Universal Cover of Non-Positively Curved Surface

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Covering map is a concept of particular interest in algebraic topology. Among them, the universal cover is of unique distinction. It is a useful tool in terms of proving global homeomorphism.

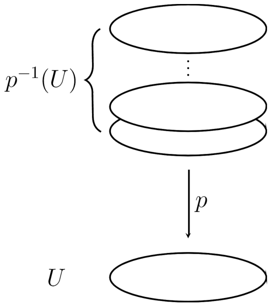
This talk aims to give simple examples of covering spaces. The first one is on some good topological spaces called Riemannian manifolds. Manifold offers handful tools that make things easier.

For simplicity, we may assume the case of dimension 2, a surface. These basic ideas should work in higher dimension as well.

Smooth Covering Map

Smooth Covering Map

Let M, N be two smooth surfaces, a surjective map $p : M \to N$ is said to be a *smooth covering map* if for any $q \in N$, there is a neighborhood V of $q \in N$ and disjoint open subsets U_α of M so that $p^{-1}(V) = \bigsqcup_\alpha U_\alpha$, and for each α, $p : U_\alpha \to V$ is a diffeomorphism. Among all covering spaces of N, the simply connected one is called the *Universal cover* of N. It is unique up to homeomorphism.



Lifting Property

Lift

One may lift one path to a path in covering space.
Given a map $\phi : X \to Y$ and $f : Z \to Y$, a lift is a map $\overline{\phi} : X \to Z$ such that $\phi = f \circ \overline{\phi}$.

$$
\begin{tikzcd}
 & Z \arrow[d, "f"] \\
X \arrow[ur, "\overline{\phi}"] \arrow[r, "\phi"] & Y
\end{tikzcd}
$$

Unique Lifting Property

Let $f : M \to N$ be a covering map. If $g, h : X \to M$ are two continuous maps such that $f \circ g = f \circ h$, then the set of points on which g and h agree is both open and closed.

Therefore, lift emanating from a fixed point is unique.

Surface

We mostly study sets and the structures on them. The structure of *surface* can be abstracted using the language of *smooth manifolds*. They satisfy several properties so that one can do calculus on it:

Surface

1. Locally Euclidean: Can be seen as the graph of a function on $\mathbb{R}^2$ locally;
2. Second Countable: Have countable topological base;
3. Hausdorff: Two points can be separated by open sets;
4. Smooth Structure: Can do change of variables smoothly.

We can talk about distance on the manifold via introducing the inner product.

Distance $d(x, y)$ on (M, g)

Distance $d(x, y) \coloneqq \inf\{\int_a^b \sqrt{\langle\dot{\gamma}, \dot{\gamma}\rangle} dt | \gamma$ is a smooth curve connecting x and $y\}$.

Geodesic

Geodesic generalizes the idea of straight lines in $\mathbb{R}^n$.

Geodesic as the curve without acceleration:

The smooth curve $\gamma : I \to M$ is called a geodesic, if for all $t \in I$, $\nabla_{\dot\gamma}\dot\gamma = 0$. Given initial position and velocity, one can determine a unique geodesic.

Geodesic as the extremal of length functional

The geodesics between p, q on M as the $\gamma : [a, b] \to M$, $\gamma(a) = p, \gamma(b) = q$, such that γ is the curve taking the extremal of this functional:

$$L(\gamma) = \int_a^b \sqrt{\langle \dot\gamma, \dot\gamma \rangle} dt.$$

When this is the length parameter, two definitions coincide.

The Euler-Lagrange equation says $\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^i} = 0$, where $\frac{\partial L}{\partial \dot q^i}$ is the generalized momentum, and $\frac{\partial L}{\partial q^i}$ is the generalized force. By "Newton's law", the geodesic is the trajectory of a free particle. Geodesic is the path a free particle follows given the initial velocity.

Completeness

Exponential Map

As mentioned earlier, we have a correspondence between initial velocity $v \in T_pM$(one can view it as the space of velocity at p) and geodesic. One can thus define the exponential map: $\exp : T_pM \to M$, which sends v to the position at time 1, namely, $\exp_p(v) = \gamma_{p,v}(1)$.

Completeness

Let M be a surface, $p \in M$. If $\exp_p$ can be defined on the whole T_pM, then we say M is *geodesically complete* at p. M is geodesically complete if and only if M is complete as a metric space.

"Local Isometry" + "Completeness" Imply Smooth Covering

Recall the definiton of smooth covering map:

Smooth Covering Map

Let M, N be two smooth surfaces, a surjective map $f : M \to N$ is said to be a *smooth covering map* if for any $q \in N$, there is a neighborhood V of $q \in N$ and disjoint open subsets U_α of M so that $f^{-1}(V) = \bigsqcup_\alpha U_\alpha$, and for each α, $f : U_\alpha \to V$ is a diffeomorphism.

Theorem

Let (M, g) and (N, h) be two surfaces, and $f : M \to N$ be a local isometry(meaning they have the same inner product at p and $f(p)$). Suppose (M, g) is complete, then f is a smooth covering map, and (N, h) is complete.

We prove this in five steps.

Proof

Step 1: Lift geodesics in $f(M)$ to geodesics in M (Use local isometry)

Step 2: $(f(M), h)$ is complete (Use the completeness of M)
Since $\gamma(t) = f \circ \tilde{\gamma}(t)$ is a geodesic.

Step 3: f is surjective (Use completeness of $f(M)$)
Completeness ensures that this is the maximal connected complete manifold.

Step 4: (N, h) is complete
By surjectivity, $N = f(M)$, so (N, h) is complete.

Proof

Step 5: f is a covering map (Verify the definition)
Fix any $q \in N$, we may assume $f^{-1}(q) = \{p_\alpha\}_{\alpha \in I}$. Choose δ small enough so that $V = B_\delta(q) \subset N$ is a geodesic ball around q. We let $U_\alpha = B_\delta(p_\alpha) \subset M$.

1). $f^{-1}(V) = \bigcup_\alpha U_\alpha$: By lifting the geodesic and its locally minimizing property.
2). $f : U_\alpha \to V$ is diffeomorphism.
This is because $f = \exp_q \circ df_{p_\alpha} \circ \exp_{p_\alpha}^{-1}$ locally.
3). For $\alpha \neq \beta$, $U_\alpha \bigcap U_\beta = \varnothing$.

$$\begin{tikzcd} T_{p_\alpha}M \arrow[r, "\text{local isometry}", "df_{p_\alpha}"'] \arrow[d, "\exp_{p_\alpha}"] & T_qN \arrow[d, "\exp_q"'] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

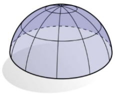
Assume there is $p \in U_\alpha \bigcap U_\beta$ and $\alpha \neq \beta$. Let $\tilde{\gamma}_\alpha$ and $\tilde{\gamma}_\beta$ be the minimal geodesic from p to p_α and p_β respectively. Then, we connect p_α and p_β with p respectively, and use the unique lifting property, and get $p_\alpha = p_\beta$.
Thus, we proved this lemma.

Theorem

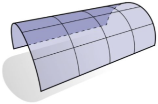
Let (M, g) and (N, h) be two surfaces, and $f : M \to N$ be a local isometry(meaning f perserves the length of curves). Suppose (M, g) is complete, then f is a smooth covering map, and (N, h) is complete.

Curvature

Gaussian curvature measures the extent that a manifold differs from flat space. One can use second fundamental form to describe it, but Gaussian curvature is intrinsically defined (dependence of parallel transport).



$K > 0$



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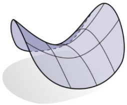
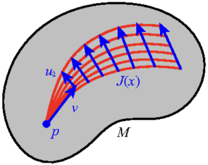


$K < 0$



Jacobi Field

Jacobi field measures how fast two infinitesimally close geodesics split. It satisfies the equation:

$$\frac{D_\gamma^2 J}{dt^2} + K(t)\|\gamma' \wedge J\|\gamma'(t) = 0.$$

When $K(t) \leq 0$, $\langle J(t), J(t) \rangle$ is a convex function of t, which implies $d\exp_p$ is non-degenerate everywhere.

Corollary: Cartan-Hadamard Theorem

Cartan-Hadamard Theorem

Let (M, g) be a complete surface. If the sectional curvature $K_M \leq 0$, then for every point $p \in M$, the exponential map

$$\exp_p : T_pM \to M$$

is a covering map.
Furthermore, if M is simply connected, then $\exp_p : T_pM \to M$ is a diffeomorphism.

It suffices to show this is a local isometry, and T_pM is complete for some $\overline{g}$.
Step 1: $\exp_p : T_pM \to M$ is a local diffeomorphism.
Step 2: Construct a metric $\overline{g}$ on T_pM to get a local isometry.
Step 3: $(T_pM, \overline{g})$ is complete.

Since $T_pM \cong \mathbb{R}^n$ is simply connected, so $\exp_p : T_pM \to M$ is a universal cover.
Furthermore, if M is simply connected, then $\exp_p : T_pM \to M$ is a global diffeomorphism.
Remark: There is no closed surface embedded in $\mathbb{R}^n$ such that curvature are non positive everywhere.

Exercise: Hadamard-Lévy's Theorem (Diffeomorphism from Local to Global)

Let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, df_p invertible for all $p \in \mathbb{R}^n$, and there exists $K \in \mathbb{R}$ such that for all $p \in \mathbb{R}^n$, $\|df_x^{-1}\| \leq K$.

Given those conditions, one can prove the injectivity and surjectivity of f, thus it becomes a global diffeomorphism.

1). Show that every line segment between any two points y, z has a lift.

2). Show that every loop $\gamma(t)$ in $\mathbb{R}^n$ has a loop as its lift.

3). Since the lift of a loop is unique, suppose by contradiction and use **(1)** and get injectivity.

Remark. Similar result can be generalized to Banach space.

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