 2 \le |\Omega| \le \infty$$

By using Butterworth Approximation:

$$|H_c(j\Omega)|^2 = \frac{1}{1 + \left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

$$(0.23306)^2 = \frac{1}{1+\left(\frac{j(1.4530)}{j\Omega_c}\right)^{2N}}$$

$$1+\left(\frac{1.4530}{\Omega_c}\right)^{2N} = \frac{1}{(0.23306)^2}$$

$$\frac{(1.4530)^{2N}}{(\Omega_c)^{2N}} = \frac{1}{(0.23306)^2} - 1$$

$$\left(\frac{1.4530}{\Omega_c}\right)^{2N} = 17.410 \quad \text{— } (A)$$

$$(0.17782)^2 = \frac{1}{1+\left(\frac{j(2)}{j\Omega_c}\right)^{2N}}$$

$$1+\left(\frac{2}{\Omega_c}\right)^{2N} = \frac{1}{(0.17782)^2}$$

$$\left(\frac{2}{\Omega_c}\right)^{2N} = \frac{1}{(0.17782)^2} - 1$$

$$\left(\frac{2}{\Omega_c}\right)^{2N} = 30.6256 \quad \text{— } (B)$$

Divide Equation (A) by Equation (B)

$$\frac{(1.4530)^{2N} / (\Omega_c)^{2N}}{(2)^{2N} / (\Omega_c)^{2N}} = \frac{17.410}{30.6256}$$

$$\left(\frac{1.4530}{2}\right)^{2N} = 0.5684$$

$$(0.7265)^{2N} = 0.5684$$

Taking $\log_{10}$ on b/s

$$\log_{10}(0.7265)^{2N} = \log_{10}(0.5684)$$

$$2N \log_{10}(0.7265) = \log_{10}(0.5684)$$

$$2N(-0.1387) = (-0.2453)$$

$$2N = 1.7688$$

$$N = \frac{1.7688}{2}$$

$$N = 0.8844$$

$$N \cong 1$$

When we put $N = 0.8844$ in eq Ⓐ and eq Ⓑ, we get the same answer of Ω_c

In Equation A:

$$\left(\frac{1.4530}{\Omega_c}\right)^{2(0.8844)} = 17.410$$

$$\left(\frac{1.4530}{\Omega_c}\right)^{1.7688} = 17.410$$

$$\left(\frac{1.4530}{\Omega_c}\right)^{1.7688/1.7688} = (17.410)^{1/1.7688}$$

$$\frac{1.4530}{\Omega_c} = 5.0291$$

$$\Omega_c = 0.2889$$

In Equation B:

$$\left(\frac{2}{\Omega_c}\right)^{2(0.8844)} = 30.6256$$

$$\left(\frac{2}{\Omega_c}\right)^{1.7688} = 30.6256$$

$$\left(\frac{2}{\Omega_c}\right)^{1.7688/1.7688} = (30.6256)^{1/1.7688}$$

$$\left(\frac{2}{\Omega_c}\right) = 6.9209$$

$\Omega_c = 0.2889$

When we put $N = 1$ in eq (A) and eq (B), we get the different answer of Ω_c.

In Equation A:

$$\left(\frac{1.4530}{\Omega_c}\right)^{2\times1} = 17.410$$

$$\left(\frac{1.4530}{\Omega_c}\right)^{2} = 17.410$$

$$\left(\frac{1.4530}{\Omega_c}\right)^{2/2} = (17.410)^{1/2}$$

$$\frac{1.4530}{\Omega_c} = 4.1725$$

$$\Omega_c = 0.3482$$

In Equation B:

$$\left(\frac{2}{\Omega_c}\right)^{2\times1} = 30.6256$$

$$\left(\frac{2}{\Omega_c}\right)^{2} = 30.6256$$

$$\left(\frac{2}{\Omega_c}\right)^{2/2} = (30.6256)^{1/2}$$

$$\frac{2}{\Omega_c} = 5.5340$$

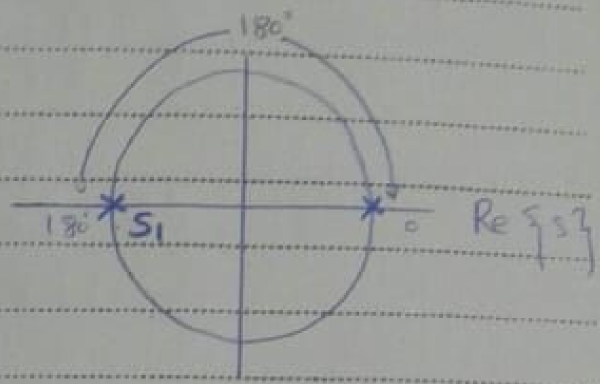
$$\Omega_c = 0.3613$$

Now, we choose $\Omega_c = 0.3482$, $N = 1$

Angular Distance b/w poles $= \dfrac{\pi}{N}$

$$= \frac{\pi}{1} = \frac{180^\circ}{1} = 180^\circ$$

N = 1 = odd; then first pole exist on x-axis



Number of poles = 2N
= 2×1 = 2

for casual and stable system choose those pole which lies on the left hand side of plane.

$$S_1 = 0.3482\, e^{j(180)}$$
$$= 0.3482 \{\cos(180^\circ) + j\sin(180^\circ)\}$$
$$= 0.3482 \{(-1) + j(0)\}$$
$$S_1 = -0.3482$$

- The resulting transfer function has the following pole
$S_1 = -0.3482$

Resulting in:

$$H(s) = \frac{(\Omega_c)^N}{(S - S_1)}$$
$$= \frac{(0.3482)^1}{(s - (-0.3482))}$$
$$H(s) = \frac{0.3482}{(s + 0.3482)} \longrightarrow (2)$$

Applying the Bilinear Transformation:

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \quad \therefore T_d = 1$$

$$s = 2\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \quad \text{put in eq (2)}$$

$$H(z) = \frac{0.3482}{\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.3482\right)}$$

$$1 = \frac{0.3482}{\left(\frac{2-2z^{-1}}{1+z^{-1}} + 0.3482\right)}$$

$$H(z) = \frac{0.3482}{\frac{2-2z^{-1}+0.3482(1+z^{-1})}{1+z^{-1}}}$$

$$H(z) = \frac{0.3482}{\frac{2-2z^{-1}+0.3482+0.3482z^{-1}}{1+z^{-1}}}$$

$$H(z) = \frac{(0.3482)(1+z^{-1})}{-1.6518z^{-1}+2.3482}$$

$$H(z) = \frac{0.3482+0.3482z^{-1}}{2.3482-1.6518z^{-1}}$$

Question: 3

Design IIR filters using impulse invariance and bilinear Transformation based on Butterworth Approximation.

$$0.78166 \le |H(e^{j\omega})| \le 1 \qquad 0 \le |\omega| \le 0.5\pi$$

$$|H(e^{j\omega})| \le 0.34409 \qquad 0.6\pi \le |\omega| \le \pi$$

Solution:

IIR filter Design Using Impulse Invariance based on Butterworth Approximation:

$$0.78166 \le |H(e^{j\omega})| \le 1 \qquad 0 \le |\omega| \le 0.5\pi$$

$$|H(e^{j\omega})| \le 0.34409 \qquad 0.6\pi \le |\omega| \le \pi$$

Discrete Time Specification $\xrightarrow{\text{convert into}}$ Continuous Time Specification

Role of T_d is cancel out by the impulse invariance method so we assume:

$T_d = 1$ $\qquad \omega = \Omega T_d \qquad \omega = \Omega$

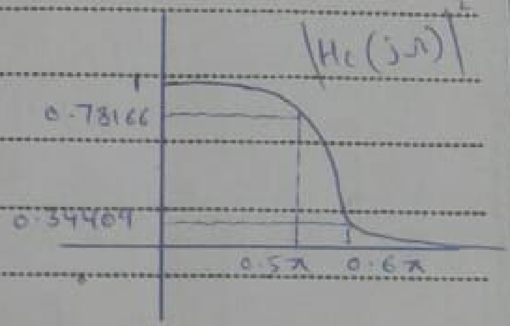
$$0.78166 \le |H(j\Omega)| \le 1 \qquad 0 \le |\Omega| \le 0.5\pi$$

$$|H(j\Omega)| \le 0.34409 \qquad 0.6\pi \le |\Omega| \le \pi$$

By using Butterworth Approximation:

$$|H_c(j\Omega)|^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

$$(0.78166)^2 = \frac{1}{1+\left(\frac{\cancel{j}\Omega}{\cancel{j}\Omega_c}\right)^{2N}}$$



$$1+\left(\frac{\Omega}{\Omega_c}\right)^{2N} = \frac{1}{(0.78166)^2}$$

$$\left(\frac{0.5\pi}{\Omega_c}\right)^{2N} = \frac{1}{(0.78166)^2} - 1$$

$$\left(\frac{0.5\pi}{\Omega_c}\right)^{2N} = 0.6366 \quad \text{— (A)}$$

$$(0.34409)^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

$$1+\left(\frac{\Omega}{\Omega_c}\right)^{2N} = \frac{1}{(0.34409)^2}$$

$$\left(\frac{0.6\pi}{\Omega_c}\right)^{2N} = \frac{1}{(0.34409)^2} - 1$$

$$\left(\frac{0.6\pi}{\Omega_c}\right)^{2N} = 7.4460 \quad \text{--- (B)}$$

Divide Equation (A) by Equation (B)

$$\frac{(0.5\pi)^{2N} / (\Omega_c)^{2N}}{(0.6\pi)^{2N} / (\Omega_c)^{2N}} = \frac{0.6366}{7.4460}$$

$$\left(\frac{0.5\pi}{0.6\pi}\right)^{2N} = 0.0854$$

$$(0.8333)^{2N} = 0.0854$$

Taking $\log_{10}$ on b/s

$$\log_{10}(0.8333)^{2N} = \log_{10}(0.0854)$$

$$2N \log_{10}(0.8333) = \log_{10}(0.0854)$$

$$2N(-0.0791) = -1.0685$$

$$2N = 13.508$$

$$N = 6.754$$

$$N \cong 7$$

When we put $N = 6.754$ in eq (A) and eq (B), we get the same answer of Ω_c

In Equation A:

$$\left(\frac{0.5\pi}{\Omega_c}\right)^{2(6.754)} = 0.6366$$

$$\left(\frac{0.5\pi}{\Omega_c}\right)^{13.508} = 0.6366$$

$$\frac{0.5\pi}{\Omega_c} = 0.9671$$

$$\Omega_c = 1.624$$

In Equation B:

$$\left(\frac{0.6\pi}{\Omega_c}\right)^{2(6.754)} = 7.4460$$

$$\left(\frac{0.6\pi}{\Omega_c}\right)^{13.508} = 7.4460$$

$$\frac{0.6\pi}{\Omega_c} = 1.1602$$

$$\Omega_c = 1.624$$

When we put $N=7$ in eq (A) and eq (B), we get the different answer of Ω_c.

In Equation A:

$$\left(\frac{0.5\pi}{\Omega_c}\right)^{2\times 7} = 0.6366$$

$$\left(\frac{0.5\pi}{\Omega_c}\right)^{14} = 0.6366$$

$$\frac{0.5\pi}{\Omega_c} = 0.9682$$

$$\Omega_c = 1.6223$$

In Equation B:

$$\left(\frac{0.6\pi}{\Omega_c}\right)^{2\times 7} = 7.4460$$

$$\left(\frac{0.6\pi}{\Omega_c}\right)^{14} = 7.4460$$

$$\Omega_c = 1.6331$$

Now, we choose $\Omega_c = 1.6331$, $N=7$

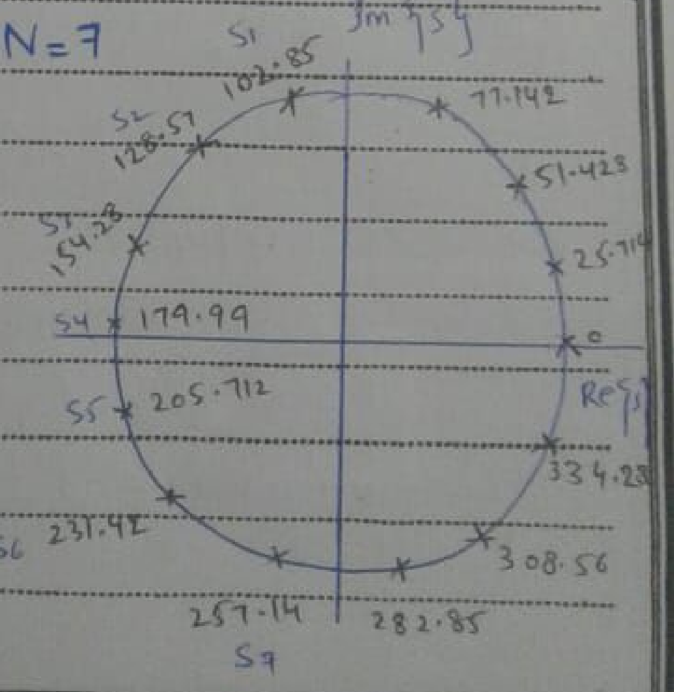
Angular Distance b/w poles $= \frac{\pi}{N}$

$$= \frac{\pi}{7} = \frac{180^\circ}{7} = 25.714$$

$N=7$ = odd = pole exist on x-axis

Number of poles $= 2N$

$= 2\times 7 = 14$



for casual and stable system choose those poles which lies on the left side of s-plane.

∴ The resulting transfer function has the following poles;

$S_1 = 1.6331\, e^{j(102.85)}$

$= 1.6331 \{\cos(102.85) + j\sin(102.85)\}$

$S_1 = -0.3625 + j1.5906$

$S_2 = 1.6331\, e^{j(128.57)}$

$= 1.6331 \{\cos(128.57) + j\sin(128.57)\}$

$S_2 = -1.0181 + j1.2768$

$S_3 = 1.6331\, e^{j(154.28)}$

$= 1.6331 \{\cos(154.28) + j\sin(154.28)\}$

$S_3 = -1.471 + j0.708$

$S_4 = 1.6331\, e^{j(179.99)}$

$= 1.6331 \{\cos(179.99) + j\sin(179.99)\}$

$S_4 = -1.6330 + 2.85\times10^{-4}\, j$

$S_5 = 1.6331\, e^{j(205.712)}$

$= 1.6331 \{\cos(205.712) + j\sin(205.712)\}$

$S_5 = -1.4714 - j0.708$

$S_6 = 1.6331\, e^{j(231.42)}$

$= 1.6331 \{\cos(231.42) + j\sin(231.42)\}$

$S_6 = -1.0184 - j1.2734$

$S_7 = 1.6331 e^{j(257.14)}$

$= 1.6331 \{\cos(257.14) + j\sin(257.14)\}$

$S_7 = -0.3634 - j1.5921$

Resulting in:

$$H(S) = \frac{(\Omega_c)^N}{(S-S_1)(S-S_2)(S-S_3)(S-S_4)(S-S_5)(S-S_6)(S-S_7)}$$

$$H(S) = \frac{(1.6331)^7}{(S-(-0.3625+j1.5906))(S-(-1.0181+j1.2768))(S-(-1.471+j0.708))(S-(-1.6330+2.85\times10^{-4}j))(S-(-1.4714-j0.708))(S-(-1.0184-j1.2734))(S-(-0.3634-j1.5921))}$$

$$H(S) = \frac{30.980}{(S+0.3625-j1.5906)(S+1.0181-j1.2768)(S+1.471-j0.708)(S+1.6330+2.85\times10^{-4}j)(S+1.4714+j0.708)(S+1.0184+j1.2734)(S+0.3634+j1.5921)}$$

Mapping in z-domain:

Poles $S = S_k$ in S-domain transform into pole at $z = e^{S_k T_d}$ in z-domain.

$$H(z) = \frac{A}{1-e^{-0.3625+j1.5906}z^{-1}} + \frac{B}{1-e^{-1.0181+j1.2768}z^{-1}} + \frac{C}{1-e^{-1.471+j0.708}z^{-1}} + \frac{D}{1-e^{-1.6330+j2.85\times10^{-4}}z^{-1}} + \frac{E}{1-e^{-1.4714-j0.708}z^{-1}} + \frac{F}{1-e^{-1.0184-j1.2734}z^{-1}} + \frac{G}{1-e^{-0.3634-j1.5921}z^{-1}}$$

IIR Filter Design using Bilinear Transformation based on Butterworth Approximation:

Given:

$$0.78166 \le |H(e^{j\omega})| \le 1 \qquad 0 \le |\omega| \le 0.5\pi$$

$$|H(e^{j\omega})| \le 0.34409 \qquad 0.6\pi \le |\omega| \le \pi$$

Solution:

Discrete Time Specification $\xrightarrow{\text{convert}}$ Continuous Time specification

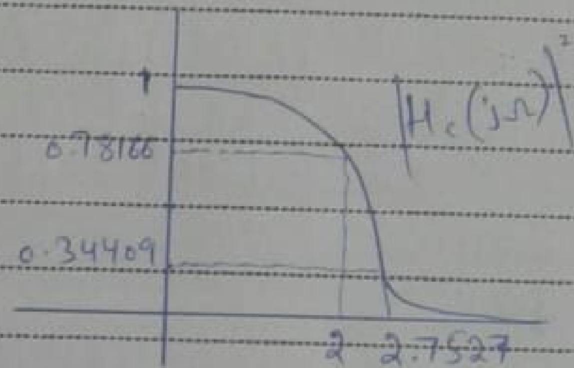
$$\Omega = \frac{2}{T_d} \tan\left(\frac{\omega}{2}\right)$$

when $\omega = 0.5\pi$

$$\Omega = \frac{2}{T_d} \tan\left(\frac{\omega}{2}\right)$$

$$= \frac{2}{1} \tan\left(\frac{0.5\pi}{2}\right)$$

$$= 2$$



when $\omega = 0.6\pi$

$$\Omega = \frac{2}{T_d} \tan\left(\frac{\omega}{2}\right)$$

$$= \frac{2}{1} \tan\left(\frac{0.6\pi}{2}\right)$$

$$= 2.7527$$

$$0.78166 \le |H(j\Omega)| \le 1 \qquad 0 \le |\Omega| \le 2$$

$$|H(j\Omega)| \le 0.34409 \qquad 2.7527 \le |\Omega| \le \infty$$

By using Butterworth Approximation:

$$|H_c(j\Omega)|^2 = \frac{1}{1 + \left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

$$(0.78166)^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

$$1+\left(\frac{\Omega}{\Omega_c}\right)^{2N} = \frac{1}{(0.78166)^2}$$

$$\left(\frac{2}{\Omega_c}\right)^{2N} = \frac{1}{(0.78166)^2} - 1$$

$$\left(\frac{2}{\Omega_c}\right)^{2N} = 0.6366 \longrightarrow \text{(A)}$$

$$(0.34409)^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

$$1+\left(\frac{\Omega}{\Omega_c}\right)^{2N} = \frac{1}{(0.34409)^2}$$

$$\left(\frac{2.7527}{\Omega_c}\right)^{2N} = \frac{1}{(0.34409)^2} - 1$$

$$\left(\frac{2.7527}{\Omega_c}\right)^{2N} = 7.4460 \longrightarrow \text{(B)}$$

Divide Equation (A) and Equation (B)

$$\frac{(2)^{2N} / (\Omega_c)^{2N}}{(2.7527)^{2N} / (\Omega_c)^{2N}} = \frac{0.6366}{7.4460}$$

$$\left(\frac{2}{2.7527}\right)^{2N} = 0.08549$$

Taking $\log_{10}$ on b/s

$$\log_{10}(0.7265)^{2N} = \log_{10}(0.08549)$$

$$2N \log_{10}(0.7265) = \log_{10}(0.08549)$$

$$2N(-0.1387) = -1.0680$$

$$2N = 7.700$$
$$N = 3.850$$
$$N \cong 4$$

When we put $N = 3.850$ in eq A and eq B, we get the same answer of Ω_c.

In Equation A:

$$\left(\frac{2}{\Omega_c}\right)^{2\times 3.850} = 0.6366$$

$$\left(\frac{2}{\Omega_c}\right)^{7.700} = 0.6366$$

$$\frac{2}{\Omega_c} = (0.6366)^{1/7.700}$$

$$\Omega_c = 2.1208$$

In Equation B:

$$\left(\frac{2.7527}{\Omega_c}\right)^{2\times 3.850} = 7.4460$$

$$\left(\frac{2.7527}{\Omega_c}\right)^{7.700} = 7.4460$$

$$\frac{2.7527}{\Omega_c} = (7.4460)^{1/7.700}$$

$$\Omega_c = 2.1209$$

When we put $N = 4$ in eq A and eq B, we get the different values of Ω_c

In Equation A:

$$\left(\frac{2}{\Omega_c}\right)^{2\times 4} = 0.6366$$

$$\left(\frac{2}{\Omega_c}\right)^{8} = 0.6366$$

$$\frac{2}{\Omega_c} = 0.6366^{1/8}$$

$\Omega_c = 2.1161$

In Equation B:

$$\left(\frac{2.7527}{\Omega_c}\right)^{2\times4} = 7.4460$$

$$\left(\frac{2.7527}{\Omega_c}\right)^{8} = 7.4460$$

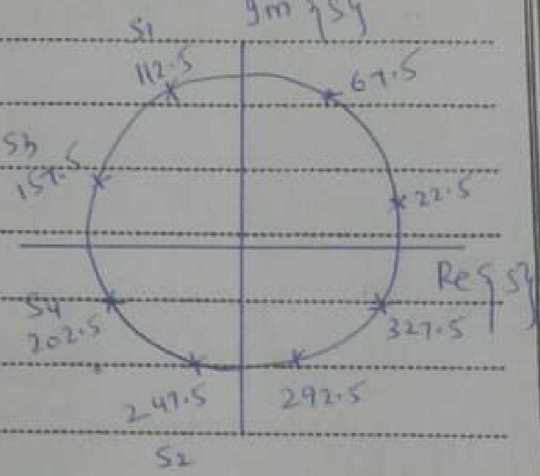
$$\frac{2.7527}{\Omega_c} = 7.4460^{1/8}$$

$$\Omega_c = 2.1417$$

Now, we choose $\Omega_c = 2.1161$, $N = 4$

Angular Distance b/w poles $= \frac{\pi}{N}$

$$= \frac{\pi}{4} = \frac{180}{4} = 45^\circ$$



$N = 4$ = even ; then No pole on x-axis

Number of poles = $2N$

$= 2\times4 = 8$

for casual and stable system choose those poles which lies on the left side of plane

The resulting transfer function has the following pole

$S_1 = 2.1161\, e^{j(112.5)}$

$= 2.1161 \{\cos(112.5) + j\sin(112.5)\}$

$S_1 = -0.8083 + j1.9548$

$S_3 = 2.1161\, e^{j(157.5)}$

$= 2.1161 \{\cos(157.5) + j\sin(157.5)\}$

$= -1.9548 + j0.8083$

$S_1 = -0.8083 + j1.9548$

$S_2 = -0.8083 - j1.9548$

$S_3 = -1.9548 + j0.8083$

$S_4 = -1.9548 - j0.8083$

Resulting in:

$$H(s) = \frac{(\Omega_c)^N}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)}$$

$$H(s) = \frac{(2.1161)^4}{(s-(-0.8083+j1.9548))(s-(-0.8083-j1.9548))(s-(-1.9548+j0.8083))(s-(-1.9548-j0.8083))}$$

$$H(s) = \frac{20.0514}{(s+0.8083-j1.9548)(s+0.8083+j1.9548)(s+1.9548-j0.8083)(s+1.9548+j0.8083)}$$

$$H(s) = \frac{20.0514}{(s^2+0.8083s+\cancel{js1.9548}+0.8083s+0.6533+\cancel{j1.5800}-\cancel{js1.9548}-\cancel{j1.5800}+3.8212)(s^2+1.9548s+\cancel{sj0.8083}+1.9548s+3.8212+\cancel{j1.5800}-\cancel{js0.8083}-\cancel{j1.5800}+0.6533)}$$

$$H(s) = \frac{20.0514}{(s^2+1.6166s+4.4745)(s^2+3.9096s+4.4745)} \longrightarrow (2)$$

Applying the Bilinear Transformation:

$$S = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \qquad \therefore T_d = 1$$

$$S = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \quad \text{put in eq } 2$$

$$= S^2 + 1.6166\,S + 4.4745$$

$$= 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1.6166\left(2\,\frac{(1-z^{-1})}{1+z^{-1}}\right) + 4.4745$$

$$= \frac{4(1-2z^{-1}+z^{-2}) + 3.2332(1-z^{-2}) + 4.4745(1-2z^{-1}+z^{-2})}{(1+z^{-1})^2}$$

$$= \frac{4-8z^{-1}+4z^{-2}+3.2332-3.2332z^{-2}+4.4745-8.949z^{-1}+4.4745z^{-2}}{(1+z^{-1})^2}$$

$$= \frac{5.2413\,z^{-2}+0.949\,z^{-1}+11.707}{(1+z^{-1})^2} \longrightarrow C$$

$$S^2 + 0.39096\,S + 4.4745$$

$$= 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.39096\left(2\,\frac{(1-z^{-1})}{1+z^{-1}}\right) + 4.4745$$

$$= \frac{4(1-2z^{-1}+z^{-2}) + 7.8192(1-z^{-2}) + 4.4745(1+2z^{-1}+z^{-2})}{(1+z^{-1})^2}$$

$$= \frac{4-8z^{-1}+4z^{-2}+7.8192-7.8192z^{-2}+4.4745+8.949z^{-1}+4.4745z^{-2}}{(1+z^{-2})^2}$$

$$= \frac{0.6553\,z^{-2}+0.949\,z^{-1}+16.2937}{(1+z^{-1})^2} \longrightarrow D$$

Put the value of eq C and eq D in eq 2

$$H(z) = \frac{20.0514}{\left(\frac{5.2413 z^{-2} + 0.949 z^{-1} + 11.707}{(1+z^{-1})^2}\right)\left(\frac{0.6553 z^{-2} + 0.949 z^{-1} + 16.293}{(1+z^{-1})^2}\right)}$$

$$H(z) = \frac{20.0514 \quad (1+z^{-1})^4}{(5.2413 z^{-2} + 0.949 z^{-1} + 11.707)(0.6553 z^{-2} + 0.949 z^{-1} + 16.293)}$$