Chapter 1

Circle split graphs containing a tent

1.1 Nested and 2-nested matrices

We begin by introducing some notation. Let A be a $(0,1)$-matrix. We define $a_{i,j}$ for $1 \leq i \leq n, 1 \leq j \leq m$, to the entry of A in row i and column j. We note $a_{.,j}$ to the column vector j of the matrix A and $a_{i,.}$ to the row vector i of the matrix A.

We define $l_i = \min_{1 \leq j \leq m}\{j \mid a_{i,j} = 1\}$ and $r_i = \max_{1 \leq j \leq m}\{j \mid a_{i,j} = 1\}$.

We say that two rows $a_{i,.}$ and $a_{k,.}$ are disjoint, if there is no column index $1 \leq j \leq m$ such that $a_{i,j} = a_{k,j} = 1$. We say that $a_{i,.}$ is nested (or included) in $a_{k,.}$ if, for every $1 \leq j \leq m$ such that $a_{i,j} = 1$, then $a_{k,j} = 1$.

Finally, we say that $a_{i,.}, a_{k,.}$ *start (respectively end) in the same column* if $l_i = l_k$ (respectively $r_i = r_k$), and we say $a_{i,.}, a_{k,.}$ *start (end) in different columns* if $l_i \neq l_k$ (respectively $r_i \neq r_k$).

Definition 1.1. *We say a $(0,1)$-matrix is* nested *if it has the consecutive ones property for the rows ($\mathsf{C1P}$) and every two rows are either disjoint or nested.*

Definition 1.2. *We say a $(0,1)$-matrix is* 2-nested *if it has the $\mathsf{C1P}$ for the rows, and there is a partition S_1, S_2 of the rows such that each submatrix obtained is nested.*

Tucker characterized all the minimal forbidden submatrices for the $\mathsf{C1P}$. See Figure 1.1 for the complete list of Tucker matrices.

Let $G = (K, S)$ a split graph, where $n = |S|, m = |K|$ let $S = \{s_1, \ldots, s_n\}$ and $K = \{v_1, \ldots, v_m\}$ be an ordering of S and K. Let $A = A(S, K)$ be the associated matrix of n rows and m columns such that

$$A\,(i,j) = \begin{cases} 1 & \text{if } s_i \text{ is adjacent to } v_j \\ 0 & \text{if } s_i \text{ is nonadjacent to } v_j \end{cases}$$

Thus the column j of A corresponds to the adjacency in S of the vertex v_j.

Correspondingly, we define nested and 2-nested graphs as follows.

Definition 1.3. *Let $G = (K, S)$ be a split graph with a linear ordering Π of K. G is said to be* nested *regarding Π if the associated matrix $A(S, K)$ is nested.*

$$M_I(k) = \begin{pmatrix} 110...00 \\ 011...00 \\ ..... \\ ..... \\ ..... \\ 000...11 \\ 100...01 \end{pmatrix} \quad M_{II}(k) = \begin{pmatrix} 011...111 \\ 110...000 \\ 011...000 \\ ..... \\ ..... \\ 000...110 \\ 111...101 \end{pmatrix} \quad M_{III}(k) = \begin{pmatrix} 110...000 \\ 011...000 \\ ..... \\ ..... \\ 000...110 \\ 011...101 \end{pmatrix}$$

$$M_{IV} = \begin{pmatrix} 110000 \\ 001100 \\ 000011 \\ 010101 \end{pmatrix} \qquad M_V = \begin{pmatrix} 11000 \\ 00110 \\ 11110 \\ 10011 \end{pmatrix}$$

Figure 1.1: Tucker matrices: $M_I(k) \in \{0,1\}^{k \times k}$, $M_{III}(k) \in \{0,1\}^{k \times k+1}$ with $k \geq 3$, and $M_{II}(k) \in \{0,1\}^{k \times k}$ with $k \geq 4$

Definition 1.4. *Let* $G = (K, S)$ *be a split graph with a linear ordering* Π *of* K. G *is said to be* nested *regarding* Π *if the associated matrix* $A(S, K)$ *is* 2*-nested.*

1.1.1 Nested matrices

Let A be a $(0,1)$-matrix. We define the auxiliary graph $H(A) = (V, E)$ where the vertex set $V = \{w_1, \dots, w_n\}$ has one vertex for each row in the matrix A, and two vertices $w_i, w_k \in V$ are adjacent if and only if the rows $a_{i,.}$ and $a_{k,.}$ are neither disjoint or nested.

When we refer to the vertex w_i, we will be speaking as needed in each case for abuse of language both of the vertex in $H(A)$ and the row array $a_{i,.}$ of A. Thus, the previous definitions apply to the vertices in $H(A)$: we say two vertices w_i and w_k in $H(A)$ are nested (respectively disjoint) if the corresponding rows $a_{i,.}, a_{k,.}$ are nested (disjoint). And two vertices w_i, w_k in $H(A)$ start (end) in the same column if the corresponding rows $a_{i,.}, a_{k,.}$ start (end) in the same column.

Suppose that the $(0,1)-$matrix A is nested. Since Tucker matrices (see Figure 1.1) are *all* the minimal forbidden submatrices for the C1P, thus the first condition of the definition of nested gives these matrices as forbidden submatrices.

Furthermore, the fact that every pair of rows are either disjoint or nested, implies that the gem matrix G_0 (see Figure 1.2) is a forbidden submatrix of A, for in this case, the two rows of the gem matrix are neither disjoint nor comparable. This matrix also represents a gem subgraph in the associated graph G. Moreover, observe that G_0 is the minimal submatrix with this property, and also every Tucker matrix has G_0 has a submatrix, thus G_0 is the only minimal submatrix for nested matrices. The following result is an inmediate consequence of the latter.

Theorem 1.1. *A* $(0, 1)$*-matrix is nested if and only if it does not contain* G_0 *as a submatrix.*

$$G_0 = \begin{pmatrix} 110 \\ 011 \end{pmatrix}$$

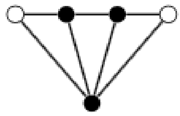


Figure 1.2: The gem matrix G_0 , and the gem subgraph in the associated graph G : black vertices are in K and white vertices in S

As a corolary, we have the following theorem for nested graphs.

Theorem 1.2. *A split graph* $G = (K, S)$ *is nested if and only if it does not contain any gem as an induced subgraph.*

1.1.2 2-nested matrices

Theorem 1.3. *A* $(0, 1)$*-matrix* A *is* $2-$*nested if and only if there is a linear ordering* Π *of the columns such that the matrix* A *with its column ordered according to* Π *does not contain any Tucker matrix, or* F_0*,* $F_1(k)$*,* $F_2(k)$ *for* $k \geq 5$ *(see Figure 1.1.2) as submatrices.*

$$F_0 = \begin{pmatrix} 11100 \\ 01110 \\ 00111 \end{pmatrix} \qquad F_1(k) = \begin{pmatrix} 011...111 \\ 111...110 \\ 000...011 \\ 000...110 \\ ..... \\ ..... \\ ..... \\ 110...000 \end{pmatrix} \qquad F_2(k) = \begin{pmatrix} 0111...10 \\ 1100...00 \\ 0110...00 \\ ..... \\ ..... \\ ..... \\ 0000...11 \end{pmatrix}$$

Figure 1.3: The matrices F_0, $F_1(k) \in \{0, 1\}^{k \times k-1}$, and $F_2(k) \in \{0, 1\}^{k \times k}$, with $k \geq 5$.

It follows from the definition of 1.4, that the existence of such partition for the rows of the matrix A is equivalent to having a bicoloring of the auxiliary graph $H(A)$ defined before. Thus, is equivalent to asking $H(A)$ to be a bipartite graph. Recall that bipartite graphs have the odd chordless cycles as minimal forbidden induced subgraphs.

Proof. $\Leftarrow$) Let Π be a linear ordering of the columns such that the matrix A does not contain any $F_0, F_1(k), F_2(k)$ or Tucker matrices as submatrices, for $k \geq 5$.

Due to Tucker's Theorem, since there are no Tucker submatrices in A, we may assert that the matrix A has the $C1P$ and hence the first condition of the definition of $2-$nested holds.

Toward a contradiction, suppose that even though A has the $C1P$, the auxiliary graph $H(A)$ does not admit a bicoloring of its vertices. Thus, there is an induced odd cycle in $H(A)$. We will show that, depending on the length of the odd cycle, we can find F_0, $F_1(k)$ or $F_2(k)$ as submatrices of A, and thus reaching a contradiction.

Suppose first that $H(A)$ has an induced odd cycle C of length 3, and suppose without loss of generality that $n = 3$, thus the only rows in A are those corresponding to the vertices of C. We will find F_0 as a submatrix of A.

Since w_1, w_2 are adjacent, w_1 and w_2 begin and end in different columns. The same holds for w_2, w_3 and w_1, w_3. We may assume without loss of generality that the vertices start in the order of the cycle, meaning that w_1 starts first, w_2 starts second and the final vertex to start is w_3, hence $l_1 < l_2 < l_3$.

Since w_1 starts first, it is clear that $a_{2,l_1} = a_{3,l_1} = 0$, thus the column $a_{.,l_1}$ of A is equal to the first column of the matrix F_0.

Since A has the $C1P$ and w_1 and w_2 are adjacent, then $a_{1,l_2} = 1$. As stated before, w_2 starts before w_3 and thus $a_{3,l_2} = 0$. Hence, the column $a_{.,l_2}$ is exactly the second column of the matrix F_0.

The third column of F_0 will be $a_{.,l_3}$: since w_1 and w_3 are adjacent and w_2, w_3 are adjacent, it is clear that $a_{1,l_3} = a_{2,l_3} = a_{3,l_3} = 1$.

For the next column of F_0, let us look at column $a_{.,r_1+1}$. Note that $r_1 + 1 > l_3$. Since w_1, w_2 are adjacent and w_1, w_3 are adjacent, and w_2, w_3 both start after w_1, then necesarily $a_{2,r_1+1} = a_{3,r_1+1} = 1$, and thus $a_{.,r_1+1}$ is exactly the fourth column of F_0.

Finally, we look at the column $a_{.,r_2+1}$. Observe that $r_2 + 1 > r_1 + 1$. Since A has the $C1P$, $a_{1,r_2+1} = 0$ and $r_2 + 1 > r_1 + 1$, using a similar argument as before, $a_{1,r_2+1} = 0$ and $a_{3,r_2+1} = 1$, which gives us the final column of F_0, and therefore a contradiction that came from assuming that there is a cycle of length 3 in $H(A)$.

Suppose now that $H(A)$ has an induced odd cycle C of length $k \geq 5$. Once again, suppose without loss of generality that the only rows in A are those corresponding to the vertices of C.

Observation 1. *Let w_i, w_k be nonadjacent vertices in $H(A)$. Then, either w_i, w_k are disjoint or nested. Also, if w_i, w_k are adjacent and w_i starts after w_k, then $a_{i,r_1} = a_{k,r_1} = 1$ and $a_{i,r_1-1} = 0, a_{k,r_1-1} = 1$*

We will now state some Lemmas which will be useful throughout the proof.

Lemma 1. *Let x, y and z be rows such that w_y and w_z are adjacent but none of them is adjacent to w_x. Then, y is contained in x if and only if z is contained in x.*

Suppose first that y is contained in x, and toward a contradiction, suppose that z is not contained in x. If z and x are disjoint, then we find a contradiction since in this case w_y and w_z cannot be adjacent for y is contained in x.

If instead z and x are not disjoint, since w_x and w_z are nonadjacent, then x is contained in z. Since y is contained in x by hypothesis, thus y is contained in z and w_y would not be adjacent to w_z, resulting again in a contradiction.

The argument is symmetric for the if case. □

Lemma 2. *Let x, y, z, and w rows such that $< w_x, w_y, w_z, w_t >$ is a chordless path such that w_x and w_z are disjoint, w_x and w_t are disjoint and w_y and w_t are disjoint.*

Then, $r_z < l_x$ if and only if $r_t < l_y$.

Suppose that $r_z < l_x$. Since w_z is adjacent to w_y, either $l_y < l_z < r_y$ or $l_z < l_y < r_z$. Since w_z and w_x are disjoint and w_y is adjacent to w_x, then necesarily $l_z < l_y < r_z$. Also, since w_z is adjacent to w_t and w_t and w_y are disjoint, $r_t < l_z$ and hence $r_t < l_z < l_y$.

The if case is symmetric. □

We split this in two possible cases.

Case 1: There is a column l such that $a_{1,j} = 0$ for $j < l$ and $a_{1,j} = 1$ for $j \geq l$.

Claim 1.1.1. *Under the previous hypothesis, w_2 and w_k are nested.*

In this case w_2 and w_k are not disjoint, for the string of 0's of row $a_{1,.}$ is placed at the beginning, and since both vertices are adjacent to w_1 in $H(A)$, there are column indices $j_1, j_2 < l$ such that $a_{1,j_1} = a_{1,j_2} = 0$ and $a_{2,j_1} = a_{k,j_2} = 1$. Thus, since w_2 and w_k are nonadjacent in $H(A)$, either w_k is nested in w_2 or viceversa. □

By Claim 1.1.1, we suppose without loss of generality that w_k is included in w_2.

Moreover, we suppose without loss of generality that $a_{1.,}$ is the row with the largest string of 1's, for every row that ends in 1, for if not, we can rearrange the rows of the cycle in such a way.

Let $j_1 = l_1 - 1$. We want to see that $a_{.,j_1} = (010...01)^t$. Since w_1 is adjacent to w_2 and w_k, and in this case w_1 starts after w_2 and w_k, hence we know that $a_{2,j_1} = a_{k,j_1} = 1$.

Since w_1, w_3 are nonadjacent, then either w_1 and w_3 are disjoint or nested.

Also, since the row $a_{1,.}$ has the longest string of 1's for every row that ends in 1, w_1 cannot be nested in w_3. Thus, we can split the proof in two cases.

Case 1.1 w_3 is included in w_1.

Observe that this implies that every 1 in the row corresponding w_3 is in a column greater than j_1, thus $a_{3,j_1} = 0$.

Since w_3, w_4 are adjacent and w_1, w_4 are nonadjacent, w_4 is included in w_1, and thus $a_{4,j_1} = 0$. By applying Lemma 1 successively to $x = w_1$, $y = w_{i-1}$ y $z = w_i$, we conclude that w_i is contained in w_1, and thus $a_{i,j_1} = 0$ for every $3 \leq i \leq k-1$. Therefore $a_{.,j_1} = (010...01)^t$, for every $k > 5$.

Let $j_2 = r_k$. We want to see that $a_{.,j_2} = (110...011)^t$.

Claim 1.1.2. *Under the previous hypothesis, w_{k-1} starts after w_k.*

If w_{k-1} starts before w_k, then for every w_i, $i = 4, \ldots, k-1$, the right end r_i is smaller that l_1, since every w_i is nonadjacent to w_1. Hence, since w_3 is included in w_1, w_4 and w_3 cannot be adjacent, which results in a contradiction. □

Observe that, since $j_2 > j_1$, then $a_{1,j_2} = 1$. Moreover, since w_k is included in w_2, $a_{2,j_2} = 1$, and since w_{k-1} and w_k are adjacent, by Claim 1.1.2 we know that $a_{k-1,j_2} = 1$.

Now, we want to see that $a_{i,j_2} = 0$ for $3 \le i \le k-2$. First, observe that since $a_{k,j_2} = 1$, then $a_{3,j_2} = 0$, for w_3 is included in w_1 and w_3 is nonadjacent to w_k, thus $1 = a_{k,j_2} \neq a_{3,j_2}$.

Claim 1.1.3. *Under the previous hypothesis, w_4 starts before w_3. The same holds for every w_i nonadjacent to w_k, $i \ge 5$.*

If w_4 starts after w_3, then either w_i is included in w_3 for $i \ge 5$ –which results in w_k being nonadjacent to w_{k-1} or w_k being adjacent to w_3, both contradictions–, or $l_i \ge r_3$ and $l_i \ge r_2$, thus w_{k-1} is nonadjacent to w_k, which again results in a contradiction. The same argument holds for w_i nonadjacent to w_k, using w_{i-2} instead of w_3. □

Since w_3 is included in w_1, by Claim 1.1.3, $a_{i,j_2} = 0$ for every $3 \le i \le k-2$ and therefore $a_{.,j_2} = (110...011)^t$.

For the steps $i = 3, \dots, k-1$, let $j_i = r_{k-i+2}$. In each step, we want to see that $a_{.,j_i} = (110...0110...0)^t$, where the last 1 in this column corresponds to row $k-i+2$.

Observation 2. *Since w_i is included in w_1 for $i = 3, \dots, k-2$, then $j_1 < j_2 < j_3 < \dots < j_{k-2}$, and thus for each step i, $a_{1,j_i} = 1$, and $a_{k,j_i} = \dots = a_{k-i+3,j_i} = 0$.*

Furthermore, since j_i is the last column (from left to right) for which the row $k-i+2$ has entry 1 and w_{k-i+2}, w_{k-i+1} are adjacent, thus $a_{k-i+1,j_i} = 1$.

Moreover, since w_3, w_{k-i+2} are nonadjacent and w_3 is included in w_1, thus $a_{3,j_i} = 0$. The same argument holds inductively for every $i \ge 3$ such that w_i, w_{k-i+2} are nonadjacent, always using the previous row to move forward with the argument.

It follows from the previous Observation that $a_{.,j_i} = (110...0110...0)^t$.

For the final step $k-1$, let $j_{k-1} = r_3$. By the definition of the previous indices and since $j_1 < j_2 < \dots < j_{k-2} < j_{k-1}$, $a_{i,j_{k-1}} = 0$ for $i = 4, \dots, k$, and $a_{1,j_{k-1}} = 1$.

Since w_2, w_3 are adjacent and j_{k-1} is the last column for which row 3 has entry 1, thus $a_{2,j_{k-1}} = 0$ and therefore $a_{.,j_{k-1}} = (1010...0)^t$, completing with this the last column to form $F_1(k)$ as a submatrix of A.

Case 1.2: w_1 and w_3 are disjoint, i.e. $a_{3,j_1} = 0$ for every $j > j_1$.

Since $a_{3,j_1+1} = 0$, $r_3 < j_1$.

In particular, there is a column index $j \le j_1$ such that $a_{3,j} = 1, a_{2,j} = 1$ and $a_{3,j-1} = 1, a_{2,j-1} = 0$.

Analogously, since w_4 is nonadjacent to w_2, then either w_2 and w_4 are disjoint or nested.

Case 1.2.1: w_2 and w_4 are disjoint.

Then, it is straightforward that $r_4 < l_2 < l_1 = j_1 + 1$.

Claim 1.1.4. *Under the previous hypothesis, w_{i+1} starts before w_i for $i = 5, \dots, k-1$.*

Let $i = 5, \dots, k-1$ such that $l_i < l_{i+1}$. In this case, w_j is included in w_{i-2} for every $j = i, \dots, k-1$. Hence, w_k and w_{i-2} are adjacent, resulting in a contradiction since $i-2 \ge 3$. □

Using Lemma 2 and Claim 1.1.4, we may assert that $r_i < l_{i-2}$ by applying the same argument inductively to the chordless path $w_{i-3}, w_{i-2}, w_{i-1}, w_i$ for

$i=5,\dots,k-1$. Thus, $r_{k-1} < l_{k-3} < \dots < l_2$, and since $w_2$and w_k are nested, $l_2 \leq l_k$. Hence, $r_{k-1} < l_k$ and therefore w_{k-1} and w_k are nonadjacent, which results in a contradiction.

Case 1.2.2: w_2 and w_4 are nested.

From the fact that $r_3 \leq j_1$, w_3, w_4 are adjacent and w_2, w_4 are nonadjacent, it follows that w_4 is included in w_2 and thus $r_4 < j_1 + 1 = l_1$. This argument can be applied inductively to assert that w_i is included in w_2 for $i=5,\dots,k-1$. Since w_3, w_k are nonadjacent, then it has to be $r_3 > r_4 > \dots > r_{k-1} > r_k$, for if not, then we would reach a contradiction.

We will rearrange the rows in A as follows:

Recall that $a_{2,j_1} = a_{2,j_1+1} = a_{1,m} = 1$, and since w_1, w_2 are adjacent, $a_{2,m} = 0$. Moreover, since w_2, w_3 are adjacent and w_1, w_3 are disjoint, then $a_{2,1} = 0$, thus w_2 starts after the first column and ends before the last column. We will place the first row at the bottom, thus making it the last row of the new matrix $A^{'}$. The other rows remain in the same position. Observe that this new matrix $A^{'}$ represents a the same cycle of length k in $H(A)$ and the rows are ordered in such a way that w_i, w_{i+1} are adjacent for $i=1,\dots,k-1$ and w_1, w_k are adjacent.

Hence, we may assume $A = A^{'}$ and reduce this subcase entirely to Case 2.2.

Case 2: $1 < l_1, r_1 < m$, i.e. the string of 1's of row $a_{1,.}$ is both preceded and followed by 0's.

Claim 1.1.5. *Since row* 1 *has 0 in the beginning and at the end, if* w_2, w_k *are nested, then we can reduce this to the previous Case.*

Suppose that $l_2, l_k < l_1$, and thus $a_{2,l_1} = a_{k,l_1} = 1$. Hence, w_2 and w_k are nested. Suppose without loss of generality that w_k is included in w_2. It follows from this fact that either w_i is nested in w_1 and w_2 for $i=4,\dots,k-1$ or $r_i < l_1$ for $i=3,\dots,k-1$. Hence, the fact that $a_{1,m} = 0$ has no influence in the proof and we procede as in the previous Case by taking $j_1,\dots,j_{k-1}$ as in Case 1.1 or Case 2.2. □

By Claim 1.1.5 we may assume without loss of generality that w_2, w_k are disjoint.

Once again, we assume that row $a_{1,.}$ has the largest string of 1's with this property.

Let $j_1 = l_1 - 1$.

We may assume that $l_2 = 0$, and $r_k = m$, meaning that the string of 1's of row 2 is at the beggining of the row, and the string of 1's of row k is at the end of the row.

We want to see that $a_{.,j_1} = (010...0)^t$. For what has been stated before, it is clear that $a_{1,j_1} = a_{k,j_1} = 0$, and $a_{2,j_1} = 1$ for w_1, w_2 are adjacent.

Case 2.1: Toward a contradiction, suppose that $a_{3,j_1} = 1$.

In this case we have two possibilities, either
w_1 and w_3 are disjoint, or w_1 is included in w_3.

Case 2.1.1: w_1 and w_3 are disjoint. Since w_2, w_4 are adjacent, either w_4 is included in w_2 or
w_2, w_4 are disjoint. If w_4 is included in w_2, given that w_2, w_i are nonadjacent for $i=4,\dots,k$, then w_i is included in w_2 and thus w_k, w_{k-1} are nonadjacent, which results in a contradiction, and hence necesarily w_2 and w_4 are disjoint.

Observe that the same argument holds for the rows i, where $i = 5, \dots, k-1$, using row $i-2$ instead of row 2. Hence, w_{i_1} and w_{i_2} are disjoint, for $i_1 = 2, \dots, i_2 - 2$ and $i_2 = 5, \dots, k-1$.

In particular, w_{k-1} and w_k are disjoint, since the string of 1's in row $k-1$ is placed at the beggining of the row, and we assumed that the string of 1's of row k is placed at the end of the row, thus reaching a contradiction.

Case 2.1.2: w_1 and w_3 are nested.

Since $a_{1,r_1} = a_{k,r_1} = 1$, thus $a_{3,r_1} = 0$, and also recall that $a_{3,j_1} = 1, a_{k,j_1} = 0$ by hypothesis. Hence, if $a_{3,j} = 0$ for $j > j_1$, then $a_{k,j} = 0$ for $j > j_1$, since otherwise w_3, w_k would be adjacent, which would be a contradiction and thus w_k is included in w_3.

Morever, either w_4 is included in w_2 –which, for a similar argument as the one stated in the previous case is not possible–, or w_4 is included in w_k and also w_4 is included in w_1. Thus, w_i is included in w_1 and w_i, w_k are disjoint for $i = 4, \dots, k-2$.

Since w_{k-1}, w_k are adjacent, there are column indices $m_1, m_2, m_3 > j_1$ such that $a_{k-1,m_1} = 1, a_{k,m_1} = 0$, $a_{k,m_2} = a_{k-1,m_2} = 1$ and $a_{k-1,m_3} = 0, a_{k,m_3} = 1$. However, since w_i is included in w_1, w_i, w_k are disjoint for $i = 4, \dots, k-2$ and $a_{k-1,m_1} = 1$, $a_{4,m_1} = 0$, and $a_{4,m_2} = a_{4,m_3} = 1$ for w_4 is included in w_k, hence w_4, w_{k-1} are adjacent and this results in a contradiction, which came from the asumption that $a_{3,j_1} = 1$.

Case 2.2: $a_{3,j_1} = 0$.

Once again, since w_1, w_3 are nonadjacent, either w_1, w_3 are disjoint or w_3 is included in w_1 (the case in which w_1 is included in w_3 is analogous by symmetry to the previous case).

Applying a similar argument to the one used in the previous cases, if w_1, w_3 are disjoint, then w_i is included in w_3 for $i = 4, \dots, k-1$, resulting in w_k, w_{k-1} nonadjacent, which is a contradiction.

Thus, necesarily w_3 is included in w_1. Since w_1, w_4 are nonadjacent and w_3, w_4 are not disjoint, in particular w_1 and w_4 are not disjoint and hence w_4 is included in w_1. We can use this argument inductively to assert that w_i is included in w_1 for $i = 3, \dots, k-1$.

We define the indices $j_l = r_l$ for $l = 2 \dots, k$. Using that w_i is included in w_1 for $i = 3, \dots, k-1$ and similar arguments as above, we can see that

$$a_{.,j_2} = \begin{pmatrix} 1\\1\\1\\0\\0\\0\\ \vdots \\0\\0\\0 \end{pmatrix} \quad a_{.,j_3} = \begin{pmatrix} 1\\0\\1\\1\\0\\0\\ \vdots \\0\\0\\0 \end{pmatrix} \quad a_{.,j_4} = \begin{pmatrix} 1\\0\\0\\1\\1\\0\\ \vdots \\0\\0\\0 \end{pmatrix} \quad \cdots \quad a_{.,j_{k-1}} = \begin{pmatrix} 1\\0\\0\\0\\0\\0\\ \vdots \\0\\1\\1 \end{pmatrix} \quad a_{.,j_k} = \begin{pmatrix} 1\\0\\0\\0\\0\\0\\ \vdots \\0\\0\\1 \end{pmatrix}$$

Therefore we have $F_2(k)$ as a submatrix of A, and this finishes this part of the proof.

$\Rightarrow$) It follows from the first part of the definition of 2−nested that the matrix A admits an ordering Π for which A has the C1P, and for the second part, it follows that A does not contain $\mathsf{F}_0, \mathsf{F}_1(\mathsf{k})$ or $\mathsf{F}_2(\mathsf{k})$ as submatrices for in that case the auxiliary graph $\mathsf{H}(\mathsf{A})$ would have an induced odd chordless cycle and this is not possible.

□

1.2 Full LRS-sortable matrices

1.2.1 Admissibility

Definition 1.5. *Let* A *be a* $(0,1)$*-matrix. We say* A *is an* enriched matrix *if some rows of* A *are marked with L or R each, and also if some of the rows of* A*, including all the rows marked with an L or an R, are colored with blue or red each.*

We define the *column extension of* A as the matrix A_{extC} obtained by adding two distinguished columns $\mathsf{c_L}$ and $\mathsf{c_R}$ to the matrix A such that, if r is a row of the matrix A, then the column $\mathsf{c_L}$ has a 1 if r is marked L and 0 if it is not marked at all, and similarly, the column $\mathsf{c_R}$ has a 1 if r is marked R and 0 if it is not marked at all. These distinguished columns will be refered to as *tag columns*.

Let B be a submatrix of A_{extC} such that B has exactly one tag column $\mathsf{c_L}$. We define *the dual matrix of* B as the sole matrix $\mathsf{B_R}$ such that $\mathsf{c_R}(\mathsf{B}^*) = \mathsf{c_L}(\mathsf{B})$. Analogously, we define $\mathsf{B_L}$ when the tag column of B is $\mathsf{c_R}$.

We use green and orange as distinct non-prescribed colors, which may be either red or blue.

Definition 1.6. *Let* A *be an enriched matrix. We say* A *is* admissible *if all of the following conditions hold:*

1. *if* r_1 *and* r_2 *are marked with the same letter, then they are nested.*
2. *If* r_1 *and* r_2 *are marked with distinct letters and have the same color, then they are disjoint.*
3. *If* r_1 *and* r_2 *are marked with distinct letters (and have distinct colors), then either they are disjoint or there is no column* j *such that* $\mathsf{r}_1(\mathsf{j}) = \mathsf{r}_2(\mathsf{j}) = 0$*.*
4. A_{extC} *does not contain any* $\mathsf{S}_1(\mathsf{k}), \mathsf{S}_2(\mathsf{k})$ *or their dual matrices as submatrices, for* $\mathsf{k} \geq 3$*.*

For each of the properties that define an admissible matrix, we will characterize every minimal forbidden induced submatrix.

First, we want to find every forbidden submatrix given by statement **??** of the definition of admissibility.

Let r_1 and r_2 be two rows marked with the same letter. Since the color of each row is irrelevant in the definition, we find the following minimal forbidden submatrix in A_{extC}, putting aside the coloring of the rows:

$$\mathsf{D}_0 = \begin{pmatrix} \mathbf{1} & 1 & 0 \\ \mathbf{1} & 0 & 1 \end{pmatrix}$$

$$S_1(2j+1) = \begin{pmatrix} \mathbf{1} & 1 & 0 & \dots & 0 & 0 & \mathbf{0} \\ \mathbf{0} & 1 & 1 & \dots & 0 & 0 & \mathbf{0} \\ . & . & . & . & . & . & \mathbf{0} \\ \mathbf{0} & 0 & 0 & \dots & 1 & 1 & \mathbf{0} \\ \mathbf{1} & 1 & 1 & \dots & 1 & 0 & \mathbf{0} \end{pmatrix} \quad S_2(2j+1) = \begin{pmatrix} \mathbf{1} & 1 & 0 & \dots & 0 & 0 & \mathbf{0} \\ \mathbf{0} & 1 & 1 & \dots & 0 & 0 & \mathbf{0} \\ . & . & . & . & . & . & . \\ \mathbf{0} & 0 & 0 & \dots & 1 & 1 & \mathbf{0} \\ \mathbf{0} & 0 & 0 & \dots & 0 & 1 & \mathbf{1} \end{pmatrix}$$

$$S_1(2j) = \begin{pmatrix} \mathbf{1} & 1 & 0 & \dots & 0 & 0 & \mathbf{0} \\ \mathbf{0} & 1 & 1 & \dots & 0 & 0 & \mathbf{0} \\ . & . & . & . & . & . & \mathbf{0} \\ \mathbf{0} & 0 & 0 & \dots & 1 & 1 & \mathbf{0} \\ \mathbf{1} & 1 & 1 & \dots & 1 & 0 & \mathbf{0} \end{pmatrix} \quad S_2(2j) = \begin{pmatrix} \mathbf{1} & 1 & 0 & \dots & 0 & 0 & \mathbf{0} \\ \mathbf{0} & 1 & 1 & \dots & 0 & 0 & \mathbf{0} \\ . & . & . & . & . & . & . \\ \mathbf{0} & 0 & 0 & \dots & 1 & 1 & \mathbf{0} \\ \mathbf{0} & 0 & 0 & \dots & 0 & 1 & \mathbf{1} \end{pmatrix}$$

Figure 1.4: Matrices $S_1(k), S_2(k) \in \{0,1\}^{k \times k+1}$ for $k \geq 3$

Let us find now every forbidden submatrix given by statement **??**. Let r_1 and r_2 be rows marked with distinct letters and colored with the same color. In this case, we find the following forbidden submatrix in A_{extC}:

$$D_1 = \begin{pmatrix} \mathbf{1} & 1 & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{1} \end{pmatrix}$$

Finally, for statement 3 of the definition of admissibility, let r_1 and r_2 be two rows marked with distinct letters and colored with distinct colors, and suppose they are not disjoint and there is a column j such that $r_1(j) = r_2(j) = 0$. Then, we find the following forbidden submatrix in A_{extC}:

$$D_2 = \begin{pmatrix} \mathbf{1} & 1 & 0 & \mathbf{0} \\ \mathbf{0} & 1 & 0 & \mathbf{1} \end{pmatrix}$$

Notice that, even though in the hypothesis the rows r_1 and r_2 are colored with distinct colors, we can assert that the uncolored matrix D_2 is forbidden, since every possible coloring of the rows is forbidden if we observe that the underlying uncolored matrix D_1 is a submatrix of D_2.

Hence, A is admissible if and only if A_{extC} is $\{D_0, D_1, D_2, S_1(k), S_2(k)\}$-free.

1.2.2 LR-sortable

Definition 1.7. *Let A be an enriched matrix. We say A is* LR-sortable *if A is admissible and there is a linear ordering Π for the columns of A such that each of the following assertions holds:*

- *Π is a consecutive-ones ordering for the rows of A.*
- *The ordering Π is such that the rows marked with L start at the first column and those marked with R end at the last column.*

Definition 1.8. *Let A be an enriched matrix. We say A is* LRS-sortable *if A is LR-sortable and each pair of rows colored with the same color are either disjoint or nested.*

Let A be an enriched matrix. We define the *extended matrix of* A as the matrix A_{ext} obtained by adding two tag columns to the matrix A as in A_{extC}, and two distinguished rows: $(1, \dots, 1, 0)$ as the first row and $(0, 1, \dots, 1)$ as the last row.

Remark 1.1. If A_{ext} has the C1P, then the distinguished rows force the tag columns c_L and c_R to be the first and last columns of A_{ext}, respectively.

Remark 1.2. An admissible matrix A is LR-sortable if and only if the extended matrix A_{ext} has the C1P for the rows.

$$M_2'(k) = \begin{pmatrix} \mathbf{0} & 1 & 1 & \dots & 1 & 1 & 1 \\ \mathbf{1} & 1 & 0 & \dots & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ \mathbf{0} & 0 & 0 & \dots & 1 & 1 & 0 \\ \mathbf{1} & 1 & 1 & \dots & 1 & 0 & 1 \end{pmatrix} \qquad M_3'(k) = \begin{pmatrix} \mathbf{1} & 1 & 0 & \dots & 0 & 0 & 0 \\ \mathbf{0} & 1 & 1 & \dots & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ \mathbf{0} & 0 & 0 & \dots & 1 & 1 & 0 \\ \mathbf{0} & 1 & 1 & \dots & 1 & 0 & 1 \end{pmatrix}$$

$$M_3''(k) = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 & \mathbf{0} \\ 0 & 1 & 1 & \dots & 0 & 0 & \mathbf{0} \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 & 1 & \mathbf{0} \\ 0 & 1 & 1 & \dots & 1 & 0 & \mathbf{1} \end{pmatrix} \qquad M_4' = \begin{pmatrix} \mathbf{1} & 1 & 0 & 0 & 0 & 0 \\ \mathbf{0} & 0 & 1 & 1 & 0 & 0 \\ \mathbf{0} & 0 & 0 & 0 & 1 & 1 \\ \mathbf{0} & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$M_4'' = \begin{pmatrix} \mathbf{1} & 1 & \mathbf{0} & 0 & 0 & 0 \\ \mathbf{0} & 0 & \mathbf{1} & 1 & 0 & 0 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & 1 & 1 \\ \mathbf{0} & 1 & \mathbf{0} & 1 & 0 & 1 \end{pmatrix} \quad M_5' = \begin{pmatrix} 1 & 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & 1 & \mathbf{0} \\ 1 & 0 & 0 & 1 & \mathbf{1} \\ 1 & 1 & 1 & 1 & \mathbf{0} \end{pmatrix} \quad M_5'' = \begin{pmatrix} 1 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{0} & 1 & 1 & 0 \\ 1 & \mathbf{0} & 0 & 1 & 1 \\ 1 & \mathbf{1} & 1 & 1 & 0 \end{pmatrix}$$

Figure 1.5: Forbidden submatrices with tag columns for LR-sortable, $M_2'(k)$, $M_2''(k)$, $M_3'(k)$, $M_3''(k)$, $M_3'''(k)$ with $k \geq 4$. If there is only one bold column, then it corresponds to the tag column c_L.

Theorem 1.4. *An admissible matrix* A *is LR-sortable if and only if the extended matrix* A_{ext} *does not contain any Tucker matrices, or* $M_2'(k)$, $M_3'(k)$, $M_3''(k)$ *for* $k \geq 3$, M_4', M_4'', M_5', M_5'' *or their dual matrices as submatrices.*

$\Rightarrow$) This follows from the last remark.

$\Leftarrow$) Suppose that the extended matrix A_{ext} does not contain any of the above listed submatrices, and still the C1P does not hold for the rows of A_{ext}.

Hence, there is a Tucker matrix M such that M is a submatrix of A_{ext}.

Suppose without loss of generality that, if M intersects only one tag column, then this tag column is c_L, since the analysis is symmetric if assumed otherwise and gives as a result in each case the dual matrix.

Case 1. Suppose first that M intersects one or both of the distinguished rows. Thus, the underlying matrix of M (i.e., the matrix without the tags) is either M_V, *or* $M_I(3)$, *or* $M_{II}(k)$ *for some* $k \geq 3$. We consider each case separately.

Case 1.1. $M_V = \begin{pmatrix} 11000 \\ 00110 \\ 11110 \\ 10011 \end{pmatrix}$

In this case, the distinguished row must be $(1,1,1,1,0)$ and thus the column with 0 is a tag column. Hence $M = M'_5$, since there are no further restrictions given by the bicoloring of the rows, which results in a contradiction.

Case 1.2. $M_I(3) = \begin{pmatrix} 110 \\ 011 \\ 101 \end{pmatrix}$

If $(1,1,0)$ is a distinguished row, then we find D_0 as a forbidden submatrix given by the second and third rows. It is symmetric if the distinguished row is either the second or the third row, and therefore this case is not possible.

Case 1.3. $M_{II}(k) = \begin{pmatrix} 011...111 \\ 110...000 \\ 011...000 \\ ..... \\ ..... \\ 000...110 \\ 111...101 \end{pmatrix}$

In this case, the only distinguished rows may be the first and the last row.

Suppose only the first row $(0,1,\dots,1)$ of M is a distinguished row. Thus, the first column is a tag column. If $k = 2j$, then the second and last row are colored with the same color, and if $k = 2j+1$, then the second and last row are colored with distinct colors. In both cases, if this coloring does not hold, then we find $S_1(k)$ as a submatrix. Hence, $M'_2(k)$ is a submatrix of A_{ext}, with a coloring of the rows given by the parity of k. The same holds if instead the last row is the sole distinguished row.

Finally, suppose both the first and the last row are distinguished.

First, notice that the first and second rows must be colored with distinct colors, for if not there is D_1 as a submatrix. The same holds for the last two rows of M. Hence, if $k = 4$, then we find D_3 as a submatrix given by the second and third rows. If instead $k > 5$, then we find $S_1(k)$ as a submatrix by taking M without the first and last rows, and therefore this case is not possible either.

Case 2. Suppose that M does not intersect any distinguished row.

If M does not have any tag column, then M is a submatrix of A. Thus, A does not have the C1P and we conclude that M is a Tucker matrix.

Suppose that instead one of the columns in M is a tag column.

Case 2.1 $M_I(k) = \begin{pmatrix} 110...00 \\ 011...00 \\ ..... \\ ..... \\ ..... \\ 000...11 \\ 100...01 \end{pmatrix}$ for some $k \geq 3$.

Notice that, if any of the columns is a tag column, then we find D_0 as a submatrix, which results in A not being admissible and thus reaching a contradiction.

Case 2.2 $M_{II}(k) = \begin{pmatrix} 011...111 \\ 110...000 \\ 011...000 \\ ..... \\ ..... \\ 000...110 \\ 111...101 \end{pmatrix}$ for some $k \geq 4$

As in the previous case, some of the columns are not elegible for being tag columns. If there is only one tag column, the only remaining possibilities for tag columns are column 1 or column $k-1$, for in any other case we find D_0 as a submatrix. Analoguosly, if instead M intersects both tag columns, then these columns are also columns 1 and $k-1$.

However, if c_L is either column 1 or column $k-1$, then we find $M'_2(k)$ as a submatrix, since we can reorder the columns of $M_{II}(k)$ to have the same disposition of the rows but having column $k-1$ as the first column. If instead the tag column is c_R, then we find the dual matrix of $M'_2(k)$.

Finally, if both columns are tag columns, then we reach the same contradiction as in Case 1. 3.

Case 2.3 $M_{III}(k) = \begin{pmatrix} 110...000 \\ 011...000 \\ ..... \\ ..... \\ 000...110 \\ 011...101 \end{pmatrix}$ for some $k \geq 3$.

In this case, the only possibilities for tag columns are column 1, column $k-1$ and column k, for if not we find D_0 as a submatrix. Once more, it is easy to see that we can reorder the columns in such a way to have the same disposition of the rows with column $k-1$ or column k replacing column 1.

Suppose first that the tag column is the first column. In that case, we find $M'_3(k)$ as a submatrix of M, which also results in a contradiction since M is admissible.

If instead the tag column is column k, then we use an analogous reasoning to find $M''_3(k)$ as a submatrix and thus reaching a contradiction.

Suppose now that both the first column and the last column of M are tag columns.

Since M is admissible, this case is not possible for the first and last row induce D_3 as a submatrix, whichever is the coloring of these rows.

Case 2.4 $M_{IV} = \begin{pmatrix} 110000 \\ 001100 \\ 000011 \\ 010101 \end{pmatrix}$

In this case, the only elegible columns for being tag columns are column 1, column 3 and column 5, since if any other column is a tag column, we find D_0 as a submatrix, thus contradicting the hypothesis of pre-admissibility for M and thus for A. Furthermore, the election of the tag column is symmetric since there is a reordering of the rows that allows us to obtain the same matrix if the tag column is either column 1, column 3 or column 5, disregarding the election of the column. Hence, we have two possibilities: when column 1 is the sole tag column of M, and when the two tag columns are columns 1 and 3. If column 1 is the only tag column, then we find M'_4 as a submatrix. If instead the columns 1 and 3 are both tag columns, then the first row and the second row are colored with the same color, for if not there is $S_2(3)$ as a submatrix and this is not possible since M is admissible. Thus, in this case we find M''_4 as a submatrix.

Case 2.5 $M_V = \begin{pmatrix} 11000 \\ 00110 \\ 11110 \\ 10011 \end{pmatrix}$

Once more and using the same argument, the only elegible columns for being tag columns are columns 2, 3 or 5. Moreover, if the second column is the sole tag column, then there is a reordering of the rows such that the matrix obtained is the same as the matrix when the third column is the tag column. If column 5 is the only tag column, then we find M'_5 as in Case 1. 1. If instead column 2 is the only tag column, then the first and second rows have the same color, for if not we find $S_1(3)$ as a submatrix of M, and thus we have $M = M''_5$. Finally, if columns 2 and 5 are both tag columns, then the first and last row induce D_2 as a submatrix, disregarding the coloring of the rows and thus this case is also not possible.

This finishes every possible case, and therefore we have reached a contradiction by assuming that A_{ext} does not contain any of the listed submatrices and still the C1P does not hold for A_{ext}. □

Corollary 1.1. *A $(0,1)$-matrix A is LRS-sortable if it does not contain any Tucker matrices, or $M'_2(k), M''_2(k), M'_3(k), M''_3(k), M'''_3(k)$, with $k \geq 4$, M'_4, M''_4, M'_5, M''_5, or $F_0, F_1(k), F_2(k)$ with $k \geq 5$ as submatrices.*

Recall that a $(0,1)$-matrix A is 2-nested if A has the C1P and there is a partition S_1, S_2 of the rows such that each obtained submatrix is nested. Equivalenty, a matrix A is 2-nested if A has the C1P and there is a proper 2-coloring of the rows such that each submatrix is nested. It follows from this remark that, if A is LRS-sortable, then A is LR-sortable and also 2-nested.

Conversely, if A does not contain any $F_0, F_1(k), F_2(k)$ or Tucker matrices as submatrices, then there are no two rows colored with the same color such that these rows are neither disjoint or nested. Moreover, since A also does not contain any $M'_2(k), M''_2(k), M'_3(k), M''_3(k), M'''_3(k), M'_4, M''_4, M'_5$ or M''_5 as submatrices, then it is also LR-sortable and therefore A is LRS-sortable. □

1.2.3 Full LRS-sortable

Definition 1.9. *Let A be an enriched matrix. We say A is* full LRS-sortable *if the given partial bicoloring of the rows of A can be extended to a bicoloring of all the rows of A such that A with the extended bicoloring is LRS-sortable.*

Lemma 3. *Let G be a graph with a partial 2-coloring of the vertices. Then, the partial 2-coloring can be extended to a proper 2-coloring of the vertices of G if and only if all of the following conditions hold:*

- *There are no even induced paths such that the only colored vertices of the path are its endpoints, and they are colored with the same color*
- *There are no odd induced paths such that the only colored vertices of the path are its endpoints, and they are colored with distinct colors*
- *There are no induced uncolored odd cycles*
- *There are no induced odd cycles with exactly one colored vertex*

Proof. □

Lemma 4. *If* A *is LRS-sortable and* B *is obtained from* A *by extending its partial coloring to a total proper 2-coloring of its rows, then* B *is full LRS-sortable if and only if* B *has no monochromatic gems.*

Theorem 1.5. *An enriched matrix* A *is full LRS-sortable if and only if* A *is LRS-sortable.*

Proof. It is clear that none of the matrices .... and their dual matrices is full LRS-sortable, for they do not admit an extension of the given bicoloring to a proper bicoloring of all the rows such that the bicolored matrix is LR-sortable. More precisely, if these matrices are fully colored, then they are no longer admissible (?) for we find ... as submatrices.

Conversely, suppose A is not full LRS-sortable. If A is not admissible, then it contains $\mathsf{D}_0, \mathsf{D}_1, \mathsf{D}_2, \mathsf{S}_1(\mathsf{k})$ or $\mathsf{S}_2(\mathsf{k})$ and thus it holds. Henceforth, we assume that A is admissible.

If A is not LRS-sortable, then there is a submatrix M such that M is one of the forbidden submatrices for LRS-sortable stated above. From now on, we assume that A is LRS-sortable.

Thus, there is a linear ordering Π of the columns of A such that A has the C1P and such that the rows marked with L start at the first column and those marked with R end at the last column, and any two rows colored with the same color are either disjoint or nested. We assume from now on that the columns of A are ordered according to Π.

El problema que queda es el coloreo total porque la matriz ya estÃ¡ 'ordenada' con la C1P.

Entonces, 2 filas que formen un gem deben tener colores distintos.

No puede tener $\mathsf{S}_1(\mathsf{k})$ ni $\mathsf{S}_2(\mathsf{k})$ porque con cualquier coloreo me queda un gem monocromÃ¡tico.

Entonces: Let H be the graph whose vertices are the rows of A and such that two vertices are adjacent in H if the underlying uncolored submatrix of A determined by these two rows contains a gem.

Si esto no pasa, es porque hay alguna de estas cosas del lema, y cada una me tiene que dar un prohibido.

We consider the vertices of H partially colored as in A. Since A is not full LRS-sortable, this partial coloring cannot be extended to a proper 2-coloring of the vertices of H.

Hence, there is either an induced odd cycle of uncolored vertices in H, or an induced path whose only colored vertices are its endpoints and whose length has the wrong parity. This is to say, if the endpoints of the induced path have the same color, then the path has even length, and if the endpoints have different colors, then the path has odd length.

Claim 1.2.1. *Let v and w be adjacent vertices in H. Then, the corresponding rows are neither disjoint nor nested, including the marked rows.*

This follows from the definition of H, for v and w are adjacent if the corresponding rows $a_{v.}$ and $a_{w.}$ induce an uncolored D_0 or D_3' as a submatrix, and thus $a_{v.}$ and $a_{w.}$ are neither disjoint or nested, regardless of whether one or two of the columns are marked or not. $\square$

Case 1. If there is an induced odd cycle C of uncolored vertices in H, then the only possibility is that the corresponding rows of any two consecutive vertices contain D_3' as a submatrix. This follows from the fact that all of the vertices in the cycle do not correspond to marked rows, for if they do, since A is admissible, then they must be colored.

Since for every pair of adjacent vertices in the odd cycle C we have a gem submatrix and we cannot bicolor C (regardless of the remaining vertices in H), then A is not a 2-nested matrix. Hence this case can be reduced to the 2-nested case.

Case 2. Suppose there are no odd uncolored cycles in H. Thus, there is an induced path $P =< v_1, v_2, \ldots, v_k >$ whose endpoints are the only colored vertices in P.

Remark 1.3. Since every inner vertex of P is uncolored, the corresponding rows in A of the vertices $v_2, \ldots, v_{k-1}$ are unmarked rows.

We have two possible cases: either P is an even path, or P is an odd path. The following claim holds for both cases.

Claim 1.2.2. *Every vertex v_i in P starts after v_{i-1}, for $i = 1, \ldots, k-1$.*

Let v_i be the first vertex in P such that v_i starts before v_{i-1}. Since v_i is nonadjacent to v_l, then v_i is either disjoint or nested with v_l, for every $l < i-1$. However, since v_i is the first vertex in P that starts before its predecesor, v_i cannot be disjoint to v_{i-2}, for they both start before v_{i-1} and thus they both intersect v_{i-1} in column l_{i-1}. Hence, either v_{i+1} is nested in v_{i-2} or viceversa.

Notice that v_{i-2} cannot be nested in v_{i+1}. This holds since v_{i+1} is adjacent in P only to v_{i-1} and v_i, hence v_{i-l} must be nested in v_{i+1} for $l = 2, \ldots, i-1$. Thus, v_0 and v_{i+1} intersect in the column l_1 and therefore they are adjacent, which results in a contradiction.

Thus, the only remaining possibility is that v_{i+1} is nested in v_{i-2}. In this case, v_{i+l} is nested in v_{i-2} for $l = 1, \ldots, k-i$, since v_i and v_{i+1} are adjacent and v_{i+l} is nonadjacent to v_{i-2}. Hence, if v_k is adjacent to v_0, then $v_1, v_2, \ldots, v_{k-1}$ are not nested in v_k. Furthermore, if v_k is nonadjacent to v_0, since v_k is adjacent to v_{k+1}, then v_{k+1} is either adjacent to v_{i-2} or is adjacent to v_1, depending on whether v_{k+1} is marked with L or R, respectively, which results once more in a contradiction. Therefore, there is no such vertex in P. $\square$

Case 2.1 Suppose that P is an induced even path, and that the endpoints v_1 and v_k are colored with the same color.

Since v_1 is colored, there is a vertex v_0 corresponding to a marked row such that v_0 and v_1 are adjacent in H and v_0 forces v_1 to be colored with the opposite color. Moreover, since the vertices $v_2, \ldots, v_{k-1}$ are uncolored, then v_0 is nonadjacent to all of them. Suppose without loss of generality that v_0 is colored red, v_1 is colored blue, and that v_0 is marked with L.

Since A is ordered according to Π, v_0 starts in the first column, which is also the L tag column. Furthermore, since $v_2, \dots, v_{k-1}$ are nonadjacent to v_0, then they all start in a column greater than r_0.

If v_k is adjacent to v_0, then $v_1, v_2, \dots, v_{k-2}$ are nested in v_k, for they are nonadjacent vertices.

If instead v_k is nonadjacent to v_0, then there is another vertex v_{k+1} adjacent to v_k such that v_{k+1} forces the color to v_k, and v_{k+1} is nonadjacent to $v_2, \dots, v_{k-1}$. We may assume v_{k+1} is also nonadjacent to v_1, since this can be reduced to the case when v_k is adjacent to v_0 by replacing v_0 by v_{k+1} and reordering the path P as $v'_i = v'_{k-i+1}$ for $i = 1, \dots, k$.

If v_k is nonadjacent to v_0, and v_{k+1} is marked with L, then we find $S_1(k)$ by choosing the columns $l_1, \dots, l_k, r_k$ and the first column of A, and the rows corresponding to $v_0, v_1, \dots, v_{k+1}$. If instead v_{k+1} is marked with R, then we find $S_2(k)$ by choosing the columns $l_1, \dots, l_k, r_k$ and the first and last column of A, and the rows corresponding to $v_0, v_1, \dots, v_{k+1}$

Finally, if v_k is adjacent to v_0, then we find $S_3(k)$ by choosing the columns $l_1, \dots, l_k, r_k$ and the first and last column of A, and the rows corresponding to $v_0, v_1, \dots, v_k$. In either case, this results in a contradiction since we assumed there are no $S_1(k), S_2(k), S_3(k)$ or their dual matrices as submatrices of A.

Case 2.2 Suppose that P is an induced odd path, and that the endpoints v_1 and v_k are colored with distinct colors.

The proof is analogous as in Case 2.1, except for the last paragraph. If P is an odd path, then v_1 and v_k cannot be adjacent to the same marked vertex v_0, for this marked vertex is colored and thus it forces both vertices to have the same color, which by hypothesis is not possible.

Therefore, we find $S_1(k)$ or $S_2(k)$ by choosing the columns $l_1, \dots, l_k, r_k$ of A, and the rows corresponding $v_0, v_1, \dots, v_{k+1}$. Once more, if v_{k+1} is marked with L, then we find $S_1(k)$, and if v_{k+1} is marked with R, then we find $S_2(k)$. □

1.3 Circle graphs containing an induced tent

1.3.1 Partitions of S and K

Let $G = (K, S)$ be a split graph where K is a clique and S is an independent set. Let T be an induced subgraph of G isomorphic to tent. Let $V(T) = \{k_1, k_3, k_5, s_{13}, s_{35}, s_{51}\}$ where $k_1, k_3, k_5 \in K$, $s_{13}, s_{35}, s_{51} \in S$, and the neighbors of s_{ij} in T are precisely k_i and k_j.

We introduce sets $K_1, K_2, \dots, K_6$ as follows.

- For each $i \in \{1, 3, 5\}$, let K_i be the set of vertices of K whose neighbors in $V(T) \cap S$ are precisely $s_{(i-2)i}$ and $s_{i(i+2)}$ (where subindices are modulo 6).
- For each $i \in \{2, 4, 6\}$, let K_i be the set of vertices of K whose only neighbor in $V(T) \cap S$ is $s_{(i-1)(i+1)}$ (where subindices are modulo 6).

Lemma 5. *If G is a circle graph, $\{K_1, K_2, \dots, K_6\}$ is a partition of K.*

Proof (Sketch). Every vertex of K is adjacent to precisely one or two vertices of $V(T) \cap S$. □

Let $i, j \in \{1, \dots, 6\}$ and let S_{ij} be the set of vertices of S that are adjacent to some vertex in K_i and some vertex in K_j, are complete to $K_{i+1}, K_{i+2}, \dots, K_{j-1}$, and are anticomplete to $K_{j+1}, K_{j+2}, \dots, K_{i-1}$ (where subindices are modulo 6).

Lemma 6. *If G is a circle graph, then all the following assertions hold:*

- $\{S_{ij}\}_{i,j\in\{1,2,\dots,6\}}$ *is a partition of* S.
- *For each* $i \in \{1, 3, 5\}$, $S_{i(i-1)}$ *and* $S_{i(i-2)}$ *are empty.*
- *For each* $i \in \{2, 4, 6\}$, $S_{i(i-1)}$ *and* $S_{i(i+2)}$ *are empty.*
- *For each* $i \in \{1, 3, 5\}$, $S_{i(i+3)}$ *and* $S_{(i+3)i}$ *are complete to* K_i.

$i \setminus j$	1	2	3	4	5	6
1	✓	✓	✓	✓	$\emptyset$	$\emptyset$
2	$\emptyset$	✓	✓	$\emptyset$	✓	✓
3	$\emptyset$	$\emptyset$	✓	✓	✓	✓
4	✓	✓	$\emptyset$	✓	✓	$\emptyset$
5	✓	✓	$\emptyset$	$\emptyset$	✓	✓
6	✓	$\emptyset$	✓	✓	$\emptyset$	✓

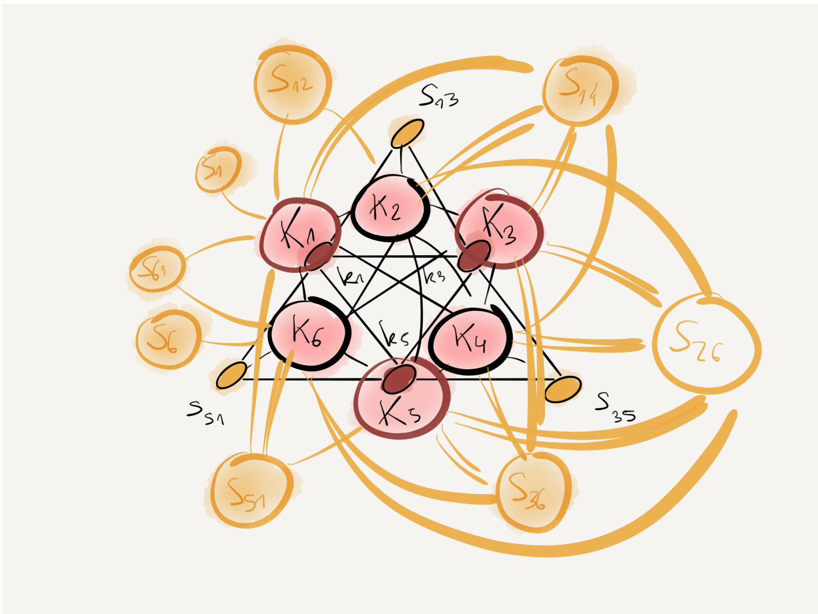


Figure 1.6: Tent T and the split graph G according to the given extensions

1.3.2 Matrices $\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_6$

Let $G = (K, S)$ and T as in the previous subsection.

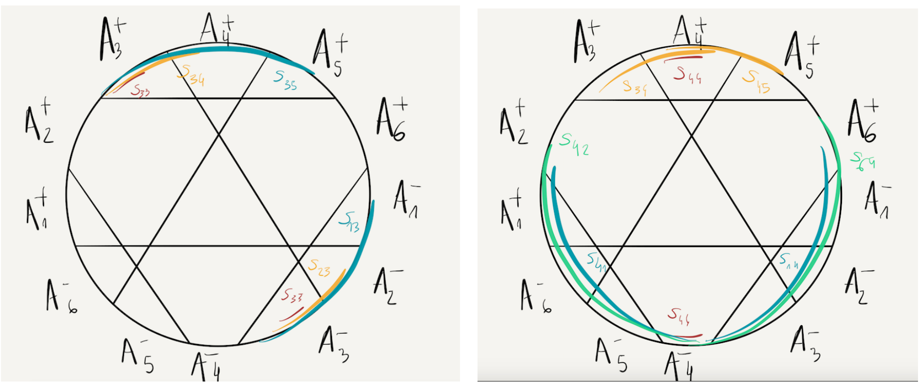
For each $i \in \{1, 2, \dots, 6\}$, let $\mathbb{A}_i$ be a $(0, 1)$-matrix having one row for each vertex $s \in S$ such that s belongs to S_{ij} or S_{ji} for some $j \in \{1, 2, \dots, 6\}$ and one column for each vertex $k \in K_i$ and such that such that the entry corresponding to row s and column k is 1 if and only if s is adjacent to k in G. For each $j \in \{1, 2, \ldots, 6\}$, we mark those rows corresponding to vertices of S_{ji} with L and those corresponding to vertices of S_{ij} with R.

Moreover, we color some of the rows of $\mathbb{A}_i$ as follows.

- If $i \in \{1, 3, 5\}$, then we color each row corresponding to a vertex $s \in S_{ij}$ for some $j \in \{1, 2, \ldots, 6\} - \{i\}$ with color red and each row corresponding to a vertex $s \in S_{ji}$ for some $j \in \{1, 2, \ldots, 6\} - \{i\}$ with color blue.
- If $i \in \{2, 4, 6\}$, then we color each row corresponding to a vertex $s \in S_{ij} \cup S_{ji}$ for some $j \in \{1, 2, \ldots, 6\}$ with color red if $j = i + 1$ or $j = i - 1$ (modulo 6) and with color blue otherwise.

Example:

$$\mathbb{A}_3 = \begin{array}{r} \\ S_{34}\ \mathbf{R} \\ S_{35}\ \mathbf{R} \\ S_{33} \\ S_{13}\ \mathbf{L} \\ S_{23}\ \mathbf{L} \end{array} \begin{array}{c} K_3 \\ \left(\begin{array}{c} \color{red}{\cdots} \\ \color{red}{\cdots} \\ \cdots \\ \color{blue}{\cdots} \\ \color{blue}{\cdots} \end{array} \right) \end{array} \qquad \mathbb{A}_4 = \begin{array}{r} \\ S_{34}\ \mathbf{L} \\ S_{45}\ \mathbf{R} \\ S_{44} \\ S_{14}\ \mathbf{L} \\ S_{64}\ \mathbf{L} \\ S_{41}\ \mathbf{R} \\ S_{45}\ \mathbf{R} \end{array} \begin{array}{c} K_4 \\ \left(\begin{array}{c} \color{red}{\cdots} \\ \color{red}{\cdots} \\ \cdots \\ \color{blue}{\cdots} \\ \color{blue}{\cdots} \\ \color{blue}{\cdots} \\ \color{blue}{\cdots} \end{array} \right) \end{array}$$



(a) $\mathbb{A}_3$ (b) $\mathbb{A}_4$

Figure 1.7: Sketch model of G with the chords associated to $\mathbb{A}_3$ and $\mathbb{A}_4$, respectively.

1.3.3 circle split equiv. full LRS-sortable

Lemma 7. *If $\mathbb{A}_i$ is not full LRS-sortable, then G contains one of the following minimal forbidden induced subgraphs for the class of circle graphs: ...*

Proof. (Sketch). Suppose first that $\mathbb{A}_i$ is not admissible. By Lema $\cdots$, $\mathbb{A}_i$ contains some submatrix D_0, D_1, D_2, D_3.

Suppose that $\mathbb{A}_i$ contains D_0. Let v_0 and v_1 in S be the vertices whose adjacency is represented by the first and second row of D_0, respectively, and let k_{i1} and k_{i2} in K_i be the vertices whose adjacency is represented by the second and third column of D_0.

Since v_1 is an unmarked vertex, then v_1 must be in $S_{(i)(i)}$. Moreover, since v_0 and v_1 are both colored with the same color and v_0 is marked, we have to analize each case depending on whether i is even or odd. Since the cases are symmetric, we assume without loss of generality that v_0 is marked with L.

If i is even, then the possibilities are $v_0 \in S_{(i-1)(i)}, v_1 \in S_{(i)(i)}$, $v_0 \in S_{(i-3)(i)}, v_1 \in S_{(i)(i)}$ and $v_0 \in S_{(i+2)(i)}, v_1 \in S_{(i)(i)}$. If i is odd, and assuming the even case proven, then the only remaining case is when $v_0 \in S_{(i-2)(i)}, v_1 \in S_{(i)(i)}$.

Case 1.1. $v_0 \in S_{(i-1)(i)}, v_1 \in S_{(i)(i)}$

Para cada una de esas matrices encontrar el subgrafo prohibido inducido de circle en G.

Suppose now that $\mathbb{A}_i$ is admissible but not LR-sortable. Then it contains ... y por cada una de ellas dar el prohibido. (ejemplo: el prohibido de la foto, Mv) □

Theorem 1.6. *Let $G = (K, S)$ be a split graph containing an induced tent. Then, G is a circle graph if and only if $\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_6$ are full LRS-sortable.*

Sketch. Necessity is clear by the previous lemma. Suppose now that each of the matrices $\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_6$ are full LRS-sortable. Let Π_i be the order of the column in a LR-ordering of $\mathbb{A}_i$ for each $i \in \{1, 2, \dots, 6\}$. Consider the circle divided into twelve pieces as in Figure .... For each $i \in \{1, 2, \dots, 6\}$ and for each vertex $k_i \in K_i$ we place a chord having one end in A_i^+ and another in in A_i^- in such a way that the ordering of the endpoints of the chords in A_i^+ and A_i^- is Pi_i. Acomodar los extremos en A_i^+ y A_i^- de las cuerdas correspondientes a los vÃ©rtices $s \in S_{ij}$ de acuerdo a Π_i. Mostrar que el modelo resultante es un modelo circle del grafo. □

Bibliography