Fundamental Theorem of Riemannian Geometry

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Abstract

We will explore a fundamental object in Riemannian geometry called the "Levi-Civita connection." We aim to explain the background necessary to understand it, as well as prove its existence and uniqueness.

What is a Riemannian Manifold?

A Riemannian manifold is a **smooth manifold** M equipped with a **Riemannian metric** g.

An n-dimensional smooth manifold M is a topological space which is locally modeled by an open set in $\mathbb{R}^n$. It is a structure locally parameterized by $\mathbb{R}^n$. We also require that the transition map between parameterizations is smooth; that is, if a manifold can be locally parameterized by two open sets in $\mathbb{R}^n$, we require that they can be transformed between one another smoothly. Examples of manifolds include $\mathbb{R}^n$, circles, the torus, the hyperbolic upper-half plane, and the Möbius strip. See *Figure 1*.

We equip our manifolds with Riemannian metrics. The intuition behind the Riemannian metric is that we want to calculate lengths of curves on a manifold. One can do this by generalizing the Euclidean inner product.

We write the **metric tensor** as $ds^2 = \sum_{i,j=1}^m g_{ij} dx_i dx_j$. Note that it is actually a quadratic form, namely a positive-definite symmetric matrix, with the (i,j)-th entry g_{ij}. Using the metric, the length of a curve is given by:

$$L(\gamma) = \int_\gamma ds = \int_a^b \sqrt{g_{ij}(\gamma(t))(\frac{d}{dt}\gamma(t), \frac{d}{dt}\gamma(t))}dt$$

An easy example of a metric is the Euclidean metric on $\mathbb{R}^n$, given by $ds^2 = dx_1^2 + \dots + dx_n^2$. If we consider the plane with polar coordinates, then another example of a Riemannian metric is $ds^2 = dr^2 + r^2 d\theta^2$.

What is a tangent space?

The **tangent space** of M at $p \in M$ is the vector space of all tangent directions at the point p. One can think of these tangent vectors as **derivations** by taking the directional derivative in the direction of the vector. Thus one can also think of the tangent space as the space of all derivations of smooth functions. We denote the tangent space by T_pM.

Notice we locally have a one-to-one correspondence between $\mathbb{R}^n$ and T_pM given by $\varphi : e_i \mapsto \frac{\partial}{\partial e_i}|_p$, where e_i is a standard basis of $\mathbb{R}^n$, given by the parameterization. Using this correspondence, one can show that $\{\frac{\partial}{\partial e_1}, \frac{\partial}{\partial e_2}, \dots, \frac{\partial}{\partial e_n}\}$ is a basis of T_pM, so every derivation can locally be written as a linear combination of this basis.

An example of the tangent space of the sphere can be seen in *Figure 2*.

A **smooth vector field** X on a manifold M is a smooth map that assigns to every point $p \in M$ a vector X_p in the tangent space T_pM. Since every tangent vector at a point p can be uniquely represented as a linear combination of the basis of the tangent space T_pM, we can always write the vector field locally as

$$X = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i},$$

where $v_i : U_\alpha \to \mathbb{R}$ are smooth real-valued functions for all i. See *Figure 3* for an example of a vector field.

What is an affine connection?

An affine connection is a generalization of a directional derivative from multivariable calculus. It allows us to take "derivatives" of vector fields with respect to other vector fields.

Recall the **directional derivative** of a function $\mathbf{f}, \mathbf{g} \in C^\infty, \mathbb{R}^n \to \mathbb{R}^m$ along $\mathbf{v}$ is given by

$$D_\mathbf{v}\mathbf{f}(\mathbf{x}) = \lim_{t\to 0} \frac{\mathbf{f}(\mathbf{x}+t\mathbf{v}) - \mathbf{f}(\mathbf{x})}{t}.$$

It has several properties:

(I.) $D_{\alpha\mathbf{v}+\beta\mathbf{w}}\mathbf{f} = \alpha D_\mathbf{v}\mathbf{f} + \beta D_\mathbf{w}\mathbf{f}$ (II.) $D_\mathbf{v}(\alpha\mathbf{f} + \beta\mathbf{g}) = \alpha D_\mathbf{v}\mathbf{f} + \beta D_\mathbf{v}\mathbf{g}$

(III.) $D_{h\mathbf{v}}\mathbf{f} = hD_\mathbf{v}\mathbf{f}$ (IV.) $D_\mathbf{v}(h\mathbf{f}) = hD_\mathbf{v}\mathbf{f} + (D_\mathbf{v}h)\mathbf{f}$

Similarly, an **affine connection** on a manifold M is a smooth map that assigns to every pair of smooth vector fields X and Y on M another smooth vector field $\nabla_X Y$ on M, satisfying the following properties:

(I.) $\nabla_{\alpha X_1 + \beta X_2} Y = \alpha\nabla_{X_1}Y + \beta\nabla_{X_2}Y$ (II.) $\nabla_X(\alpha Y_1 + \beta Y_2) = \alpha\nabla_X Y_1 + \beta\nabla_X Y_2$

(III.) $\nabla_{fX}(Y) = f\nabla_X Y$ (IV.) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$

where $f \in C^\infty(M)$ and X, Y smooth on M.

The Christoffel symbol Γ_{ij}^k is defined by the relationship

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^m \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

What is a geodesic?

Geodesics are curves on manifolds that generalize the idea of straight line in Euclidean space. One characteristic of straight lines is that they realize the shortest path between points. We can generalize it as g-geodesic. We define the g-geodesic between p, q on M as the $\gamma : [a,b] \to M$, $\gamma(a) = p, \gamma(b) = q$, such that γ take the minima of this energy functional:

$$E(\gamma) = \int_a^b g_{ij}(\gamma(t))(\frac{d}{dt}\gamma(t), \frac{d}{dt}\gamma(t))dt$$

This implies that the length of the curve γ on M connecting a and b has minimal length.

Another characteristic of straight in Euclidean space is that the tangents at all points are parallel, we can generalize it as ∇-geodesic. Intuitively, this is the trajectory of an object given zero acceleration. The smooth curve $\gamma : I \to M$ is called a ∇-geodesic, if for all $t \in I$,

$$\nabla_{\frac{d\gamma(t)}{dt}} \frac{d\gamma(t)}{dt} = 0$$

Surprisingly, two definitions of geodesics coincide under the Levi-Civita connection. They both reduce to the same form, namely a curve is either a g-geodesic or a ∇-geodesic if

$$\frac{d^2}{dt^2}\gamma^k + \Gamma_{ij}^k \frac{d}{dt}\gamma^i \frac{d}{dt}\gamma^j = 0$$

Notice that this is an ordinary differential equation. The solution exists and is unique locally, hence two definitions of geodesic coincide under the Levi-Civita connection.

Fundamental Theorem of Riemannian Geometry

This theorem says that given the Riemannian metric, we can uniquely define an ideal affine connection, among all the possible connections, called the **Levi-Civita connection**. In the following, $\langle\cdot,\cdot\rangle$ denotes the Riemannian metric on smooth manifold M, $\langle X, Y\rangle = g(X,Y)$.

Theorem: If $(M, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold, then there exists a unique affine connection ∇ s.t.

(I). It is compatible with the metric, meaning that for all vector fields X, Y, Z, we have

$$X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

(II). It is torsion-free, meaning that for all vector fields X, Y

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

We refer to this connection as the Levi-Civita connection.

Here, $X\langle Y, Z\rangle$ denotes the derivative of the function $\langle Y, Z\rangle$ along the vector field X, and $[X,Y]$ denotes the Lie bracket of X and Y:

$$[X,Y](f)X(Y(f)) - Y(X(f)) = \nabla_X\nabla_Y f - \nabla_Y\nabla_X f, \text{ where } f \in C^\infty(M)$$

Proof of Theorem

The goal of the proof is to find something which determines the connection ∇. If we can find a formula that characterizes it, then we can use that characterization to build the connection.

We break this up into two steps. We prove uniqueness first, then we prove existence.

Uniqueness: We show that if such a connection exists, then it is unique. We do this by finding a formula which characterizes the connection. Notice that if ∇ satisfies the assumed properties, then

$$X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = \langle\nabla_X Y, Z\rangle + \langle[X,Z], Y\rangle + \langle\nabla_Z X, Y\rangle \quad (1)$$

$$Y\langle Z, X\rangle = \langle\nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle = \langle\nabla_Y Z, X\rangle + \langle[Y,X], Z\rangle + \langle\nabla_X Y, Z\rangle \quad (2)$$

$$Z\langle X, Y\rangle = \langle\nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle = \langle\nabla_Z X, Y\rangle + \langle[Z,Y], X\rangle + \langle\nabla_Y Z, X\rangle \quad (3)$$

Adding Equations (1) and (2) and subtracting Equation (3), we are left with

$$X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle = \langle\nabla_X Y, Z\rangle + \langle[X,Z], Y\rangle + \langle\nabla_Z X, Y\rangle +$$
$$\langle\nabla_Y Z, X\rangle + \langle[Y,X], Z\rangle + \langle\nabla_X Y, Z\rangle - (\langle\nabla_Z X, Y\rangle + \langle[Z,Y], X\rangle + \langle\nabla_Y Z, X\rangle)$$

We get:

$$\frac{1}{2}(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle[X,Z], Y\rangle - \langle[Y,X], Z\rangle - \langle[Y,Z], X\rangle) = \langle\nabla_X Y, Z\rangle \quad (4)$$

Since $\langle\cdot,\cdot\rangle$ is bilinear and Z is arbitrary, this equation actually defines a mapping

$$\langle\nabla_X Y, \cdot\rangle : TM \to \mathbb{R}$$

Using the properties of the Riemannian metric, we see that this uniquely defines the vector field $\nabla_X Y$.

Existence: We show that such a connection must exist. As promised earlier, we will do this by using the formula derived in the uniqueness part.

Let ∇ be the connection defined by Equation (4). We need to check that it satisfies the compatibility and torsion-free conditions. We first show compatibility:

$$\langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = \frac{1}{2}(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle[X,Z], Y\rangle - \langle[Y,X], Z\rangle - \langle[Y,Z], X\rangle)$$
$$+\frac{1}{2}(X\langle Z, Y\rangle + Z\langle Y, X\rangle - Y\langle X, Z\rangle - \langle[X,Y], Z\rangle - \langle[Z,X], Y\rangle - \langle[Z,Y], X\rangle) = X\langle Y, Z\rangle$$

For torsion-free, we observe that

$$\langle\nabla_X Y - \nabla_Y X, Z\rangle = \frac{1}{2}(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle[X,Z], Y\rangle - \langle[Y,X], Z\rangle - \langle[Y,Z], X\rangle) -$$
$$\frac{1}{2}(Y\langle X, Z\rangle + X\langle Z, Y\rangle - Z\langle Y, X\rangle - \langle[Y,Z], X\rangle - \langle[X,Y], Z\rangle - \langle[X,Z], Y\rangle) = \langle[X,Y], Z\rangle$$

Hence,

$$\langle\nabla_X Y - \nabla_Y X - [X,Y], Z\rangle \equiv 0 \implies \nabla_X Y - \nabla_Y X = [X,Y]$$

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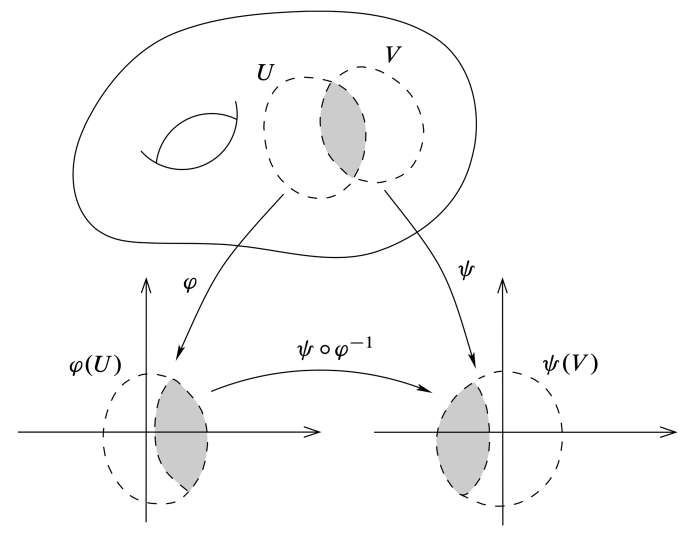

Figure 1. Manifold and the transition map

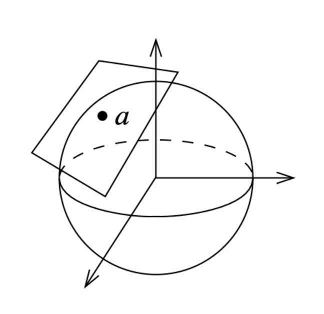

Figure 2. Tangent space

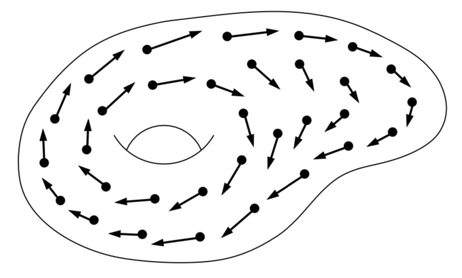

Figure 3. Vector field

What comes next?

A Riemannian metric requires a positive definite metric, but from linear algebra we know that quadratic forms may not necessarily be positive definite. We can generalize the notion of a Riemannian manifold to something called a pseudo-Riemannian manifold, which is a manifold M with a metric g which may not be positive definite; for example, maybe the metric is of the form $g = dx_1^2 + \dots + dx_p^2 - dx_{p+1}^2 - \dots - dx_{p+q}^2$. It turns out that the fundamental theorem of Riemannian geometry is still doable in pseudo-Riemannian geometry, hence one can still have Levi-Civita connection. It would be interesting to explore what other results also hold true in the pseudo-Riemannian case.

References

[1] Burns, Keith, Gidea, Marian, CRC Press, Differential Geometry and Topology With a View to Dynamical Systems

[2] Lee, John M., Springer, Introduction to Smooth Manifolds