Astronomía Avanzada I (Semester 1 2025)

Stellar Atmospheres (3)

Radiative Transfer Effects

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May 30, 2025

Recap: Flux and Brightness

The **(specific) intensity** I_λ is a measure of brightness:

$$I_\lambda = \frac{E_\lambda}{\cos\theta \, \mathrm{d}\lambda \, \mathrm{d}\sigma \, \mathrm{d}\omega \, \mathrm{d}t}$$

The **flux** F_λ is the projection of the specific intensity in the radial direction, integrated over all solid angles:

$$F_\lambda = \oint I_\lambda \cos\theta \, \mathrm{d}\omega$$

I_λ is independent of the distance from the source. F_λ obeys the inverse square law.

Recap: Flux and Brightness

Flux is related to the **intensity**:

The flux F_λ is a measure of the net energy flow across an area $\mathrm{d}\sigma$, over a time $\mathrm{d}t$, in a $\mathrm{d}\lambda$. The only directional significance is whether the energy crosses $\mathrm{d}\sigma$ from the top or from the bottom.

Then we can write:

$$F_\lambda = \frac{\oint \mathrm{d}E_\lambda}{\mathrm{d}\lambda\,\mathrm{d}\sigma\,\mathrm{d}t} = \oint I_\lambda \cos\theta\,\mathrm{d}\omega$$

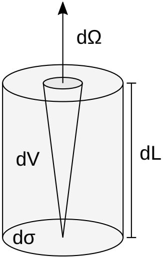
The solid angle $\mathrm{d}\omega$ appears for I_λ but not for F_λ.

Mean Intensity and Energy Density

The **mean intensity**

$$J_\lambda = \frac{1}{4\pi} \oint I_\lambda \, \mathrm{d}\Omega$$

is related to the energy density u_λ:



Energy radiated through area element $\mathrm{d}\sigma$ during time $\mathrm{d}t$:

$$\mathrm{d}E_\lambda = I_\lambda \, \mathrm{d}\lambda \, \mathrm{d}\sigma \, \mathrm{d}\Omega \, \mathrm{d}t$$

with $l = c \, \mathrm{d}t$: $\mathrm{d}V = l \, \mathrm{d}\sigma = c \, \mathrm{d}t \, \mathrm{d}\sigma$

Hence, the energy contained in the volume element $\mathrm{d}V$ per wavelength interval, the **energy density**, is

$$u_\lambda \, \mathrm{d}V \, \mathrm{d}\lambda = \oint I_\lambda \, \mathrm{d}\Omega \, \mathrm{d}\lambda \, \mathrm{d}\sigma \, \mathrm{d}t = 4\pi J_\lambda \frac{\mathrm{d}V}{c} \, \mathrm{d}\lambda$$

$$u_\lambda = \frac{4\pi}{c} J_\lambda \left[\frac{\mathrm{erg}}{\mathrm{cm}^3 \, \mathrm{\AA}} \right]$$

Radiation Pressure

Because a photon possesses an energy E, it also carries a **momentum** $p_\lambda = E_\lambda / c$ and can exert a **radiation pressure**:

Consider photons transfering momentum to a solid wall. We get the force:

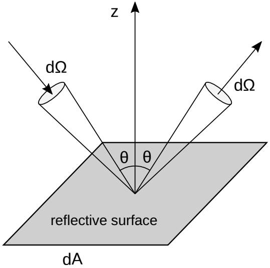
$$F = \frac{\mathrm{d}p_{\lambda,\perp}}{\mathrm{d}t} = \frac{1}{c}\frac{\mathrm{d}E_\lambda}{\mathrm{d}t}\cos\theta$$

The pressure is:

$$\mathrm{d}P_\lambda = \frac{F}{\mathrm{d}\sigma} = \frac{1}{c}\frac{\mathrm{d}E_\lambda}{\mathrm{d}t}\frac{\cos\theta}{\mathrm{d}\sigma} = \frac{1}{c} I_\lambda \cos^2\theta \,\mathrm{d}\Omega\,\mathrm{d}\lambda$$

with

$$I_\lambda = \frac{\mathrm{d}E_\lambda}{\cos\theta\,\mathrm{d}\lambda\,\mathrm{d}\sigma\,\mathrm{d}\Omega\,\mathrm{d}t}$$



Radiation Pressure

The change in the z component of the momentum of photons with wavelengths between λ and $\lambda + \mathrm{d}\lambda$ that are reflected from the area $\mathrm{d}A$ in a time interval $\mathrm{d}t$ is

$$
\begin{aligned}
\Delta p_\lambda \,\mathrm{d}\lambda &= [p_{\lambda,z}^{\mathrm{final}} - p_{\lambda,z}^{\mathrm{initial}}]\,\mathrm{d}\lambda \\
&= \left[\frac{E_\lambda \cos\theta}{c} - \left(-\frac{E_\lambda \cos\theta}{c}\right)\right]\,\mathrm{d}\lambda \\
&= \frac{2E_\lambda \cos\theta}{c}\,\mathrm{d}\lambda \\
&= \frac{2}{c} I_\lambda \,\mathrm{d}\lambda\,\mathrm{d}t\,\mathrm{d}A\,\cos^2\theta\,\mathrm{d}\Omega
\end{aligned}
$$

Radiation Pressure

From Newton's 2nd and 3rd laws, the radiation pressure exerted by these photons with wavelengths between λ and $\lambda + \mathrm{d}\lambda$ in the case of reflection:

$$P_\lambda^{\mathrm{rad}} = \frac{2}{c} \int_{\mathrm{hemisphere}} I_\lambda \, \mathrm{d}\lambda \, \cos^2\theta \, \mathrm{d}\Omega = \frac{2}{c} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_\lambda \, \mathrm{d}\lambda \cos^2\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

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Imagine removing the reflecting surface $\mathrm{d}A$ and replacing it with a mathematical surface. Instead of being reflected, photons will be streaming through $\mathrm{d}A$ from the other side. We have:

$$\begin{aligned} P_\lambda^{\mathrm{rad}} \, \mathrm{d}\lambda &= \frac{1}{c} \int I_\lambda \, \mathrm{d}\lambda \, \cos^2\theta \, \mathrm{d}\Omega \quad \text{(transmission)} \\ &= \frac{1}{c} \int_{\pi=0}^{2\pi} \int_{\theta=0}^{\pi} I_\lambda \, \mathrm{d}\lambda \, \cos^2\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= \frac{4\pi}{3c} I_\lambda \, \mathrm{d}\lambda \quad \text{(isotropic radiation field)} \end{aligned}$$

Radiation Pressure

The total radiation pressure produced by photons with all wavelengths is

$$P_{\mathrm{rad}} = \int_0^\infty P_\lambda^{\mathrm{rad}}\,\mathrm{d}\lambda.$$

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$$P_{\mathrm{rad}} = \int_0^\infty P_\lambda^{\mathrm{rad}}\,\mathrm{d}\lambda.$$

For blackbody radiation, the total radiation pressure can be expressed as

$$\begin{aligned} P_{\mathrm{rad}} &= \frac{4\pi}{3c}\int_0^\infty B_\lambda(T)\,\mathrm{d}\lambda \\ &= \frac{4\sigma_{\mathrm{SB}}T^4}{3c} = \frac{1}{3}u \end{aligned}$$

Radiation Pressure Gradient

Despite photon's *random walk* journey to the stellar surface, the energy from the deep interior of the star manages to escape into space.

The temperature in the stellar atmosphere decreases outward, the radiation pressure is smaller at larger radii. This gradient in the radiation pressure causes a **net flow of radiation energy moving outward** from the center of the star and eventually emerging at the stellar surface. This process can be described by

$$\frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}r} = -\frac{\rho\bar{\kappa}}{c}F_{\mathrm{rad}}$$

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The transfer of radiation energy is a subtle process involving the slow outward diffusion of randomly walking photons, drifting towards the surface in response to small differences in the radiation pressure.

Radiation Pressure Gradient

Assume a **grey atmosphere** ($=$ opacity is independent of wavelength and is described by the Rosseland mean opacity $\bar{\kappa}$), and ignore all the remaining wavelength dependencies by integrating over all wavelengths

$$I = \int_0^\infty I_\lambda \, \mathrm{d}\lambda$$

and

$$S = \int_0^\infty S_\lambda \, \mathrm{d}\lambda$$

Radiation Pressure Gradient

The transfer equation for a plane-parallel gray atmosphere is

$$\mu \frac{\mathrm{d}I}{\tau_\perp} = I - S$$

with $\mu = \cos\theta$.

Integrating over all solid angles and recalling that S is isotropic, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\perp} \int I\mu \,\mathrm{d}\Omega = \int I \,\mathrm{d}\Omega - \int S \,\mathrm{d} = \Omega$$

$$\frac{\mathrm{d}F_{\mathrm{rad}}}{\mathrm{d}\tau_\perp} = 4\pi(\langle I\rangle - S)$$

Radiation Pressure Gradient

Furthermore, multiplying the transfer equation by $\mu = \cos\theta$ and integrating over all solid angles, we find

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\perp} \underbrace{\int I\mu^2 \,\mathrm{d}\Omega}_{=cP_{\mathrm{rad}}} = \underbrace{\int I\mu \,\mathrm{d}\Omega}_{=F_{\mathrm{rad}}} - \underbrace{S\int \mu \,\mathrm{d}\Omega}_{=0}$$

Recall that

$$F_\lambda \,\mathrm{d}\lambda = \int I_\lambda \,\mathrm{d}\lambda \,\mu \,\mathrm{d}\Omega$$

and

$$P_{\lambda,\mathrm{rad}} \,\mathrm{d}\lambda = \frac{1}{c} \int I_\lambda \,\mathrm{d}\lambda \mu^2 \,\mathrm{d}\Omega$$

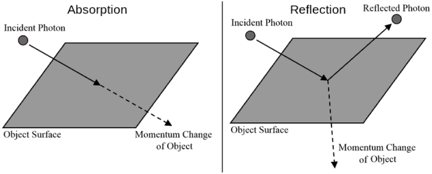
Therefore, we have

$$\frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}\tau_\perp} = \frac{1}{c} F_{\mathrm{rad}}.$$

Radiation Pressure

A photon has **momentum** $p_\lambda = E_\lambda / c$.
We consider photons hittig a solid wall, transferring momentum. We have the **force**

$$F = \frac{\mathrm{d}p_{\lambda\perp}}{\mathrm{d}t} = \frac{1}{c}\frac{\mathrm{d}E_\lambda}{\mathrm{d}t}\cos\theta$$



This generates **radiation pressure**:

$$\mathrm{d}P_\lambda = \frac{F}{\mathrm{d}\sigma} = \frac{1}{c}\frac{\mathrm{d}E_\lambda}{\mathrm{d}t}\frac{\cos\theta}{\mathrm{d}\sigma} = \frac{1}{c} I_\lambda \cos^2\theta \,\mathrm{d}\omega\,\mathrm{d}\lambda$$

$$\Rightarrow P(\lambda) = \frac{1}{c}\oint_{4\pi} I_\lambda \cos^2\theta \,\mathrm{d}\omega = \frac{4\pi}{c} K_\lambda$$

Where K_λ is the K-**integral**

$$K_\lambda = \frac{1}{4\pi}\oint I_\lambda \cos^2\theta \,\mathrm{d}\omega.$$

Spherical vs. Plane-Parallel Geometry

When calculating the transfer equations in stars, in principal, we would have to consider spherical geometry.

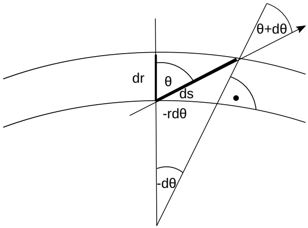
Fortunately, the **geometrical thickness** of most photospheres is small compared to their radii, thus permitting the plane-parallel approximation $r \to \infty$:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\cos\theta \frac{\partial I_\nu}{\partial t}$$

which we can derive from

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = \frac{\partial I_\nu}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}s} + \frac{\partial I_\nu}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}s}$$

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \cos\theta, \quad \frac{\mathrm{d}\theta}{\mathrm{d}s} = -\frac{\sin\theta}{r}$$



Plane-Parallel Atmosphere Approximation

Consider the atmosphere as a plane-parallel medium, define a vertical optical depth, $\tau_{\lambda,\perp}(z)$, as

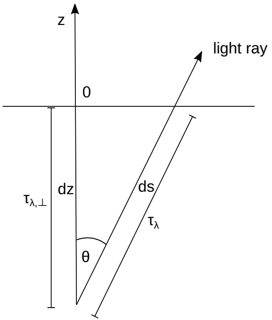
$$\tau_{\lambda,\perp}(z) \equiv \int_z^0 \rho\kappa_\lambda(z)\,\mathrm{d}z$$

Since $\mathrm{d}z = \cos\theta\,\mathrm{d}s$, the optical depth of a ray traveling upward at an angle θ from the same initial position z can be related with $\tau_{\lambda,\perp}(z)$ by

$$\tau_\lambda = \frac{\tau_{\lambda\perp}}{\cos\theta} = \frac{\tau_{\lambda,\perp}}{\mu}$$

Hence:

$$\mu\frac{\mathrm{d}I_\lambda}{\tau_{\lambda\perp}} = I_\lambda - S_\lambda$$



Plane-Parallel Atmosphere Approximation

In most cases, stellar atmospheres are in an equilibrium state, no net energy is subtracted from or added to the radiation field. For a plane-parallel atmosphere, this means that the radiative flux must have the same value at any level, including its surface.
Hence,

$$F_{\mathrm{rad}} = F_{\mathrm{surf}} = \sigma_{\mathrm{SB}} T_e^4 = \mathrm{const}$$

Also, since the flux is a constant, we have

$$\frac{\mathrm{d}F_{\mathrm{rad}}}{\mathrm{d}\tau_{\perp}} = 0 = 4\pi(\langle I \rangle - S) \Rightarrow \langle I \rangle = S$$

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The radiation pressure can be found by integrating

$$\frac{\mathrm{d}F_{\mathrm{rad}}}{\mathrm{d}r} = -\frac{\rho\bar{\kappa}}{c} F_{\mathrm{rad}}$$

$$P_{\mathrm{rad}} = \frac{1}{c} F_{\mathrm{rad}} \tau_\perp + C$$

where C is the constant of integration. A **boundary condition** is needed to solve C.

The Plane-Parallel Transfer Equation

The plane-parallel transfer equation for stars with thin photospheres

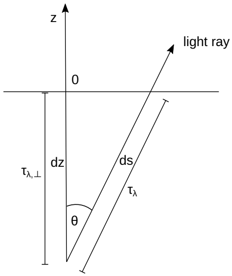
$$\cos\theta \frac{\mathrm{d}I_\lambda(\theta)}{\mathrm{d}\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

is identical to the parallel-ray transfer equation (for ISM studies)

$$\frac{\mathrm{d}I_\theta}{\mathrm{d}\tau_\theta} = -I_\lambda + S_\lambda$$

except for:

- the $\cos\theta$ term, because the optical depth is measured along the radial direction x and not along the line of sight, e.g. $\mathrm{d}\tau_\lambda = -\kappa_\lambda \rho \mathrm{d}x$
- sign change, since we are now looking from the outside in, along the direction x.



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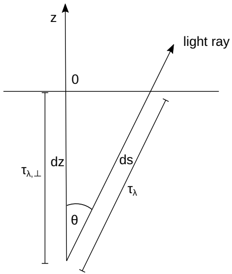

The full spherical geometry transfer equation is necessary for **supergiants**. There is no flux going into the star, i.e.

Surface Intensity

To derive the intensity at the surface, we multiplicate the plane-parallel transfer equation by an integrating factor $e^{-\tau/\cos\theta} = e^{-u}$,

$$\frac{\mathrm{d}I_\lambda(\theta)}{\mathrm{d}u} e^{-u} - I_\lambda(\theta) e^{-u} = -S_\lambda e^{-u}$$

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This can be written as

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This can be written as

$$\frac{\mathrm{d}(I_\lambda(\theta) e^{-u})}{\mathrm{d}u} = -S_\lambda e^{-u}$$

Integrating $\mathrm{d}u$ from 0 to ∞:

$$\left[I_\lambda(\theta) e^{-u}\right]_0^\infty = -\int_0^\infty S_\lambda(\tau_\lambda e^{-u})\,\mathrm{d}u$$

$$I_\lambda(0,\theta) = \int_0^\infty S_\lambda(\tau_\lambda) e^{-u} \mathrm{d}u$$

Grey Atmosphere

In the previous section, we assumed that the opacity can be independent of λ, i.e. $\kappa_\lambda = \kappa$. We call such a hypothetical atmosphere a **grey atmosphere**.

In the theory of stellar atmospheres, much of the technical effort goes into iteration schemes using equations of radiative equilibrium (such as discussed today) to find the source function S_λ.

Often, a starting point for such iterations is the grey atmosphere.

Thermal (Radiative) Equilibrium

In stellar atmospheres, radiation dominates transfer of energy, so we can discuss three conditions of radiative equilibrium which can be used to derive the **temperature structure in the photosphere**.

The radiation we see from the Sun comes from a layer of geometrical height of a few hundred km.

In a column of 100 km height and 1 cm^2 cross-section there are 10^{24} particles (since $\rho \sim 10^{17}/\mathrm{cm}^3$ in the Sun), each of which has a thermal energy of this column is therefore $10^{12}\ \mathrm{erg/cm^2}$. The observed radiative energy loss (per cm^2) of the Solar surface is $F_\odot = 6.3 \times 10^{10}\ \mathrm{erg/cm^2/s}$.

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If the Sun shines at a constant rate, the **energy content** of the Solar photosphere can last only for 15 seconds without being resupplied from below. Exact that amount of energy has to be supplied or else the photosphere would quickly change temperature.

First Equation of Radiative Equilibrium

As the photosphere is not changing temperature, $\mathrm{d}F/\mathrm{d}t = 0$ or $\mathrm{d}F/\mathrm{d}x = 0$ or $\mathrm{d}F/\mathrm{d}\tau = 0$, i.e. the total flux ust be constant at all depth of the photosphere because of conservation of energy. This is the **first equation of radiative equilibrium**:

$$F(x) = F(0) = \mathrm{const} = \sigma_{\mathrm{SB}}\, T_{\mathrm{eff}}^4$$

When all the energy is carried by radiation, we have

$$F(x) = \int_0^\infty F_\lambda(\tau_\lambda)\mathrm{d}\lambda = F(0)$$

Although the shape of F_λ can be expected to change very significantly with depth, its integral remains invariant.

If **other sources of energy transport** must be taken into account, then a more general expression of flux constancy must be applied:

$$\Phi(x) + \int_0^\infty F_\lambda(\tau_\lambda)\mathrm{d}\lambda = F(0)$$

Here, $\Phi(x)$ is for example the convective flux.

Second Equation of Radiative Equilibrium

We integrate the plane-parallel transfer equation over the solid angle ω:

$$\int \cos\theta \frac{\mathrm{d}I_\lambda(\tau_\lambda,\theta)}{\mathrm{d}\tau_\lambda}\,\mathrm{d}\Omega = \int I_\lambda(\tau_\lambda,\theta)\,\mathrm{d}\Omega - \int S_\lambda(\tau_\lambda)\,\mathrm{d}\Omega$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}(F_\lambda(\tau_\lambda)) = 4\pi(J_\lambda(\tau_\lambda)) - \int S_\lambda(\tau_\lambda)\,\mathrm{d}\Omega$$

Based on the definition of the mean intensity:

$$J_\lambda = \frac{1}{4\pi}\oint I_\lambda \mathrm{d}\Omega$$

Finally, assuming S_λ to be isotropic, we obtain

$$\frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}(F_\lambda(\tau_\lambda)) = J_\lambda(\tau_\lambda) - S_\lambda(\tau_\lambda)$$

In the case of a grey atmosphere (opacity κ independent of wavelength)

$$\frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}\tau}F(\tau) = -S(\tau) + J(\tau) = 0$$

Since $\mathrm{d}F/\mathrm{d}\tau = 0$, the source function must equal the mean intensity J.

Second Equation of Radiative Equilibrium

If the atmosphere is not grey, which is the situation for most stars, let's incorporate the opacity κ into the RHS, and integrating over wavelength

$$\frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\infty F(\tau_\lambda)\,\mathrm{d}\lambda\right) = \int_0^\infty (\kappa_\lambda S_\lambda - \kappa_\lambda J_\lambda)\,\mathrm{d}\lambda = 0$$

Since $\mathrm{d}F/\mathrm{d}t = 0$, we can get the radiative balance equation (energy conservation)

$$\int_0^\infty \kappa_\lambda J_\lambda\,\mathrm{d}\lambda = \int_0^\infty \kappa_\lambda S_\lambda\,\mathrm{d}\lambda$$

Ths is the **second equation of radiative equilibrium** and can be understood as the total energy absorbed (LHS) must equal the total energy re-emitted (RHS) if no heating or cooling is taking place.

Third Equation of Radiative Equilibrium

The third equation is obtained by multiplying the transfer equation by $\cos\theta$ and integrating over the solid angle and the over the wavelength:

$$\cos\theta \frac{\mathrm{d}I_\lambda(\theta)}{\mathrm{d}\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

$$\oint \cos^2\theta \frac{\mathrm{d}I_\lambda(\tau_\lambda,\theta)}{\mathrm{d}\tau_\lambda}\,\mathrm{d}\Omega = \oint \cos\theta\,\mathrm{d}I_\lambda(\tau_\lambda,\theta)\Omega - \int \cos\theta S_\lambda(\tau_\lambda,\theta)\,\mathrm{d}\Omega$$

$$K_\lambda(\tau_\lambda) = \frac{1}{4\pi}\oint I_\lambda \cos^2\theta\,\mathrm{d}\Omega \qquad F_\lambda = \oint I_\lambda \cos\theta\,\mathrm{d}\Omega \qquad 0\ (S_\lambda \text{ is isotropic})$$

$$4\pi \int \frac{\mathrm{d}K_\lambda}{\mathrm{d}\tau_\lambda}\,\mathrm{d}\lambda = \int F_\lambda\,\mathrm{d}\lambda = F(\tau)$$

We get the **third radiative equilibrium condition**:

$$\int_0^\infty \frac{\mathrm{d}K_\lambda}{\mathrm{d}\tau_\lambda}\,\mathrm{d}\lambda = \frac{F(\tau)}{4\pi}$$

Equations of Radiative Equilibrium

All three radiative equilibrium conditions are not independent.

A S_λ that is a solution of one will be the solution of all three.

The **flux constant** $F(0)$ is often expressed in terms of an effective temperature $F(0) = \sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4$.

When model photospheres are constructed using flux constancy as a condition to be fulfilled by the model, the effective temperature becomes one of the fundamental parameters characterizing the model.

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In **real stars**, energy is created or lost from the radiation field through e.g. convection, magnetic fields, plus in supernovae atmospheres energy conservation is not valid (radioactive decay of Ni to Fe), so the energy constraints are more complicated in reality.

The Depth Dependence of the Source Function

In a **grey atmosphere**, with $K(\tau) = \int_0^\infty K_\lambda \, \mathrm{d}\lambda$, the third equation implies:

$$\frac{\mathrm{d}K(\tau)}{\mathrm{d}\tau} = \frac{F(\tau)}{4\pi}$$

We can differentiate this, and insert our earlier result:

$$\frac{\mathrm{d}^2K(\tau)}{\mathrm{d}\tau^2} = \frac{1}{4\pi}\frac{\mathrm{d}F(\tau)}{\tau} = J(\tau) - S(\tau) = 0$$

Integration of the equation w.r.t. τ gives $K(\tau) = c_1\tau + c_2$ where $\mathrm{d}K/\mathrm{d}\tau = c_1 = F/(4\pi)$.

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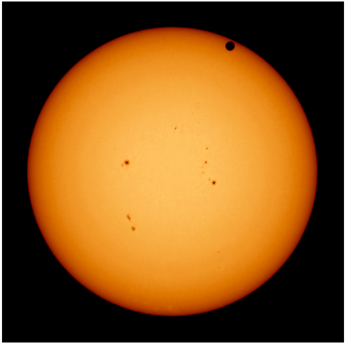
For a given F, we now have two equations to determine the three unknowns J, S, and K (or c_2). We need an additional relation between two of these variables in order to determine all three.

Limb Darkening



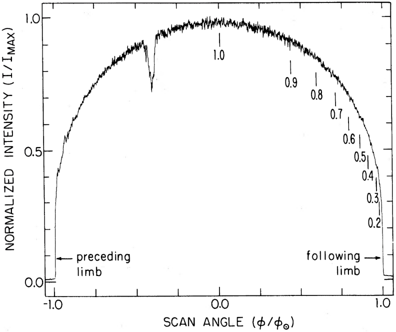


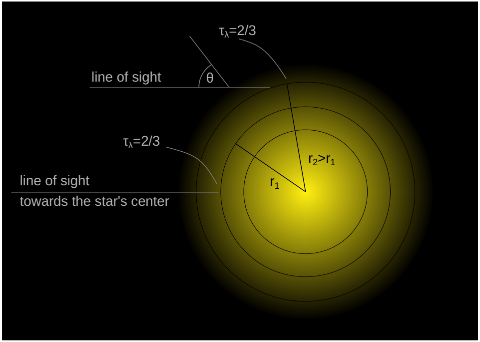
FIG. 1.—A typical drift scan plotted against scan angle $\phi/\phi_\odot$. The intensities plotted here have been corrected for zero point and normalized. Representative values of $\mu = \cos\theta$ are marked along the following half of the scan.

left: Sun, observed at Venus transit
right: angular dependence of Solar intensity, credit: Petro et al. (1984)

Limb Darkening

Limb darkening is the gradual decrease in brightness of the disk of a star as observed from its centre to its edge, or limb. It can be easily seen in photographs of the Sun.

Its understanding offered astronomers an opportunity to construct models with such gradients. This encouraged the development of the **theory of radiative transfer**.



Limb Darkening

Let us assume a linear source function, $S_\lambda(\tau_\lambda) = a_\lambda + b_\lambda \tau_\lambda$. We derive

$$\begin{aligned} I_\lambda(0,\theta) &= \int_0^\infty a_\lambda e^{-u} \mathrm{d}u + \int_0^\infty b_\lambda \tau_\lambda e^{-u} \mathrm{d}u \\ &= a_\lambda \int_0^\infty e^{-u} \mathrm{d}u + b_\lambda \cos\theta \int_0^\infty u e^{-u} \mathrm{d}u \end{aligned}$$

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Recall $u = \tau / \cos(\theta)$, so $\tau = u \cos(\theta)$ and we can use the standard integral

$$\int_0^\infty u^n e^{-u} \mathrm{d}u = n!$$

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Limb darkening occurs: less light from the limb vs. from the center if $b_\lambda > 0$.

Limb Darkening

We convert the solution of the transfer equation by using the vertical optical depth, $\tau_\perp$:

$$I_\lambda(0) = I_{\lambda,0} e^{-\tau_{\lambda,\perp}/\mu} - \int_{\tau_{\lambda,\perp}/\mu}^{0} \frac{S_\lambda}{\mu} e^{-\tau_{\lambda,\perp}/\mu} \, \mathrm{d}\tau_{\lambda,\perp}$$

Taking the initial position of the rays to be at the center, where $\tau_{\lambda,\perp} = \infty$, we have

$$I_\lambda(0) = \int_0^\infty \frac{S_\lambda}{\mu} e^{-\tau_{\lambda,\perp}/\mu} \, \mathrm{d}\tau_{\lambda,\perp}$$

The source function is often assumed to have the form

$$S_\lambda = a_\lambda + b_\lambda \tau_{\lambda,\perp}$$

Limb Darkening

Applying our model of plane-parallel gray atmosphere in LTE with the Eddington approximation, we can obtain

$$S = \langle I \rangle = \frac{3\sigma_{\rm SB}}{4\pi} T_e^4 \left(\tau_\perp + \frac{2}{3} \right) = a + b\tau_\perp$$

Therefore,

$$a = \frac{\sigma_{\rm SB}}{2\pi} T_e^4, \quad b = \frac{3\sigma_{\rm SB}}{4\pi} T_e^4$$

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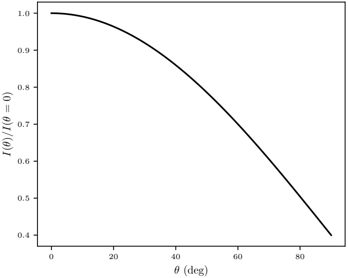
The emergent intensity will have the form $I(0) = a + b\mu$, and the ratio of the emergent intensity is

$$\frac{I(\theta)}{I(\theta=0)} = \frac{a + b\cos\theta}{a+b} = \frac{2}{5} + \frac{3}{5}\cos\theta.$$



The Quadratic Source Function

Until now: We assumed a **linear source function**.

More generally: If

$$S_\lambda(\tau_\lambda) = \sum_n a_{\lambda n} \tau_\lambda^n$$

then

$$I_\lambda(0,\theta) = \sum_n A_n \cos^n \theta, \quad A_n = a_{\lambda n} \int_0^\infty u^n e^{-u} \, \mathrm{d}u = a_{\lambda n} n$$

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We still get $S_\lambda(0)$ at the limb, but a more complicated result at the center. For example, a quadratic term requires the solution of

$$S(\tau_\lambda) = a_{0\lambda} + a_{1\lambda}\tau_\lambda + a_{2\lambda}\tau_\lambda^2, \quad I_\lambda(0,\theta) = a_{0\lambda} + a_{1\lambda}\cos\theta + 2a_{2\lambda}\cos^2\theta$$

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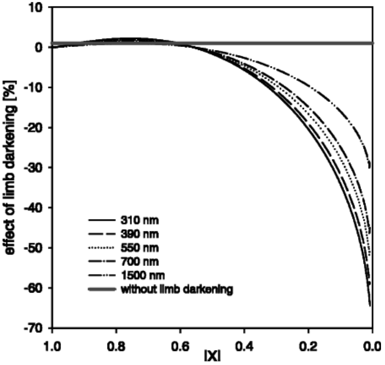
At $\theta = 90^\circ$, $\tau_\lambda = 0$, whilst at $\theta = 0^\circ$, $\tau_\lambda \sim 1 + 2a_{1\lambda}/a_{2\lambda}$ providing $a_{2\lambda} \ll a_{1\lambda}$. This gives a **ratio of the limb-to-center intensity** of

$$I_\lambda(0,90^\circ)/I_\lambda(0,0^\circ) = a_{0\lambda}/(a_{0\lambda} + a_{1\lambda} + 2a_{2\lambda})$$

Wavelength Dependency of Limb Darkening

Limb darkening is observed to be greatest at shorter wavelengths. The temperature distribution of the upper atmosphere of the Sun can be obtained from limb darkening measurements, carried out via e.g. multi-filter images of the Solar continuum.



Limb darkening in various wavelengths. The variable x decreases with the distance from the center. Credit: Koepke et al. (2001)

Limb Darkening for other Stars

Until recently, the Sun was the only star for which limb darkening was observed, as one needs to spacially resolve the disk to measure it. Other methods are now possible.

Limb darkening for other stars than our Sun can be measured by using the following methods:

1. Direct interferometry, using a high spatial resolution imaging - e.g. COAST array, providing the star is very large and nearby (a cool supergiant).

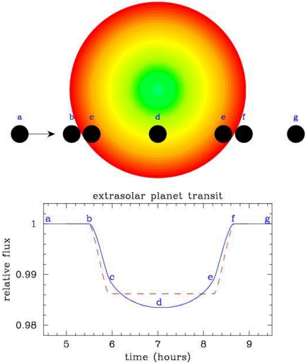
2. The light curve of an eclipsing binary system during secondary eclipse allows us to stuuy limb darkening of the primary, although this is not trivial. A similar approach followed by extrasolar planets occulting their star (e.g. HD209458).

3. Analyzing the light curve from gravitational microlensing of a background star by a foreground source (e.g. PLANET team).

Limb Darkening and Transit Light Curves

Limb darkening must be taken into account when working with transit curves from such as binary systems or exoplanets.

Limb darkening is a second-order, but key effect for determining the exact radius of the planet, because limb darkening not only modifies the shape of the transit light curve, but it also significantly affects the true transit depth.

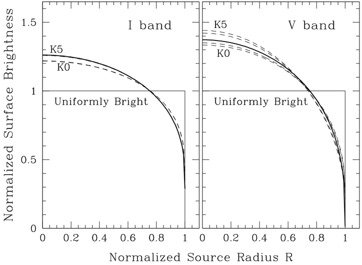


credit: juliaastro.org

Limb Darkening from Microlensing

Galactic gravitational microlensing occurs when a foreground object (lens) passes in front of a background star (source). The gravitational deflection of light by the lens causes the flux from the source to be amplified.

One such event, MACHO 97-BLG-28, was studied to reveal limb darkening information for the background K giant (Albrow et al. 1999).



Thick lines show how much fainter the K giant becomes at its limb in the red I (left) and blue-green V filter (right). If the star would emit a uniform amount of light across its whole stellar disk, the profile would look like the "Uniformly Bright" profile. credit: Albrow et al. (1999)

Eddington Approximation

A plane-parallel gray atmosphere needs a description of the angular distribution of the intensity.
The Eddington approximation assumes an isotropic radiation field,

$$
\begin{aligned}
\langle I \rangle &= \frac{1}{2}(I_{\rm out} + I_{\rm in}) \\
F_{\rm rad} &= \pi(I_{\rm out} - I_{\rm in}) \\
F_{\rm rad} &= \frac{2\pi}{3c}(I_{\rm out} - I_{\rm in}) = \frac{4\pi}{3c}\langle I \rangle
\end{aligned}
$$

Applying $P_{\rm rad}$ to the plane-parallel gray atmosphere, we find

$$\frac{4\pi}{3c}\langle I \rangle = \frac{1}{c}F_{\rm rad}\tau_{\perp} + C$$

Eddington Approximation

The constant C can be determined by evaluating the first two equations of the Eddington approximation at the stellar surface, where $\tau_\perp = 0$ and $I_{\rm in} = 0$. Given $\langle I(\tau_\perp = 0)\rangle = \frac{F_{\rm rad}}{2\pi}$, we get

$$C = \frac{2}{3c} F_{\rm rad}.$$

Therefore,

$$\frac{4\pi}{3}\langle I\rangle = F_{\rm rad}\left(\tau_\perp + \frac{2}{3}\right).$$

Substituting $F_{\rm rad} = \sigma_{\rm SB} T_e$, one obtains

$$\langle I\rangle = \frac{3\sigma_{SB}}{4\pi} T_e^4 \left(\tau_\perp + \frac{2}{3}\right)$$

Eddington Approximation

Given

$$\langle I \rangle = \frac{3\sigma_{SB}}{4\pi} T_e^4 \left(\tau_\perp + \frac{2}{3} \right).$$

If the atmosphere is assumed to be in LTE, the soure function is equal to the Planck function, $S_\lambda = B_\lambda$. Integrating over all wavelengths, we have

$$\langle I \rangle = S = B = \frac{\sigma_{SB} T^4}{\pi}$$

Finally, we obtain a description for the variation of the temperature in a plane-parallel gray atmosphere in LTE with the Eddington approximation

$$T^4 = \frac{3}{4} T_e^4 \left(\tau_\perp + \frac{2}{3} \right)$$

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$$T^4 = \frac{3}{4} T_e^4 \left(\tau_\perp + \frac{2}{3} \right)$$

Note that $T = T_e$ at $\tau_\perp = \frac{2}{3}$.
By definition, the surface of the star is where $T = T_e$, i.e. $\tau_\perp = 2/3$ since $L = 4\pi R_*^2 \sigma_{\rm SB} T_e^4$.

Eddington Approximation

Previously we had seen that for the determination of the flux the anisotropy in the radiation field is very important: in the flux integral the inward-going intensities are subtracted from the outward-going ones, due to the factor $\cos\theta$.

But for K, a small anisotropy is unimportant: the intensities are multiplied by $\cos^2\theta$, which does not change sign for inward and outward radiation.

In order to evaluate K or c_2 we can approximate the radiation field by an isotropic radiation field of the mean intensity J: $I = J$ (by definition). From the definition of K_λ we obtain

$$4\pi K_\lambda = \oint I_\lambda(\tau_\lambda, \theta) \cos^2\theta \, \mathrm{d}\omega = J_\lambda(\tau_\lambda) \oint \cos^2\theta \, \mathrm{d}\omega = \frac{4\pi}{3} J_\lambda(\tau_\lambda)$$

or after division by 4π:

$$K_\lambda(\tau_\lambda) = \frac{1}{3} J_\lambda(\tau_\lambda)$$

This approximation for the K function is known as the **Eddington approximation**.

Eddington Approximation

Inserting the Eddington approximation into the above equation

$$\frac{\mathrm{d}K(\tau)}{\mathrm{d}\tau} = \frac{F(\tau)}{4\pi}$$

we find

$$\frac{\mathrm{dK}(\tau)}{\mathrm{d}\tau} = \frac{1}{3}\frac{\mathrm{d}J(\tau)}{\mathrm{d}\tau} = \frac{F(\tau)}{4\pi} = c_1, \qquad \frac{\mathrm{d}J(\tau)}{\mathrm{d}\tau} = \frac{3}{4\pi}F(\tau)$$

Since the mean intensity J equals the source function S in a grey atmosphere, when integrating the latter result we obtain

$$S(\tau) = \frac{3}{4\pi}\tau F(0) + C = J(\tau)$$

From the conditions of radiative equilibrium, we finally obtained the law for the depth dependence of the source function (for a grey atmosphere assuming the Eddington approximation).
We evaluate C using boundary conditions for the known emerging flux (there is no flux going into the star), i.e. $I(0,\theta) = I^- = 0$ for $\pi/2 < \theta < \pi$, plus assuming the outward intensity is independent of θ, i.e. $I(0,\theta) = I^+$ for $0 < \theta < \pi/2$.

Eddington Approximation

Boundary condition: There is no flux going into the star, i.e. $I(0, \theta) = I^- = 0$ for $\pi/2 < \theta < \pi$.
We also assume that the outward intensity does not depend upon θ, i.e. $I(0, \theta) = I^+ = \mathrm{const}$ for $0 < \theta < \pi/2$.

From this, we get

$$J(0) = \frac{1}{2\pi} I^+ = \frac{1}{2\pi} F(0)$$

Hence $C = J(0) = F(0)/2\pi$, so

$$S(\tau) = \frac{1}{\pi} \left(\frac{3}{4}\tau + \frac{1}{2} \right) F(0)$$

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To find the depth dependence of T, we also need to assume LTE.

Temperature Structure of the Grey Atmosphere

In LTE, the source function is the Planck function,
$S(\tau) = B(\tau) = \sigma_{\mathrm{SB}} T^4/\pi$.

$$B(\tau) = \frac{\sigma_{\mathrm{SB}}}{\pi} T^4(\tau) = \frac{3}{4\pi}\left(\tau + \frac{2}{3}\right) F(0)$$

Recall that $F(0) = \sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4$, by definition, so

$$\frac{1}{\pi}\sigma_{\mathrm{SB}} T^4(\tau) = \frac{3}{4\pi}\left(\tau + \frac{2}{3}\right)\sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4 \quad \text{or} \quad T^4(\tau) = \frac{3}{4}\left(\tau + \frac{2}{3}\right) T_{\mathrm{eff}}^4$$

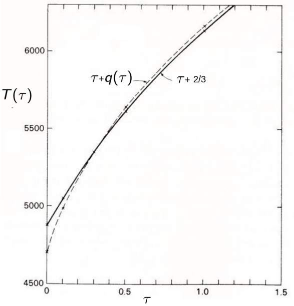
With this, we derived the **temperature dependence on optical depth**.
Note $T(\tau = 2/3) = T_{\mathrm{eff}}$ as we obtained earlier, and $T^4(\tau = 0) = T_{\mathrm{eff}}^4/2$

Temperature Structure of the Grey Atmosphere

A **complete solution** of the grey case, using accurate boundary conditions, without Eddington approximation, leads to a solution only slightly different from this, usually expressed as

$$T^4(\tau) = \frac{3}{4}\left(\tau + q(\tau)\right) T_{\rm eff}^4$$

Here, $q(\tau)$ is a slowly varying function, called the Hopf function, with $q = 1/\sqrt{3} = 0.577$ at $\tau = 0$ to $= 0.710$ at $\tau = \infty$.



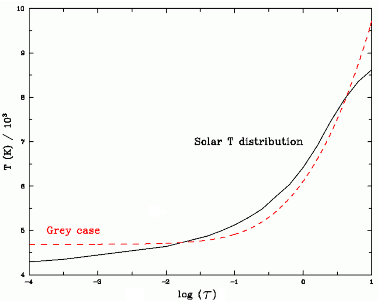
Comparison between $T(\tau)$ in the Solar atmosphere using the simplified Eddington approximation (solid) versus the exact grey atmosphere (dashed) using the Hopf function, $q(\tau) \sim 0.0710 - 0.133 \times e^{-2\tau}$.

Grey Temperature Structure

How good is the approximation of a grey atmosphere?

We must next look at the frequency dependence of the sources of opacity. The grey temperature distribution is shown here versus the observed Solar temperature distribution as a function of optical depth τ at 5000Å (D. Gray, Table 9.2).

The poor match is because the opacity is wavelength dependent.



Summary

Three equations of radiative equilibrium can be derived:

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Limb darkening in other stars can be estimated from interferometry, eclipsing binaries, microlensing.