REFERENCES

[1] H. W. Bode, "Feedback—The history of an idea," *Proc. Symp. on Active Networks and Feedback Systems, MRI*, Polytechnic Press, Brooklyn, New York, N. Y., vol. 10, pp. 1–7; 1960.

[2] C. J. Savant, "Basic Feedback Control System Design," McGraw-Hill Book Co., Inc., New York, N. Y.; 1958.

[3] B. Friedland, "Optimum space guidance and control," *Technical News Bulletin*, vol. 6, no. 2, pp. 11–21; 1963. (A much more complete, but unfortunately less available, treatise by this author on the same subject is contained in Technical Note 62/3, Melper Inc., Watertown, Mass.; June, 1962.)

[4] R. E. Kalman, "The Theory of Optimal Control and the Calculus of Variations," Research Institute for Advanced Studies, Baltimore, Md., Technical Rept. No. 61-63; 1961.

[5] L. I. Rozonoer, "L. S. Pontryagin's maxium principle in the theory of optimum systems," *Automation and Remote Control*, vol. 20, pp. 1288–1302, pp. 1405–1421, pp. 1517–1532; June, July, August, 1960.

[6] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mischenko, "The Mathematical Theory of Optimal Processes," Interscience Publishers, a division of John Wiley and Sons, Inc., New York, N. Y.; 1962.

[7] L. A. Zadeh, "Optimality and nonscaler-valued performance criteria," IEEE TRANS. ON AUTOMATIC CONTROL, (*Correspondence*), vol. AC-8, pp. 59–60; January, 1963.

[8] R. Bellman, "Dynamic Programming," Princeton University Press, Princeton, N. J.; 1957.

[9] C. A. Desoer, "Pontryagin's maximum principle and the principle of optimality," *J. Franklin Inst.*, vol. 271, pp. 361–367; May, 1961.

[10] S. E. Dreyfus, "Dynamic programming and the calculus of variations," *Jour. of Math. Analysis and Application*, vol. 1, pp. 228–239; September, 1960.

[11] I. Flügge-Lotz and H. Marbach, "The optimal control of some attitude control systems for different performance criteria," *J. Basic Engrg, ASME Trans.*, Series D, vol. 85, pp. 165–176; June, 1963.

[12] W. H. Foy, "Fuel minimization in flight vehicle attitude control," IEEE TRANS. ON AUTOMATIC CONTROL, vol. AC-8, pp. 84–88; April, 1963.

[13] J. S. Meditch, "On minimal-fuel satellite attitude controls," IEEE TRANS. ON APPLICATIONS AND INDUSTRY, vol. 83, pp. 120–128; March, 1964.

[14] M. Athans, "Minimum-fuel control systems: second-order case," IEEE TRANS. ON APPLICATIONS AND INDUSTRY, vol. 82, pp. 8–17; March, 1963.

[15] O. J. M. Smith, Discussion on "Minimum-fuel control systems: second-order case," IEEE TRANS. ON APPLICATIONS AND INDUSTRY, vol. 82, pp. 17; March, 1963.

[16] B. Friedland and P. E. Sarachik, "Indifference regions in optimum attitude control," IEEE TRANS. ON AUTOMATIC CONTROL, vol. AC-9, pp. 180–181; April, 1964.


# On the Problem of Optimal Thrust Programming For a Lunar Soft Landing

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*Summary*—**The problem of minimal fuel thrust programming for the terminal phase of a lunar soft landing mission is shown to be equivalent to the minimal time problem for the mission. The existence of an optimal (minimal fuel) thrust program for the problem is then assured by appealing to existence theorems for time optimal controls, and the optimal thrust program is developed by application of the Pontryagin maximum principle.**

**It is shown that the optimal thrust program consists of either full thrust from the initiation of the mission until touchdown, or a period of zero thrust (free-fall) followed by full thrust until touchdown. An approximate switching function which is adequate for a large number of cases is derived, and a preliminary system design is presented.**

## INTRODUCTION

IN THIS REPORT the Pontryagin maximum principle[1] is applied to develop an optimal (minimal fuel) thrust program for the terminal phase of a lunar soft-landing mission. The problem consists essentially in specifying the thrust program for the vertical flight of a vehicle in vacuo in a constant gravitational field. This problem has previously been treated by the calculus of variations[2] and Miele's theory for the extremization of linear integrals.[3,4]

The present report is not only concerned with specifying an optimal thrust program for the mission,[2,3,4] but also with the question of the existence of an optimal thrust program as well as some of the problems inherent in implementing it once it has been established. In particular, the problem of actually synthesizing an optimal thrust program will be treated. Moreover, a preliminary system design, including computer and other hardware requirements, will be presented.

## PROBLEM FORMULATION

Assume that a space vehicle is in the terminal phase of a lunar soft landing mission as shown in Fig. 1. Moreover, assume that the motion of the vehicle is vertical

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[1] S. L. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, "The Mathematical Theory of Optimal Processes," Interscience Publishers, a division of John Wiley and Sons, Inc., New York, N. Y.; 1962.

[2] A. Miele, "Optimization Techniques; With Applications to Aerospace Systems," Academic Press, New York, N. Y., G. Leitmann, ed., ch. 4; 1960.

[3] *Ibid.*, ch. 3.

[4] D. G. Hull, "Thrust Programs for Minimum Propellant Consumption During Vertical Take-Off and Landing Maneuvers of a Rocket Vehicle in a Vacuum," Boeing Scientific Research Laboratories, Flight Sciences Laboratory, Rept. No. 59; July, 1962.

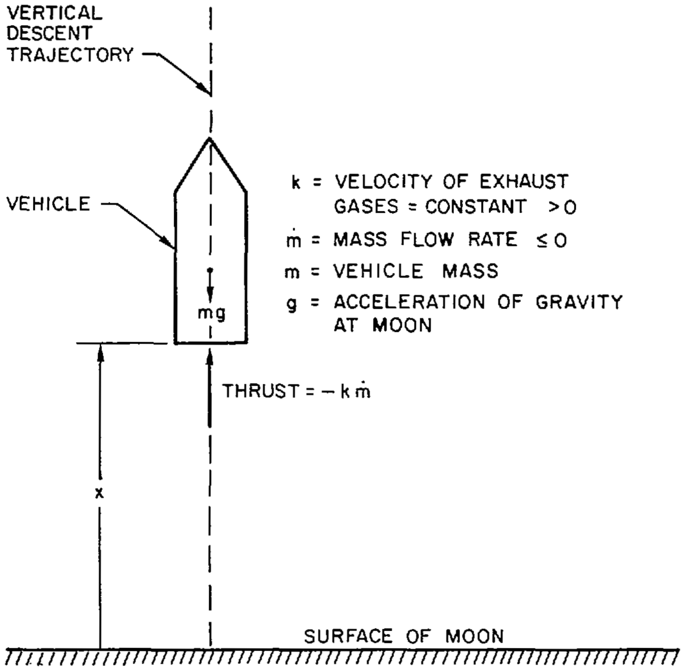


Fig. 1—Terminal phase of lunar soft landing mission.

and subject to the following conditions: a) the only forces acting on the vehicle are its own weight and the thrust which acts as a braking force, *i.e.*, aerodynamic forces are negligible, b) the thrust is tangent to the descent trajectory, c) the moon is flat in the vicinity of the desired landing point, d) the vehicle is in the near field of the moon so that the acceleration of gravity is constant, e) the velocity of the exhaust gases with respect to the vehicle is constant, and f) the propulsion system is capable of mass flow rates between zero and a fixed upper limit.

Under these assumptions, it is well known[5] that the motion of the vehicle is governed by the relation

$$\ddot{x} = -\frac{k\dot{m}}{m} - g \tag{1}$$

where $x$ is the altitude, $m$ the total mass, $\dot{m}$ the mass flow rate, $k$ the velocity of the exhaust gases with respect to the vehicle, and $g$ the acceleration of gravity at the surface of the moon. The single and double dots denote the first and second derivatives with respect to time, respectively. In (1), $k>0$, and $\dot{m}\leq 0$.

Assume that the mission is to be performed during the time interval $0\leq t\leq\tau$ where $\tau$ is free. (The reason for allowing the terminal time $\tau$ to be free will become apparent later.)

Assume that altitude and altitude rate can be measured and are the only external inputs available to the thrust propgram. They might be obtained, for example, using a radar altimeter and Doppler radar, respectively.

In order to achieve a lunar soft landing, it is necessary that the vehicle arrive at the surface of the moon with zero velocity. Hence, it is required that $(x(\tau), \dot{x}(\tau))=0$.

It is assumed that the ratio of the maximum thrust to the initial mass is greater than the acceleration of gravity, and that the propulsion system possesses the capability of stopping the vehicle above or just at the surface of the moon for the range of initial altitudes and altitude rates of interest.

Since the vehicle is in the terminal descent phase of the mission, it follows that $x(0)>0$ and $\dot{x}(0)<0$.

System performance is measured by specifying

$$S = -\int_0^T \dot{m}(t)dt = m(0) - m(\tau) \tag{2}$$

as the cost function be minimized. Observe that the value of (2) is simply the change in mass during the mission, and is, therefore, equal to the fuel consumption. It is clear from physical considerations that $m(0)>0$.

In minimizing (2), only thrust (mass flow rate) programs for which a) $\dot{m}(t)$ is a measurable function[6] and b) $-\alpha\leq\dot{m}(t)\leq 0$ where $\alpha$ is a positive constant and $0\leq t\leq\tau$ are considered. Thrust programs which satisfy these two conditions are termed admissible.

An admissible thrust program which transfers the vehicle from an initial state $(x(0), \dot{x}(0))$ to the terminal state $(x(\tau), \dot{x}(\tau))=0$ and minimizes (2) will be called an optimal thrust program. The corresponding motion of the vehicle will be termed an optimal trajectory.

## Optimal Thrust Program

Since $\dot{m}/m = d/dt(\ln m)$, (1) may also be written as

$$\ddot{x} = -k\frac{d}{dt}(\ln m) - g. \tag{3}$$

Integrating (3) between the limits of 0 and $t$,

$$\dot{x}(t) = -k\ln\frac{m(t)}{m(0)} - gt + \dot{x}(0). \tag{4}$$

It then follows that $\dot{x}(\tau)=0$ if, and only if,

$$k\ln\frac{m(\tau)}{m(0)} = \dot{x}(0) - g\tau.$$

Solving for $m(\tau)$,

$$m(\tau) = m(0)\exp\left[\frac{\dot{x}(0)-g\tau}{k}\right]. \tag{5}$$

Substituting (5) into (2),

$$S = m(0)\left\{1 - \exp\left[\frac{\dot{x}(0)-g\tau}{k}\right]\right\}.$$

Hence, for a given $m(0)$, $\dot{x}(0)$, $g$, and $k$, the amount of fuel required to stop the vehicle, *i.e.*, to force $x(\tau)=0$,

[5] Miele, *op. cit.*, p. 128.

[6] The assumption that $\dot{m}(t)$ is a measurable function is made here so that we may appeal to certain existence theorems for optimal controls. In the Appendix, we shall show that the optimal thrust program is piecewise constant.

is a monotonic strictly increasing function of the terminal time $\tau$. Therefore, minimizing the terminal time $\tau$ is equivalent to minimizing the fuel consumption. The reason for allowing $\tau$ to be free in the problem formulation is clear. The minimal time problem is now considered.

With $x = x_1$, $\dot{x}_1 = x_2$, $x_3 = m$, and $u = \dot{m}$, (1) is represented by the system

$$\dot{x}_1 = x_2; \qquad \dot{x}_2 = -\frac{k}{x_3} u - g; \qquad \dot{x}_3 = u. \tag{6}$$

In (6), $u$ is the control variable and is constrained by the relation $-\alpha \leq u(t) \leq 0$ for $0 \leq t \leq \tau$. Note that the boundary conditions on (6) are $x_1(0) = x(0)$, $x_2(0) = \dot{x}(0)$, $x_3(0) = m(0)$, $x_1(\tau) = 0$, $x_2(\tau) = 0$, and that $x_3(\tau)$, the terminal mass, is free.

Physically, it is desired that $x_1(t) \geq 0$ and $x_3(t) > 0$, $0 \leq t \leq \tau$. However, these requirements lead to a problem with restricted phase coordinates.[7] Rather than treat this latter, more difficult problem, it is assumed that $x_1(t)$ and $x_3(t)$ are unrestricted (although subject to the assumption that $x_1(0) > 0$ and $x_3(0) > 0$) for $0 \leq t \leq \tau$, and shown that $x_1(t) \geq 0$ and $x_3(t) > 0$ along an optimal trajectory for $0 \leq t \leq \tau$.

Now assume that there exists at least one admissible thrust program which can perform the desired transfer. (This is consistent with the assumption that the propulsion system possesses the capability of stopping the vehicle above or just at the surface of the moon.) Then, since the minimal fuel problem is equivalent to the minimum time problem, one can appeal to theorems on the existence of time optimal controls,[8,9] to establish the existence of minimal fuel controls for the problem. Hence, the existence of a minimal fuel thrust program within the class of measurable and bounded thrust programs is assured. Since it is assumed that not all of the initial mass is propellant, it then follows from above that $x_3(t) > 0$, $0 \leq t \leq \tau$, along an optimal trajectory. The fact that $x_1(t) \geq 0$, $0 \leq t \leq \tau$, along an optimal trajectory is shown in the Appendix.

The Pontryagin maximum principle is now applied to obtain the form of the optimal thrust program. The Hamiltonian for the minimal time problem as formulated above is

$$H = \psi_1 x_2 - \psi_2 \frac{k}{x_3} u - \psi_2 g + \psi_3 u \tag{7}$$

where the $\psi_i$, $i = 1, 2, 3$, are certain nontrivial solutions of the system of equations,

$$\dot{\psi}_1 = 0; \qquad \dot{\psi}_2 = -\psi_1; \qquad \dot{\psi}_3 = -\psi_2 \frac{k}{(x_3)^2} u. \tag{8}$$

The optimal thrust (mass flow rate) program is obtained by determining an admissible $u(t)$ which maximizes the Hamiltonian. From (7), $H$ is maximized if

$$u(t) = \begin{cases} -\alpha & \text{whenever } \psi_3(t) - \dfrac{k}{x_3(t)} \psi_2(t) < 0 \\ 0 & \text{whenever } \psi_3(t) - \dfrac{k}{x_3(t)} \psi_2(t) > 0 \end{cases},$$

or, equivalently,

$$u(t) = \begin{cases} -\alpha & \text{whenever } \psi_2(t) > \dfrac{x_3(t)}{k} \psi_3(t) \\ 0 & \text{whenever } \psi_2(t) < \dfrac{x_3(t)}{k} \psi_3(t) \end{cases}, \tag{9}$$

$0 \leq t \leq \tau$. Note that $u(t)$ is indeterminate whenever

$$\psi_3(t) - \frac{k}{x_3(t)} \psi_2(t) = 0,$$

or, equivalently, whenever

$$\psi_2(t) - \frac{x_3(t)\psi_3(t)}{k} = 0.$$

The condition expressed by this relation is called the singularity condition.

It is shown in the Appendix that the singularity condition cannot hold on any finite closed interval in $[0, \tau]$. Hence, there exist no singular optimal controls[10] for the problem. It is also shown that there is at most one switching in $[0, \tau]$, and that the optimal control consists of either full thrust from the initiation of the mission until touchdown, or a period of zero thrust (free-fall), followed by full thrust until touchdown.

## Synthesis of the Optimal Thrust Program

Because of its simplicity, the optimal thrust program can be synthesized by determining an appropriate switching function. Development of the switching function consists in determining a relation $f(x_1, x_2) = 0$ such that if full thrust is applied continuously from the time when this relation is first satisfied until touchdown, a soft landing is achieved.

The existence of this switching function is demonstrated by actually constructing it. In addition, it is shown that the switching time, *i.e.*, the instant at which thrusting is initiated, is unique.

The switching function is obtained by integrating the equations of motion under the assumption $u(t) = -\alpha$, and determining the relation which must exist between the initial altitude and altitude rate in order to achieve a soft landing in a time $t_1$ using full thrust. Let $[0, t_1]$ be the interval over which thrusting occurs. Let

[7] Pontryagin, *et al.*, *op. cit.*, ch. 6.
[8] E. B. Lee and L. Markus, "Optimal control for nonlinear processes," *Arch. for Rat. Mech. and Anal.*, vol. 8, pp. 36–58; 1961.
[9] E. Roxin, "The existence of optimal controls," *Mich. Math. J.*, vol. 9, pp. 109–119; 1962.
[10] C. D. Johnson and J. E. Gibson, "Singular solutions in problems of optimal control," IEEE Trans. on Automatic Control, vol. AC-8, pp. 4–15; January, 1963.

$x_1^*$, $x_2^*$, and $M_0$ be the altitude, altitude rate, and mass, respectively, at the initiation of thrusting. Note that $x_3(t) = M_0 - \alpha t$ for $0 \leq t \leq t_1$.

From (4),

$$x_2(t) = -k \ln\left(1 - \frac{\alpha}{M_0} t\right) - gt + x_2^*, \quad 0 \leq t \leq t_1. \tag{10}$$

It then follows from the first equation of (6) that

$$\begin{aligned} x_1(t) &= \int_0^t x_2(s)ds + x_1^* \\ &= \frac{kM_0}{\alpha}\left(1 - \frac{\alpha}{M_0}t\right) \ln\left(1 - \frac{\alpha}{M_0}t\right) + kt - \tfrac{1}{2}gt^2 \\ &\quad + x_2^* t + x_1^*, \qquad 0 \leq t \leq t_1. \end{aligned} \tag{11}$$

For a soft landing, it is required that $x_1(t_1) = x_2(t_1) = 0$. Hence, at $t = t_1$, (10) and (11) become

$$0 = -k \ln\left(1 - \frac{\alpha}{M_0} t_1\right) - gt_1 + x_2^* \tag{12}$$

and

$$\begin{aligned} 0 = \frac{kM_0}{\alpha}\left(1 - \frac{\alpha}{M_0}t_1\right) \ln\left(1 - \frac{\alpha}{M_0}t_1\right) + kt_1 - \tfrac{1}{2}gt_1^2 \\ + x_2^* t_1 + x_1^*, \end{aligned} \tag{13}$$

respectively.

Solving (12) for $x_2^*$, substituting the result into (13), and simplifying, one obtains

$$x_1^* = -\frac{kM_0}{\alpha} \ln\left(1 - \frac{\alpha}{M_0} t_1\right) - kt_1 - \tfrac{1}{2}gt_1^2 \tag{14}$$

and

$$x_2^* = k \ln\left(1 - \frac{\alpha}{M_0} t_1\right) + gt_1 \tag{15}$$

as the relations which determine the initial state (altitude and altitude rate) from which a soft landing can be achieved in time $t_1$ using full thrust.

Rather than eliminate the parameter $t_1$ from (14) and (15) to obtain an exact expression $f(x_1^*, x_2^*) = 0$ for the switching function, an approximate relation which is adequate for a large number of cases is now derived.

Observe first that $\alpha t_1/M_0$ is the fraction of the initial mass which is consumed during thrusting. If the mission can be accomplished with no more than 25 per cent of the initial mass as propellant, then the approximation

$$\ln\left(1 - \frac{\alpha}{M_0} t_1\right) \cong -\frac{\alpha t_1}{M_0} - \frac{\alpha^2 t_1^2}{2M_0^2}$$

may be used. The maximum error in this approximation is 2.23 per cent.

Substituting the approximation of the ln function into (14) and (15), and simplifying,

$$x_1^* \cong a t_1^2 \tag{16}$$

$$x_2^* \cong -2at_1 - bt_1^2, \tag{17}$$

respectively, where

$$a = \frac{1}{2}\left(\frac{k\alpha - gM_0}{M_0}\right) \tag{18}$$

and

$$b = \frac{k\alpha^2}{2M_0^2} \cdot \tag{19}$$

Since the maximum thrust to initial mass ratio is greater than $g$, $k\alpha > gM_0$, and, therefore, $a > 0$. Moreover, $x_1^*$ which is the initial altitude, is positive. Hence, the only value of $t_1$ which satisfies (16) and is physically meaningful is

$$t_1 = \sqrt{\frac{x_1^*}{a}} \cdot \tag{20}$$

Substituting (20) into (17) and simplifying,

$$f(x_1^*, x_2^*) = \frac{b}{a} x_1^* + 2a\sqrt{\frac{x_1^*}{a}} + x_2^* = 0. \tag{21}$$

Eq. (21) is the desired switching function. It is remarked that the set of all initial states from which it is possible to reach the origin using full thrust is comprised of those states $(x_1^*, x_2^*)$ which satisfy (21). However, for physical reasons, (21) is of interest only for a particular region of the fourth quadrant of the $x_1$-$x_2$ plane. This is seen by noting that $0 \leq t_1 \leq 0.25 M_0/\alpha$ where the lower bound results if $x_1^* = x_2^* = 0$, *i.e.*, the vehicle is initially at the landing site, and the upper bound results if 25 per cent of the initial mass is propellant which may be used for the mission. It then follows from (18) and (19) that only the region of the $x_1$-$x_2$ plane defined by the relations

$$0 \leq x_1 \leq 0.0625a\left(\frac{M_0}{\alpha}\right)^2$$

and

$$-0.5a\left(\frac{M_0}{\alpha}\right) - 0.0625b\left(\frac{M_0}{\alpha}\right)^2 \leq x_2 \leq 0.0, \tag{22}$$

is of interest.

If one solves (21) for $x_2^*$ and considers the region of the $x_1$-$x_2$ plane defined by the relations in (22), one can obtain a plot of (21) as shown in Fig. 2.

For a given initial altitude $\xi_1$, and initial altitude rate $\xi_2$, it is easy to show that the free-fall trajectory of the space vehicle is given by

$$x_1 = \xi_1 - \frac{1}{2g}\left[(x_2)^2 - (\xi_2)^2\right].$$

The free-fall trajectory for a given initial state $(\xi_1, \xi_2)$ in the fourth quadrant of the $x_1$-$x_2$ plane is shown in

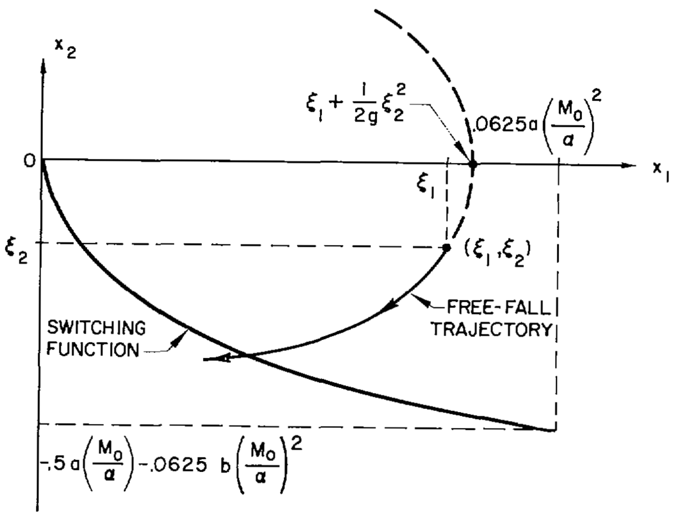


Fig. 2—Plot of switching function and vehicle free-fall trajectory.

Fig. 2. It is clear that the curves of the switching function and a free-fall trajectory cannot intersect more than once in the fourth quadrant.

Now let $\xi_1$ and $\xi_2$ be the altitude and altitude rate, respectively, at the initiation of the mission, and assume that $f(\xi_1, \xi_2) > 0$, *i.e.*, the point $(\xi_1, \xi_2)$ lies above the switching function curve. The optimal thrust program for this case is clear. The vehicle is allowed to free-fall until its altitude and altitude rate are such that $f(x_1, x_2) = 0$ at which time thrusting is initiated and continues until touchdown. Since the free-fall trajectory and the switching curve intersect only once, it is clear that the switching time is unique and, therefore, must be the optimal one.

This scheme is easily realized in practice. Measurements of altitude and altitude rate are taken during free fall and substituted into the relation

$$f(x_1, x_2) = \frac{b}{a} x_1 + 2a\sqrt{\frac{x_1}{a}} + x_2.$$

As long as $f(x_1, x_2) > 0$, the vehicle is allowed to free-fall. As soon as $f(x_1, x_2) = 0$, thrusting is initiated. The computation involved is simple enough that it can be performed in real time by a small special-purpose digital computer. A block diagram of the optimal system is given in Fig. 3. Once thrusting is initiated, there is no longer a need to measure altitude and altitude rate, and to compute $f(x_1, x_2)$ unless they are needed for other purposes.

If the altitude and altitude rate at the initiation of the mission are such that $f(\xi_1, \xi_2) = 0$, then thrusting is initiated immediately and the optimal system assumes the same form as above.

However, difficulty arises if $f(\xi_1, \xi_2) < 0$. In particular, it can be shown in this case that even if full thrust is applied, the vehicle will still arrive at the surface of the moon with a finite downward velocity. In other words, a soft landing is beyond the capability of the assumed

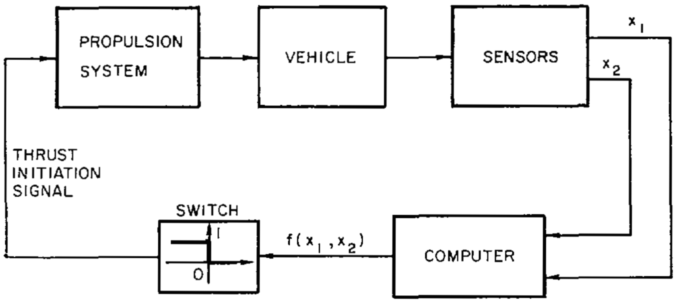


Fig. 3—Block diagram of optimal system.

propulsion system and a higher thrust level is needed to accomplish the mission. To avoid this difficulty, it has been assumed that $f(\xi_1, \xi_2) \geq 0$, *i.e.*, that the mission and the propulsion system are compatible.

## Discussion of Results

Because of its simplicity, the systen proposed here is especially appealing from the viewpoints of size, weight, power, and reliability. As noted earlier, the computer functions are very simple and only involve computation of the function $f(x_1, x_2)$ and the generation of the thrust initiation signal. Hence, computer design and development should pose no major difficulties.

Since the altitude and altitude rate during the free-fall phase of the descent cannot be determined exactly due to noise and bias in their measurement, the measured and free-fall trajectories will, in general, differ from one another. Hence, it follows that the thrust will not, in general, be initiated at the correct time to achieve a soft landing. In other words, successful execution of the mission depends heavily on the accuracy with which the altitude and altitude rate are known at the initiation of thrusting. It is seen from (10) that uncertainty in the initial altitude rate appears directly in the altitude rate during descent. As can be seen from (11), the initial altitude error appears directly in the altitude during descent, but the initial altitude rate error contributes a component which increases linearly with time during the powered descent.

The problems associated with designing a thrust program which will compensate for error propagation such as that described above can be circumvented by introducing a "biased landing point." For example, the design presented in this paper could equally well be effected by choosing a terminal altitude of, say, 50 to 100 feet and a terminal altitude rate of $-1$ to $-5$ feet per second. When the sensed velocity reaches the desired value, thrust acceleration is reduced to one lunar $g$, and the vehicle descends to the surface of the moon at a small constant velocity with thrust cutoff occurring at touchdown. It should be noted that the altitude error is not critical in this scheme as long as the requisite terminal velocity is achieved at some altitude in the specified range. In a manned mission where safety is a primary consideration, it may be desirable to incorporate a slow descent "closure phase," such as that described here, rather than attempt a direct descent. This problem as well as a number of other problems associated with implementing a lunar soft landing system are discussed elsewhere. (See References.)

In this study, the terminal mass was assumed to be free. A more useful result would be the determination of the minimal fuel thrust program required to deliver a fixed payload to the surface of the moon. However, a parametric study based on the results presented here could be conducted to determine such fuel requirements. For example, one might consider a representative set of initial conditions and determine the various masses at touchdown corresponding to this set. These data would then give one some idea of the payload capabilities of the system.

In any practical mission, one would naturally place more than the minimum amount of fuel onboard the vehicle in order to allow for maneuvering and mission abort if the latter becomes necessary. One purpose of a fuel optimization study such as this one is to develop a system which accomplishes the mission with an efficient utilization of fuel. Computation of the minimum fuel required for a given mission, say, for example, by simulation studies, may then be used as a guide in specifying fuel requirements.

Having assumed that the thrust could only act as a braking force, *i.e.*, $k>0$, and $\dot{m}\leq 0$ in (1), it was shown earlier that the minimal fuel and minimal time problems are equivalent. Now if impulsive thrusting is permitted, it is clear that the mission can be accomplished in minimum time (and, therefore, with minimum fuel) by allowing the vehicle to free-fall until touchdown at which time an impulsive thrust is applied to reduce the vehicle's velocity to zero instantaneously. This result follows immediately from the fact that any thrusting prior to touchdown will cause touchdown to occur later than it would if the vehicle were allowed to free-fall. Hence, the mission would not be accomplished in minimum time.

From elementary energy considerations, the velocity impulse required at touchdown is

$$\Delta V = \sqrt{2gx(0) + \dot{x}^2(0)}$$

where $x(0)$ and $\dot{x}(0)$ are the initial altitude and altitude rate, respectively, and $\Delta V$ is the required velocity impulse. The fuel consumption in this case is given by the well-known relation

$$S = m(0)[1 - \exp(-k\Delta V)], \tag{23}$$

where all of the terms have been defined previously. It is clear that (23) gives the greatest lower bound on the fuel required to perform the soft landing under the assumption that the thrust can only act as a braking force. It is remarked that it can be shown that this result does not necessarily hold if the thrust is permitted to accelerate as well as brake the vehicle's motion downward.

From the results derived above, it follows that fuel consumption is inversely dependent on the maximum thrust level attainable. That is, as the maximum thrust level is increased, the optimal thrust program, which was derived earlier, will more closely approximate the impulsive thrust program.

## Appendix

For time optimal control of the autonomous system (6), it is well known[11] that the Hamiltonian (7) must satisfy the relation

$$\psi_1(t)x_2(t) - \psi_2(t)\frac{k}{x_3(t)}u(t) - \psi_2(t)g + \psi_3(t)u(t) = \text{constant} \geq 0 \tag{24}$$

for each $t$, $0\leq t\leq\tau$, along an optimal trajectory.

It is shown first that the singularity condition

$$\psi_3(t) - \frac{k}{x_3(t)}\psi_2(t) = 0 \tag{25}$$

cannot hold along an optimal trajectory on any closed interval $[t', t'']\subset[0, \tau]$, $t'<t''$.

Assume the contrary. Then from (24) and (25),

$$\psi_1(t)x_2(t) - \psi_2(t)g = \text{constant} \tag{26}$$

for $t\in[t', t'']$. Differentiating (26) and substituting from (6) and (8),

$$\psi_1(t)\frac{ku(t)}{x_3(t)} = 0$$

for $t\in[t', t'']$. Since $k$ is a positive constant and $x_3(t)>0$ for all $t\in[0, \tau]$, it follows that

$$\psi_1(t)u(t) = 0 \tag{27}$$

for $t\in[t', t'']$.

From the first equation in (8), $\dot{\psi}_1=0$. Therefore, $\psi_1(t)=\text{constant}$ for $t\in[0, \tau]$. Hence, (27) holds for $\psi_1(t)=\text{constant}\neq 0$ only if $u(t)\equiv 0$ for $t\in[t', t'']$. The other possibility is that $\psi_1(t)\equiv 0$ for $t\in[t', t'']$.

Consider first the case where $\psi_1(t)\equiv 0$ for $t\in[t', t'']$. It then follows from the first equation in (8) that $\psi_1(t)\equiv 0$ for $t\in[0, \tau]$, and from the second equation in (8) that $\psi_2(t)=\text{constant}$ for $t\in[0, \tau]$. Hence, the following three subcases must be considered. $\psi_2(t)\equiv 0$, $\psi_2(t)=\text{constant}<0$, and $\psi_2(t)=\text{constant}>0$, all for $t\in[0, \tau]$.

If $\psi_2(t)\equiv 0$ for $t\in[0, \tau]$, then $\dot{\psi}_3(t)\equiv 0$ for $t\in[0, \tau]$ from the last equation in (8). Since the terminal mass $x_3(\tau)$ is free, the transversality condition[12] gives $\psi_3(\tau)=0$. Hence, $\psi_1(t)=\psi_2(t)=\psi_3(t)\equiv 0$ for $t\in[0, \tau]$. However, the maximum principle[13] asserts that, for optimality, there must exist a nonzero solution of the system of (8). Hence, this subcase cannot arise.

[11] Pontryagin, *et al.*, *op. cit.*, pp. 20–21.
[12] *Ibid.*, pp. 45–58.
[13] *Ibid.*, pp. 20–21.

Now if $\psi_2(t) = \text{constant} < 0$ for $t \in [t', t'']$, and, therefore, for $t \in [0, \tau]$, it follows from the last equation in (8) that

$$\dot{\psi}_3(t) \leq 0$$

for $t \in [0, \tau]$ since $-\alpha \leq u(t) \leq 0$ for $t \in [0, \tau]$. Since $\psi_3(\tau) = 0$, this means that $\psi_3(t) \geq 0$ for $t \in [0, \tau]$. But

$$\frac{k}{x_3(t)} \psi_2(t) < 0$$

for $t \in [0, \tau]$. Hence,

$$\psi_3(t) - \frac{k}{x_3(t)} \psi_2(t) > 0$$

for $t \in [t', t'']$ which contradicts (25).

In a like manner, it can be shown that if $\psi_2(t) = \text{constant} > 0$ for $t \in [t', t'']$, then

$$\psi_3(t) - \frac{k}{x_3(t)} \psi_2(t) < 0$$

for $t \in [t', t'']$ which also contradicts (25). Consequently, the singularity condition cannot hold on any closed interval $[t', t'']$ for the case where $\psi_1(t) \equiv 0$, $t \in [t', t'']$.

Secondly, consider the case where $u(t) \equiv 0$ for $t \in [t', t'']$. From the last equation of (8), it follows that $\dot{\psi}_3(t) = 0$, and therefore, that $\psi_3(t) = \text{constant}$ for $t \in [t', t'']$. Also, since $u(t) = 0$ for $t \in [t', t'']$, it is clear that $x_3(t) = \text{constant}$ for $t \in [t', t'']$. Hence, (25) can only hold on $[t', t'']$ if $\psi_2(t) = \text{constant}$ on this interval. But $\psi_2(t) = \text{constant}$ only if $\dot{\psi}_2(t) = -\psi_1(t) \equiv 0$ for $t \in [t', t'']$. However, it was shown in the preceding case that the singularity condition cannot hold on the interval $[t', t'']$ if $\psi_1(t) \equiv 0$ on this interval. Hence, along an optimal trajectory, the singularity condition cannot hold on any closed interval $[t', t''] \subset [0, \tau]$, $t' < t''$.

It is now shown that the singularity condition can, at most, hold only for a single value of $t \in [0, \tau]$. Before proceeding, it will be convenient to develop two results which are fundamental to the sequel.

First, note from (6) and (8) that

$$\begin{aligned} \frac{d}{dt}(\psi_3 x_3) &= \psi_3 \dot{x}_3 + \dot{\psi}_3 x_3 \\ &= \psi_3 u - \psi_2 \frac{k}{x_3} u \\ &= -\frac{ku}{x_3}\left(\psi_2 - \frac{x_3 \psi_3}{k}\right) \end{aligned} \tag{28}$$

almost everywhere in $[0, \tau]$.

Second, assume that an optimal solution for the problem is given, and define

$$h(t) = \psi_2(t) - \frac{x_3(t)\psi_3(t)}{k} \tag{29}$$

for $t \in [0, \tau]$ where $\psi_2(t)$ and $\psi_3(t)$ are solutions of (8) which correspond to the optimal solution. Then, from the maximum principle[14] $\psi_2(t)$, $\psi_3(t)$ and $x_3(t)$ are continuous functions of $t$ for $t \in [0, \tau]$, and it follows that $h(t)$ is also. Hence, the set of all points $t \in [0, \tau]$ where $h(t) = 0$ is closed. Call this set $I_0$.

Let $(t', t'')$ be an open interval in the complement of $I_0$, $t' < t''$, such that $t', t'' \in I_0$. Thus, $h(t') = h(t'') = 0$. Let

$$I = \{t : t \in (t', t'')\}.$$

It is clear that, for $t \in I$, there are two cases: $h(t) > 0$ or $h(t) < 0$.

As the second result, it is asserted that cases where $h(t') = h(t'') = 0$ and $h(t) > 0$ or $h(t) < 0$ for $t \in I$ cannot occur.

Assume the contrary, and consider first the case where $h(t) > 0$ for $t \in I$. Then from (9), $u(t) = -\alpha$ for $t \in I$, and it follows from (28) and (29) that

$$\frac{d}{dt}(\psi_3 x_3) > 0 \tag{30}$$

for $t \in I$. From (8), recall that $\dot{\psi}_2(t) = -\psi_1(t) = \text{constant}$ for $t \in [0, \tau]$. If $\dot{\psi}_2(t) = \text{constant} \leq 0$ for $t \in [t', t'']$, it follows that

$$\frac{dh}{dt} = \dot{\psi}_2 - \frac{1}{k}\frac{d}{dt}(\psi_3 x_3) < 0 \tag{31}$$

for $t \in I$. But this means that $h(t) < h(t') = 0$ for $t \in I$ which contradicts the assumption that $h(t) > 0$ for $t \in I$. Therefore, $\dot{\psi}_2(t) = \text{constant} > 0$ for $t \in I$.

From (28) and (31), it follows that

$$\begin{aligned} \frac{dh}{dt} &= \dot{\psi}_2 + \frac{u}{x_3}\left(\psi_2 - \frac{x_3\psi_3}{k}\right) \\ &= \dot{\psi}_2 + \frac{u}{x_3} h. \end{aligned} \tag{32}$$

Using the facts that $x_3(t)$ and $h(t)$ are continuous functions of $t$ for $t \in [t', t'']$, and, indeed, for $t \in [0, \tau]$ along an optimal trajectory, and that $u(t)$ is indeterminate only when $h(t)$ vanishes, it can be shown that $dh/dt$ is a continuous function of $t$ for $t \in [0, \tau]$. Hence,

$$\left.\frac{dh}{dt}\right|_{\substack{t=t'\\t=t''}} = \dot{\psi}_2 = \text{constant} > 0 \tag{33}$$

for the case under consideration.

Now let $\delta > 0$ be any real number such that $(t'' - \delta) \in I$. Then, since $h(t) > 0$ for $t \in I$ and $h(t'') = 0$,

$$\left.\frac{dh}{dt}\right|_{t=t''} = \lim_{\delta \to 0-} \frac{h(t'') - h(t'' - \delta)}{\delta} \leq 0$$

which contradicts (33). Therefore, the case where $h(t') = h(t'') = 0$ and $h(t) > 0$ for $t \in I$ cannot occur.

Now consider the other possibility where $h(t') = h(t'') = 0$ and $h(t) < 0$ for $t \in I$. From (9), $u(t) = 0$ for $t \in I$.

[14] *Ibid.*, pp. 20–21.

Since $h(t')=h(t'')=0$, it follows from (32) and the continuity of $dh/dt$ that

$$\frac{dh}{dt} = \dot{\psi}_2 = \text{constant} \tag{34}$$

for $t\in[t', t'']$. But the boundary conditions $h(t')=h(t'')=0$ can only be satisfied simultaneously if $\dot{\psi}_2=0$ in (34) which implies that $h(t)\equiv 0$ for $t\in[t', t'']$. But this contradicts the hypothesis that $h(t)\neq 0$ for $t\in I$. Hence, the case where $h(t')=h(t'')=0$ and $h(t)<0$ for $t\in I$ cannot occur. This completes the proof of the second result.

It is now easy to show that the singularity condition can, at most, hold only for a single value of $t\in[0, \tau]$. Assume the contrary, and let $t_1, t_2\in[0, \tau]$ be some two distinct points at which $h(t)$ vanishes. Since the "singularity" condition cannot hold on the entire closed interval $[t_1, t_2]$, there exists at least one point $t_0\in[t_1, t_2]$ at which $h(t_0)>0$ or $h(t_0)<0$. Consider the case where $h(t_0)>0$. The proof for the case where $h(t_0)<0$ is identical.

Since $h(t)$ is continuous on $[0, \tau]$, the set of all points $t>t_0$, $t\in[0, \tau]$, at which $h(t)=0$ is closed. Let $t''>t_0$ be the g.l.b. of this set. Similarly, the set of all points $t<t_0$, $t\in[0, \tau]$, at which $h(t)=0$ is also closed. Let $t'<t_0$ be the l.u.b. of this set. Hence, the open interval $(t', t'')$, $t'<t''$, containing $t_0$, is such that $h(t')=h(t'')=0$ with $h(t)>0$. But this is impossible, as shown above. Therefore, $h(t)$ can vanish at most once in the interval $[0, \tau]$. As a result, there is at most one switching in the interval $[0, \tau]$ and the optimal thrust program is piecewise constant.

Now consider the four possible forms which $h(t)$ can assume on $[0, \tau]$ and show that the optimal thrust program consists of either full thrust from the initiation of the mission until touchdown or a period of zero thrust (free-fall), followed by full thrust until touchdown.

*Case 1*

$h(t)>0$ for $t\in[0, \tau]$. It immediately follows from the definition of $h(t)$ and (9) that $u(t)=-\alpha$ for $t\in[0, \tau]$. In this case, full thrust is used from the initiation of the mission until touchdown. Since it is assumed that the propulsion system is capable of stopping the vehicle above or just at the surface of the moon for the initial altitudes and altitude rates of interest, it is clear that there will exist initial states $(x_1(0), x_2(0))$ from which a soft landing can be achieved using full thrust and for which $x_1(t)\geq 0$ for $t\in[0, \tau]$. Hence, this case is physically meaningful.

It is clear that the subcases where $h(t)$ vanishes at either $t=0$ or $t=\tau$, but is positive everywhere else in $[0, \tau]$, lead to the same thrust program. In addition, it is easy to show that $h(t)$ cannot vanish in $[0, \tau]$ for this case.

*Case 2*

$h(t)<0$ for $t\in[0, \tau]$. From the definition of $h(t)$ and (9), it follows that $u(t)=0$ for $t\in[0, \tau]$. It is clear that this case cannot lead to a soft landing unless $x(0)=\dot{x}(0)=0$, or $x(0)<0$ and $\dot{x}(0)>0$. But it was assumed that $x(0)>0$ and $\dot{x}(0)<0$. Hence, this case is not relevant to our problem.

The above argument also applies to the subcases where $h(t)$ vanishes at either $t=0$ or $t=\tau$, but is negative everywhere else in $[0, \tau]$.

*Case 3*

$h(t)>0$ for $t\in[0, t_1)$, $h(t)<0$ for $t\in(t_1, \tau]$, and $h(t_1)=0$ where $t_1\in[0, \tau]$. For this case,

$$u(t) = \begin{cases} -\alpha & \text{for } t\in[0, t_1) \\ 0 & \text{for } t\in(t_1, \tau]. \end{cases}$$

In order to achieve a soft landing, *i.e.*, $x(\tau)=\dot{x}(\tau)=0$ in this case, it is necessary to have a particular $x(t_1)<0$ and $\dot{x}(t_1)>0$ so that the vehicle will "coast" to the desired terminal state. However, since full thrust is used for $t\in[0, t_1)$ and it has been assumed that the propulsion system is capable of stopping the vehicle above or just at the surface of the moon for the range of initial altitudes and altitude rates of interest, this case cannot occur.

*Case 4*

$h(t)<0$ for $t\in[0, t_1)$, $h(t)>0$ for $t\in(t_1, \tau]$, and $h(t_1)=0$ where $t_1\in(0, \tau)$. For this case,

$$u(t) = \begin{cases} 0 & \text{for } t\in[0, t_1) \\ -\alpha & \text{for } t\in(t_1, \tau]. \end{cases}$$

In this case, the optimal thrust program consists of a period of zero thrust (free-fall) followed by full thrust until touchdown. It is clear that this case leads to a soft landing with $x(t)\geq 0$ for $t\in[0, \tau]$.

Hence, only Cases 1 and 4 are relevant to the problem, and the optimal thrust program has been established.

## References

[1] R. K. Cheng and I. Pfeffer, "Terminal guidance system for soft lunar landing," in "Guidance and Control—Progress in Astronautics and Rocketry," vol. 8, R. E. Robertson and J. S. Farrior, Eds., Academic Press, New York, N. Y.; 1962.

[2] B. A. Kriegsman and M. H. Reiss, "Terminal guidance and control techniques for soft lunar landing," *ARS J.*, vol. 32, pp. 401–413; March, 1962.

[3] B. A. Hall, R. G. Dietrich, and K. E. Tierman, "A Minimum Fuel Vertical Touchdown Lunar Landing Guidance Technique," presented at the AIAA Guidance and Control Conference, Cambridge, Mass.; August, 1963.