Additivity to the Rescue?

Triangulating n points in d dimensions is computationally expensive, with complexity $O(n^{d/2})$.

An additivity assumption will require us to instead use only $d-1$ two-dimensional triangulations for each variable, with complexity $O(d^2 n \log n)$ overall.

Write the negative log-conditional-likelihood for a data point (X_i, Y_i) as

$$\log\Bigl(\int \exp(f(X_i, y))\, dy\Bigr) - f(X_i, Y_i) \tag{1.1}$$

where $f(x, y)$ is jointly concave in x and y. Under an additive assumption this is

$$\log\Bigl\{\int \exp\Bigl(\sum_{j=1}^{d-1} f_j(X_{ij}, y)\Bigr)\, dy\Bigr\} - \sum_{j=1}^{d-1} f_j(X_{ij}, Y_i). \tag{1.2}$$

Assume that each f_j is concave. For identifiability we will also assume $\mathbb{E}(f_j(X_j, Y)) = 0$ for $j = 1, \ldots, d-1$.

Introduce tent pole parameters $z_{ij} \equiv f_j(X_{ij}, Y_i)$. Let z denote the collection of these $n(d-1)$ parameters, and define $z_{\bullet j} = (z_{1j}, \ldots, z_{nj})^T$. The identifiability condition becomes the constraint

$$\sum_{i=1}^{n} z_{ij} = 0 \tag{1.3}$$

for each $j = 1, \ldots, d-1$.

Define, as before for the d-dimensional case, the height function

$$t_{x_j, y}(z_{\bullet j}) = \sup\Bigl\{\lambda^T z_{Sj} \;:\; \lambda^T (X_{Sj}, Y_S) = (x_j, y)\Bigr\} \tag{1.4}$$

where the $\sup$ is over all convex combinations of subsets $S \subset \{1, \ldots, n\}$ of $d+1$ elements. This is convex as a function of $z_{\bullet j}$, and defines the jth concave function as $f_j(x_j, y) = t_{x_j, y}(z_{\bullet j})$. The heights $z_{\bullet j}$ determine a two-dimensional triangulation of $\mathbb{R}^2 = \{(x_j, y)\}$.

Our objective function, without regularization, becomes

$$\ell(z) = \sum_{i=1}^{n}\Bigl[\log\Bigl\{\int \exp\Bigl(\sum_{j=1}^{d-1} t_{X_{ij}, y}(z_{\bullet j})\Bigr)\, dy\Bigr\} - \sum_{j=1}^{d-1} z_{ij}\Bigr]. \tag{1.5}$$

The optimization can be carried out by backfitting, optimizing over each $z_{\bullet j}$ in turn.