Geometry of Classical Dynamics

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Abstract

We introduce the mathematics underlying classical mechanics using the theory of smooth manifolds with their canonical structures. We also provide examples and explain the connections to other fields of mathematics and physics to illuminate the theory.

1. Configuration Spaces and Riemannian Manifolds

It turns out that **Manifold** is the correct model for configuration space. A manifold is a topological space locally modeled by Euclidean space. Intuitively, an n-dimensional manifold is a graph of n-variable function. We locally parameterize the space and sew the coordinate patches together; that is, if a manifold can be locally parameterized by two open sets in $\mathbb{R}^n$, we require that they can be transformed between one another smoothly. An example is the trajectory of a pendulum assuming the rod is non-elastic, then the trajectory is a circle. This is a one-dimensional manifold S^1. Another example is the double pendulum. The configuration space of a double pendulum is the torus $\mathbb{T}^2 := S^1 \times S^1$, since we use two parameters θ_1 and θ_2 to parametrize the location. See **figure 1**.

We equip our manifolds with Riemannian metrics. The intuition behind the Riemannian metric is that we want to calculate lengths of curves on a manifold. One can do this by generalizing the Euclidean inner product. We write the **metric tensor** as $ds^2 = g_{ij}dx^i dx^j$. Using the metric, the length of a curve is defined by:

$$L(\gamma) = \int_\gamma ds = \int_a^b \sqrt{g_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)}dt$$

An easy example of a metric is the Euclidean metric on $\mathbb{R}^n$, given by $ds^2 = dx_1^2 + ... + dx_n^2$. If we consider the plane with polar coordinates, then another example of a Riemannian metric is $ds^2 = dr^2 + r^2 d\theta^2$. We denote by (M, g) a **Riemannian manifold**, i.e. a smooth manifold M together with a metric tensor g.

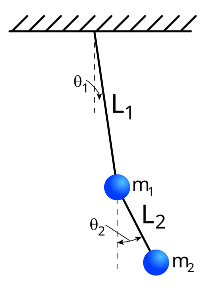

Figure 1. Double Pendulum

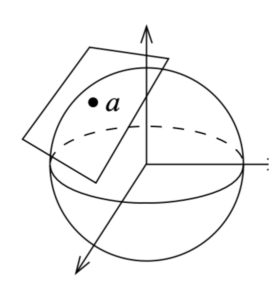

Figure 2. Tangent space

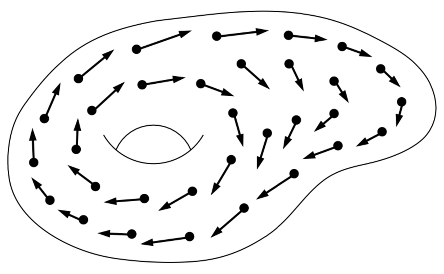

Figure 3. Vector field

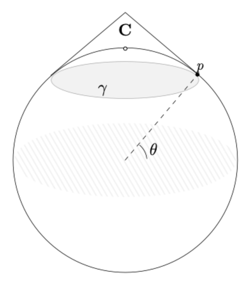

Figure 4. Cone on θ-latitude

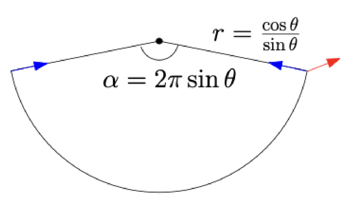

Figure 5. Expanded Cone

2. Velocity Spaces and Tangent Bundles

To describe the velocity along a path and to define the Riemannian metrics, one needs additional structures on a manifold. Mathematically, we introduce the tangent bundle. At every point p of a manifold M, we can have different velocities, these span a vector space called tangent space of M at p, denoted by T_pM. See **Figure 2**. One can identify these tangent vectors with **derivations** by taking the directional derivative in the direction of the vector.

Notice we locally have a one-to-one correspondence between $\mathbb{R}^n$ and T_pM given by $\varphi : e_i \mapsto \frac{\partial}{\partial e_i}|_p$, where e_i is a standard basis of $\mathbb{R}^n$, given by the parameterization. Using this correspondence, one can show that $\{\frac{\partial}{\partial e_1}, \frac{\partial}{\partial e_2}, ..., \frac{\partial}{\partial e_n}\}$ is a basis of T_pM, so every derivation can locally be written as a linear combination of this basis.

A **smooth vector field** X on a manifold M is a smooth map that assigns to every point $p \in M$ a vector X_p in the tangent space T_pM. Since every tangent vector at a point p can be uniquely represented as a linear combination of the basis of the tangent space T_pM, we can always write the vector field locally as $X = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$, where $v_i : U_\alpha \to \mathbb{R}$ are smooth real-valued functions for all i. See **Figure 3** for an example of a vector field.

3. Minimal Action Principle and the Euler-Lagrange's Equation

The minimal action principle dominates the evolution of a dynamical system locally. Given a Lagrangian $L \in C^\infty(TM, \mathbb{R})$, the Euler-Lagrange equation gives a sufficient condition for an action functional to obtain its extremum. Given a path $\gamma(t)$, where $\gamma(t_0)$ and $\gamma(t_1)$ is fixed, the action functional is defined as:

$$\int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t))dt.$$

Consider a smooth variation of $\gamma(t)$, denoted by $\gamma_s(t)$ where $s \in (-\varepsilon, \varepsilon)$. Plugging the variation into the functional, one gets a smooth function from $(-\varepsilon, \varepsilon)$ to $\mathbb{R}$, given by

$$\Gamma(s) = \int_{t_0}^{t_1} L(\gamma_s(t), \dot{\gamma}_s(t))dt.$$

A necessary condition for $\Gamma(s)$ to obtain its extremum is $\frac{\partial \Gamma}{\partial s} = 0$, therefore

$$\begin{aligned}\frac{\partial \Gamma}{\partial s} &= \frac{\partial}{\partial s}\int_{t_0}^{t_1} L(\gamma_s(t), \dot{\gamma}_s(t))dt = \int_{t_0}^{t_1} \frac{\partial}{\partial s} L(\gamma_s(t), \dot{\gamma}_s(t))dt \\ &= \int_{t_0}^{t_1} \frac{\partial L}{\partial x^i}\frac{\partial \gamma^i}{\partial s} + \frac{\partial L}{\partial \dot{x}^i}\frac{\partial \dot{\gamma}^i}{\partial s}dt = \int_{t_0}^{t_1} (\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i})\frac{\partial \gamma^i}{\partial s}dt = 0.\end{aligned}$$

Since $\frac{\partial \gamma^i}{\partial s}$ is arbitrary variation, the fixed endpoints variation gives the following Euler-Lagrange's Equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \frac{\partial L}{\partial x^i}.$$

4. Momentum, Phase Spaces and Cotangent Bundles

To take the momentum into consideration, one needs to go to the phase space. It turns out that the cotangent bundle is where the momentum lives naturally. Cotangent space is the dual space of tangent space, namely, all linear functionals that $T_pM \to \mathbb{R}$, denoted by T_p^*M.

The generalized momentum is defined as $\frac{\partial L}{\partial \dot{x}^i}$. This is in fact the coordinate of momentum in the phase space, so a more precise description would be $p := \frac{\partial L}{\partial \dot{x}^i}dp^i$, which can be viewed as a function from T_pM to $\mathbb{R}$.

The cotangent space carries a canonical symplectic structure. For any smooth manifold M, define a 1-form θ on the cotangent bundle T_p^*M by

$$\theta(X)_{x,\phi} = (p^j dx_j)(\pi_*(X))$$

for each tangent vector $X \in T_{(x,\phi)}T^*M$, where $\pi : T^*M \to M$ is the canonical projection. The antisymmetric 2-form $\omega := d\theta$ is closed and non-degenerated. We refer to θ and ω as the canonical 1-form and canonical 2-form on T^*M, respectively. In the local coordinate chart, we have $\omega = d\theta = dp_j \wedge dx^j$.

5. Hamiltonian Vector fields and Hamiltonian Equations

The Lagrangian mechanics is the geometry on TM, while the Hamiltonian mechanics is the geometry on T^*M.

Usually, people define Hamiltonian as the Legendre transformation of Lagrangian, or the first integral of Euler-Lagrange's equation. Slightly more generally, Hamiltonian can be viewed as a smooth function on the cotangent bundle.

Now, we characterize the motion of a particle using **Hamiltonian vector field**, denoted by X_H. Given a Hamiltonian $H \in C^\infty(T^*M)$, X_H is defined as the vector field so that

$$dH = \omega(\cdot, X_H).$$

This is similar to the gradient of a function in the Riemannian case, which is defined as

$$df = g(\cdot, \nabla f),$$

so X_H is also called the symplectic gradient, and people sometimes denote it by $\nabla_\omega H$.

The **Hamiltonian equation** is written as

$$\dot{\gamma} = X_H,$$

where $\gamma(t)$ is a curve in T^*M. Equivalently, the equation can be written as

$$\dot{q}(t) = \frac{\partial H}{\partial p}, \quad \dot{p}(t) = -\frac{\partial H}{\partial q}.$$

An alternative description of these equations is in terms of an observable Q, given by

$$\frac{d}{dt}Q(\gamma(t)) = \{Q, H\}(\gamma(t)),$$

where $\{Q, H\} := \omega(X_Q, X_H)$ is called **Poisson bracket**.

6. Conservation of Energy and Noether's Theorem, Symmetry and Conserved Quantity

To characterize the conservation of energy, one defines the **Lie Derivative** of a scalar-valued function Q along a vector field X:

$$\mathcal{L}_X Q = \frac{d}{dt}\phi_t^*(Q)\Big|_{t=0}$$

which characterize the change of Q along the flow direction. In particular, a smooth ODE defines a smooth vector field on a configuration space, which defines a flow. Therefore, if $\mathcal{L}_X Q \equiv 0$, Q is a conserved quantity of this system. More generally, Q can be differentiable tensor fields. In particular, when Q is the symplectic form itself, we have by Cartan's magic formula:

$$\mathcal{L}_{X_H}\omega = d \circ \iota_{X_H}\omega + \iota_{X_H} \circ d\omega = 0.$$

Symmetry is a group structure. Assume $L(q, \dot{q})$ is invariant under flow $g_s, s \in \mathbb{R}$, and let $q(t)$ be the solution to the E-L equations, then

$$L(g_s(q), dg_s(\dot{q})) = L(q, \dot{q}).$$

This implies $\frac{\partial}{\partial s}L(g_s(q), dg_s(\dot{q})) = 0$, which implies

$$\frac{\partial L}{\partial q^i}\frac{\partial g_s^i}{\partial s}\Big|_{s=0} + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{g}_s^i}{\partial s}\Big|_{s=0} = 0.$$

We plug E-L equations in, and get

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}^i})\frac{\partial g_s^i}{\partial s}\Big|_{s=0} + \frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}\frac{\partial g_s^i}{\partial s}\Big|_{s=0} = 0,$$

and hence

$$\frac{d}{dt}\left(\sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i}\frac{\partial g_{s,i}}{\partial s}\Big|_{s=0}\right) = 0.$$

Therefore, a symmetry corresponds to a conservative.

7. Acceleration and Affine Connection

One wishes to calculate the acceleration, however, that requires the structure of vector space in order to subtract one vector from another. We introduce the concept of the affine connection, as known as **parallel transport** or **covariant derivative** on a manifold.

An **affine connection** on a manifold M is a smooth map that assigns to every pair of smooth vector fields X and Y on M another smooth vector field $\nabla_X Y$ on M, satisfying the following properties:

(I.) $\nabla_{\alpha X_1 + \beta X_2}Y = \alpha\nabla_{X_1}Y + \beta\nabla_{X_2}Y$ (II.) $\nabla_X(\alpha Y_1 + \beta Y_2) = \alpha\nabla_X Y_1 + \beta\nabla_X Y_2$

(III.) $\nabla_{fX}(Y) = f\nabla_X Y$ (IV.) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$

where $f \in C^\infty(M)$ and X, Y smooth on M.

The Christoffel symbol Γ_{ij}^k is defined by the relationship $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = \Gamma_{ij}^k\frac{\partial}{\partial x_k}$.

If (M, g) is a Riemannian manifold, then there exists a unique affine connection ∇ such that

(I). It is compatible with the metric, meaning that for all vector fields X, Y, Z, we have

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

(II). It is torsion-free, meaning that for all vector fields X, Y, we have

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

We refer to this connection as the Levi-Civita connection.

In particular, the acceleration of a particle in the Euclidean space is the derivative of its velocity along a trajectory γ, which, in the general setting, is

$$\frac{D_{\dot{\gamma}}\dot{\gamma}}{dt} := \nabla_{\dot{\gamma}}\dot{\gamma} = \nabla_{(\dot{\gamma}^i\partial_i)}\dot{\gamma}^j\partial_j = \Gamma_{ij}^k\dot{\gamma}^i\dot{\gamma}^j\partial_k + \ddot{\gamma}^k\partial_k.$$

8. Dynamics of Free Particles and Geodesics

As an example, we demonstrate the Newton's Mechanics of a free particle.

The smooth curve $\gamma : I \to M$ is called a geodesic, if for all $t \in I$,

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0.$$

It is saying that If we take $\dot{\gamma}(t)$ as a vector field along $\gamma(t)$, then it stay constant on γ. So, the geodesics are those curves whose tangent vectors $\dot{\gamma}$ are parallel transport on themselves.

Then, take a local coordinate system on M, we can write the geodesic equation as

$$\ddot{\gamma}^k + \Gamma_{ij}^k\dot{\gamma}^i\dot{\gamma}^j = 0.$$

This equation has a unique solution.

Similarly, if we define the geodesics between p, q on M as the $\gamma : [a, b] \to M$, $\gamma(a) = p, \gamma(b) = q$, such that γ take the minima of the length functional:

$$L(\gamma) = \int_a^b \sqrt{g_{ij}\dot{\gamma}^i\dot{\gamma}^j}dt.$$

This means the length between p, q on M is minimal. The solution is equivalent to the following action functional:

$$E(\gamma) = \int_a^b g_{ij}\dot{\gamma}^i\dot{\gamma}^j dt$$

due to Cauchy-Schwarz inequality, which states

$$L^2 \leq (b - a)E$$

and the equality holds if and only if we choose the natural parametrization.

Then, we plug in the Euler-Lagrange equation, where the Lagrangian is (the kinetic energy) $L = g_{ij}\dot{\gamma}^i\dot{\gamma}^j$,

$$\frac{\partial L}{\partial x_k} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} \implies \ddot{\gamma}^k + \frac{1}{2}(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l})g^{kl}\dot{\gamma}^i\dot{\gamma}^j = 0.$$

Notice that in the Levi-Civita connection,

$$\Gamma_{ij}^k = \frac{1}{2}g^{lk}(\frac{\partial g_{li}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l}),$$

two definitions of geodesics coincide. By our calculation, we know the free particle follows the trajectory of minimal length.

9. Parallel Transport and Foucault's Pendulum

Given a path γ connecting $p, q \in M$, the parallel transport of a vector v along γ defines a vector field along γ, given by the following equation:

$$\nabla_{\dot{\gamma}}X = 0, \quad X(p) = v.$$

The canonical embedding $i : S^2 \to \mathbb{R}^3$ is given by parametrization (x, y, z), $x^2 + y^2 + z^2 = 1$, so S^2 bears a induced metric from $\mathbb{R}^3$. We build a cone **C** stuck along γ, the θ-latitude, see **Figure 4**. We can furthermore flatten it as shown in **Figure 5** by an isometry to $\mathbb{R}^2$. As mentioned above, metric determines connection. So we may calculate the connection on the sphere along the θ-latitude. Obviously, the parallel transport on $\mathbb{R}^2$ is trivial, hence the parallel transport along γ is trivial. This is why we need Foucault's pendulum to swing because we are parallel transporting the velocity vector.

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