Excursus: Partial Autocorrelation

According to the partial correlation coefficient in gross sectional regression, the partial autocorrelation coefficient π describes the supplementary information, which is provided by the additional lag. Thereby, the existing information, derived from the previous lags, is taken into consideration.

From the lag 0 one can achieve all the present information and hence X_t is fully specified:

$$\pi_0 = 1$$

If an equation has only one explanatory variable, it makes no sense to test for additional information of this one variable. Therefore, the autocorrelation coefficient ρ and the partial autocorrelation coefficient π of the first lag are the same:

$$\pi_1 = \rho_1$$

However, if there exist more than one explanatory variable, as for example in the following model,

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \alpha_3 X_{t-3} + u_t,$$

then one could ask, which additional influence the variable X_{t-2} or the variable X_{t-3} has, on the explanation of X_t. For example, if we want to know which additional information can be derived from X_{t-3}, we control for information already obtained from X_{t-1} and X_{t-2}. One can calculate the partial autocorrelation of lag τ by,

$$\pi_\tau = \frac{\rho_\tau - \sum_{j=1}^{\tau-1} \pi_{\tau-1,j} \cdot \rho_{\tau-j}}{1 - \sum_{j=1}^{\tau-1} \pi_{\tau-1,j} \cdot \rho_{\tau-j}},$$

with $\tau > 1$.

Thereby, ρ_τ is the autocorrelation coefficient of lag τ. For $\pi_{\tau,j}$ is given by the recursion equation, discovered by Durbin (1960):

$$\pi_{\tau,j} = \pi_{\tau-1,j} - \pi_\tau \pi_{\tau-1,\tau-j},$$

where $\pi_{\tau,\tau} = \pi_\tau$.

Example:

Correlogramm of dlmib:

Date: 11/05/07 Time: 13:16
Sample: 10/01/1997 9/28/2007
Included observations: 2607

	AC	PAC	Q-Stat	Prob
1	0.003	0.003	0.0176	0.894
2	0.024	0.024	1.5762	0.455
3	-0.033	-0.034	4.4863	0.214
4	0.069	0.069	16.957	0.002
5	-0.036	-0.035	20.396	0.001
6	0.021	0.018	21.583	0.001
7	-0.018	-0.012	22.403	0.002
8	0.038	0.031	26.260	0.001
9	0.024	0.030	27.708	0.001
10	0.005	-0.002	27.773	0.002

Calculation of PACF:

Lag 1:

$\pi_{1,1} = \rho_1 = 0.003$

Lag 2:

$$\pi_{2,2} = \frac{\rho_2 - \pi_{1,1}\rho_1}{1 - \pi_{1,1}\rho_1} = \frac{\rho_2 - {\rho_1}^2}{1 - {\rho_1}^2} = \frac{0.024 - 0.003^2}{1 - 0.003^2} = 0.023991 \approx 0.024$$

Lag 3:

$$\pi_{2,1} = \frac{\rho_1 - \rho_2\rho_1}{1 - \pi_{1,1}\rho_1} = \frac{\rho_1(1 - \rho_2)}{1 - {\rho_1}^2} = \frac{0.003\,(1 - 0.024)}{1 - 0.003^2} = 0.002928 \approx 0.003$$

The partial autocorrelation coefficient $\pi_{2,1}$ is required to calculate $\pi_{3,3}$:

$$\pi_{3,3} = \frac{\rho_3 - \pi_{2,1}\rho_2 - \pi_{2,2}\rho_1}{1 - \pi_{2,1}\rho_2 - \pi_{2,2}\rho_1} = \frac{-0.033 - 0.003 \cdot 0.024 - 0.024 \cdot 0.003}{1 - 0.003 \cdot 0.024 - 0.024 \cdot 0.003}$$
$$= -0.03314877 \approx -0.034$$

Lag 4:

In the same way, one could calculate $\pi_{3,1}$ and $\pi_{3,2}$ now, in order to derive $\pi_{4,4}$. Therefore, one can use Durbin's recursion equation:

$$\pi_{3,1} = \pi_{2,1} - \pi_{3,3}\pi_{2,2} = 0.003 - (-0.034) \cdot 0.024 = 0.003816 \approx 0.0038$$

$$\pi_{3,2} = \pi_{2,2} - \pi_{3,3}\pi_{2,1} = 0.024 - (-0.034) \cdot 0.003 = 0.024102 \approx 0.0241$$

$$\pi_{4,4} = \frac{\rho_4 - \pi_{3,1}\rho_3 - \pi_{3,2}\rho_2 - \pi_{3,3}\rho_1}{1 - \pi_{3,1}\rho_3 - \pi_{3,2}\rho_2 - \pi_{3,3}\rho_1}$$
$$= \frac{0.069 - 0.0038 \cdot (-0.033) - 0.0241 \cdot 0.024 - (-0.034) \cdot 0.003}{1 - 0.0038 \cdot (-0.033) - 0.0241 \cdot 0.024 - (-0.034) \cdot 0.003}$$
$$= 0.0686249 \approx 0.069$$