Homework 11

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1. Simple is NP-complete.

Proof. Simple is NP-complete if Simple $\in$ NP and Simple $\in$ NP-Hard.

We first want to prove that Simple $\in$ NP by showing that VerifySimple $\in P$ such that Simple can be verified in polynomial time. Given a graph G and a path p of length n where $n \geq k$, we can iterate over the length of the path in a nested `for` loop to check that the path touches every vertex at most once and the relevant edges between the list of vertices are present such that no two vertices in the path are the same. This can be done in $O(n^2)$ time such that Simple $\in$ NP

Next, we want to prove that Simple $\in$ NP-Hard by showing that Ham-Path $\leq_p$ Simple because Ham-Path is a known NP-hard problem. Ham-Path(G) is true if and only if the subroutine CheckSimple(G, k) is true, with $k = |V| - 1$ since we know that a Hamiltonian path has a length of $|V| - 1$. By definition, a Hamiltonian path is any path on a graph where each vertex is visited exactly once such that if there exists a Hamiltonian path of length $k = |V| - 1$, there also exists a simple paths of length k. Therefore, Ham- Path $\leq_p$ Simple such that Simple $\in$ NP-Hard.

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2. 4-Space is NP-complete.

Proof. 4-Space is NP-complete if 4-Space $\in$ NP and 4-Space $\in$ NP-Hard.

We first want to prove that 4-Space $\in$ NP by showing that Verify4-Space $\in P$ such that 4-Space can be verified in polynomial time. Given a graph $G = (V, E)$, we can have a nested `for` loop to iterate through all of the edges and vertices to check the distance between two vertices u and v has a distance of 4. This can be done in $O(n^2)$.

Next, we want to prove that 4-Space $\in$ NP-Hard by showing that IND-SET $\leq_p$ 4-Space because IND-SET is a known NP-hard problem. Given a graph G, we will make a duplicate graph G'. In G', we will have divide the edges (u, v) into two and add a vertex w, adding edges (u, w) and (w, v). We will give each edge in G' a weight of $|E| + 1$ and give every vertex a weight of 1. Call Verify4-Space$(G', k(|E| + 1)$ to solve IND-SET. Once we have IND-SET, which we will denote as I, we will check the distance between (u, v) for all u of $v \in I$. This will result in the distance between $(u, v) \geq 4$. We take the sum of the vertex weights from G' of all $v \in I$, which equals to $k(|E| + 1)$. If Verify4-Space$(G', k(|E| + 1)$ returns `TRUE`, then we know that there exists a set of vertices that have a distance of ≥ 4, meaning that the sum of the vertices was within the threshold of $k(|E| + 1)$. Verify4-Space will return `FALSE` otherwise.

With this information, we will show that given an input from 4-Space, 4-Space$f(G, k)$, will be $\geq k(|E| + 1)$. Using the same weights of 1 for our vertices, we know it will never satisfy the condition $\geq k(|E| + 1)$. This means that these vertices will never be included in the set 4-Space$f(G, k)$. Therefore, every node in G' is also in G. This also means that there has to be a set k vertices that added up can make the total weight at most $|E|$. Since we doubled the distance between all the vertices in G', the original distance of (u, v) must have been ≥ 2, making these k vertices adjacent. Therefore, IND-SET $\leq_p$ 4-SPACE, such that 4-SPACE $\in$ NP-COMPLETE.

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3. BIG-VC is NP-complete

Proof. BIG-VC is NP-complete if BIG-VC $\in$ NP and BIG-VC $\in$ NP-HARD.

We first want to prove that SIMPLE $\in$ NP by showing that VERIFYBIG-VC $\in P$ such that BIG-VC can be verified in polynomial time. Suppose we input a graph G and a set of vertices S as a certificate into VERIFYBIG-VC where $|S| = k$. We iterate over the length of S to check that $k \geq |V|/5$ and return FALSE in the case that it doesn't. We check that for each edge $(u, v) \in V$, $u \in S$ such that the set of all vertices in S touches every edge. Since $|E| = O(|V|^2)$, VERIFYBIG-VC will run in $O(|V|^2)$ time such that BIG-VC $\in$ NP

Next, we want to prove that BIG-VC $\in$ NP-HARD by showing that VERTEXCOVER $\leq_p$ BIG-VC because VERTEXCOVER is a known NP-hard problem. Suppose f is a function that takes k and a graph G and outputs a modified graph G' that contains an additional $|V|$ dummy vertices of degree zero and the integer $k + |V|$. BIG-VC is NP-Hard if VERTEXCOVER(G, k) $\iff$ BIG-VC$(G', k + |V|)$.

When S is a vertex cover outputted from VERTEXCOVER(G, k), the set of vertices that constitute a vertex cover in the graph G will be a connected component in the graph G' that contains the same vertices and edges. Thus, S' will be a vertex cover that includes all k vertices in S in addition to the $|V|$ additional disconnected vertices in the graph G'. Because $|V'| = 2|V|$ and $|S'| = k + |V| \geq \frac{2|V|}{5} = \frac{|V'|}{5}$, we know $k \geq |V|/5$ will always be true. Since BIG-VC will return TRUE because there exists a set S' with $k + |V|$ vertices for the graph G' when VERTEXCOVER returns TRUE for a graph G with a set S of k vertices, VERTEXCOVER $\implies$ BIG-VC.

When S' is a vertex cover outputted for BIG-VC$(G', k + |V|)$ and $k \geq |V'|/5$, $|S'| = k + |V|$. Let D be the set of $|V|$ isolated dummy vertices in S' such that $|S' \backslash D| = (k + |V|) - |V| = k$. Removing D from S' will give us the vertex cover S from the graph G.

Since VERTEXCOVER will return TRUE because there exists a set S with k vertices for the graph G' when BIG-VC returns TRUE for a graph G' with a set S' of $k + |V|$ vertices, BIG-VC $\implies$ VERTEXCOVER.

Therefore, BIG-VC $\in$ NP and BIG-VC $\in$ NP-HARD such that BIG-VC is NP-complete.

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