Astronomía Avanzada I (Semester 1 2024)

# Stellar Atmospheres (2)

## Introduction to Radiative Transfer

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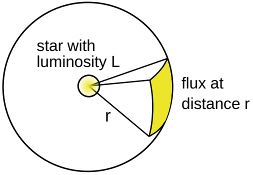
# Recap: Flux

Flux (or radiant flux), $F$, is the total amount of energy that crosses a unit area per unit time. Flux is usually given in watts per square meter $(\mathrm{W/m^2})$.

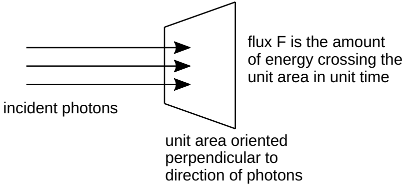
The flux of an astronomical source depends on the luminosity of the object and its distance from the Earth, according to the inverse square law:

$$F = \frac{L}{4\pi r^2}$$

where $F$ = flux measured at distance $r$,
$L$ = luminosity of the source,
$r$= distance to the source.




surface area of sphere 4πd²



# Recap: Flux

**example:** The luminosity of the Sun is $L_\odot = 3.839 \times 10^{26}\ \mathrm{W}$.
At a distance of $1\ \mathrm{AU} = 1.496 \times 10^{11}\ \mathrm{m}$, Earth receives a radiant flux above its absorbing atmosphere of

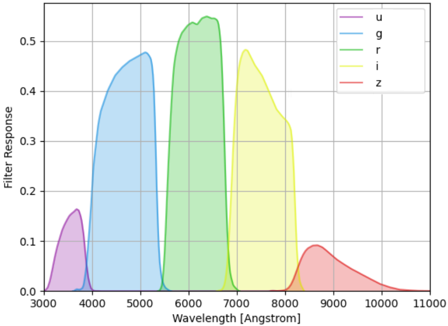
$$F = \frac{L}{4\pi r^2} = 1365\ \mathrm{W\,m^{-2}}.$$

# Recap: Magnitudes and Broad-Band Filters



We often measure the flux $F$ from astronomical objects via a **logarithmic magnitude scale**.

Magnitudes almost universally involve a set of **broad-band filters**, e.g. Johnson *UBVRI* or Sloan *ugriz*:



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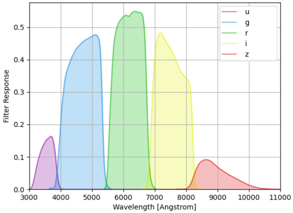



We then calculate:

$$m = -2.5 \log \int_0^\infty F_\nu W(\nu)\, \mathrm{d}\nu + \mathrm{const}$$

with:
$F_\nu$ a star's spectral energy distribution (SED)
$W(\nu)$ a filter passband

# Recap: The Black Body



We have already seen:

**Black-body radiation** is the thermal electromagnetic radiation within, or surrounding, a body in thermodynamic equilibrium with its environment, emitted by an idealized black body (opaque, non-reflective).

For a black body, the **Stefan-Boltzmann law** states that the total energy radiated per unit area per unit time (also known as the flux) is directly proportional to the fourth power of the black body's temperature:

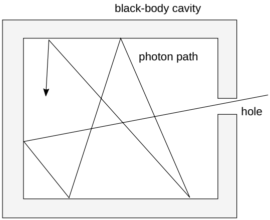
$$F = \sigma_{\mathrm{SB}}\, T^4.$$

# The Black Body



Imagine a box which is completely closed except for a small hole.

Any light entering the box will have a very small likelihood of escaping, and will eventually be absorbed by the gas or walls.
For constant temperature walls, this is in **thermodynamic equilibrium**.



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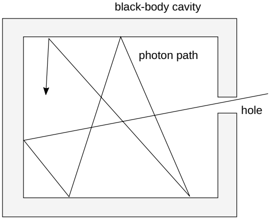
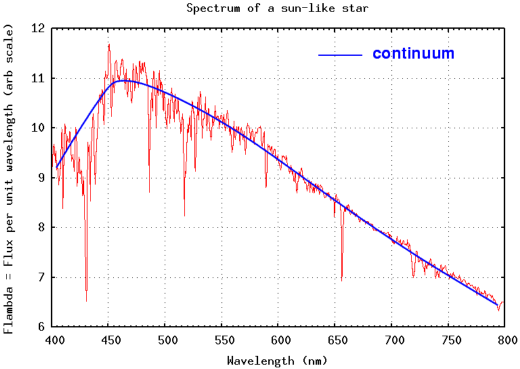



If this box is **heated**, the walls will emit photons, filling the inside with radiation. A small fraction of the radiation will leak out of the hole, but so little that the gas within it remains in equilibrium.
The emitted radiation is that of a **black body**.
Stars share properties of the black-body emitter, in the sense that a negligibly small fraction of the radiation escapes from each.

# Stars as Black Bodies

Comparison of black body spectrum vs. stellar spectrum:




This spectrum of a solar-like star shows just how far a typical star's spectrum (red) deviates from the ideal blackbody (blue).

credit: Michael Richmond,

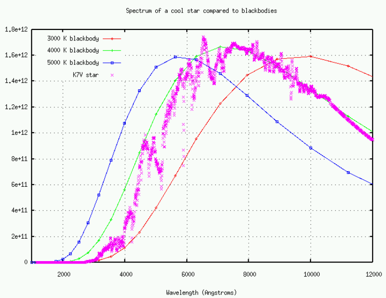
`https://aasnova.org/2018/10/31/perfect-blackbodies-in-the-sky/`

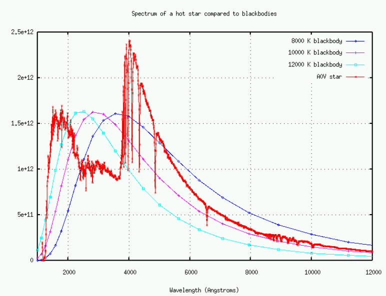
# Stars as Black Bodies



How close to black bodies are real stars? It depends...

For relatively cool stars (e.g. a K7 dwarf), a black body is a pretty good model, whereas for hot stars, the spectrum differs very strongly from a black body in the near-UV:



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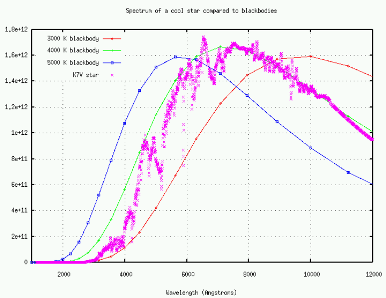


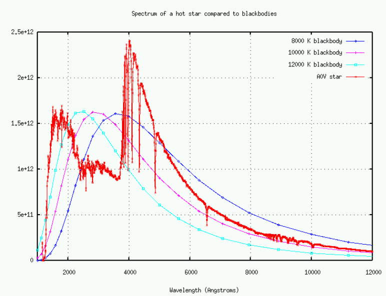


The reason for this difference: sources of continuous and line opacity in the stellar photospheres.

# Stars as Black Bodies

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What's the difference?

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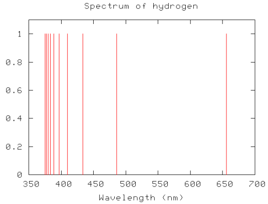
The outer parts of stellar atmospheres are largely H. In the red of the optical, H is nearly transparent, allowing blackbody radiation to escape. However, in the blue and near-UV, strong H absorption lines lead to absorption below 3650 Å. The energy is re-emitted as less-energetic photons at longer wavelengths.



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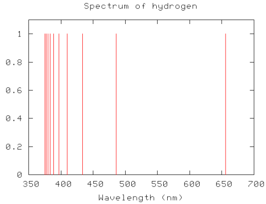



This effect, called **line-blanketing**, redistributes much of the UV energy of the star into the visible and IR.

# Recap: The Stefan-Boltzmann Law

Blackbody radiation is continuous and isotropic, with the intensity only varying with wavelength and temperature.

Following empirical (Josef Stefan in 1879) and theoretical (Ludwig Boltzmann in 1884) studies of black bodies, there is a relation between flux and temperature, known as the Stefan-Boltzmann law:

$$F = \sigma T^4$$

with the Boltzmann constant $\sigma_{\rm SB} = 5.6705 \times 10^{-5}$ erg/cm$^2$/s/K$^4$

Despite stars not being black bodies, the **effective temperature** is calculated using the above equation:

$$L = 4\pi R^2 = \sigma_{\rm SB} T^4$$

The effective temperature is the temperature which a black body would need to radiate the same amount of energy as the star.

# The Planck Formula

The **black body intensity** is defined (following discovery by Max Planck in 1900) as either

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp(hc/\lambda kT) - 1}$$

or

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1}$$

where $c = 2.99 \times 10^{10}\ \mathrm{cm}$, $h = 6.67 \times 10^{-27}\ \mathrm{erg\,s}$, $k = 1.38 \times 10^{-16}\ \mathrm{erg/s}$.

We can use this to compute the bolometric flux:

$$F = \pi \int_0^\infty B_\nu(T)\,\mathrm{d}\nu = \pi \int_0^\infty \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1}\,\mathrm{d}\nu$$
$$= \pi \frac{2h}{c^2} \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1}\,\mathrm{d}x = \pi \frac{2h}{c^5} \left(\frac{kT}{h}\right)^4 \frac{\pi^4}{15} = \sigma_{\mathrm{SB}}\, T^4$$

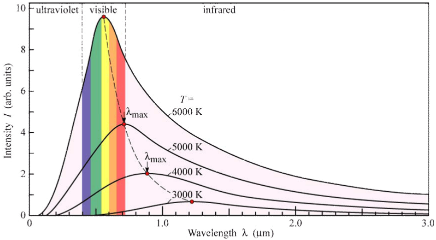
# Wien's Displacement Law

For increasing temperatures, the black body intensity increases over all $\lambda$, and the maximum in the energy distribution shifts to shorter $\lambda$.

**Wien's displacement law** states that the spectral radiance of black-body radiation per unit wavelength peaks at the wavelength $\lambda_{\max}$ given by:

$$\lambda_{\max} = \frac{b}{T}$$

where $T$ is the absolute temperature and $b$ is Wien's displacement constant, $b = 2.897771955 \times 10^{3}\ \mathrm{m\,K}$.



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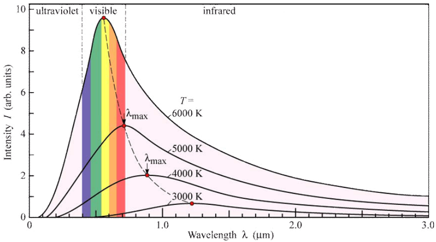


**example:** $\lambda_{\max} = 5175\text{Å}$ for the Sun

# Rayleigh-Jeans and Wien approximations

At **long wavelengths** $\lambda \gg \lambda_{\max}$ (small frequencies $\nu \ll \nu_{\max}$), the Planck equation can be approximated by the **Rayleigh-Jeans law**:

$$B_\nu(T) \sim 2\frac{\nu^2}{c^2}kT, \quad B_\lambda(T) \sim 2ckT\lambda^{-4}$$

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At **short wavelengths** $\lambda \leq \lambda_{\max}$ (large frequencies $\nu \geq \nu_{\max}$) the **Wien law** is a good approximation:

$$B_\nu(T) \sim 2\frac{h\nu^3}{c^2}\exp\left(-\frac{h\nu}{kT}\right), \quad B_\lambda(T) \sim 2\frac{hc^2}{\lambda^5}\exp\left(-\frac{hc}{\lambda kT}\right)$$

# Interaction Radiation - Matter

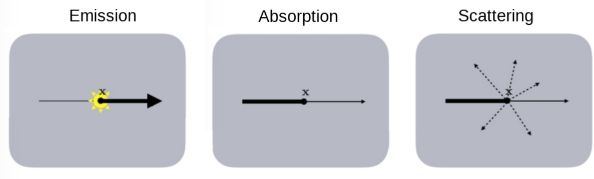
When radiation interacts with matter, energy can be removed from, or delivered to, the radiation field.

We can distinguish different forms of interaction:

**True emission:** The photon is generated, it extracts kinetic energy from the gas.

**True absorption:** The photon is destroyed (thermalized), energy is transferred into kinetic energy of the gas.

**Scattering:** The photon interacts with a scatterer - the direction is changed, the energy slightly changed, there is no energy exchange with the gas.

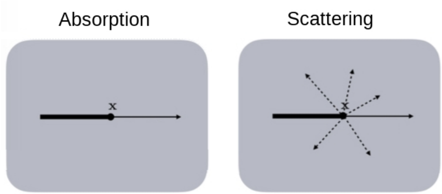


# Interaction Radiation - Matter



Among them, there are two physical processes that contribute to the opacity $\kappa_\lambda$ (the subscript means the absorption is depending on the photon wavelength):

true absorption and scattering.



# True Absorption and True Emission



The following are examples for **true absorption**:

**photoionization** (bound-free): Excess energy is transferred into kinetic energy of the released electron $\Rightarrow$ effect on local temperature
The electron will interact with photons by means of electronic transitions (and spectral lines) only if is bound to the atom. If the medium is able to deliver enough energy to separate it from the nucleus, "photo- ionization" will occur.

**photoexcitation** (bound-bound): Followed by electron collisional de-excitation; excitation energy is transferred to the electron $\Rightarrow$ effect on local temperature

**photoionization** (bound-bound): Followed by collisional ionization.

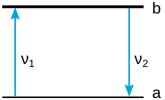
The **reverse** processes are examples for true emission.

# Scattering

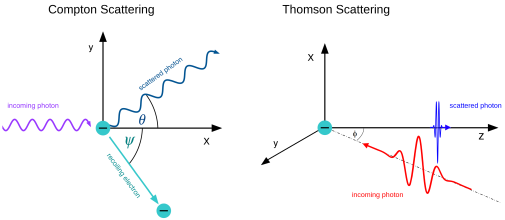
The following are examples for **scattering processes**:

A **2-level atom** absorbs photon with frequency $\nu_1$, re-emits photon with frequency $\nu_2$. The frequencies differ slightly because:

- levels $a$ and $b$ have non-vanishing energy width
- Doppler effect because atom moves



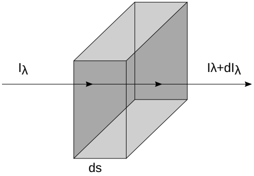

The scattering of photons **by free electrons**:
Compton or Thomson scattering (inelastic vs. elastic), collision of a photon with free electron.



# The Absorption Coefficient



**Absorption:** any process that removes photons from a beam of light.




The change in the intensity, $\mathrm{d}I_\lambda$, of a ray of wavelength $\lambda$ as it travels through a gas (i.e.: stellar atmosphere) can be expressed as

$$\mathrm{d}I_\lambda = -\kappa_\lambda \rho I_\lambda \, \mathrm{d}s$$

where $\kappa_\lambda$ is the so-called **absorption coefficient (opacity)** $\left[\mathrm{cm}^2\, g^{-1}\right]$, $\rho$ is the density (in mass per unit volume), and $\mathrm{d}s$ is a length. The distance $s$ is measured along the path traveled by the ray and increases in the direction that the ray travels.

It can also be interpreted as $\alpha_\lambda = \kappa_\lambda \rho$ where $\alpha_\lambda$ is the absorption coefficient $\left[\mathrm{cm}^{-1}\right]$.

# Optical Depth





The **mean free path** is the average distance a particle (such as an atom, a molecule, or a photon) travels before substantially changing its direction or energy, typically as a result of one or more collisions with other particles.

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We can calculate the mean free path of the photons:

$$\ell = \frac{1}{\rho \kappa_\lambda} = \frac{1}{n \sigma_\lambda}$$

Define an optical depth, $\tau_\lambda$, back along a light ray by

$$\mathrm{d}\tau_\lambda = -\rho \kappa_\lambda \, \mathrm{d}s$$

where $s$ is the distance measured along a photon's path in its direction of motion. Note that when observing the light from a star, we are looking back along the path traveled by the photon.

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The difference in optical depth between a light ray's initial position ($s = 0$) and its final position after traveling a distance $s$ is

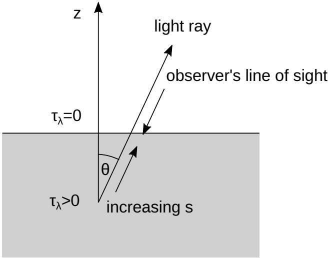
$$\Delta\tau_\lambda = \tau_{\lambda,f} - \tau_{\lambda,0} = -\int_0^s \rho\kappa_\lambda\,\mathrm{d}s$$

# Optical Depth

Let the outermost layers of a star to be at $\tau_\lambda = 0$ for all wavelengths. We then have

$$0 - \tau_{\lambda,0} = -\int_0^s \rho\kappa_\lambda \,\mathrm{d}s$$

$$\tau_\lambda = \int_0^s \rho\kappa_\lambda \,\mathrm{d}s$$




with

$$\tau_\lambda \begin{cases} \ll 1, & \text{(optical thin)} \\ \gg 1, & \text{(optical thick).} \end{cases}$$

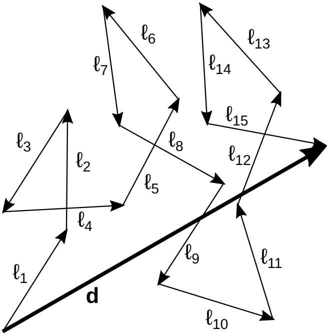
Note that $\tau_\lambda$ is the optical depth of the ray's initial position, a distance $s > 0$ from the top of the stellar atmosphere. Furthermore,

$$I_\lambda = I_{\lambda,0} = e^{-\tau_\lambda}$$

# Opacity and Random Walk

The photon diffusion process follows a path called **random walk**, which can be described by a net vector displacement **d** as the result of making a large number $N$ of randomly directed steps, each of length $\ell$ (the mean free path):

$$\mathbf{d} = \ell_1 + \ell_2 + \ldots + \ell_N$$



# Opacity and Random Walk



We find that

$$
\begin{aligned}
\mathbf{d} \odot \mathbf{d} = & \ell_1 \odot \ell_1 + \ell_2 \odot \ell_2 + \ldots + \ell_1 \odot \ell_N + \\
& \ell_2 \odot \ell_1 + \ell_2 \odot \ell_2 + \ldots + \ell_2 \odot \ell_N + \\
& \ldots + \\
& \ell_N \odot \ell_1 + \ell_N \odot \ell_2 + \ldots + \ell_N \odot \ell_N \\
= & \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_i \odot \ell_j
\end{aligned}
$$

or

$$
d^2 = N\ell^2 + \ell^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \cos \theta_{ij}
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& \ldots + \\
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Since

$$d^2 = N\ell^2 + \ell^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \cos\theta_{ij} \simeq N\ell^2$$

when $N \ll 1$, the displacement $d$ for a random walk is related to the length of the mean free path $l$ by $d = \ell\sqrt{N}$.

# Opacity and Random Walk



Because the optical depth $\tau$ is roughly the number of photon mean free paths to the stellar surface, the distance to the surface can also be written as

$$d = \tau_\lambda \ell = \ell\sqrt{N}$$

with $N = \tau_\lambda^2$.

When $\tau_\lambda \sim 1$, a photon may escape from that level of the star.

More careful analysis shows that the average level from which photons of wavelength $\lambda$ can escape is at a characteristic optical depth of $\tau \simeq 2/3$. Looking into a star at any angle, we look back to an optical depth of about $\tau_\lambda \simeq 2/3$, as measured back along the line of sight.

# Opacity and Optical Depth



An optical depth of $\tau = 0$ corresponds to no reduction in intensity, i.e. the top of a star's photosphere.

An optical depth of $\tau = 1$ corresponds to a reduction in intensity by a factor of $e = 2.7$.

If the optical depth is large ($\tau \gg 1$), negligible intensity reaches the observer.

In stellar atmospheres, typical photons originate from $\tau = 2/3$.

# Radiation: Terms



For correctly describing how light interacts with the material in a stellar atmosphere, we need a carefully defined **terminology**.

What we will consider here is the **net flow of energy** in a given direction, instead of the specific path taken by individual photons.

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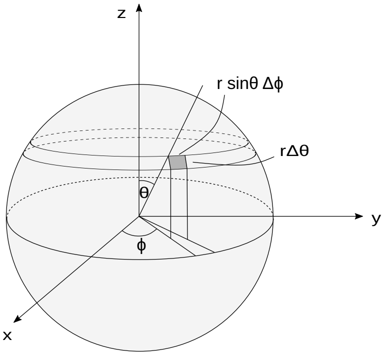
We start with defining **Specific and Mean Intensity**.

# Solid Angle in Spherical Coordinates





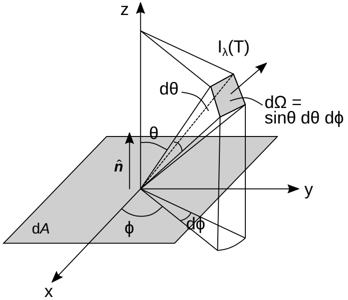

Put the center at a given point of a stellar atmosphere, with the polar axis along the star's radius.
Area of a patch on a unit radius sphere limited by $(\theta, \theta + \Delta\theta)$ and $(\phi, \phi + \Delta\phi)$ is $\Delta\Omega = \sin\theta \Delta\theta \Delta\phi$.

# Specific Intensity

Define $\mathrm{d}E_\lambda$ as the amount of energy carried by a ray of light with a wavelength between $\lambda$ and $\lambda + \mathrm{d}\lambda$ (frequency between $\nu$ and $\nu + \mathrm{d}\nu$) passing through a surface of area $\mathrm{d}A$ at an angle $\theta$ into a cone of solid angle $\mathrm{d}\Omega$ in a time interval $\mathrm{d}t$.

The **specific intensity** of the ray is defined as

$$I_\lambda \equiv \frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda\,\mathrm{d}t\,\mathrm{d}A\,\hat{\mathbf{n}}\cdot\mathrm{d}\Omega}, \quad I_\nu \equiv \frac{\mathrm{d}E_\nu}{\mathrm{d}\nu\,\mathrm{d}t\,\mathrm{d}A\,\hat{\mathbf{n}}\cdot\mathrm{d}\Omega}$$



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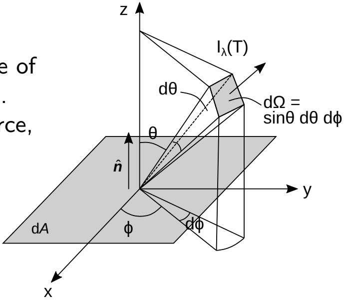
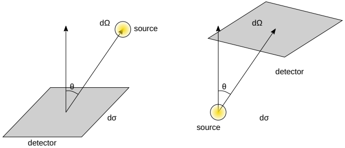
The **specific intensity** $I_\lambda$ is then a measure of brightness with units of $\mathrm{erg}/(\mathrm{s\,cm^2\,rad^2\,Å})$.
$I_\lambda$ is independent of distance from the source, and can only be measured directly if we resolve the radiating surface (e.g. Sun, nebulae, planets).



# Specific Intensity

**Theorem:** Specific intensity is conserved (is constant) along any ray in empty space.

This follows directly from geometry. Let $\mathrm{d}\,\sigma_1$ and $\mathrm{d}\,\sigma_2$ be two infinitesimal surfaces along a ray of length $r$:




Let $\mathrm{d}\Omega_1 \ll 1\,\mathrm{rad}$ be the solid angle subtended by $\mathrm{d}\sigma_2$ as seen from the center of the surface $\mathrm{d}\sigma_1$ and $\mathrm{d}\Omega_2 \ll 1\,\mathrm{rad}$ be the solid angle subtended by $\mathrm{d}\sigma_1$ as seen from the center of the surface $\mathrm{d}\sigma_2$. Then

$$\mathrm{d}\Omega_1 = \frac{\cos\theta_2\,\mathrm{d}\sigma_2}{r^2}, \quad \mathrm{d}\Omega_2 = \frac{\cos\theta_1\,\mathrm{d}\sigma_1}{r^2}$$

# Specific Intensity



The power $\mathrm{d}P_1$ in the frequency range $\nu$ to $\nu + \mathrm{d}\nu$ flowing through the area $\mathrm{d}\sigma_1$ in solid angle $\mathrm{d}\Omega_1$ is

$$\begin{aligned} \mathrm{d}P_1 &= \frac{dE_1}{dt} \\ &= (I_\nu)_1 \cos\theta_1 \mathrm{d}\Omega_1 \mathrm{d}\sigma_1 \mathrm{d}\nu \\ &= (I_\nu)_1 \cos\theta_1 \left( \frac{\cos\theta_2 \, \mathrm{d}\sigma_2}{r^2} \right) \mathrm{d}\sigma_1 \, \mathrm{d}\nu \\ &= (I_\nu)_1 \left( \frac{\cos\theta_1 \cos\theta_2 \, \mathrm{d}\sigma_1 \, \mathrm{d}\sigma_2}{r^2} \right) \, \mathrm{d}\nu \end{aligned}$$

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&= (I_\nu)_1 \cos\theta_1 \left(\frac{\cos\theta_2\,\mathrm{d}\sigma_2}{r^2}\right) \mathrm{d}\sigma_1\,\mathrm{d}\nu \\
&= (I_\nu)_1 \left(\frac{\cos\theta_1 \cos\theta_2\,\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2}{r^2}\right) \mathrm{d}\nu
\end{aligned}$$

Likewise

$$\begin{aligned}
\mathrm{d}P_2 = \frac{dE_2}{\mathrm{d}t} &= (I_\nu)_2 \cos \left(\frac{\cos\theta_1\,\mathrm{d}\sigma_1}{r^2}\right) \mathrm{d}\sigma_2\,\mathrm{d}\nu \\
&= (I_\nu)_2 \left(\frac{\cos\theta_1 \cos\theta_2\,\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2}{r^2}\right) \mathrm{d}\nu
\end{aligned}$$

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$$\begin{aligned} \mathrm{d}P_1 &= \frac{dE_1}{dt} \\ &= (I_\nu)_1 \cos\theta_1 \mathrm{d}\Omega_1 \mathrm{d}\sigma_1 \mathrm{d}\nu \\ &= (I_\nu)_1 \cos\theta_1 \left(\frac{\cos\theta_2\, \mathrm{d}\sigma_2}{r^2}\right) \mathrm{d}\sigma_1\, \mathrm{d}\nu \\ &= (I_\nu)_1 \left(\frac{\cos\theta_1 \cos\theta_2\, \mathrm{d}\sigma_1\, \mathrm{d}\sigma_2}{r^2}\right) \mathrm{d}\nu \end{aligned}$$

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Radiation energy is conserved in free space (where there is no absorption or emission), so $\mathrm{d}E_1 = \mathrm{d}E_2$ and $(I_\nu)_1 = (I_\nu)_2$. Q.E.D.

# Specific Intensity

The conservation of specific intensity has two important consequences:

1. Brightness is independent of distance. Thus the camera setting for a good exposure of the Sun would be the same, regardless of being close to the Sun (e.g. near Venus) or far away from the Sun (e.g. near Mars).

# Specific Intensity

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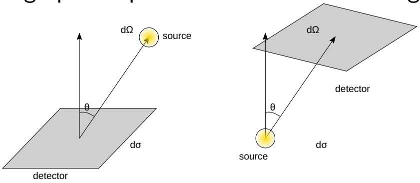
1. Brightness is independent of distance. Thus the camera setting for a good exposure of the Sun would be the same, regardless of being close to the Sun (e.g. near Venus) or far away from the Sun (e.g. near Mars).

2. Brightness is the same at the source and at the detector. Thus you can think of brightness in terms of energy flowing out of the source or as energy flowing into the detector.
No passive optical system (e.g., a telescope) can increase the specific intensity of an extended source. Astronomical objects appear much brighter on photographs than to the eye (with or without a telescope) only as a long photographic exposure accumulates more light.



# Flux



We can express $\mathrm{d}\Omega$ by means of $\theta$ and $\phi$:

$$F_\lambda = \oint I_\lambda \cos\theta \,\mathrm{d}\omega = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} I_\lambda \cos\theta \sin\theta \,\mathrm{d}\theta$$

If no flux enters the surface, and if there is no azimuthal dependence for $I_\lambda$, then

$$F_\lambda = \oint I_\lambda \cos\theta \,\mathrm{d}\omega = 2\pi \int_0^{\pi} I_\lambda \cos\theta \sin\theta \,\mathrm{d}\theta = -2\pi \int_0^{\pi} I_\lambda \cos\theta \,\mathrm{d}(\cos\theta)$$

# Intensity vs. Flux



The **mean intensity** $J_\lambda$ (sometimes also written $\langle I_\lambda \rangle$) is the directional average of the specific intensity (over $4\pi$ steradians):

$$
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J_\lambda &\equiv \frac{1}{4\pi} \oint I_\lambda \, \mathrm{d}\omega \\
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integrated over the whole unit sphere centered on the point of interest

The **flux** $F_\lambda$ is the projection of the specific intensity in the radial direction (integrated over all solid angles):

$$J_\lambda = \oint \underbrace{I_\lambda \cos\theta \, \mathrm{d}\Omega}$$

The amount of energy going through 1 $\mathrm{cm}^2$ per second per 1 Å into the solid angle $\mathrm{d}\Omega$ in the direction inclined by the angle $\theta$ to the normal of the area.

# Specific Radiation Flux

**Specific radiation flux** or **flux density** $F_\lambda \, \mathrm{d}\lambda$ is the net energy having a wavelength between $\lambda$ and $\lambda + \mathrm{d}\lambda$ that passes each second through a unit area in the direction of the $z$ axis:

$$\begin{aligned} F_\lambda \, \mathrm{d}\lambda &= \int I_\lambda \, \mathrm{d}\lambda \hat{\mathbf{n}} \cdot \mathrm{d}\Omega \\ &= \int I_\lambda \, \mathrm{d}\lambda \, \cos\theta \, \mathrm{d}\Omega \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I_\lambda \, \mathrm{d}\lambda \, \cos\theta \, \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \end{aligned}$$

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**Note:** The factor $\cos\theta$ allows the **cancelation** of oppositely directed rays.

For an isotropic radiation field, there is no net transport of energy and $F_\lambda = 0$.

# Specific Radiation Flux

**Radiation flux** is the netto energy going through the area perpendicular to the z-axis.

It can be decomposed into two half-spaces:

$$
\begin{aligned}
F &= -2\pi \int_0^{\pi} I_\lambda \cos\theta \, \mathrm{d}(\cos\theta) = 2\pi \int_{-1}^{1} I(\mu)\mu \, \mathrm{d}\mu \\
&= 2\pi \int_0^1 I(\mu)\mu \, \mathrm{d}\mu + 2\pi \int_{-1}^{0} I(\mu)\mu \, \mathrm{d}\mu \\
&= 2\pi \int_0^1 I(\mu)\mu \, \mathrm{d}\mu - 2\pi \int_0^{1} I(-\mu)\mu \, \mathrm{d}\mu \\
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Netto = Outwards - Inwards
**Special case:** isotropic radiation field: $F = 0$
(An isotropic radiator is a theoretical point source of waves which radiates the same intensity of radiation in all directions.)

# Specific Energy Density



How much energy is contained within the radiation field?

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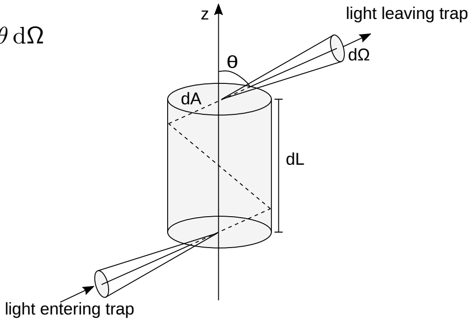
How much energy is contained within the radiation field?

Use a "trap" consisting of a small cyliner of length $\mathrm{d}L$, open at both ends, with perfect reflecting inside walls. The energy inside the trap is the same as what would be present at that location if the trap were removed.

The radiation travels through the trap in a time $\mathrm{d}t = \mathrm{d}L/(c\cos\theta)$.

$$
\begin{aligned}
E_\lambda\,\mathrm{d}\lambda &= I_\lambda\,\mathrm{d}\lambda\,\mathrm{d}t\,\mathrm{d}A\cos\theta\,\mathrm{d}\Omega \\
&= I_\lambda\,\mathrm{d}\lambda\,\mathrm{d}A\,\mathrm{d}\Omega\,\frac{\mathrm{d}L}{c}
\end{aligned}
$$



# Specific Energy Density



The **specific energy density** $u_\lambda$ is the energy per unit volume with a wavelength between $\lambda$ and $\lambda + \mathrm{d}\lambda$

$$u_\lambda \, \mathrm{d}\lambda = \frac{E_\lambda \, \mathrm{d}\lambda}{\mathrm{d}A \, \mathrm{d}L} = \frac{I_\lambda \, \mathrm{d}\lambda \, \mathrm{d}A \, \mathrm{d}\Omega \, \mathrm{d}L}{c \, \mathrm{d}A \, \mathrm{d}L} = \frac{1}{c} \int I_\lambda \, \lambda \, \mathrm{d}\Omega$$

$$= \frac{1}{c} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I_\lambda \, \mathrm{d}\lambda \, \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = \frac{4\pi}{c} \langle I_\lambda \rangle \, \mathrm{d}\lambda$$

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For an isotropic radiation field,

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$$u_\lambda\,\mathrm{d}\lambda = \frac{4\pi}{c} I_\lambda\,\mathrm{d}\lambda$$

and for blackbody radiation

$$u_\lambda\,\mathrm{d}\lambda = \frac{4\pi}{c} B_\lambda\,\mathrm{d}\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}\,\mathrm{d}\lambda$$

# Total Energy Density



The **total energy density** of a radiation field is found by integrating over all wavelengths or frequencies:

$$
\begin{aligned}
u &= \int_0^\infty u_\lambda \, \mathrm{d}\lambda \\
&= \int_0^\infty u_\nu \, \mathrm{d}\nu .
\end{aligned}
$$

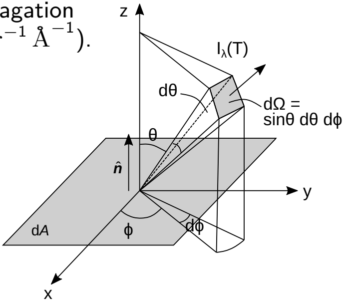
# Radiative Transfer

We define the (mass) emission coefficient as the amount of radiation emitted per second, per unit wavelength interval, per unit mass, per unit solid angle, in certain direction.

So in the case of **pure emission**, one finds the change in specific intensity

$$\mathrm{d}I_\lambda = \rho j_\lambda \, \mathrm{d}s$$

with
$I_\lambda$ the specific intensity
$s$ the distance traveled in direction of propagation
$j_\lambda$ the emission coefficient $(\mathrm{erg\,cm^{-3}\,s^{-1}\,sr^{-1}\,\mathring{A}^{-1}})$.



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**on units:**

An $\mathrm{erg}$ is equal to one gram centimeter-squared per second-squared $(\mathrm{g\,cm^2\,s^{-2}})$. This is equal to $10^{-7}$ joules or 100 nanojoules $(\mathrm{nJ})$ in SI units.

Steradian $(\mathrm{ster})$ is the unit of solid angle in the International System of Units (SI). The entire sphere has a solid angle of $4\pi$ $\mathrm{sr}$.

# Radiative Transfer

In the case of **pure absorption**, we find for the number of photons in a beam $N_p$ to obey the following differential equation:

$$\frac{\mathrm{d}N_p}{\mathrm{d}s} = -N_p n\sigma = -N_p \kappa\rho$$

where $s$ is the distance traveled in the direction of propagation.

# Radiative Transfer

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where $s$ is the distance traveled in the direction of propagation.

The same applies to specific intensity:

$$\mathrm{d}I_\lambda = -I_\lambda \kappa_\lambda \rho \, \mathrm{d}s$$

with the **wavelength-dependent** monochromatic opacity $\kappa_\lambda$.

This is the equation of radiative transfer in a purely absorbing medium.

# Radiative Transfer

The primary mode of energy transport through the surface layers of a star is by radiation.
The **radiative transfer equation** describes how the physical properties of the material are coupled to the spectrum we ultimately measure.

As a beam of light moves through the gas in the stellar atmosphere, photons are not only added by emission from the surrounding material (the case we saw so far), but are also removed from the beam by absorption or scattering.

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Incorporating both **absorption and emission**, one obtains

$$\mathrm{d}I_\lambda = -\rho\kappa_\lambda I_\lambda \,\mathrm{d}s + \rho j_\lambda \,\mathrm{d}s$$

The **opacity** $\kappa_\lambda$ has units of $\mathrm{cm}^2\,\mathrm{g}^{-1}$. The change in the intensity of the beam is determined by the rates at which the competing process of emission and absorption occur.

# Radiative Transfer Equation

The **competition** between the rates at which photons are removed from the beam by absorption, and added to the beam by emission describes how rapidly the intensity of the light beam changes. To introduce the **ratio of emission to absorption**, we rewrite the previous equation

$$-\frac{1}{\rho\kappa_\lambda}\frac{\mathrm{d}I_\lambda}{\mathrm{d}s} = I_\lambda - \frac{j_\lambda}{\kappa_\lambda}.$$

Now, define the ratio of the emission coefficient to the absorption coefficient as the **source function**, $S_\lambda \equiv j_\lambda/\kappa_\lambda$, which has the same units as the intensity, $\mathrm{ergs\,s^{-1}\,cm^{-3}\,ster^{-1}}$.

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We obtain the **radiative transfer equation (RTE)**:

$$-\frac{1}{\rho\kappa_\lambda}\frac{\mathrm{d}I_\lambda}{\mathrm{d}s} = I_\lambda - S_\lambda$$

The intensity of the light tends to become equal to the local value of the source function.

# Physical Interpretation of Source Function $S_\lambda$

In the Thermodynamic Equilibrium (TE), nothing changes with time.

A beam of light passing through such a gas volume will not change either:

$$\mathrm{d}I_\lambda/\mathrm{d}\tau_\lambda = 0 \Rightarrow S_\lambda = I_\lambda = B_\lambda$$

In TE, the source function equals the Planck function.

In the theory of stellar atmospheres, we make the assumption of **local thermodynamic equilibrium (LTE)** where $S_\lambda = B_\lambda$. This does not necessarily mean that $S_\lambda = I_\lambda$.

# Optical Depth



Often, when describing radiative transfer, we use the **optical depth**.

The optical depth is a dimensionless quantity related to the absorption coefficient or opacity.

We find it by answering the **question**: What is the ratio of the final intensity to the original intensity?

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There are several factors which must affect the intensity in obvious ways:

- the greater the distance the beam travels, the more light will be removed from it
- the greater the density of material $\rho$, the more light will be scattered or absorbed
- the composition of the material; some atoms are much more efficient at absorbing light than others

# Optical Depth



We can put all this together in a mathematical equation:

$$\mathrm{d}I = -\rho\kappa I \,\mathrm{d}s$$

with
$I$ the incoming intensity of the light
$\mathrm{d}I$ the amount of light added to the beam (hence the negative sign)
$\mathrm{d}s$ the distance the light travels
$\rho$ the density of the material
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$\mathrm{d}s$ the distance the light travels
$\rho$ the density of the material
$\kappa$ the opacity or absorption coefficient.

$\kappa$ serves two purposes: first, it is a constant of proportionality which serves to make the units match on both sides of the equation; second, it expresses the efficiency with which this particular sort of material (hydrogen, or helium, or electrons, or ...) absorbs and scatters light.
The wavelength-depending **spectral optical depth** $\tau_\lambda$ is given similarly by

$$\mathrm{d}\tau_\lambda = -\rho\kappa_\lambda \, \mathrm{d}s$$

$$\mathrm{d}I = -\rho\kappa_\lambda I \, \mathrm{d}s$$

# Radiative Transfer Equation



Let's **solve** the equation of radiative transfer.

Consider radiation passing through material in which opacity and emissivity are known functions of $s$. We attempt to solve the RTE through use of an integrating factor:

Rewrite the transfer equation with the optical depth, $\tau_\lambda$,

$$\frac{\mathrm{d}\, I_\lambda}{\mathrm{d}\, \tau_\lambda} = I_\lambda - S_\lambda$$

$$\frac{\mathrm{d}}{\mathrm{d}\, \tau_\lambda} I_\lambda e^{-\tau_\lambda} = S_\lambda e^{-\tau_\lambda}$$

# Radiative Transfer Equation



Integrate from front of slab ($s = 0$) to $s$:

$$I_\lambda e^{-\tau_\lambda} = I_{\lambda,0} e^{-\tau_{\lambda,0}} + \int_{\tau_\lambda}^{\tau_{\lambda,0}} S_\lambda e^{-t}\,\mathrm{d}\tau$$

where $I_{\lambda,0}$ and $\tau_{\lambda,0}$ are intensity and optical depth at front of slab (optical depth decreases as we travel through slab), and $t$ is a dummy variable.
In terms of $I_\lambda$ and $S_\lambda$, the solution can be written as

$$I_\lambda = I_{\lambda,0} e^{\tau_\lambda - \tau_{\lambda,0}} + \int_{\tau_\lambda}^{\tau_{\lambda,0}} S_\lambda e^{\tau_\lambda - t}\,\mathrm{d}t$$

# Radiative Transfer Equation

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**The problem:** Despite it looks simple - in many situations $S_\lambda$ (and $\kappa_\lambda$) depend on $I_\lambda$!

# Radiative Transfer Equation



There are some **simple** RTE cases:

**Emission, no absorption:** $\kappa_\lambda \to 0$

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*interpretation:* The intensity at $s$ is the incoming intensity plus the sum of contributions from emitting material in the interval $(0, s)$.

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**Constant source function:**

$$I_\lambda = I_{\lambda,0} e^{-\Delta\tau} + S_\lambda (1 - e^{-\Delta\tau})$$

*interpretation:* The intensity at $s$ is the incoming intensity attenuated by factor $e^{-\Delta\tau}$, plus contributions from emitting material, also attenuated.

# Radiative Transfer Equation





There are some **simple** RTE cases:

**Homogeneous radiation field:**

$$I_\lambda = I_{\lambda,0} = S_\lambda$$

In the special case of a **black body**:
Since $I_\lambda = B_\lambda$ (Planck function), $S_\lambda = B_\lambda$, and $j_\lambda = \kappa B_\lambda$.

*interpretation:* most emission at wavelengths where opacity is high; good absorber is also good emitter

# Summary



The Stefan-Boltzmann law ($F = \sigma_{\mathrm{SB}} T^4$) defines a star's effective temperature, i.e. the temperature which a black body would need in order to radiate the same amount of energy as the star.

The specific intensity ($I_\lambda$ or $I_\nu$) is distance independent.

The flux ($F_\lambda$ or $F_\nu$) is the angular integral over the projection of the specific intensity in the radial direction and obeys he inverse square law.

The Planck function is monotropic in temperature.