ANOTHER PROOF THAT CONVEX FUNCTIONS ARE LOCALLY LIPSCHITZ

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The Wayne State Mathematics Department Coffee Room recently brewed the following result [this MONTHLY, vol. 79 (1972), 1121–1124]. *Every convex function f defined on an open convex set in R^n is locally Lipschitz.* A different recipe yields the same result with less work and applies in much more general spaces. It goes like this: (1) control the size of f by showing (local) boundedness, (2) mix boundedness with convexity to obtain a Lipschitz condition, (3) embellish with desired generalizations. Here are the details.

LEMMA A. *A convex function f, defined on an open convex set U in R^n, is locally bounded; that is, it is bounded in a neighborhood of each point $\boldsymbol{x}_0$ in U.*

Proof. Choose a cube K in U centered at $\boldsymbol{x}_0$ and with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m$ $(m=2^n)$. Since a cube is the convex hull of its vertices, we may for any $\boldsymbol{x}$ in K find scalars λ_i satisfying

$$\boldsymbol{x} = \sum_1^m \lambda_i \boldsymbol{v}_i, \qquad \lambda_i \geqq 0, \qquad \sum_1^m \lambda_i = 1.$$

By convexity (Jensen's inequality for convex functions),

$$f(\boldsymbol{x}) \leqq \sum_1^m \lambda_i f(\boldsymbol{v}_i) \leqq \max_{1 \leqq i \leqq m} f(\boldsymbol{v}_i) \equiv M,$$

so f is bounded above on K.

On the other hand, for $\boldsymbol{x}$ in K we may choose $\boldsymbol{y}$ in K so that $\boldsymbol{x}_0 = \frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}$. Thus,

$$f(\boldsymbol{x}_0) \leqq \tfrac{1}{2} f(\boldsymbol{x}) + \tfrac{1}{2} f(\boldsymbol{y}),$$

or

$$f(\boldsymbol{x}) \geqq 2f(\boldsymbol{x}_0) - f(\boldsymbol{y}) \geqq 2f(\boldsymbol{x}_0) - M,$$

and f is also bounded below on K. ∎

THEOREM A. *Let f be convex on an open convex set U in R^n. Then f is locally Lipschitz on U; that is, it is Lipschitz on a neighborhood of each point $\boldsymbol{x}_0$ of U. Consequently, f is Lipschitz on any compact subset of U.*

Proof. According to the lemma, f is locally bounded; so given $\boldsymbol{x}_0$, we may find a spherical neighborhood $N_{2\varepsilon}(\boldsymbol{x}_0)$ of radius 2ε on which f is bounded, say by M. For distinct $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in $N_\varepsilon(\boldsymbol{x}_0)$, set $\boldsymbol{x}_3 = \boldsymbol{x}_2 + (\varepsilon/\alpha)(\boldsymbol{x}_2 - \boldsymbol{x}_1)$ where $\alpha = \|\boldsymbol{x}_2 - \boldsymbol{x}_1\|$ and note that $\boldsymbol{x}_3$ is in $N_{2\varepsilon}(\boldsymbol{x}_0)$. If we solve for $\boldsymbol{x}_2$, we obtain

$$\boldsymbol{x}_2 = \frac{\varepsilon}{\alpha + \varepsilon} \boldsymbol{x}_1 + \frac{\alpha}{\alpha + \varepsilon} \boldsymbol{x}_3$$

and so by convexity,

$$f(x_2) \leqq \frac{\varepsilon}{\alpha+\varepsilon} f(x_1) + \frac{\alpha}{\alpha+\varepsilon} f(x_3).$$

Then

$$f(x_2) - f(x_1) \leqq \frac{\alpha}{\alpha+\varepsilon}\,[f(x_3) - f(x_1)] \leqq \frac{\alpha}{\varepsilon}\left|f(x_3) - f(x_1)\right|,$$

which combined with $|f| \leqq M$ and $\alpha = \|x_2 - x_1\|$ yields

$$f(x_2) - f(x_1) \leqq (2M/\varepsilon)\|x_2 - x_1\|.$$

Since the roles of x_1 and x_2 can be interchanged, we have

$$\left|f(x_2) - f(x_1)\right| \leqq (2M/\varepsilon)\|x_2 - x_1\|,$$

that is, f is Lipschitz on $N_\varepsilon(x_0)$. We conclude that f is locally Lipschitz on U.

Now let D be a compact subset of U. The collection $\{N_\varepsilon(x_0)\}$ of neighborhoods obtained above covers D, as does some finite subcollection $N_1, N_2, \cdots, N_m$. Let $K = \max\{K_1, K_2, \cdots, K_m\}$ where K_i is the Lipschitz constant corresponding to N_i, $i = 1, 2, \cdots, m$. Finally let $x \in N_i$ and $y \in N_j$ be any two distinct points of D and choose a segment $[w, z]$ containing segment $[x, y]$ in its interior so that $w \in N_i$ and $z \in N_j$. From the convexity of f on segment $[w, z]$,

$$-K \leqq \frac{f(x) - f(w)}{\|x - w\|} \leqq \frac{f(y) - f(x)}{\|y - x\|} \leqq \frac{f(z) - f(y)}{\|z - y\|} \leqq K$$

which yields the conclusion $\left|f(y) - f(x)\right| \leqq K\|y - x\|$. ∎

Now for the embellishments. The definitions of convex, bounded, and Lipschitz all extend without modification to an arbitrary normed linear space. So does the proof of Theorem A; only the lemma offers any difficulties, but they are real. A convex function on an infinite dimensional normed linear space may be locally unbounded. For example, the linear functional $f: p \to p'(0)$ on the space of polynomials normed by

$$\|p\| = \max_{-1 \leqq x \leqq 1} |p(x)|$$

has this property. A slight additional condition fixes everything up.

LEMMA B. *Let f be convex on an open convex set U in a normed linear space. If f is bounded above in a neighborhood of just one point, then f is locally bounded on U.*

Proof. For convenience of notation, we suppose that the given point is the origin and that f is bounded above by M on a spherical neighborhood $N = N_\varepsilon(0)$. Let y be any other point of U and choose $\rho > 1$ so that $z = \rho y$ is in U. If $\lambda = 1/\rho$, then

$$V = \{v : v = (1 - \lambda)x + \lambda z, x \text{ in } N\}$$

is a neighborhood of $\boldsymbol{y} = \lambda \boldsymbol{z}$ with radius $(1 - \lambda)\varepsilon$. Moreover,

$$f(\boldsymbol{v}) \leqq (1 - \lambda) f(\boldsymbol{x}) + \lambda f(\boldsymbol{z}) \leqq M + f(\boldsymbol{z}).$$

Thus, f is bounded above in some neighborhood of each point $\boldsymbol{y}$ in U. A repetition of the second paragraph in the proof of Lemma A shows that it is also bounded below on each such neighborhood. ∎

We have all the ingredients for a tangy generalization.

THEOREM B. *Let f be convex on an open convex set U in a normed linear space. If f is bounded above in a neighborhood of one point of U, then f is locally Lipschitz on U, hence Lipschitz on any compact subset of U.*

Compactness is a strong requirement, often missing, especially for sets in infinite dimensional spaces. We can make a substitute for it; and the proof of the resulting theorem is still essentially that of Theorem A.

THEOREM C. *Let f be convex with $|f| \leqq M$ on an open convex set U in a normed linear space. If U contains an ε-neighborhood of a subset V, then f is Lipschitz (with Lipschitz constant $2M/\varepsilon$) on V.*

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ON POLARS OF CONVEX POLYGONS

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In discussions concerning convexity and linear inequalities, it is often necessary to find the polar of a convex set in Euclidean space. The purpose of this note is to give a very elementary method for completely determining the polars of certain convex polygons in R^2. We feel this is worthwhile for two reasons. First, it is an interesting geometric result that can be easily understood by students with a minimal background in geometry. Second, while it is usually stated that the polar of a convex polyhedron is a convex polyhedron (cf. [**1**, p. 174]), no mention is made of how the vertices of the polar can be explicitly found, and this is the content of our result.

Given a set U in the real linear space R^2, the polar of U is defined by

$$U^\circ = \{(u, v) \in R^2 : |ux + vy| \leqq 1 \text{ for all } (x, y) \in U\}.$$

If $z = (a, b)$ and $(a, b) \neq (0,0)$, it is simple to show that $\{z\}^\circ$ is the infinite strip