Sorting Algorithms

Ananth Grama, Anshul Gupta, George Karypis, and Vipin Kumar

To accompany the text "Introduction to Parallel Computing",
Addison Wesley, 2003.

Topic Overview

- Issues in Sorting on Parallel Computers
- Sorting Networks
- Bubble Sort and its Variants
- Quicksort
- Bucket and Sample Sort
- Other Sorting Algorithms

Sorting: Overview

- One of the most commonly used and well-studied kernels.

- Sorting can be *comparison-based* or *noncomparison-based*.

- The fundamental operation of comparison-based sorting is *compare-exchange*.

- The lower bound on any comparison-based sort of n numbers is $\Theta(n \log n)$.

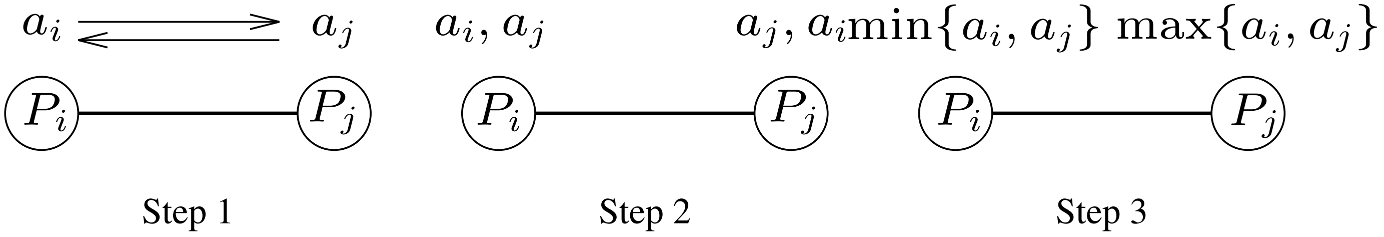
- We focus here on comparison-based sorting algorithms.

Sorting: Basics

What is a parallel sorted sequence? Where are the input and output lists stored?

- We assume that the input and output lists are distributed.
- The sorted list is partitioned with the property that each partitioned list is sorted and each element in processor P_i's list is less than that in P_j's list if $i < j$.

Sorting: Parallel Compare Exchange Operation



A parallel compare-exchange operation. Processes P_i and P_j send their elements to each other. Process P_i keeps $\min\{a_i, a_j\}$, and P_j keeps $\max\{a_i, a_j\}$.

Sorting: Basics

What is the parallel counterpart to a sequential comparator?

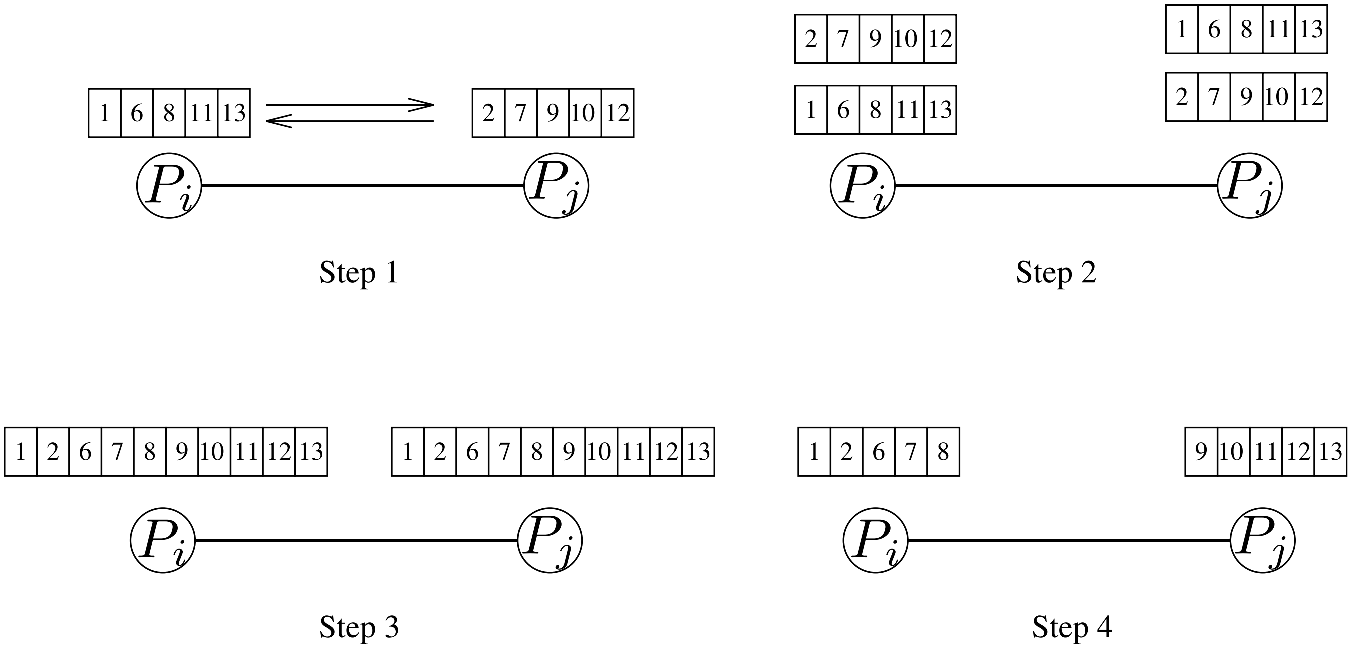
- If each processor has one element, the compare exchange operation stores the smaller element at the processor with smaller id. This can be done in $t_s + t_w$ time.

- If we have more than one element per processor, we call this operation a compare split. Assume each of two processors have n/p elements.

- After the compare-split operation, the smaller n/p elements are at processor P_i and the larger n/p elements at P_j, where $i < j$.

- The time for a compare-split operation is $(t_s + t_w n/p)$, assuming that the two partial lists were initially sorted.

Sorting: Parallel Compare Split Operation

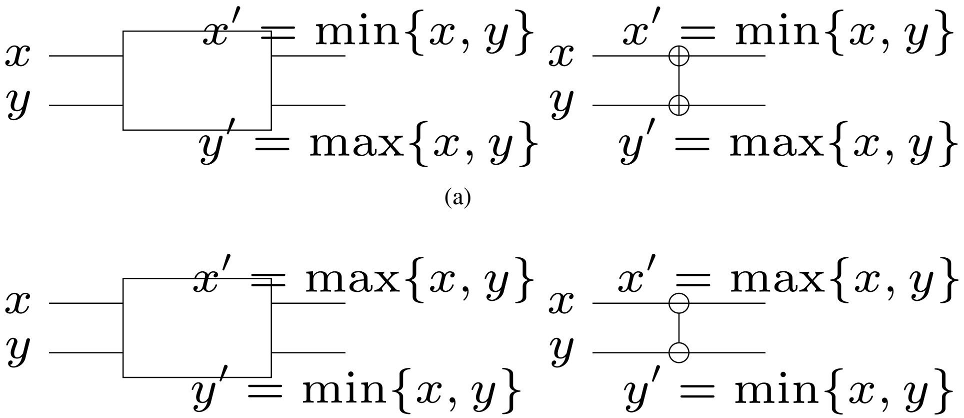


A compare-split operation. Each process sends its block of size n/p to the other process. Each process merges the received block with its own block and retains only the appropriate half of the merged block. In this example, process P_i retains the smaller elements and process P_j retains the larger elements.

Sorting Networks

- Networks of comparators designed specifically for sorting.
- A comparator is a device with two inputs x and y and two outputs x' and y'. For an *increasing comparator*, $x' = \min\{x, y\}$ and $y' = \max\{x, y\}$; and vice-versa.
- We denote an increasing comparator by $\oplus$ and a decreasing comparator by $\ominus$.
- The speed of the network is proportional to its depth.

Sorting Networks: Comparators

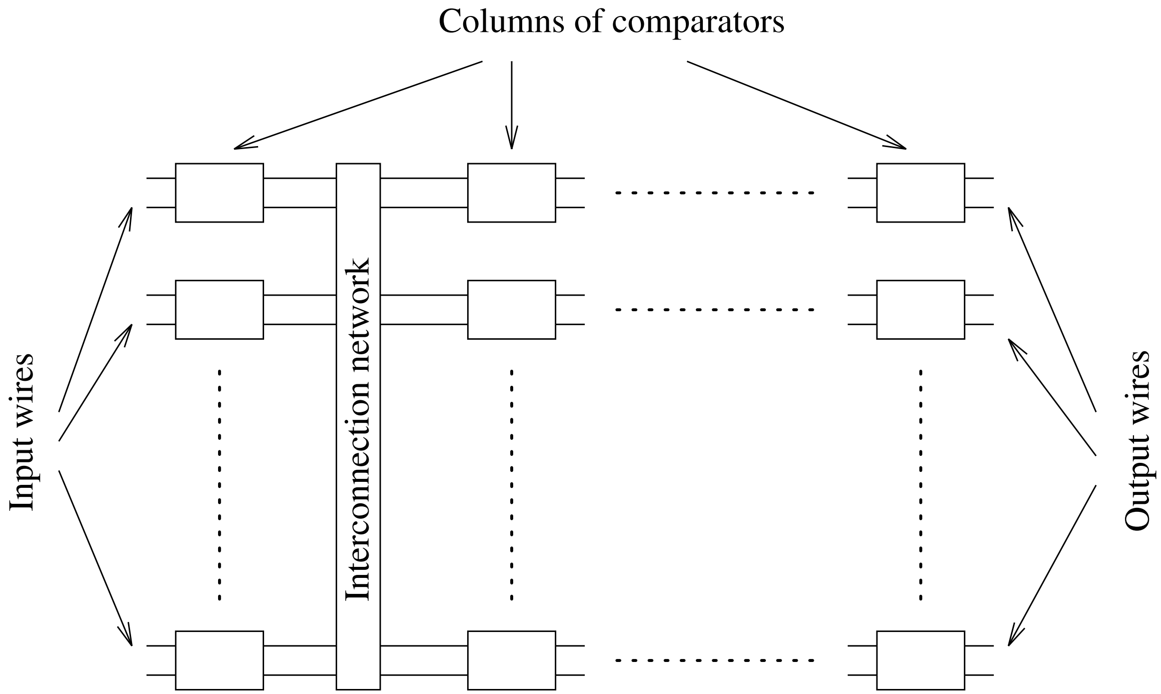


(a)

(b)

A schematic representation of comparators: (a) an increasing comparator, and (b) a decreasing comparator.

Sorting Networks



A typical sorting network. Every sorting network is made up of a series of columns, and each column contains a number of comparators connected in parallel.

Sorting Networks: Bitonic Sort

- A bitonic sorting network sorts n elements in $\Theta(\log^2 n)$ time.

- A bitonic sequence has two tones – increasing and decreasing, or vice versa. Any cyclic rotation of such networks is also considered bitonic.

- $\langle 1, 2, 4, 7, 6, 0 \rangle$ is a bitonic sequence, because it first increases and then decreases. $\langle 8, 9, 2, 1, 0, 4 \rangle$ is another bitonic sequence, because it is a cyclic shift of $\langle 0, 4, 8, 9, 2, 1 \rangle$.

- The kernel of the network is the rearrangement of a bitonic sequence into a sorted sequence.

Sorting Networks: Bitonic Sort

- Let $s = \langle a_0, a_1, \ldots, a_{n-1} \rangle$ be a bitonic sequence such that $a_0 \leq a_1 \leq \ldots \leq a_{n/2-1}$ and $a_{n/2} \geq a_{n/2+1} \geq \ldots \geq a_{n-1}$.

- Consider the following subsequences of s:

$$\begin{aligned} s_1 &= \langle \min\{a_0, a_{n/2}\}, \min\{a_1, a_{n/2+1}\}, \ldots, \min\{a_{n/2-1}, a_{n-1}\} \rangle \\ s_2 &= \langle \max\{a_0, a_{n/2}\}, \max\{a_1, a_{n/2+1}\}, \ldots, \max\{a_{n/2-1}, a_{n-1}\} \rangle \end{aligned} \tag{1}$$

- Note that s_1 and s_2 are both bitonic and each element of s_1 is less that every element in s_2.

- We can apply the procedure recursively on s_1 and s_2 to get the sorted sequence.

Sorting Networks: Bitonic Sort

Original sequence	3	5	8	9	10	12	14	20	95	90	60	40	35	23	18	0
1st Split	3	5	8	9	10	12	14	0	95	90	60	40	35	23	18	20
2nd Split	3	5	8	0	10	12	14	9	35	23	18	20	95	90	60	40
3rd Split	3	0	8	5	10	9	14	12	18	20	35	23	60	40	95	90
4th Split	0	3	5	8	9	10	12	14	18	20	23	35	40	60	90	95

Merging a 16-element bitonic sequence through a series of $\log 16$ bitonic splits.

Sorting Networks: Bitonic Sort

- We can easily build a sorting network to implement this bitonic merge algorithm.

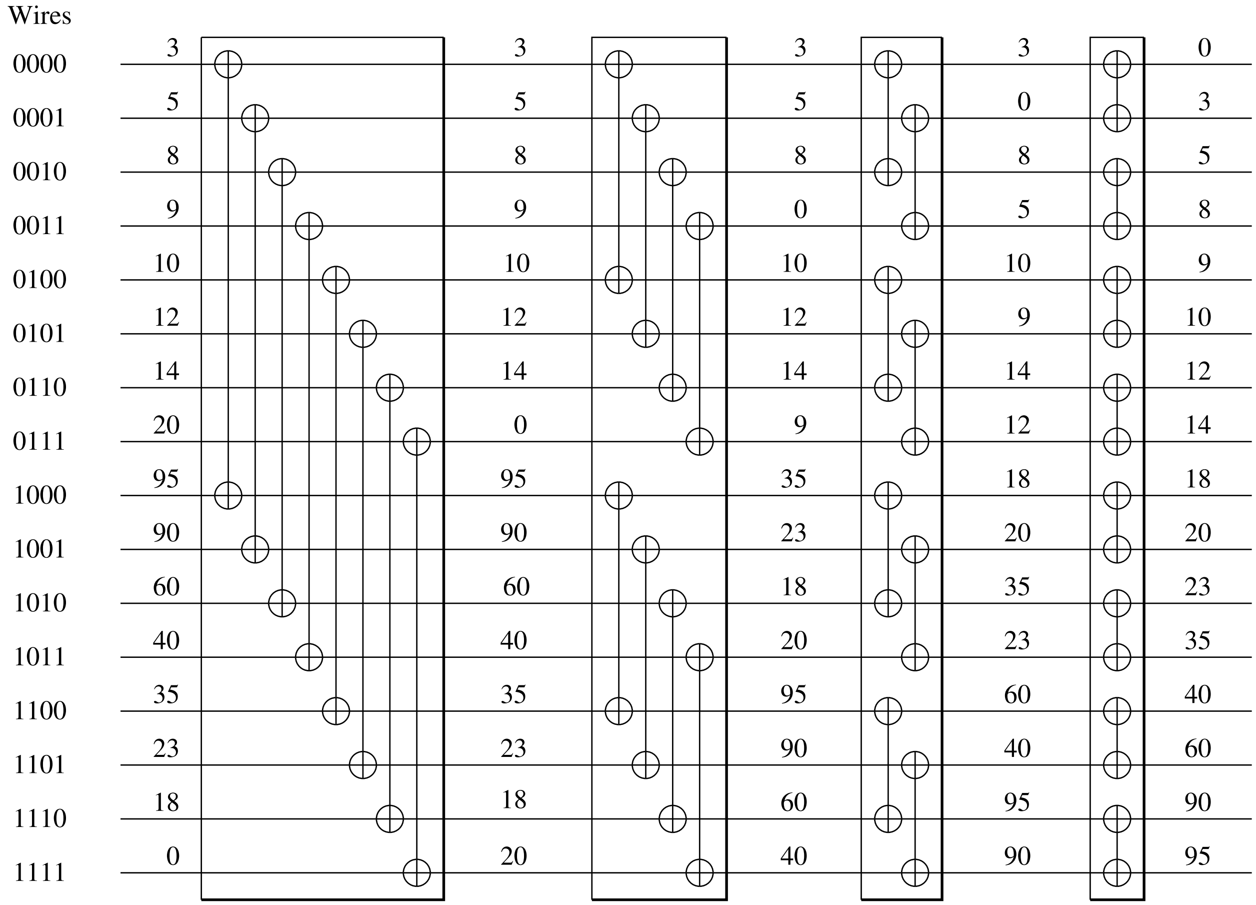
- Such a network is called a *bitonic merging network*.

- The network contains $\log n$ columns. Each column contains $n/2$ comparators and performs one step of the bitonic merge.

- We denote a bitonic merging network with n inputs by $\oplus$BM(n).

- Replacing the $\oplus$ comparators by $\ominus$ comparators results in a decreasing output sequence; such a network is denoted by $\ominus$BM(n).

Sorting Networks: Bitonic Sort



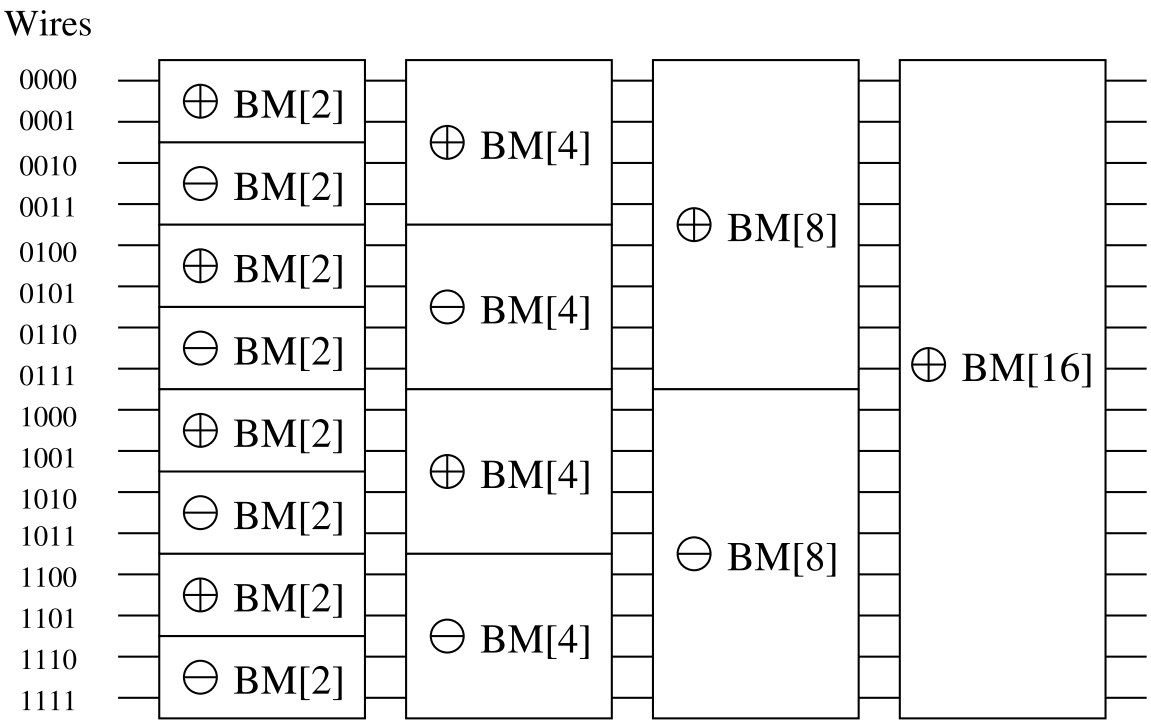
A bitonic merging network for $n = 16$. The input wires are numbered $0, 1 \ldots, n-1$, and the binary representation of these numbers is shown. Each column of comparators is drawn separately; the entire figure represents a $\oplus$BM(16) bitonic merging network. The network takes a bitonic sequence and outputs it in sorted order.

Sorting Networks: Bitonic Sort

How do we sort an unsorted sequence using a bitonic merge?

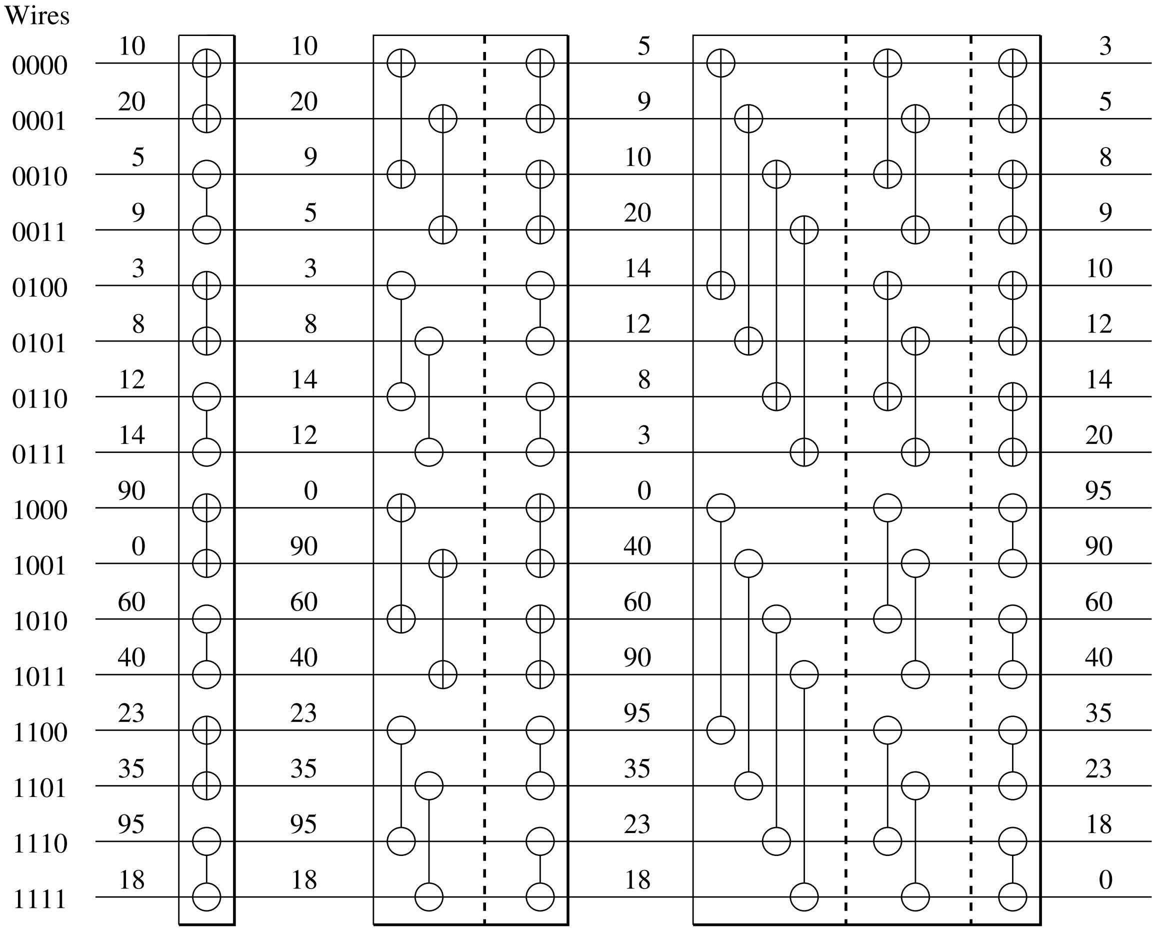
- We must first build a single bitonic sequence from the given sequence.
- A sequence of length 2 is a bitonic sequence.
- A bitonic sequence of length 4 can be built by sorting the first two elements using $\oplus$BM(2) and next two, using $\ominus$BM(2).
- This process can be repeated to generate larger bitonic sequences.

Sorting Networks: Bitonic Sort



A schematic representation of a network that converts an input sequence into a bitonic sequence. In this example, $\oplus$BM(k) and $\ominus$BM(k) denote bitonic merging networks of input size k that use $\oplus$ and $\ominus$ comparators, respectively. The last merging network ($\oplus$BM(16)) sorts the input. In this example, $n = 16$.

Sorting Networks: Bitonic Sort



The comparator network that transforms an input sequence of 16 unordered numbers into a bitonic sequence.

Sorting Networks: Bitonic Sort

- The depth of the network is $\Theta(\log^2 n)$.

- Each stage of the network contains $n/2$ comparators. A serial implementation of the network would have complexity $\Theta(n \log^2 n)$.

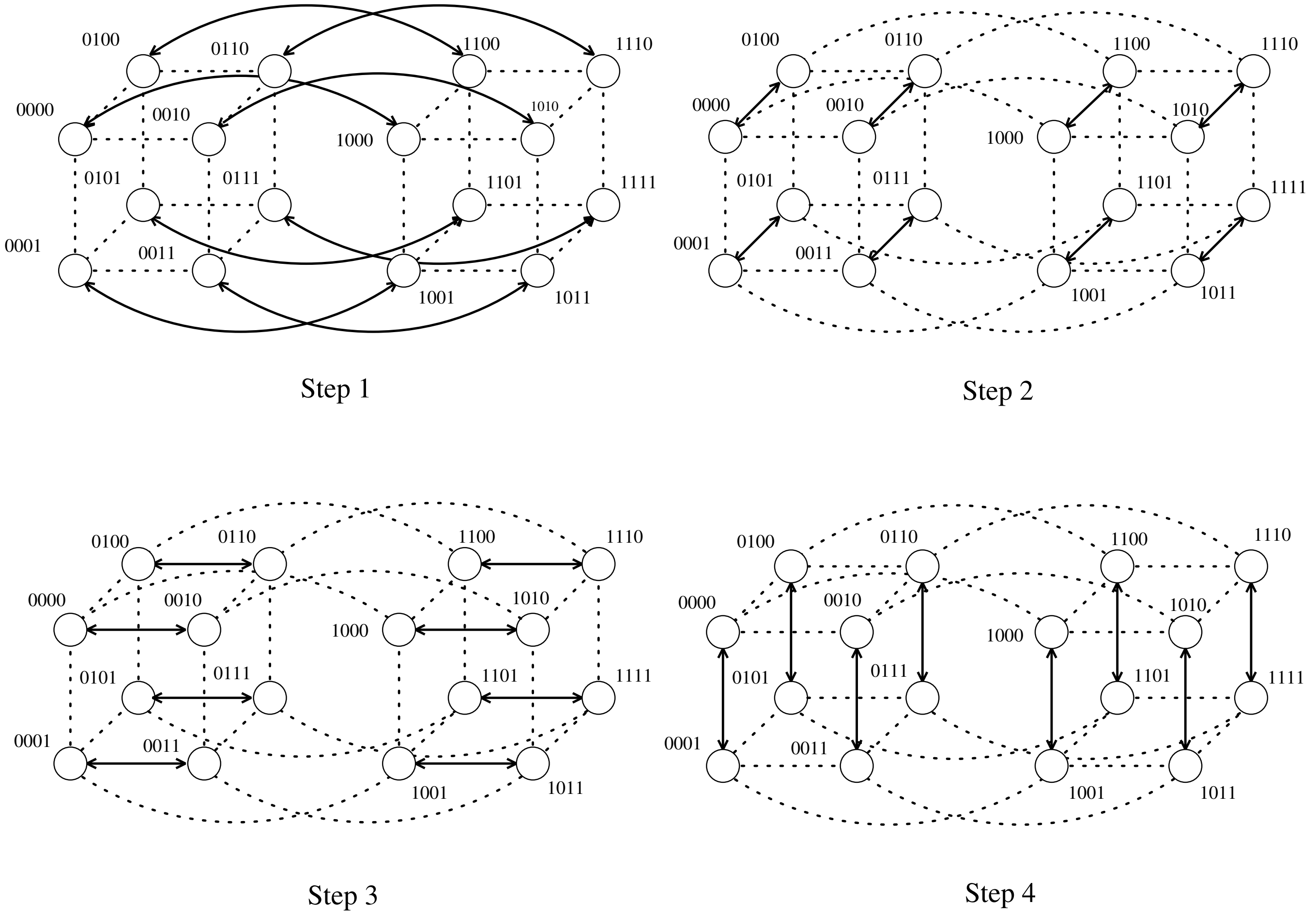
Mapping Bitonic Sort to Hypercubes

- Consider the case of one item per processor. The question becomes one of how the wires in the bitonic network should be mapped to the hypercube interconnect.

- Note from our earlier examples that the compare-exchange operation is performed between two wires only if their labels differ in exactly one bit!

- This implies a direct mapping of wires to processors. All communication is nearest neighbor!

Mapping Bitonic Sort to Hypercubes



Step 1

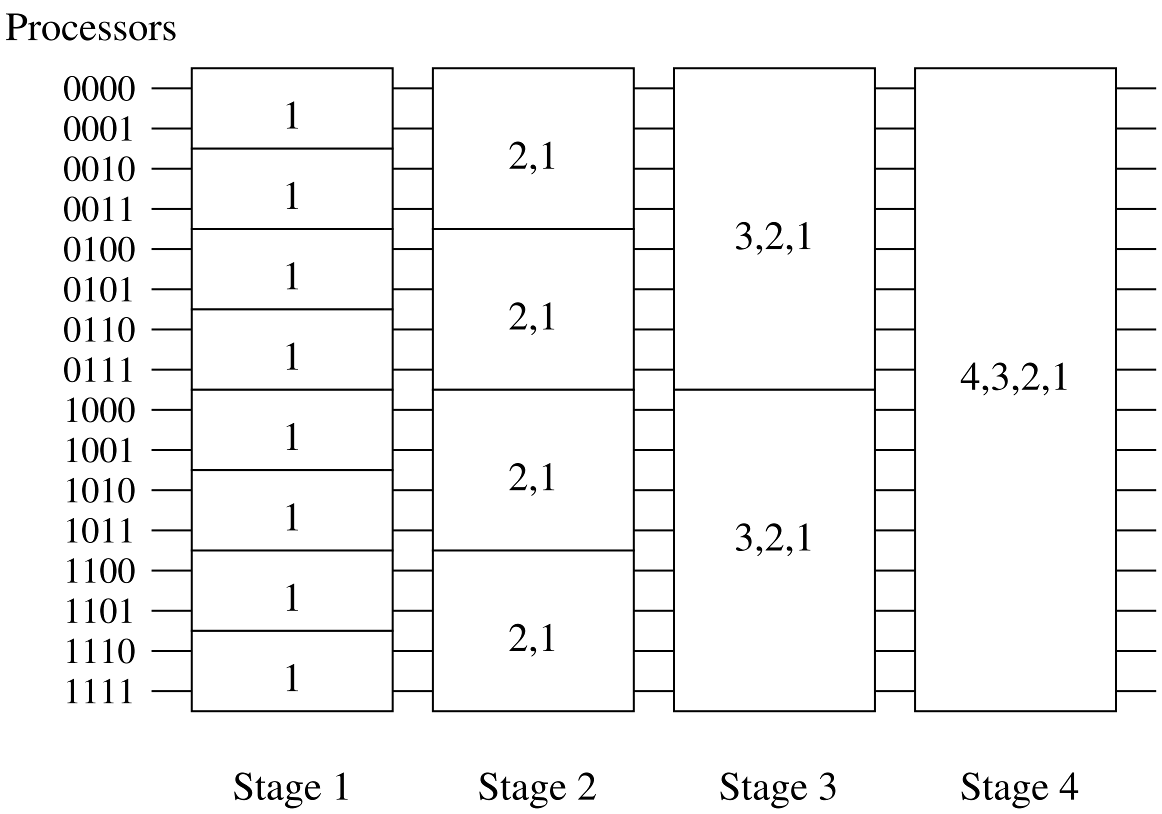
Step 2

Step 3

Step 4

Communication during the last stage of bitonic sort. Each wire is mapped to a hypercube process; each connection represents a compare-exchange between processes.

Mapping Bitonic Sort to Hypercubes



Communication characteristics of bitonic sort on a hypercube. During each stage of the algorithm, processes communicate along the dimensions shown.

Mapping Bitonic Sort to Hypercubes

```
procedure BITONIC_SORT($label, d$)
begin
    for $i := 0$ to $d - 1$ do
        for $j := i$ downto $0$ do
            if $(i + 1)^{st}$ bit of $label \neq j^{th}$ bit of $label$ then
                comp_exchange_max(j);
            else
                comp_exchange_min(j);
end BITONIC_SORT
```

Parallel formulation of bitonic sort on a hypercube with $n = 2^d$ processes.

Mapping Bitonic Sort to Hypercubes

- During each step of the algorithm, every process performs a compare-exchange operation (single nearest neighbor communication of one word).

- Since each step takes $Theta(1)$ time, the parallel time is

$$T_P = \Theta(\log^2 n) \tag{2}$$

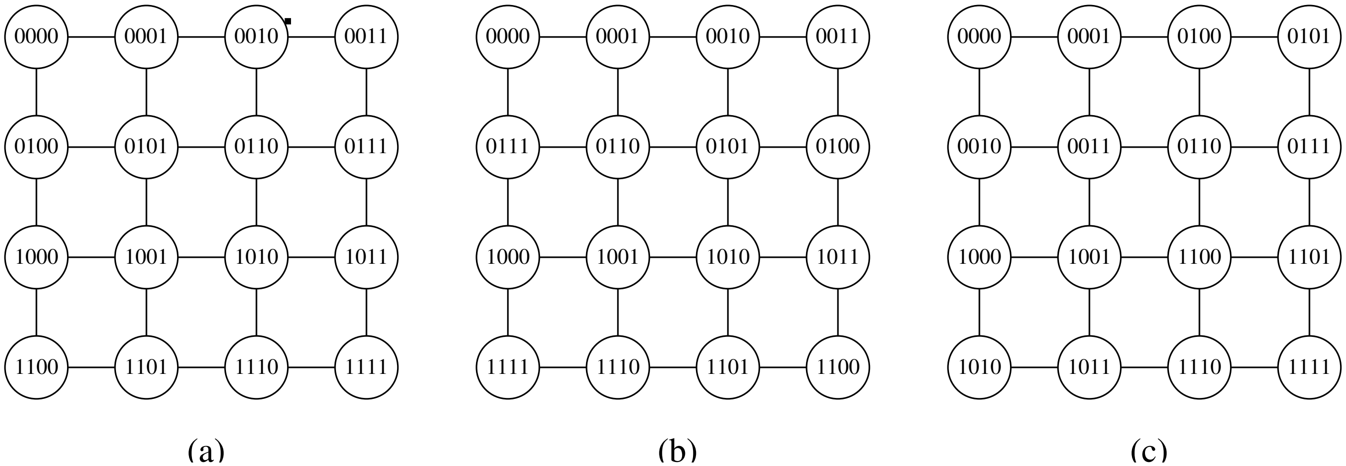
- This algorithm is cost optimal w.r.t. its serial counterpart, but not w.r.t. the best sorting algorithm.

Mapping Bitonic Sort to Meshes

- The connectivity of a mesh is lower than that of a hypercube, so we must expect some overhead in this mapping.

- Consider the row-major shuffled mapping of wires to processors.

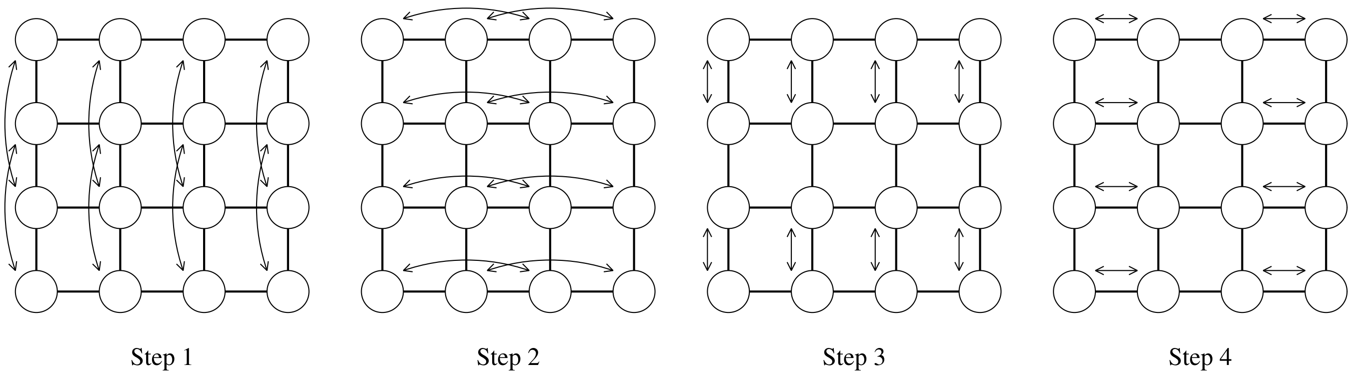
Mapping Bitonic Sort to Meshes



Different ways of mapping the input wires of the bitonic sorting network to a mesh of processes: (a) row-major mapping, (b) row-major snakelike mapping, and (c) row-major shuffled mapping.

Mapping Bitonic Sort to Meshes

Stage 4



The last stage of the bitonic sort algorithm for $n = 16$ on a mesh, using the row-major shuffled mapping. During each step, process pairs compare-exchange their elements. Arrows indicate the pairs of processes that perform compare-exchange operations.

Mapping Bitonic Sort to Meshes

- In the row-major shuffled mapping, wires that differ at the i^{th} least-significant bit are mapped onto mesh processes that are $2^{\lfloor (i-1)/2 \rfloor}$ communication links away.

- The total amount of communication performed by each process is $\sum_{i=1}^{\log n} \sum_{j=1}^{i} 2^{\lfloor (j-1)/2 \rfloor} \approx 7\sqrt{n}$, or $\Theta(\sqrt{n})$. The total computation performed by each process is $\Theta(\log^2 n)$.

- The parallel runtime is:

$$T_P = \overbrace{\Theta(\log^2 n)}^{\text{comparisons}} + \overbrace{\Theta(\sqrt{n})}^{\text{communication}}.$$

- This is not cost optimal.

Block of Elements Per Processor

- Each process is assigned a block of n/p elements.

- The first step is a local sort of the local block.

- Each subsequent compare-exchange operation is replaced by a compare-split operation.

- We can effectively view the bitonic network as having $(1 + \log p)(\log p)/2$ steps.

Block of Elements Per Processor: Hypercube

- Initially the processes sort their n/p elements (using merge sort) in time $\Theta((n/p)\log(n/p))$ and then perform $\Theta(\log^2 p)$ compare-split steps.

- The parallel run time of this formulation is

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(\frac{n}{p}\log^2 p\right)}^{\text{comparisons}} + \overbrace{\Theta\left(\frac{n}{p}\log^2 p\right)}^{\text{communication}}.$$

- Comparing to an optimal sort, the algorithm can efficiently use up to $p = \Theta(2^{\sqrt{\log n}})$ processes.

- The isoefficiency function due to both communication and extra work is $\Theta(p^{\log p}\log^2 p)$.

Block of Elements Per Processor: Mesh

- The parallel runtime in this case is given by:

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(\frac{n}{p}\log^2 p\right)}^{\text{comparisons}} + \overbrace{\Theta\left(\frac{n}{\sqrt{p}}\right)}^{\text{communication}}$$

- This formulation can efficiently use up to $p = \Theta(\log^2 n)$ processes.

- The isoefficiency function is $\Theta(2^{\sqrt{p}}\sqrt{p})$.

Performance of Parallel Bitonic Sort

The performance of parallel formulations of bitonic sort for n elements on p processes.

Architecture	Maximum Number of Processes for $E = \Theta(1)$	Corresponding Parallel Run Time	Isoefficiency Function
Hypercube	$\Theta(2^{\sqrt{\log n}})$	$\Theta(n/(2^{\sqrt{\log n}}) \log n)$	$\Theta(p^{\log p} \log^2 p)$
Mesh	$\Theta(\log^2 n)$	$\Theta(n/ \log n)$	$\Theta(2^{\sqrt{p}} \sqrt{p})$
Ring	$\Theta(\log n)$	$\Theta(n)$	$\Theta(2^p p)$

Bubble Sort and its Variants

The sequential bubble sort algorithm compares and exchanges adjacent elements in the sequence to be sorted:

```
procedure BUBBLE_SORT($n$)
begin
    for $i := n - 1$ downto $1$ do
        for $j := 1$ to $i$ do
            compare-exchange($a_j, a_{j+1}$);
end BUBBLE_SORT
```

Sequential bubble sort algorithm.

Bubble Sort and its Variants

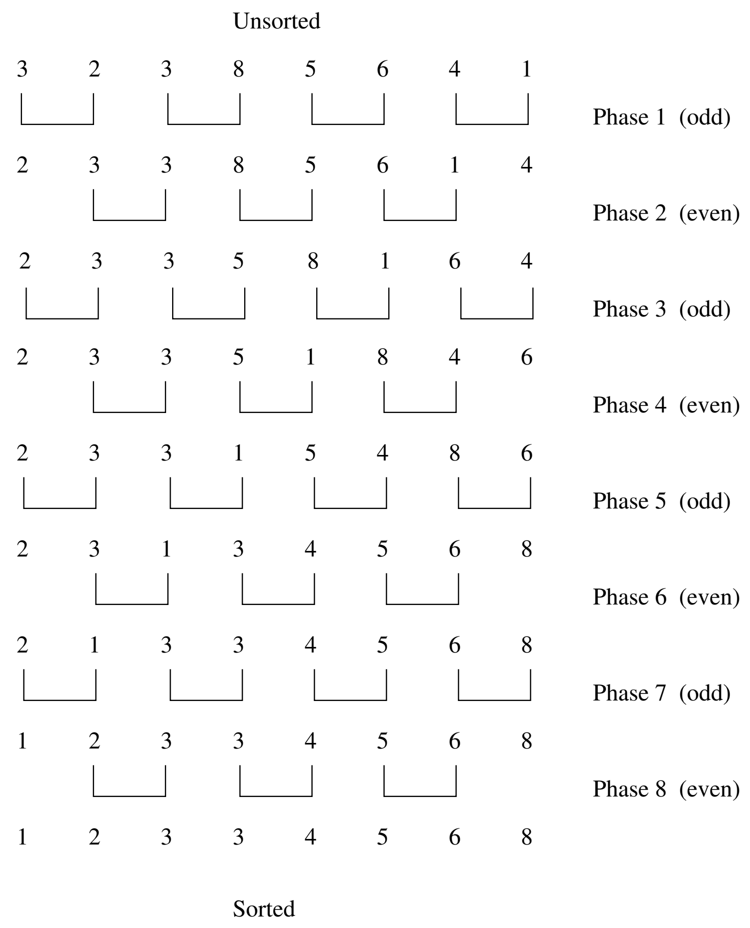
- The complexity of bubble sort is $\Theta(n^2)$.
- Bubble sort is difficult to parallelize since the algorithm has no concurrency.
- A simple variant, though, uncovers the concurrency.

Odd-Even Transposition

```
procedure ODD-EVEN($n$)
begin
    for $i := 1$ to $n$ do
    begin
        if $i$ is odd then
            for $j := 0$ to $n/2 - 1$ do
                compare-exchange($a_{2j+1}, a_{2j+2}$);
        if $i$ is even then
            for $j := 1$ to $n/2 - 1$ do
                compare-exchange($a_{2j}, a_{2j+1}$);
    end for
end ODD-EVEN
```

Sequential odd-even transposition sort algorithm.

Odd-Even Transposition



Unsorted

3 2 3 8 5 6 4 1

Phase 1 (odd)

2 3 3 8 5 6 1 4

Phase 2 (even)

2 3 3 5 8 1 6 4

Phase 3 (odd)

2 3 3 5 1 8 4 6

Phase 4 (even)

2 3 3 1 5 4 8 6

Phase 5 (odd)

2 3 1 3 4 5 6 8

Phase 6 (even)

2 1 3 3 4 5 6 8

Phase 7 (odd)

1 2 3 3 4 5 6 8

Phase 8 (even)

1 2 3 3 4 5 6 8

Sorted

Sorting $n = 8$ elements, using the odd-even transposition sort algorithm. During each phase, $n = 8$ elements are compared.

Odd-Even Transposition

- After n phases of odd-even exchanges, the sequence is sorted.
- Each phase of the algorithm (either odd or even) requires $\Theta(n)$ comparisons.
- Serial complexity is $\Theta(n^2)$.

Parallel Odd-Even Transposition

- Consider the one item per processor case.
- There are n iterations, in each iteration, each processor does one compare-exchange.
- The parallel run time of this formulation is $\Theta(n)$.
- This is cost optimal with respect to the base serial algorithm but not the optimal one.

Parallel Odd-Even Transposition

```
procedure ODD-EVEN_PAR($n$)
begin
    $id$ := process's label
    for $i := 1$ to $n$ do
    begin
        if $i$ is odd then
            if $id$ is odd then
                compare-exchange_min($id + 1$);
            else
                compare-exchange_max($id - 1$);
        if $i$ is even then
            if $id$ is even then
                compare-exchange_min($id + 1$);
            else
                compare-exchange_max($id - 1$);
    end for
end ODD-EVEN_PAR
```

Parallel formulation of odd-even transposition.

Parallel Odd-Even Transposition

- Consider a block of n/p elements per processor.
- The first step is a local sort.
- In each subsequent step, the compare exchange operation is replaced by the compare split operation.
- The parallel run time of the formulation is

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta(n)}^{\text{comparisons}} + \overbrace{\Theta(n)}^{\text{communication}}.$$

Parallel Odd-Even Transposition

- The parallel formulation is cost-optimal for $p = O(\log n)$.
- The isoefficiency function of this parallel formulation is $\Theta(p\,2^p)$.

Shellsort

- Let n be the number of elements to be sorted and p be the number of processes.

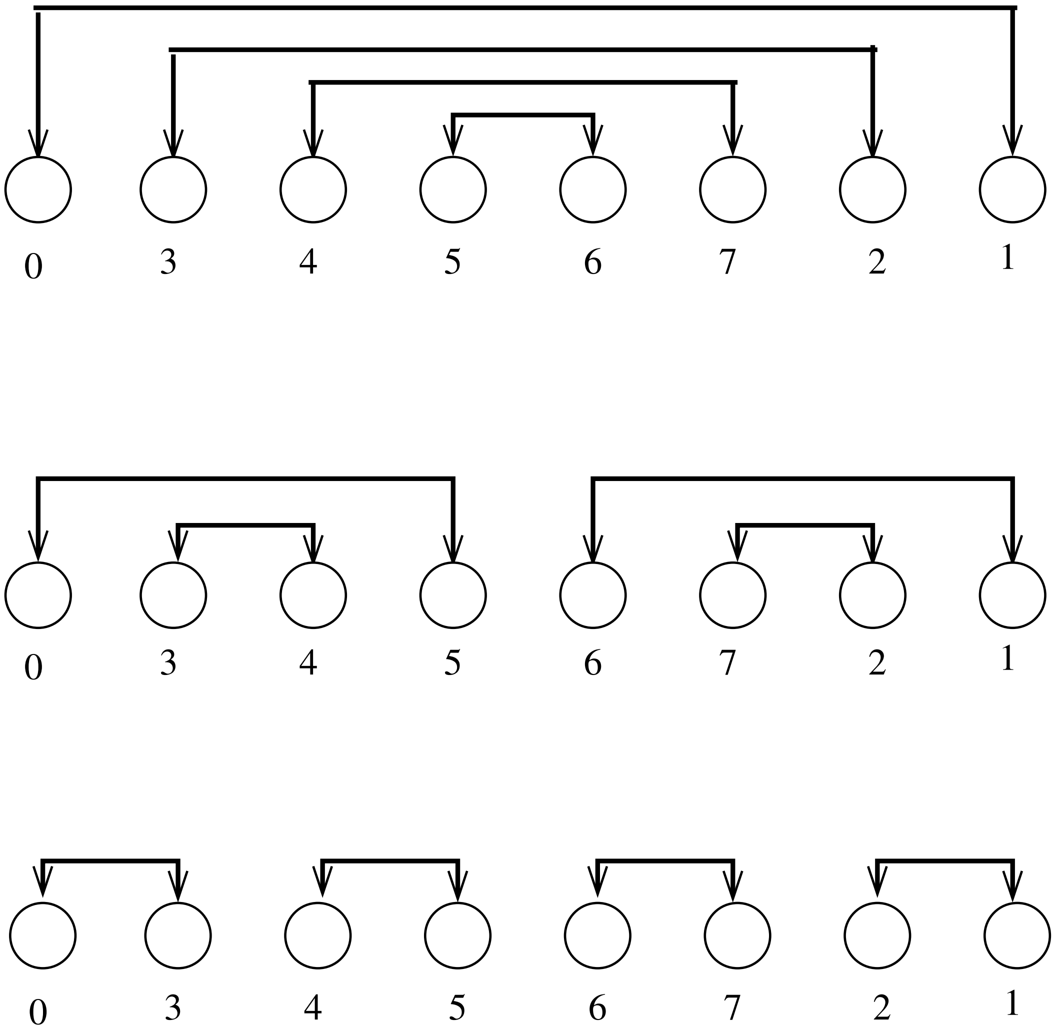
- During the first phase, processes that are far away from each other in the array compare-split their elements.

- During the second phase, the algorithm switches to an odd-even transposition sort.

Parallel Shellsort

- Initially, each process sorts its block of n/p elements internally.
- Each process is now paired with its corresponding process in the reverse order of the array. That is, process P_i, where $i < p/2$, is paired with process P_{p-i-1}.
- A compare-split operation is performed.
- The processes are split into two groups of size $p/2$ each and the process repeated in each group.

Parallel Shellsort



An example of the first phase of parallel shellsort on an eight-process array.

Parallel Shellsort

- Each process performs $d = \log p$ compare-split operations.

- With $O(p)$ bisection width, the each communication can be performed in time $\Theta(n/p)$ for a total time of $\Theta((n \log p)/p)$.

- In the second phase, l odd and even phases are performed, each requiring time $\Theta(n/p)$.

- The parallel run time of the algorithm is:

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(\frac{n}{p}\log p\right)}^{\text{first phase}} + \overbrace{\Theta\left(l\frac{n}{p}\right)}^{\text{second phase}}. \qquad (3)$$

Quicksort

- Quicksort is one of the most common sorting algorithms for sequential computers because of its simplicity, low overhead, and optimal average complexity.

- Quicksort selects one of the entries in the sequence to be the pivot and divides the sequence into two – one with all elements less than the pivot and other greater.

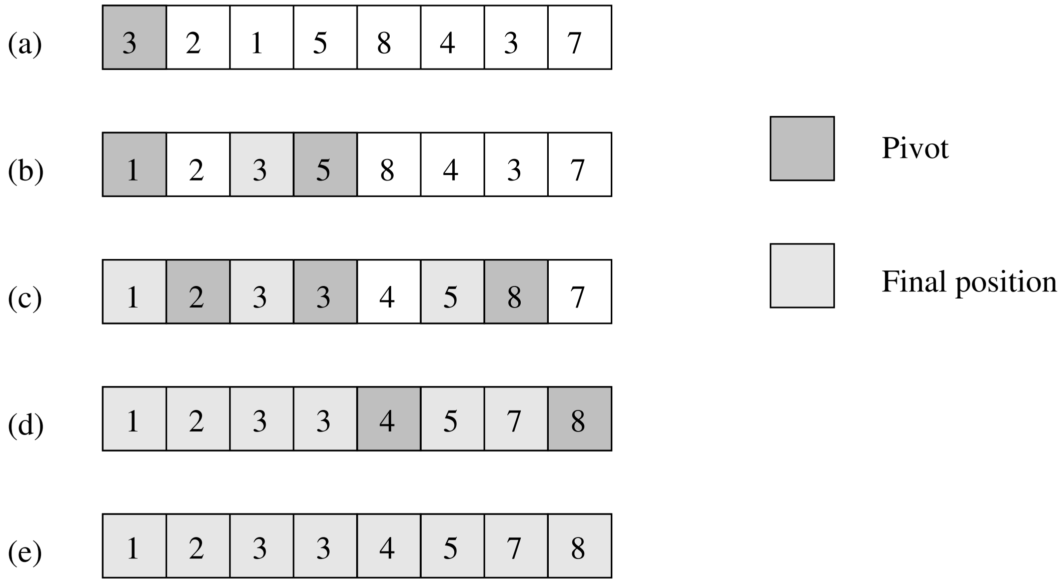
- The process is recursively applied to each of the sublists.

Quicksort

```
procedure QUICKSORT $(A, q, r)$
begin
    if $q < r$ then
    begin
        $x := A[q]$;
        $s := q$;
        for $i := q + 1$ to $r$ do
            if $A[i] \leq x$ then
            begin
                $s := s + 1$;
                swap($A[s], A[i]$);
            end if
        swap($A[q], A[s]$);
        QUICKSORT $(A, q, s)$;
        QUICKSORT $(A, s + 1, r)$;
    end if
end QUICKSORT
```

The sequential quicksort algorithm.

Quicksort



(a)	3	2	1	5	8	4	3	7
(b)	1	2	3	5	8	4	3	7
(c)	1	2	3	3	4	5	8	7
(d)	1	2	3	3	4	5	7	8
(e)	1	2	3	3	4	5	7	8

Example of the quicksort algorithm sorting a sequence of size $n = 8$.

Quicksort

- The performance of quicksort depends critically on the quality of the pivot.

- In the best case, the pivot divides the list in such a way that the larger of the two lists does not have more than αn elements (for some constant α).

- In this case, the complexity of quicksort is $O(n \log n)$.

Parallelizing Quicksort

- Lets start with recursive desomposition – the list is partitioned serially and each of the subproblems is handled by a different processor.

- The time for this algorithm is lower-bounded by $\Omega(n)$!

- Can we parallelize the partitioning step – in particular, if we can use n processors to partition a list of length n around a pivot in $O(1)$ time, we have a winner.

- This is difficult to do on real machines, though.

Parallelizing Quicksort: PRAM Formulation

- We assume a CRCW (concurrent read, concurrent write) PRAM with concurrent writes resulting in an arbitrary write succeeding.

- The formulation works by creating pools of processors. Every processor is assigned to the same pool initially and has one element.

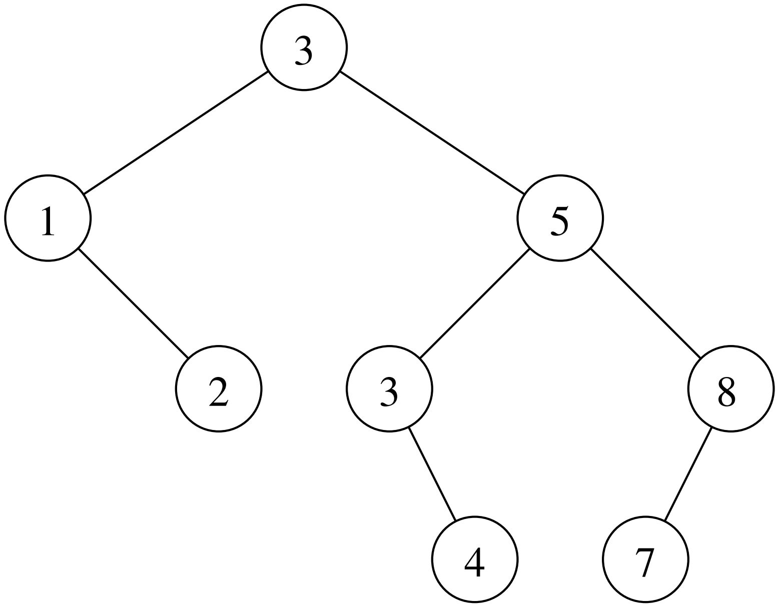
- Each processor attempts to write its element to a common location (for the pool).

- Each processor tries to read back the location. If the value read back is greater than the processor's value, it assigns itself to the 'left' pool, else, it assigns itself to the 'right' pool.

- Each pool performs this operation recursively.

- Note that the algorithm generates a tree of pivots. The depth of the tree is the expected parallel runtime. The average value is $O(\log n)$.

Parallelizing Quicksort: PRAM Formulation



A binary tree generated by the execution of the quicksort algorithm. Each level of the tree represents a different array-partitioning iteration. If pivot selection is optimal, then the height of the tree is $\Theta(\log n)$, which is also the number of iterations.

Parallelizing Quicksort: PRAM Formulation

(a)

1	2	3	4	5	6	7	8
33	21	13	54	82	33	40	72

(b) root = 4

(c)

	1	2	3	4	5	6	7	8
leftchild				1				
rightchild				5				

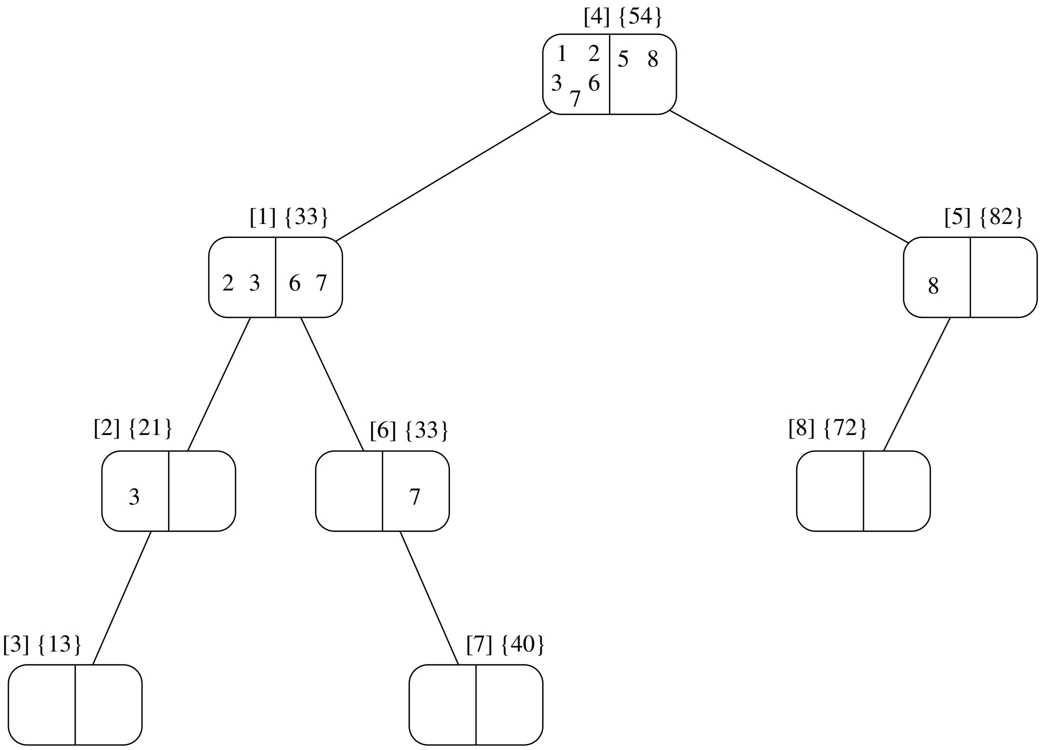
(d)

	1	2	3	4	5	6	7	8
leftchild	2			1	8			
rightchild	6			5				

(e)

	1	2	3	4	5	6	7	8
leftchild	2	3		1	8			
rightchild	6			5		7		

(f)

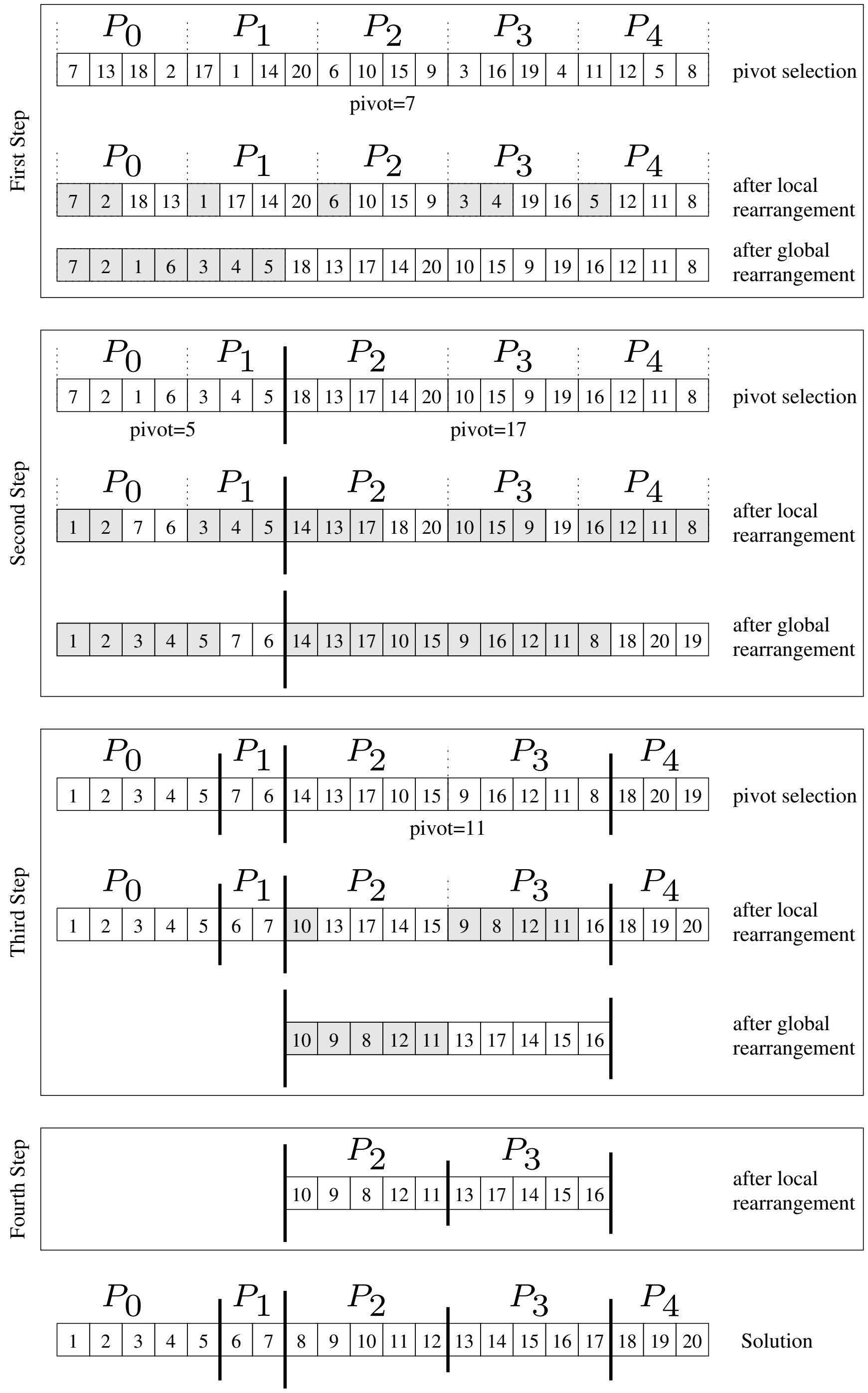


The execution of the PRAM algorithm on the array shown in (a).

Parallelizing Quicksort: Shared Address Space Formulation

- Consider a list of size n equally divided across p processors.
- A pivot is selected by one of the processors and made known to all processors.
- Each processor partitions its list into two, say L_i and U_i, based on the selected pivot.
- All of the L_i lists are merged and all of the U_i lists are merged separately.
- The set of processors is partitioned into two (in proportion of the size of lists L and U). The process is recursively applied to each of the lists.

Shared Address Space Formulation



First Step

P_0 P_1 P_2 P_3 P_4

7 13 18 2 17 1 14 20 6 10 15 9 3 16 19 4 11 12 5 8 — pivot selection

pivot=7

P_0 P_1 P_2 P_3 P_4

7 2 18 13 1 17 14 20 6 10 15 9 3 4 19 16 5 12 11 8 — after local rearrangement

7 2 1 6 3 4 5 18 13 17 14 20 10 15 9 19 16 12 11 8 — after global rearrangement

Second Step

P_0 P_1 P_2 P_3 P_4

7 2 1 6 3 4 5 | 18 13 17 14 20 10 15 9 19 16 12 11 8 — pivot selection

pivot=5 pivot=17

P_0 P_1 P_2 P_3 P_4

1 2 7 6 3 4 5 | 14 13 17 18 20 10 15 9 19 16 12 11 8 — after local rearrangement

1 2 3 4 5 7 6 | 14 13 17 10 15 9 16 12 11 8 18 20 19 — after global rearrangement

Third Step

P_0 P_1 P_2 P_3 P_4

1 2 3 4 5 | 7 6 | 14 13 17 10 15 9 16 12 11 8 | 18 20 19 — pivot selection

pivot=11

P_0 P_1 P_2 P_3 P_4

1 2 3 4 5 | 6 7 | 10 13 17 14 15 9 8 12 11 16 | 18 19 20 — after local rearrangement

| 10 9 8 12 11 13 17 14 15 16 | — after global rearrangement

Fourth Step

P_2 P_3

| 10 9 8 12 11 | 13 17 14 15 16 | — after local rearrangement

P_0 P_1 P_2 P_3 P_4

1 2 3 4 5 | 6 7 | 8 9 10 11 12 | 13 14 15 16 17 | 18 19 20 — Solution

Parallelizing Quicksort: Shared Address Space Formulation

- The only thing we have not described is the global reorganization (merging) of local lists to form L and U.

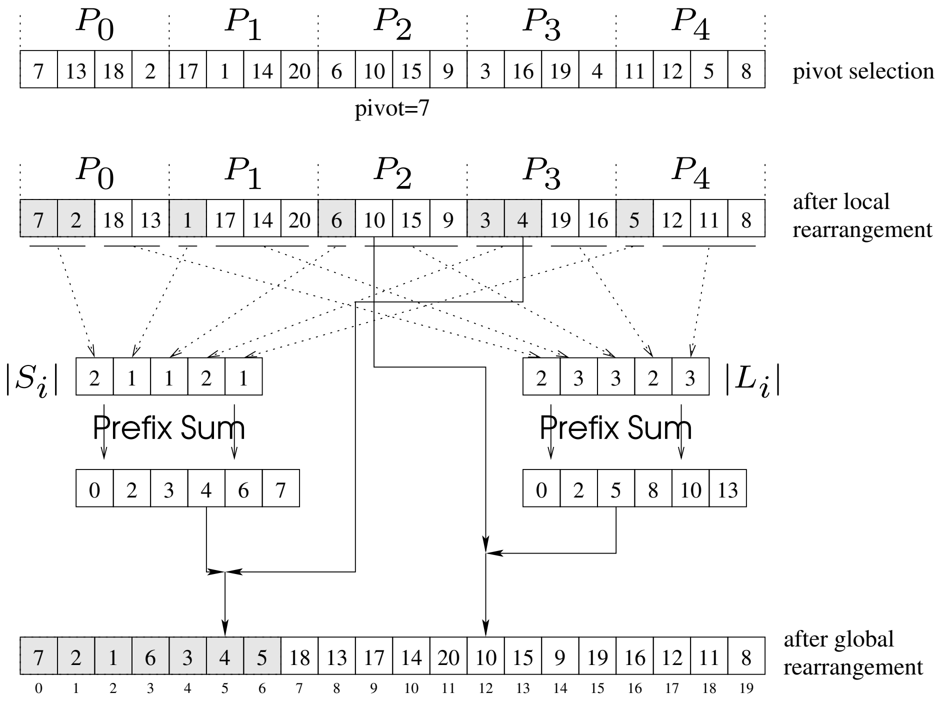
- The problem is one of determining the right location for each element in the merged list.

- Each processor computes the number of elements locally less than and greater than pivot.

- It computes two sum-scans to determine the starting location for its elements in the merged L and U lists.

- Once it knows the starting locations, it can write its elements safely.

Parallelizing Quicksort: Shared Address Space Formulation


P_0 P_1 P_2 P_3 P_4
pivot selection
pivot=7
P_0 P_1 P_2 P_3 P_4
after local rearrangement
$|S_i|$
$|L_i|$
Prefix Sum
Prefix Sum
after global rearrangement

7 13 18 2 17 1 14 20 6 10 15 9 3 16 19 4 11 12 5 8

7 2 18 13 1 17 14 20 6 10 15 9 3 4 19 16 5 12 11 8

$|S_i|$: 2 1 1 2 1

$|L_i|$: 2 3 3 2 3

0 2 3 4 6 7

0 2 5 8 10 13

7 2 1 6 3 4 5 18 13 17 14 20 10 15 9 19 16 12 11 8

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

Efficient global rearrangement of the array.

Parallelizing Quicksort: Shared Address Space Formulation

- The parallel time depends on the split and merge time, and the quality of the pivot.

- The latter is an issue independent of parallelism, so we focus on the first aspect, assuming ideal pivot selection.

- The algorithm executes in four steps: (i) determine and broadcast the pivot; (ii) locally rearrange the array assigned to each process; (iii) determine the locations in the globally rearranged array that the local elements will go to; and (iv) perform the global rearrangement.

- The first step takes time $\Theta(\log p)$, the second, $\Theta(n/p)$, the third, $\Theta(\log p)$, and the fourth, $\Theta(n/p)$.

- The overall complexity of splitting an n-element array is $\Theta(n/p) + \Theta(\log p)$.

Parallelizing Quicksort: Shared Address Space Formulation

- The process recurses until there are p lists, at which point, the lists are sorted locally.

- Therefore, the total parallel time is:

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(\frac{n}{p}\log p\right) + \Theta(\log^2 p)}^{\text{array splits}}. \tag{4}$$

- The corresponding isoefficiency is $\Theta(p\log^2 p)$ due to broadcast and scan operations.

Parallelizing Quicksort: Message Passing Formulation

- A simple message passing formulation is based on the recursive halving of the machine.

- Assume that each processor in the lower half of a p processor ensemble is paired with a corresponding processor in the upper half.

- A designated processor selects and broadcasts the pivot.

- Each processor splits its local list into two lists, one less (L_i), and other greater (U_i) than the pivot.

- A processor in the low half of the machine sends its list U_i to the paired processor in the other half. The paired processor sends its list L_i.

- It is easy to see that after this step, all elements less than the pivot are in the low half of the machine and all elements greater than the pivot are in the high half.

Parallelizing Quicksort: Message Passing Formulation

- The above process is recursed until each processor has its own local list, which is sorted locally.

- The time for a single reorganization is $\Theta(\log p)$ for broadcasting the pivot element, $\Theta(n/p)$ for splitting the locally assigned portion of the array, $\Theta(n/p)$ for exchange and local reorganization.

- We note that this time is identical to that of the corresponding shared address space formulation.

- It is important to remember that the reorganization of elements is a bandwidth sensitive operation.

Bucket and Sample Sort

- In Bucket sort, the range $[a, b]$ of input numbers is divided into m equal sized intervals, called buckets.

- Each element is placed in its appropriate bucket.

- If the numbers are uniformly divided in the range, the buckets can be expected to have roughly identical number of elements.

- Elements in the buckets are locally sorted.

- The run time of this algorithm is $\Theta(n \log(n/m))$.

Parallel Bucket Sort

- Parallelizing bucket sort is relatively simple. We can select $m = p$.
- In this case, each processor has a range of values it is responsible for.
- Each processor runs through its local list and assigns each of its elements to the appropriate processor.
- The elements are sent to the destination processors using a single all-to-all personalized communication.
- Each processor sorts all the elements it receives.

Parallel Bucket and Sample Sort

- The critical aspect of the above algorithm is one of assigning ranges to processors. This is done by suitable splitter selection.

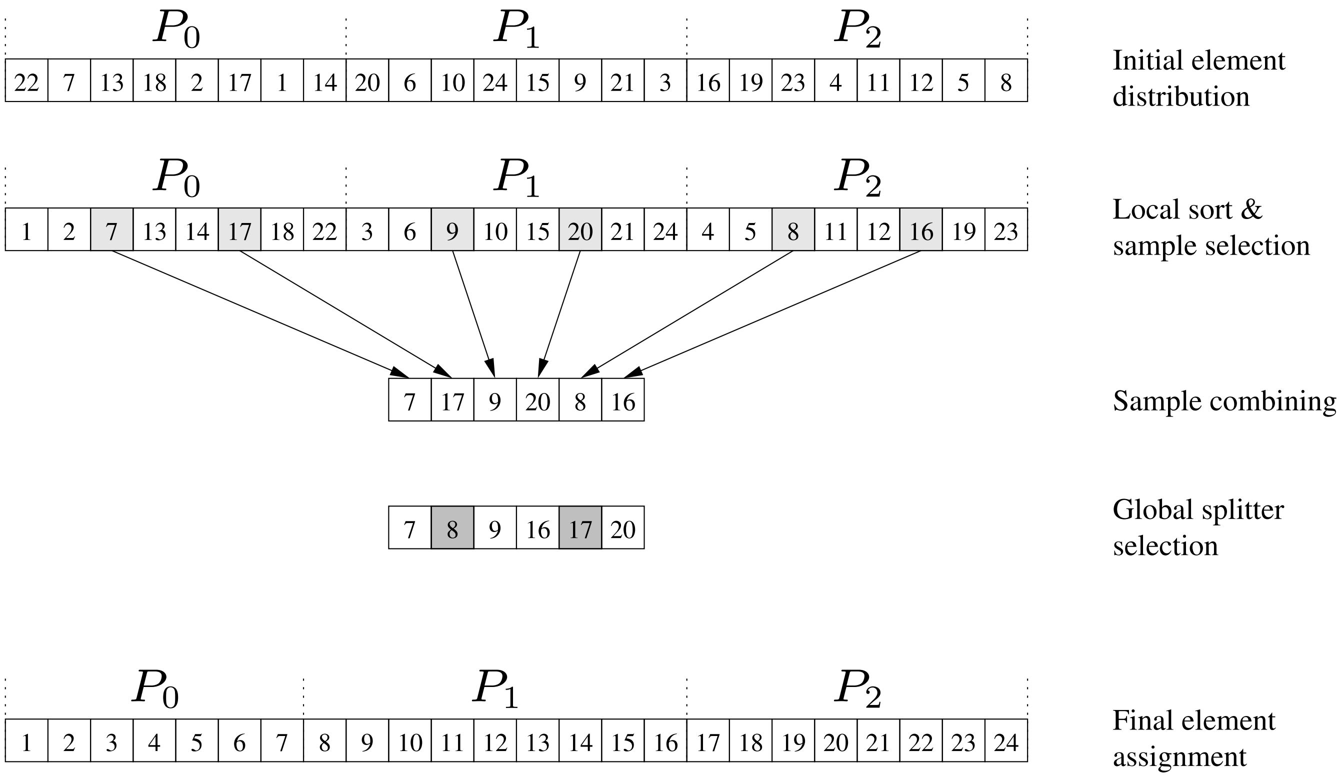
- The splitter selection method divides the n elements into m blocks of size n/m each, and sorts each block by using quicksort.

- From each sorted block it chooses $m - 1$ evenly spaced elements.

- The $m(m - 1)$ elements selected from all the blocks represent the sample used to determine the buckets.

- This scheme guarantees that the number of elements ending up in each bucket is less than $2n/m$.

Parallel Bucket and Sample Sort



P_0 P_1 P_2

22 7 13 18 2 17 1 14 20 6 10 24 15 9 21 3 16 19 23 4 11 12 5 8

Initial element distribution

P_0 P_1 P_2

1 2 7 13 14 17 18 22 3 6 9 10 15 20 21 24 4 5 8 11 12 16 19 23

Local sort & sample selection

7 17 9 20 8 16

Sample combining

7 8 9 16 17 20

Global splitter selection

P_0 P_1 P_2

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

Final element assignment

An example of the execution of sample sort on an array with 24 elements on three processes.

Parallel Bucket and Sample Sort

- The splitter selection scheme can itself be parallelized.
- Each processor generates the $p-1$ local splitters in parallel.
- All processors share their splitters using a single all-to-all broadcast operation.
- Each processor sorts the $p(p-1)$ elements it receives and selects $p-1$ uniformly spaces splitters from them.

Parallel Bucket and Sample Sort: Analysis

- The internal sort of n/p elements requires time $\Theta((n/p)\log(n/p))$, and the selection of $p-1$ sample elements requires time $\Theta(p)$.

- The time for an all-to-all broadcast is $\Theta(p^2)$, the time to internally sort the $p(p-1)$ sample elements is $\Theta(p^2 \log p)$, and selecting $p-1$ evenly spaced splitters takes time $\Theta(p)$.

- Each process can *insert* these $p-1$ splitters in its local sorted block of size n/p by performing $p-1$ binary searches in time $\Theta(p\log(n/p))$.

- The time for reorganization of the elements is $O(n/p)$.

Parallel Bucket and Sample Sort: Analysis

- The total time is given by:

$$T_P = \overbrace{\Theta\left(\frac{n}{p}\log\frac{n}{p}\right)}^{\text{local sort}} + \overbrace{\Theta\left(p^2\log p\right)}^{\text{sort sample}} + \overbrace{\Theta\left(p\log\frac{n}{p}\right)}^{\text{block partition}} + \overbrace{\Theta(n/p).}^{\text{communication}} \tag{5}$$

- The isoefficiency of the formulation is $\Theta(p^3 \log p)$.