We construct a simulation graph with log-concave density as follows. Consider the following joint distribution

$$p(\boldsymbol{x}, s) = p(\boldsymbol{x}|s) \times p(s) = \mathcal{N}(\boldsymbol{x}; \mathbf{0}, s\boldsymbol{I}_m) \times p(s) \quad (s > 0)$$

where $\boldsymbol{x} \in \mathbb{R}^m$ is drawn from a multivariate Gaussian with variance $s \in \mathbb{R}_+$ drawn from $p(s)$. Since

$$\log p(\boldsymbol{x}, s) = \log \mathcal{N}(\boldsymbol{x}; \mathbf{0}, s\boldsymbol{I}_m) + \log p(s) = -\frac{m}{2}\log(2\pi) - \frac{m}{2}\log s - \frac{\boldsymbol{x}^\top \boldsymbol{x}}{2s} + \log p(s)$$

and $\frac{\boldsymbol{x}^\top \boldsymbol{x}}{2s}$ is convex jointly for $(\boldsymbol{x}, s)$, it can be concluded that $p(\boldsymbol{x}, s)$ is log-concave if $\log p(s) - \frac{m}{2}\log s$ is concave [1]. Consider a Gamma prior on s:

$$p(s) = \mathrm{Ga}(s; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} s^{\alpha-1}\exp(-\beta s) \quad (s > 0).$$

Then according to the above analysis, $p(\boldsymbol{x}, s) = \mathcal{N}(\boldsymbol{x}; \mathbf{0}, s\boldsymbol{I}_m) \times \mathrm{Ga}(s; \alpha, \beta)$ is log-concave if $\alpha \geq 1 + \frac{m}{2}$. The conditional distribution $p(s|\boldsymbol{x})$ can be derived as

$$p(s|\boldsymbol{x}) \propto s^{-\frac{m}{2}}\exp(-\frac{\boldsymbol{x}^\top \boldsymbol{x}}{2s}) s^{\alpha-1}\exp(-\beta s) = \mathrm{GIG}(s; 2\beta, \boldsymbol{x}^\top\boldsymbol{x}, \alpha - \frac{m}{2})$$

where $\mathrm{GIG}(s; a, b, c) = \frac{(a/b)^{c/2}}{2K_c(\sqrt{ab})} s^{c-1}\exp(-\frac{1}{2}(as + \frac{b}{s}))$ $(s > 0)$ is the Generalized Inverse Gaussian (GIG) distribution with mean $E(s) = \frac{\sqrt{b}}{\sqrt{a}}\frac{K_{c+1}(\sqrt{ab})}{K_c(\sqrt{ab})}$, and $K_c(\cdot)$ is the modified Bessel function of the second kind. Hence the conditional mean can be expressed as

$$E(s|\boldsymbol{x}) = \frac{\|\boldsymbol{x}\|_2}{\sqrt{2\beta}} \frac{K_{\alpha-\frac{m}{2}+1}(\sqrt{2\beta}\|\boldsymbol{x}\|_2)}{K_{\alpha-\frac{m}{2}}(\sqrt{2\beta}\|\boldsymbol{x}\|_2)}$$

which is a nonlinear and non-additive function of $\boldsymbol{x}$.

Motivated by the above analysis, we generate data as

$$s_{1i} \sim \mathrm{Ga}(\alpha, \beta), \boldsymbol{x}_{1i} \sim \mathcal{N}(\mathbf{0}, s_{1i}\boldsymbol{I}_m); \quad s_{2i} \sim \mathrm{Ga}(\alpha, \beta), \boldsymbol{x}_{2i} \sim \mathcal{N}(\mathbf{0}, s_{2i}\boldsymbol{I}_m); \quad \cdots\cdots \quad (i = 1, 2, \cdots, n)$$

Then the vector $(s_{1i}, \boldsymbol{x}_{1i}, s_{2i}, \boldsymbol{x}_{2i}, \cdots) \in \mathbb{R}^p$ is drawn from a log-concave graph with linkage structure depicted in Figure 6. In the experiment we set $p = 20, n = 500, m = 4, \alpha = 5, \beta = 0.5$. We then do neighborhood regression using SCCAM, Meinshausen-Bhlmann, and nonparanormal to identify the graph structure. We declare a edge between two nodes if an edge exists in either direction. As we observe in the ROC curve in Figure 6, SCCAM performs much better than the baselines for this experiment.

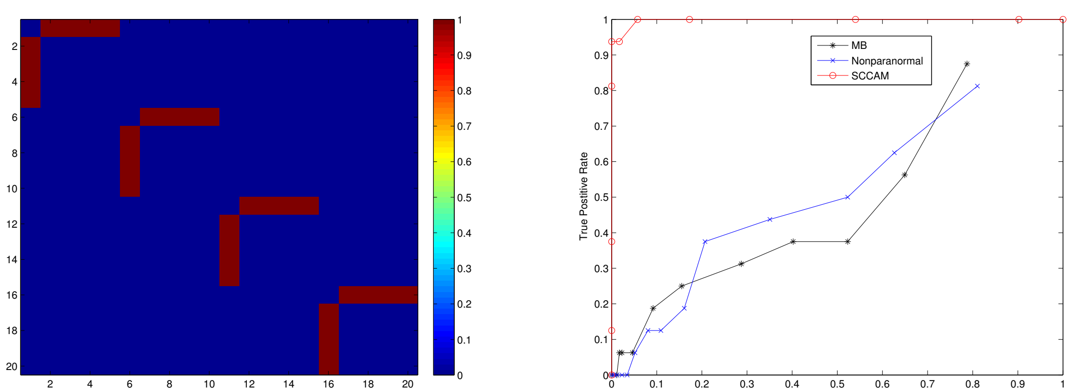

Figure 6: Left: The true graph structure. Right: ROC curve for graph estimation.

[1]Matthias Seeger, A Note on Log-Concavity, 2007.