Conditional Log-Concave Density Estimation

Write the negative log-conditional-likelihood for a data point (X_i, Y_i) as

$$\log\Big(\int \exp(f(X_i, y))\, dy\Big) - f(X_i, Y_i) \tag{1.1}$$

where $f(x, y)$ is jointly concave in x and y. Let $z_i \equiv f(X_i, Y_i)$ and write

$$f(x, y) = \min_k \Big\{ z_k + \beta_k^T (x - X_k) + \alpha_k (y - Y_k) \Big\}. \tag{1.2}$$

Define

$$I_k(X_i) = \Big\{ y \ :\ k = \arg\min_j \Big\{ z_j + \beta_j^T (X_i - X_j) + \alpha_j (y - Y_j) \Big\} \Big\}. \tag{1.3}$$

Then

$$\log \int \exp(f(X_i, y)) dy = \log\Big(\sum_k e^{z_k + \beta_k^T (X_i - X_k)} \int_{I_k(X_i)} e^{\alpha_k (y - Y_k)}\, dy \Big). \tag{1.4}$$

By convexity of log-sum-exp, this is a convex function of z, α, and β. Note that, assuming I_k is an interval, the above integral over I_k can be computed in closed form.

We are thus led to the convex program

$$\begin{aligned} \underset{z,\alpha,\beta}{\text{minimize}} \quad & \sum_{i=1}^n \Big\{ \log\Big(\sum_k e^{z_k + \beta_k^T (X_i - X_k)} \int_{I_k(X_i)} e^{\alpha_k (y - Y_k)}\, dy \Big) - z_i \Big\} & (1.5) \\ \text{subject to} \quad & z_j \le z_i + \beta_i^T (X_j - X_i) + \alpha_i (Y_j - Y_i), \quad 1 \le i, j \le n. & (1.6) \end{aligned}$$