

# Variable Selection in Convex Function Estimation

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## Abstract

We consider the problem of estimating a sparse convex function of many variables. In contrast to classical nonparametric regression with smoothness constraints, we show that convexity is additively faithful—it suffices to estimate a convex additive model for variable selection. We develop algorithms for estimating sparse convex additive models, including an approach using iterative quadratic programming. Supporting experiments and statistical theory are presented, showing variable selection consistency in dimensions that can scale exponentially in the sample size. An attractive feature of this framework is the lack of tuning parameters for smoothness.

## 1 Introduction

We consider the problem of estimating a convex function of several variables from noisy values of the function at a finite sample of input points. Recent work [5, 6] shows that the minimax rate for convex function estimation in  $p$  dimensions is  $n^{-4/(4+p)}$ . Loosely speaking, this shows that the geometric convexity constraint is statistically equivalent to requiring two derivatives of the function, and thus is subject to the same curse of dimensionality. However, if the function is sparse, with  $s \ll p$  relevant variables, then the faster rate  $n^{-4/(4+s)}$  may be achievable if the  $s$  variables can be identified. To determine the relevant variables, we show that it suffices to estimate a sum of  $p$  one-dimensional convex functions, leading to significant computational and statistical advantages. In addition, we introduce algorithms and supporting statistical theory for a practical, effective approach to this variable selection problem.

The sparse nonparametric regression problem is considered in [9], where it is shown that computationally efficient, near minimax-optimal estimation is possible, but in ambient dimensions that scale only as  $p = O(\log n)$ . This is in stark contrast to the exponential scaling  $p = O(e^{n^c})$  enjoyed by sparse linear models [19]. Sparse additive models [15] provide a practical alternative to fully nonparametric function estimation. But the additive assumption is limited. In particular, the natural idea of first selecting the single variable effects, then the pairwise effects, and so on, does not in general lead to consistent variable selection. In other words, the general nonparametric model is not additively faithful. Remarkably, the additional assumption of convexity does lead to additive faithfulness, as we show here. In addition, we show that exponential scaling is achievable for sparse convex additive models. Thus, the geometric convexity constraint is quite different from the smoothness constraints imposed in traditional nonparametric regression.

A key to our approach is the observation that least squares nonparametric estimation under convexity constraints is equivalent to a finite dimensional quadratic program. Specifically, the infinite

dimensional optimization

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^n (Y_i - m(x_i))^2 \\ & \text{subject to} && m : \mathbb{R}^p \rightarrow \mathbb{R} \text{ is convex} \end{aligned} \quad (1.1)$$

is precisely equivalent to the finite dimensional quadratic program

$$\begin{aligned} & \text{minimize}_{h, \beta} && \sum_{i=1}^n (Y_i - h_i)^2 \\ & \text{subject to} && h_j \geq h_i + \beta_i^T (x_j - x_i), \text{ for all } i, j. \end{aligned} \quad (1.3)$$

See [1], Section 6.5.5. Here  $h_i$  is the estimated function value  $m(x_i)$ , and the vectors  $\beta_i \in \mathbb{R}^d$  represent supporting hyperplanes to the epigraph of  $m$ . Importantly, this finite dimensional quadratic program does not have tuning parameters for smoothing the function. Such parameters are the bane of nonparametric estimation.

Estimation of convex functions arises naturally in several applications. Examples include geometric programming [1], computed tomography [14], target reconstruction [10], image analysis [4] and circuit design [7]. Other applications include queuing theory [3] and economics, where it is of interest to estimate concave utility functions [13]. See [11] for other applications. Beyond cases where the assumption of convexity is natural, we offer that the convexity assumption is attractive as a tractable, nonparametric relaxation of the linear model. In addition to the lack of tuning parameters, other than the regularization parameter  $\lambda$  to control the level of sparsity, the global convexity leads to effective, scalable algorithms. We demonstrate use of our approach on experiments with standard regression data sets, in a comparison with sparse linear models (lasso).

**Notation.** If  $\mathbf{x}$  is a vector, we use  $\mathbf{x}_{-k}$  to denote the vector with  $k$ -th coordinate removed. Let  $\mathbf{v} \in \mathbb{R}^n$ , then  $v_{(1)}$  denotes the smallest coordinate of  $\mathbf{v}$  in magnitude, and  $v_{(j)}$  denotes the  $j$ -th smallest;  $\mathbf{1}_n \in \mathbb{R}^n$  is all ones. If  $X \in \mathbb{R}^p$  and  $S \subset \{1, \dots, p\}$ , then  $X_S$  is the subvector of  $X$  restricted to coordinates in  $S$ . Given  $n$  samples  $X^{(1)}, \dots, X^{(n)}$ , we use  $\bar{X}$  to denote empirical average.

## 2 Additive Faithfulness

The additive approximation may mark a relevant variable as irrelevant. Such mistakes are inherent to the approximation and may persist even with infinite samples. Lemma 2.1 below provides examples of errors made by the additive approximations that minimize the  $L_2$  distance.

**Lemma 2.1.** *Let  $F$  be a probability distribution on  $\mathbf{C} = [0, 1]^s$  and let  $X = (X_1, \dots, X_s) \sim F$ . Let  $f : \mathbf{C} \rightarrow \mathbb{R}$  be an integrable function and suppose, without loss of generality, that  $\mathbb{E}f(X) = 0$ . Let*

$$f_1^*, \dots, f_s^* := \arg \min \left\{ \mathbb{E} \left( f(X) - \sum_{k=1}^s f_k(X_k) \right)^2 : \mathbb{E} f_k(X_k) = 0 \right\}. \text{ Then } f_k^*(x_k) = \mathbb{E}[f(X) | x_k].$$

Lemma 2.1 follows from the stationarity condition of the optimal solution. If  $F$  is the uniform distribution, then  $f_k^*(x_k) = \int_{\mathbf{x}_{-k}} f(x_k, \mathbf{x}_{-k}) d\mathbf{x}_{-k}$ .

**Example 2.1.** We give two examples of additive unfaithfulness under the uniform distribution. First, consider the following function:

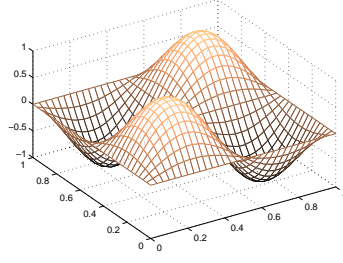
$$(\text{egg carton}) \quad f(x_1, x_2) = \sin(2\pi x_1) \sin(2\pi x_2) \quad \text{for } (x_1, x_2) \in [0, 1]^2$$

For all  $x_1$ ,  $\int_{x_2} f(x_1, x_2) dx_2 = 0$  and also, for all  $x_2$ ,  $\int_{x_1} f(x_1, x_2) dx_1 = 0$ . An additive approximation would set  $f_1 = 0$  and  $f_2 = 0$ . Next, consider the function

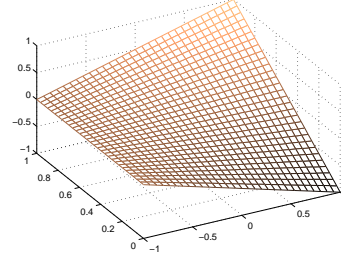
$$(\text{tilting slope}) \quad f(x_1, x_2) = x_1 x_2 \quad \text{for } x_1 \in [-1, 1], x_2 \in [0, 1].$$

For all  $x_2$ ,  $\int_{x_1} f(x_1, x_2) dx_1 = 0$ , therefore, we expect  $f_2 = 0$  under the additive approximation. This function, for every fixed  $x_2$ , is a zero-intercept linear function of  $x_1$  with slope exactly  $x_2$ .

It is important to understand the kind of functions for which the additive approximation can accurately capture all of the relevant variables. We call this property *additive faithfulness*.



(a) egg carton



(b) tilting slope

Figure 1: Two additively unfaithful functions. Relevant variables are zeroed out under an additive approximation because every “slice” of the function integrates to zero.

**Definition 2.1.** Let  $C = [0, 1]^s$ , and  $f : C \rightarrow \mathbb{R}$ . We say that  $f$  depends on coordinate  $k$  if there exist  $x'_k \neq x_k$  such that  $f(x'_k, \mathbf{x}_{-k})$  and  $f(x_k, \mathbf{x}_{-k})$  are different functions of  $\mathbf{x}_{-k}$ .

Let  $F$  be a probability distribution on  $C$  and assume without loss of generality that  $\mathbb{E}f(X) = 0$ . Let

$$f_1^*, \dots, f_s^* := \arg \min \left\{ \mathbb{E}(f(X) - \sum_{k=1}^s f_k(X_k))^2 : \mathbb{E}f_k(X_k) = 0, \forall k \right\}.$$

We say that  $f$  is *additively faithful* under  $F$  in case  $f_k^* = 0$  iff  $f$  does not depend on coordinate  $k$ .

Remarkably, under many distributions, a convex multivariate function can always be faithfully approximated by an additive function.

**Theorem 2.1.** Let  $F$  be a product distribution supported and positive on  $C = [0, 1]^s$  so that  $X_1, \dots, X_s$  are independent. If  $f$  is convex and twice differentiable, then  $f$  is additively faithful under  $F$ .

We give the full proof in Section 6.1 of the Appendix, but pause here to provide some intuition. From Lemma 2.1, we know intuitively that the additive approximation zeroes out  $k$  when, fixing  $x_k$ , every “slice” of  $f$  integrates to zero. We prove Theorem 2.1 by showing that “slices” of convex functions that integrate to zero cannot be “glued” together while still maintaining convexity.

Theorem 2.1 plays an important role in our sparsistency analysis, in which we prove that the additive approximation is variable selection consistent (or “sparsistent”) even when the true function is not additive.

**Remark 2.1.** We assume twice differentiability in Theorem 2.1 to simplify the proof. We believe this smoothness condition is not necessary because every non-smooth convex function can be approximated arbitrarily well by a smooth one. Without restrictions on the distribution, a convex function may not be additively faithful. Intuitively, an arbitrarily shaped density  $\rho$  may “undo” the convexity of  $f$  so that the product  $\rho(\mathbf{x}) f(\mathbf{x})$  resembles an egg carton or a tilting slope. With appropriate conditions on the density  $\rho$ , however, it is possible to relax the independence assumption. We leave this to future work.

### 3 Optimization for Sparse Convex Additive Models

We consider the following nonparametric regression problem

$$y_i = f(\mathbf{x}_i) + \epsilon_i = \sum_{k=1}^p f_k(x_{ki}) + \epsilon_i \quad i = 1, 2, \dots, n$$

where  $\mathbf{x}_i \in \mathbb{R}^p$  is the covariate,  $y_i$  is the response and  $\epsilon_i$  is mean zero noise. The regression function  $f(\cdot)$  is the summation of functions  $f_k(\cdot)$  in each variable dimension. We impose an additional constraint that each  $f_k(\cdot)$  is an univariate convex function, which can be represented by its supporting hyperplanes, i.e.,

$$h_{kj} \geq h_{ki} + \beta_{ki}(x_{kj} - x_{ki}) \quad (\forall i, j) \quad (3.1)$$

where  $h_{ki} := f_k(x_{ki})$  and  $\beta_{ki}$  is the subgradient at point  $x_{ki}$ . We apparently need  $O(n^2 p)$  constraints to impose the supporting hyperplane constraints, which is computationally expensive for

large scale problems. In fact, only  $O(np)$  constraints suffice, since univariate convex functions are characterized by the condition that the subgradient, which is a scalar, must increase monotonically. This observation leads to our optimization program:

$$\begin{aligned} \min_{\mathbf{h}, \boldsymbol{\beta}} \quad & \frac{1}{2n} \sum_{i=1}^n \left( y_i - \sum_{k=1}^p h_{ki} \right)^2 + \lambda \sum_{k=1}^p \|\boldsymbol{\beta}_k\|_{\infty} \\ \text{s.t.} \quad & \sum_{i=1}^n h_{ki} = 0, \quad h_{k(i+1)} = h_{k(i)} + \beta_{k(i)}(x_{k(i+1)} - x_{k(i)}), \quad \beta_{k(i+1)} \geq \beta_{k(i)} \quad (\forall k, i) \end{aligned} \quad (3.2)$$

Let  $\{(1), (2), \dots, (n)\}$  be a reordering of  $\{1, 2, \dots, n\}$  such that  $x_{k(1)} \leq x_{k(2)} \leq \dots \leq x_{k(n)}$ . It is easy to verify that the constraints in (3.2) satisfy the supporting hyperplane constraints, as

$$\begin{aligned} \forall j \geq i, \quad h_{k(j)} - h_{k(i)} &= \sum_{t=i}^{j-1} (h_{k(t+1)} - h_{k(t)}) = \sum_{t=i}^{j-1} \beta_{k(t)}(x_{k(t+1)} - x_{k(t)}) \\ &\geq \beta_{k(i)} \sum_{t=i}^{j-1} (x_{k(t+1)} - x_{k(t)}) = \beta_{k(i)}(x_{k(j)} - x_{k(i)}) \\ \forall j < i, \quad h_{k(j)} - h_{k(i)} &= \sum_{t=j}^{i-1} (h_{k(t)} - h_{k(t+1)}) = \sum_{t=j}^{i-1} \beta_{k(t)}(x_{k(t)} - x_{k(t+1)}) \\ &\geq \beta_{k(i)} \sum_{t=j}^{i-1} (x_{k(t)} - x_{k(t+1)}) = \beta_{k(i)}(x_{k(j)} - x_{k(i)}), \end{aligned}$$

The  $\ell_{\infty}/\ell_1$  penalty  $\sum_{k=1}^p \|\boldsymbol{\beta}_k\|_{\infty}$  encourages group sparsity of the vectors  $\boldsymbol{\beta}_k$ , and thus performs variable selection. We refer to this framework as the sparse convex additive model (SCAM). Notice that if we replace  $\beta_{k(i+1)} \geq \beta_{k(i)}$  with  $\beta_{k(i+1)} = \beta_{k(i)}$ , the optimization reduces to the lasso. Note while one can use supporting hyperplanes to the epigraph as in (1.4), SCAM uses the *inner piece-wise linear function* that approximates the graph with secant lines.

The SCAM optimization in (3.2) is a quadratic program (QP) with  $O(np)$  variables and  $O(np)$  constraints. Directly applying a QP solver would be computationally expensive for relatively large  $n$  and  $p$ . However, notice that variables in different feature dimensions are only coupled in the term  $(y_i - \sum_{k=1}^p h_{ki})^2$ . Hence, we can apply the block coordinate descent method, where in each step we solve the following QP subproblem for  $\{\mathbf{h}_{k\cdot}, \boldsymbol{\beta}_{k\cdot}\}$  with the other variables fixed:

$$\begin{aligned} \min_{\mathbf{h}_{k\cdot}, \boldsymbol{\beta}_{k\cdot}, \gamma_k} \quad & \frac{1}{2n} \sum_{i=1}^n \left( (y_i - \sum_{r \neq k} h_{ri}) - h_{ki} \right)^2 + \lambda \gamma_k \\ \text{s.t.} \quad & \sum_{i=1}^n h_{ki} = 0, \quad h_{k(i+1)} = h_{k(i)} + \beta_{k(i)}(x_{k(i+1)} - x_{k(i)}), \quad \beta_{k(i+1)} \geq \beta_{k(i)}, \quad -\gamma_k \leq \beta_{k(i)} \leq \gamma_k \quad (\forall i). \end{aligned}$$

The extra variable  $\gamma_k$  is introduced to deal with the  $\ell_{\infty}$  norm. This QP subproblem involves  $O(n)$  variables,  $O(n)$  constraints and a sparse structure, which can be solved efficiently using optimization packages (e.g., MOSEK: <http://www.mosek.com/>). We cycle through all feature dimensions ( $k$ ) from 1 to  $p$  multiple times until convergence. Empirically, we observe that the algorithm converges in only a few cycles. We also implemented an ADMM solver for (3.2), but found that it is not as efficient as this QP solver.

After optimization, the function estimator for any input data  $\mathbf{x}_j$  is, according to (3.1),

$$f(\mathbf{x}_j) = \sum_{k=1}^p f_k(x_{kj}) = \sum_{k=1}^p \max_i \{h_{ki} + \beta_{k(i)}(x_{kj} - x_{k(i)})\}.$$

### 3.1 Alternative Formulation

Optimization (3.2) can be reformulated in terms of the 2nd derivatives. The alternative formulation replaces the ordering constraints  $\beta_{k(i+1)} \geq \beta_{k(i)}$  with positivity constraints, which simplifies theoretical analysis. Define  $d_{k(i)}$  as the gradient increment:  $d_{k(1)} = \beta_{k(1)}$ , and  $d_{k(2)} = \beta_{k(2)} - \beta_{k(1)}$ . The convexity constraint is equivalent to the constraint that  $d_{k(i)} \geq 0$  for all  $i > 1$ .

It is easy to verify that  $\beta_{k(i)} = \sum_{j \leq i} d_{k(i)}$  and

$$f_k(x_{k(i)}) = f_k(x_{k(1)}) + d_{k(1)}(x_{k(i)} - x_{k(1)}) + d_{k(2)}(x_{k(i)} - x_{k(2)}) + \cdots + d_{k(i-1)}(x_{k(i)} - x_{k(i-1)})$$

Define  $\Delta(j, x_{ki}) = 0$  if  $\text{ord}(i) \leq j$ , and  $\Delta(j, x_{ki}) = x_{ki} - x_{k(j)}$  otherwise, where  $j$  ranges from 1 to  $n - 1$ . Also define  $\Delta(x_{ki}) = \{\Delta(j, x_{ki})\}_{j=1, \dots, n-1}$  as an  $n - 1$  dimensional vector. Then  $f_k(x_{ki}) = \Delta(x_{ki})^\top d_k$  where  $\Delta(x_{ki}) \in \mathbb{R}^{n-1}$ . Given  $n$  samples  $\{x_{ki}\}_{i=1}^n$ , we can further define  $\Delta_k \in \mathbb{R}^{n \times n-1}$  such that the  $i$ -th row is  $\Delta(x_{ki})$ . Finally, we reformulate (3.2) as an equivalent optimization program with only centering and positivity constraints:

$$\begin{aligned} \min_{\{d_k \in \mathbb{R}^{n-1}, c_k \in \mathbb{R}\}} & \frac{1}{2n} \left\| Y - \sum_{k=1}^p (\Delta_k d_k - c_k \mathbf{1}_n) \right\|_2^2 + \lambda_n \sum_{k=1}^p \|d_k\|_1 \\ \text{s.t. } & \forall k, d_{k2}, \dots, d_{k(n-1)} \geq 0 & (\text{convexity}) \\ & c_k = \frac{1}{n} \mathbf{1}_n^\top \Delta_k d_k & (\text{centering}) \end{aligned} \quad (3.3)$$

$\|d_k\|_1$  is not identical to  $\|\beta_{k\cdot}\|_\infty$ , but it is easy to verify that  $\|\beta_{k\cdot}\|_\infty \leq \|d_k\|_1 \leq 2\|\beta_{k\cdot}\|_\infty$ .

**Remark 3.1.** For parts of our theoretical analysis, we will also impose onto (3.3) a boundness constraint  $-B\mathbf{1}_n \leq \Delta_k d_k + c_k \mathbf{1}_n \leq B\mathbf{1}_n$  which constrains that  $\|f_k\|_\infty \leq B$ , or a Lipschitz constraint  $\|d_k\|_1 \leq L$  which constrains that  $f_k$  must be  $L$ -Lipschitz. We use these constraints only in the proof for technical reasons; we never need nor use these constraints in our experiments.

## 4 Analysis of Variable Selection Consistency

We divide our analysis into two parts. We first establish a sufficient *deterministic* condition for sparsistency. We then consider the stochastic setting and argue that the deterministic conditions hold with high probability.

### 4.1 Deterministic Setting

We follow Wainwright [19] and define a purely theoretical construct called *restricted regression*.

**Definition 4.1.** In *restricted regression*, we restrict the indices  $k$  in optimization (3.3) to lie in the support  $S$  instead of ranging from  $1, \dots, p$ .

Our analysis then diverges from the now-standard “primal-dual witness technique” [19]. Primal-dual witness explicitly solves all the dual variables. Because our optimization is more complex, we do not solve the dual variables on  $S$ , we instead write the dual variables on  $S^c$  as a function of the restricted regression *residual*, which is implicitly a function of the dual variables on  $S$ .

**Theorem 4.1.** (*Deterministic setting*) Let  $\{\hat{d}_k, \hat{c}_k\}_{k \in S}$  be the minimizer of the restricted regression, that is, the solution to optimization (3.3) where we restrict  $k \in S$ . Let  $\hat{d}_k = 0$  and  $\hat{c}_k = 0$  for  $k \in S^c$ . Let  $\hat{r} := Y - \sum_{k \in S} (\Delta_k \hat{d}_k - \hat{c}_k \mathbf{1})$  be the restricted regression residual. For  $k \in \{1, \dots, p\}$ ,  $\Delta_{k,j}$ , the  $j$ -th column of  $\Delta_k$ , is the  $n$ -dimensional vector  $\max(X_k - X_{k(j)} \mathbf{1}, 0)$ . Suppose for all  $j$  and all  $k \in S^c$ ,  $\lambda_n > |\frac{1}{n} \hat{r}^\top \Delta_{k,j}|$ . Then  $\hat{d}_k, \hat{c}_k$  for  $k = 1, \dots, p$  is an optimal solution to the full regression 3.3. Furthermore, any solution to the optimization program 3.3 must be zero on  $S^c$ .

This result holds regardless of whether we impose the boundness and Lipschitz conditions in optimization 3.3. The full proof of Theorem 4.1 is in Section 6.2 of the Appendix.

**Remark 4.1.** The incoherence condition of [19] is implicitly encoded in our condition on  $\lambda_n, \hat{r}, \Delta_{k,j}$ . We can reconstruct the incoherence condition if we assume that the true function  $f_0$  is linear and that our fitted functions  $\hat{f}_k$  are linear as well.

Theorem 4.1 allows us to analyze false negative rates and false positive rates separately. To control false positives, we study when the condition  $\lambda_n > |\frac{1}{n} \hat{r}^\top \Delta_{k,j}|$  is fulfilled for all  $j$  and all  $k \in S^c$ . To control false negatives, we study the restricted regression.

## 4.2 Probabilistic Setting

We use the following statistical setting:

1. Let  $\mathcal{X} = [-b, b]^p$ . Let  $F$  be a distribution supported and positive on  $\mathcal{X}$ . Let  $X^{(1)}, \dots, X^{(n)}$  be independent samples from  $F$ .
2. Let  $Y' = f_0(X) + \epsilon$  where  $f_0$  is the true function and  $\epsilon$  is noise. Given  $n$  samples  $Y'^{(1)}, \dots, Y'^{(n)}$ , we input into our optimization  $Y = Y' - \bar{Y}'$ .
3. Let  $S \subset \{1, \dots, p\}$  denote the relevant variables, i.e.,  $f_0(X) = f_0(X_S)$ , with  $s := |S|$ .

Each of our statistical theorems will use a subset of the following assumptions:

- A1:  $X_S, X_{S^c}$  are independent. A1':  $\{X_k\}_{k \in S}$  are independent.  
A2:  $\|f_0\|_\infty \leq B$ . A2':  $f_0$  is convex, twice-differentiable,  $L$ -Lipschitz, and  $\text{supp}(f_0) = S$ .  
A3: Suppose  $\epsilon$  is mean-zero sub-Gaussian, independent of  $X$ , with sub-Gaussian scale  $\sigma$ .

We will use assumptions A1, A2, A3 to control the probability of false positives and the stronger assumptions A1', A2', A3 to control the probability of false negatives.

**Remark 4.2.** We make strong assumptions on the covariates in A1 in order to make very weak assumptions on the true regression function  $f_0$  in A2. In particular, we do not assume that  $f_0$  is additive. Relaxing these assumptions is an interesting direction for future work.

**Theorem 4.2.** (Controlling false positives) Suppose assumptions A1, A2, A3 hold. Suppose also that we run optimization (3.3) with the  $B$ -boundness constraint. Let  $c, C$  be absolute constants. Suppose  $\lambda_n \geq cb(sB + \sigma)\sqrt{\frac{s}{n} \log n \log(pn)}$ . Then with probability at least  $1 - \frac{C}{n}$ , for all  $j, k$ ,  $\lambda_n > |\frac{1}{n} \hat{r}^\top \Delta_{k,j}|$ . Therefore, any solution to the full regression (3.3), with boundedness constraint, is zero on  $S^c$ .

The proof of Theorem 4.2 exploits independence of  $\hat{r}$  and  $\Delta_{k,j}$  from A1, and then uses concentration of measure results to argue that  $|\frac{1}{n} \hat{r}^\top \Delta_{k,j}|$  concentrates around zero at a desired rate. The fact that  $\hat{r}$  is a centered vector is crucial to our proof, and our theory thus further illustrates the importance of imposing the centering constraints in optimization (3.3). Our proof uses the concentration of the average of data sampled *without* replacement [17], illustrating that the proof method is not a trivial application of existing techniques. The full proof of Theorem 4.2 is in Section 6.3 of the Appendix.

**Theorem 4.3.** (Controlling false negatives) Suppose assumptions A1', A2', A3 hold. Let  $\hat{f} = \{\hat{d}_k, \hat{c}_k\}_{k \in S}$  be any solution to the restricted regression with both the  $B$ -boundedness and  $L$ -Lipschitz constraint. Let  $c, C$  be absolute constants. Suppose  $L \max \left( \lambda_n, b(B + \sigma)B\sigma\sqrt{\frac{s^5}{n^{4/5} \log sn}} \right) \rightarrow 0$ . Then, for sufficiently large  $n$ ,  $\hat{f}_k = (\hat{d}_k, \hat{c}_k) \neq 0$  for all  $k \in S$  with probability at least  $1 - \frac{C}{n}$ .

This is a finite sample version of Theorem 2.1. We need stronger assumptions in Theorem 4.3 to use our additive faithfulness result, Theorem 2.1. We also include an extra Lipschitz constraint so that we can use existing covering number results [2]. Recent work [6] shows that the Lipschitz constraint is not required with more advanced empirical process theory techniques. We give the full proof of Theorem 4.3 in Section 6.4 of the Appendix.

Combining Theorem 4.2 and 4.3 and ignoring dependencies on  $b, B, L, \sigma$ , we have the following result.

**Corollary 4.1.** Assume A1', A2', A3. Let  $\lambda_n = \Theta \left( \sqrt{\frac{s^3}{n} \log n \log(pn)} \right)$ . Suppose  $\lambda_n \rightarrow 0$  and  $\sqrt{\frac{s^5}{n^{4/5} \log sn}} \rightarrow 0$ . Let  $\hat{f}_n$  be a solution to (3.3) with boundedness and Lipschitz constraints. Then  $\mathbb{P}(\text{supp}(\hat{f}_n) = \text{supp}(f_0)) \rightarrow 1$ .

The above corollary implies that sparsistency is achievable at the same exponential scaling of the ambient dimension  $p = O(\exp(n^c))$ ,  $c < 1$  rate as parametric models. The cost of nonparametric modeling is reflected in the scaling with respect to  $s$ , which can only scale at  $o(n^{4/25})$ .

## 5 Experiments

We first illustrate our methods using a simulation of the following regression problem

$$y_i = \mathbf{x}_{iS}^\top \mathbf{Q} \mathbf{x}_{iS} + \epsilon_i \quad (i = 1, 2, \dots, n).$$

Here  $\mathbf{x}_i$  denotes data sample  $i$  drawn from  $\mathcal{N}(\mathbf{0}, \mathbf{I}_p)$ ,  $\mathbf{x}_{iS}$  is a subset of  $\mathbf{x}_i$  with dimension  $|S| = 5$ , where  $S$  represents the active feature set, and  $\epsilon_i$  is the additive noise drawn from  $\mathcal{N}(0, 1)$ .  $\mathbf{Q}$  is a symmetric positive definite matrix of dimension  $|S| \times |S|$ . Notice that if  $\mathbf{Q}$  is diagonal, then the true function is convex additive; otherwise the true function is convex but not additive. For all the simulations in this section, we set  $\lambda = 4\sqrt{\log(np)/n}$ .

In the first simulation, we set  $\mathbf{Q} = \mathbf{I}_{|S|}$  (the additive case), and choose  $n = 100, 200, \dots, 1000$  and  $p = 64, 128, 256, 512$ . For each  $(n, p)$  combination, we generate 200 independent data sets. For each data set we use SCAM to infer the model parameterized by  $\mathbf{h}$  and  $\beta$ ; see equation (3.2). If  $\|\beta_k\|_\infty < 10^{-8}$  ( $\forall k \notin S$ ) and  $\|\beta_k\|_\infty > 10^{-8}$  ( $\forall k \in S$ ), then we declare correct support recovery. We then plot the probability of support recovery over the 200 data sets in Figure 2(a). We observe that SCAM performs consistent variable selection when the true function is convex additive. To give the reader a sense of the running speed, the code runs in about 2 minutes on one data set with  $n = 1000$  and  $p = 512$ , on a MacBook with 2.3 GHz Intel Core i5 CPU and 4 GB memory.

In the second simulation, we study the case in which the true function is convex but not additive. We generate four  $\mathbf{Q}$  matrices plotted in Figure 2(b), where the diagonal elements are all 1 and the off-diagonal elements are 0.5 with probability  $\alpha$  ( $\alpha = 0, 0.2, 0.5, 1$  for the four cases). We fix  $p = 128$  and choose  $n = 100, 200, \dots, 1000$ . We again run the SCAM optimization on 200 independently generated data sets and plot the probability of recovery in Figure 2(c). The results demonstrate that SCAM performs consistent variable selection even if the true function is not additive (but still convex).

In the third simulation, we study the case of correlated design, where  $\mathbf{x}_i$  is drawn from  $\mathcal{N}(\mathbf{0}, \Sigma)$  instead of  $\mathcal{N}(\mathbf{0}, \mathbf{I}_p)$ , with  $\Sigma_{ij} = \nu^{|i-j|}$ . We use the non-additive  $\mathbf{Q}$  with  $\alpha = 0.5$  and fix  $p = 128$ . The recovery curves for  $\nu = 0.2, 0.4, 0.6, 0.8$  are depicted in Figure 2(d). As can be seen, for design of moderate correlation, SCAM can still select relevant variables well.

We next use the Boston housing data rather than simulated data. This data set contains 13 covariates, 506 samples and one response variable indicating housing values in suburbs of Boston. The data and detailed description can be found on the UCI Machine Learning Repository website <http://archive.ics.uci.edu/ml/datasets/Housing>.

We first use all  $n = 506$  samples (with normalization) to train SCAM, using a set of candidate  $\{\lambda^{(t)}\}$  with  $\lambda^{(1)} = 0$  (no regularization). For each  $\lambda^{(t)}$  we obtain a subgradient matrix  $\beta^{(t)}$  with  $p = 13$  rows. The non-zero rows in this matrix indicate the variables selected using  $\lambda^{(t)}$ . We plot  $\|\beta^{(t)}\|_\infty$  and the row-wise mean of  $\beta^{(t)}$  versus the normalized norm  $\frac{\|\beta^{(t)}\|_{\infty,1}}{\|\beta^{(1)}\|_{\infty,1}}$  in Figures 3(a) and 3(b). As a comparison we plot the LASSO/LARS result in a similar way in Figure 3(c). From the figures we observe that the first three variables selected by SCAM and LASSO are the same: LSTAT, RM and PTRATIO, which is consistent with previous findings [16]. The fourth variable selected by SCAM is TAX (with  $\lambda^{(t)} = 0.09$ ). We then refit SCAM with only these four variables without regularization, and plot the inferred additive functions in Figure 3(e). As can be seen, these functions contain clear nonlinear effects which cannot be captured by LASSO. The shapes of these functions are in agreement with those obtained by SpAM [16].

Next, in order to quantitatively study the predictive performance, we run 10 times 5-fold cross validation, following the same procedure described above (training, variable selection and refitting). A plot of the mean and standard deviation of the predictive Mean Squared Error (MSE) in Figure 3(d). Since for SCAM the same  $\lambda^{(t)}$  may lead to slightly different number of selected features in different folds and runs, the values on the x-axis (average number of selected features) for SCAM are not necessarily integers. Nevertheless, the figure clearly shows that SCAM has a much lower predictive MSE than LASSO. We also compared the performance of SCAM with that of Additive Forward Regression (AFR) presented in [12], and found that they are similar. The main advantages of SCAM compared with AFR and SpAM are 1) there are no other tuning parameters (such as bandwidth) besides  $\lambda$ ; 2) SCAM is formulated as a convex program, which guarantees a global optimum.

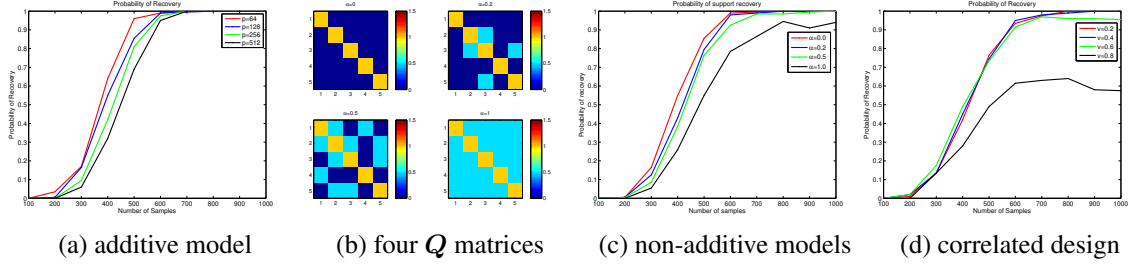


Figure 2: Support recovery results where the additive assumption is correct (a), incorrect (b), (c), and with correlated design (d).

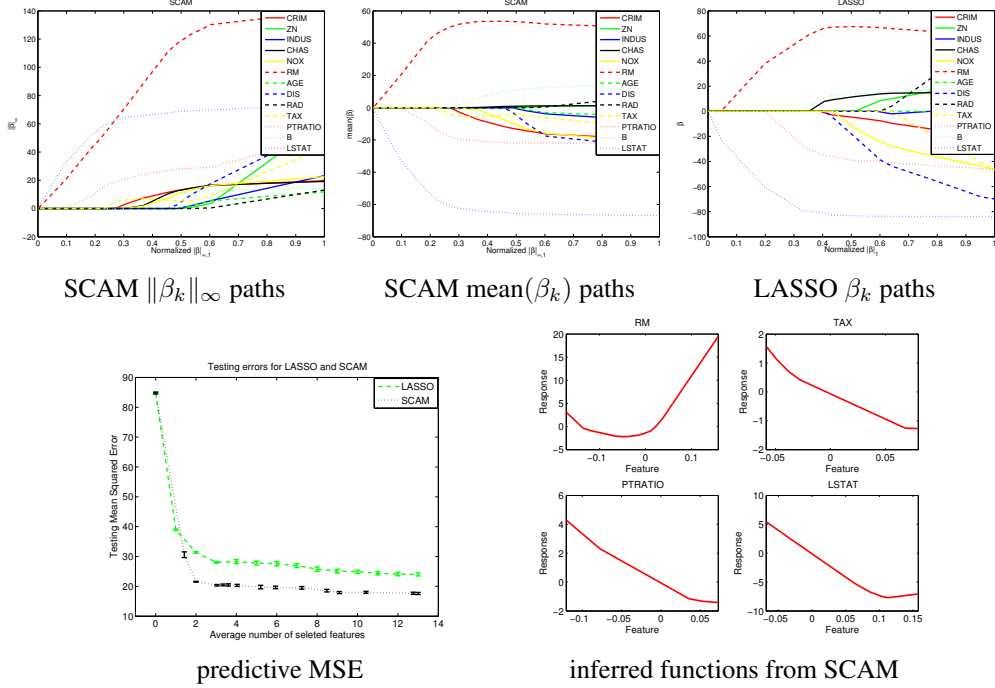


Figure 3: Results on Boston housing data, showing regularization paths, MSE and fitted functions.

## 6 Discussion

We have introduced a framework for estimating high dimensional but sparse convex functions. Because of the special properties of convexity, variable selection for convex functions enjoys additive faithfulness—it suffices to carry out variable selection over an additive model, in spite of the approximation error this introduces. Sparse convex additive models can be optimized using block coordinate quadratic programming, which we have found to be effective and scalable. We established variable selection consistency results, allowing exponential scaling in the ambient dimension. We expect that the technical assumptions we have used in these analyses can be weakened; this is one direction for future work. Another interesting direction for building on this work is to allow for additive models that are a combination of convex and concave components. If the convexity/concavity of each component function is known, this again yields a convex program. The challenge is to develop a method to automatically detect the concavity or convexity pattern of the variables.



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