Greedy Algorithms | Set 2 (Kruskal's Minimum Spanning Tree Algorithm)

What is Minimum Spanning Tree?
Given a connected and undirected graph, a *spanning tree* of that graph is a subgraph that is a tree and connects all the vertices together. A single graph can have many different spanning trees. A *minimum spanning tree (MST)* or minimum weight spanning tree for a weighted, connected and undirected graph is a spanning tree with weight less than or equal to the weight of every other spanning tree. The weight of a spanning tree is the sum of weights given to each edge of the spanning tree.

How many edges does a minimum spanning tree has?
A minimum spanning tree has (V – 1) edges where V is the number of vertices in the given graph.

What are the applications of Minimum Spanning Tree?
See this for applications of MST.

Below are the steps for finding MST using Kruskal's algorithm

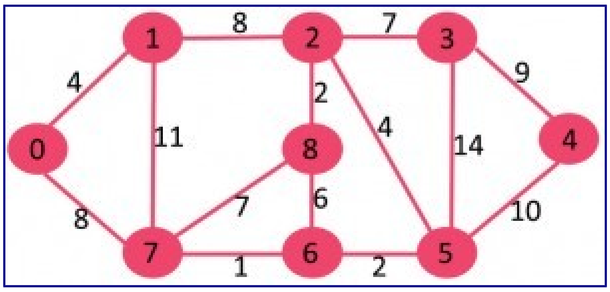
```
1. Sort all the edges in non-decreasing order of their weight.

2. Pick the smallest edge. Check if it forms a cycle with the spanning tree
formed so far. If cycle is not formed, include this edge. Else, discard it.

3. Repeat step#2 until there are (V-1) edges in the spanning tree.
```

The step#2 uses Union-Find algorithm to detect cycle. So we recommend to read following post as a prerequisite.
Union-Find Algorithm | Set 1 (Detect Cycle in a Graph)
Union-Find Algorithm | Set 2 (Union By Rank and Path Compression)

The algorithm is a Greedy Algorithm. The Greedy Choice is to pick the smallest weight edge that does not cause a cycle in the MST constructed so far. Let us understand it with an example: Consider the below input graph.



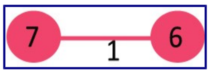
The graph contains 9 vertices and 14 edges. So, the minimum spanning tree formed will be having (9 – 1) = 8 edges.

```
After sorting:
Weight   Src    Dest
1         7      6
2         8      2
2         6      5
```

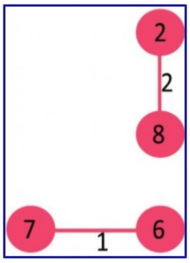
4	0	1
4	2	5
6	8	6
7	2	3
7	7	8
8	0	7
8	1	2
9	3	4
10	5	4
11	1	7
14	3	5

Now pick all edges one by one from sorted list of edges

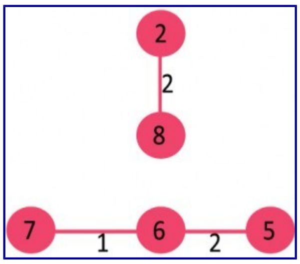
1. *Pick edge 7-6:* No cycle is formed, include it.



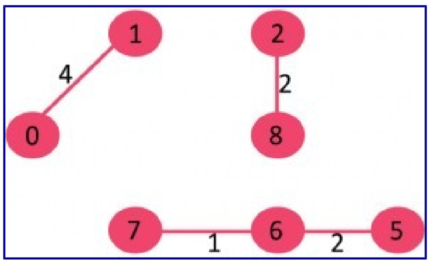
2. *Pick edge 8-2:* No cycle is formed, include it.



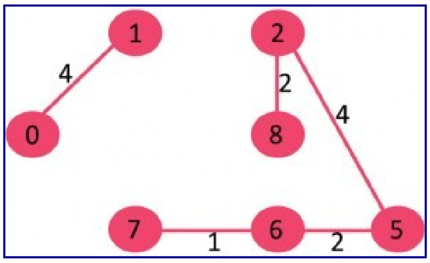
3. *Pick edge 6-5:* No cycle is formed, include it.



4. *Pick edge 0-1:* No cycle is formed, include it.

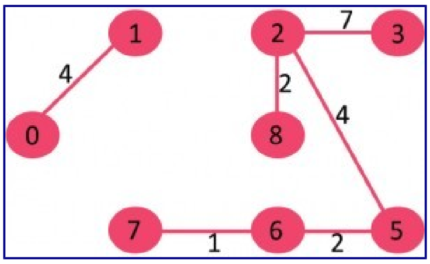


5. *Pick edge 2-5:* No cycle is formed, include it.



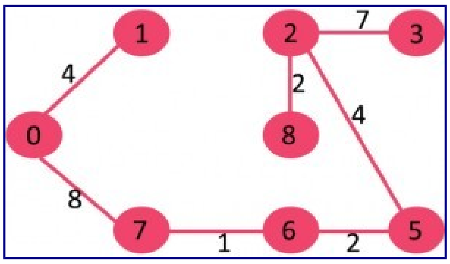
6. *Pick edge 8-6:* Since including this edge results in cycle, discard it.

7. *Pick edge 2-3:* No cycle is formed, include it.



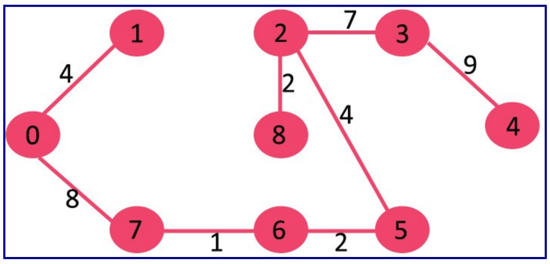
8. *Pick edge 7-8:* Since including this edge results in cycle, discard it.

9. *Pick edge 0-7:* No cycle is formed, include it.



10. *Pick edge 1-2:* Since including this edge results in cycle, discard it.

11. *Pick edge 3-4:* No cycle is formed, include it.



Since the number of edges included equals (V – 1), the algorithm stops here.

We strongly recommend you to minimize your browser and try this yourself first.

```
// C++ program for Kruskal's algorithm to find Minimum Spanning
Tree
```

```
// of a given connected, undirected and weighted graph
#include <stdio.h>
#include <stdlib.h>
#include <string.h>

// a structure to represent a weighted edge in graph
struct Edge
{
    int src, dest, weight;
};

// a structure to represent a connected, undirected and weighted
graph
struct Graph
{
    // V-> Number of vertices, E-> Number of edges
    int V, E;

    // graph is represented as an array of edges. Since the graph
is
    // undirected, the edge from src to dest is also edge from
dest
    // to src. Both are counted as 1 edge here.
    struct Edge* edge;
};

// Creates a graph with V vertices and E edges
struct Graph* createGraph(int V, int E)
{
    struct Graph* graph = (struct Graph*) malloc( sizeof(struct
Graph) );
    graph->V = V;
    graph->E = E;

    graph->edge = (struct Edge*) malloc( graph->E * sizeof( struct
Edge ) );

    return graph;
}

// A structure to represent a subset for union-find
struct subset
{
    int parent;
    int rank;
};

// A utility function to find set of an element i
// (uses path compression technique)
int find(struct subset subsets[], int i)
{
    // find root and make root as parent of i (path compression)
```

```
    if (subsets[i].parent != i)
        subsets[i].parent = find(subsets, subsets[i].parent);

    return subsets[i].parent;
}

// A function that does union of two sets of x and y
// (uses union by rank)
void Union(struct subset subsets[], int x, int y)
{
    int xroot = find(subsets, x);
    int yroot = find(subsets, y);

    // Attach smaller rank tree under root of high rank tree
    // (Union by Rank)
    if (subsets[xroot].rank < subsets[yroot].rank)
        subsets[xroot].parent = yroot;
    else if (subsets[xroot].rank > subsets[yroot].rank)
        subsets[yroot].parent = xroot;

    // If ranks are same, then make one as root and increment
    // its rank by one
    else
    {
        subsets[yroot].parent = xroot;
        subsets[xroot].rank++;
    }
}

// Compare two edges according to their weights.
// Used in qsort() for sorting an array of edges
int myComp(const void* a, const void* b)
{
    struct Edge* a1 = (struct Edge*)a;
    struct Edge* b1 = (struct Edge*)b;
    return a1->weight > b1->weight;
}

// The main function to construct MST using Kruskal's algorithm
void KruskalMST(struct Graph* graph)
{
    int V = graph->V;
    struct Edge result[V];  // Tnis will store the resultant MST
    int e = 0;  // An index variable, used for result[]
    int i = 0;  // An index variable, used for sorted edges

    // Step 1:  Sort all the edges in non-decreasing order of
their weight
    // If we are not allowed to change the given graph, we can
create a copy of
    // array of edges
    qsort(graph->edge, graph->E, sizeof(graph->edge[0]), myComp);
```

```
    // Allocate memory for creating V ssubsets
    struct subset *subsets =
        (struct subset*) malloc( V * sizeof(struct subset) );

    // Create V subsets with single elements
    for (int v = 0; v < V; ++v)
    {
        subsets[v].parent = v;
        subsets[v].rank = 0;
    }

    // Number of edges to be taken is equal to V-1
    while (e < V - 1)
    {
        // Step 2: Pick the smallest edge. And increment the index
        // for next iteration
        struct Edge next_edge = graph->edge[i++];

        int x = find(subsets, next_edge.src);
        int y = find(subsets, next_edge.dest);

        // If including this edge does't cause cycle, include it
        // in result and increment the index of result for next
edge
        if (x != y)
        {
            result[e++] = next_edge;
            Union(subsets, x, y);
        }
        // Else discard the next_edge
    }

    // print the contents of result[] to display the built MST
    printf("Following are the edges in the constructed MST\n");
    for (i = 0; i < e; ++i)
        printf("%d -- %d == %d\n", result[i].src, result[i].dest,
                                                   result[i].weigh
t);
    return;
}

// Driver program to test above functions
int main()
{
    /* Let us create following weighted graph
             10
        0--------1
        |  \     |
       6|   5\   |15
        |      \ |
        2--------3
```

```
         4       */
    int V = 4;  // Number of vertices in graph
    int E = 5;  // Number of edges in graph
    struct Graph* graph = createGraph(V, E);


    // add edge 0-1
    graph->edge[0].src = 0;
    graph->edge[0].dest = 1;
    graph->edge[0].weight = 10;

    // add edge 0-2
    graph->edge[1].src = 0;
    graph->edge[1].dest = 2;
    graph->edge[1].weight = 6;

    // add edge 0-3
    graph->edge[2].src = 0;
    graph->edge[2].dest = 3;
    graph->edge[2].weight = 5;

    // add edge 1-3
    graph->edge[3].src = 1;
    graph->edge[3].dest = 3;
    graph->edge[3].weight = 15;

    // add edge 2-3
    graph->edge[4].src = 2;
    graph->edge[4].dest = 3;
    graph->edge[4].weight = 4;

    KruskalMST(graph);

    return 0;
}
```

```
Following are the edges in the constructed MST
2 -- 3 == 4
0 -- 3 == 5
0 -- 1 == 10
```

Time Complexity: O(ElogE) or O(ElogV). Sorting of edges takes O(ELogE) time. After sorting, we iterate through all edges and apply find-union algorithm. The find and union operations can take atmost O(LogV) time. So overall complexity is O(ELogE + ELogV) time. The value of E can be atmost $O(V^2)$, so O(LogV) are O(LogE) same. Therefore, overall time complexity is O(ElogE) or O(ElogV)

Greedy Algorithms | Set 5 (Prim's Minimum Spanning Tree (MST))

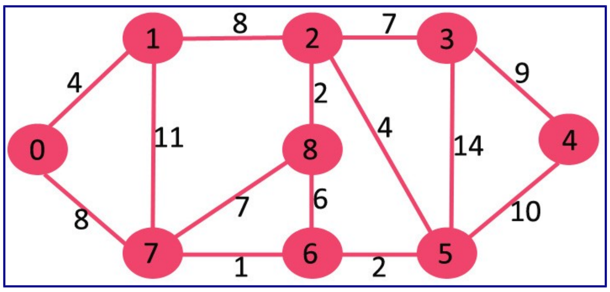
We have discussed Kruskal's algorithm for Minimum Spanning Tree. Like Kruskal's algorithm, Prim's algorithm is also a Greedy algorithm. It starts with an empty spanning tree. The idea is to maintain two sets of vertices. The first set contains the vertices already included in the MST, the other set contains the vertices not yet included. At every step, it considers all the edges that connect the two sets, and picks the minimum weight edge from these edges. After picking the edge, it moves the other endpoint of the edge to the set containing MST.
A group of edges that connects two set of vertices in a graph is called cut in graph theory. *So, at every step of Prim's algorithm, we find a cut (of two sets, one contains the vertices already included in MST and other contains rest of the verices), pick the minimum weight edge from the cut and include this vertex to MST Set (the set that contains already included vertices).*

How does Prim's Algorithm Work? The idea behind Prim's algorithm is simple, a spanning tree means all vertices must be connected. So the two disjoint subsets (discussed above) of vertices must be connected to make a *Spanning* Tree. And they must be connected with the minimum weight edge to make it a *Minimum* Spanning Tree.

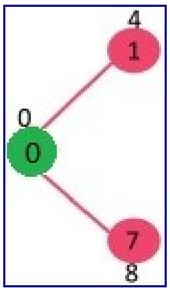
Algorithm
1) Create a set *mstSet* that keeps track of vertices already included in MST.
2) Assign a key value to all vertices in the input graph. Initialize all key values as INFINITE. Assign key value as 0 for the first vertex so that it is picked first.
3) While mstSet doesn't include all vertices
….**a)** Pick a vertex *u* which is not there in *mstSet* and has minimum key value.
….**b)** Include *u* to mstSet.
….**c)** Update key value of all adjacent vertices of *u*. To update the key values, iterate through all adjacent vertices. For every adjacent vertex *v*, if weight of edge *u*-*v* is less than the previous key value of *v*, update the key value as weight of *u*-*v*

The idea of using key values is to pick the minimum weight edge from cut. The key values are used only for vertices which are not yet included in MST, the key value for these vertices indicate the minimum weight edges connecting them to the set of vertices included in MST.

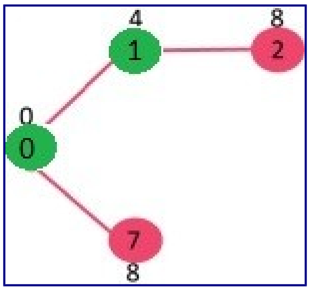
Let us understand with the following example:



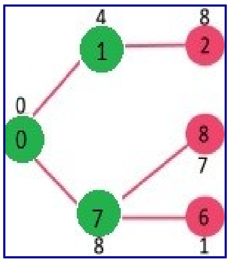
The set *mstSet* is initially empty and keys assigned to vertices are {0, INF, INF, INF, INF, INF, INF, INF} where INF indicates infinite. Now pick the vertex with minimum key value. The vertex 0 is picked, include it in *mstSet*. So *mstSet* becomes {0}. After including to *mstSet*, update key values of adjacent vertices. Adjacent vertices of 0 are 1 and 7. The key values of 1 and 7 are updated as 4 and 8. Following subgraph shows vertices and their key values, only the vertices with finite key values are shown. The vertices included in MST are shown in green color.



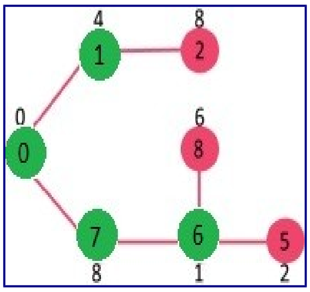
Pick the vertex with minimum key value and not already included in MST (not in mstSET). The vertex 1 is picked and added to mstSet. So mstSet now becomes {0, 1}. Update the key values of adjacent vertices of 1. The key value of vertex 2 becomes 8.



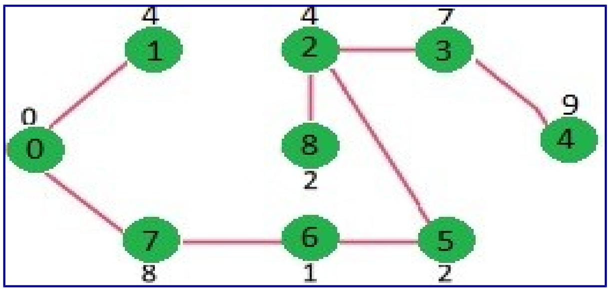
Pick the vertex with minimum key value and not already included in MST (not in mstSET). We can either pick vertex 7 or vertex 2, let vertex 7 is picked. So mstSet now becomes {0, 1, 7}. Update the key values of adjacent vertices of 7. The key value of vertex 6 and 8 becomes finite (7 and 1 respectively).



Pick the vertex with minimum key value and not already included in MST (not in mstSET). Vertex 6 is picked. So mstSet now becomes {0, 1, 7, 6}. Update the key values of adjacent vertices of 6. The key value of vertex 5 and 8 are updated.



We repeat the above steps until *mstSet* includes all vertices of given graph. Finally, we get the following graph.



[We strongly recommend that you click here and practice it, before moving on to the solution.]

How to implement the above algorithm?

We use a boolean array mstSet[] to represent the set of vertices included in MST. If a value mstSet[v] is true, then vertex v is included in MST, otherwise not. Array key[] is used to store key values of all vertices. Another array parent[] to store indexes of parent nodes in MST. The parent array is the output array which is used to show the constructed MST.

```
// A C / C++ program for Prim's Minimum Spanning Tree (MST)
algorithm.
// The program is for adjacency matrix representation of the graph

#include <stdio.h>
#include <limits.h>

// Number of vertices in the graph
#define V 5

// A utility function to find the vertex with minimum key value,
from
// the set of vertices not yet included in MST
int minKey(int key[], bool mstSet[])
{
   // Initialize min value
   int min = INT_MAX, min_index;

   for (int v = 0; v < V; v++)
     if (mstSet[v] == false && key[v] < min)
```

```
         min = key[v], min_index = v;

   return min_index;
}

// A utility function to print the constructed MST stored in
parent[]
int printMST(int parent[], int n, int graph[V][V])
{
   printf("Edge   Weight\n");
   for (int i = 1; i < V; i++)
      printf("%d - %d    %d \n", parent[i], i, graph[i]
[parent[i]]);
}

// Function to construct and print MST for a graph represented
using adjacency
// matrix representation
void primMST(int graph[V][V])
{
     int parent[V]; // Array to store constructed MST
     int key[V];   // Key values used to pick minimum weight edge
in cut
     bool mstSet[V];  // To represent set of vertices not yet
included in MST

     // Initialize all keys as INFINITE
     for (int i = 0; i < V; i++)
        key[i] = INT_MAX, mstSet[i] = false;

     // Always include first 1st vertex in MST.
     key[0] = 0;     // Make key 0 so that this vertex is picked
as first vertex
     parent[0] = -1; // First node is always root of MST

     // The MST will have V vertices
     for (int count = 0; count < V-1; count++)
     {
        // Pick the minimum key vertex from the set of vertices
        // not yet included in MST
        int u = minKey(key, mstSet);

        // Add the picked vertex to the MST Set
        mstSet[u] = true;

        // Update key value and parent index of the adjacent
vertices of
        // the picked vertex. Consider only those vertices which
are not yet
        // included in MST
        for (int v = 0; v < V; v++)
```

```
            // graph[u][v] is non zero only for adjacent vertices
of m
            // mstSet[v] is false for vertices not yet included in
MST
            // Update the key only if graph[u][v] is smaller than
key[v]
          if (graph[u][v] && mstSet[v] == false && graph[u][v] <
key[v])
             parent[v]  = u, key[v] = graph[u][v];
     }

     // print the constructed MST
     printMST(parent, V, graph);
}


// driver program to test above function
int main()
{
   /* Let us create the following graph
          2    3
      (0)--(1)--(2)
       |   / \   |
      6| 8/   \5 |7
       | /     \ |
      (3)-------(4)
            9          */
   int graph[V][V] = {{0, 2, 0, 6, 0},
                      {2, 0, 3, 8, 5},
                      {0, 3, 0, 0, 7},
                      {6, 8, 0, 0, 9},
                      {0, 5, 7, 9, 0},
                     };

    // Print the solution
    primMST(graph);

    return 0;
}
```

Output:

```
Edge   Weight
0 - 1    2
1 - 2    3
0 - 3    6
1 - 4    5
```

Time Complexity of the above program is O(V^2). If the input graph is represented using adjacency list, then the time complexity of Prim's algorithm can be reduced to O(E log V) with the help of binary heap. Please see Prim's MST for Adjacency List Representation for more details.

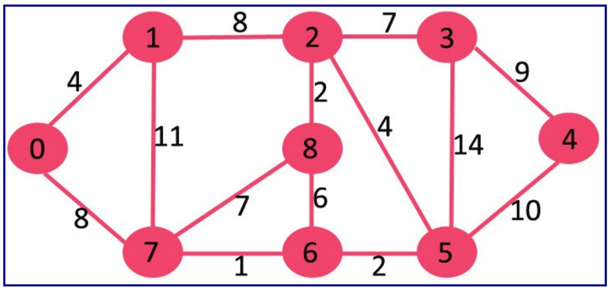
Greedy Algorithms | Set 7 (Dijkstra's shortest path algorithm)

Given a graph and a source vertex in graph, find shortest paths from source to all vertices in the given graph.

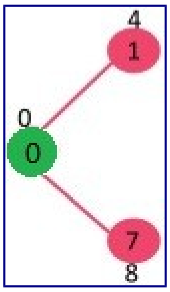
Dijkstra's algorithm is very similar to Prim's algorithm for minimum spanning tree. Like Prim's MST, we generate a *SPT (shortest path tree)* with given source as root. We maintain two sets, one set contains vertices included in shortest path tree, other set includes vertices not yet included in shortest path tree. At every step of the algorithm, we find a vertex which is in the other set (set of not yet included) and has minimum distance from source.

Below are the detailed steps used in Dijkstra's algorithm to find the shortest path from a single source vertex to all other vertices in the given graph.
Algorithm
1) Create a set *sptSet* (shortest path tree set) that keeps track of vertices included in shortest path tree, i.e., whose minimum distance from source is calculated and finalized. Initially, this set is empty.
2) Assign a distance value to all vertices in the input graph. Initialize all distance values as INFINITE. Assign distance value as 0 for the source vertex so that it is picked first.
3) While *sptSet* doesn't include all vertices
….**a)** Pick a vertex u which is not there in *sptSet*and has minimum distance value.
….**b)** Include u to *sptSet*.
….**c)** Update distance value of all adjacent vertices of u. To update the distance values, iterate through all adjacent vertices. For every adjacent vertex v, if sum of distance value of u (from source) and weight of edge u-v, is less than the distance value of v, then update the distance value of v.

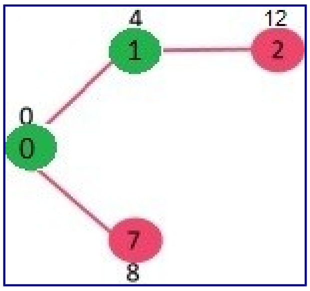
Let us understand with the following example:



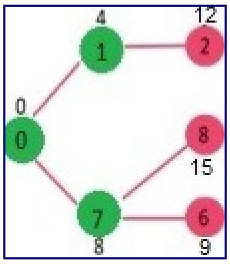
The set *sptSet*is initially empty and distances assigned to vertices are {0, INF, INF, INF, INF, INF, INF, INF} where INF indicates infinite. Now pick the vertex with minimum distance value. The vertex 0 is picked, include it in *sptSet*. So *sptSet* becomes {0}. After including 0 to *sptSet*, update distance values of its adjacent vertices. Adjacent vertices of 0 are 1 and 7. The distance values of 1 and 7 are updated as 4 and 8. Following subgraph shows vertices and their distance values, only the vertices with finite distance values are shown. The vertices included in SPT are shown in green color.



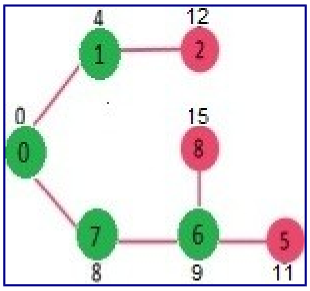
Pick the vertex with minimum distance value and not already included in SPT (not in sptSET). The vertex 1 is picked and added to sptSet. So sptSet now becomes {0, 1}. Update the distance values of adjacent vertices of 1. The distance value of vertex 2 becomes 12.



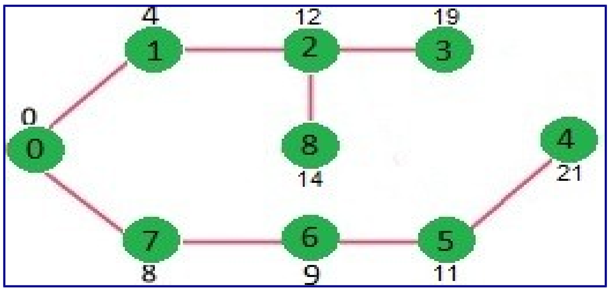
Pick the vertex with minimum distance value and not already included in SPT (not in sptSET). Vertex 7 is picked. So sptSet now becomes {0, 1, 7}. Update the distance values of adjacent vertices of 7. The distance value of vertex 6 and 8 becomes finite (15 and 9 respectively).



Pick the vertex with minimum distance value and not already included in SPT (not in sptSET). Vertex 6 is picked. So sptSet now becomes {0, 1, 7, 6}. Update the distance values of adjacent vertices of 6. The distance value of vertex 5 and 8 are updated.



We repeat the above steps until *sptSet* doesn't include all vertices of given graph. Finally, we get the following Shortest Path Tree (SPT).



How to implement the above algorithm?

[We strongly recommend that you click here and practice it, before moving on to the solution.]

We use a boolean array sptSet[] to represent the set of vertices included in SPT. If a value sptSet[v] is true, then vertex v is included in SPT, otherwise not. Array dist[] is used to store shortest distance values of all vertices.

```
// A C / C++ program for Dijkstra's single source shortest path
algorithm.
// The program is for adjacency matrix representation of the graph

#include <stdio.h>
#include <limits.h>

// Number of vertices in the graph
#define V 9

// A utility function to find the vertex with minimum distance
value, from
// the set of vertices not yet included in shortest path tree
int minDistance(int dist[], bool sptSet[])
{
   // Initialize min value
   int min = INT_MAX, min_index;

   for (int v = 0; v < V; v++)
     if (sptSet[v] == false && dist[v] <= min)
         min = dist[v], min_index = v;

   return min_index;
}

// A utility function to print the constructed distance array
int printSolution(int dist[], int n)
{
   printf("Vertex   Distance from Source\n");
   for (int i = 0; i < V; i++)
      printf("%d \t\t %d\n", i, dist[i]);
}
```

```
// Funtion that implements Dijkstra's single source shortest path
algorithm
// for a graph represented using adjacency matrix representation
void dijkstra(int graph[V][V], int src)
{
     int dist[V];     // The output array.  dist[i] will hold the
shortest
                      // distance from src to i

     bool sptSet[V]; // sptSet[i] will true if vertex i is included
in shortest
                     // path tree or shortest distance from src to
i is finalized

     // Initialize all distances as INFINITE and stpSet[] as false
     for (int i = 0; i < V; i++)
        dist[i] = INT_MAX, sptSet[i] = false;

     // Distance of source vertex from itself is always 0
     dist[src] = 0;

     // Find shortest path for all vertices
     for (int count = 0; count < V-1; count++)
     {
       // Pick the minimum distance vertex from the set of
vertices not
       // yet processed. u is always equal to src in first
iteration.
       int u = minDistance(dist, sptSet);

       // Mark the picked vertex as processed
       sptSet[u] = true;

       // Update dist value of the adjacent vertices of the picked
vertex.
       for (int v = 0; v < V; v++)

         // Update dist[v] only if is not in sptSet, there is an
edge from
         // u to v, and total weight of path from src to  v
through u is
         // smaller than current value of dist[v]
         if (!sptSet[v] && graph[u][v] && dist[u] != INT_MAX
                                       && dist[u]+graph[u][v] <
dist[v])
            dist[v] = dist[u] + graph[u][v];
     }

     // print the constructed distance array
     printSolution(dist, V);
}
```

```
// driver program to test above function
int main()
{
   /* Let us create the example graph discussed above */
   int graph[V][V] = {{0, 4, 0, 0, 0, 0, 0, 8, 0},
                      {4, 0, 8, 0, 0, 0, 0, 11, 0},
                      {0, 8, 0, 7, 0, 4, 0, 0, 2},
                      {0, 0, 7, 0, 9, 14, 0, 0, 0},
                      {0, 0, 0, 9, 0, 10, 0, 0, 0},
                      {0, 0, 4, 14, 10, 0, 2, 0, 0},
                      {0, 0, 0, 0, 0, 2, 0, 1, 6},
                      {8, 11, 0, 0, 0, 0, 1, 0, 7},
                      {0, 0, 2, 0, 0, 0, 6, 7, 0}
                     };

    dijkstra(graph, 0);

    return 0;
}
```

Output:

```
Vertex   Distance from Source
0                0
1                4
2                12
3                19
4                21
5                11
6                9
7                8
8                14
```

Notes:

1) The code calculates shortest distance, but doesn't calculate the path information. We can create a parent array, update the parent array when distance is updated (like [prim's implementation]) and use it show the shortest path from source to different vertices.

2) The code is for undirected graph, same dijekstra function can be used for directed graphs also.

3) The code finds shortest distances from source to all vertices. If we are interested only in shortest distance from source to a single target, we can break the for loop when the picked minimum distance vertex is equal to target (Step 3.a of algorithm).

4) Time Complexity of the implementation is O(V^2). If the input graph is represented using adjacency list, it can be reduced to O(E log V) with the help of binary heap. Please see Dijkstra's Algorithm for Adjacency List Representation for more details.

5) Dijkstra's algorithm doesn't work for graphs with negative weight edges. For graphs with negative weight edges, Bellman–Ford algorithm can be used, we will soon be discussing it as a separate post.

Dynamic Programming | Set 23 (Bellman–Ford Algorithm)

Given a graph and a source vertex *src* in graph, find shortest paths from *src* to all vertices in the given graph. The graph may contain negative weight edges.
We have discussed [Dijkstra's algorithm] for this problem. Dijksra's algorithm is a Greedy algorithm and time complexity is O(VLogV) (with the use of Fibonacci heap). *Dijkstra doesn't work for Graphs with negative weight edges, Bellman-Ford works for such graphs. Bellman-Ford is also simpler than Dijkstra and suites well for distributed systems. But time complexity of Bellman-Ford is O(VE), which is more than Dijkstra.*

Algorithm
Following are the detailed steps.

Input: Graph and a source vertex *src*
Output: Shortest distance to all vertices from *src*. If there is a negative weight cycle, then shortest distances are not calculated, negative weight cycle is reported.

1) This step initializes distances from source to all vertices as infinite and distance to source itself as 0. Create an array dist[] of size |V| with all values as infinite except dist[src] where src is source vertex.

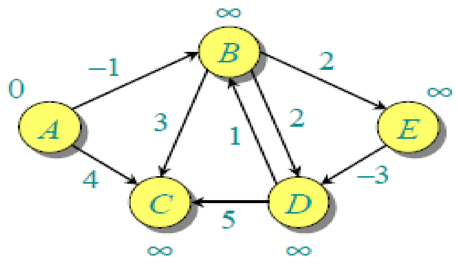
2) This step calculates shortest distances. Do following |V|-1 times where |V| is the number of vertices in given graph.
…..**a)** Do following for each edge u-v
………………If dist[v] > dist[u] + weight of edge uv, then update dist[v]
………………….dist[v] = dist[u] + weight of edge uv

3) This step reports if there is a negative weight cycle in graph. Do following for each edge u-v
……If dist[v] > dist[u] + weight of edge uv, then "Graph contains negative weight cycle"
The idea of step 3 is, step 2 guarantees shortest distances if graph doesn't contain negative weight cycle. If we iterate through all edges one more time and get a shorter path for any vertex, then there is a negative weight cycle

How does this work? Like other Dynamic Programming Problems, the algorithm calculate shortest paths in bottom-up manner. It first calculates the shortest distances for the shortest paths which have at-most one edge in the path. Then, it calculates shortest paths with at-nost 2 edges, and so on. After the ith iteration of outer loop, the shortest paths with at most i edges are calculated. There can be maximum |V| – 1 edges in any simple path, that is why the outer loop runs |v| – 1 times. The idea is, assuming that there is no negative weight cycle, if we have calculated shortest paths with at most i edges, then an iteration over all edges guarantees to give shortest path with at-most (i+1) edges (Proof is simple, you can refer this or MIT Video Lecture)

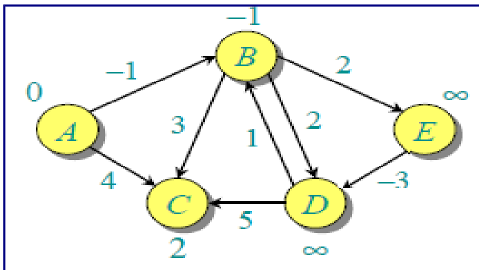
Example
Let us understand the algorithm with following example graph. The images are taken from this source.

Let the given source vertex be 0. Initialize all distances as infinite, except the distance to source itself. Total number of vertices in the graph is 5, so *all edges must be processed 4 times.*



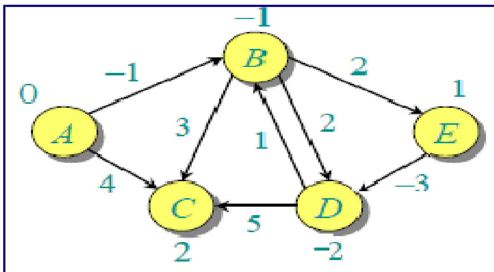
A	B	C	D	E
0	∞	∞	∞	∞

Let all edges are processed in following order: (B,E), (D,B), (B,D), (A,B), (A,C), (D,C), (B,C), (E,D). We get following distances when all edges are processed first time. The first row in shows initial distances. The second row shows distances when edges (B,E), (D,B), (B,D) and (A,B) are processed. The third row shows distances when (A,C) is processed. The fourth row shows when (D,C), (B,C) and (E,D) are processed.



A	B	C	D	E
0	∞	∞	∞	∞
0	−1	∞	∞	∞
0	−1	4	∞	∞
0	−1	2	∞	∞

The first iteration guarantees to give all shortest paths which are at most 1 edge long. We get following distances when all edges are processed second time (The last row shows final values).



A	B	C	D	E
0	∞	∞	∞	∞
0	−1	∞	∞	∞
0	−1	4	∞	∞
0	−1	2	∞	∞
0	−1	2	∞	1
0	−1	2	1	1
0	−1	2	−2	1

The second iteration guarantees to give all shortest paths which are at most 2 edges long. The algorithm processes all edges 2 more times. The distances are minimized after the second iteration, so third and fourth iterations don’t update the distances.

Implementation:

```
// A C / C++ program for Bellman-Ford's single source
// shortest path algorithm.

#include <stdio.h>
#include <stdlib.h>
```

```
#include <string.h>
#include <limits.h>

// a structure to represent a weighted edge in graph
struct Edge
{
    int src, dest, weight;
};

// a structure to represent a connected, directed and
// weighted graph
struct Graph
{
    // V-> Number of vertices, E-> Number of edges
    int V, E;

    // graph is represented as an array of edges.
    struct Edge* edge;
};

// Creates a graph with V vertices and E edges
struct Graph* createGraph(int V, int E)
{
    struct Graph* graph =
         (struct Graph*) malloc( sizeof(struct Graph) );
    graph->V = V;
    graph->E = E;

    graph->edge =
       (struct Edge*) malloc( graph->E * sizeof( struct Edge ) );

    return graph;
}

// A utility function used to print the solution
void printArr(int dist[], int n)
{
    printf("Vertex   Distance from Source\n");
    for (int i = 0; i < n; ++i)
        printf("%d \t\t %d\n", i, dist[i]);
}

// The main function that finds shortest distances from src to
// all other vertices using Bellman-Ford algorithm.  The function
// also detects negative weight cycle
void BellmanFord(struct Graph* graph, int src)
{
    int V = graph->V;
    int E = graph->E;
    int dist[V];

    // Step 1: Initialize distances from src to all other vertices
```

```
    // as INFINITE
    for (int i = 0; i < V; i++)
        dist[i]   = INT_MAX;
    dist[src] = 0;

    // Step 2: Relax all edges |V| - 1 times. A simple shortest
    // path from src to any other vertex can have at-most |V| - 1
    // edges
    for (int i = 1; i <= V-1; i++)
    {
        for (int j = 0; j < E; j++)
        {
            int u = graph->edge[j].src;
            int v = graph->edge[j].dest;
            int weight = graph->edge[j].weight;
            if (dist[u] != INT_MAX && dist[u] + weight < dist[v])
                dist[v] = dist[u] + weight;
        }
    }

    // Step 3: check for negative-weight cycles.  The above step
    // guarantees shortest distances if graph doesn't contain
    // negative weight cycle.  If we get a shorter path, then
there
    // is a cycle.
    for (int i = 0; i < E; i++)
    {
        int u = graph->edge[i].src;
        int v = graph->edge[i].dest;
        int weight = graph->edge[i].weight;
        if (dist[u] != INT_MAX && dist[u] + weight < dist[v])
            printf("Graph contains negative weight cycle");
    }

    printArr(dist, V);

    return;
}

// Driver program to test above functions
int main()
{
    /* Let us create the graph given in above example */
    int V = 5;  // Number of vertices in graph
    int E = 8;  // Number of edges in graph
    struct Graph* graph = createGraph(V, E);

    // add edge 0-1 (or A-B in above figure)
    graph->edge[0].src = 0;
    graph->edge[0].dest = 1;
    graph->edge[0].weight = -1;
```

```
    // add edge 0-2 (or A-C in above figure)
    graph->edge[1].src = 0;
    graph->edge[1].dest = 2;
    graph->edge[1].weight = 4;

    // add edge 1-2 (or B-C in above figure)
    graph->edge[2].src = 1;
    graph->edge[2].dest = 2;
    graph->edge[2].weight = 3;

    // add edge 1-3 (or B-D in above figure)
    graph->edge[3].src = 1;
    graph->edge[3].dest = 3;
    graph->edge[3].weight = 2;

    // add edge 1-4 (or A-E in above figure)
    graph->edge[4].src = 1;
    graph->edge[4].dest = 4;
    graph->edge[4].weight = 2;

    // add edge 3-2 (or D-C in above figure)
    graph->edge[5].src = 3;
    graph->edge[5].dest = 2;
    graph->edge[5].weight = 5;

    // add edge 3-1 (or D-B in above figure)
    graph->edge[6].src = 3;
    graph->edge[6].dest = 1;
    graph->edge[6].weight = 1;

    // add edge 4-3 (or E-D in above figure)
    graph->edge[7].src = 4;
    graph->edge[7].dest = 3;
    graph->edge[7].weight = -3;

    BellmanFord(graph, 0);

    return 0;
}
```

Output:

```
Vertex   Distance from Source
0                0
1                -1
2                2
3                -2
4                1
```

Notes

1) Negative weights are found in various applications of graphs. For example, instead of paying cost for a path, we may get some advantage if we follow the path.

2) Bellman-Ford works better (better than Dijksra's) for distributed systems. Unlike Dijksra's where we need to find minimum value of all vertices, in Bellman-Ford, edges are considered one by one.

Exercise

1) The standard Bellman-Ford algorithm reports shortest path only if there is no negative weight cycles. Modify it so that it reports minimum distances even if there is a negative weight cycle.

2) Can we use Dijksra's algorithm for shortest paths for graphs with negative weights – one idea can be, calculate the minimum weight value, add a positive value (equal to absolute value of minimum weight value) to all weights and run the Dijksra's algorithm for the modified graph. Will this algorithm work?