

Astronomía Avanzada I (Semester 1 2024)

Stellar Atmospheres (4)

Local Thermodynamic Equilibrium

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Recap

Three equations of radiative equilibrium can be derived:

- a constant flux with depth
- energy absorbed equals energy emitted
- the K -integral is linear in τ .

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

Recap

Stellar
Atmospheres
(4)

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Summary

Three equations of radiative equilibrium can be derived:

- a constant flux with depth
- energy absorbed equals energy emitted
- the K -integral is linear in τ .

From these, the grey temperature distribution $T(\tau)$ may be derived, assuming

- the Eddington approximation and
- LTE, in reasonable agreement with the exact science case.

Many Temperatures

Stellar
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(4)

We can describe the **temperature of a star's photosphere** in many ways:

Effective Temperature

Measure the star's radius and luminosity. Find the temperature of a blackbody producing the same amount of energy: $L = 4\pi R^2 \sigma T_{\text{eff}}^4$

Color Temperature

Measure the spectrum of a star, or, more commonly, its color via the difference of magnitudes in two filters. Find the temperature of a blackbody which would produce a spectrum of the same shape, or the same difference in magnitudes when observed through those filters.

Excitation temperature

Measure the ratio of atoms in two energy levels of some neutral atom; say, the $n = 2$ to $n = 1$ ratio of hydrogen. Using the Boltzmann equation, $\frac{N_b}{N_a} = \frac{g_b}{g_a} \exp(-(E_b - E_a)/kT)$ determine the temperature which would reproduce the observed ratio.

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

Many Temperatures

Stellar
Atmospheres
(4)

We can describe the **temperature of a star's photosphere** in many ways:

Ionization temperature

Measure the ratio of atoms in two ionization stages; say, ionized hydrogen to neutral hydrogen. We determine the temperature which would reproduce the observed ratio from the Saha equation,

$$\frac{N_{II}}{N_I} = \frac{2Z_{II}}{n_e Z_I} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} \exp(-\chi/kT).$$

Kinetic temperature

Measure the speed of particles in the gas. If they follow a Maxwell-Boltzmann distribution, $f(v, v + \Delta v) = \frac{m}{2\pi kT} \exp(-mv^2/2kT) 4\pi v^2 \Delta v$, determine the temperature which would reproduce the observed distribution. This is often simplified to "measure the average particle speed and compare to the average speed of a Maxwell-Boltzmann distribution."

Recap

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dynamic
Equilibrium
(LTE)

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Equation

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Stellar
Atmospheres
(4)

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For gas confined within an "ideal box", the excitation temperature, the ionization temperature, the kinetic temperature, and the color temperature are the same.

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

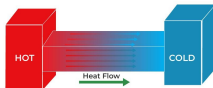
Summary

Thermal Equilibrium vs. Thermodynamic Equilibrium

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Atmospheres
(4)

Equilibrium is a state where there are no unbalanced potentials.

Thermal equilibrium is when no temperature gradient exists between the systems and also between the systems and surroundings (no heat transfer).



Chemical equilibrium is the state in which both reactants and products are present in concentrations which have no further tendency to change with time (no reactions) and no mass transfer by diffusion.

Mechanical equilibrium is where the pressure forces are balanced in a system and between the system and the surroundings.

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

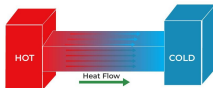
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The system is in **thermodynamic equilibrium** if the conditions for all three equilibrium is satisfied.

Recap

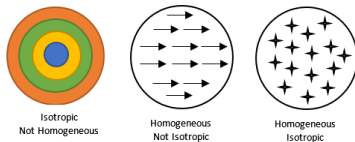
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dynamic
Equilibrium
(LTE)

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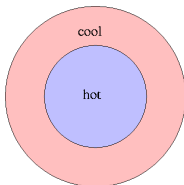
Thermodynamic Equilibrium

A thermodynamic equilibrium is isotrop (the same in all directions) and homogeneous (the same in all places):



For an entire star as a whole, neither condition is true: the center of a star is much hotter than its outer regions, for example.

A star can not be in perfect thermodynamic equilibrium because a net outward flow of energy occurs.



Thermodynamic Equilibrium

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Atmospheres
(4)

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Local Thermo-
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Equilibrium
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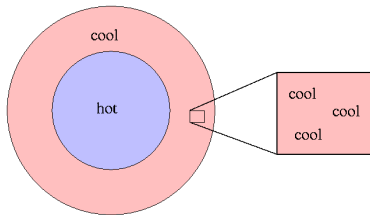
The Saha
Equation

Summary

The **Local Thermodynamic Equilibrium (LTE)**:

Particles and photons are effectively confined to a limited volume of nearly constant temperature.

The idealized case of a single temperature can still be employed if the distance over which the temperature changes is much larger than the distance traveled by the particles and photons between their collisions.



Thermodynamic Equilibrium

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

In the **Thermodynamic Equilibrium**,

1. All particles have Maxwellian distribution in velocities (with the same temperature T).
2. The atom populations follow the Boltzmann law (same T).
3. The ionization is described by the Saha equation (same T).
4. The radiation intensity is given by the Planck function (same T).
5. The principle of detailed equilibrium is valid (the number of direct processes = number of inverse processes).

In a **Local Thermodynamic Equilibrium (LTE)**, 1 to 3 are applied locally. The radiation spectrum can be very far from a Planck function.

Local Thermodynamic Equilibrium (LTE)

Stellar
Atmospheres
(4)

The interaction of radiation and matter is the most important physical process in stellar atmospheres.

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

To find I_λ we need to know α_λ and ϵ_λ (or k_λ and j_λ), the absorption and emission coefficients:

$\alpha_\lambda = \kappa_\lambda \rho$ where α_λ is the absorption coefficient $[\text{cm}^{-1}]$.

To find α_λ and ϵ_λ , knowing the density ρ , temperature T , and chemical composition X are not enough. We need to know **distributions of atoms over levels and ionization states**, which depend on radiation I_λ .

Local Thermodynamic Equilibrium (LTE)

Stellar
Atmospheres
(4)

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In LTE, ρ , T and X fully determine α_λ and ϵ_λ .

Local Thermodynamic Equilibrium (LTE)

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

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Equation

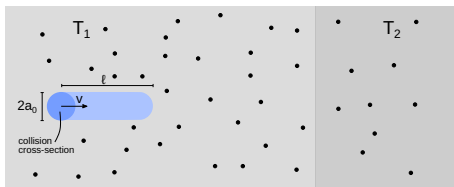
Summary

In the study of stellar atmospheres, the assumption of LTE is described by:

1. Electron and ion velocity distributions are Maxwellian.
2. Excitation equilibrium is given by Boltzmann equation.
3. Ionization equilibrium is given by Saha equation (which we will see later on).
4. The source function is given by the Planck function.

Is LTE a valid assumption?

For LTE, the photon and particle mean free paths need to be much smaller than the length scale over which the temperature changes significantly.



Radiation cannot play a role in defining atom populations and ionization states. Collisions should dominate.

Generally, when collisional processes dominate over radiative processes in the excitation and ionization of atoms, the state of the gas is close to LTE.

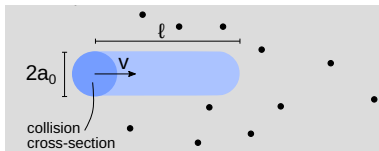
LTE and the Photosphere

Stellar
Atmospheres
(4)

The mean free path between collisions is

$$\ell = \frac{vt}{n\sigma vt} = \frac{1}{n\sigma}$$

where $\sigma \equiv \pi(2a_0)^2$ is the collision cross-section.



The density of the Solar photosphere is $\rho = 2.5 \times 10^{-7} \text{ g/cm}^3$, so the density of H atoms is $n(H) = \rho/m_H = 1.5 \times 10^{17} \text{ cm}^{-3}$.

Two atoms will collide if their centers pass within a radius of 2 Bohr radii ($2a_0$) of each other. The collision cross-section of the atom is $\sigma = \pi(2a_0)^2 = 3.52 \times 10^{-20} \text{ m}^2$.

The mean free path between collisions is $1/\sigma_{SB} \times 1/n(H) = 2.27 \times 10^{-4} \text{ m}$, i.e. atoms are confined within a limited volume of space in the photosphere at effectively fixed temperature (relative to the temperature scale height).

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

LTE and the Photosphere

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

According to a model solar atmosphere, the temperature in the photosphere varies from 5580 K to 5790 K over a distance of 25 km. The **temperature scale height**, H_T , is given by

$$H_T \equiv \frac{T}{|dT/dr|} = \frac{5685 \text{ K}}{(5790 - 5580) \text{ K}/25 \text{ km}} = 677 \text{ km}$$

Compared to the free path, $\ell = 2.27 \times 10^{-4} \text{ m} \ll H_T = 667 \text{ km}$. The gas number density of the photosphere at that level is about $n = 1.25 \times 10^{17} \text{ cm}^{-3}$.

As a result, the **atoms** in the gas see an essential constant kinetic temperature between collisions and are effectively confined within a limited volume of space in the photosphere. However, this is not the case for the **photons** as well.

LTE and the Photosphere

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

Consider a beam of light traveling through a gas

$$\int_{I_{\lambda,i}}^{I_{\lambda,f}} \frac{dI_{\lambda}}{I_{\lambda}} = - \int_0^s \rho \kappa_{\lambda} ds \Rightarrow I_{\lambda} = I_{\lambda,0} \exp\left(- \int_0^s \rho \kappa_{\lambda} ds\right)$$

For a uniform gas of constant opacity and density

$$I_{\lambda} = I_{\lambda,0} e^{-\rho \kappa_{\lambda} s}$$

The intensity declines exponentially, falling by a factor of e^{-1} over a characteristic distance of $\ell = 1/\rho \kappa_{\lambda}$.

In the case of the solar photosphere, the density is $\rho = 2.1 \times 10^{-7} \text{ g cm}^{-3}$ and the opacity is $\kappa_{500\text{nm}} = 0.3 \text{ cm}^2 \text{ g}^{-1}$.

Therefore, the characteristic distance traveled by a beam is

$$\ell = \frac{1}{\rho \kappa_{500\text{nm}}} = 160 \text{ km} \sim H_T = 677 \text{ km}$$

LTE and the Photosphere

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$$\ell = \frac{1}{\rho \kappa_{500\text{nm}}} = 160 \text{ km} \sim H_T = 677 \text{ km}$$

Consequently: Radiation dominates over collision in the photosphere (otherwise no light would escape!). LTE is not strictly valid in the photosphere, despite it is a good assumption in stellar interiors. If LTE is no longer valid, all processes must be calculated in detail with non-LTE.

Kinetic Gas Theory

Stellar
Atmospheres
(4)

The model proposes that the atmosphere can be described as an ideal gas. In most of the photospheres the pressure is low ($< 10^4 \text{ N/m}^2$), allowing this assumption.

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dynamic
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(LTE)

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Equation

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Stellar
Atmospheres
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Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

To describe the gas particles statistically, we describe it as a system in which energy and momentum is distributed among the particles by collisions. The collisions are elastic and lead to the system to an equilibrium, in which the statistical properties do not change.

Boltzmann Equation

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

Ludwig Boltzmann first found the distribution of the number of particles at a given energy given a thermodynamic system at a fixed temperature.

Assuming that energy takes continuous values we can say that the number of molecules per unit volume with energies in the range E to $E + dE$ is:

$$N_{[E, E+dE]} = \int_E^{E+dE} n_\nu(E) dE$$

where

$$n_\nu(E) = n_0 e^{-E/k_B T}$$

and n_0 is a constant setting the scale: when $E = 0$, then $n_\nu(E) = n_0$.

This frequency distribution

$$n_\nu(E) \propto e^{-E/k_B T}$$

is called the **Boltzmann distribution**.

Maxwell-Boltzmann Velocity Distribution

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

If the gas is in thermal equilibrium, we can describe the **velocity distribution** of its particles using the Boltzmann distribution in an ideal scenario:

1. Velocities are not relativistic ($v \ll c$).
2. Particles interact only by collisions.
3. The collisions do not change the internal structure of the particles (elastic collisions).

With the first condition, we can say that the kinetic energy of the particle is $K = mv^2/2$, while conditions 2 and 3 allow us to ensure that all energy is kinetic (there is no potential).

If the motion is isotropic, g_i will be constant.

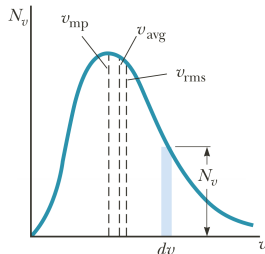
Maxwell-Boltzmann Velocity Distribution

Stellar
Atmospheres
(4)

The Boltzmann distribution for energy can be used to find a distribution of the speeds of the molecules, the **Maxwell-Boltzmann speed (or velocity) distribution** describing the number of molecules with speeds between v and $v + dv$:

$$\int_v^{v+dv} N_v dv = \int_v^{v+dv} 4\pi N \left(\frac{m_0}{2\pi k_B T} \right)^{3/2} v^2 \exp(-m_0 v^2 / 2k_B T) dv$$

(The detailed derivation can be found in textbooks on Thermodynamics.)



The number of molecules having speeds ranging from v to $v + dv$ equals the area of the rectangle, $N_v dv$.

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

Maxwell-Boltzmann Velocity Distribution

Stellar
Atmospheres
(4)

Because the particles produce **Doppler shifts**, the line of sight velocities have a distribution that is an important special case for **spectroscopy**:

$$\frac{dN(v_R)}{N_{\text{total}}} = \left(\frac{m_0}{2\pi k_B T} \right)^{3/2} \exp(-m_0 v_R^2 / 2k_B T) dv_R$$

where v_R is the radial (line of sight) velocity component.

Recap

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dynamic
Equilibrium
(LTE)

The Saha
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Stellar
Atmospheres
(4)

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where v_R is the radial (line of sight) velocity component.

Consider an atom emitting at frequency ν_0 , and moving with velocity v_x along the line of sight to the observer. The observed frequency is ν .

Doppler shift equation:

$$\frac{\nu - \nu_0}{\nu_0} = \frac{v_x}{c}$$

We combine this with the velocity distribution, and get the **Doppler width of the line** as:

$$\Delta\nu_D = \frac{\nu_0}{c} \sqrt{\frac{2kT}{m}}$$

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

Maxwell-Boltzmann Velocity Distribution

Stellar
Atmospheres
(4)

We have the Maxwell-Boltzmann velocity distribution:

$$\int_v^{v+dv} N_v dv = \int_v^{v+dv} 4\pi N \left(\frac{m_0}{2\pi k_B T} \right)^{3/2} v^2 \exp(-m_0 v^2 / 2k_B T) dv$$

The term in the exponential is the ratio between the kinetic energy of the particles ($mv^2/2$) and the characteristic thermal energy (kT).

The distribution peaks at the most probable speed of

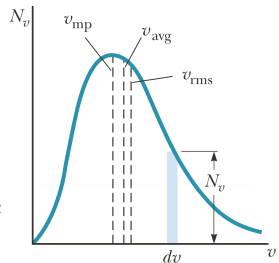
$$v_{mp} = \sqrt{\frac{3kT}{m}}$$

The average velocity, v_{avg} , is

$$v_{avg} = \left(\frac{8kT}{\pi m} \right)^{1/2}$$

The root mean square velocity, v_{rms} , is

$$v_{rms} = \left(\frac{3kT}{m} \right)^{1/2}$$



Boltzmann Equation

Stellar
Atmospheres
(4)

The temperature of the gas causes a distribution of particle energies: velocities and orbital positions of the electrons.

Recap

More energetic orbitals are less likely/more difficult to populate.

Local Thermo-
dynamic
Equilibrium
(LTE)

The Boltzmann equation describes the distribution of electrons among the atomic (or molecular) orbits resulted from collisions (thermal motions) between atoms.

The Saha
Equation

Summary

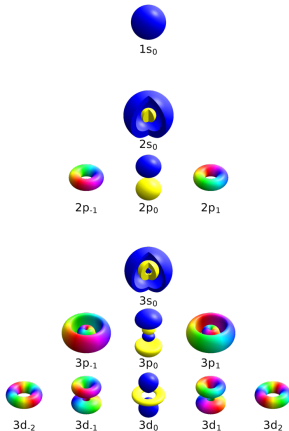
Quantum Numbers

In addition to the orbit n , we have other **quantum numbers**:

l : azimuthal quantum number with $l = 0, \dots, n - 1$

m : magnetic quantum number with $-l \leq m_l \leq l$

s : electron spin angular momentum $\pm 1/2$



Single-electron orbitals for hydrogen-like atoms with quantum numbers $n = 1, 2, 3$ (blocks), l (rows) and m (columns). The spin s is not shown, because it has no spatial dependence.

Degeneracy

More than one quantum state may have the same energy: this is called **degeneracy**.

For each n the allowed values of l are $0, 1, 2, \dots, n-1$. For each l , there are $2l+1$ allowed values for m . From this, the degeneracy of each energy level is $\sum_{l=1}^{n-1} (2l+1) = n^2$.

Adding in the electron spin which is an intrinsic degree of freedom, the degeneracy of each energy level is $g_n = 2n^2$.

Thus, for H, orbital n has a statistical weight of $g_n = 2n^2$.

Transition energy between levels u (upper) and l (lower):

$$\chi_{ul} = \chi_{\text{ion}} \left(\frac{1}{n_l^2} - \frac{1}{n_u^2} \right)$$

where $\chi_{\text{ion}} = -13.6 \text{ eV}$

Boltzmann Equation

Let the statistical weight, g_a , stand for the number of states with energy E_a . Similarly, let g_b be the number of states with energy E_b .

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

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Stellar
Atmospheres
(4)

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dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

The ratio of the probability $P(s_b)$ that the system is in state s_b to the probability $P(s_a)$ that the system is in state s_a is:

$$\frac{P(s_b)}{P(s_a)} = \frac{e^{-E_b/kT}}{e^{-E_a/kT}} = e^{-(E_b-E_a)/kT}$$

This is the **Boltzmann factor**.

Boltzmann Equation

Stellar
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This is the **Boltzmann factor**.

The **Boltzmann equation** gives the ratio of the probability $P(E_b)$ that the system will be found in any of the degenerate state with energy E_b to the probability $P(E_a)$:

$$\frac{P(E_b)}{P(E_a)} = \frac{g_b}{g_a} e^{-(E_b-E_a)/kT} = \frac{g_b P(s_b)}{g_a P(s_a)}$$

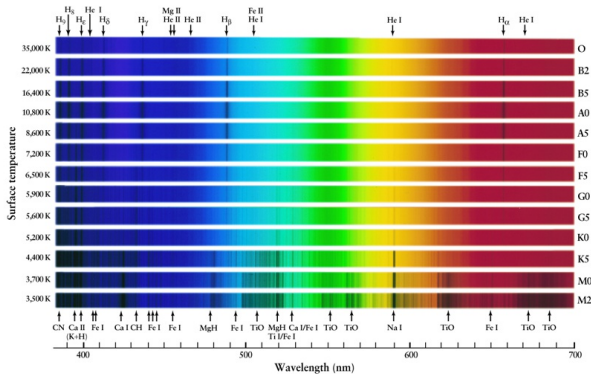
As the number of atoms in the stellar atmosphere is very high, we can understand that the ratio of the probabilities represents well the ratio of the populations: $P(E_b)/P(E_a) = N_b/N_a$.

Balmer Lines

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Atmospheres
(4)

An exceptionally high T is required for a significant number of H atoms to have electrons in their 1^{st} excited states.

The Balmer lines (involving an upward transition from $n = 2$ orbital) reach a peak strength at spectral class A ($\sim 10000K$).



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Equilibrium
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Equation

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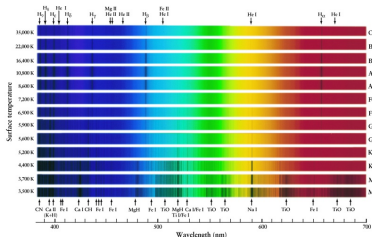
Stellar Atmospheres (4)

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Recap

Local Thermo-dynamic Equilibrium (LTE)

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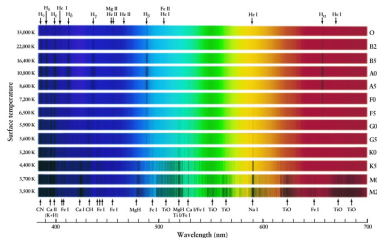


So why do the Balmer lines diminish in strength at higher temperatures?

Stellar Atmospheres (4)

Local Thermo-dynamic Equilibrium (LTE)

The Balmer lines (involving an upward transition from $n = 2$ orbital) reach a peak strength at spectral class A ($\sim 10000K$).



So why do the Balmer lines diminish in strength at higher temperatures?

We need the **Saha equation** to answer this question.

The Partition Function

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

The possibility of finding the initial and final ions in states other than the ground state is described by the **Z-partition function**:

The partition function Z is the weighted sum of the numbers of states the atom can arrange its electrons with the same energy.

If E_j is the energy of the j^{th} energy level and g_j is the degeneracy of that level, the partition function is defined as

$$Z = \sum_{j=1}^{\infty} g_j e^{-(E_j - E_1)/kT}$$

Therefore, the number of atoms N_i in the i^{th} energy level with energy E_i and degeneracy g_i is given by

$$\frac{N_i}{N_{\text{tot}}} = \frac{g_i e^{-(E_i - E_1)/kT}}{Z}$$

A Thought on Ionization

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

To get a hydrogen atom's electron into level 2, it required a collision with at least 10.2 eV of energy. However, once it is in level 2, it requires only 3.4 eV more energy to knock it completely away from the atom, so the first excited state is a very precarious place for an electron! The actual number of atoms in level 2 at any one time is a balance between collisions to get it there in the first place, and collisions to ionize the atom.

The Saha Equation

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

The Saha equation (Meghnad Saha, 1920) considers the **relationship between ionization and the electron density** in the gas. As the number density increases, this lowers the number of atoms in higher ionization states.

In the following, m_e is the electron mass, χ_{ion} is the energy needed to remove an electron from an atom in the ground state, N is the number of ions in the respective states.

Let Z_i and Z_{i+1} be the partition functions for the atom in its initial and final stages of ionization. The ratio of the number of atoms in stage $i + 1$ to the number of atoms in stage i is:

$$\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i} \left(\frac{2\pi m_e kT}{h^2} \right) e^{-\chi_i/kT}$$

The factor 2 reflects the two possible spins of the free electron. The number density of free electrons, n_e , can affect the ionization stage of the electrons.

The Saha Equation

Stellar
Atmospheres
(4)

Sometimes we prefer to use the pressure of free electrons, P_e , instead of their number density, n_e ; the two are related with $P_e = n_e kT$.

The Saha equation takes an alternative form

$$\frac{N_{i+1}}{N_i} = \frac{2kTZ_{i+1}}{P_e Z_i} \left(\frac{2\pi m_e kT}{h^2} \right) e^{-\chi_i/kT}$$

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

The Saha Equation

Stellar
Atmospheres
(4)

If we multiply N_1^+/N_1^0 from earlier by N^+/N_1^+ and N_1^0/N_0 , we again obtain the Saha equation:

$$\begin{aligned}\frac{N^+ N_e}{N_0} &= \frac{2u^+}{u^0} \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_{\text{ion}}/kT} \\ &= 4.83 \times 10^{15} \frac{u^+}{u^0} T^{3/2} e^{-\chi_{\text{ion}}/kT}\end{aligned}$$

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

In logarithmic form the Saha equation can be written as

$$\log \frac{N^+}{N^0} = \log \frac{u^+}{u^0} + \log 2 + \frac{5}{2} \log T - \chi_{\text{ion}} \Theta - \log P_e - 0.48$$

where χ_{ion} is measured in eV, $\Theta = 5040/T$, and the electron pressure P_e is related to the electron density via the ideal gas law ($P_e = N_e kT$).

The Saha Equation

Stellar
Atmospheres
(4)

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Recap

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The Saha
Equation

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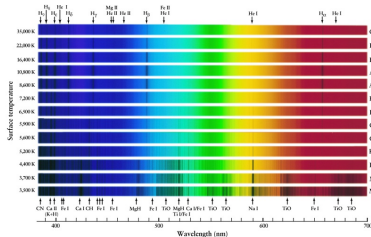
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What we see:

High temperatures favorize ionization, high pressure favors recombination.

Stellar Atmospheres (4)

The Balmer lines (involving an upward transition from $n = 2$ orbital) reach a peak strength at spectral class A ($\sim 10000K$).



So why do the Balmer lines diminish in strength at higher temperatures?

Strength of Balmer Lines

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

We look at the Saha equation for H between 5000 – 25,000 K.

The partition function for H II is $Z_{II} = 1$ (no degeneracy for a bare proton).

The energy of the first excited state of hydrogen is $E_2 - E_1 = 10.2 \text{ eV}$

above the ground state energy. Because $10.2 \text{ eV} \gg kT$ for this temperature regime, the Boltzmann factor $\exp(-(E_2 - E_1/kT)) \ll 1$.

Nearly all of the H I atoms are therefore in the ground state. So the partition function simplifies to $Z_I \simeq g_1 = 2 \times 1^2 = 2$.

We can find the variation of the strength of the $H\alpha$ line with temperature by combining the Boltzmann and Saha equations to the **Saha-Boltzmann Equation**:

We insert these values into the Saha equation with $\chi_I = 13.6 \text{ eV}$. This gives the ratio of ionized to neutral H, N_{II}/N_I , for a given temperature T . This ratio is then used to find the fraction of ionized hydrogen,

$$\frac{N_{II}}{N_{\text{total}}} = \frac{N_{II}}{N_I + N_{II}} = \frac{N_{II}/N_I}{1 + N_{II}/N_I}.$$

Strength of Balmer Lines

Stellar
Atmospheres
(4)

The Balmer line strength depends on N_2/N_{total} , the fraction of *all* hydrogen atoms that are in the first excited state.

As almost all of the neutral H atoms are either in ground state or first excited state, we can use $N_1 + N_2 \simeq N_I$:

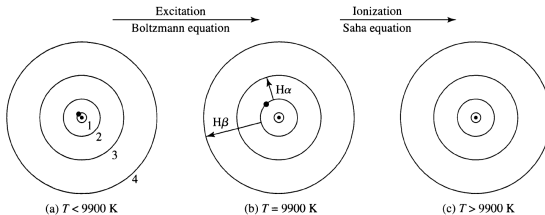
$$\frac{N_2}{N_{\text{total}}} = \left(\frac{N_2}{N_1 + N_2} \right) \left(\frac{N_1}{N_{\text{total}}} \right) = \left(\frac{N_2/N_1}{1 + N_2/N_1} \right) \left(\frac{1}{1 + N_{II}/N_I} \right)$$

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

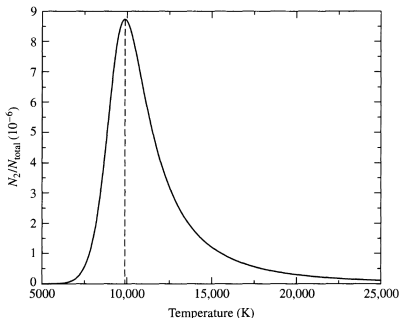
Summary



Strength of Balmer Lines

Stellar
Atmospheres
(4)

A plot vs. T confirms our observation of the strength of the H lines: The curve peaks right near 10,000 K as required to match the observations.



Ratio of H atoms in the $n = 2$ level to the total number of H atoms as a function of temperature. Figure from Carroll and Ostlie, *Modern Astrophysics*, Addison-Wesley Publishing Co., 1996.

At $T = 5000$ K, essentially no H is ionized. Almost all H will be ionized by $\sim 10,000$ K. This limits the strength of the H Balmer lines above 9500 K.

The narrow region inside a star where H is partially ionized is called a hydrogen **partial ionization zone** and has a characteristic temperature of approx. 10,000 K for a wide range of stellar parameters.

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

Comments on the Saha-Boltzmann Equation

Stellar
Atmospheres
(4)

Stellar atmospheres are not only composed of H. Typically there are 10 H atoms for every He atom. Ionized He brings more electrons into the atmosphere, but requires a higher temperature to be ionized.

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The model based on the Saha and Boltzmann laws only describe atmospheres in thermodynamic equilibrium (i.e. they can be described by the Maxwell- Boltzmann velocity distribution).

The Saha
Equation

The density of the gas must be low ($< 1 \text{ kg m}^{-3}$) so that the influence of nearby ions on the atomic orbitals is low (and the ionization energy of ionization decreases).

Summary

Comments on Temperature

Stellar
Atmospheres
(4)

According to the associated physical process, we can use different definitions of temperature in the medium to be studied:

- Effective temperature (Stefan-Boltzmann law, luminosity).
- Excitation temperature (Boltzmann equation).
- Ionization temperature (Saha equation).
- Kinetic temperature (Maxwell-Boltzmann distribution).
- Color temperature (Planck function setting).

In the ideal conditions of a closed box, all temperatures except the effective temperature describe the same parameter. In this There is, of course, no net transfer of energy between the "box", the matter and the radiation. Absorptions and emissions of photons occur at similar rates and the system is in thermodynamic equilibrium (or local thermodynamic equilibrium, LTE).

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

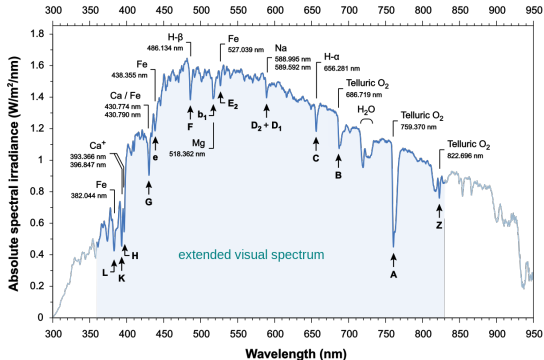
The Saha
Equation

Summary

Strength of Metal Lines

The doublet of CaII (produced by the ground state of singly ionized Calcium, Ca^+) represents the strongest lines in the visible solar spectrum, the H line at 3968.469 Å and the K line at 3933.663 Å. They are much stronger than the Balmer lines (produced by 1^{st} excited state of neutral H).

Photosphere of the Sun: two Ca atoms for every million H atoms - why then CaII H and K lines are much stronger than Balmer lines in the Sun?



Strength of Metal Lines

Stellar
Atmospheres
(4)

Let's consider **hydrogen** first.

With the Saha equation and the Boltzmann equation, one gets

$$\left[\frac{N_{II}}{N_I} \right]_H = \frac{2kTZ_{HII}}{P_e Z_{HI}} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi_i/kT} = 7.70 \times 10^{-5}$$

\Rightarrow there is one HII ion for every $\sim 13,000$ neutral H atoms (HI)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

Strength of Metal Lines

Stellar
Atmospheres
(4)

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\Rightarrow there is one HII ion for every $\sim 13,000$ neutral H atoms (HI)

From the Boltzmann equation we get the fraction of HI atoms in the first excited state. Using $g_n = 2n^2$ (implying $g_1 = 2$ and $g_2 = 8$), the ratio of the H atoms that are capable of producing Balmer absorption lines is

$$\left[\frac{N_2}{N_1} \right]_H = \frac{g_2}{g_1} e^{-(E_2-E_1)/kT} = 5.06 \times 10^{-9}$$

$$\left[\frac{N_2}{N_{\text{tot}}} \right]_{\text{Balmer}} = \left(\frac{N_2/N_1}{1 + N_2/N_1} \right) \left(\frac{1}{1 + N_{II}/N_I} \right) = 5.06 \times 10^{-9}$$

\Rightarrow only one of about 200 million H atoms is in the first excited state and capable of producing Balmer absorption lines

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

Strength of Metal Lines

Stellar
Atmospheres
(4)

Now, let's look at the **calcium** atoms.

The ionization energy of CaI is $\chi_I = 6.11 \text{ eV}$, about half of the ionization energy of H (13.6 eV).

The Saha equation is very sensitive to the ionization energy as χ/kT appears as an exponent and $kT \sim 0.5 \text{ eV} \ll \chi$.

Again, apply the Saha equation and the Boltzmann equation to get

$$\left[\frac{N_{II}}{N_I} \right]_{\text{Ca}} = \frac{2kTZ_{\text{CaII}}}{P_e Z_{\text{CaI}}} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi_I/kT} = 9.18 \times 10^2 \gg 1$$

\Rightarrow Practically all of the Ca atoms are in the form of CaII; only one atom out of 900 remains neutral.

We now estimate how many of these are in the ground state, capable of forming the CaII H and K absorption lines.

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

Strength of Metal Lines

Stellar
Atmospheres
(4)

The K ($\lambda = 393.3$ nm) line is produced by the first excited state, which has $E_2 - E_1 = 3.12$ eV above the ground state. Degeneracies are $g_1 = 2$ and $g_2 = 4$. The ratio of the number of CaII ions in the first excited state to those in the ground state is:

$$\left[\frac{N_2}{N_1} \right]_{\text{CaII}} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/kT} = 3.79 \times 10^{-3} \ll 1$$

This implies that almost all the Ca atoms in the Sun's photosphere are singly ionized and in the ground state, and capable of forming the H and K lines of Ca:

$$\begin{aligned} \left[\frac{N_1}{N_{\text{tot}}} \right]_{\text{CaII}} &\simeq \left[\frac{N_1}{N_1 + N_2} \right]_{\text{CaII}} \left[\frac{N_{\text{CaII}}}{N_{\text{tot}}} \right]_{\text{Ca}} \\ &= \left(\frac{1}{1 + [N_2/N_1]_{\text{CaII}}} \right) \left(\frac{[N_{\text{II}}/N_{\text{I}}]_{\text{Ca}}}{1 + [N_{\text{II}}/N_{\text{I}}]_{\text{Ca}}} \right) \\ &= 0.995 \end{aligned}$$

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

Strength of Metal Lines

Stellar
Atmospheres
(4)

Now, it becomes clear why the CaII H and K lines are so much stronger in the Sun than the Balmer lines.

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

We have $N_{\text{H}}/N_{\text{Ca}} = 5 \times 10^5$, but only an extremely small fraction, 5.06×10^{-9} , of these H atoms are un-ionized and in the first excited state, capable of producing Balmer lines. Multiplying these two factors:

$$5 \times 10^5 \times 5.06 \times 10^{-9} = 2.53 \times 10^{-3} \ll 0.995$$

shows the strength of these CaII lines reflects the sensitive temperature dependence of the atomic states of excitation and ionization.

Strength of Metal Lines

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

To **summarize** this:

Ca II absorption lines are stronger than H Balmer lines in the Sun's spectrum due to their temperature sensitivity, multiple ionization stages of Calcium, and lower energy level transitions:

1. Temperature sensitivity: CaII lines are more sensitive to the photospheric temperature range than H Balmer lines. In the Sun's photosphere, where the temperature is around 5,500 K, the CaII ions are more easily excited, making the CaII absorption lines stronger.
2. Ionization stages: Most H in the Sun is fully ionized (H^+), while Ca has multiple ionization stages (Ca^+ , Ca^{++} , etc.). The CaII absorption lines represent the transition between the first and second ionization stages, which occurs more easily in the photosphere than H Balmer transitions.
3. Energy level transitions: The CaII lines are the result of transitions between lower energy levels, while Hydrogen Balmer lines involve transitions from higher energy levels. Since lower energy level transitions are more common, the CaII lines appear stronger.

Strength of Metal Lines

From this, we have seen that the strength of an absorption line depends on various factors, e.g.:

- number of atoms present
- their energy levels
- temperature and pressure of the environment

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

Stellar Temperatures

Stellar
Atmospheres
(4)

Another observational effect that can be understood using the Saha equation is that supergiants and giants have lower temperatures than dwarfs of the same spectral type.

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

Spectral classes are defined by line ratios of different ions, e.g. $\text{HeI } 4542\text{\AA} / \text{HeI } 4471$ for O stars. At higher temperatures the fraction of HeI will increase relative to HeI so the above ratio will increase.

However, supergiants have lower surface gravities (or pressure) than main-sequence stars, so from the Saha equation a lower P_e at the same temperature will give a higher ion fraction, N^+/N^0 .

Assuming a given spectral class corresponds to a fixed ratio N^+/N^0 , a star with a lower pressure can have a lower T_{eff} for the same ratio and spectral class.

Summary

Stellar
Atmospheres
(4)

Recap

Local Thermo-
dynamic
Equilibrium
(LTE)

The Saha
Equation

Summary

LTE = Maxwell + Boltzmann + Saha

The **Boltzmann equation** describes the degree of excitation of an atom or ion, e.g. $N(H_{n=2})/N(H_{n=1})$.

The **Saha equation** describes the degree of ionization of successive ions, e.g. $N(\text{He}^+)/N(\text{He}^0)$ or $N(\text{He}^{2+})/N(\text{He}^+)$.

The **Partition function** is the weighted sum of the number of ways an atom or ion can arrange its electrons with the same energy.

Ionization is an extremely energy-consuming process. Ionization happens within a very small temperature interval.

Saha-Boltzmann explains the spectral type (or temperature) dependence of lines in stellar atmospheres, e.g. strongest Balmer series at spectral type A and strong CaII lines in Solar-type stars.