AST 381 Homework 3: Random Numbers, Monte Carlo, and Ray Tracing

Nina Filippova

February 25, 2022

1. Integral Processes: MC Edition *Numerically integrate the following function using the MC approach discussed in class (i.e., using points randomly sampled from the interval [1, 5]):*

$$I = \int_1^5 \frac{1}{x^{3/2}} dx.$$

Which approach do you like better, and why?

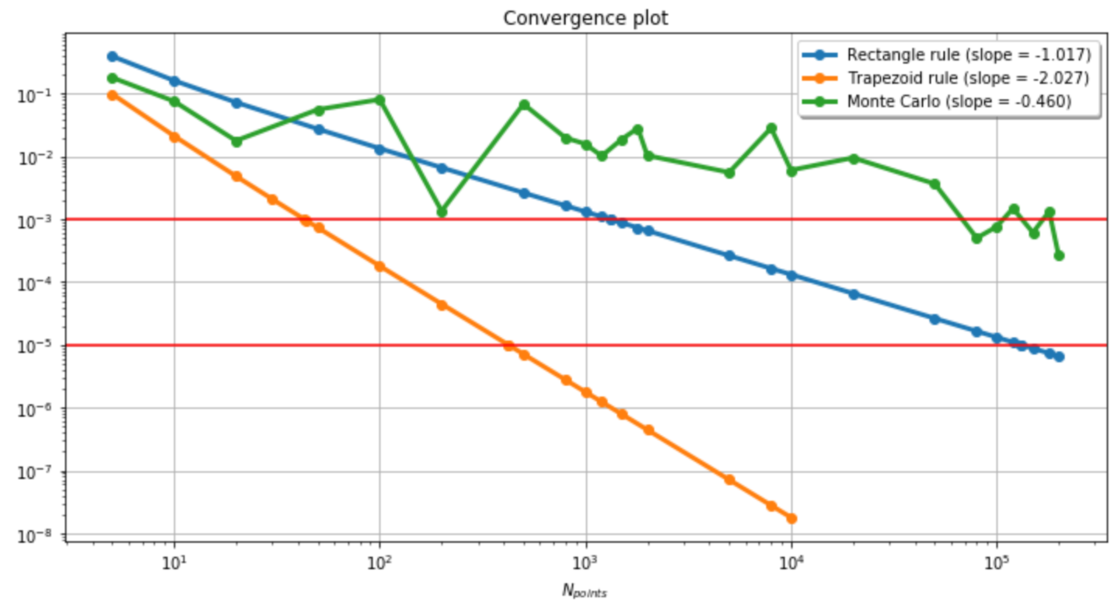


Figure 1: Convergence plot for first-order rectangle rule, second-order trapezoid rule, and Monte Carlo integration.

Monte Carlo integration converges much more noisily than the rectangle and trapezoid rules due to the stochastic nature of the method. The overall convergence goes as $\sim N^{1/2}$, as expected. I think it's very cool that Monte Carlo integration works, but for this problem, the other methods approximate the exact solution better using fewer grid cells.

2. A random stroll from the center of the Sun *Assume the typical mean free path of a photon is* $\ell \simeq 4\times10^{-1}$ *cm. Let's consider a theorist's Sun, which is a 2D circle with radius* $1R_\odot$*. Every time a photon hits an electron (after going 1 mean free path), it is scattered in an arbitrary direction* θ*. Roughly how long does it take for a photon to escape the Sun after being emitted? (You can round to the nearest power of 10.)*

Use your own random number generator to complete the Monte Carlo. If you are curious, compare this to one of the built-in "black boxes" in Python.

(Note: You cannot do this problem by using the actual solar radius - the code will never finish. Instead, assume a radius of 7 cm, 70 cm, and extrapolate accordingly.)

Use a timer function to figure out how long the wall clock time is for your code. What general lessons do you learn from this exercise?

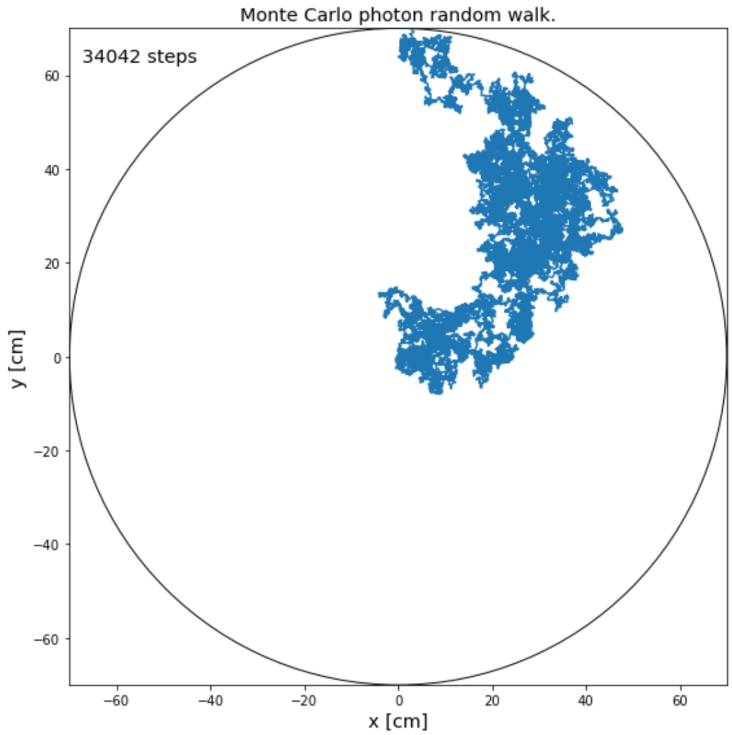


Figure 2: Sample random walk (for an $R = 70$ cm Sun).

Running 20 samples each for $R = 7$, 70, and 700 cm:

R (cm)	Number of steps	Elapsed time (s)
7.0	$3.681 \times 10^2 \pm 2.147 \times 10^2$	$3.633 \times 10^{-3} \pm 2.106 \times 10^{-3}$
70.0	$2.564 \times 10^4 \pm 1.742 \times 10^4$	$2.476 \times 10^{-1} \pm 1.689 \times 10^{-1}$
700.0	$3.245 \times 10^6 \pm 1.381 \times 10^6$	33.12 ± 13.83

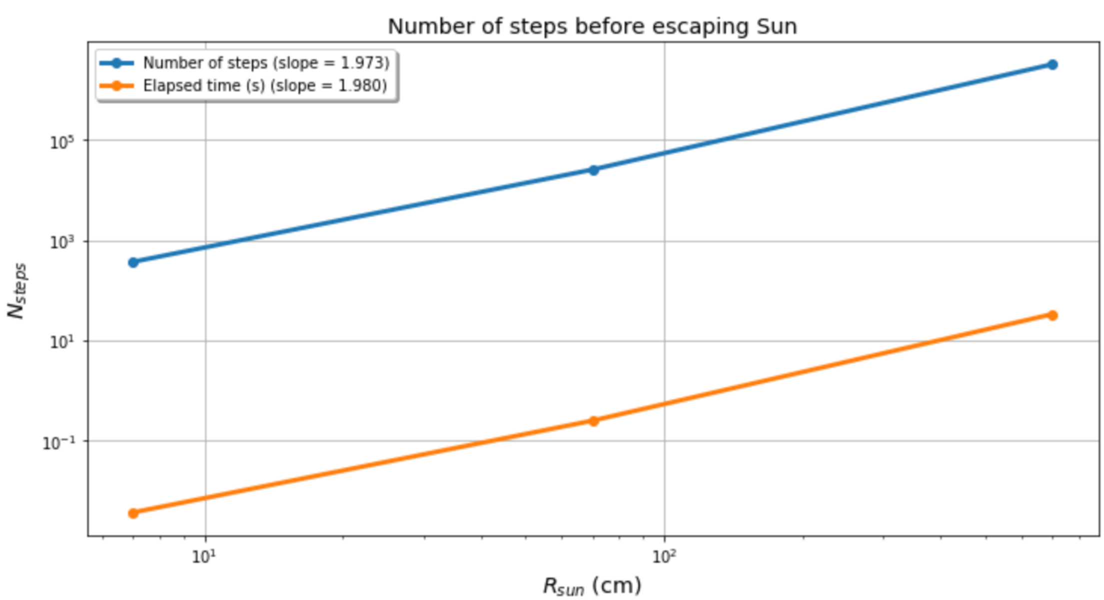


Figure 3: Average (over 20 trials) number of steps, elapsed clock time, required for the photon to escape the Sun, as a function of solar radius.

The number of steps seems to be proportional to the radius squared, so

$$\frac{N_{\text{sun}}}{N_{700\text{cm}}} = \left(\frac{R_{\text{sun}}}{R_{700\text{cm}}}\right)^2 \quad \longrightarrow \quad N_{\text{sun}} = N_{700\text{cm}} \left(\frac{R_{\text{sun}}}{R_{700\text{cm}}}\right)^2 \sim 10^6 \times \left(\frac{10^{10}}{10^2}\right)^2 \sim 10^{22} \text{ steps.}$$

If the photon travels 0.4 cm with each step, then it will take about

$$\frac{(0.4 \text{ cm/step}) \times (10^{22} \text{ steps})}{3.0 \times 10^{10} \text{ cm/s}} \sim 10^{11} \text{ seconds } \sim 3000 \text{ years}$$

for the photon to escape the 2D Sun.

The amount of time it would take to run the code using the actual radius of the Sun would be much too long, so interpolating is helpful for this type of problem.

3. Observing Dusty Protostars *In this problem you will use the radiative transfer code,* HYPERION, *to model an embedded low-mass protostar.*

(a) *Search through the code and documentation and determine how and where random numbers are being used in* HYPERION *(in particular, focus on the core algorithm and physics, and ignore the random numbers appearing in test problems). Where and why are they invoked? What generation method is being used? For a given run with identical parameters, will the results be deterministic?*

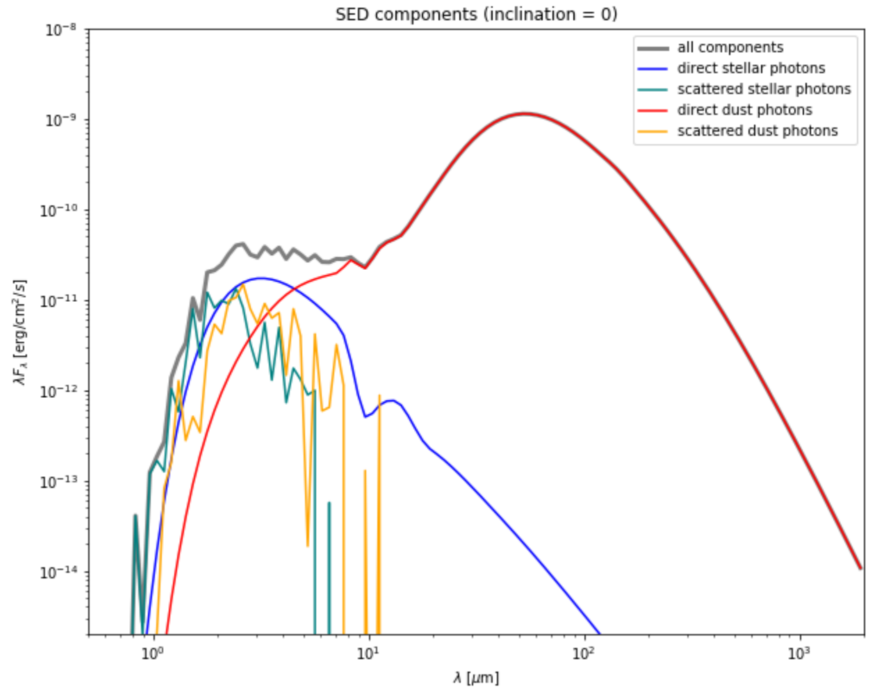
Random numbers are used in order to select:

- the source from which the photon packet (PP) is emitted (even sampling or luminosity-weighted probability distribution function);
- the direction and frequency of the PP, accounting for the type and spectrum of the source (e.g., draw frequency from Planck distribution);
- the optical depth to extinction (follows an exponential distribution: uniformly sample $\xi \in [0, 1]$, then $\tau = -\ln(\xi)$);
- the type of interaction (absorption or scattering) with the dust ($\xi \in [0, 1]$; if ξ is greater than the albedo of the dust, the PP is absorbed; else it is scattered);
- the direction of the PP after the interaction (and the frequency, if the PP is being absorbed and re-emitted).

(b) *Use the documentation example for setting up "Analytical YSO Models" to initialize a model of a star with envelope and disk with the following properties:*

- *YSO:* $L = 5L_\odot$, $R = 2R_\odot$, $T = 6200$ K.
- *Flared disk:* $M_\text{d} = 0.01M_\odot$, $R_{\text{d,min}} = 10R_\odot$, $R_{\text{d,max}} = 200$ AU, $\rho \propto r^{-1}$, *and flaring power* $\beta = 1.25$.
- *Envelope:* $M_\text{e} = 0.4M_\odot$, $R_{\text{e,min}} = 200$ AU, $R_{\text{e,max}} = 10^4$ AU, $\rho \propto r^{-2}$.
- *Dust model for all: Kim, Martin, & Hendry 1994.*
- *Spherical grid: 200 x 100 x 5.*

Start with 10,000 photons (initial, imaging, and raytracing). Make a plot showing the SEDs of this source viewed from several different angles, assuming a distance of 300 pc. Explain any features of the SED. How does the SED change with viewing angle and why?



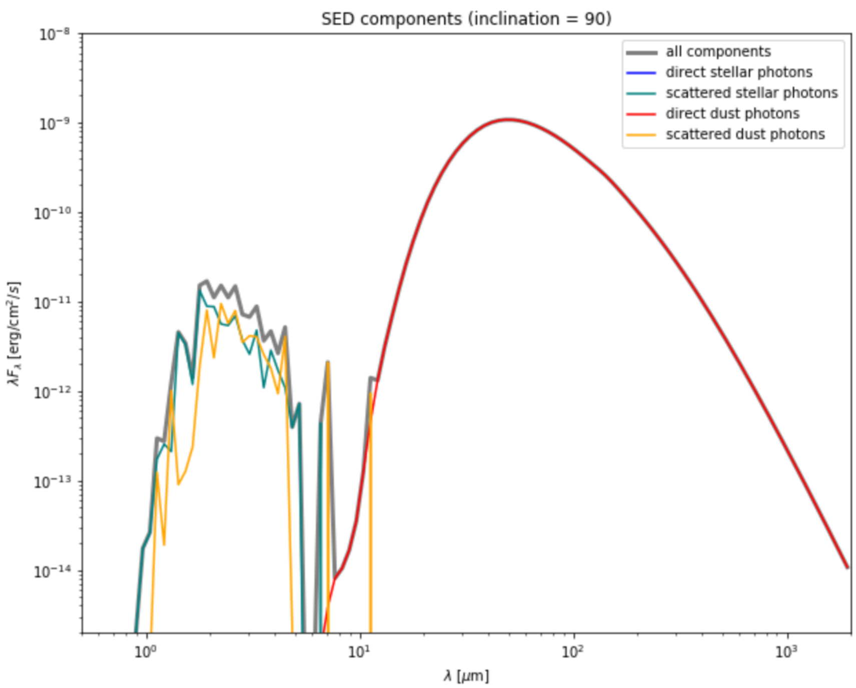


Figure 4: SEDs for 10^4 photons, $200 \times 100 \times 5$ grid resolution, viewed from inclinations of 0 degreed (pole-on) and 90 degrees (edge-on). Additional viewing angles in Jupyter notebook.

Dust re-emission dominates the longer-wavelength end of the SED. Viewed edge-on, the dusty disk blocks observtions of photons directly emitted by the YSO; the shorter-wavelength end of the SED is dominated by photons scattered by the dust. This leads to a double-peaked SED, with the gap between the peaks becoming deeper as the viewing angle increases.

(c) *Vary the number of photon packets and grid resolution. How many photons and what resolution would you need to produce a converged SED? Describe your convergence criteria and make a plot showing the dependence of the SED on different values.*

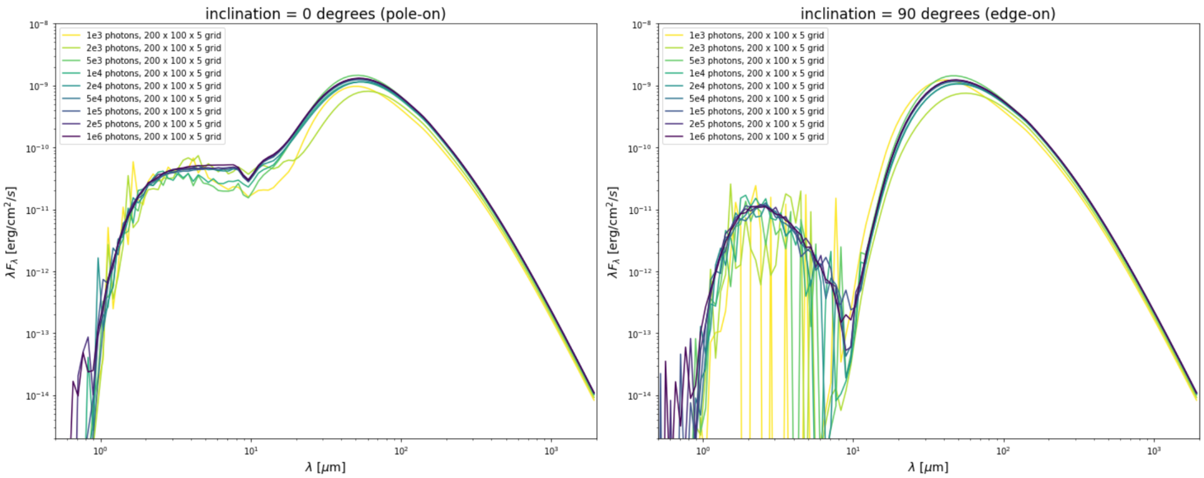


Figure 5: SED comparison plot (viewed pole-on and edge-on) for a fixed $200 \times 100 \times 5$ grid, varying the number of photon packets. For the edge-on view, the peak at shorter wavelengths is noticeably noisy for $\lesssim 2 \times 10^5$ photon packets.

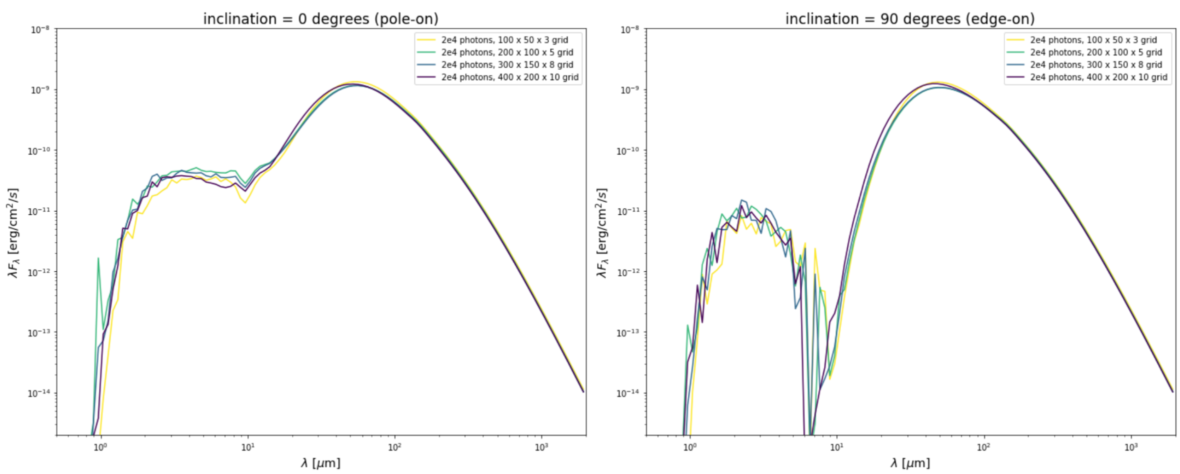


Figure 6: SED comparison plot (viewed pole-on and edge-on) for 2×10^4 photon packets, varying grid resolution.

For the $200 \times 100 \times 5$ grid, the SED seems well-resolved (i.e., no delta function-like spikes in the $\lambda \lesssim 10\mu$m region) for $N_{\rm photons} \gtrsim 2 \times 10^5$. As the grid resolution increases, the number of photon packets must be increased as well; this is particularly noticeable when looking at temperature plots for, e.g., 2×10^4 vs. 1×10^6 photon packets for a $400 \times 200 \times 10$ grid.

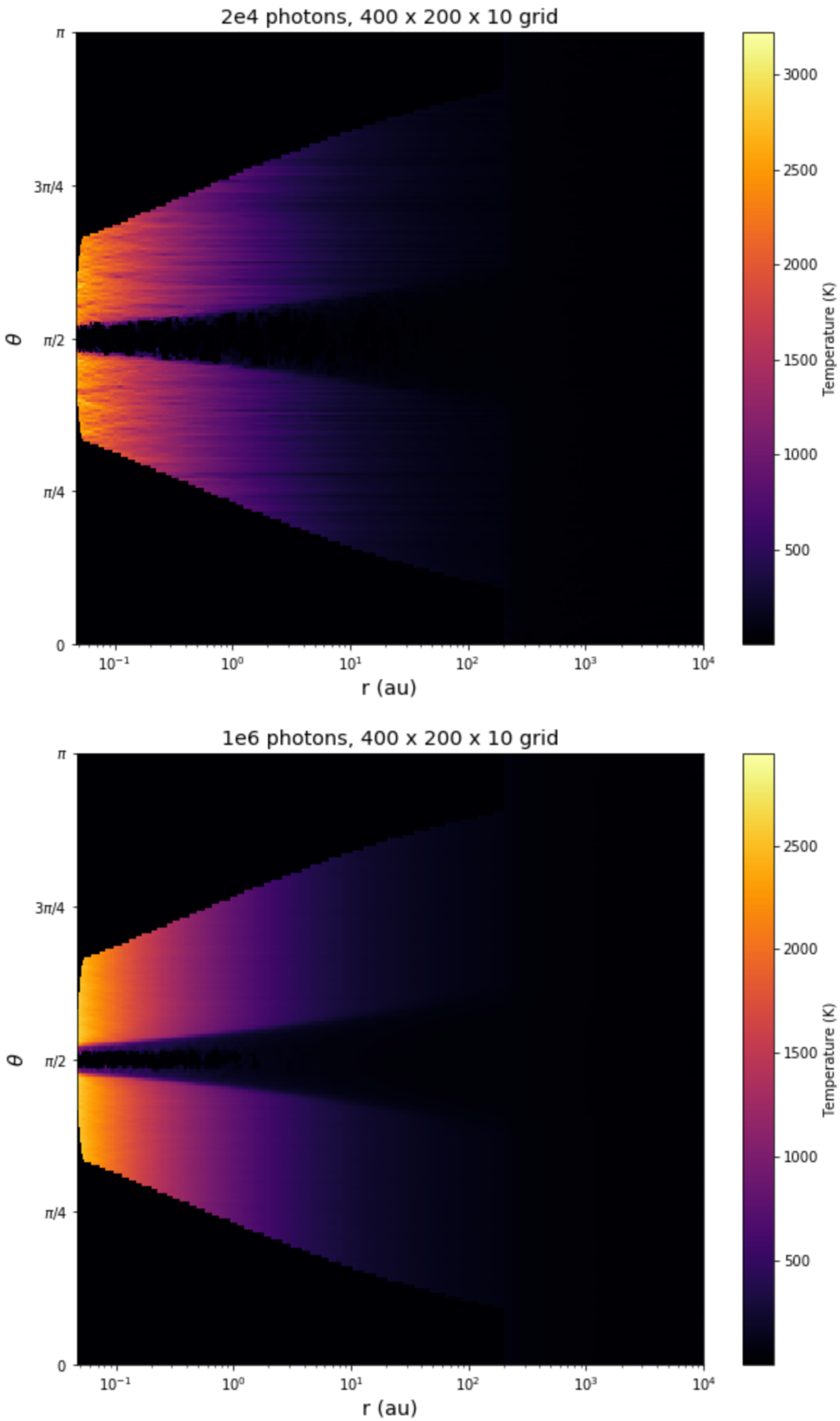


Figure 7: Temperature as a function of r, θ for a $400 \times 200 \times 10$ grid with 2×10^4 vs. 1×10^6 photon packets. With 2×10^4 photons, individual photon packet tracks can be observed; the plot becomes much smoother with 1×10^6 photon packets.