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Discrete Time - Examples for frequency warping

Signals and Control Systems 1

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Abstract

In DSP, it is necessary to convert one signal spectrum into another. Because current digital circuits, such as microprocessor-controlled circuits, Arduinos, Raspberry Pis, and other digital logic circuits, require digital input for processing, the most common conversion is from analog to digital. ADC is used to convert an analog signal to a digital signal. However, utilising merely an ADC is insufficient because there are still some undesirable noise signals present during the conversion. Filters are used both before and after the ADC to remove these undesired elements. There are different techniques to convert analog filters to the digital filter. Among them, bilinear transformation is discussed in this report. While applying the bilinear transformation to analog filter, a non-linear distortion occurs at higher frequencies when an analog signal is converted to a digital signal. This phenomenon, known as frequency warping, is explained in this study. The overall implementation showing the occurrence of warping and how it is eliminated is done using Python.

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List of Abbreviations

IIR	Infinite Impulse Response
FIR	Finite Impulse Response
ADC	Analog to Digital Converter
DSP	Digital Signal Processing
Op-amp	Operational Amplifiers
MCU	Micro Controller Unit
SoC	System on Chip

1. Introduction

In signal processing, filter is a device or process that removes, amplifies or attenuates some unwanted components or features from the signal or spectrum [7][8]. An example is shown in Fig.1

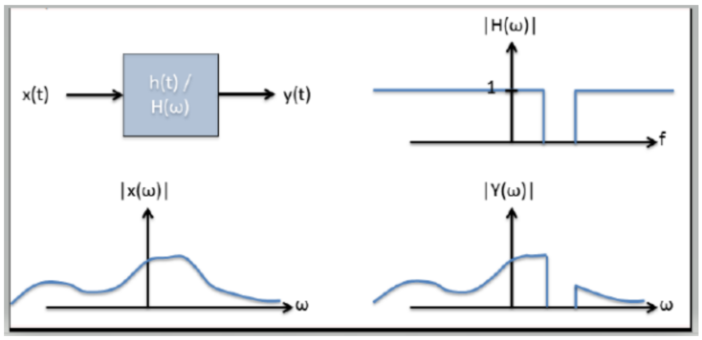

Fig. 1 Filter Example [8]

Filtering in signal processing is defined as a process or technique that remove certain band of frequencies and allow other frequencies to pass through. Which frequencies are removed and which frequencies are left depends on the type of filter which is used. There are numerous bases of classifying filters, all of which may overlap in various ways, but there is no clear hierarchical classification. Filters may be linear or non-linear, time variant or time invariant, causal or non causal and so on. One of the important bases of classifying filter is analog or digital filter. From simple inductor-capacitor networks to more complex active filters with op-amps, analog filters have been around over the history of electronics. Due to the performance need of latest application and decreasing product costs, digital filters embedded into micro-controllers's application codes are becoming the norm.

Filters in both continuous and discrete domain removes unwanted noise or signal components but they work differently in each domain. Rather than making new digital filter, an existing analog filter can be converted into digital filters. There have been several different methods of converting analog prototype filter to digital filter.

Bilinear transform is one of the simplest and most universally used transformation method for converting an analog prototype filter into desired digital filter. In the process of transforming analog filter into digital filter an undesirable phenomenon known as frequency warpping is introduced. But the effect of frequeny warping can be overcome by tweaking the frequency component of digital filter which is known as pre-warping. This reports aims to discuss need of digital filter, various conversion techniques and other aspects in brief. Bilinear transform is discussed in great depth as a part of the report. The concept is realized and implemented in python using jupyter notebook. Quick observation has been made which is supported by graphs to have better overview.

2. Advantage of Digital Filter over Analog Filter

Analog filters are circuits made up of analog components such resistors, capacitors, inductors, and op-amps. A 2^{nd} order analog lowpass filter is shown in Fig.2 , and in its simplest form, only 5 components are needed to build the filter. Analog filters have an excellent resolution, good electromagnetic capability properties as there is no clock generating noise. Analog filters are inexpensive, quick, and have a wide dynamic range in frequency and amplitude. Analog filters have several significant drwabacks that affect filter performance such as components aging, temperature drift, and component tolerance. These disadvantages necessitates need of digital filter. Frequency response of analog filter is always fixed, for example a butterworth filter will always be a butterworth filter until and unless components are physically changed.

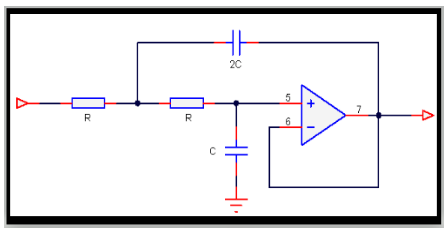

Fig. 2 Analog low-pass filter [9]

Digital filters are frequently found in digital signal processing chips such as MCUs, SoCs, processors, and DSPs. Digital Filters are adaptive and flexible, we can design and implement a filter with any frequency response we want, deploy it, and then change the filter coefficients without changing any physical components. These davantages make digital filter much more attractive choice for modern applications.

3. Analog Filter to Digital Filter Conversion Techniques

There are various filter conversion techniques available for analog to digital filter transformations. Different aspects of analog and digital filters are preserved for these transitions to happen. If we want that the shape of impulse response of analog filter remains same from analog to digital filter, then impulse invariance transformation can be used. If we want to map the differential equation representation of analog filter to corresponding difference equation representation of digital filter, then we can use the finite difference approximation techniques. There are numerous other methods that are also possible. Step Invariance technique is proposed if we want to preserve the shape of the step response. Like impulse invariance is the matched-z transform, which matches the pole zero representation. The simplest, popular and universally used transformation method is Bilinear Transformation. Some of the conversion techinuies are briefly explained as :

3. 1 Impulse Invariance Method

It is a technique of designing a discrete-time-infinite-response (IIR) filter from continuous-time filters in which the impulse response of the continuous-time system is sampled to produce the impulse response of the discrete-time system. The frequency response of the discrete-time system will be a sum of shifted copies of the frequency response of the continuous-time system. This method maps the continuous filter transfer function into the *z*-domain to have similar impulse response. After H(s) is expanded using a partial fraction, analog poles can be transformed into digital poles to obtain H(z). The advantages of this method are that it produces a steady transformation and that the analog and angular frequencies are linearly connected. However, it comes with the drawback of aliasing. As a result, this approach is only employed when the analog filter is of the lowpass or bandpass variety.

Advantages of Impulse Invariance method :

1. Preserve shape of filter frequency response in digital filter.
2. There is no frequency warping

Disadvantages of bi-linear transformation method :

1. There is aliasing effect.
2. The mapping is many to one.
3. Suitable only for lowpass and bandpass filter

3.2 Bilinear Transform

The method of filter design by impulse invariance suffers from aliasing. Hence in order to overcome this drawback Bilinear transformation method is designed. The bilinear transformation method is an alternative to impulse invariance that uses a different mapping that maps the continuous-time system's frequency response in the discrete-time case. The bilinear transformation is the method of squashing the infinite straight analog frequency axis so that it becomes finite.

Advantages of bi-linear transformation method :

1. The mapping is one to one
2. There is no aliasing effect
3. Stable analog filter is transformed into the stable digital filter
4. There is no restriction one type of filter that can be transformed
5. There is one to one transformation from the s-domain to the Z- domain

Disadvantages of bi-linear transformation method :

The mapping is non linear in this method because of this frequency warping effect takes place

3.3 Window Method

The window approach for digital filter design is quick, easy, and reliable, but it's not always the best option. It's simple to understand in terms of the convolution theorem for Fourier transforms, therefore it is a good follow-up to Fourier theorems and windows for spectrum analysis. It can be used in conjunction with the frequency sampling approach to great advantage. The duration of window functions is always restricted. In contrast to an infinite-impulse-response (IIR) digital filter, the window approach always creates a finite-impulse-response (FIR) digital filter.

There are many other techniques both with advantages and disadvantages. Based on application the techniques can be selected. Some of pros and cons of few transformations methods have been illustrated in Table 1.

Techniques	Mapping	Advantages	Disadvantages
Impulse Invariance	$z = e^{st_0}$	Preserves Shape of filter	Aliasing
Approximation of Derivatives	$s = \frac{1 - z^{-1}}{t_0}$	No Aliasing	Restricted pole location, shape distortion
Matched Z transform	$(s-a) \to (1 - e^{at_0})$	Directly maps pole and zero locations	Aliasing
Bilinear Transform	$s = \frac{2}{t_0}\frac{1 - z^{-1}}{1 + z^{-1}}$	No aliasing	Shape distortion

Table 1, Pros and Cons of Transformation Methods

4. Bilinear Transform

Bilinear Transformation also known as linear fractional transformation, is used for converting an analog prototype filter into a desired digital filter. The basic concept of bilinear transform is to map a given transfer function $H(s) = L\{h(t)\}$ of linear time invariant (LTI) filter given in a continuous time domain to the transfer function $H(z) = Z\{h[k]\}$ of a linear shift invariant filter in discrete time domain. The transform ensures to preserve certain properties of the analog filter into digital filter. Particularly,

1. A rational function $H(s)$ in s should be mapped into a rational function $H(z)$ in z^{-1} of the same order.
2. Poles and Zeros should be mapped from the left s-half plane into the unit circle in the z-plane.

The above mentioned first criterion ensures that the difference equation can be realized by a difference equation while the second assures to preserve the important system properties (such as causality and stability).

4.1 Derivation

We shall derive the relationship between the transfer function of analog filter in Laplace transform and transfer function of digital filter in Z-transform.

An analog system can be represented by differential equation in time domain.

$$\frac{dy}{dt} = u(\tau)$$

Integrating both sides with respect to time

$$y(t) = \int_0^t u(\tau)d\tau$$

where $y(t)$ is the output for the input $u(t)$

In Laplace domain, we can represent the input and output signal as follows:

$$U(s) \triangleq L\{u(t)\}$$

and

$$Y(s) \triangleq L\{y(t)\}$$

$$= L\{\int_0^t u(\tau)\, d\tau$$

$$= \frac{1}{s} U(s)$$

Therefore, the transfer function is

$$H_a(s) \triangleq \frac{Y(s)}{U(s)} = \frac{1}{s}\frac{U(s)}{U(s)} = \frac{1}{s} \quad (1)$$

Evaluating $y(t)$ at specified time $t = kT_s$, $y(t)$ can be written as

$$y[k] \triangleq y(kT_s)$$

$$= \int_0^{kT_s} u(\tau)d\tau$$

By splitting up the integral:

$$y[k] = \int_0^{(k-1)T_s} u(\tau)d\tau + \int_{(k-1)T_s}^{kT_s} u(\tau)d\tau$$

Resulting in the recursive formula for $y[k]$:

$$y[k] = y[k-1] + \int_{(k-1)T_S}^{kT_S} u(\tau)d\tau \qquad (2)$$

The second term expresses the area under the curve of $u(t)$ between $t=(k-1)T_s$ and $t=kT_s$. It can be approximated in several different ways, using numerical integration techniques. In the case of the bilinear transform, the trapezoidal rule is used, with a step size (sample time) of T_s . This is illustrated in the following Fig. 1.

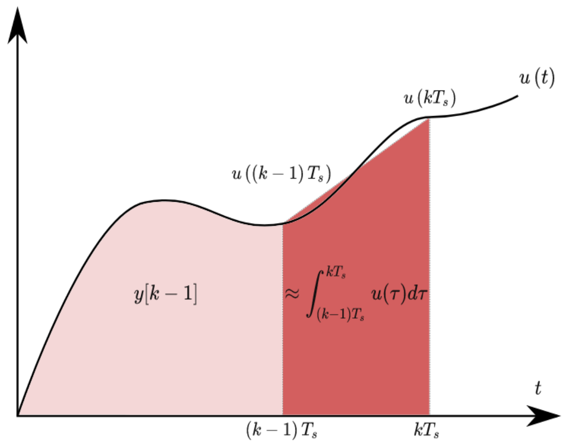

Fig. 3 Area under the curve of $u(t)$

The left of Fig 1 represents the first term of equation (2), and the right darker red trapezoid on the right approximates the second term of the equation.

The area of this trapezoid is given by:

$$A = T_s \frac{u((k-1)T_S)+u(kT_S)}{2} = \frac{T_S}{2}(u[k-1]+u(k))$$

$$u[k] \triangleq \mathrm{u}[kT_S]$$

Therefore, we can approximate Equation (2):

$$y(k) \approx y[k-1] + \frac{T_S}{2}(u[k-1]+u(k)) \qquad (3)$$

We now have an approximation of the recurrence relation for the integrator.

In Equation (1), we found that the transfer function of an integrator in the Laplace domain was $H_a(s) = \frac{1}{s}$. We can now apply the Z-transform to the recurrence relation in Equation (3) to relate the Laplace domain to the Z-domain.

$$Z\{y[k] \approx y[k-1] + \tfrac{T_s}{2}(u[k-1] + u(k))\}$$

$$\Leftrightarrow \; Y(z) \approx z^{-1}Y(z) + \tfrac{T_s}{2}(z^{-1}U(z) + U(z))$$

$$\Leftrightarrow \; (1 - z^{-1})Y(z) \approx \tfrac{T_s}{2}(1 + z^{-1})U(z)$$

$$H_d(z) \triangleq \frac{Y(z)}{U(z)} \approx \frac{T_s}{2}\frac{z+1}{z-1} \tag{4}$$

If we compare the continuous-time transfer function $H_a(s)$ from Equation (1) to the approximated discrete-time transfer function $H_d(z)$ from equation (4) we can find the approximation of *s* in the function of *z*, that can be used to discretize a continuous-time transfer function:

$$s = \frac{2}{T_s}\frac{z-1}{z+1}$$

We can also express *z* in the function of *s*:

$$z = \frac{2 + sT}{2 - sT}$$

The above derivation reveals the link between the complex frequencies *s* and *z* of the Laplace and *z*-transformation, respectively. It is given as:

$$z = e^{sT}$$

where T denotes the sampling interval.

For sampled signals, the resulting mapping from the s-plane into the z-plane is shown in Fig 2

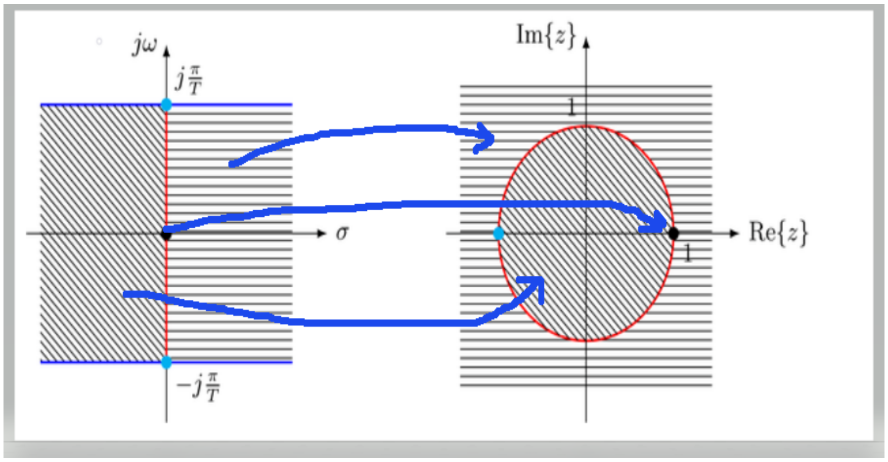

Fig. 4 Relationship between s plane and z plane

The shading indicates how the different areas are mapped. The imaginary axis *s=jω* is mapped onto the unit circle $z = e^{j\Omega}$ representing the frequency response of the continuous and discrete system. The left half-plane of the *s*-plane is mapped into the unit circle of the *z*-plane.

4.2 Frequency Response

Let's consider the mapping of the frequency response $\boldsymbol{H(j\omega) = H(s)|_{s=(j\omega)}}$ of a continuous system to the frequency response $\boldsymbol{H_d(e^{j\omega T}) = H_d|_{z=e^{j\Omega}}}$ of a discrete system. Introducing the bilinear transform into the continuous system to yield its discrete counterpart results in

$$H_d(e^{j\Omega}) = H\left(\frac{2}{T}\frac{z-1}{z+1}\right)|_{z=e^{j\Omega}}$$

$$= H(j\frac{2}{T}tan\frac{\Omega}{2})$$

The imaginary axis *s=jω* of the *s*-plane is mapped onto the unit circle $\boldsymbol{e^{j\Omega}}$ of the z-plane. It is observed that for sampled signals the mapping between the continuous frequency axis *ω* and the frequency axis Ω of the discrete system is Ω=ωT. However, for the bilinear transform the mapping is non-linear

$$\omega = \frac{2}{T}tan\frac{\Omega}{2}$$

$$\Omega = 2tan^{-1}(\frac{\omega T}{2})$$

In the Fig. 3 the mapping between the continuous frequency axis *ω* and the frequency axis Ω of the discrete system is shown for T = 1.

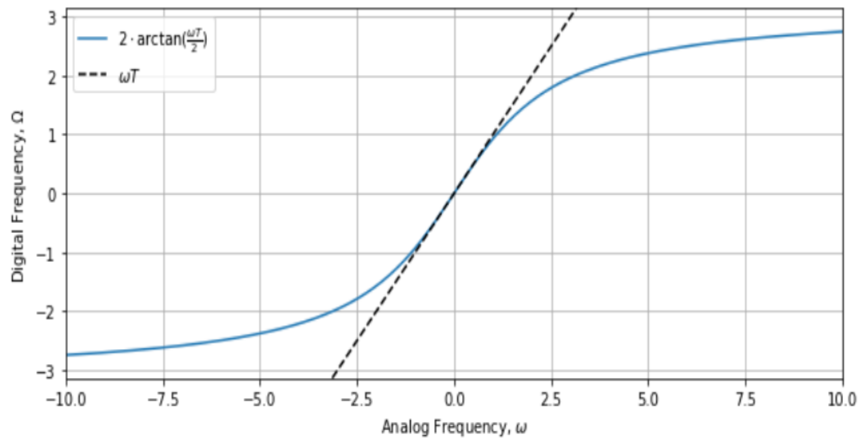

Fig. 5 Relation between Analog Frequency and Digital Frequency

It is evident from the Fig. 3 that the frequency axis deviates from the linear mapping $\Omega = \omega T$, especially for high frequencies. The conversion of $H(s)$ to $H(z)$ with the bilinear transformation does not change the value of frequency response, but it changes the frequencies at which the value changes. The frequency response of the digital filter $H_d(e^{j\Omega})$ therefore deviates from the desired continuous frequency response $H(j\omega)$. This is due to the first-order approximation of the mapping from the *s*-plane to the *z*-plane.

It is observed that near zero frequency the relation of angular frequency ω and digital frequency, $\Omega = 2tan^{-1}(\frac{\omega T}{2})$ is linear. The compression increases as the digital frequency, Ω reaches $\frac{\pi}{2}$. This nonlinear compression effect is known as frequency warping.

Besides this drawback, the bilinear transform has several benefits:

- Stability and minimum phase of the continuous filter is preserved. This is due to mapping of the left half-space of the *s*-plane into the unit-circle of the *z*-plane.
- The order of the continuous filter is preserved. This is due to the linear mapping rule.
- No aliasing distortion as for instance as observed for impulse invariance method.

In the design of a digital filter, the effects of the frequency warping must be considered. The prototype filter frequency scale must be pre-warped so that after the bilinear transform, the critical frequencies are in the correct places.

4.3 Characteristic of Bilinear Transform

The characteristics of the bilinear transform are following

1. The digital filter has the same arrangement as the prototype filter.
2. On the z-plane, the left-half s-plane is mapped into the unit circle. This ensures that stability is maintained.

3. Optimal approximations to piecewise constant prototype filters, such as the four cases in Continuous Frequency Definition of Error [2], become optimal digital filters.
4. The bilinear transform creates a cascade of sections that is identical to the complete system transformation.

The bilinear transform is most used to convert a prototype continuous Laplace transform transfer function to a digital transfer function. Because of the above-mentioned characteristics in point 3 which specifies that optimality is preserved, it is the one used in most popular filter design programs. The maximally flat prototype is transformed into a maximally flat digital filter. This property only holds for approximations to piecewise constant ideal frequency responses because the frequency warping does not change the shape of a constant. If the prototype is an optimal approximation to a differentiator or to a linear-phase characteristic, the bilinear transform will destroy the optimality. Those approximations must be made directly in the digital frequency domain.

5. Design of Digital Filter using Bilinear Transform

Let us design a digital filter $H_d(z)$ that approximates a given analog filter $H_a(s)$ using bilinear transform. Transfer function $H_a(s)$ can be obtained from the analysis of an analog circuit or the filter design methods. Transfer function in the digital domain $H_d(z)$ can be obtained as:

$$H_d(z) = H(s)\big|_{s=\frac{2}{T}\frac{z-1}{z+1}}$$

The coefficients of digital filters are obtained by representing the numerator and denominator of discrete-time rational transfer function $H_d(z)$ as a polynomial with respect to z^{-1}.

For example, the Transfer function for a continuous system of second-order is given as

$$H(s) = \frac{\beta_0 + \beta_1 s + \beta_2 s^2}{\alpha_0 + \alpha_1 s + \alpha_2 s^2}$$

and the bilinear transform results in

$$H_d(z) = \frac{(\beta_2 K^2 - \beta_1 K + \beta_0)z^{-2} + (2\beta_0 - 2\beta_2 K^2)z^{-1} + (\beta_2 K^2 + \beta_1 K + \beta_0)}{(\alpha_2 K^2 - \alpha_1 K + \alpha_0)z^{-2} + (2\alpha_0 - 2\alpha_2 K^2)z^{-1} + (\alpha_2 K^2 + \alpha_1 K + \alpha_0)} \qquad (5)$$

where $K = \frac{2}{T}$

As mentioned in section 3.2 Frequency response, the frequency response of the digital filter $H_d(e^{j\Omega})$ will differ for high frequencies from the desired analog frequency response $H(j\omega)$.

In conclusion, if we want our digital filter to have specific characteristics at a given corner frequency ω_c we have to use different frequency ω_{cw}, when designing the filter in the analog domain. This technique is known as pre-warping and the frequency ω_{cw}, is known as warped corner frequency. The pre-warped frequency is given by:

$$\omega_{cw} = \frac{2}{T}\, tan\,(\frac{\omega_c T}{2})$$

where ω_c is in radians per second. The actual frequency f_c in Hertz (cycles per second) is given by

$$f_c = \frac{2\pi}{\omega_c}$$

In the case of an elliptic-function filter where there are two critical frequencies that determine the transition region, then both the frequencies must be pre-warped.

5.1 Digital Realization of an Analog System

A second-order analog filter can be realized by the following passive circuit as illustrated in Fig. 4

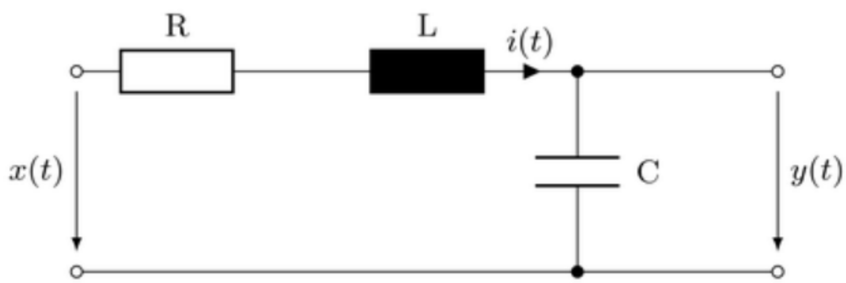

Fig. 6 Low pass filter realization by RLC circuit

Where $x(t)$ denotes the input signal and $y(t)$ denotes the output signal. The transfer function of the circuit is given as:

$$H(s) = \frac{Y(s)}{X(s)}$$

$$= \frac{1}{LCs^2 + RCs + 1}$$

Using bilinear transform of the second-order as mentioned in equation 5, transfer function in the digital domain is given by:

$$H_d(z) = \frac{T^2z^{-2} + 2T^2z^{-1} + T^2}{(4LC - 2TRC + T^2)z^{-2} + (-8LC + 2T^2)z^{-1} + (4LC - 2TRC + T^2)}$$

Considering two different cut-off frequencies of low pass filter:

Case I:

Sampling frequency, f_s = 44100 Hz

Cut-off frequency, f_c = 1000 Hz

The low pass analog filter and the corresponding digital filter is realized in python, and the graph is plotted as shown in Fig. 5

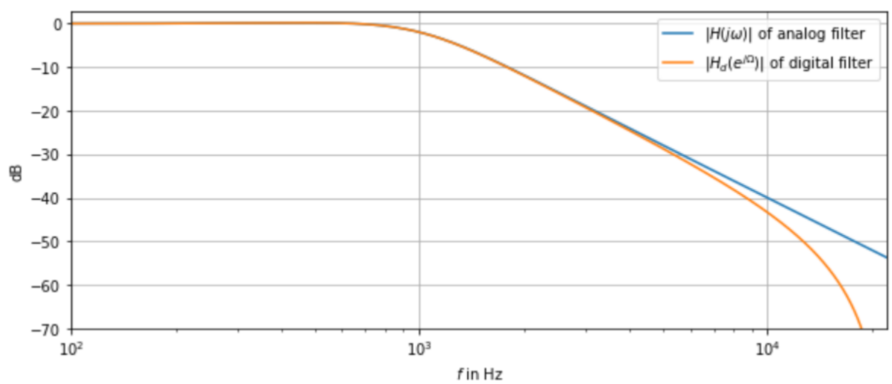

Fig. 7 Low pass filter with a cut off frequency 1000 Hz

Case II:

Sampling frequency, f_s = 44100 Hz

Cut-off frequency, f_c = 5000 Hz

Again, analog filter will be realized in python and graph shall be plotted as shown in Fig. 6

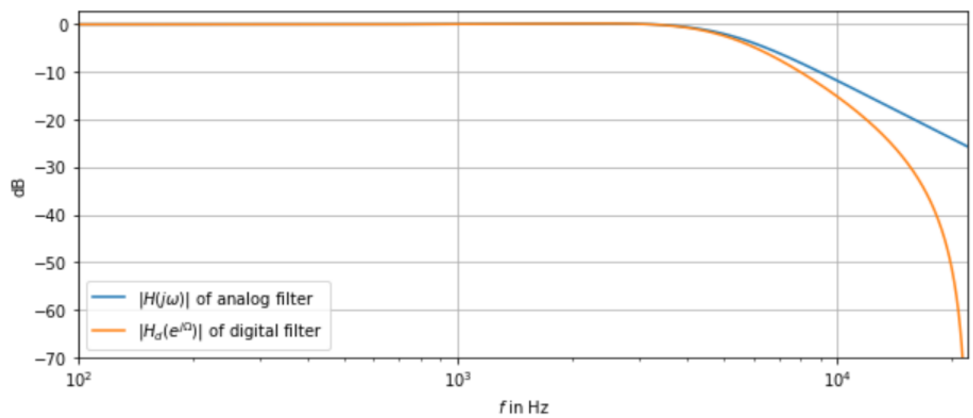

Fig. 8 Low pass filter with cut off frequency 5000 Hz

Observation:

Increasing the corner frequency leads to larger deviations for the higher frequencies as the width of the stop-band of the filter decreases. These deviations may even lead to a different corner or cut-off frequency as the attenuation in the transition region between the pass- and the stop-band changes.

5.2 Design of Digital Filter from Butterworth Filter

The Butterworth filter is realized in python and is approximated in the digital domain using bilinear transform. The filter transfer function in the analog domain and digital domain with and without pre-warping technique is illustrated in Fig. 7, Fig. 8 and Fig. 9 for different ranges of cut off frequencies.

Considering three different cut-off frequencies of Butterworth filter:

Case I:

Sampling frequency, f_s = 44100 Hz
Cut-off frequency, f_{c1} = 5000 Hz
Cut-off frequency, f_{c2} = 6000 Hz

Pre warped frequency is calculated as,

$$f_{cw1} = 2f_s \,.\, \tan\frac{2\pi f_{c1}}{f_s}$$

$$f_{cw2} = 2f_s \,.\, \tan\frac{2\pi f_{c2}}{f_s}$$

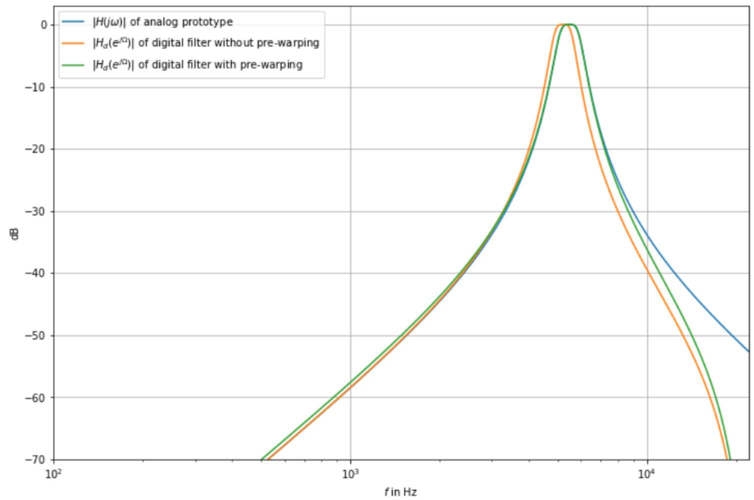

Fig. 9 Butterworth Filter approximation in Digital Filter

Case II:

Sampling frequency, f_s = 44100 Hz
Cut-off frequency, f_{c1} = 1000 Hz
Cut-off frequency, f_{c2} = 3000 Hz

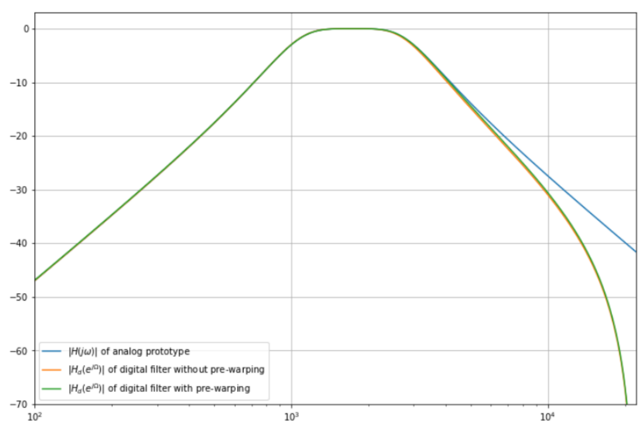

Fig. 10 Butterworth Filter approximation in Digital Filter

Case III:

Sampling frequency, f_s = 44100 Hz
Cut-off frequency, f_{c1} = 9000 Hz
Cut-off frequency, f_{c2} = 13000 Hz

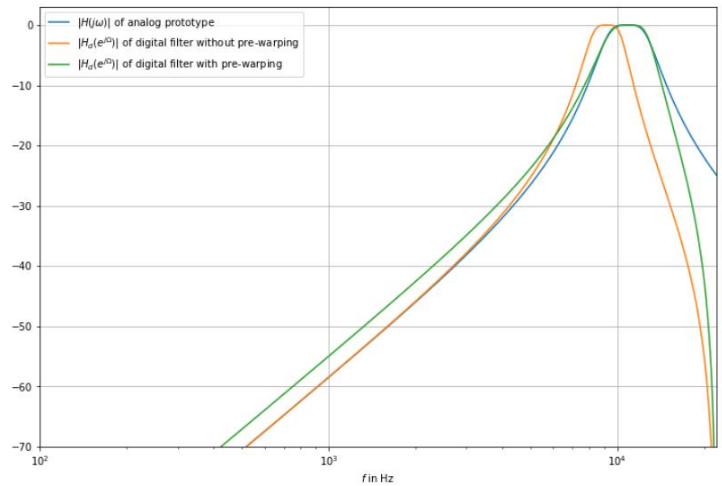

Fig. 11 Butterworth Filter approximation in Digital Filter

Observation:

With the introduction of the pre-warping technique in the design process, the location and the width of the digital filter's pass-band are closer to the expected pass-band. The effect of pre-warping can be observed in the case of higher pass-band frequencies as illustrated in Case III because the deviation caused by bilinear transformation is less prominent for lower frequencies as shown in Case II.

6. Summary and Conclusion

One of the most powerful DSP tools is digital filtering. Aside from the obvious benefits of virtually eliminating errors in the filter caused by passive component fluctuations over time and temperature, op-amp drift (active filters), and other factors, digital filters can achieve performance specifications that would be difficult, if not impossible, to achieve with an analog implementation. Furthermore, the features of a digital filter can be altered quickly using the software. As a result, they are common in adaptive filtering applications in communications including modem echo cancellation, noise cancellation, and voice recognition.

Different approaches, such as impulse invariance and bilinear transformation, can be used to create digital filters. At higher frequencies, when using the Bilinear Transformation technique for conversion, a non-linearity known as frequency warping occurs. Pre-warping must be implemented when constructing digital filters to avoid this.

The Butterworth analog filter is transformed into a digital filter to illustrate pre-warping and warping. The frequency response is displayed before and after pre-warping is implemented. It is demonstrated that before pre-warping, the frequency response of filter exhibits non-linearity at higher frequencies. It is further demonstrated that by using the pre-warping technique, this non-linearity can be eradicated.

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