Game Theory and Mechanism Design

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Practice Problems

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Problem Set 1

Warm-up

1. Given a strategic form game and a player, show that a strongly dominant strategy, if one exists, must be unique.

2. Show that every dominant strategy equilibrium (strong or weak or very weak) is also a pure strategy Nash equilibrium.

3. Give an example of a simple game (two players, two strategies each) having a pure strategy Nash equilibrium that is not a very weakly dominant strategy equilibrium.

4. A strategic form game has 2 players having 3 strategies each. What is the minimum number and maximum number of pure strategy Nash equilibria for such a game.

5. Give examples of two player pure strategy games for the following situations:
   (a) The game has a unique Nash equilibrium which is not a weakly dominant strategy equilibrium.
   (b) The game has a unique Nash equilibrium which is a weakly dominant strategy equilibrium but not a strongly dominant strategy equilibrium.
   (c) The game has one strongly dominant or one weakly dominant strategy equilibrium and a second one which is only a Nash equilibrium

Workhorse

1. In the Braess paradox example with n=1000 given in the book, compute the minimum delay and maximum delay that could be incurred by a vehicle in Case 1 (without link AB) and in case 2 (with link AB).

2. There are n players. Each player announces a number in the set 1,2, ..., m where m is a fixed positive integer. A prize of One Rupee is split equally between all the people whose number is closes to two thirds of the average number. Formulate this as a strategic form game.

3. Develop the strategic form game for the Pigou network game (page 52) for n=4.

4. There are n departments in an organization. Each department can try to convince the central authority (of the organization) to get a certain budget allocated. If h_i is the number of hours of work put in by a department to make the proposal, let $c_i = w_i * h_i^2$ be the cost of this effort to the department, where w_i is a constant. When the effort levels of the departments are $(h_1, h_2, ..., h_n)$, the total budget that gets allocated to all the departments is:

$$\alpha \sum_{i=1}^{n} h_i + \beta \prod_{i=1}^{n} h_i$$

   where α and β are constants.
   Consider a game where the departments simultaneously and independently decide how many hours to spend on this effort. Show that a strongly dominant strategy equilibrium exists if and only if $\beta = 0$. Compute this equilibrium.

5. A two player symmetric strategic form game is one in which $S_1 = S_2$ and $u_1(s_1, s_2) = u_2(s_2, s_1)\ \forall s_1 \in S_1\ \forall s_2 \in S_2$. Show in such a game that the strategy profile (s_1^*, s_2^*) is a pure strategy Nash equilibrium if and only if the profile (s_2^*, s_1^*) is also a pure strategy Nash equilibrium.

6. Compute strongly or weakly dominant strategy equilibria of the Braess paradox game when the number 25 is replaced by the number 20 (Example 5.5) from 'Game Theory and Mechanism Design' by Y.Narahari.

7. Consider the following instance of the prisoners' dilemma problem.

	NC	C
NC	-4,-4	-2,-x
C	-x,-2	-x,-x

   Find the values of x for which:

   (a) the profile (C, C) is a strongly dominant strategy equilibrium.
   (b) the profile (C, C) is a weakly dominant strategy equilibrium but not a strongly dominant strategy equilibrium.
   (c) the profile (C, C) is a not even a weakly dominant strategy equilibrium.

   In each case, say whether it is possible to find such an x. Justify your answer in each case.

8. First Price Auction: Assume two bidders with valuations v_1 and v_2 for an object. Their bids are in multiples of some unit (that is, discrete). The bidder with higher bid wins the auction and pays the amount that he has bid. If both bid the same amount, one of them gets the object with equal probability $\frac{1}{2}$. In this game, compute a pure strategy Nash equilibrium of the game.

Thought Provoking

1. Common knowledge example - Give your own intuitive explanation as to why all the five mothers cry only on the fifth day (6 lines max)

2. Can there exist multiple WDSE in a strategic form game? Prove or disprove.

3. Do you think $(v_1, ...v_n)$ is the unique WDSE for the Vickrey auction game?

4. We showed that the strategy profile (AB, AB, ..., AB) is a SDSE for the Braess Paradox game. Try to derive conditions under which this strategy profile will not be a SDSE.

5. We showed for Vickrey auction that the profile $(v_1, v_2, ..., v_n)$ satisfies the first condition for a weakly dominant strategy equilibrium. Prove the second condition.

6. Compute a Nash equilibrium for the two person game with $S_1 = \{0, 1\}$, $S_2 = \{3, 4\}$ $u_1(x, y) = -u_2(x, y) = |x - y| \ \forall \ (x, y) \in \{0, 1\} \times \{3, 4\}$

7. Consider a strategic form game with $N = \{1, 2\}$; $S_1 = S_2 = [a, b] \times [a, b]$ where a and b are positive real numbers such that $a < b$. That is, each player picks simultaneously a point in the square $[a, b] \times [a, b]$. Define the payoff functions:

$$u_1(s_1, s_2) = -u_2(s_1, s_2) = d(s_1, s_2)$$

where $d(s_1, s_2)$ is the Euclidean distance between the points s_1 and s_2 . For this above game, compute all pure strategy Nash equilibria.