1 Inductive Proofs

Prove each of the following claims by induction

Claim 1. *The sum of the first n even numbers is n^2+n. That is, $\sum_{i=1}^{n} 2i = n^2+n$.*

Proof. 1. We set the base case at $n = 1$ and solve each side of the equation to see if they are equal to each other.

$$\begin{aligned}\sum_{i=1}^{1} 2i &= (2 \cdot 1)\\ &= 2\end{aligned}$$

$$\begin{aligned}n^2 + n &= 1^2 + 1\\ &= 1 + 1\\ &= 2\end{aligned}$$

2. Assume that for some integer k, we get the inductive hypothesis

$$\sum_{i=1}^{k} 2i = k^2 + k$$

3. If the formula assumed in step 2 is true, it will also be true for $n = k+1$. We can test this by plugging $k + 1$ into the formula giving us

$$\sum_{i=1}^{k+1} 2i = (k+1)^2 + (k+1)$$

$$\begin{aligned}\sum_{i=1}^{k+1} 2i &= \sum_{i=1}^{k} 2i + 2(k+1) && \text{separate the last term of the sum}\\ &= (k^2 + k) + 2(k+1) && \text{by applying inductive hypothesis}\\ &= k^2 + 3k + 2 && \text{by math}\end{aligned}$$

$$\begin{aligned}(k+1)^2 + (k+1) &= (k^2 + 2k + 1) + (k+1)\\ &= k^2 + 3k + 2\end{aligned}$$

Given the above, we can assume by induction that the equation is true for all n where $n > 1$.

$\square$

Claim 2. $\sum_{i=1}^{n} 3^i = \frac{3}{2}(3^n - 1)$

Proof. 1. We set the base case at $n = 1$ where 1 is the lower bound and solve each side of the equation to see if they are equal to each other.

$\sum_{i=1}^{1} 3^i = 3^1$

$= 3$

$\frac{3}{2}(3^n - 1) = \frac{3}{2}(3^1 - 1)$

$= \frac{3}{2}(2)$

$= \frac{6}{2}$

$= 3$

2. Assume that for some integer k, we get the inductive hypothesis

$$\sum_{i=1}^{k} 3^i = \frac{3}{2}(3^k - 1)$$

3. If the formula assumed in step 2 is true, it will also be true for $n = k + 1$. We can test this by plugging $k + 1$ into the formula giving us

$$\sum_{i=1}^{k+1} 3^i = \frac{3}{2}(3^{k+1} - 1)$$

$\sum_{i=1}^{k+1} 3^i = \sum_{i=1}^{k} 3^i + 3^{k+1}$ separate the last term of the sum

$= \frac{3}{2}(3^k - 1) + 3^{k+1}$ by applying inductive hypothesis

$= \frac{3}{2} \cdot 3^k - \frac{3}{2} + 3^k \cdot 3$ by math

$= \frac{9}{2} \cdot 3^k - \frac{3}{2}$

$= \frac{3}{2}(3 \cdot 3^k - 1)$

$= \frac{3}{2}(3^{k+1} - 1)$

Given that the above is equal to the right hand side of the equation, we can assume by induction that the equation in step 2 true for all n where $n > 1$.

$\square$

Claim 3. *For any integer $n \geq 1$, $5^n - 1$ is divisible by 4. In other words, for every positive integer n there exists some constant z_n such that $5^n - 1 = 4z_n$. (Note that z_n denotes a different z for each power of 5; that is, $5^1 - 1 = 4z_1$, $5^2 - 1 = 4z_2$, and so on for a series of z_n values.) You may write your proof in general terms of divisibility by four or in specific terms by solving for z_n in the inductive case.*

Proof.

1. We set the base case at $n = 1$ where 1 is the lower bound and $z_n = 4n$ solve each side of the equation to see if they are equal to each other.

$$\begin{aligned} 5^1 - 1 &= 5 - 1 \\ &= 4 \end{aligned}$$

$$\begin{aligned} 4z_n &= 4(1) \\ &= 4 \end{aligned}$$

2. Assume that for some integer k, we get the inductive hypothesis

$$5^k - 1 = 4z_k$$

3. If the formula assumed in step 2 is true, it will also be true for $n = k + 1$. We can test this by plugging $k + 1$ into the formula giving us

$$5^{k+1} - 1 = 4z_{k+1}$$

$$\begin{aligned} 5^{k+1} &= 5^k \cdot 5 && \text{rewrite the } k+1 \text{ term} \\ &= 5(4z_k + 1) && \text{by inductive hypothesis} \end{aligned}$$

$$\begin{aligned} \text{so, } 5(4z_k + 1) - 1 &= 20z_k + 4 && \text{substitute back into inductive step} \\ &= 4(5z_k + 1) && \text{by math} \end{aligned}$$

Because the above is divisible by 4, then $5^{k+1} - 1$ is also divisible by 4.

□

2 Recursive Invariants

The function `maxOdd`, given below in pseudocode, takes as input an array A of size n of numbers. It returns the largest *odd* number in the array. If no odd numbers appear in the array, it returns negative infinity ($-\infty$). Using induction, prove that the `maxOdd` function works correctly. Clearly state your recursive invariant at the beginning of your proof.

```
Function maxOdd(A,n)
  If n = 0 Then
    Return -infinity
  Else
  Set best To maxOdd(A,n-1)
    If A[n-1] > best And A[n-1] is odd Then
      Set best To A[n-1]
    EndIf
    Return best
  EndIf
EndFunction
```

Proof. Our recursive invariant is $P(A, n)$ where if the array A contains an odd number, the function `maxOdd` will return *best* as the largest odd number in array A up to index $n-1$ where n is the length of array A. If no odd number exists in array A, `maxOdd` will return negative infinity.

1. We set the base case to $n = 0$ where the array A is an empty array containing no elements.

   According to the pseudocode, if $n = 0$, the function will return negative infinity so we know that `maxOdd` works for the base case.

2. Assume that, given an array of up to size k, the function `maxOdd` will always return the greatest odd number in the array A up to index $k - 1$.

3. If the inductive hypothesis made in step 2 is true, it will also be true for $k + 1$. We can test this by running $P(A, k + 1)$.

   In the case that the array A contains $k + 1$ number of elements and the array at index $k+1-1$ or k is an odd number that is greater than *best* up to index $k-1$, we know that for $P(A, k)$, `maxOdd` will return the greatest odd number in array A up to index $k - 1$ because of the inductive hypothesis. Adding an additional element that is an odd number greater than *best* up to index $k-1$ will reset *best* to the greatest odd number at index $(k+1)-1$ or k.

   However, if the array A contains $k + 1$ number of elements and the array at index $(k + 1) - 1$ is not an odd number greater than *best* up to index $k - 1$, we know that we know that for $P(A, k)$, `maxOdd` will return the greatest odd number in array A up to index $k-1$ because of the inductive hypothesis. Adding an additional element that is not an odd number greater than $best$ up to index $k-1$ will not change the value of $best$.

$\square$