Homework 4

CS41, Spring — Due Sunday, February 19

1. The goal of this problem is to formalize the intuition that if two nodes in a network are very far apart, then their connection is "fragile," in the sense that all routes between them could be completely severed by removing a small number of other nodes.

   Notation. Below, we use the set subtraction operation $\setminus$ (`\setminus`). If A and B are sets, then $A \setminus B$ is just the set of all elements that belong to A but do not belong to B.

   (a) Use a counterexample to disprove the following statement: If $G = (V, E)$ is any graph and $u, v \in V$ are two vertices with $d(u, v) \geq |V|/2$, then there is some vertex $w \in V \setminus \{u, v\}$ such that every u–v path includes w.

   (b) Prove that if $G = (V, E)$ is a graph and $u, v \in V$ satisfy $d(u, v) > |V|/2$, then there is some vertex $w \in V \setminus \{u, v\}$ such that every u–v path includes w.

   (c) Generalize part (b) by proving that if $G = (V, E)$ is a graph, $k < |V|$ is a positive integer, and $u, v \in V$ satisfy $d(u, v) > (|V| - 2 + k)/k$, then there is some set $S \subseteq V \setminus \{u, v\}$ such that $|S| < k$ and every u–v path includes at least one vertex from S.

   (d) Give an algorithm that runs in $O(|V| + |E|)$ time and finds such a set S whenever $d(u, v) > (|V| - 2 + k)/k$.

2. (from Kleinberg & Tardos) You're helping a group of ethnographers analyze some oral history data they've collected by interviewing members of a village to learn about the lives of people who've lived there over the past two hundred years.

   From these interviews, they've learned about a set of n people (all of them now deceased), whom we'll denote $P_1, P_2, \ldots, P_n$. They've also collected facts about when these people lived relative to one another. Each fact has one of the following two forms:

   - For some i and j, person P_i died before person P_j was born; or
   - for some i and j, the life spans of P_i and P_j overlapped at least partially.

   Naturally, they're not sure that all these facts are correct; memories are not so good, and a lot of this was passed down by word of mouth. So what they'd like you to determine is whether the data they've collected is at least internally consistent, in the sense that there could have existed a set of people for which all the facts they've learned simultaneously hold.

   Give an efficient algorithm to do this: either it should produce proposed dates of birth and death for each of the n people so that all the facts hold true, or it should report (correctly) that no such dates can exist—that is, the facts collected by the ethnographers are not internally consistent.