Homework 10

CS41, Spring 2023 — Due Sunday, April 16

1. Consider the following two problems.

   i. The decision problem of determining whether there is a Hamiltonian cycle in a digraph.
   Input: A directed graph $G = (V, E)$, in adjacency list form.
   Output: TRUE, if G contains a Hamiltonian cycle; otherwise, FALSE.

   ii. Finding a Hamiltonian cycle in a digraph, if one exists.
   Input: A directed graph $G = (V, E)$, in adjacency list form.
   Output: A Hamiltonian cycle in G, if G contains a Hamiltonian cycle; otherwise, FALSE.

   Suppose that there is an algorithm CHECKHAM for problem (i) that runs in time $O(n^k)$, for some constant k and $n = |V|$. Using CHECKHAM as a subroutine, design an algorithm FINDHAM that solves problem (ii) and runs in time $O(n^\ell)$, for some constant ℓ. Explain why your algorithm and its running time.

2. The famous *traveling salesman problem (TSP)* is about finding short routes that visit each city exactly once, or equivalently, finding light Hamiltonian cycles in a weighted graph. Consider two variants of this problem:

   The Budget-TSP Search Problem
   Input: A digraph $G = (V, E)$, a weight function $w : E \to \mathbb{N}$, and a budget $b \geq 0$.
   Output: A Hamiltonian cycle of total weight at most b, if such a cycle exists in G, or FALSE.

   The TSP Optimization Problem
   Input: An $n \times n$ matrix D of non-negative integers, all of which are $\leq 2^n$.
   Output: A permutation (reordering) $(a_1, \ldots, a_n)$ of $\{1, \ldots, n\}$ that minimizes the sum

   $$D[a_n, a_1] + \sum_{i=1}^{n-1} D[a_i, a_{i+1}] .$$

   Suppose that you have an algorithm BUDGETTSP that solves the budget-TSP search problem and runs in $O((n \log b)^k)$ time, for some constant k.

   Using the algorithm BUDGETTSP as a subroutine, design an algorithm that solves the TSP optimization problem and runs in $O(n^m)$ time, for some constant m (which you should specify). Explain your algorithm and its running time.

3. For any $k \geq 0$, we define a *k-spanning tree* for an undirected graph $G = (V, E)$ to be a spanning tree $T \subseteq E$ such that every vertex in the tree (V, T) has degree $\leq k$.

   Consider the problem BDD-SPAN, of determining, given an input graph G and a parameter $k \in \mathbb{N}$, whether or not G has a k-spanning tree. Prove that HAM-PATH $\leq_p$ BDD-SPAN.