Detuning on Multi Bunch

Let M be the 2x2 one-turn map in the horizontal plane which is assumed to be linear and uncoupled. Then,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_j = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{j-1} \tag{1}$$

where $j \equiv$ turn number.

Throughout the analysis, only rigid bunches with one particle per bunch are considered and the transverse planes are assumed to be decoupled. If we consider no longitudinal motion, the longitudinal coordinates will be fixed. In that case we can assume $z_0 = 0$ without loss of generality. If we consider the positive Z axis in the direction opposite to direction of beam propagation, the Z coordinates of the bunches will be positive. The wakes produce only a kick and affect the momentum. If $\Delta x'_i$ denotes the wake kick due to the impedance in x for bunch i, the equation for the kick from [?] can be written as

$$\Delta x'_i = C \sum_{z_k < z_i} W_x(z_i - z_k, x_k, x_i) \tag{2}$$

for the conditions described, where $C = \frac{e^2}{E_0 \beta^2 \gamma}$ with $\beta = \sqrt{1-\gamma^{-2}}$, E_0 being the rest mass of the elementary particles and e the elementary charge. The wake field W_x is given by

$$W_x(z, x_k, x_i) = W_x^{dip}(z) x_k + W_x^{quad}(z) x_i \tag{3}$$

The wakes are assumed to be single turn, i.e particles from previous turns do not create any wakes, and ultra-relativistic which gives $W(z=0) = 0$. Substituting for the wake function gives,

$$\Delta x'_i = C \sum_{z_k < z_i} \left\{ W_x^{dip}(z_i - z_k) x_k + W_x^{quad}(z_i - z_k) x_i \right\} \tag{4}$$

Initially, 2 bunches are considered. The leading bunch can be denoted by the subscript 0 and the trailing bunch by the subscript 1. The wake kick on the trailing particle can then be written as,

$$\Delta x'_{1_j} = C\{W_x^{dip}(z_1) x_{0_j} + W_x^{quad}(z_1) x_{1_j}\} \tag{5}$$

where the subscript j denotes the turn number. It should be noted that the wake kick depends on the coordinates of the same turn. Because of the absence of multi-turn effect, there is no kick felt by the leading particle. The wake kick is felt once during the turn and causes a change in the momentum. Hence the wake kick should be added to the x' term in equation 1. The one turn equation for turn 1 including the wake kick can be written as,

$$\begin{aligned} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}_1 &= M * \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}_0 + \begin{pmatrix} 0 \\ \Delta x'_1 \end{pmatrix} \\ &= M * \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}_0 + \begin{pmatrix} 0 \\ C\{W_x^{dip}(z_1) x_{0_1} + W_x^{quad}(z_1) x_{1_1}\} \end{pmatrix} \\ &= M * \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}_0 + \begin{pmatrix} 0 & 0 \\ C W_x^{dip}(z_1) & C W_x^{quad}(z_1) \end{pmatrix} \begin{pmatrix} x_{0_1} \\ x_{1_1} \end{pmatrix} \end{aligned} \tag{6}$$

For convenience, consider an intermediate coordinate vector for turn 1 resulting from the transfer matrix. Then,

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix}_{1,int} = M * \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}_0 \tag{7}$$

The final coordinate vector for turn 1 can be obtained by applying the wake kick to the intermediate coordinate vector for turn 1.

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix}_1 = \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}_{1,int} + \begin{pmatrix} 0 \\ CW_x^{dip}(z_1)x_{0_{1,int}} + CW_x^{quad}(z_1)x_{1_{1,int}} \end{pmatrix} \tag{8}$$

It should be noted that equation 8 holds because the wake produces only a kick and does not affect the bunch position. As the wake kick for bunch 0 is $\Delta x_0' = 0$, the equation for particle 0 can be written as,

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}_1 = M * \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}_0 \tag{9}$$

This can be generalised to include both bunches to give the equation,

$$\begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{pmatrix}_{1,int} = T * \begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{pmatrix}_0 \tag{10}$$

where $T \equiv$ transfer matrix $= \begin{pmatrix} M & O \\ O & M \end{pmatrix}$, where M is the 2×2 one-turn map and O is a 2x2 null matrix. Using equation 10, the final turn 1 coordinates are given by,

$$\begin{aligned}
\begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{pmatrix}_1 &= \begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{pmatrix}_{1,int} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CW_x^{dip}(z_1) & 0 & CW_x^{quad}(z_1) & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{pmatrix}_{1,int} \\
&= \left\{ I + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CW_x^{dip}(z_1) & 0 & CW_x^{quad}(z_1) & 0 \end{pmatrix} \right\} \begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{pmatrix}_{1,int} \\
&= \left\{ I + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CW_x^{dip}(z_1) & 0 & CW_x^{quad}(z_1) & 0 \end{pmatrix} \right\} \{T\} \begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{pmatrix}_0
\end{aligned} \tag{11}$$

with I being an identity matrix of appropriate dimensions.

The same model can be extended to include 3 bunches. The wake kick for bunch 2 in turn j can be obtained from equation 4.

$$\Delta x_{2_j}' = C\{W_x^{dip}(z_2)x_{0_j} + W_x^{dip}(z_2 - z_1)x_{1_j} + \left[W_x^{quad}(z_2) + W_x^{quad}(z_2 - z_1)\right] x_{2_j}\} \tag{12}$$

$$\begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}_1 = \{I + W\}\{T\} \begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}_0 \tag{13}$$

$$= \{T + WT\} \begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}_0$$

Where, $W = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ CW_x^{dip}(z_1) & 0 & CW_x^{quad}(z_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ CW_x^{dip}(z_2) & 0 & CW_x^{dip}(z_2 - z_1) & 0 & CW_x^{quad}(z_2) + CW_x^{quad}(z_2 - z_1) & 0 \end{pmatrix}$ is the wake matrix, and

$T = \begin{pmatrix} M & O & O \\ O & M & O \\ O & O & M \end{pmatrix}$ similar to that defined when considering two bunches.

Extending to n bunches but still maintaining the same assumptions, the matrices can be defined as,

$$T_{n \times n} = \begin{pmatrix} M & O & \dots & O \\ O & M & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & M \end{pmatrix} \tag{14}$$

where every element is a 2x2 matrix, and

$$W_{2n \times 2n} = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ CW_x^{dip}(z_1) & 0 & CW_x^{quad}(z_1) & 0 & \dots & 0 \\ 0 & 0 & \vdots & \ddots & \dots & 0 \\ \vdots & & & & & \vdots \\ CW_x^{dip}(z_n) & 0 & CW_x^{dip}(z_n - z_1) & \dots & C\sum_{z_k < z_n} W_x^{quad}(z_n - z_k) & 0 \end{pmatrix} \tag{15}$$

The 2x2 transfer matrix M is given by

$$M = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix} \tag{16}$$

$$\det M = 1 \Rightarrow \beta\gamma - \alpha^2 = 1 \tag{17}$$

Assuming a smooth approximation, $\alpha = 0$. The β is given by,

$$\beta = \frac{R}{Q} \tag{18}$$

where $R \equiv$ machine radius and $Q \equiv$ the tune.
For the SPS with $R = \frac{6911}{2\pi}$ and assuming Q-20 optics, $Q = 20.13$ for the X axis, $\beta = 54.64$.

$$\alpha = 0 \Rightarrow \gamma = \frac{1}{\beta} = 0.0183$$

The parameter $\mu \equiv$ phase advance of each turn and is given by,

$$\mu = 2\pi Q \tag{19}$$

which for the SPS computes to $\mu = 126.48$. Substituting the values we get,

$$M = \begin{pmatrix} \cos 126.48 & 54.55 \sin 126.48 \\ -0.0183 \sin 126.48 & \cos 126.48 \end{pmatrix} = \begin{pmatrix} 0.6845 & 39.8314 \\ -0.0133 & 0.6845 \end{pmatrix} \tag{20}$$

α_X	0	α_Y	0
β_X	54.64	β_Y	54.50
γ_X	0.0183	γ_Y	0.0183
μ_X	126.48	μ_Y	126.79
Q_X	20.13	Q_Y	20.18

Table 1: Parameter values for the SPS with Q20 optics

Let $v_1, v_2, ..., v_{2n}$ be the eigenvectors and $\lambda_1, \lambda_2, ..., \lambda_{2n}$ be the corresponding eigenvalues of $\{(I+W)T\}$ where n is the number of rigid bunches. Assuming the matrix can be diagonalised, any vector can be expressed as a linear combination of the eigenvectors. Hence,

$$\begin{pmatrix} x_0 \\ x'_0 \\ \vdots \\ x_{n-1} \\ x'_{n-1} \end{pmatrix}_0 = Av_1 + Bv_2 + ... + Nv_{2n} \tag{21}$$

for vector of positions and momenta corresponding to turn 0 and

$$\{(I+W)T\}^m \begin{pmatrix} x_0 \\ x'_0 \\ \vdots \\ x_{n-1} \\ x'_{n-1} \end{pmatrix}_0 = A\lambda_1^m v_1 + B\lambda_2^m v_2 + ... + N\lambda_{2n}^m v_{2n} \tag{22}$$

for that of turn m. As $\{(I+W)T\} \in \mathbb{R}$, the eigenvalues are complex conjugates. Hence the pairs of eigenvalues for mode i can be represented by $\lambda_{i_{1,2}} = r_i e^{\pm j\mu_i}$. From the properties of eigenvalues and from our representation,

$$\begin{aligned} tr\,\{(I+W)T\} &= \sum_{i=1}^{n} \lambda_{i_1} + \lambda_{i_2} \\ &= 2\sum_{i=1}^{n} r_i \cos\mu_i \end{aligned} \tag{23}$$

and

$$
\begin{aligned}
det\left\{(I+W)T\right\} &= \prod_{i=1}^{n}\lambda_{i_1}\lambda_{i_2} \\
&= \prod_{i=1}^{n} r_i^2
\end{aligned}
\tag{24}
$$

It should be noted that in absence of wakes, the determinant is 1 and $r_i = 1$ for all i. For stability, $\lambda_{i_{1,2}}^m$ must not grow with m. Hence we get the condition $|r_i| \le 1$ from equation 22. Now if we consider the mode $i = 1$, with the eigen values $\lambda_{1_1} = r_1 e^{j\mu_1}$ and $\lambda_{1_2} = r_1 e^{-j\mu_1}$ we get,

$$
\begin{aligned}
\lambda_{1_1} + \lambda_{1_2} &= 2r_1 \cos\mu_1 \\
&= 2\sqrt{\lambda_{1_1}\lambda_{1_2}}\cos\mu_1 \\
\Rightarrow \mu_1 &= \arccos\frac{\lambda_{1_1} + \lambda_{1_2}}{2\sqrt{\lambda_{1_1}\lambda_{1_2}}}
\end{aligned}
\tag{25}
$$

The equation 25 is useful for calculating the tune of each mode from the eigenvalues. A plot of the tunes thus obtained can be seen in Fig. **??**. The obtained tunes have been sorted according to the imaginary part of the eigenvalues. The plots make it clear that in the presence of both driving and detuning wakes, it is the detuning wakes that dominate the tune shift.

To include the effect of a damper, an additional term related to the damper gain needs to be added to the matrix. An ideal damper can be seen as reducing the bunch positions by a finite value by applying a negative kick without affecting the bunch momentum. As the term due to the damper is independent of the wakes, the damper term can be added to the transfer matrix. If we consider a single bunch with the coordinates $\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$ with no wakes, then the coordinates for turn j in terms of turn $(j-1)$ are given by

$$
\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}_j = M * \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}_{j-1}
\tag{26}
$$

where $M \equiv$ transfer. Let the transfer matrix be given by

$$
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\tag{27}
$$

We can write the bunch coordinates as,

$$
\begin{aligned}
x_{0_j} &= a x_{0_{j-1}} + b x'_{0_{j-1}} \\
x'_{0_j} &= c x_{0_{j-1}} + d x'_{0_{j-1}}
\end{aligned}
\tag{28}
$$

As the damper acts on and affects only the bunch position and not the momentum, after introduction of the damper, the bunch position is given by,

$$
\begin{aligned}
x_{0_j} &= a x_{0_{j-1}} + b x'_{0_{j-1}} - g x_{0_{j-1}} \\
&= (a-g) x_{0_{j-1}} + b x'_{0_{j-1}}
\end{aligned}
\tag{29}
$$

Where $g \equiv$ damper gain. The new transfer matrix is then given by,

$$
M_{damper} = \begin{pmatrix} a-g & b \\ c & d \end{pmatrix}
\tag{30}
$$

All the above analysis is valid only when considering single-turn wakes as the equation 4 is holds only for single-turn wakes. In practice, wakes are often multi-turn as they decay over several turns and not just a single turn. When considering wake memory, all the bunches need to be considered over n_{wake} turns. It is convenient to develop a model by separating the dipolar and quadrupolar wake kicks. The new expressions for the kick experienced by bunch i in turn j are given by,

$$\begin{aligned}
\Delta^{dip} x_{i_j} &= C \sum_{k=0}^{n_{wake}-1} \sum_{s=0}^{n-1} W_x^{dip}(kS + (z_i - z_s)) x_{s_{j-k}} \\
\Delta^{quad} x_{i_j} &= C \sum_{k=0}^{n_{wake}-1} \sum_{s=0}^{n-1} W_x^{quad}(kS + (z_i - z_s)) x_{i_j}
\end{aligned} \tag{31}$$

where $\Delta^{dip} x_{i_j}$ is the dipolar kick, $\Delta^{quad} x_{i_j}$ is the quadrupolar kick and $S \equiv$ circumference of the machine. It should be noted that because of the multi-turn effect, a particular bunch i will be affected by the wakes of all other bunches and not just the ones ahead of it. From equation 31, the wake matrix of equation 15 needs to be rewritten as a function of the turn number k. As the dipolar wake kick depends on previous turn coordinates too, while the quadrupolar wake kick depends only on the present turn, it is convenient to define two different matrices, $W_{2n\times 2n}^{dip}(k)$ for the dipolar wakes and $W_{2n\times 2n}^{quad}(k)$ for the quadrupolar wakes

$$\mathrm{W}_{2n\times 2n}^{dip}(k) = \begin{pmatrix}
0 & 0 & \dots & \dots & \dots & 0 \\
CW_x^{dip}(kS+0) & 0 & CW_x^{dip}(kS+(-z_1)) & \dots & CW_x^{dip}(kS+(-z_{n-1})) & 0 \\
0 & 0 & \dots & \dots & \dots & 0 \\
CW_x^{dip}(kS+z_1) & 0 & CW_x^{dip}(kS+0) & \dots & CW_x^{dip}(kS+(z_1-z_{n-1}) & 0 \\
0 & 0 & \vdots & \ddots & \dots & 0 \\
\vdots & & & & & \vdots \\
CW_x^{dip}(kS+z_{n-1}) & 0 & CW_x^{dip}(kS+(z_{n-1}-z_1)) & \dots & CW_x^{dip}(kS+0) & 0
\end{pmatrix} \tag{32}$$

$$\mathrm{W}_{2n\times 2n}^{quad}(k) = \begin{pmatrix}
0 & 0 & \dots & \dots & 0 \\
\sum_{s=0}^{n-1} W_x^{quad}(kS+(-z_s)) & 0 & \dots & \dots & 0 \\
0 & 0 & \dots & \dots & 0 \\
0 & 0 & \sum_{s=0}^{n-1} W_x^{quad}(kS+(z_1-z_s)) & \dots & 0 \\
0 & 0 & \ddots & \dots & 0 \\
\vdots & & & & \vdots \\
0 & 0 & 0 & \sum_{s=0}^{n-1} W_x^{quad}(kS+(z_{n-1}-z_s)) & 0
\end{pmatrix} \tag{33}$$

Let the vector of coordinates for turn j be defined by X_j where $X_j = \begin{pmatrix} x_0 \\ x'_0 \\ \vdots \\ x_{n-1} \\ x'_{n-1} \end{pmatrix}_j$. The dipolar and quadrupolar kicks can then be written in matrix form as,

$$\Delta^{dip} X_j = \sum_{k=0}^{n_{wake}-1} W_{2n\times 2n}^{dip}(k) X_{j-k} \tag{34}$$

$$\Delta^{quad} X_j = \sum_{k=0}^{n_{wake}-1} W_{2n\times 2n}^{quad}(k) X_j \tag{35}$$

respectively. The intermediate coordinate vector defined similar to equation 7 after applying the transfer matrix is then given by,

$$X_{j,int} = TX_{j-1} \tag{36}$$

The final coordinates for the j^{th} turn are obtained by applying the kick,

$$\begin{aligned} X_j &= X_{j,int} + \sum_{k=0}^{n_{wake}-1} W^{quad}_{2n\times 2n}(k) X_{j,int} + W^{dip}_{2n\times 2n}(0) X_{j,int} + \sum_{k=1}^{n_{wake}-1} W^{dip}_{2n\times 2n}(k) X_{j-k} \\ &= \left\{ I + \sum_{k=0}^{n_{wake}-1} W^{quad}_{2n\times 2n}(k) + W^{dip}_{2n\times 2n}(0) \right\} TX_{j-1} + \sum_{k=1}^{n_{wake}-1} W^{dip}_{2n\times 2n}(k) X_{j-k} \end{aligned} \tag{37}$$

Of course, the equation 37 is valid only because the quadrupolar kick always depends on the instantaneous position of the bunch and not on the positions in previous turns. Let $Y_{j-1} = \begin{pmatrix} X_{j-1} \\ X_{j-2} \\ \vdots \\ X_{j-(n_{wake}-1)} \end{pmatrix}$ be defined as the vector of the coordinates of all previous $n_{wake}-1$ turns. The equation 37 can then be written in a matrix form as,

$$Y_j = \{A + B\} Y_{j-1} \tag{38}$$

where

$$A = \begin{pmatrix} \left\{ I + \sum_{k=0}^{n_{wake}-1} W^{quad}_{2n\times 2n}(k) + W^{dip}_{2n\times 2n}(0) \right\} T & O & \dots & O & O \\ I & O & \dots & O & O \\ O & I & \dots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \dots & I & O \end{pmatrix}$$

and

$$B = \begin{pmatrix} W^{dip}_{2n\times 2n}(1) & W^{dip}_{2n\times 2n}(2) & \dots & W^{dip}_{2n\times 2n}(n_{wake}-1) \\ O & O & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & O \end{pmatrix}$$

with $I \equiv 2n \times 2n$ identity matrix and $O \equiv 2n \times 2n$ null matrix.