Invariant Subspace Problem

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Introduction: What is invariant subspace problem?

Let T be a linear transformation on an $\mathbb{F}$-linear space V, i.e. a linear mapping from V to V.

If W is a linear subspace of V, $T(W) \subset W$, then we say W is a T-invariant subspace of V.

Invariant space in finite dimension:

Linear algebra

Eigenvalues

Let T be a linear transformation on an $\mathbb{C}$-vector space V. Pick a basis, $A \in \mathbb{C}^{n \times n}$ is the matrix of T on this basis. The eigenvalue of T on $\mathbb{C}$ is a number λ in $\mathbb{C}$ such that:

$$Av = \lambda v \text{ for some } v \in \mathbb{C}^n$$

In finite dimension, this is equivalent to solve the equation:

$$det(A - \lambda I) = 0$$

The solution to the degree n polynomial in $\mathbb{C}[x]$: $det(A - xI)$, is the eigenvalue of T.

Notes

It is important to talk about the eigenvalue in $\mathbb{C}$. We can have some linear transformations like rotation that do not have any eigenvalue in $\mathbb{R}$. See Example:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$$

Therefore, let's restrict our base field of vector space as $\mathbb{C}$.

Invariant space in finite dimension

Eigenspaces

The λ-eigenspaces is the invariant subspace of T with respect to an eigenvalue λ. Non-trivial invariant subspace exists, since

$$(A - \lambda I)x = 0$$

must have solution.

$$\dim Ker(A - \lambda I) = n - rank(A - \lambda I) > 0$$

where $Ker(A - \lambda I) = \{x | x \in V, (A - \lambda I)x = 0\}$, hence non-trivial.

Then,

$$\forall x \in Ker(A - \lambda I), Tx = \lambda x$$

invariant subspace problem on finite dimensional vector space

$Ker(A - \lambda I)$ is an non-trivial T-invariant subspace.

Banach space

Review: Normed space

Let E be a non-empty set. If $\forall x \in E$, we assign a real number $p(x) \geq 0$ correspond to this x, and it satisfies:

1): $p(x+y) \leq p(x) + p(y)$
2): $p(\alpha x) = |\alpha| p(x)$
3): $p(x) = 0$ iff $x = 0$

Examples of normed space

$\mathbb{R}^n$, with norm: $|x_1| + |x_2| + ... + |x_n|$
$C[a,b]$, with norm: $\sup_{[a,b]} |f| \quad f \in C[a,b]$
$L^1[a,b]$, with norm: $\int_{[a,b]} |f|$

Definition of Banach Space

Banach space is a complete normed linear space.
(Completeness means every Cauchy sequence converges.)

What is a Bounded linear Operator

In functional analysis, we usually call a mapping an **operator**.

Let $\mathbb{F}$ be real or complex number field, X, Y be two $\mathbb{F}$-vector spaces, $T : X \to Y$.

If $\forall x, y \in X, \alpha, \beta \in \mathbb{F}$, we have:

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

Then we call T a **linear operator**. (Just linear transformation, but we don't have matrix now)

If a linear operator T map every bounded set in X to bounded set in Y, then we call T a **bounded linear operator**.

The set of all such T is denoted $B(X, Y)$.

We can also define the norm on operator:

$$||T|| = \sup_{||x||=1} ||Tx||$$

With norm topology, we can define the **continuity of operator**, that is: $\forall n \in \mathbb{N}, x_n \in X$. T is continuous at x If for any $x_n \to x$, $T(x_n) \to Tx$.

Bounded Linear operator

Theorem: Continuity of Linear operator

A linear operator T is bounded iff T is a continuous operator.

Compact Set

Now we need to introduce the formal definition of compactness:

A set E is called **compact** if we can always take finite open subcover from any open cover of E.

In finite dimensional $\mathbb{R}^n$, compact set is equivalent to bounded and closed set. However, in infinite dimension, a closed unit ball $B := \{x \mid ||x|| \leq 1\}$ will be a counterexample.

In metric space, sequence compact is equivalent to compact.

Compact operator

Let $T \in B(X, Y)$. T is called compact operator if $\overline{T(\Omega)}$ (i.e. the closure of $T(\Omega)$) is compact in Y, for all bounded $\Omega \in X$.

Banach algebra

Algebra

Roughly speaking, an algebra is a vector space equipped with multiplication.

(Inner product is not a multiplication.)

About all the bounded linear operators

$B(X, Y)$, i.e. the space of all the bounded linear operators from X to Y, is a complete, normed complex algebra.

Let $A, B \in B(X, Y)$,

$$||AB|| \leq ||A|| \; ||B||$$

Here, the multiplication is defined as the composition of two operators, and the norm is defined as the norm on operator.

$$||T|| = \sup_{||x||=1} ||Tx||$$

Definition of spectrum and eigenvalue
(of a bounded linear operator)

Spectrum of a bounded linear operator

Let $T \in B(X, X)$, the spectrum $\sigma(T)$ of T is the set of all complex number λ such that $\lambda I - T$ non-invertible.

$$\sigma(T) := \{\lambda \in \mathbb{C} \text{ s.t. } \lambda I - T \text{ non-invertible}\}$$

We can prove that $\sigma(T)$ is compact and non-empty.

Eigenvalue of an operator

Note that $\lambda \in \sigma(T)$ is not enough for λ to be an eigenvalue.
Some operator are non-invertible but injective.
λ is an eigenvalue of T if $T - \lambda I$ that is not injective, namely, $Ker(T - \lambda I) \neq 0$.

Some Results of Spectrum Theory

Spectrum radius formula

Let $T \in B(X, X)$, then

$$r(T) = \sup_{\lambda \in \sigma(T)} ||\lambda|| = \lim_{n \to \infty} ||T^n||^{\frac{1}{n}}$$

Eigenvalue of compact operator

Every $0 \neq \lambda \in \sigma(T)$ is an eigenvalue of T if T is a compact operator. also, $\dim Ker(T - \lambda I) < \infty$

Lomonosov invariant subspace theorem

Lomonosov invariant subspace theorem

Suppose that X is an infinite-dimensional complex Banach space and that $0 \neq T \in B(X, X)$ is compact operator, then there is a closed subspace M of X such that $M \neq \{0\}, M \neq X$, and

$$S(M) \subset M$$

for every $S \in B(X, X)$ such that $ST = TS$.

As a corollary, since $TT = TT$, of course T has a non-trivial invariant subspace.

Sketch of Proof

Our goal is to construct an eigenvalue, then we will have a non-trivial invariant subspace.

The proof, although being simplified for several times, are technical and long.

We break it up into 3 steps.

Step 1: We use compactness to prove that $r(T) > 0$

Step 2: We use the fact that "Every non-zero $\lambda \in \sigma(T)$ is an eigenvalue of T if T is a compact operator" to get an eigenvalue.

Step 3: We use the fact that $\dim Ker(T - \lambda I) < \infty$ to get a non-trivial eigenspace.

Proof:

Assume the conclusion is false, construct a non-zero invariant subspace, then it has to be trivial.

Let

$$\Gamma = \{S \in B(X, X) | S \circ T = T \circ S\}$$

Γ is an unital Banach subalgebra of $B(X, X)$.
For each $y \in X$,

$$\Gamma(y) = \{S(y) | S \in \Gamma\}$$

$\Gamma(y)$ is a subspace of X containing y. Hence, as long as $y \neq 0$, $\Gamma(y) \neq \{0\}$.
Moreover, $S(\Gamma(y)) \subset \Gamma(y)$ since $S \circ \Gamma \subset \Gamma$.
Then, $\Gamma(y)$ is an S-invariant subspace for all $y \in X, S \in \Gamma$.
Now, assume the conclusion is not true, it follows that $\Gamma(y)$ has to be dense in X for all $y \in X$, otherwise we will have a non-trivial invariant subspace.

Proof:

Use the fact that this invariant subspace is trivial, and that $ST = TS$, to make up a situation to use spectrum radius formula.

We can pick $x_0 \in X$ such that $Tx_0 \neq 0$, then $x_0 \neq 0$, and the boundedness of T implies that there exists an $\varepsilon \geq 0$ s.t.

$$\forall x \in B = \overline{B(x_0, \varepsilon)}, ||Tx|| > \frac{1}{2}||Tx_0||, ||x|| > \frac{1}{2}||x_0|$$

Hence, $0 \notin K = \overline{T(B)}$.

In particular, we can have $\forall y \in K$, $\exists$ a neighbourhood W_y and $S_y \in \Gamma$,

$$S_y(W_y) \subset B$$

Such W_y covers compact set K, hence finite collection suffices.

Proof:

Use the fact that this invariant subspace is trivial, and that $ST = TS$, to make up a situation to use spectrum radius formula.

Hence, there exists $S_1, S_2, ..., S_n$ such that for every $y \in K$, $|S_i(y) - x_0| < \frac{1}{2}|x_0|$ holds.

Then, we have $W_1, W_2, ..., W_n$, whose union covers K, such that $S_i(W_i) \subset B$ for some $S_i \in \Gamma, 1 \leq i \leq n$

Put $\mu = max\{||S_1||, ||S_2||, ..., ||S_n||\}$, starting with x_0, we have Tx_0 lies in K, hence in some W_{i_1}, and $S_{i_1}Tx_0 \in B$.

Therefore $TS_{i_1}Tx_0$ lies in K, hence in some W_{i_2}, and $S_{i_2}TS_{i_1}Tx_0 \in B$. Then,

$$x_N = S_{i_N}T...S_{i_1}Tx_0 = S_{i_N}S_{i_{N-1}}...S_{i_1}T^N x_0 \in B$$

Proof:

We use spectrum radius formula to prove that $r(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}}$ is not zero.

Then, since $x_N \in B$,

$$\frac{1}{2}||x_0|| \leq ||x_N|| \leq \mu^N \; ||T^N|| \; ||x_0||$$

We then have

$$\frac{1}{2} \leq \mu^N \; ||T||^N$$

Hence,

$$r(T) = \lim_{N\to\infty} ||T||^{\frac{1}{N}} \geq \frac{1}{\mu} > 0$$

That is, $r(T) \neq 0$

Proof:

We use the fact that "Every $0 \neq \lambda \in \sigma(T)$ is an eigenvalue of T if T is a compact operator" to get an eigenvalue., and use the fact that $\dim Ker(T - \lambda I) < \infty$ to get a non-trivial eigenspace.

Since

$$r(T) = \sup_{\lambda \in \sigma(T)} ||\lambda|| \neq 0$$

We must have some $\lambda_0 \in \sigma(T), \lambda_0 \neq 0$

Then, the result follows trivially from the fact that every non-zero $\lambda \in \sigma(T)$ is an eigenvalue of T since T is a compact operator.

Then, λ_0 is an eigenvalue, and the corresponding eigenspace is:

$$M_{\lambda_0} = \{x \in X | Tx = \lambda_0 x\}$$

This is finite-dimensional, hence is not the whole space X. If $S \in \Gamma, x \in M_{\lambda_0}$, then

$$T(Sx) = S(Tx) = S(\lambda_0 x) = \lambda_0(Sx)$$

Thus we can find a non-trivial S-invariant subspace.

$\square$

Question to be explored

Formally, the invariant subspace problem for a complex Banach space H of dimension > 1 is the question whether every bounded linear operator $T : H \to H$ has a non-trivial closed T-invariant subspace.

For finite and non-separable Hilbert space, the answer is yes; for some Banach spaces, there are some counterexamples. However, for separable Hilbert space, the problem is still open.

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