Detuning on Multi Bunch

Let M be a 2x2 transfer matrix. Then,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_j = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{j-1} \tag{1}$$

where $j \equiv$ turn number.
Wake kick in x for particle i, $\Delta x'_i$ due to the impedance is given by

$$\Delta x'_i = C \sum_{z_s > z_{S_i}} n_S W_x(z_{Si} - z_S, x_S, x_{S_i}) \tag{2}$$

assuming decoupled planes. The wake field W_x is given by

$$W_x(z, x_S, x_{S_i}) = W_x^{dip}(z) x_S + W_x^{quad}(z) x_{S_i} \tag{3}$$

Substituting for the wake function gives,

$$\Delta x'_i = C \sum_{z_s > z_{S_i}} \left\{ n_S W_x^{dip}(z_{Si} - z_S) x_S + n_S W_x^{quad}(z_{Si} - z_S) x_{S_i} \right\} \tag{4}$$

If we consider only two macroparticles and rigid bunches, $n_S = 1$ and the S subscript can be dropped. The leading particle can be denoted by the subscript 0 and the trailing particle by the subscript 1. Further, if we consider no longitudinal motion, the longitudinal coordinates will be fixed. In that case we can assume $z_0 = 0$ without loss of generality. If we consider the positive Z axis in the direction opposite to direction of beam propagation, the Z coordinates of particles will be positive. Considering these simplifications, the wake kick on the trailing particle can be written as,

$$\Delta x'_1 = C\{W_x^{dip}(z_1) x_0 + W_x^{quad}(z_1) x_1\} \tag{5}$$

For maintaining causality, there is no kick felt by the leading particle. The wake kick is felt once during the turn and causes a change in the momentum. Hence the wake kick should be added to the x' term in equation 1. The one turn equation for turn 1 including the wake kick can be written as,

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}_1 &= M * \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}_0 + \begin{pmatrix} 0 \\ \Delta x'_1 \end{pmatrix}_0 \\
&= M * \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}_0 + \begin{pmatrix} 0 \\ C\{W_x^{dip}(z_1) x_0 + W_x^{quad}(z_1) x_1\} \end{pmatrix}_0 \\
&= M * \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}_0 + \begin{pmatrix} 0 & 0 \\ C W_x^{dip}(z_1) & C W_x^{quad}(z_1) \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}_0
\end{aligned} \tag{6}$$

The wakes for particle 0 are very small and can be neglected. Hence, the wake kick is $\Delta x_0' = 0$. Accordingly, the equation for particle 0 can be written as,

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}_1 = M * \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}_0 \tag{7}$$

Combining equations 6 and 7, and rearranging gives,

$$\begin{aligned}
\begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{pmatrix}_1 &= T * \begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{pmatrix}_0 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CW_x^{dip}(z_1) & 0 & CW_x^{quad}(z_1) & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{pmatrix}_0 \\
&= \left\{ T + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CW_x^{dip}(z_1) & 0 & CW_x^{quad}(z_1) & 0 \end{pmatrix} \right\} \begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \end{pmatrix}_0
\end{aligned} \tag{8}$$

In equation 8, $T = \begin{pmatrix} M & O \\ O & M \end{pmatrix}$, where M is the 2x2 transfer matrix and O is a 2x2 null matrix. The same model can be extended to include 3 particles. The wake kick for particle 2 can be obtained from equation 4.

$$\Delta x_2' = C\{W_x^{dip}(z_2)x_0 + W_x^{dip}(z_2 - z_1)x_1 + \left[W_x^{quad}(z_2) + W_x^{quad}(z_2 - z_1)\right] x_2\} \tag{9}$$

$$\begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}_1 = \{T + W\} \begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}_0 \tag{10}$$

Where, $W = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ CW_x^{dip}(z_1) & 0 & CW_x^{quad}(z_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ CW_x^{dip}(z_2) & 0 & CW_x^{dip}(z_2 - z_1) & 0 & CW_x^{quad}(z_2) + CW_x^{quad}(z_2 - z_1) & 0 \end{pmatrix}$ is the wake matrix and $T = \begin{pmatrix} M & O & O \\ O & M & O \\ O & O & M \end{pmatrix}$ similar to that defined when considering two particles.

Extending to n macroparticles but still maintaining 1 particle per slice, the matrices can be defined as,

$$T_{n \times n} = \begin{pmatrix} M & O & \dots & O \\ O & M & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & M \end{pmatrix} \tag{11}$$

where every element is a 2x2 matrix and

$$W_{2n\times 2n} = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ CW_x^{dip}(z_1) & 0 & CW_x^{quad}(z_1) & 0 & \dots & 0 \\ 0 & 0 & \vdots & \ddots & \dots & 0 \\ \vdots & & & & & \vdots \\ CW_x^{dip}(z_n) & 0 & CW_x^{dip}(z_n - z_1) & \dots & C\sum_{z_n > z_S} W_x^{quad}(z_n - z_s) & 0 \end{pmatrix} \tag{12}$$

The 2x2 transfer matrix M is given by

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \tag{13}$$

$$\det M = 1 \Rightarrow \beta\gamma - \alpha^2 = 1 \tag{14}$$

Assuming a smooth approximation, $\alpha = 0$. The β is given by,

$$\beta = \frac{R}{Q} \tag{15}$$

where $R \equiv$ machine radius and $Q \equiv$ the tune.
For the SPS with $R = \frac{6911}{2\pi}$ and assuming Q-20 optics, $Q = 20.13$ for the X axis, $\beta = 54.64$.

$$\alpha = 0 \Rightarrow \gamma = \frac{1}{\beta} = 0.0183$$

The parameter $\mu \equiv$ phase advance of each turn and is given by,

$$\mu = 2\pi Q \tag{16}$$

which for the SPS computes to $\mu = 126.48$. Substituting the values we get,

$$M = \begin{pmatrix} \cos 126.48 & 54.55\sin 126.48 \\ -0.0183\sin 126.48 & \cos 126.48 \end{pmatrix} \tag{17}$$

α_X	0	α_Y	0
β_X	54.64	β_Y	54.50
γ_X	0.0183	γ_Y	0.0183
μ_X	126.48	μ_Y	126.79
Q_X	20.13	Q_Y	20.18

Table 1: Parameter values for the SPS with Q20 optics

Let $v_1, v_2, ..., v_{2n}$ be the eigenvectors and $\lambda_1, \lambda_2, ..., \lambda_{2n}$ be the corresponding eigenvalues of $T+W$ where n is the number of rigid bunches. Then,

$$\begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \\ x_2 \\ x_2' \\ \vdots \\ x_n \\ x_n' \end{pmatrix}_0 = Av_1 + Bv_2 + ... + Nv_{2n} \tag{18}$$

for one turn and

$$\{T+W\}^m \begin{pmatrix} x_0 \\ x_0' \\ x_1 \\ x_1' \\ x_2 \\ x_2' \\ \vdots \\ x_n \\ x_n' \end{pmatrix}_0 = A\lambda_1^m v_1 + B\lambda_2^m v_2 + ... + N\lambda_n^m v_n \tag{19}$$

for m turns. As $T+W \in \mathbb{R}$, the eigen values are complex conjugates. Hence the pairs of eigenvalues for mode i can be represented by $\lambda_{i_{1,2}} = r_i e^{\pm j\mu_i}$. From the properties of eigenvalues,

$$\begin{aligned} tr(T+W) &= \sum_{i=1}^{2n} \lambda_i \\ &= 2\sum_{i=1}^{n} r_i \cos\mu_i \end{aligned} \tag{20}$$

and

$$\begin{aligned} det(T+W) &= \prod_{i=1}^{2n} \lambda_i \\ &= \prod_{i=1}^{n} r_i \end{aligned} \tag{21}$$

It should be noted that in absence of the wakes, the determinant is 1 and $r_i = 1$ for all i. For stability, $\lambda_{i_{1,2}}^m$ must not grow with m. Hence we get the condition $|r_i| \leq 1$. Now if we consider the mode $i = 1$, with the eigen values $\lambda_{1_1} = r_1 e^{j\mu_1}$ and $\lambda_{1_2} = r_1 e^{-j\mu_1}$ we get,

$$\begin{aligned} \lambda_{1_1} + \lambda_{1_2} &= 2r_1 \cos\mu_1 \\ &= 2\sqrt{\lambda_{1_1}\lambda_{1_2}} \cos\mu_1 \\ \Rightarrow \mu_1 &= \arccos\frac{\lambda_{1_1} + \lambda_{1_2}}{2\sqrt{\lambda_{1_1}\lambda_{1_2}}} \end{aligned} \tag{22}$$

To include the effect of a damper, an additional term related to the damper gain needs to be added to the matrix. The damper reduces the bunch positions by a finite value by applying a negative kick but does not affect the bunch momentum. As the term due to the damper is independent of the wakes, the damper term can be added to the transfer matrix. If we consider a single bunch with the coordinates $\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$ with no wakes, then the coordinates for turn i in terms of turn $(i-1)$ are given by

$$\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}_i = M * \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}_{i-1} \tag{23}$$

where $M \equiv$ transfer. Let the transfer matrix be given by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{24}$$

We can write the partcle coordinates as,

$$\begin{aligned} x_{0_i} &= a x_{0_{i-1}} + b x'_{0_{i-1}} \\ x'_{0_i} &= c x_{0_{i-1}} + d x'_{0_{i-1}} \end{aligned} \tag{25}$$

As the damper acts on and affects only the bunch position and not the momentum, after introduction of the damper, the bunch position is given by,

$$\begin{aligned} x_{0_i} &= a x_{0_{i-1}} + b x'_{0_{i-1}} - g x_{0_{i-1}} \\ &= (a-g) x_{0_{i-1}} + b x'_{0_{i-1}} \end{aligned} \tag{26}$$

Where $g \equiv$ damper gain. The new transfer matrix is then given by,

$$M_{damper} = \begin{pmatrix} a-g & b \\ c & d \end{pmatrix} \tag{27}$$