Astronomía Avanzada I (Semester 1 2024)

Stellar Atmospheres (6)

Model Atmosphere

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May 24, 2024

Recap

Bound-bound transitions contribute to the line absorption.

Bound-free and free-free transitions (plus scattering) contribute to the continuous absorption, mostly by H and He.

The atomic H absorption coefficient is highly T sensitive. For late-type stars in the optical and IR, bound-free and free-free transitions of the H^- ion dominate the continuous opacity, since the population of the atomic H in $n = 3$ (Paschen series) is so low.

For early-type stars, atomic H dominates, producing strong jumps in the opacity to the Lyman, Balmer and Paschen edges.

Dominant Sources of Opacity

The most important transitions for the **continuous absorption** are those which ionize atoms (with a continuum of final states). For H and He the line spectra do not greatly affect radiative transport.

Some metals with very complex line spectra contribute to the continuum.

New stellar opacities have been recalcultaed in the past ~ 20 years by two groups: OPAL (Iglesias et al.) and The Opacity Project/ OP (Seaton et al.) which have led to a factor of 3 increase in opacity under some temperature-density condictions via improved treatment of atomic data.

Line Absorption

A **bound-bound transition** absorbs or emits at $h\nu = hc/\lambda = \chi_u - \chi_l$ where χ is the excitation of the upper and lower levels above the ground state. Such transitions contribute to the line absorption.

For H, the excitation energy for a level n of H is

$$\chi_n = \chi_{\rm ion}\left(1 - \frac{1}{n^2}\right)$$

where $\chi_{\rm ion} = 13.6\ \rm eV$.

A bound-bound transition between $n_{\rm low}$ and $n_{\rm high}$ occurs at the wavelength

$$\frac{1}{\lambda} = R_{\rm H}\left(\frac{1}{n_{\rm low}^2} - \frac{1}{n_{\rm high}^2}\right)$$

$$h\nu = \chi_{\rm ion}\left(\frac{1}{n_{\rm low}^2} - \frac{1}{n_{\rm high}^2}\right)$$

where $R_{\rm H} = 109677.5\ \rm cm^{-1}$ is the Rydberg constant for H.
The line **Lyman** α at 1215 Å is the transition between the ground-state ($n = 1$) and the first excited state ($n = 2$).

Continuous Absorption

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There are two possibilities:

1. A transition from a bound state (level n) to a free state with velocity v. The energy of the absorbed bound-free photon is given by

$$h\nu = hc/\lambda = (\chi_{\mathrm{ion}} - \chi_n) + mv^2/2$$

Each bound-free transition corresponds to an ionization process (since the electron is free afterwards). The emission of a photon by a free-bound transition corresponds to a recombination process.

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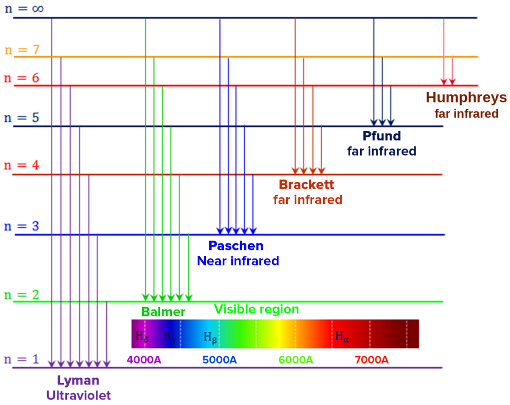
2. Finally, one can get a continuum of transitions if the electron goes from one free-state (with velocity v_1) to another free-state (with velocity v_2). The energy of the **free-free transition** is:

$$h\nu = \frac{hc}{\lambda} = \frac{mv_2^2}{2} - \frac{mv_1^2}{2}$$

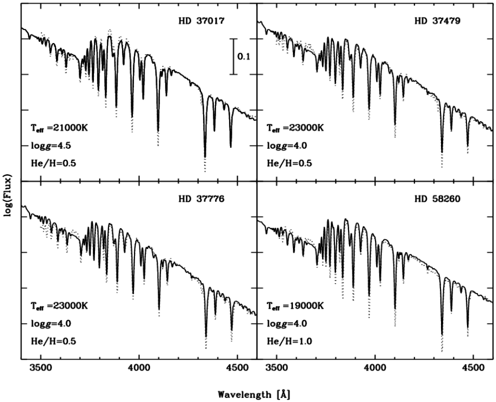
Lyman, Balmer, Paschen Continua

For H, transitions occurring between $n = 1$ and another bound state $n \geq 2$ etc. are known as the Lyman series (observed in the UV); between $n = 2$ and higher are the Balmer series (in the optical), and higher series observed in the IR: Paschen ($n = 3$), Brackett ($n = 4$), Pfund ($n = 5$), etc.



Balmer Jump

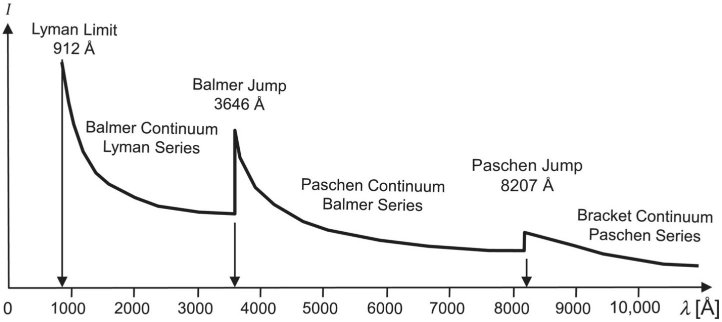
The **Balmer jump** or Balmer discontinuity is intensity difference in stellar continua on either side of the limit of the Balmer series of H, at $\sim$ 364.5 nm. It is caused by electrons being completely ionized directly from the second energy level of a H atom (bound-free absorption), which creates a continuum absorption at wavelengths shorter than 364.5 nm.



credit:
Cidale et al. (2007)

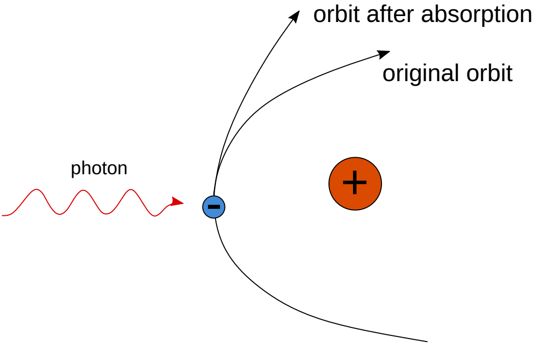
Balmer Jump

The contribution of level n will start at $\lambda_n = hc/(\chi_{\rm ion} - \chi_n)$ and continue for shorter λ. There is a discontinuity at λ_n because of a sudden change in the number of absorbing atoms, e.g.:
Lyman jump (912Å) due to the contribution of $n = 1$
Balmer jump (3647Å) due to the contribution of $n = 2$.



Free-Free Absorption Coefficient

When a free electron collides with a proton, its orbit (unbound) is altered. A photon may be absorbed during such a collision, the orbital energy of the electron being increased by the photon energy.



The strength of the absorption depends on the electron velocity: slower electrons are more likely to absorb a photon because a slow encounter increases the probability of a photon passing by during the collision.

Free-Free Absorption Coefficient

Adopting a Maxwellian distribution, the cross-section for the fraction of electrons in the velocity interval $\mathrm{d}v$ is:

$$\mathrm{d}\sigma_{ff}(\mathrm{H}) = \frac{2}{3\sqrt{3}} \frac{h^2 e^2 R}{\pi m_e^3} \frac{1}{\nu^3 v} \, \mathrm{d}v$$

We integrate over the velocity:

$$\sigma_{ff}(\mathrm{H}) = \frac{2}{3\sqrt{3}} \frac{h^2 e^2 R}{\pi m_e^3} \frac{1}{\nu^3} \left(\frac{2m_e}{\pi \kappa t}\right)^{1/2}$$

The total absorption coefficient for H is:

$$\kappa_{\mathrm{ff}}^{H} = \frac{\sigma_{\mathrm{ff}} G_{\mathrm{ff}} N_i N_e}{N}$$

where the number density of electrons, ions and neutral H are N_e, N_i and N, respectively.

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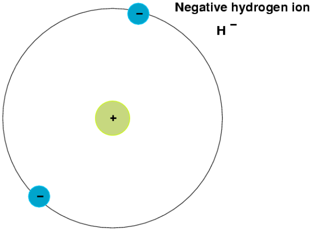
The free-free continuous absorption coefficient for H is much smaller than the bound-free coefficient.

Negative Hydrogen Ion H^-

The H atom is capable of holding a second electron in a bound state (binding energy 0.754 eV). All photons with $\lambda < 1.64\mu m$ have sufficient energy to ionize the H^- ion back to neutral H atom plus a free electron.



The extra electrons to form H^- come from ionized metals (e.g. Ca^+).

For Solar-like stars, it turns out that H^- is the dominant continuum opacity source at optical wavelengths. In early-type stars, H^- is too highly ionized to play a role, whilst in late-type stars there are too few free electrons (since no ionized metals).

Negative Hydrogen Ion H^- in the Sun

We can use the Saha equation to derive the relative population of H^- in the Sun.

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? Why is then the H^- absorption coefficient so important?

Negative Hydrogen Ion H^- in the Sun

Recall: Only H atoms in the 3^{rd} quantum level ($n = 3$, Paschen continuum) can contribute to the visual continuous opacity. From the Boltzmann equation:

$$\log N(\mathrm{H}_{n=3})/N(\mathrm{H}_{n=1}) = \log 2(3)^2 - 5040/5777 \times 12.1 = -9.6$$

i.e. $N_H(n = 3)/N_{\mathrm{H}(n=1)} = 2.4 \times 10^{-10}$ for the Sun.

We can now compare the number of H^- ions and H atoms in the Paschen continuum:

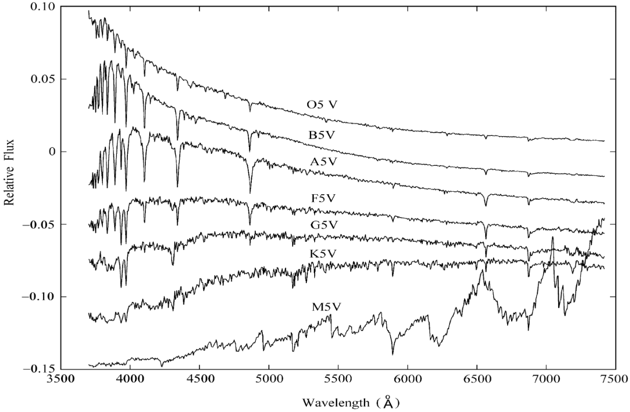
$$\log N(\mathrm{H}_{n=3})/N(\mathrm{H}^-) = 2.4 \times 10^{-10}/2.1 \times 10^{-8} = 0.01$$

The atomic absorption coefficients per absorbing atom are comparable, so we expect H^- bound-free absorption to be 100 times more important than the H Paschen continuum for the Sun.

The Balmer continuum ($n = 2$) cannot so easily be neglected, and does contribute to the opacity at shorter wavelengths.

Negative Hydrogen Ion H^- in the Sun

Note: For early type stars (A and earlier) we find $N_{\mathrm{H}}(n=3)/N(\mathrm{H}^- \ll 1$ so absorption of neutral H is much more impartant than H^-.
This is why such stars have very strong discontinuities in the Balmer and Paschen limits.



H^- Continuous Opacity

We have identified H^- (bound-free) in the visual and H^- (free-free) in the IR as principal sources of opacity in the Sun.

The H Balmer continuum shortward of the 3647 ÅBalmer jump is an additional contributor.

What other forms of opacity play a role in other stars?

Physical Processes Contributing to Opacity

Many physical processes are contributing to opacity:

Bound-bound transitions: absorption or emission of radiation from electrons moving between bound energy levels.

Bound-free transitions: the energy of the higher level electron state lies in the continuum or is unbound.

Free-free transitions: they change the motion of an electron from one free state to another.

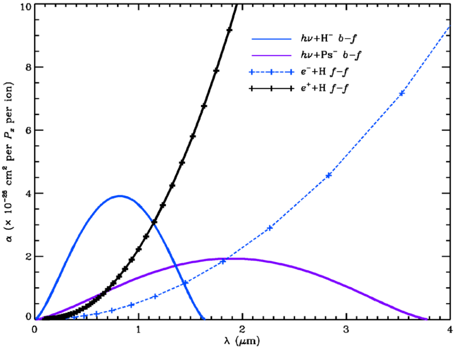
Electron scattering: the deflection of a photon from its original path by a particle, without changing its wavelength

- Rayleigh scattering: photons scatter off bound electrons (varies with λ^{-4})
- Thomson scattering: photons scatter off free electrons (independent of λ)

Photodissociation may occur for molecules.

Confirmation of H$^-$

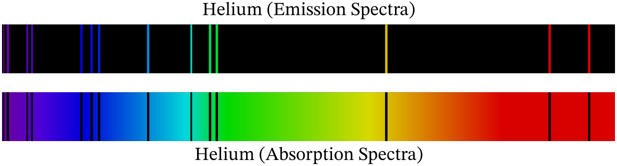
The wavelength dependence of the optical depth τ_λ (and hence absorption coefficient κ_λ) can be observationally derived for the Sun - the optical and IR dependence agrees remarkably well with the theoretical absorption coefficient for bound-free and free-free H$^-$.



Absorption coefficients of four scattering processes, calculated with $T = 6300K$.

He Opacity

Helium is the next most abundant element after H. Helium was discovered in 1868 by Jules Janssen when observing the spectrum of the Sun's chromosphere and discovering a bright yellow line at 587.49 nanometers.



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Helium (Emission Spectra)

Helium (Absorption Spectra)

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Is Helium important for the continuous absorption in the Sun or other stars?

He Opacity

From the Boltzmann equation, one can get that only 10^{-17} of the He atoms can contribute to the absorption, and since He is 10 % as abundant as H (by number), only one in 10^{-18} atomas are He atoms in the 1^{st} excited state.

Consequently, He opacity plays a negligible role for the Sun. The bound-free absorption from He^- is generally negligible, whilst free-free He^- (with a form similar to free-free H^-) can be significant at long wavelengths in cool stars.

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Photoionization (bound-free processes) from He only play a significant role for the hottest, O-type, stars.

Iron Opacity

If He only plays a role for very hot stars, do any metals contribute to the continuous opacity in cool stars?

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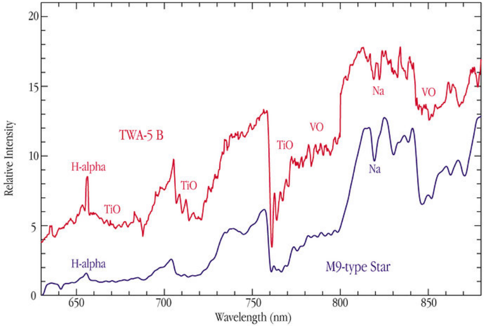
Iron ($Fe/H = 10^{-4}$) is generally the dominant metal continuous opacity source in stellar atmospheres.

In the Sun, let's consider absorption by atomic Fe in the UV (1000Å) for which an excitation energy of $\sim 1.7\,\mathrm{eV}$ is required. The fraction of excited Fe atoms is 4×10^{-2} (from Boltzmann equation), whilst the fraction of ionized to neutral Fe is approximately 6 (from Saha equation).

Accounting for the abundance of Fe, we obtain the fraction of atomic Fe atoms absorbing at 2000 Å relative to the total number of H atoms to be $4 \times 10^{-2} \times 10^{-4} \times 1/6 = 6 \times 10^{-7}$.
We already obtained 2×10^{-8} for H^-, so metallic lines in the UV are much more important for absorption than the H^- ion, or the neutral H atom. Even more important is the absorption by the metal atoms in the ground level, which is < 1570 Å for Fe, < 1520 Å for Si.

Molecular Opacity

CN^-, C_2^-, $H_2\ O^-$, CH_3, TiO are important sources of opacity in late (K-type) and very late (M-type) stars, as well as brown dwarfs.

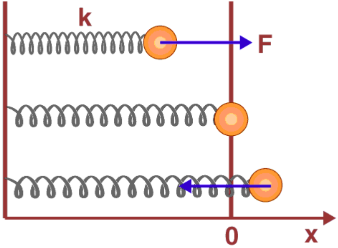


Optical spectrum of brown dwarf TWA-5 B. Also shown is the optical spectrum of a typical M9-type star. The spectra are very similar, with broad molecular absorption bands from TiO and VO. TWA-5 B also shows strong hydrogen emission ($H\alpha$) and weak sodium (Na) absorption, both indicative of its comparatively young age. Credit: ESO

Scattering

A classical (harmonic) oscillator is a system that, when displaced from its equilibrium position, experiences a restoring force F proportional to the displacement x: $\vec{F} = -k\vec{x}$, where k is a positive constant.

The Eigenfrequency (*eigen* = its own) is one of the natural resonant frequencies of a system.



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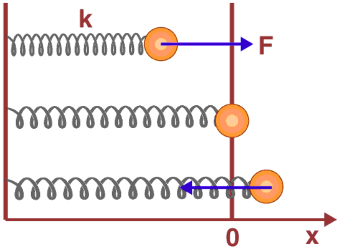
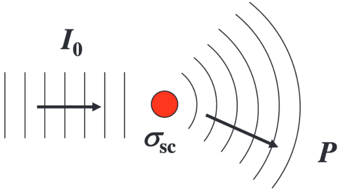


We can make use of this model to describe **scattering**.

Scattering

In the classical picture of an atom, we consider the electron as being bound to the atom. If a force were to pull on the electron and then let go, it would oscillate with Eigenfrequencies $\omega = 2\pi\nu$.



The cross-section σ_T is defined as the total scattered power P (energy per unit time) divided by the incident intensity I_0 (energy per unit time per unit area).
The scattering cross-section for a **classical oscillator** can be written as

$$\sigma_T = P\frac{8\pi}{c|\mathbf{E}_0|^2} = \frac{8\pi}{3}\frac{e^4}{m_e^2 c^4}\left[\frac{\nu^4}{(\nu^2-\nu_0^2)^2+\gamma^2\omega^2}\right], \quad \omega = 2\pi\nu$$

where ν_0 is the **Eigenfrequency** of an atom and γ is the damping constant.

Thomson and Rayleigh Scattering

Two cases are of interest:

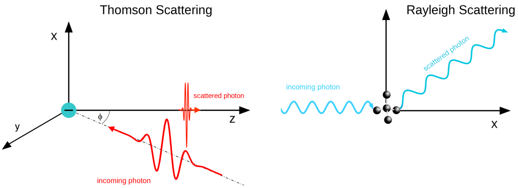
1. **Thomson (electron) scattering** ($\nu_0 = 0$, $\gamma = 0$)
Photons scatter off a free electron, change in direction

$$\sigma_T = \frac{8\pi}{3}\frac{e^4}{m_e^2 c^4} = \frac{8\pi}{3} r_e^2 = 6.65 \times 10^{-25}\ \mathrm{cm}^2/\mathrm{electron}$$

with r_e: classical electron radius

2. **Rayleigh scattering** by atoms and molecules ($\nu \leq \nu_0$, $\gamma \leq \nu_0$)

$$\sigma_R(\nu) \propto \sigma_T \nu^4 = \sigma_T \lambda^{-4}$$



Thomson Scattering

In electron scattering (Thomson scattering), the path of the photon is altered, but not the energy.

Since an electron is tiny, it makes a poor target for an incident photon, so the cross-section for Thomson scattering is very small ($\sigma_T = 6.65 \times 10^{-25}\ \mathrm{cm}^2$), and has te same value for all photons of all wavelength: As such, electron scattering is the only grey opacity source.

Although electrons are very abundant in the **Solar photoshere**, the small cros-section makes it unimportant.

Electron scattering is most effective as a source of opacity at high temperatures. In **atmospheres of OB** stars where most of the as is completely ionized, other sources of opacit involving bound electrons are excluded. In this regime, α_T dominates the continuum opacity.

Rayleigh Scattering

Rayleigh scattering by H atoms in **Solar-type stars** is more relevant than electron scattering, as atoms are much more common ($N(\mathrm{H}) \ll N(\mathrm{H}^+)$).

For **M stars**, H_2 becomes the dominant form of hydrogen, with strong electronic transitions in the UV, so Rayleigh scattering by H_2 can be important.

The cross-section for Rayleigh scattering is much smaller than σ_T and is proportional to λ^{-4} so increases steeply towards the blue. (In the same way the sky appears blue, due to a steep increase in the scattering cross-section of sunlight scattered by molecules in our atmosphere.)

The cross-section is sufficiently small relative to metallic absorption coefficients that Rayleigh scattering only plays a dominant role in extended envelopes of supergiants.

Total Extinction Coefficient κ

The total extinction coefficient is given by:

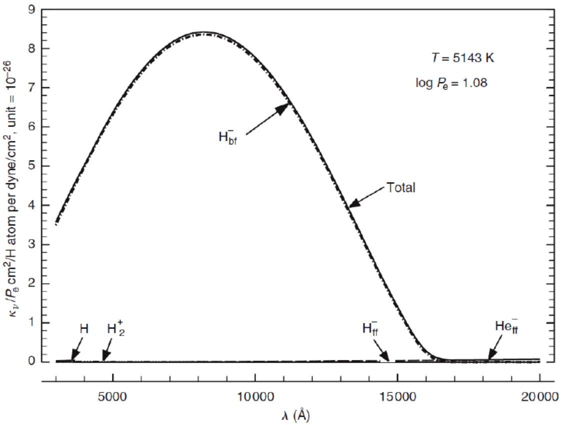
$$\kappa_\nu = \left(1 - e^{-h\nu/kT}\right) \sum_j x_j (\kappa_j^{bb} + \kappa_j^{bf} + \kappa_j^{ff}) + \kappa^s$$

where the sum is over all elements j of numer fraction x_j.

The $e^{-h\nu/kT}$ term accouns for stimulated emission (incident photon stimulates electron to de-excite and emit photon with identical energy, as in a laser).

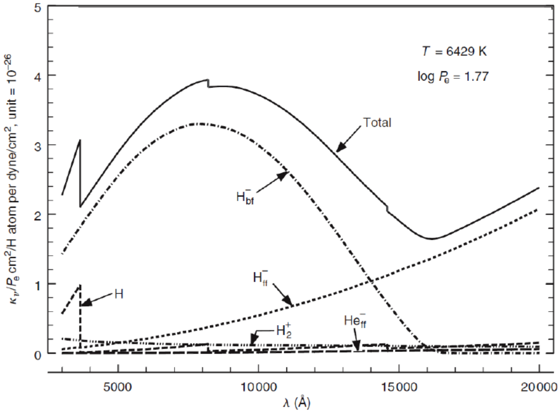
Extinction for G-type Stars

For G stars, the H^- ion (bound-free) dominates for optical wavelengths.



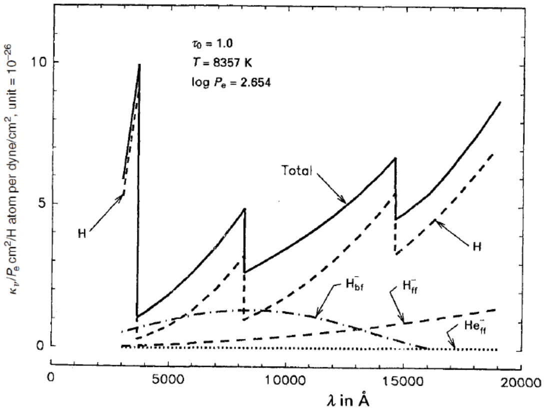
Extinction for F-type Stars

For F stars, the absorption is dominated by the two components of the H^- ion (bound-free and free-free), with a contribution from the Balmer continua below 3647 Å.



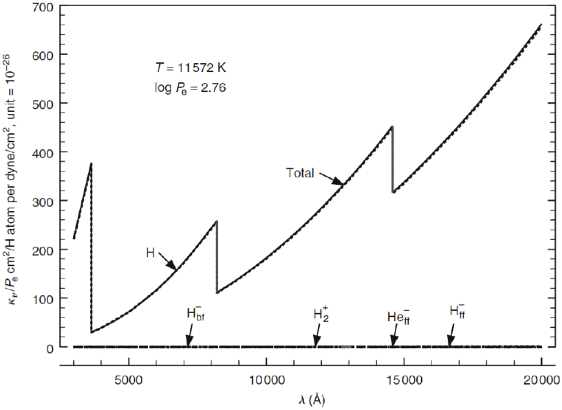
Extinction for A-type Stars

For late A stars, absorption from the H^- ion is dropping back compared to the cooler cases, while neutral hydrogen has grown with increasing temperature. H (bound-free) Balmer, Paschen and Brackett continua start to dominate.

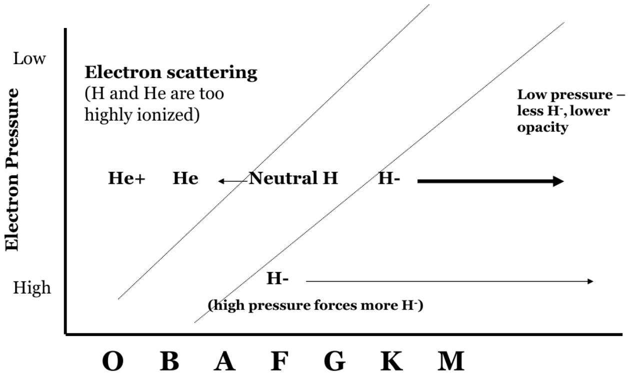


Extinction for B-type Stars

For late B type stars, H (bound-free) Balmer, Paschen and Brackett continua completely dominate.
For O stars, electron scattering is the primary opacity source.



Dominant Opacity vs. Spectra Type



Rosseland Approximation

The interiors of stars usually reach a high degree of local homogeneity. The **energy flux** can be described by a simple expression related with the local temperature gradient and is often referred to as the **Rosseland approximation**.

We start with the plane-parallel assumption

$$\frac{\mu}{\rho\kappa_\nu}\frac{\partial I_\nu(z,\mu)}{\partial z} = -I_\nu(z,\mu) + S_\nu .$$

We can write this as

$$I_\nu(z,\mu) = S_\nu - \frac{\mu}{\rho\kappa_\nu}\frac{\partial I_\nu(z,\mu)}{\partial z}$$

The "zeroth" approximation may be expressed as

$$I_\nu^{(0)}(z,\mu) \sim S_\nu^{(0)} = B_\nu(T)$$

which is independent of μ, that is, isotropic.

Rosseland Approximation

Now, get the first order approximation in μ by using $I_\nu^{(0)} = B_\nu$:

$$I_\nu^{(1)}(z, \mu) \sim B_\nu(T) - \frac{\mu}{\rho\kappa_\nu} \frac{\partial B_\nu}{\partial z}$$

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Compute the emerging flux $F_\nu(z)$ for the above intensity:

$$\begin{aligned} F_\nu(z) &= \int I_\nu^{(1)}(z,\mu)\cos\theta\,\mathrm{d}\Omega \\ &= 2\pi\int_{-1}^{+1} I_\nu^{(1)}(z,\mu)\mu\,\mathrm{d}\mu \\ &= \underbrace{2\pi\int_{-1}^{+1} B_\nu(T)\mu\,\mathrm{d}\mu}_{=0} - 2\pi\int_{-1}^{+1}\frac{\mu^2}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial z}\,\mathrm{d}\mu \end{aligned}$$

The angle-independent term does not contribute to the flux.

Rosseland Approximation

The emerging flux can be described by

$$
\begin{aligned}
F_\nu &= -\frac{2\pi}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial \nu}\int_{-1}^{+1}\nu^2\,\mathrm{d}\mu \\
&= -\frac{4\pi}{3\rho\kappa_\nu}\frac{\partial B_\nu(T)}{\partial z} \\
&= -\frac{4\pi}{3\rho\kappa_\nu}\frac{\partial B_\nu(T)}{\partial T}\frac{\partial T}{\partial z}
\end{aligned}
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&= -\frac{4\pi}{3\rho\kappa_\nu}\frac{\partial B_\nu(T)}{\partial T}\frac{\partial T}{\partial z}
\end{aligned}
$$

To obtain the flux, integrate over all frequencies

$$
F(z) = \int_0^\infty F_\nu\,\mathrm{d}\nu
$$

Rosseland Approximation

This is the **definition for the Rosseland mean opacity** than we encountered earlier on:

$$\frac{1}{\bar{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu(T)}{\partial T}\, \mathrm{d}\nu}{\int_0^\infty \frac{\partial B_\nu(T)}{\partial T}\, \mathrm{d}\nu}$$

A more convenient form for this result is

$$\int_0^\infty \frac{\partial B_\nu}{\partial T}\, \mathrm{d}\nu = \frac{\partial}{\partial T} \int_0^\infty B_\nu\, \mathrm{d}\nu = \frac{\partial B(T)}{\partial T} = \frac{4aT^3}{\pi}$$

Therefore, we have:

$$F(z) = -\frac{16\sigma_{\mathrm{SB}} T^3}{3\rho\bar{\kappa}} \frac{\partial T}{\partial z}$$

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4. Magnetic fields are excluded.

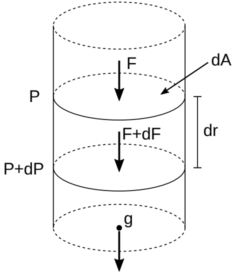
Ideal Gas

We require a knowledge of the electron pressure in order to use the Saha equation, which is related to the gas pressure. How can we calculate this in stellar atmospheres?

Ideal Gas

We start with the hydrostatic equilibrium.
Forces acting upon the volume element of density $\rho(r)$ are gravity

$$\mathrm{d}F_g = -\frac{GM_r\,\mathrm{d}m}{r^2} = -\frac{GM_r\rho(r)}{r^2}\mathrm{d}A\,\mathrm{d}r$$



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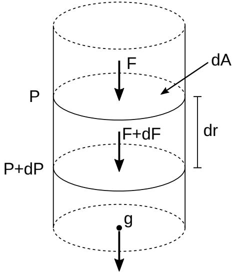

plus buoyancy (pressure difference $\times$ area):

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Since the mass of the atmosphere is negligible compared to the stellar mass and the radius of the photosphere is negligible vs. the stellar radius:

$$\mathrm{d}F_g = -\frac{GM_r\,\mathrm{d}m}{R^2}\mathrm{d}A\,\mathrm{d}r = -g\rho(r)\,\mathrm{d}A\,\mathrm{d}r$$

since

$$g = \frac{GM}{R^2}$$

Hydrostatic Equilibrium

The Earth's atmosphere is neither lost to space nor pulled to the surface of the Earth via balance between gravity ($\mathrm{d}F_g = -g\rho\,\mathrm{d}A\,\mathrm{d}r$) and gas pressure ($\mathrm{d}F_p = -\mathrm{d}P\,\mathrm{d}A$), i.e. **hydrostatic equilibrium** which is the balance between gravitational and pressure forces ($\mathrm{d}F_g + \mathrm{d}F_p = 0$).

If the gas pressure is P_g, the differential form of the hydrostatic equilibrium equation for an ideal gas is then

$$\frac{\mathrm{d}P_g}{\mathrm{d}r} = -g\rho(r)$$

We can eliminate $\rho(r)$ with the ideal gas equation, $P_g = \rho kT/\mu m_H$:

$$\frac{\mathrm{d}P_g}{\mathrm{d}r} = -g\frac{\mu(r)m_H}{kT(r)}P_g(r) = -g\frac{\mu(r)}{R_g T(r)}P_g(r)$$

In the above we have introduced the **gas constant** $R_g = k/m_H = 8.3 \times 10^7\ \mathrm{erg/mol/K}$ with the mean molecular weight μ.

Pressure Scale Height

We obtain

$$\frac{1}{P_g}\frac{\mathrm{d}P_g}{\mathrm{d}r} = \frac{\mathrm{d}\ln P_g}{\mathrm{d}r} = -\frac{g\mu(r)}{R_g T(r)}$$

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For an idealized isothermal ($T(r) = \mathrm{const}$) atmosphere with $\mu(r) = \mathrm{const}$, we can integrate this expression

$$P_g(r) = P_g(r_0)\exp(-(r-r_0)g\mu/R_g T) = P_g(r_0)\exp(-(r-r_0)/H)$$

where we have introduced the **scale height**

$$H = \frac{kT}{g\mu m_H} = \frac{R_g T}{g\mu}$$

I.e. gas pressure changes by a factor of e over a scale height.

For a (ficticious) atmosphere of constant density, corresponding to the gas pressure at the base of the real atmosphere, we can put the total mass of the real atmosphere into a layer of height H.

Example Atmospheres

Star	μ	$T[K]$	$\log g$	H
Betelgeuse	1	3600	0	3 $\mathrm{R}_\odot$
Sun	1	6000	4.4	180 km
White Dwarf	0.5	1.5×10^4	8	0.25 km
Neutron Star	0.5 $(\mathrm{H}^+, \mathrm{N_e})$	10^6 - 10^7	14	1.7 mm

Gas Pressure $P_g(\rho)$

For the Saha equation, we need T and P_g in a particular layer of the atmosphere, which can be described by optical depth τ.

The temperature dependence on average optical depth is known as

$$T^4(\bar{\tau}) \sim \frac{3}{4}(\bar{\tau} + \frac{2}{3})T_{\mathrm{eff}}^4$$

The average optical depth $\mathrm{d}\bar{\tau} = -\kappa_R \rho \, \mathrm{d}r$ may be expressed via the Rosseland mean opacity per unit mass (cm^2 / g), κ_R.
Thus we generally express the gas pressure as a functin of optical depth.
From hydrostatic equilibrium, we obtain

$$\frac{\mathrm{d}P_s}{\mathrm{d}r} = -g\rho(r) \Rightarrow \frac{\mathrm{d}P_g}{\mathrm{d}\bar{\tau}} = \frac{g}{\kappa_R}$$

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The gas pressure can now be obtained by integrating this differential equation, although in general, κ_R is a complicated function of temperature and pressure.

Integration of Hydrostatic Equation

In the **simplest case**, assuming a constant mean opacity (which is not a very sensible approximation, but okay for electron scattering), with $\tau = 0$ and $P_g = 0$ at the surface:

$$P_g = \frac{g}{\kappa_R}\bar{\tau}$$

Knowing $T(\tau)$ for a given $T_{\rm eff}$, we can assume a value for κ_R. We then insert this into the above equation and compute a value for the gas pressure.

More realistically, we can approximate the mean opacity with a power-law expression,

$$\kappa_R = \kappa_{R,0}(P_g/P_{g,0})^n$$

with $\kappa_{R,0}$ the opactiy at a reference wavelength, and $n = 1$ usually acceptable (especially if H^- is responsilefor the continuous opacity).

Gravity Dependence of P_g

We can integrate this expression ($P = 0$ at surface):

$$P_g^n \, \mathrm{d}P_g = \frac{g}{\kappa_{g,0}} P_{g,0}^n \, \mathrm{d}\tau$$

to get:

$$\frac{1}{n+1} P_g^{n+1} \, \mathrm{d}P_g = \frac{g}{\kappa_{g,0}} P_{g,0}^n \bar{\tau}$$

For $n = 1$:

$$P_g = \sqrt{\frac{2g\bar{\tau}P_{g,0}}{\kappa_{g,0}}}$$

i.e. the gas pressure for a given optical depth increases with $g^{1/2}$.
For different stars we see down to $\tau = 2/3$, whose pressure varies approximately as $g^{1/2}$.

The greater the pressure, the greater the Rosseland mean opacity, so we see geometrically higher layers in stars with higher gravity.
Giants have deep atmospheres (this is why the plane-parallel approximation cannot be used for giants!), dwarfs thin ones.

Electron Pressure

So far we have dealt with the gas pressure, but it is the electron pressure that is relevant for the Saha equation.
We can generally say

$$P_g = NkT$$

where N is the sum of all particles/ cm^3, and

$$P_e = n_e kT$$

with n_e being the number of electrons / cm^3.

Of course, $n_e = n^+ + 2n^{2+} + 3n^{3+}$ etc.
In the simplest case of pure H,

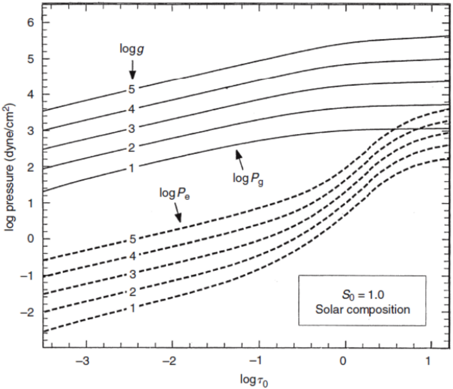
$$N = N(\mathrm{H}) + N(\mathrm{H}^+) + n_e = N(\mathrm{H}) + 2N(\mathrm{H}^+) = P_g/kT$$

since from charge conservation, $n_e = N(\mathrm{H}^+)$.

Given $N(\mathrm{H}^+)n_e/N(\mathrm{H}) = f(T)$ from the Saha equation, we may solve for $N(\mathrm{H}^+) = n_e$ and $N(\mathrm{H})$, if T and P_g are known.

Numerical Examples

In fact, the gas pressure exponent is not $1/2$, but numerical results show it ranges from 0.57 to 0.64 from shallow to deep layers.
The electron pressure dependence on gravity is not so linear, with an exponent of $1/3$ predicted ($P_e \propto P_g$ for solar-type stars). Numerical calculations show 0.45 to 0.3 from shallow to deep layers.



credit: Gray, Fig. 9.13

Role of Metals

For a pure H atmosphere in the case of the Solar photosphere, the gas pressure greatly exceeds the electron pressure.

Although metals are few in number, some are very easily ionized, e.g. $\mathrm{Na/H} = 2 \times 10^{-6}$, $\mathrm{Mg/H} = 3 \times 10^{-5}$, $\mathrm{Al/H} = 2.7 \times 10^{-6}$, $\mathrm{Ca/H} = 2 \times 10^{-6}$, $\mathrm{Si/H} = 3 \times 10^{-5}$.

These will contribute electrons to the atmosphere, increasing P_e and suppressing ionization.

Gas and Electron Pressure

To properly calculate the electron density, all low ionization energy species and their corresponding abundances should be included.

For ionized hydrogen, we find $P_e = 0.5\, P_g$.

For doubly ionized helium, we find $P_e = 2/3\, P_g$.

Summary

The atmosphere of a star contains less than one billionth of its total mass, so why do we care at all?

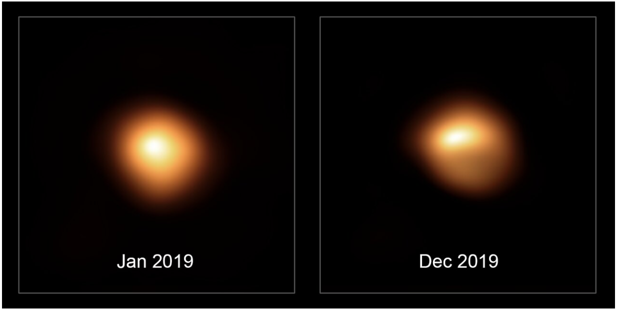
The amosphere of a star is what we can see, measure and analyze.

The stellar atmosphere is therefore the source of information in order to put a star from the color-magnitude diagram (e.g. $B - V, m_v$) of the observer into the HRD (L, T_{eff}) of the theoretician and, hence, to derive the theory of stellar evolution.

Atmosphere analyses reveal element abundance and show us results of cosmo-chemistry, starting from the earliest moments of the formation of the Universe.

Hence, working with stellar atmospheres enables a test for big-bang theory.

Variable Stars



Comparison of VLT-SPHERE images of Betelgeuse taken in January 2019 and December 2019, showing the changes in brightness and shape. Betelgeuse is an intrinsically variable star.