Biswas-Saha and Tevisan
Week 2

Nina Moorman

May 21, 2020

1 Notes from discussion on May 18th, 19th,...

It should be easy to show that

$$1 + \lambda_n \leq \beta(G)\,,$$

where (following Trevisan) λ_n is the smallest eigenvalue of the normalized adjacency matrix, $M := \frac{1}{d}A$, with $A := A(G)$ being the adjacency matrix of a d-regular graph G. Thus M is a symmetric matrix, with $M(i,j) = 1/d$ if there is an edge between the vertices i, j and zero otherwise.

Let us investigate $1 + \lambda_n$, since $\lambda_n = -1 + \delta$, for some small $\delta > 0$, as long as G is non-bipartite. For example, if we let h be the vertex expansion, then Biswas and Saha show that $\delta \geq h^4/d^8$, up to some constant.

Additionally, if we assume $\delta < 1$, then $|\lambda_n| = 1 - \delta$, so that $1 - |\lambda_n| = \delta$, the quantity that Trevisan comments on (in Section 5 (eq 8) of his paper). Thus it makes sense to study $\delta = 1 + \lambda_n$.

1.1 Easy side of the Inequalities in (8) of Trevisan

Let $G = (V, E)$ be a graph and let

$$\lambda_n = \min_{x \in \Re^n} \frac{\langle x, Mx \rangle}{\langle x, x \rangle},$$

where

$$\langle x, Mx \rangle = (2/d) \sum_{\{i,j\} \in E} x_i x_j\,, \quad \text{and} \quad \langle x, x \rangle = \sum_{i \in V} x_i^2\,.$$

the summation is over unordered pairs $\{i, j\}$ of edges, hence the factor 2 in front. (**Check?**) So any real "test function" x when plugged into $\frac{\langle x, Mx \rangle}{\langle x, x \rangle}$, gives an upper bound for λ_n, since λ_n involves minimization over all real-valued functions. Choosing x as below,

$$\begin{cases} i \in R & x_i = 1 \\ i \in V - (L \cup R) & x_i = 0 \\ i \in L & x_i = -1 \end{cases}$$

gives the following bound.

Claim. $1 + \lambda_n \leq \beta(G)$.

Indeed, for the above choice of x, we have that

$$1 + \frac{\langle x, Mx \rangle}{\langle x, x \rangle} = 1 + \frac{2}{d}(\sum_{i,j \in E} x_i x_j)/(\sum_i x_i^2) \,.$$

Let $S = (L \cup R)$. Then the RHS can be evaluated as

$$\begin{aligned} 1 + \frac{\langle x, Mx \rangle}{\langle x, x \rangle} &= \frac{d|S| + 2\#\text{edges}(L) + 2\#\text{edges}(R) - 2[\#\text{edges between } L \text{ and } R]}{d|S|} \\ &= \frac{2\#\text{edges}(L) + 2\#\text{edges}(R) + [\#\text{edges between } S \text{ and } V \setminus S]}{d|S|} \,. \end{aligned}$$

Now the optimal choice of S with the partition L and R achieving $\beta(G)$, together with the corresponding x as above, gives us the claim.

1.2 Moving Forward

To-do 1. We can try to lower bound $1 + \lambda_n$ by $\beta^2(G)$ (up to a constant) as Trevisan did. But perhaps we can "clean up" his proof some?

To-do 2. Rather than reproving his result, we can bound $\beta(G)$ from below by a suitable function of vertex or edge expansion and d, and use Trevisan's result, to derive hopefully a better result - hopefully for general non-bipartite graphs, and also better qualitatively than what Biswas and Saha obtained for Cayley (sum) graphs.

To-do 3. Let us first recall that under an assumption of "non-negative" curvature (in the sense of Bochner-Bakry-Émery), Klartag-Kozma-Ralli-T. showed that the Cheeger inequality – relating the smallest (nonzero) eigenvalue of the Laplacian of a graph to the edge expansion constant – is tight, up to a factor of d. In particular, such an inequality holds for Abelian Cayley graphs, since this class of graphs was shown by KKRT to have non-negative curvature.

The above suggests that perhaps one can also investigate sufficient criteria under which Trevisan's inequality - relating the smallest eigenvalue λ_n of the normalized adjacency matrix M of a graph G to $\beta(G)$ - is tight. So for example, what class of non-bipartite graphs satisfy $1 + \lambda_n(G) \leq \beta^2(G)$, up to a factors of d (say)?

Pure speculation. Perhaps this has something to do with the size of the odd cycle in G or the number of them, as a measure of how far from being partite a graph is? At least going over Trevisan's proof carefully might suggest a way of identifying some sufficient criterion, perhaps not the best possible?