Homework 8

CS41, Spring 2023

Alina Palacios and Nina Zhuo

1. Our algorithm will operate as follows:

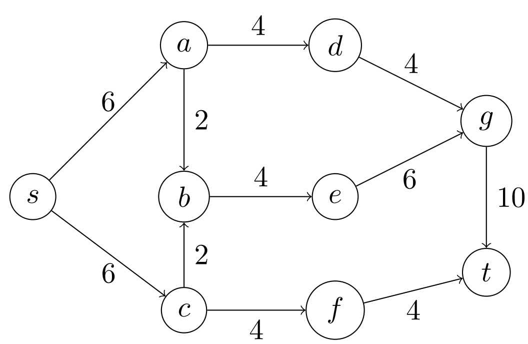
   i. We start by running Kruskal's in $O(|E|log|V|)$ on a modified graph G where $V = V \setminus U$, resulting in an MST that excludes all vertices $u \in U$.

   ii. Iterate over all sorted edges to find lowest cost edges that connect the MST to all $u \in U$, keeping track of what vertices have been already added in from $V \subseteq T$ to ensure that we do not connect a vertex of the set U to another vertex in the set U such that that any $u \in U$ included in the MST will be a leaf.

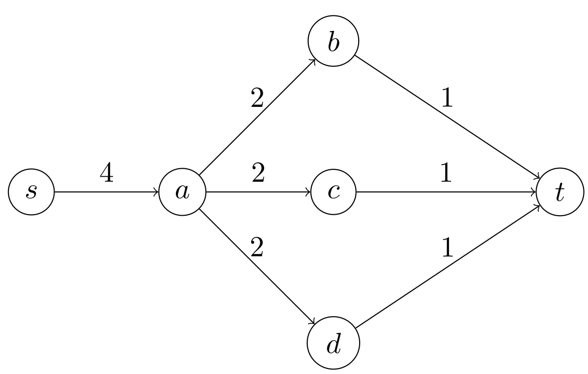
   iii. Iterate over $|E|$ number of vertices in the MST to check that $|V| = |E| - 1$, ensuring that all $u \in U$ included in the graph is a leaf in the MST.

   Our algorithm runs in $O(|E|log|V|)$ time because we traverse through the number of vertices and check that $|V| = |E| - 1$, thus $O(|E|log|V|)$.

2. The max flow for the given flow network is 12 as shown in the diagram below.

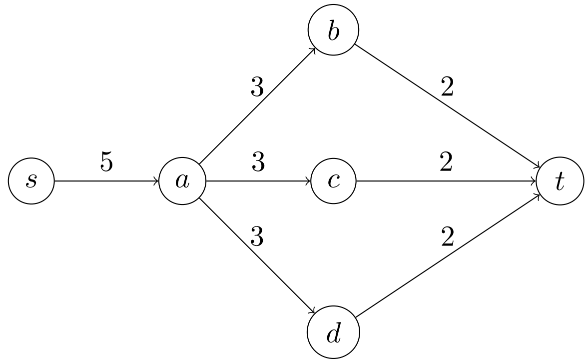


3. *Proof.* False, by counterexample. Let G be a flow network with the following vertices and edge capacities:



   Here is the max flow is 3 and the minimum cut includes the edges $b-t$, $c-t$, and $d-t$, whose capacities add up to 3.

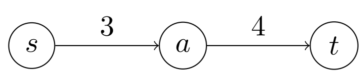
When we add 1 to the capacity of each edge, we get the resulting flow network:



We see that max flow now is 5 such that the minimum cut 5 also. However, the edges from the previous minimum cut add up to 6, which exceeds the max flow value. Now, the minimum cut for G includes only the edge from $s - a$. Thus, adding 1 to the capacity of every edge in G will not result in the cut (L, R) still being the same min cut.

$\square$

4. False; Ford-Fulkerson only requires traversing some augmenting path P such that the number of times F-F is run is not dependent on the number of distinct $s - t$ paths. Further, each distinct $s-t$ path will have edges with flow capacity of at least 1 such that, in a flow network with only one distinct path P, the flow can be augmented along P multiple times. Consider the following flow network:



Though there is only one distinct $s - t$ path, we can send an augmenting flow of 1 along that $s - t$ path for each iteration of F-F for 3 iterations, which is not equal to the number of distinct $s - t$ paths.

Thus, for any given flow network, we send a flow of at least 1 from the source vertex s to the sink vertex t along any augmenting path P for each iteration of F-F until we reach some max flow value $f*$. Since the value of a max flow $f*$ is finite, the number of times the F-F algorithm can incrementally augment the flow by 1 is at most the value of a max flow $f*$.