What is Quantum Computation?

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Abstract

Quantum computation is a rapidly progressing field today. What are its principles? In what sense is it distinct from conventional computation? What are its advantages and disadvantages? What type of problems can it address? How practical is it to make a quantum computer? I summarise some of the important concepts of quantum computation, in an attempt to answer these questions. A deeper understanding of them would pave the way for future development.

1 Motivation

"Because the nature isn't classical, damn it … " —Richard Feynman
Let us begin by analysing what a computer is and what it actually does. Computation is processing of information. The processing may be carried out by a living entity or an inanimate machine, but ultimately the processor is a physical device, and not just a mathematical construct to implement algorithms. It follows that what is computable and what is not is limited by the laws of physics [1].

Traditional computer science is based on Boolean logic and algorithms. Its basic variable is a bit, with two possible values, 0 or 1. These values are represented in the computer as stable saturated states, `off` or `on`. Quantum mechanics offers a new set of rules that go beyond this classical paradigm. The basic variable is now a qubit, represented as a normalised vector in a two dimensional complex Hilbert space. $|0\rangle$ and $|1\rangle$ form a basis in this space, and are physically represented as two eigenstates of a two-level quantum system. The logic that can be implemented with such qubits is quite distinct from Boolean logic, and this is what has made quantum computation exciting by opening up new possibilities [2].

Quantum computation is thus not a question of merely implementing the old Boolean logic rules at a different physical level with a different set of components.

We can take advantage of the novel quantum features to devise new type of software as well as hardware. In my view, having novel quantum devices will not make the traditional computers obsolete, rather they would improve and enhance what is classically possible to do.

2   Digital versus analogue computation

Computers can be broadly classified into two types, digital and analogue, based on the type of variables that carry information. In digital computers the variables take only a discrete set of values, while in analogue computers the variables belong to a continuous parameter range.

It is worthwhile to look at two specific examples, electronic computers and living organisms (where the genetic code and the nervous system form centres of computation). Most physical parameters performing computation are essentially continuous, be it the voltages and currents in a circuit or concentrations of chemicals and enzymes in a cell. A continuous parameter can be given a sufficiently accurate digital representation by choosing a large enough number of bits. Whether this is desirable or not depends on the optimisation criteria of the task to be accomplished.

Major advantages of digitisation are high precision and high speed. The former arises from the breakup of a single continuous value into a sequence of discrete digits, while the latter is a result of a simple instruction set. Small fluctuations and noise can be quickly corrected by resetting the signal to the nearest known discrete value. But if a large error does occur due to bit-flip, then its consequences typically grow exponentially, leading the whole calculation completely astray. Also high speed in elementary operations is accompanied by an increase in the depth of the calculation and in power consumption.

Analogue computation has limited precision and speed. But it has a higher tolerance against errors, in the sense that a local error would not lead the whole computation totally off the mark. It also has the flexibility to handle complex instructions (e.g. integration and differentiation) reducing the depth of computation. Moreover, its low power consumption permits high density packing of components.

Electronic computers use digital instructions on digital variables. This strategy is optimised for high precision arithmetic and data management. Designated components for basic Boolean operations simplify the hardware, and make the instructions programmable into a universal machine. Sufficient temporary memory is required to store intermediate results.

Priorities of living organisms are different, and over millions of years they have evolved a scheme which can be dubbed digital instructions on analogue variables. A neuron does or does not fire depending on whether the potential from its various synapses exceeds or falls below a certain threshold. The genetic code is composed of discrete combinations of DNA base pairs. The instructions are often highly complex and use designated organs for specialised tasks. They control molecular concentrations and chemical reactions performing the desired

task, and errors are weeded out by punctuating analogue processes by digital steps. Such a combination (even the instruction language has statistical features) is convenient for pattern recognition problems [3]. For living organisms, it is far more important to guard against runaway errors than to carry out high precision arithmetic. A brain cannot compete in arithmetic with today's electronic computers, but in contrast the electronic computers are no match for the brain in pattern recognition (derivative evaluation) problems.

The reason for all this elaboration is my belief that quantum computers are more akin to living organisms than to electronic computers. Their best use would come from digital instructions on analogue variables, with errors being cleaned up by digital steps punctuating analogue processes. As we will see below, most of the research so far in quantum computation is concentrated on how a quantum computer can beat its classical counterpart at some specific problems in arithmetic. But the future of quantum computation is likely to be in a different domain, as yet unexplored.

3   Classical information theory and computation

Computation converts a given input to a specific output. The process is usually deterministic, but it can be made probabilistic and still be labeled reproducible in the sense of an ensemble. Clearly all the information about the problem must be contained in the input (instructions for processing data are also a type of input) in some form or the other; the computer carries out the tedious task of making the desired information explicit and presents it as the output. It is not always possible to reconstruct the input from the output, and knowledge is lost in such processes. The limiting case is the one where all the information in the input is retained in the output, albeit in a different form.

This description is reminiscent of the second law of thermodynamics, with the identification of information with entropy. It is convenient to use the language of communications to quantify information. If a message X randomly takes values x with probabilities $p(x)$, then the information conveyed by it is [4]

$$ S(X) \equiv S(\{p(x)\}) = -\sum_x p(x) \, \log p(x) \ . \tag{1} $$

Information is thus a measure of surprise one has upon receiving a message. A message of repeated bits carries no information, since there is nothing more to learn after receiving the first bit. In contrast, a message of uniformly random bits carries maximum information, since one has no idea of what will come next.

Shannon proved two limiting theorems based on this definition of $S(X)$. The noiseless coding theorem gives the data compression limit: to communicate n values of X, one need only send only $nS(X)$ bits. The noisy coding theorem asserts the existence of efficient error correcting codes: over a binary symmetric channel with bit-flip error probability p, a coded message of n bits can transmit upto $n(1 - S(\{p, 1-p\}))$ bits with an arbitrarily small error probability.

Information contained in correlations between two parts X and Y of a system is described in terms of mutual entropy

$$I(X:Y) = S(X) + S(Y) - S(X,Y) = \sum_x \sum_y p(x,y) \log[p(x,y)/p(x)p(y)] \ , \ (2)$$

where $p(x,y)$ is the joint probability for $X = x$ and $Y = y$. In absence of correlations, $p(x,y) = p(x)p(y)$ and $I(X:Y) = 0$.

The conventional paradigm for a universal digital computer is a Turing machine. It has a finite number of internal states, a memory with unlimited storage capacity in the form of an unbounded tape divided into cells, a read/write head, left/right movement capability, and unconstrained computing time at its disposal. It starts off in a certain state, looking at the contents of a cell. Each subsequent step is determined by the current state and the cell contents. At each step, the machine updates the current cell contents, moves one cell to the left or right, and changes to a new internal state. One of the states must be "halt" signifying the end of computation. (Generation of random numbers is, strictly speaking, not a computation. But with an added coin-toss instruction, a Turing machine can perform probabilistic computation.)

Given a specific problem to solve, the programmer devises an effective procedure or algorithm—a set of instructions for the computer to carry out starting with some initial data. One can investigate what type of problems can be tackled in this manner. The answer illustrates the power of the universal Turing machine, and is summarised by the Church-Turing hypothesis: "Every function which would be naturally regarded as computable can be computed by the universal Turing machine."

From the practical point of view, it is important to find out the extent of resources needed (hardware and time) for a specific computation, i.e. to find out what is not just computable but also efficiently computable. Another property of the universal Turing machine is useful here—it can simulate any other computer with at most a polynomial overhead. The complexity of problems can thus be classified independent of the model of computation: those which can be solved with resources polynomial in the input size (P), and those which require superpolynomial resources (hard). An important subset of hard problems are those for which solutions can be verified in polynomial time (NP). On classical computers, there exist many superpolynomial problems: prime factorisation of large numbers, global extremisation problems such as the travelling salesman, Boolean circuit satisfiability problem, and so on.

Gödel demonstrated that there exist uncomputable functions—questions that cannot be answered at all by a consistent system of axioms and rules. Such an incompleteness of mathematical logic/arithmetic has no place in the physical realm—physical realisations produce physical results. Shifting the emphasis from computation to simulation, Deutsch therefore proposed a Church-Turing principle [5, 6]: "Every finitely realisable physical system can be simulated arbitrarily closely by a universal quantum computer operating by finite means." This statement can be taken to be the definition of what a quantum computer is and what it can do [7].

4 Quantum information theory

There is no direct comparison between information content of a classical bit that can take two discrete values and a qubit that can take any value in a two dimensional complex Hilbert space. The best one can do is to quantify the information of an arbitrary quantum message in units of that of a qubit. Let us carefully consider a general quantum state and the information that can be extracted from it.

A n-bit classical variable can take any of the 2^n discrete values. Similarly a n-qubit state is a vector in the 2^n dimensional complex Hilbert space. Keeping in mind that the overall phase of a quantum state is not measurable, a normalised n-qubit state can be specified as

$$|x\rangle \;=\; \sum_{i=0}^{2^n-1} c_i|e_i\rangle \;\;,\;\; \sum_{i=0}^{2^n-1} |c_i|^2 \;=\; 1 \;\;. \tag{3}$$

The basis vectors of the Hilbert space can be identified with the classical states, while the freedom to vary c_i allows superposed quantum states. With complex amplitudes c_i, these states are much more general than interpolation between the extreme classical values that can be represented by analogue devices. The superposition principle exists in classical wave mechanics too, but it finds a much more emphatic realisation in unusual quantum phenomena, e.g. "Schrödinger's cat" that is dead and alive at the same time.

The concept of quantum measurement adds another subtlety. It can only be defined in probabilistic language (in the sense of an ensemble). Consider a measurement operator with eigenstates $\{|e_i\rangle\}$. Measuring it in the quantum state $|x\rangle$ produces an eigenvalue λ_j with probability $|c_j|^2$, and collapses the state to $|e_j\rangle$ [8]. Thus it is not possible to determine all the c_j in general; the relative phase information of different c_j is lost forever. The characterisation of physical quantum information has to be in terms of what can be extracted out of a state, and not in terms of all the parameters that define the state.

All the properties of a quantum state are fully specified by the density matrix ρ. For a pure state $|x\rangle$, it is just the projection operator $\rho_{\text{pure}} = |x\rangle\langle x|$. The measurement process converts the pure state into a mixed state, and ρ becomes diagonal—a weighted average of pure state density matrices

$$\rho_{\text{mixed}} \;=\; \sum_{j=0}^{2^n-1} |c_j|^2 |e_j\rangle\langle e_j| \;\;. \tag{4}$$

The accessible quantum information is then the von Neumann entropy,

$$S(\rho) \;=\; -\operatorname{Tr}(\rho \log \rho) \;\;. \tag{5}$$

It is zero for a pure quantum state, while it reduces to the Shannon entropy $S(\{|c_j|^2\})$ for a diagonal density matrix. $S(\rho)$ also describes the quantum analogue of the noiseless coding theorem: the quantum data compression limit for

n qubits chosen at random from an ensemble of pure states is $nS(\rho)$. Dealing with ensembles of mixed states is trickier (but inevitable when studying quantum error correction), however, and although various bounds exist, firm theorems regarding them are still to be established.

The most interesting feature of quantum information is contained not in individual qubits, but in correlations amongst them, often referred to as entanglements. Bell showed that there exist quantum entanglements which cannot be realised by any classical probabilistic local hidden variable theory [9]. Such entanglements originate from complex superposition coefficients, and exemplify physically possible tasks which no classical computer can perform. With the reduced density matrices defined as partial traces, the entanglement entropy for pure quantum states is

$$E(X:Y) \;=\; S(\rho_X) + S(\rho_Y) - S(\rho_{XY}) \;\;,\;\; \rho_A \;=\; \mathrm{Tr}_B(\rho_{AB}) \;\;. \qquad (6)$$

The well-known spin singlet state is a case where the reduced density matrices are proportional to identity and all the non-trivial properties of the state reside in the correlation between the two spins.

5   Quantum dynamics

Quantum dynamics is exactly linear and unitary, unlike the non-linear and dissipative behaviour often seen in its classical counterpart. Time evolution of quantum states in discrete steps is conveniently expressed using Heisenberg's matrix mechanics (in contrast to Schrödinger's wave mechanics). The quantum state is represented as a column vector with components c_i, and multiplying it with a $2^n \times 2^n$ unitary matrix from left preserves its norm. In terms of the Hermitian Hamiltonian H, the unitary matrix is $U = \exp(i \int H dt)$. This dynamics is precise and reversible ($U^{-1} \equiv U^\dagger$ reverses the evolution) [10].

With the superposition principle, operation of U on an n-qubit state is just a single quantum step, although the corresponding matrix multiplication amounts to 2^n steps on an n-bit classical computer (provided that the matrix U is dense enough). This feature has been exploited to convert classically superpolynomial problems into quantum polynomial ones. For example, before performing an operation, set the initial state for each qubit in an n-qubit string to $(|0\rangle + |1\rangle)/\sqrt{2}$. Then the final state is a superposition of all the outcomes corresponding to every possible classical input to the operation—there is complete parallel processing. Of course, quantum measurement does not allow individual determination of each outcome. But if only one property of the possible outcomes is desired, then a cleverly designed measurement can pick it up, and we obtain an exponential speed-up through quantum parallelism.

The linear unitary evolution makes it impossible to copy an arbitrary unspecified state in the Hilbert space. Consider the copying operation for two distinct states $|u\rangle$ and $|v\rangle$:

$$U_{copy}|u\rangle|0\rangle \;=\; |u\rangle|u\rangle \;,\; U_{copy}|v\rangle|0\rangle \;=\; |v\rangle|v\rangle \;\Longrightarrow\; \langle u|v\rangle \;=\; \langle u|v\rangle^2 \;\;. \qquad (7)$$

This is impossible to satisfy for non-orthogonal $|u\rangle$ and $|v\rangle$. Note that there is no problem with copying a specific state:

$$|x\rangle|0\rangle \;\not\longrightarrow\; |x\rangle|x\rangle \;,\; \text{but}\;\; (a|0\rangle + b|1\rangle)|0\rangle \stackrel{\text{C}-\text{not}}{\longrightarrow} a|0\rangle|0\rangle + b|1\rangle|1\rangle \qquad (8)$$

is allowed. In fact the latter is the "controlled not" operation, described in more detail below, converting superposition into entanglement. A simple extension is the allowed operation $\sum_i c_i|e_i\rangle|0\rangle \to \sum_i c_i|e_i\rangle|f(e_i)\rangle$.

It is also impossible to extract any information from a quantum state without disturbing it. Consider a detector in initial state $|\psi\rangle$ interacting with a signal:

$$U_{int}|\psi\rangle|u\rangle \;=\; |\psi_u\rangle|u\rangle \,,\, U_{int}|\psi\rangle|v\rangle \;=\; |\psi_v\rangle|v\rangle \;\Longrightarrow\; \langle u|v\rangle \;=\; \langle\psi_u|\psi_v\rangle\langle u|v\rangle \;. \; (9)$$

For non-orthogonal $|u\rangle$ and $|v\rangle$, $\langle\psi_u|\psi_v\rangle = 1$ means that the final state of the detector is the same irrespective of the state of the signal; the states must be disturbed if something is to be learnt about them. Moreover, a measurement operation can leave the state undisturbed, only when the state is an eigenstate of the measurement operator (eigenstates of Hermitian operators are orthogonal). In other situations, the act of measurement is non-unitary, probabilistic and irreversible—the information orthogonal to the projected eigenstate is lost.

Putting all these features together, a quantum computer can be described as a device that is a set of qubits (a) initialised in some known state, (b) evolved only by a succession of selected unitary operations, and (c) measured in a specific basis. Although quantum states and unitary evolution have entered, the classical concepts of initialisation, deterministic control of the evolution sequence and measurement are still part of it. These classical concepts are not physical limitations, rather they are the limitations of our traditional framework of knowledge acquisition through experiments.

6   Reversibility and thermodynamics

Historically, reversibility of computation was first investigated to understand the limits imposed by thermodynamical laws on energy consumption and speed of computers. According to the second law of thermodynamics, entropy can never decrease. So the only processes that can be fully reversed are the adiabatic ones where entropy is held fixed. Such processes, carried out at an infinitesimal speed, consume essentially no energy.

Most of the physical processes—in mechanics, electrodynamics, chemical reactions, atomic physics etc.—are reversible at the microscopic level. So a reversible model of computation only needs implementation of elementary computational steps (i.e. logic gates) with these processes. Landauer emphasised that increase in entropy can be eliminated from information processing, except from the irreversible act of erasure [11]. Erasure is inevitable only when readying a blank tape for input/output, in preparation for the next computational step. A generic reversible computer converts (input + blank tape) to (input + output). There is no creation or destruction of information in such processing;

there is only creation of correlation/entanglement between different parts of the system. (e.g., measurement is establishment of a perfect correlation between the signal and the detector.) This analysis forms the basis of a unified description of information and entropy—they can be freely converted into each other.

Reversibility implies that nothing will happen on the average in an equilibrium situation. To achieve something, a reversible dynamics must start from a non-equilibrium situation or have a driving force. Even then a reversible process will execute much like a ballistic wave, oscillating back and forth. One must pick the correct boundary conditions to obtain results, i.e. know precisely when and where to start and stop.

Obviously each reversible computational element has the same number of inputs and outputs; given the output, the input can always be reproduced by running the computer backwards. Bennett constructed an explicit model of reversible computation using Boolean logic operations [12]. All reversible logic gates can be represented as square permutation matrices, with each row as well as each column having only one occurence of 1 and the rest of the elements being 0. The only one bit reversible gates are "identity" and "not". A convenient two bit reversible gate is the "exclusive or", $(x, y) \rightarrow (x, x \oplus y)$. It is also called "controlled not", because the second bit is flipped if the first bit is 1 and left unchanged otherwise. A universal set of classical reversible gates is complete with a three bit gate, and a common choice is the "(controlled)^2 not" gate. All these gates are their own inverses, and any Boolean logic circuit can be constructed with these gates as building blocks. For example, take three bits with the third one initialised to zero. Application of C^2-not to the three bits, followed by application of C-not to the first two bits produces a simple binary adder—the second output bit is the "sum" of the first two input bits while the third one is the "carry".

Note that the copy and measurement operations have no conflict with reversible computation (e.g., if y is initialised to 0, C-not just copies x into y.). The constraints of Eqs.(8-9) are easily solved, when the only choices for $|u\rangle$ and $|v\rangle$ are to be the same or to be orthogonal. Models involving elastically colliding billiard balls have been constructed to demonstrate fully reversible copy and measurement operations. These two operations are forbidden not by the laws of thermodynamics, but by the superposition principle of quantum mechanics.

7   Quantum gates and circuits

In order to implement an algorithm, it is much more convenient to study gate arrays that will simulate it than to find appropriate instructions for a Turing machine. Erase, fork (requiring copy) and feedback (requiring non-linearity) operations have no place in quantum circuits. It has been shown that any quantum Turing machine can be simulated by acyclic quantum gate arrays.

Quantum gates represent general unitary transformations in the Hilbert space, describing interactions amongst qubits. Reversible Boolean logic gates are easily generalised to quantum circuits by interpreting them as the trans-

formation rules for the basis states. In addition there are gates representing continuous transformations in the Hilbert space. Almost any two qubit quantum gate (i.e. one where the complex phases are not rational multiples of π) is universal [13]. From the practical point of view, it is convenient to simplify matters further and choose a combination of quantum and classical gates as universal building blocks. One easy choice happens to be a general single qubit gate (represented as a $U(2)$ matrix) and the two qubit C-not gate [14].

It is important to check that the number of elementary quantum gates needed to build up a general n-qubit unitary operation does not grow exponentially with n. For many useful operations (including n-qubit controlled gates), this number is bounded from above by a quadratic function of n. Furthermore, with continuous variables and their limited resolution in any practical implementation, we can only carry out bounded error calculations and not arbitrary precision calculations. This is not a limitation provided that the resource demand (e.g. the accuracy in the specification of the complex phases) does not grow exponentially with the reduction in error bounds. The problems that can be tackled in this manner form the complexity class BQP—bounded error quantum probabilistic decision problems that can be solved with polynomial resources.

In principle, the quantum amplitudes may be encoded in space, time or internal degrees of freedom (e.g., respectively standing waves, travelling waves or polarisation in case of light). The quantum elements have to couple to both the external driving force and the desired interaction. The driving force typically carries irreversible effects with it, and then it becomes desirable to separate it from the quantum signal. In such a case, two different properties of the same physical object can be used for computation, e.g. couple the space-time degrees of freedom to the driving force and encode the amplitudes in the internal degrees of freedom. This choice is often also dictated by the fact that one has a better control over the internal degrees of freedom than on the space-time ones. Thus it has become customary to realise quantum gates in an S-matrix framework— there are well-defined incoming and outgoing asymptotic states, and inbetween the interaction Hamiltonian acts for a finite amount of time. The full quantum algorithm is depicted as a sequentially evolving network of elementary gates.

8   Quantum algorithms

Quantum algorithms that beat their classical counterparts exploit two specific features: superposition and entanglement. Superposition can transfer the complexity of the problem from a large number of sequential steps to a large number of coherently superposed quantum states. After parallel processing, a clever interference can extract the desired feature from the result. Entanglement is used to create complicated correlations that permit the desired interference. A typical quantum algorithm starts with a highly superposed state, builds up entanglement, and then eliminates the undesired components providing a compact result. A uniform spread of the initial state over a large number of basis states is easily achieved using the Walsh-Hadamard transform, $H = (\sigma_x + \sigma_z)/\sqrt{2}$,

where the σ's are Pauli matrices.

Shor used quantum parallelism to construct a quantum Fourier transform (QFT) [15]. The discrete Fourier transform is a unitary operation ($N = 2^n$):

$$|x\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_y e^{2\pi ixy/N}|y\rangle \ , \ \ x = x_{n-1} \cdot 2^{n-1} + \ldots + x_0 \ . \qquad (10)$$

This is a polynomial in $\exp(2\pi ix/N)$. The classical fast Fourier transform reduces the total number of operations from $O(N^2)$ to $O(N \log N)$ by fully factorising this polynomial over the field of complex numbers. In binary notation,

$$|x\rangle \longrightarrow \frac{1}{\sqrt{N}}(|0\rangle + e^{2\pi i(.x_0)}|1\rangle)\ldots(|0\rangle + e^{2\pi i(.x_{n-1}\ldots x_0)}|1\rangle) \ . \qquad (11)$$

Since each x-basis state goes to a factorised unentangled state, by superposing all of them, QFT reduces the number of operations to $O((\log N)^2)$. Only one property of the heavily superposed state can be determined at the end; it is periodicity in Shor's algorithm (classical randomised algorithms reduce the prime factorisation problem to finding the period of a function). QFT is a versatile algorithm with applications expected in many pattern finding problems.

In general, the gain extractable using quantum parallelism depends on the structure of the problem. Grover's algorithm for finding an item in an unsorted database is an example where the gain is quadratic [16]. The algorithm starts with the maximally superposed state uniformly spread over n qubits, representing $N = 2^n$ items. It uses a quantum oracle (i.e. a black box routine that gives immediate answer to a query) that takes the superposed state as an input, and outputs the state after flipping the sign of the amplitude corresponding to the desired item while leaving all the other amplitudes unaltered. The next step is to invert all the amplitudes about their average value. Both these steps are unitary operations, and the procedure is iterated. After I iterations, such that $(2I + 1)\sin^{-1}(1/\sqrt{N}) = \pi/2$, the admixture of all the unwanted terms in the initial superposition is eliminated and one obtains the desired item. A classical algorithm would take $O(N/2)$ queries on average to find the desired item. The crucial quantum ingredient here is that maximal interference allows one to make $O(N)$ queries in $O(\sqrt{N})$ steps. The iterations can keep cycling forever, and they have to be stopped precisely at the right place to get the correct answer. One knows exactly when to stop, because the overlap of the initial state with the desired one is known, even though the desired state is unknown.

The problem of finding parity of n bits is one of the toughest ones, where the gain provided by a quantum algorithm is only a factor of two.

Quantum computers can of course be used to simulate quantum models, i.e. systems which are idealised parts of the real world but which still carry characteristic quantum correlations [6]. The important property in these problems is quantum entanglement. Bell's theorem proves that certain types of quantum entangled states cannot be easily simulated with classical resources. Non-local correlations in quantum entangled states have been used to provide quantum teleportation (a quantum state is destroyed in one place, and after transmission

of a classical signal, recreated somewhere else). They exploit the peculiarity that, given a specific quantum state, an operation on one of its parts can be interpreted as a (possibly different) operation on the other part. For example,

$$|1\rangle(|0\rangle - |1\rangle) \overset{\text{C}-\text{not}}{\longrightarrow} |1\rangle(|1\rangle - |0\rangle) \equiv -|1\rangle(|0\rangle - |1\rangle) \ . \qquad (12)$$

In the first interpretation, C-not leaves the first qubit untouched and changes the second one, while in the second interpretation, the first qubit undergoes a phase change and the second one remains unaltered.

Another application of entanglement is in dense coding, where the components of a spin-singlet state are separated, then certain encoding steps are performed on one of the halves and it is sent to the location of the other half. The decoding analysis of the reunited halves provides information worth two classical bits, while only one encoded qubit was transmitted. This example emphasises the importance of entanglement as an information resource, and also shows that a bit and a qubit cannot be naively equated in information content. (Quantum teleportation is essentially the inverse operation of dense coding.)

Quantum cryptography is based on the property that extraction of even partial quantum information from a signal leaves a signature behind. This possibility of detecting eavesdropping is a unique quantum feature that is absent in classical cryptography. Bennett and Brassard proposed a protocol based on two mutually non-orthogonal sets of basis states [17]. After transmission of a quantum key, an exchange of classical bits over a public channel detects eavesdropping, eliminates noise and increases security by distilling a smaller key composed of parity bits. The fact that only superposition principle is used in this method has made it easy to implement; it has already been tested over a distance of 24 kms with existing fibre optic cables.

It should be noted that quantum algorithms are often probabilistic, but just as in the case of classical randomised algorithms, if the success probability is greater than half then one can hope to get to the answer with a few trials. Also practical quantum algorithms have to be stable against small errors, in the sense that despite round-off errors in the continuous amplitudes, one should obtain the right answer with a bit of extra work.

9   Decoherence and quantum error correction

The third law of thermodynamics forbids a finite system to have zero entropy (or temperature). A pure quantum state having zero entropy is therefore an idealisation. Any physical state cannot be made perfectly pure, it will always have some interaction with its environment. Even a reasonably well isolated quantum computer has to interact with its surroundings for the preparation of the initial state, for receiving instructions and for displaying the results. The inevitable external disturbances destroy exact unitary evolution and reversibility, changing the pure state into a mixed state. This process is called decoherence.

Physically, decoherence is a noise due to unwanted scatterings, diffusion and localisation in the Hilbert space encoding the quantum signal. It can be thought

of as a process which entangles the quantum signal with the environment. The evolution for the joint state of the signal and the environment is still unitary, but the environmental degrees of freedom are not observed. The reduced density matrix for the signal is obtained by performing a trace over all the unobserved degrees of freedom. This averaging typically takes place over a large number of incoherent variables, suppressing the off-diagonal elements of the density matrix. The signal is reduced to a mixed state, increasing its entropy due to neglect of information contained in the entanglement with the environment.

The precision of a practical quantum device depends on its sensitivity to environmental disturbances, and decoherence must be controlled. Just on dimensional grounds, the decoherence time scale due to random thermal noise is $\hbar/kT \approx 0.76 \cdot 10^{-11} \sec/T(^\circ K)$. It can be increased M-fold by combined inertia, if M coherently coupled quanta are used to represent a signal (e.g. superconductors). It can also be increased by reducing the coupling between the quantum state and the environment; for instance, the nuclear spins are so well shielded by the electron cloud from their surroundings that the typical relaxation time in NMR experiments is $O(10)$ seconds.

Generally, superposition is much more stable against decoherence than entanglement. What is an eigenstate with a specific choice of the basis becomes a superposed state with another choice of basis, making it easy to manipulate superposition. Entanglement, however, is highly fragile and must be carefully protected. Quantum algorithms that make less use of entanglement, in space and time, are easier to implement. (As already mentioned, quantum cryptography protocol that doesn't use entanglement has been tested to good accuracy.) Moreover, if it is known that some intermediate state should have a particular feature, that can be exploited to improve the stability of an algorithm.

To make a quantum computer work longer than the decoherence time scale, it is mandatory that checks and error corrections are built into the system. In classical computers, bit flip errors are corrected with redundancy, parity checks, and sophisticated Hamming codes. In the latter case, the encoding is performed by embedding a k-bit code in an n-bit word. By maintaining a minimum Hamming distance (i.e. the number of bits that differ between two words) d between any two codewords, any binary vector in the 2^n-dimensional space is within Hamming distance $l = [(d-1)/2]$ of at most one encoded word. Local errors flip only one bit at a time, and the code structure thus allows upto l local errors to be corrected.

These classical codes have been generalised to quantum ones, by interpreting the words as basis vectors of the expanded Hilbert space. With independent errors for each qubit, the quantum error operators are direct products of $\{1, \sigma_x, \sigma_y, \sigma_z\}$ for each qubit. Linearity of quantum mechanics then allows for their sequential elimination. The unitary error correction step is to entangle the signal with another string of qubits called the ancilla, such that the signal becomes pure by transferring the error to the ancilla. The ancilla is thereafter decoupled and discarded, restoring the signal to its proper state. Observation of the ancilla can give a clue to the nature of the environmental disturbances, but it gives no information regarding the encoded state.

In a physical computer, an error may occur at several different stages—in implementation of the logic gates, in memory, in transmission of the data, or in the encoding/decoding steps themselves. To take care of all of them, the computation must remain encoded throughout, from the input to the output. In addition, a discretisation of the continuous phases helps in correcting the errors, even at the cost of increasing the depth of computation. Such a scheme is called fault tolerant computation [18]. It makes arbitrary length quantum computation possible once the error rate per qubit per gate falls below a certain finite threshold (present estimates are about $10^{-4} - 10^{-5}$).

The error correcting schemes correct only local errors, by making the encoded information non-local. There is no way to correct global errors. To make the errors affecting different qubits uncorrelated, we must have the graininess (or correlation length) of the environment smaller than the coding scale size of the qubits. This is a tough proposition to fulfill in practice, with atomic dimensions characterising both the environment and the qubits.

10 Quantum hardware

Practical requirements for a system that can be used as a quantum computer are quite stringent: (a) The quantum degrees of freedom have to be defined precisely (e.g. one can't have 100 ± 5 qubits doing a computation), and there has to be a sufficient number of qubits to do a reasonable calculation; (b) It should be possible to initialise the system in any desired state (this may need low temperatures to remove thermal noise); (c) High degree of isolation from environment should be available to reduce decoherence; (d) It should be possible to subject the system to a precisely controlled sequence of unitary transformations; (e) The measurement process should be able to detect the state of the system with high degree of certainty. All these features need to be implemented with a stiff tolerance criterion; errors more than a fraction of a percent are unacceptable. Needless to say, this is too far away in the future.

What has been realised so far is quite modest: $O(10)$ logic operations on a few qubit systems with an accuracy of a few percent. Most quantum components are found at the atomic scale, amongst atoms and photons. Various proposals involve trapped ions, nuclear spins, long-lived atomic energy levels, quantum dots, Cooper pairs, and so on. Alternatives involving macroscopic quantum states as components (e.g. coherent states of lasers or superconductors) have also been suggested. The important constraint on such devices, for their quantum nature to be manifest, is that the action for transitions between various states should be $O(\hbar)$. At the atomic scale, movable parts are difficult to control. So in a typical quantum device, qubits are held in place and instructions are supplied to them by external pulses of electromagnetic radiation.

Implementation of single bit unitary operations (i.e. phase rotations) is relatively easy, in contrast to the two bit C-not operation and entanglement. Controlled quantum operations use a multicomponent system whose interaction with the external fields depends on the interaction amongst the compo-

nents. (e.g. quantum energy levels can shift depending on interactions amongst the atoms, which in turn determines whether there will be or will not be any resonant interaction with a photon of a particular frequency.)

At present there are two experimentally realised systems that can be labeled quantum information processors: ions in a linear trap manipulated using laser beams [19], and nuclear spin chains handled using bulk NMR methods [20]. In the first system, a string of ions are confined by electric fields in a high vacuum. Each ion has two long-lived states which act as the two states of a qubit. Laser beams can illuminate individual ions, allowing transitions between the two states. The coulomb interaction between the ions provides coupling to the vibrational modes of the ions in the trap. The phonons in the centre-of-mass vibrational mode are excited/absorbed by the laser photon momentum. This permits transfer of information (e.g. the C-not operation) between any two ions. The initial state is prepared by optical pumping and laser cooling, while the final detection is achieved through laser fluorescence. A control over decoherence requires submicroKelvin temperatures and high shielding from noise voltages. That is the main limitation to enlarging the computational capacity.

The qubits in NMR experiments are the nuclear spins of individual atoms in a molecule, coupled together by their magnetic moments. Each spin has its characteristic resonance frequency in an oscillating magnetic field. It can therefore be rotated by applying a pulse of appropriate frequency and duration. Due to magnetic dipole-dipole interactions, the resonance frequencies depend on the orientations of the neighbouring spins, and so one can perform conditional C-not operations. NMR experiments are carried out at room temperature with liquid compounds containing $O(10^{23})$ molecules, however, and the desired signal has to be cleverly extracted. The coherent quantum signal appears only as a small deviation from an incoherent thermal background, much like a quasiparticle. Subtracting from the density matrix the part that is proportional to identity picks out this signal (identity remains invariant under unitary transformations). The traceless part of the density matrix can be initialised, manipulated and measured just like a pure quantum state, and so it becomes the quantum processor. The main limitation here is that with increasing number of spins the fraction of molecules participating in a particular signal falls exponentially, making it harder to pick out the signal from the thermal ensemble.

11 Future directions

The foundations of the subject of quantum computation have become well established, but everything else required for its future growth is under exploration. That covers quantum algorithms, logic gate operations, error correction, understanding dynamics and control of decoherence, atomic scale technology and worthwhile applications. I describe some possibilities below.

Reversibility of quantum computation may help in solving NP problems, which are easy in one direction but hard in the opposite sense. Global minimisation problems may benefit from interference effects (as seen in Fermat's

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principle in wave mechanics). Simulated annealing methods may improve due to quantum tunneling through barriers. Powerful properties of complex numbers (analytic functions, conformal mappings) may provide new algorithms.

Quantum field theory can extend quantum computation to allow for creation and destruction of quanta. The natural setting for such operations is in quantum optics. For example, the traditional double slit experiment (or beam splitter) can be viewed as the copy operation. It is permitted in quantum theory because the intensity of the two copies is half the previous value. Inclusion of such particle number non-conserving operations may speed up some algorithms [21].

Theoretical tools for handling many-body quantum entanglement are not well developed. Its improved characterisation may produce better implementation of quantum logic gates and possibilities to correct correlated errors.

Though decoherence can be described as an effective process, its dynamics is not understood. To be able to control decoherence, one should be able to figure out the eigenstates favoured by the environment in a given setup.

The dynamics of measurement process is not understood either, even after several decades of quantum mechanics. Measurement is just described as a a non-unitary projection operator in an otherwise unitary quantum theory. Ultimately both the system and the observer are made up of quantum building blocks, and a unified quantum description of both measurement and decoherence must be developed. Apart from theoretical gain, it would help in improving the detectors that operate close to the quantum limit of observation.

For a physicist, it is of great interest to study the transition from classical to quantum regime. Enlargement of the system from microscopic to mesoscopic levels, and reduction of the environment from macroscopic to mesoscopic levels, can take us there. If there is something beyond quantum theory lurking there, it would be noticed in the struggle for making quantum devices. We may discover new limitations of quantum theory in trying to conquer decoherence.

Theoretical developments alone will be no good without a matching technology. Nowadays, the race for miniaturisation of electronic circuits is not too far away from the quantum reality of nature. To devise new types of instruments, we must change our view-point from scientific to technological—quantum effects are not for only observation, we should learn how to control them for practical use. The future is not foreseen yet, but it is definitely promising.

Acknowledgements

References

[1] This was forcefully brought out by Richard Feynman: *Feynman Lectures on Computation*, A.J.G. Hey and R.W. Allen (Eds.), Perseus Books, 1996.

[2] Geometry has a well-known example of what can happen when one alters the basic axioms of a theoretical framework. When mathematicians put aside Euclid's fifth postulate regarding parallel lines, they came up with curved space geometry. This did not make the old flat space geometry useless, but paved the way for new applications such as general relativity.

[3] If I meet a friend after a gap of several years, I would still like to recognise him despite the fact that his face may have changed somewhat. To allow for this flexibility, I don't mind paying the penalty of goofing up when trying to tell apart identical twins.

[4] Shannon's choice of 2 as the base of logarithm is convenient for binary arithmetic. A different choice for the base can be compensated by an appropriate value for the Boltzmann's constant in physical considerations involving entropy.

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[6] R. Feynman, *Simulating Physics with Computers*, Int. J. Theo. Phys. 21 (1982) 467; *Quantum Mechanical Computers*, Found. Phys. 16 (1986) 507.

[7] The label "quantum computers" is used here because, to the best of our present knowledge, laws of quantum mechanics describe the behaviour of various components of the computers we design. Were still more fundamental laws of physics to be discovered some day, and were we able to use them to design components of a future computer, that computer will be named accordingly.

[8] We restrict ourselves to the orthogonal projective measurements considered by von Neumann. The most general quantum measurement, however, can be defined in terms of "positive operator valued measures". See for example: A. Peres, *Quantum Theory: Concepts and Methods*, Kluwer Academic Publishers, 1995.

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[10] Quantum dynamics can be labeled Markovian only in terms of amplitudes, in the sense that the amplitude and its time derivative fix the future evolution of the state. But this dynamics is not Markovian in terms of probabilities, which have far less information than the amplitudes. Concepts of reaching an equilibrium or thermalisation cannot be defined.

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