On bounding the spectrum of graphs using isoperimetry

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Project Overview

Eigenvalues of the adjacency matrix provide a lot of information about a graph and are easier to compute than some related combinatorial, isoperimetric quantities such as the Max Cut, sparsest cut, and vertex/edge expansion. For this reason, inequalities relating (linear) algebraic information to geometric information about graphs are of considerable interest to researchers in combinatorics, computer science, engineering, and recently, in additive and analytic combinatorics.

In *Max Cut and the Smallest Eigenvalue* (2008), Luca Trevisan describes an approximation algorithm, a variant to spectral partitioning, for the Max Cut Problem. Trevisan's algorithm can be implemented in nearly linear time and tightens the bounds on the cost of the cut. In order to determine the bottleneck of the graph, Trevisan bounds the smallest eigenvalue of the adjacency matrix (with bounds analogous to Cheeger's inequality).

To bound this eigenvalue, Trevisan defines a new constant $\beta(G)$ that describes the "bipartiteness ratio" of the graph. Prasad Tetali, Peter Ralli, and I propose to improve $\beta(G)$ to tighten the bounds of the associated spectrum minima, and examine applications of this result to Biswas-Saha's *A Cheeger Type Inequality in Finite Cayley Sum Graphs* (2019).

Background

We will first define $\beta(G)$. Take V to be the set of vertices of the graph G. We wish to minimize the part of the graph that isn't bipartite. To do so, we take all subsets of V which includes at most half of the total vertices (any more would trivialize the problem). We split a given subset S into two sets, L and R, to determine how bipartite S is. We can do this by considering which edges are not in between L and R, i.e. the "non-bipartiteness" of L and R. We minimize this non-bipartiteness over all subsets S, giving us the minimum non-bipartiteness, or equivalently the maximum bipartiteness of the graph. The minimization of the ratio of non-bipartite edges between L and R to all edges in S defines $\beta(G)$, the "bipartiteness ratio".

Trevisan relates $\beta(G)$ to the smallest eigenvalue of the adjacency matrix (largest eigenvalue of the normalized Laplacian) as this eigenvalue has relevant connectivity properties. As our research focuses on bounding the smallest eigenvalue, we will briefly describe how knowledge of the minima of the spectrum gives us valuable information about a graph. We first arrange the eigenvalues of a graph to obtain its ordered spectrum

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n.$$

There is much research regarding the second smallest eigenvalue λ_{n-1} in spectral graph theory. Thus we know that if λ_{n-1} is small, then the graph is close to being bipartite and if λ_{n-1} is large, then it is in fact an expander graph (sparse with strong connectivity properties). The smallest eigenvalue λ_n has interesting connectivity properties as well. A graph is bipartite when $\lambda_n = -1$, hence the distance between -1 and λ_n is often thought of as the "non-bipartiteness" of the graph. Thus the smallest eigenvalue has a natural connection to $\beta(G)$, and our work will focus primarily on bounding λ_n. We will therefore analyze the distance between λ_n and -1. Additionally, since λ_{n-1} is also useful in quantifying a graph's bipartiteness, quantifying the distance between λ_{n-1} and λ_n may be helpful in improving characterizations of $\beta(G)$.

Finally, we note that under the assumption of non-negative curvature, Klartag-Kozma-Ralli-Tetali (KKRT) showed in *Discrete Curvature and Abelian Groups* that the Cheeger inequality, relating the smallest (nonzero) eigenvalue of the Laplacian of a graph to the edge expansion constant, is tight (up to a factor of d). This could be applied to Biswas-Saha's result, to improve the bound of the smallest eigenvalue.

Semester Objectives

In this research we will improve Trevisan's lower bound of λ_n. We may do so by bounding $\beta(G)$ from below by a suitable function of vertex or edge expansion and d. We will also investigate sufficient criteria under which Trevisan's inequality, relating the smallest eigenvalue λ_n of a graph G to $\beta(G)$, is tight.

Additionally, we will improve upon Biswas-Saha's bounds on λ_n. We can do so two different ways. Applying Trevisan's results can improve the lower bound on λ_n. Also, Abelian Cayley graphs have been shown in KKRT to have non-negative curvature, and thus we can tightly bound their non-trivial eigenvalues using edge expansion. Using this approach, we will tightly bound the smallest eigenvalue of Cayley Sum Graphs. Lastly, we wish to investigate generalizing this result to general non-bipartite graphs.

Methods and Techniques

In order to redefine Trevisan's $\beta(G)$ in terms of edge/vertex expansion we may apply analysis techniques employed in Prasad Tetali's *On the Complexity and Approximation of λ_∞, Spread Constant and Maximum Variance Embedding*. We may also employ Trevisan's algorithm, a variant spectral partitioning, whose analysis involves semi-definite programming.

To improve upon Biswas-Saha's bounds on λ_n, we will employ the technique previously used in KKRT which tightly bounds the eigenvalues of the Laplacian with edge expansions. There are various other techniques to bound the λ_n that we may use, such as the Path Resistance Method for Bounding the Smallest Nontrivial Eigenvalue of a Laplacian. We may also leverage bounds on volume growth as discussed in Peter Ralli's thesis *Curvature and Isoperimetry in Graphs* while studying bounds on the spectrum of the Laplacian.