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CONSTRUCTIVE APPROXIMATION

Kolmogorov Entropy for Classes of Convex Functions

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Abstract Kolmogorov ε-entropy of a compact set in a metric space measures its *metric massivity* and thus replaces its dimension which is usually infinite. The notion quantifies the compactness property of sets in metric spaces, and it is widely applied in pure and applied mathematics. The ε-entropy of a compact set is the most economic quantity of information that permits a recovery of elements of this set with accuracy ε. In the present article we study the problem of asymptotic behavior of the ε-entropy for uniformly bounded classes of convex functions in L_p-metric proposed by A.I. Shnirelman. The asymptotic of the Kolmogorov ε-entropy for the compact metric space of convex and uniformly bounded functions equipped with L_p-metric is $\varepsilon^{-1/2}$, $\varepsilon \to 0_+$.

Keywords Kolmogorov ε-entropy · Kolmogorov ε-capacity · Compact metric space · Massivity of a set · Convex function · L_p-metric · Hamming distance · Asymptotic behavior · Multistep approximation procedure

Mathematics Subject Classification (2000) 41A46 · 46B10 · 52A10 · 52A41 · 54C70

1 Introduction

In his research on ideal incompressible fluids (fluid dynamics) A.I. Shnirelman [10] arrived at the following question: *What is the asymptotic of the Kolmogorov ε-entropy of the metric space of convex and uniformly bounded functions equipped with L_p-metric*? In the present paper we give an answer to this question by establishing

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the precise asymptotic of Kolmogorov ε-entropy for the L_p-metric space of convex and uniformly bounded functions (*without prescribed uniform Lipschitz behavior*). Our considerations extend the geometric approach to the problem of Kolmogorov ε-entropy for sets of convex functions initiated by Bronshtein [1]. We begin by giving a brief description of the concept of Kolmogorov ε-entropy.

1.1 Kolmogorov ε-Entropy for Compact Sets in Metric Spaces

Following Fréchet, a metric space is a tuple $\mathfrak{M}_\rho$, where $\mathfrak{M}$ is a set and ρ is a metric on $\mathfrak{M}$. Hence, a compact set in a metric space can be considered as a compact metric space.

Definition Let $\mathfrak{M}_\rho$ be a compact metric space. Given $\varepsilon > 0$, the set $A \subset \mathfrak{M}_\rho$ is called an ε-net for $\mathfrak{M}_\rho$ if for each $w \in \mathfrak{M}_\rho$ there is $a \in A$ such that $\rho(w, a) \leq \varepsilon$.

Definition (Kolmogorov ε-entropy) Let $\varepsilon > 0$. If a minimal ε-net of a compact metric space $\mathfrak{M}_\rho$ contains $\mathbb{N}_{\mathfrak{M}_\rho}(\varepsilon)$ elements, then the Kolmogorov ε-entropy $H_{\mathfrak{M}_\rho}(\varepsilon)$ of $\mathfrak{M}_\rho$ is defined by $H_{\mathfrak{M}_\rho}(\varepsilon) = \log_2 \mathbb{N}_{\mathfrak{M}_\rho}(\varepsilon)$.

Kolmogorov ε-entropy quantifies the compactness property of sets in metric spaces. Compact sets can be ordered with respect to their massivity by using as a measure of their massivity the asymptotic of their ε-entropy, $\varepsilon \to 0_+$. In information theory the ε-entropy of a compact set is the most economic quantity of information that permits a recovery (a binary record) of each element of this set with accuracy ε. Studying the asymptotic behavior of Kolmogorov ε-entropy for compact metric spaces is a branch of approximation theory.

Definition Let $\mathfrak{M}_\rho$ be a compact metric space. Given $\varepsilon > 0$, the set $A \subset \mathfrak{M}_\rho$ is called an ε-distant set for $\mathfrak{M}_\rho$ if for each pair $a_1, a_2 \in A$, $\rho(a_1, a_2) \geq \varepsilon$.

Definition (Kolmogorov ε-capacity) Let $\mathfrak{M}_\rho$ be a compact metric space. Given $\varepsilon > 0$, if a maximal ε-distant set of a compact metric space $\mathfrak{M}_\rho$ contains $\mathbb{M}_{\mathfrak{M}_\rho}(\varepsilon)$ elements, then the Kolmogorov ε-capacity $C_{\mathfrak{M}_\rho}(\varepsilon)$ of $\mathfrak{M}_\rho$ is defined by $C_{\mathfrak{M}_\rho}(\varepsilon) = \log_2 \mathbb{M}_{\mathfrak{M}_\rho}(\varepsilon)$.

Lemma A (Kolmogorov-Tikhomirov [7, 8]) *The following relations between the ε-entropy and the ε-capacity of $\mathfrak{M}_\rho$ hold*: $C_{\mathfrak{M}_\rho}(2\varepsilon) \leq H_{\mathfrak{M}_\rho}(\varepsilon) \leq C_{\mathfrak{M}_\rho}(\varepsilon)$.

The concept of Kolmogorov ε-entropy is widely applied in many branches of pure and applied mathematics, see [2, 4, 7, 8, 11–13].

1.2 Kolmogorov ε-Entropy for C-Metric Compact Space of Convex, Uniformly Bounded and Uniformly Lipschitz Functions

Let $\Lambda_{C,\mathrm{Lip}}$ be the metric space of convex functions defined on an interval, uniformly Lipschitz, uniformly bounded and equipped with C-metric. Denote by $H_{\Lambda_{L_C,\mathrm{Lip}}}(\varepsilon)$ the Kolmogorov ε-entropy of the compact metric space $\Lambda_{C,\mathrm{Lip}}$. The following result holds.

Theorem B (Bronshtein [1], Dudley [3]) *There exist constants* $c_1 > 0$ *and* $c_2 > 0$ *such that* $c_1\varepsilon^{-1/2} \le H_{\Lambda_{C,\mathrm{Lip}}}(\varepsilon) \le c_2\varepsilon^{-1/2}$, $\varepsilon \le \varepsilon_0$.

1.3 Main Results

In Sect. 2 we obtain the precise asymptotic of the Kolmogorov ε-entropy for the L_p-metric space of convex, uniformly bounded and uniformly Lipschitz functions (Theorem 2.1). We attempt to see the result of Theorem 2.1 in light of a recent general statement on quantifying of the Krein-Milman Theorem (see Comparison Explanations in Sect. 2). At the beginning of Sect. 3 we show (Lemma 3.2) that the metric space of convex and uniformly bounded functions equipped with L_p-metric, $1 \le p < \infty$, is compact (*without a condition for uniform Lipschitz behavior*). This makes the problem of estimating the massivity of such sets by using Kolmogorov ε-entropy meaningful. Next in Sect. 3, we answer (Theorem 3.1) a question asked by Shnirelman [10]: *What is the asymptotic behavior of the Kolmogorov ε-entropy for the L_p-metric space of convex and uniformly bounded functions*? By quantifying the compactness property of the L_p-metric space of convex and uniformly bounded functions in terms of Kolmogorov ε-entropy, we evaluate its massivity. The proof of Theorem 3.1 is based on Lemma 3.1 and *a multistep approximation procedure*.

2 Kolmogorov ε-Entropy for Lipschitz Classes of Convex Functions in L_p-Metric

We establish the precise asymptotic of the Kolmogorov ε-entropy for the L_p-metric space of convex and uniformly bounded functions under the additional condition of uniform Lipschitz behavior. According to Arzelà-Ascoli's Theorem and Lebesgue's dominated convergence theorem, the L_p-metric space of convex uniformly bounded and uniformly Lipschitz functions is compact, taking into account that the pointwise limit of a sequence of convex functions is a convex function.

Remark 2.1 Let $\Lambda_{L_p[a,b],\mathrm{Lip}_M,B}$ be the L_p-metric space of convex functions, defined on an interval $[a,b]$, uniformly Lipschitz on $[a,b]$ with a Lipschitz constant M and uniformly bounded on $[a,b]$ with a uniform bound B. Let $f, g \in \Lambda_{L_p[a,b],\mathrm{Lip}_M,B}$. Then trivially, $\|f-g\|_{L_p[a,b]} = c(b-a, M, B, p)\|\tilde{f}-\tilde{g}\|_{L_p[0,1]}$, where $\tilde{f}, \tilde{g}$ are convex on $[0, 1]$; $|\tilde{f}|, |\tilde{g}|$ are bounded by 1; and $\tilde{f}, \tilde{g}$ are Lipschitz on $[0, 1]$ with a Lipschitz constant 1. We denote by $\Lambda_{L_p,\mathrm{Lip}}$ the L_p-metric space of convex functions defined on an interval $[0, 1]$, uniformly Lipschitz on $[0, 1]$ with a Lipschitz constant 1 and uniformly bounded on $[0, 1]$ with a uniform bound 1. Then the number of elements in a minimal $\Lambda_{L_p[a,b],\mathrm{Lip}_M,B}(\varepsilon)$-net is equal to the number of the elements in a minimal $\Lambda_{L_p,\mathrm{Lip}}(\varepsilon/c(b-a, M, B, p))$-net. Hence the order of the ε-asymptotic of the entropy for $\Lambda_{L_p[a,b],\mathrm{Lip}_M,B}$ is equal to the order of the ε-asymptotic of the entropy for $\Lambda_{L_p,\mathrm{Lip}}$. Analogously, the order of the ε-asymptotic of the capacity for $\Lambda_{L_p[a,b],\mathrm{Lip}_M,B}$ is equal to the order of the ε-asymptotic of the capacity for $\Lambda_{L_p,\mathrm{Lip}}$.

In view of the above remark, without any restriction, we study Kolmogorov ε-entropy of the metric space $\Lambda_{L_p,\mathrm{Lip}}$ of convex functions, uniformly Lipschitz on [0, 1] with a Lipschitz constant 1, uniformly bounded on [0, 1] with a uniform bound 1 and equipped with $L_p[0,1]$-metric.

Lemma 2.1 *For a fixed $\varepsilon > 0$, the ε-entropy $H_{\Lambda_{L_p,\mathrm{Lip}}}(\varepsilon)$ and the ε-capacity $C_{\Lambda_{L_p,\mathrm{Lip}}}(\varepsilon)$ of $\Lambda_{L_p,\mathrm{Lip}}$ are increasing functions of p, $1 \le p < \infty$. In addition,*

$$H_{\Lambda_{L_p,\mathrm{Lip}}}(\varepsilon) \le H_{\Lambda_{C,\mathrm{Lip}}}(\varepsilon) \le c\,\varepsilon^{-1/2} \quad \text{and} \quad C_{\Lambda_{L_p,\mathrm{Lip}}}(\varepsilon) \le C_{\Lambda_{C,\mathrm{Lip}}}(\varepsilon).$$

Proof The proof is a trivial consequence of Theorem B and the norm inequalities

$$\|f-g\|_{L_p[0,1]} \le \|f-g\|_{L_q[0,1]} \le \|f-g\|_{C[0,1]} \quad (1 \le p < q < \infty). \qquad \square$$

How shall we proceed with the study of Kolmogorov $\boldsymbol{\varepsilon}$-entropy for $\boldsymbol{\Lambda_{L_p,\mathrm{Lip}}}$? *We establish a lower bound for the capacity $C_{\Lambda_{L_1,\mathrm{Lip}}}(\varepsilon)$ (Lemma 2.5). According to Lemma 2.1, this will be a lower bound for the capacity $C_{\Lambda_{L_p,\mathrm{Lip}}}(\varepsilon)$, $1 < p < \infty$, and by using Lemma A we obtain a lower bound for the ε-entropy of $\Lambda_{L_p,\mathrm{Lip}}$. In order to obtain an estimate from below for the ε-capacity of $\Lambda_{L_1,\mathrm{Lip}}$, we use a combinatorial result (Lemma 2.4), a polynomial inequality between different metrics (Lemma 2.2), an appropriate construction and geometric considerations by interpreting a definite integral in terms of area (see Fig. 1 and the proof of Lemma 2.5). The estimate from above of the ε-entropy for $\Lambda_{L_p,\mathrm{Lip}}$ is a corollary of Lemma 2.1.*

Lemma 2.2 [9] *Let q be a polynomial of degree ≤ 1 defined on the interval $[x_k, x_{k+1}]$. Then $\|q\|_{C[x_k,x_{k+1}]} \le 4(x_{k+1}-x_k)^{-1}\|q\|_{L_1[x_k,x_{k+1}]}$.*

Proof Taking into account that $\max_{x\in[x_k,x_{k+1}]}|q(x)|$ is at one of the endpoints of $[x_k, x_{k+1}]$, we obtain $\|q\|_{L_1[x_k,x_{k+1}]} \ge \frac{1}{2}\frac{x_{k+1}-x_k}{2}\|q\|_{C[x_k,x_{k+1}]}$. $\square$

Lemma 2.3 *Let A_{k-1}, A_k, A_{k+1} be three different points on the circle $\partial B(0,r)$ centered at 0 of radius r. Let $\mathrm{dist}(A_{k-1}, A_k) = \mathrm{dist}(A_k, A_{k+1}) = h$. Then $\mathrm{dist}(A_k, P) = h^2/(2r)$, where P is the orthogonal projection of A_k on $[A_{k-1}, A_{k+1}]$ and $\mathrm{dist}(A_{k-1}, A_{k+1}) = 2\sqrt{h^2 - h^4/(4r^2)}$.*

Proof The proof is a straight application of the Pythagorean Theorem. $\square$

Lemma 2.4 (Combinatorial Lemma) *Consider the set Ω_n of all binary vectors $\mathbf{v} = (v_1, v_2, \dots, v_n)$ of a fixed length n, i.e., $v_k = 1$ or $v_k = 0$, $k = 1, 2, \dots, n$. Then for sufficiently large $n \ge n_0$ there exists a subset Ω_n^* of Ω_n containing $\ge (4/3)^n$ vectors such that any two vectors $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ from Ω_n^* are different in more than $\lfloor n/10 \rfloor$ places, where $\lfloor n/10 \rfloor$ is the integer part of $n/10$.*

Remark 2.2 The metric space Ω_n of binary strings of length n equipped with Hamming distance is known as the Hamming cube [5, 6]. Let us point out that Lemma 2.4

shows the existence of a $\lfloor n/10 \rfloor$-distant set in Ω_n with respect to the Hamming distance containing not less than $(4/3)^n$ elements (strings).

Proof of Lemma 2.4 Let $\mathbf{v} \in \Omega_n$, and let $A(\mathbf{v})$ be the set of all vectors in Ω_n which are different from $\mathbf{v}$ at $\leq \lfloor n/10 \rfloor$ places. Substitute $s = \lfloor n/10 \rfloor$, and obviously $s \geq 1$ for $n \geq 10$. For the number $\mathbb{N}[A(\mathbf{v})]$ of vectors in $A(\mathbf{v})$ the following estimate holds:

$$\mathbb{N}[A(\mathbf{v})] = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{s} \leq (s+1)\binom{n}{s}, \quad (s < n/2).$$

By Stirling's approximation $\sqrt{2\pi} s^{s+1/2} e^{-s+1/(12s+1)} < s! < \sqrt{2\pi} s^{s+1/2} e^{-s+1/(12s)}$,

$$\begin{aligned}\binom{n}{s} &= \frac{n(n-1)\cdots(n-(s-1))}{s!} \leq \left(\frac{n}{s}\right)^s \frac{s^s}{s!} \\ &\leq \left(\frac{n}{s}\right)^s \frac{s^s}{\sqrt{2\pi} s^{s+1/2} e^{-s+1/(12s+1)}} = \left(\frac{ne}{s}\right)^s \frac{1}{\sqrt{2\pi} s^{1/2} e^{1/(12s+1)}} \leq \left(\frac{ne}{s}\right)^s.\end{aligned}$$

Obviously, $n/11 \leq n/10 - 1 < [n/10] = s \leq n/10$ if $n \geq 110$, and therefore, $(\frac{ne}{s})^s \leq (11e)^{[n/10]} \leq (11e)^{n/10} = [(11e)^{1/10}]^n < (1.41)^n$. Hence for $n \geq n_0$ the following estimate holds: $\mathbb{N}[A(\mathbf{v})] < (s+1)\binom{n}{s} < (n/10+1)(\frac{1.41}{1.5})^n (3/2)^n < (3/2)^n$.

Construct a set of vectors as follows: Take an arbitrary $\mathbf{v^{(1)}} \in \Omega_\mathbf{n}$. Then choose $\mathbf{v^{(2)}} \in \Omega_\mathbf{n} \backslash \mathbf{A}(\mathbf{v^{(1)}})$. Next, choose $\mathbf{v^{(3)}} \in \Omega_\mathbf{n} \backslash [\mathbf{A}(\mathbf{v^{(1)}}) \cup \mathbf{A}(\mathbf{v^{(2)}})]$, and continue this procedure until Ω_n is entirely exhausted. In this way we obtain a set of vectors $\Omega_n^* = \{\mathbf{v^{(1)}}, \mathbf{v^{(2)}}, \ldots, \mathbf{v^{(m)}}\}$ such that each two vectors from Ω_n^* are different in more than $[n/10]$ places. The total number $\mathbb{N}[\Omega_n]$ of vectors in Ω_n is 2^n. We estimate from below the number m of vectors in Ω_n^*: $\mathbb{N}[\Omega_n] = 2^n \leq \sum_{k=1}^m \mathbb{N}[A(\mathbf{v^{(k)}})] < m(3/2)^n$. From here, $m > (4/3)^n$. This completes the proof. □

Lemma 2.5 *The ε-capacity $C_{\Lambda_{L_p,\mathrm{Lip}}}(\varepsilon)$ of $\Lambda_{L_p,\mathrm{Lip}}$ is an increasing function of p, $1 \leq p < \infty$. Moreover, there exists a constant $c > 0$ such that*

$$c\,\varepsilon^{-1/2} \leq C_{\Lambda_{L_1,\mathrm{Lip}}}(\varepsilon) \leq C_{\Lambda_{L_p,\mathrm{Lip}}}(\varepsilon) \leq C_{\Lambda_{C,\mathrm{Lip}}}(\varepsilon).$$

Proof By Lemma 2.1, $C_{\Lambda_{L_1,\mathrm{Lip}}}(\varepsilon) \leq C_{\Lambda_{L_p,\mathrm{Lip}}}(\varepsilon) \leq C_{\Lambda_{C,\mathrm{Lip}}}(\varepsilon)$. We prove the estimate $c\,\varepsilon^{-1/2} \leq C_{\Lambda_{L_1,\mathrm{Lip}}}(\varepsilon)$. Without any restriction and for geometric simplicity, our considerations will be in the interval $[-1,1]$ (see Remark 2.1). For $\delta > 0$ consider the part $\partial B_\delta := \{(x,y) : (y-\delta)^2 + x^2 = 1 + \delta^2;\ y \leq 0\}$ of the circle $B((0,\delta), \sqrt{1+\delta^2})$ centered at $(0,\delta)$ of radius $\sqrt{1+\delta^2}$ (see Fig. 1).

Let $f(x)$ denote the convex function $y = f(x)$ defined on $[-1,1]$ whose graph coincides with ∂B_δ. Take in a consecutive order $(n+2)$ points $(-1,0) = A_0, A_1, \ldots, A_n, A_{n+1} = (1,0)$ on the graph of $f(x)$ such that $\mathrm{dist}(A_k, A_{k+1}) = \sqrt{\varepsilon}$, $k = 0, 1, \ldots, n-1$, and $\sqrt{\varepsilon} \leq \mathrm{dist}(A_n, A_{n+1}) \leq 2\sqrt{\varepsilon}$. The function $y = f(x)$ is Lipschitz with Lipschitz constant $M(\delta)$. Denote by $-1 = x_0, x_1, \ldots, x_n, x_{n+1} = 1$ the orthogonal projections of the points $A_0, A_1, \ldots, A_n, A_{n+1}$ on the x-axis. The points $x_0, x_1, \ldots, x_n, x_{n+1}$ satisfy $\mathrm{dist}(A_k, A_{k+1})/\sqrt{1+M(\delta)^2} \leq x_{k+1} - x_k \leq \mathrm{dist}(A_k, A_{k+1})$, $k = 0, 1, \ldots, n$, where $\mathrm{dist}(\circ,\circ)$ is the Euclidean distance. Hence,

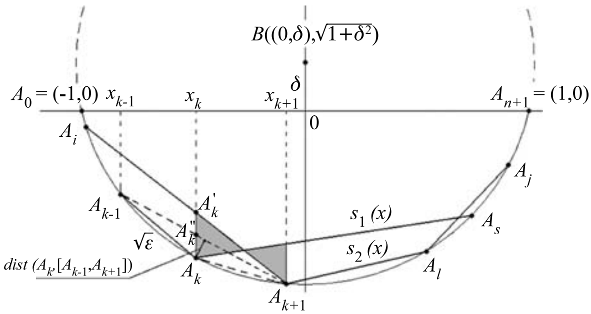

Fig. 1 The graph of the *generating* function for the spline set Spl_n is the part of the circle $B((0,\delta),\sqrt{1+\delta^2})$ below the x-axis. The first degree splines $s_1(x) \in Spl_n^*$, $s_2(x) \in Spl_n^*$ have end-knots at A_0 and A_{n+1} and as other knots some of the points $A_1, A_2, \dots, A_n$. The point A_k is a knot of $s_1(x)$ but not a knot of $s_2(x)$

$c_1\sqrt{\varepsilon} \le x_{k+1} - x_k \le c_2\sqrt{\varepsilon}$, $c_1 > 0, c_2 > 0$. Construct all first-degree splines (polygons) having as end-knots A_0 and A_{n+1} and as other knots some of the n points A_k, $k = 1, 2, \dots, n$, on the graph of $y = f(x)$. Denote the set of all such splines by Spl_n. Take $s(x) \in Spl_n$ and form a vector $\mathbf{v} = (v_1, v_2, \dots, v_n)$ corresponding to $s(x)$ as follows: If a point A_k, $1 \le k \le n$, is a knot of $s(x)$, then assign $v_k = 1$; if A_k is not a knot of $s(x)$, then assign $v_k = 0$. *In this way we obtain a one-to-one correspondence between the set Ω_n (see Lemma 2.4) and the set of all splines Spl_n constructed by using the function $y = f(x)$, $x \in [-1, 1]$, as a generating (mother) function. Both sets Ω_n and Spl_n have 2^n elements, where $n \ge c/\sqrt{\varepsilon}, c > 0$.* Consider the subset $Spl_n^* \subset Spl_n$ corresponding to the set $\Omega_n^* \subset \Omega_n$ of Lemma 2.4. Take two polygons s_1 and s_2 from the set Spl_n^* (see Fig. 1). According to Lemma 2.4, *at more than $\lfloor n/10 \rfloor$ points $A_k, k = 1, 2, \dots, n$, we have the following property: If A_k is a knot of $s_1(x)$, then A_k is not a knot of $s_2(x)$, and if A_k is a knot of $s_2(x)$, then A_k is not a knot of $s_1(x)$.* Without any restriction, suppose that A_k is a knot of $s_1(x)$ but not a knot of $s_2(x)$. Obviously, the triangle with vertices at A_{k-1}, A_k, A_{k+1} does not contain points from the graph of $s_2(x)$ except possibly either A_{k-1}, or A_{k+1}, or the entire line segment $[A_{k-1}, A_{k+1}]$ if both A_{k-1} and A_{k+1} are knots of $s_2(x)$. We continue with more details from our geometric construction (see Fig. 1) in order to complete the proof.

Consider $s_1(x) - s_2(x)$ in the interval $[x_k, x_{k+1}]$. Let A_k be a knot of $s_1(x)$ but not of $s_2(x)$. For the knot A_k we count the interval to the right of A_k, namely $[x_k, x_{k+1}]$. Denote by A'_k the point of intersection of $s_2(x)$ and $x = x_k$. The point $(x_k, 0)$ is the orthogonal projection of A_k on the x-axis. The point A'_k is on the line segment $[A_k, (x_k, 0)]$. The triangle $\triangle A_{k-1}, A_k, A_{k+1}$ with vertices at A_{k-1}, A_k, A_{k+1} does not contain points from the graph of $s_2(x)$ except possibly the point A_{k-1}, or A_{k+1}, or the entire line segment $[A_{k-1}, A_{k+1}]$. If we denote by A''_k the point of intersection of the line segment $[A_k, (x_k, 0)]$ with the line segment $[A_{k-1}, A_{k+1}]$, then by Lemma 2.3

$$\operatorname{dist}(A_k, A'_k) \ge \operatorname{dist}(A_k, A''_k) \ge \operatorname{dist}\bigl(A_k, [A_{k-1}, A_{k+1}]\bigr) \ge \frac{\varepsilon}{2\sqrt{1+\delta^2}}.$$

Therefore, $\operatorname{dist}(A_k, A'_k) \ge \varepsilon/(2\sqrt{1+\delta^2})$. Hence the first-degree polynomial $s_1(x) - s_2(x)$ on $[x_k, x_{k+1}]$ has at x_k a functional value $\ge \varepsilon/(2\sqrt{1+\delta^2})$. By interpreting the L_1-norm $\|s_1 - s_2\|_{L_1[x_k, x_{k+1}]}$ in terms of the area between the graphs of $s_1(x)$ and $s_2(x)$ on the interval $[x_k, x_{k+1}]$, by Lemma 2.2 and by the fact that $x_{k+1} - x_k \ge$

$c(\delta)\sqrt{\epsilon}$, we conclude that $\|s_1 - s_2\|_{L_1[x_k, x_{k+1}]} \geq c(\delta)\varepsilon^{3/2}$. According to Lemma 2.4, for the spline pair $\{s_1, s_2\}$, we have at more than $\lfloor n/10 \rfloor$ points A_k, $k = 1, 2, \ldots, n$, the following property: *If A_k is a knot of $s_1(x)$, then A_k is not a knot of $s_2(x)$, and if A_k is a knot of $s_2(x)$, then A_k is not a knot of $s_1(x)$.* In view of this, at more than $\lfloor n/10 \rfloor$ intervals $[x_k, x_{k+1}]$ we have $\|s_1 - s_2\|_{L_1[x_k, x_{k+1}]} \geq c(\delta)\,\varepsilon^{3/2}$. From here,

$$\|s_1 - s_2\|_{L_1[-1,1]} \;\geq\; c(\delta) \left\lfloor \frac{n}{10} \right\rfloor \varepsilon^{3/2} \;\geq\; \frac{c(\delta)}{\sqrt{\varepsilon}} \varepsilon^{3/2} = c(\delta)\,\varepsilon.$$

The number of splines (polygons) in the set Spl_n^* is $\geq (4/3)^n \geq 2^{c/\sqrt{\varepsilon}}$, $c > 0$, according to Lemma 2.4. Hence, Spl_n^* is a $c(\delta)\varepsilon$-distant set in $\Lambda_{L_p[-1,1],\mathrm{Lip}_{M(\delta)},1}$ containing at least $2^{c/\sqrt{\varepsilon}}$ elements (polygons). Substituting $\varepsilon = \varepsilon/c(\delta)$, we obtain an ε-distant set in the metric space $\Lambda_{L_p[-1,1],\mathrm{Lip}_{M(\delta)},1}$ containing at least $2^{c\sqrt{c(\delta)}/\sqrt{\varepsilon}}$ elements (polygons). According to Remark 2.1 and the definition of Kolmogorov ε-capacity, we conclude that $C_{\Lambda_{L_1,\mathrm{Lip}}}(\varepsilon) \geq c\,\varepsilon^{-1/2}$. This completes the proof. □

Theorem 2.1 *Let $H_{\Lambda_{L_p,\mathrm{Lip}}}(\varepsilon)$ denote the Kolmogorov ε-entropy of the compact metric space $\Lambda_{L_p,\mathrm{Lip}}$. Then there exist constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1\,\varepsilon^{-1/2} \leq H_{\Lambda_{L_p,\mathrm{Lip}}}(\varepsilon) \leq c_2\,\varepsilon^{-1/2} \quad (1 \leq p < \infty,\; \varepsilon \leq \varepsilon_0).$$

Conclusion *Theorem 2.1 is an L_p-metric extension of Theorem B. By Theorem 2.1 and Remark 2.1, we obtain*

$$c_1(b-a, M, B, p)\varepsilon^{-1/2} \leq H_{\Lambda_{L_p[a,b],\mathrm{Lip}_M,B}}(\varepsilon) \leq c_2(b-a, M, B, p)\varepsilon^{-1/2}.$$

Hence the precise asymptotic of the entropy $H_{\Lambda_{L_p[a,b],\mathrm{Lip}_M,B}}(\varepsilon)$ for the compact metric space $\Lambda_{L_p[a,b],\mathrm{Lip}_M,B}$ is $\varepsilon^{-1/2}$, $\varepsilon \to 0_+$.

Proof of Theorem 2.1 The proof is a corollary of Lemma 2.1 (estimate from above) and Lemma 2.5, Lemma A (estimate from below). □

Comparison Explanations We discuss Theorem 2.1 in the light of the following statement due to Carl [2]:
Let X be a set in Hilbert space. Let $\mathbb{N}_X(\varepsilon)$ be the number of elements in a minimal ε-net of X such that $\mathbb{N}_X(\varepsilon) = O(\varepsilon^{-1/\sigma})$, $\varepsilon \to 0_+$. Let $\mathrm{conv}(X)$ *be the convex hull of X. Then the number of elements in a minimal ε-net of* $\mathrm{conv}(X)$ *satisfies the asymptotic estimate* $\mathbb{N}_{\mathrm{conv}(X)}(\varepsilon) = O(\exp[\varepsilon^{-2/(1+2\sigma)}])$, $\varepsilon \to 0_+$.

The result is a quantification of the well-known Krein–Milman Theorem and gives a final solution of a problem initiated by Dudley [4]. In fact, Carl's result [2] has been formulated in terms of a minimal ε-covering by balls, but it is easily seen that the number of elements in a minimal ε-covering by balls is exactly equal to the number of elements in a minimal ε-net of a given compact metric space.

Consider the compact metric space $\Lambda_{L_2,\mathrm{Lip}}$ of convex, uniformly bounded, uniformly Lipschitz functions f on $[0, 1]$ normalized with $f(0) = 0$ for simplicity. We perturb each function $f \in \Lambda_{L_2,\mathrm{Lip}}$ with a polynomial $p_K(x) = Kx$ ($K > 0$, $K \gg$)

to obtain a metric space $\Lambda_{L_2,\mathrm{Lip},+}$ of increasing functions. The space $\Lambda_{L_2,\mathrm{Lip},+}$ has the same number of elements in a minimal ε-net as $\Lambda_{L_2,\mathrm{Lip}}$. Take the Lipschitz constant to be 1. We present a construction to estimate the asymptotic of Kolmogorov ε-entropy for $\Lambda_{L_2,\mathrm{Lip},+}$ by using Carl's result. Consider the pre-compact metric space $\Lambda_{L_2,\mathrm{Lip},\mathrm{Pol},+}$ of convex, increasing polygons s defined on $[0, 1]$, uniformly Lipschitz, uniformly bounded, having knots in $[0, 1]$ and normalized with $s(0) = 0$. Cauchy L_2-completion of $\Lambda_{L_2,\mathrm{Lip},\mathrm{Pol},+}$ is the metric space $\Lambda_{L_2,\mathrm{Lip},+}$. Define the set $X := \{t_{x_*}(x) = (x - x_*)_+,\ x_* \in [0, 1]\}$. Note that X is the set of all first-degree truncated power functions with knots in $[0, 1]$.

Let $s(x) \in \Lambda_{L_2,\mathrm{Lip},\mathrm{Pol},+}$ with knots $0 = x_0 < x_1 < x_2 < \cdots < x_m < x_{m+1} = 1$. Then it is well known and easily seen that $s(x) = \sum_{k=0}^{m} a_k\, t_{x_k}(x), a_0 + a_1 + \cdots + a_m \le 1, a_k \ge 0$. In view of this $\mathrm{conv}(X) = \Lambda_{L_2,\mathrm{Lip},\mathrm{Pol},+}$. Given $\varepsilon > 0$, an ε-net of X contains not less then $O(1/\varepsilon)$ elements. Explicitly, such an ε-net is the set of truncated first-degree power functions $\{t_{k\frac{\varepsilon}{2}}(x),\ 0 \le k \le \lfloor 1/(2\varepsilon)\rfloor + 1\}$. From here, $\sigma = 1$ and $\mathbb{N}_{\Lambda_{L_2,\mathrm{Lip}}}(\varepsilon) = \mathbb{N}_{\Lambda_{L_2,\mathrm{Lip},+}}(\varepsilon) = \mathbb{N}_{\Lambda_{L_2,\mathrm{Lip},\mathrm{Pol},+}}(\varepsilon) = \mathbb{N}_{\mathrm{conv}(X)}(\varepsilon) = O(\exp(\varepsilon^{-2/(1+2)})) = O(\exp(\varepsilon^{-1/1.5}))$, $\varepsilon \to 0_+$. However, $\varepsilon^{-1/1.5} \gg \varepsilon^{-1/2}, \varepsilon \to 0_+$. Hence the general result formulated above does not imply the precise asymptotic of $H_{\Lambda_{L_2,\mathrm{Lip}}}(\varepsilon)$ given by Theorem 2.1.

3 Kolmogorov $\boldsymbol{\varepsilon}$-Entropy for $\boldsymbol{L_p}$-Metric Space of Convex and Uniformly Bounded Functions

By using *a multistep approximation procedure*, we obtain the precise asymptotic of Kolmogorov ε-entropy for the L_p-metric space of convex and uniformly bounded functions *without a condition for uniform Lipschitz behavior*.

Let $\Lambda_{C[a,b],\mathrm{Lip}_M,B}$ be the metric space of convex functions, defined on an interval $[a, b]$, uniformly Lipschitz on $[a, b]$ with a Lipschitz constant M, uniformly bounded on $[a, b]$ with a uniform bound B and equipped with C-metric. Denote by $H_{\Lambda_{C[a,b],\mathrm{Lip}_M,B}}(\varepsilon)$ the ε-entropy of $\Lambda_{C[a,b],\mathrm{Lip}_M,B}$. The following result is based on considerations due to Bronshtein [1].

Lemma 3.1 *There exists an absolute constant c such that*

$$H_{\Lambda_{C[a,b],\mathrm{Lip}_M,B}}(\varepsilon) \le c\,\frac{[(b-a)^2 + B^2]^{1/4}(1 + M^2)^{1/4}}{\varepsilon^{1/2}}.$$

Proof Consider the disk $B(0, r)$ of radius r, centered at the origin. Denote by $W_{\rho,r}$ the compact metric space of convex closed subsets in the disk $B(0, r)$ equipped with Hausdorff metric ρ. According to a result due to Bronshtein [1], the following estimates hold for the ε-entropy $H_{W_{\rho,r}}(\varepsilon)$ of $W_{\rho,r}$: $c_1\sqrt{r}/\sqrt{\varepsilon} \le H_{W_{\rho,r}}(\varepsilon) \le c_2\sqrt{r}/\sqrt{\varepsilon}$. For $f \in \Lambda_{C[a,b],\mathrm{Lip}_M,B}$ let $\mathrm{conv}(f) := \{(x, y) : a \le x \le b,\ f(x) \le y \le B\}$ be the convex set having as a boundary the graph of f on $[a, b]$, $x = a$, $x = b$ and $y = B$. Then for each $f, g \in \Lambda_{C[a,b],\mathrm{Lip}_M,B}$ we have

$$\frac{\|f - g\|_{C[a,b]}}{(1 + M^2)^{1/2}} \le \rho\bigl(\mathrm{conv}(f), \mathrm{conv}(g)\bigr) \le \|f - g\|_{C[a,b]}.$$

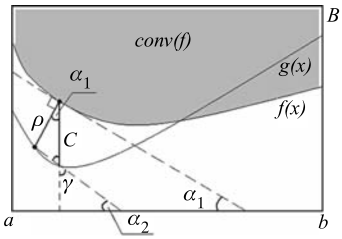

Fig. 2 The functions $y = f(x)$ and $y = g(x)$ are convex on the interval $[a, b]$. The shadow region $\mathrm{conv}(f)$ is a convex set bounded by the graph of $y = f(x)$, $x = a$, $x = b$, and $y = B$. The Hausdorff metric for sets is denoted by ρ and C is the uniform metric for functions

The right-hand side of the above inequality is trivial. The proof of the left-hand side is based on geometric considerations (see Fig. 2): At each point $(x, f(x))$ on the boundary of $\mathrm{conv}(f)$ we can construct a line orthogonal to the support line of $\partial[\mathrm{conv}(f)]$ at the same point such that the constructed line forms with the y-axis an angle having a tangent not exceeding M by absolute value. We have $\rho/\sin\gamma = C/\sin(\alpha_1 + \gamma)$, and from here, applying Cauchy–Buniakowski inequality, we obtain: $C = \rho\frac{\sin(\alpha_1+\gamma)}{\sin\gamma} = \rho(\cos\alpha_1 + \sin\alpha_1 \cot\gamma) = \rho(\cos\alpha_1 + \sin\alpha_1 \tan\alpha_2)$, $\gamma = \pi/2 - \alpha_2$. Since $|\tan\alpha_2| \le M$, we obtain: $C \le \rho(|\cos\alpha_1| \times 1 + |\sin\alpha_1| \times |\tan\alpha_2|) \le \rho(\cos^2\alpha_1 + \sin^2\alpha_1)^{1/2}(1 + M^2)^{1/2} = \rho(1 + M^2)^{1/2}$.

Let $f \in \Lambda_{C[a,b],\mathrm{Lip}_M,B}$. The set $\mathrm{conv}(f)$ is in the disk $B(0, r)$, $r = ((b - a)^2 + B^2)^{1/2}$. Hence the number of elements in a minimal ε-net $\mathbb{N}_{\Lambda_{C[a,b],\mathrm{Lip}_M,B}}(\varepsilon)$ of $\Lambda_{C[a,b],\mathrm{Lip}_M,B}$ is not greater than the number of elements in a minimal $\varepsilon/(1+M^2)^{1/2}$-net of the compact metric space $W_{\rho,r}$. In view of this, the result follows. □

Remark 3.1 Let $\Lambda_{L_p[a,b],B}$ be the L_p-metric space of convex functions, uniformly bounded on $[a, b]$ with a uniform bound B. Let $f, g \in \Lambda_{L_p[a,b],B}$. Then trivially, $\|f - g\|_{L_p[a,b]} = c(b - a, B, p)\|\tilde{f} - \tilde{g}\|_{L_p[0,1]}$, where $\tilde{f}, \tilde{g}$ are convex on $[0, 1]$ and bounded by 1. Denote by Λ_{L_p} the L_p-metric space of convex functions defined on the interval $[0, 1]$ and uniformly bounded by 1. Then the number of elements in a minimal $\Lambda_{L_p[a,b],B}(\varepsilon)$-net is equal to the number of the elements in a minimal $\Lambda_{L_p}(\varepsilon/c(b - a, B, p))$-net. Hence, the ε-asymptotic of the entropy for $\Lambda_{L_p[a,b],B}$ is equal to the ε-asymptotic of the entropy for Λ_{L_p}.

In view of Remark 3.1, without any restriction, we study the metric space Λ_{L_p} of functions defined on an interval $[0, 1]$, convex and uniformly bounded on $[0, 1]$ with a uniform bound 1 and equipped with $L_p[0, 1]$-metric.

Lemma 3.2 *The metric space Λ_{L_p}, $1 \le p < \infty$, is compact.*

Proof Let $\{\varepsilon_k\}_{k=0}^{\infty}$ be a sequence of numbers such that $\varepsilon_{k+1} < \varepsilon_k, \varepsilon_k > 0$ and $\lim_{k\to\infty} \varepsilon_k = 0$. Let $\{f_j\}_{j=0}^{\infty} \in \Lambda_{L_p}$. The convexity of f_j implies that the sequence $\{f_j\}_{j=0}^{\infty}$ is uniformly Lipschitz in $[\varepsilon_0, 1 - \varepsilon_0]$ with a Lipschitz constant $1/\varepsilon_0$. According to Arzelà-Ascoli's Theorem, $\{f_j\}_{j=0}^{\infty}$ is a compact metric space with respect to $C[\varepsilon_0, 1 - \varepsilon_0]$-metric and by Lebesgue's dominated convergence theorem there is a subsequence $\{f_j^{[0]}\}_{j=0}^{\infty}$ convergent in $L_p[\varepsilon_0, 1 - \varepsilon_0]$-metric to a convex in $[\varepsilon_0, 1 - \varepsilon_0]$ function f_{ε_0}. Now consider the sequence $\{f_j^{[0]}\}_{j=0}^{\infty}$ and ε_1. It is uniformly Lipschitz in $[\varepsilon_1, 1-\varepsilon_1]$ with a Lipschitz constant $1/\varepsilon_1$. Analogously, we can choose a subsequence $\{f_j^{[1]}\}_{j=0}^{\infty}$ convergent in $L_p[\varepsilon_1, 1-\varepsilon_1]$-metric to a convex function f_{ε_1} in $[\varepsilon_1, 1-\varepsilon_1]$. Moreover, $f_{\varepsilon_0}(x) = f_{\varepsilon_1}(x)$ for $\varepsilon_0 \le x \le 1-\varepsilon_0$. Now choose ε_2 and apply the same procedure to the sequence $\{f_j^{[1]}\}_{j=0}^{\infty}$, and so on. We obtain an infinite matrix of functions $f_j^{[k]}$, $j = 0, 1, \ldots;\ k = 0, 1, 2, \ldots$; such that for each fixed k the sequence $\{f_j^{[k]}\}_{j=0}^{\infty}$ is pointwise (even uniformly) convergent to a convex function f_{ε_k} in $[\varepsilon_k, 1-\varepsilon_k]$, and $f_{\varepsilon_k}(x) = f_{\varepsilon_{k-1}}(x)$ for $x \in [\varepsilon_{k-1}, 1-\varepsilon_{k-1}]$, $k = 1, 2, \ldots$. Applying the Diagonal Principle, we obtain a sequence $\{f_k^{[k]}\}_{k=0}^{\infty}$ convergent in a pointwise sense to a convex function f in the open interval $(0,1)$ and by Lebesgue's dominated convergence theorem $\lim_{k\to\infty} \|f_k^{[k]} - f\|_{L_p[0,1]} = 0$. In order to determine the limit function $f(x)$ by convexity at 0 and 1, we use the fact that $f_k^{[k]}(x)$, $k = 0, 1, \ldots$, are uniformly bounded for $x \in [0,1]$ and choose a convergent at $x = 0$ and $x = 1$ subsequence $\{f_{k_j}^{[k_j]}\}_{j=0}^{\infty}$ of $\{f_k^{[k]}\}_{k=0}^{\infty}$ that is pointwise convergent to f for all $x \in [0,1]$. $\square$

Remark 3.2 The sequence $f(x) = x^n$, $0 \le x \le 1$, shows that Lemma 3.2 does not hold in the C-metric space of convex and uniformly bounded functions.

Theorem 3.1 *Consider the ε-entropy $H_{\Lambda_{L_p}}(\varepsilon)$ of the metric space Λ_{L_p}. There exist constants $c_1 > 0$, $c_2 > 0$ such that*

$$\frac{c_1}{\varepsilon^{1/2}} \le H_{\Lambda_{L_p}}(\varepsilon) \le \frac{c_2}{\varepsilon^{1/2}}, \quad \varepsilon \le \varepsilon_0,\ 1 \le p < \infty.$$

Conclusion *Denote by $H_{\Lambda_{L_p[a,b],B}}(\varepsilon)$ the Kolmogorov ε-entropy of the metric space $\Lambda_{L_p[a,b],B}$. By Remark 3.1 and Theorem 3.1, it follows that*

$$c_1(p, b-a, B)\varepsilon^{-1/2} \le H_{\Lambda_{L_p[a,b],B}}(\varepsilon) \le c_2(p, b-a, B)\varepsilon^{-1/2},$$

where the constants $c_1(p, b-a, B) > 0$, $c_2(p, b-a, B) > 0$ depend on p, on the length of the interval $[a,b]$, and on the uniform bound B. Hence the ε-entropy $H_{\Lambda_{L_p[a,b],B}}(\varepsilon)$ of the compact metric space $\Lambda_{L_p[a,b],B}$ has the precise asymptotic $\varepsilon^{-1/2}$, $\varepsilon \to 0_+$.

Remark 3.3 By Theorem 2.1 and the results on Kolmogorov entropy for Sobololev's spaces [7, 8], we observe that the entropy of $\Lambda_{L_p,\mathrm{Lip}}$ is equal to the entropy of twice differentiable functions with uniformly bounded second derivative. In view of this, one might expect the following asymptotic behavior of the entropy for convex functions under some additional restrictions: $H_{\Lambda_{L_p,\mathrm{Lip}}}(\varepsilon) \asymp \varepsilon^{-1/2} \Rightarrow H_{\Lambda_{L_p,\text{Höld}(\alpha)}}(\varepsilon) \asymp \varepsilon^{-1/(1+\alpha)} \Rightarrow H_{\Lambda_{L_p}}(\varepsilon) \asymp \varepsilon^{-1}$ $(\varepsilon \to 0_+)$. Hence the trivial expectation is that the asymptotic behavior of the entropy $H_{\Lambda_{L_p}}(\varepsilon)$ for the metric space Λ_{L_p} should be ε^{-1}, $\varepsilon \to 0_+$. However, Theorem 3.1 shows that the metric space Λ_{L_p} has its own metric massivity in terms of Kolmogorov ε-entropy that does not match the metric massivity of uniformly Lipschitz functions.

Proof of Theorem 3.1 Estimate from above in the particular case $p = 1$. We consider the metric space Λ_{L_1}. Let α, β be such that $0 < \alpha < \beta < 1$. Denote by $\Lambda_{C[\alpha,\beta]}$

the C-metric compact space consisting of the restrictions of functions from Λ_{L_1} on the interval $[\alpha, \beta]$. Then an obvious but important observation is that $\Lambda_{C[\alpha,\beta]}$ is uniformly Lipschitz with Lipschitz constant $1/\min[\alpha, 1-\beta]$ and hence compact by Arzelà-Ascoli's Theorem.

Fix $\varepsilon > 0$ and consider $\bigcup_{k:k\geq 1;\, \varepsilon^{(2/3)^k}\leq 1/4}[\varepsilon^{(2/3)^{k-1}}, \varepsilon^{(2/3)^k}] \subset [\varepsilon, 1/4]$. We start from ε towards the right endpoint 1 of the interval $[\varepsilon, 1]$ *a multistep approximation procedure* consisting of a finite number (depending on the chosen $\varepsilon > 0$) of approximation steps on the local compact metric spaces $\Lambda_{L_1[\varepsilon^{(2/3)^{k-1}}, \varepsilon^{(2/3)^k}]}$, $k = 1, 2, \ldots$, such that $\varepsilon^{(2/3)^k} \leq 1/4$.

Step $k = 1$ *of the multistep approximation procedure*: Consider the interval $[\varepsilon, \varepsilon^{2/3}]$ and the corresponding compact metric space $\Lambda_{L_1[\varepsilon,\varepsilon^{2/3}]}$. Obviously, this is a family of uniformly Lipschitz functions with a uniform Lipschitz constant $1/\varepsilon$ and uniformly bounded by 1. Let $f \in \Lambda_{L_1[\varepsilon,\varepsilon^{2/3}]}$. Consider the normalized function $\varepsilon^{1/2} f(\varepsilon^{1/2}v)$ for $\varepsilon \leq \varepsilon^{1/2}v \leq \varepsilon^{2/3}$, i.e., $\varepsilon^{1/2} \leq v \leq \varepsilon^{1/6}$. We have $|f(x) - f(y)| \leq \frac{1}{\varepsilon}|x-y|\, (x, y \in [\varepsilon, \varepsilon^{2/3}])$, and in view of this: $|\varepsilon^{1/2} f(\varepsilon^{1/2}v) - \varepsilon^{1/2} f(\varepsilon^{1/2}w)| \leq \frac{1}{\varepsilon}\varepsilon^{1/2}|\varepsilon^{1/2}v - \varepsilon^{1/2}w| = |v-w|\ (v, w \in [\varepsilon^{1/2}, \varepsilon^{1/6}])$. Hence the function $\varepsilon^{1/2} f(\varepsilon^{1/2}v)$, $v \in [\varepsilon^{1/2}, \varepsilon^{1/6}]$, is Lipschitz with a Lipschitz constant $M = 1$ and bounded by $B = \varepsilon^{1/2}$. Now consider $f, g \in \Lambda_{L_1[\varepsilon,\varepsilon^{2/3}]}$ and the corresponding normalizations $\varepsilon^{1/2} f(\varepsilon^{1/2}v)$, $\varepsilon^{1/2} g(\varepsilon^{1/2}v)$, $\varepsilon^{1/2} \leq v \leq \varepsilon^{1/6}$. We have:

$$\begin{aligned}
&\int_{\varepsilon}^{\varepsilon^{2/3}} \big|f(x) - g(x)\big|\, dx \\
&\quad = \int_{\varepsilon^{1/2}}^{\varepsilon^{1/6}} \big|\varepsilon^{1/2} f\big(\varepsilon^{1/2}v\big) - \varepsilon^{1/2} g\big(\varepsilon^{1/2}v\big)\big|\, dv \\
&\quad \leq \big(\varepsilon^{1/6} - \varepsilon^{1/2}\big)\big\|\varepsilon^{1/2} f\big(\varepsilon^{1/2}v\big) - \varepsilon^{1/2} g\big(\varepsilon^{1/2}v\big)\big\|_{C[\varepsilon^{1/2},\varepsilon^{1/6}]} \leq \varepsilon\big(\varepsilon^{1/6} - \varepsilon^{1/2}\big)
\end{aligned}$$

if $\|\varepsilon^{1/2} f(\varepsilon^{1/2}v) - \varepsilon^{1/2} g(\varepsilon^{1/2}v)\|_{C[\varepsilon^{1/2},\varepsilon^{1/6}]} \leq \varepsilon$.

According to Lemma 3.1, the number of elements in a minimal ε-net of the metric space $\Lambda_{C[\varepsilon^{1/2},\varepsilon^{1/6}]}$ of uniformly Lipschitz and uniformly bounded functions with a uniform Lipschitz constant 1 and a uniform bound $\varepsilon^{1/2}$ is bounded from above as follows:

$$\mathbb{N}_{\Lambda_{C,\mathrm{Lip}}[\varepsilon^{1/2},\varepsilon^{1/6}]}(\varepsilon) \leq c_1 2^{\frac{c_2}{\varepsilon^{1/2}}[(\varepsilon^{1/6}-\varepsilon^{1/2})^2+(\varepsilon^{1/2})^2]^{1/4}} \leq c_1 2^{\frac{c_2}{\varepsilon^{1/2}}(\varepsilon^{1/6}-\varepsilon^{1/2})^{1/2}}$$

if $\varepsilon^{1/6} - \varepsilon^{1/2} \geq \varepsilon^{1/2}$ that is equivalent to $1/4 \geq \varepsilon^{2/3}$.

Step k of the multistep approximation procedure ($k = 1, 2, 3, \ldots$): Consider the interval $[\varepsilon^{(2/3)^{k-1}}, \varepsilon^{(2/3)^k}]$ and the corresponding compact metric space $\Lambda_{L_1[\varepsilon^{(2/3)^{k-1}}, \varepsilon^{(2/3)^k}]}$. This is a family of uniformly Lipschitz functions with a uniform Lipschitz constant $1/\varepsilon^{(2/3)^{k-1}}$ and uniformly bounded by 1. Let $f \in \Lambda_{L_1[\varepsilon^{(2/3)^{k-1}}, \varepsilon^{(2/3)^k}]}$. Consider the normalized functions $\varepsilon^{2^{k-2}/3^{k-1}} f(\varepsilon^{2^{k-2}/3^{k-1}}v)$ for $\varepsilon^{2^{k-1}/3^{k-1}} \leq \varepsilon^{2^{k-2}/3^{k-1}}v \leq \varepsilon^{2^k/3^k}$, i.e., $\varepsilon^{2^{k-2}/3^{k-1}} \leq v \leq \varepsilon^{2^{k-2}/3^k}$. We have $|f(x) - f(y)| \leq$

$\frac{1}{\varepsilon^{2^{k-1}/3^{k-1}}}|x-y|$ $(x,\ y \in [\varepsilon^{(2/3)^{k-1}}, \varepsilon^{(2/3)^k}])$ and in view of this:

$$\begin{aligned}&\left|\varepsilon^{2^{k-2}/3^{k-1}} f\left(\varepsilon^{2^{k-2}/3^{k-1}} v\right) - \varepsilon^{2^{k-2}/3^{k-1}} f\left(\varepsilon^{2^{k-2}/3^{k-1}} w\right)\right| \\ &\quad \le \frac{1}{\varepsilon^{2^{k-1}/3^{k-1}}} \varepsilon^{2^{k-2}/3^{k-1}} \left|\varepsilon^{2^{k-2}/3^{k-1}} v - \varepsilon^{2^{k-2}/3^{k-1}} w\right| = |v-w| \\ &\quad \left(v,\ w \in \left[\varepsilon^{2^{k-2}/3^{k-1}}, \varepsilon^{2^{k-2}/3^k}\right]\right).\end{aligned}$$

Hence the *normalized* functions $\varepsilon^{2^{k-2}/3^{k-1}} f(\varepsilon^{2^{k-2}/3^{k-1}} v)$, $v \in [\varepsilon^{2^{k-2}/3^{k-1}}, \varepsilon^{2^{k-2}/3^k}]$, are Lipschitz with a Lipschitz constant $M=1$ and bounded by $B = \varepsilon^{2^{k-2}/3^{k-1}}$.

We have for $f, g \in \Lambda_{L_1[\varepsilon^{(2/3)^{k-1}}, \varepsilon^{(2/3)^k}]}$ and the corresponding normalizations $\varepsilon^{2^{k-2}/3^{k-1}} f(\varepsilon^{2^{k-2}/3^{k-1}} v)$, $\varepsilon^{2^{k-2}/3^{k-1}} g(\varepsilon^{2^{k-2}/3^{k-1}} v)$ for $\varepsilon^{2^{k-2}/3^{k-1}} \le v \le \varepsilon^{2^{k-2}/3^k}$:

$$\begin{aligned}&\int_{\varepsilon^{2^{k-1}/3^{k-1}}}^{\varepsilon^{2^k/3^k}} \left|f(x)-g(x)\right| dx \\ &= \int_{\varepsilon^{2^{k-2}/3^{k-1}}}^{\varepsilon^{2^{k-2}/3^k}} \left|\varepsilon^{2^{k-2}/3^{k-1}} f\left(\varepsilon^{2^{k-2}/3^{k-1}} v\right) - \varepsilon^{2^{k-2}/3^{k-1}} g\left(\varepsilon^{2^{k-2}/3^{k-1}} v\right)\right| dv \\ &\le \left(\varepsilon^{2^{k-2}/3^k} - \varepsilon^{2^{k-2}/3^{k-1}}\right) \\ &\quad \times \left\|\varepsilon^{2^{k-2}/3^{k-1}} f\left(\varepsilon^{2^{k-2}/3^{k-1}} v\right) - \varepsilon^{2^{k-2}/3^{k-1}} g\left(\varepsilon^{2^{k-2}/3^{k-1}} v\right)\right\|_{C[\varepsilon^{2^{k-2}/3^{k-1}}, \varepsilon^{2^{k-2}/3^k}]} \\ &\le \varepsilon\left(\varepsilon^{2^{k-2}/3^k} - \varepsilon^{2^{k-2}/3^{k-1}}\right)\end{aligned}$$

if $\|\varepsilon^{2^{k-2}/3^{k-1}} f(\varepsilon^{2^{k-2}/3^{k-1}} v) - \varepsilon^{2^{k-2}/3^{k-1}} g(\varepsilon^{2^{k-2}/3^{k-1}} v)\|_{C[\varepsilon^{2^{k-2}/3^{k-1}}, \varepsilon^{2^{k-2}/3^k}]} \le \varepsilon$.

By Lemma 3.1, the number of elements in a minimal ε-net for the metric space $\Lambda_{C[\varepsilon^{2^{k-2}/3^{k-1}}, \varepsilon^{2^{k-2}/3^k}]}$ of uniformly Lipschitz functions with a uniform Lipschitz constant 1 and uniformly bounded with a uniform bound $\varepsilon^{2^{k-2}/3^{k-1}}$ is bounded from above as follows:

$$\begin{aligned}&\mathbb{N}_{\Lambda_{C,\mathrm{Lip}}[\varepsilon^{2^{k-2}/3^{k-1}}, \varepsilon^{2^{k-2}/3^k}]}(\varepsilon) \\ &\quad \le c_1 2^{\frac{c_2}{\varepsilon^{1/2}}[(\varepsilon^{2^{k-2}/3^k} - \varepsilon^{2^{k-2}/3^{k-1}})^2 + (\varepsilon^{2^{k-2}/3^{k-1}})^2]^{1/4}} \\ &\quad \le c_1 2^{\frac{c_2}{\varepsilon^{1/2}}(\varepsilon^{2^{k-2}/3^k} - \varepsilon^{2^{k-2}/3^{k-1}})^{1/2}}\end{aligned}$$

if $\varepsilon^{2^{k-2}/3^k} - \varepsilon^{2^{k-2}/3^{k-1}} \ge \varepsilon^{2^{k-2}/3^{k-1}}$ that is equivalent to $1/4 \ge \varepsilon^{(2/3)^k}$.

We follow the above approximation procedure for $k = 1, 2, 3, 4, \ldots, k^*$ until the first $k = k^*(\varepsilon) = k^*$ such that $3/16 \le \varepsilon^{(2/3)^{k^*}} \le 1/4$. Note that $\varepsilon^{(2/3)^{k^*}} \le 1/4$ is equivalent to $k^* \le \ln(\ln(1/\varepsilon))/\ln(3/2) + \ln(\ln(4))/\ln(2/3) = O(\ln\ln(1/\varepsilon))$, $\varepsilon \to 0_+$. Next, we perform the same procedure from $1-\varepsilon$ towards 0 (the left endpoint of the interval $[0,1]$) up to the iteration step for which the left endpoint of the corresponding interval is in $[1-1/4, 1-3/16]$. By Remark 2.1 and Theorem 2.1, the local metric space $\Lambda_{L_1[3/16, 1-3/16]}$ of functions, uniformly Lipschitz with a uniform

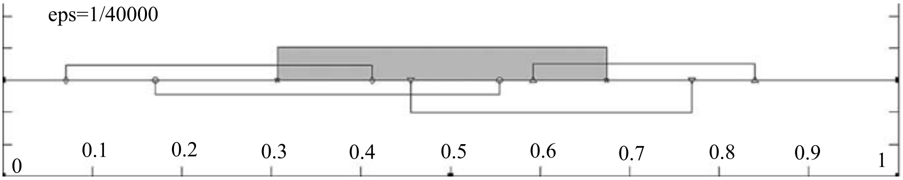

Fig. 3 For a fixed positive integer k_0, the shadow interval $[\varepsilon^{2^{k_0-3}/3^{k_0-1}}, \varepsilon^{2^{k_0-3}/3^{k_0}}]$ overlaps with at most 4 other intervals from the interval sequence $\{[\varepsilon^{2^{k-3}/3^{k-1}}, \varepsilon^{2^{k-3}/3^{k}}]\}_{k=1}^{\infty}$

Lipschitz constant $M = 16/3$ and uniformly bounded with a uniform bound $B = 1$, has an ε-net containing not more than $c_1\, 2^{c_2/\sqrt{\varepsilon}}$ elements.

Summing up, a minimal $\varepsilon[1 + 2\sum_{k=1}^{k^*}(\varepsilon^{2^{k-2}/3^k} - \varepsilon^{2^{k-2}/3^{k-1}})]$*-net of* $\Lambda_{L_1[\varepsilon,1-\varepsilon]}$ *contains at most* $\tilde{c}_1\, 2^{\frac{\tilde{c}_2}{\sqrt{\varepsilon}}[1+2\sum_{k=1}^{k^*}(\varepsilon^{2^{k-2}/3^k} - \varepsilon^{2^{k-2}/3^{k-1}})^{1/2}]}$ *elements* (*functions*).

Furthermore: $(\varepsilon^{2^{k-2}/3^k} - \varepsilon^{2^{k-2}/3^{k-1}})^{1/2} + (\varepsilon^{2^{k-2}/3^{k-1}})^{1/2} \le \sqrt{2}(\varepsilon^{2^{k-2}/3^k})^{1/2}$, and from here, $(\varepsilon^{2^{k-2}/3^k} - \varepsilon^{2^{k-2}/3^{k-1}})^{1/2} \le (\sqrt{2}+1)[\varepsilon^{2^{k-3}/3^k} - \varepsilon^{2^{k-3}/3^{k-1}}]$ if $\varepsilon^{2^{k-2}/3^k} > 2\varepsilon^{2^{k-2}/3^{k-1}}$ which is equivalent to $1/4 \ge \varepsilon^{(2/3)^k}$.

In the final step of the proof of an estimate from above in the particular case $p = 1$, we show that the series:

$$\sum_{k=1}^{\infty}\left(\varepsilon^{2^{k-2}/3^k} - \varepsilon^{2^{k-2}/3^{k-1}}\right) \quad \text{and} \quad \sum_{k=1}^{\infty}\left(\varepsilon^{2^{k-3}/3^k} - \varepsilon^{2^{k-3}/3^{k-1}}\right)$$

are uniformly bounded for each $\varepsilon < 1$. Obviously,

$$\begin{aligned}\varepsilon^{2^{k-2}/3^k} - \varepsilon^{2^{k-2}/3^{k-1}} &= \left(\varepsilon^{2^{k-3}/3^k} - \varepsilon^{2^{k-3}/3^{k-1}}\right)\left(\varepsilon^{2^{k-3}/3^k} + \varepsilon^{2^{k-3}/3^{k-1}}\right) \\ &\le 2\left(\varepsilon^{2^{k-3}/3^k} - \varepsilon^{2^{k-3}/3^{k-1}}\right).\end{aligned}$$

We prove that $\sum_{k=1}^{\infty}(\varepsilon^{2^{k-3}/3^k} - \varepsilon^{2^{k-3}/3^{k-1}})$ is uniformly bounded from above for $0 < \varepsilon < 1$. Consider the interval sequence $\{[\varepsilon^{2^{k-3}/3^{k-1}}, \varepsilon^{2^{k-3}/3^k}]\}_{k=1}^{\infty}$ and for a fixed arbitrary chosen positive integer k_0, the corresponding to k_0 interval: $[\varepsilon^{2^{k_0-3}/3^{k_0-1}}, \varepsilon^{2^{k_0-3}/3^{k_0}}]$. Asking how many other intervals from the above interval sequence overlap with the chosen interval is equivalent to solving with respect to the positive integer l the inequalities: $\varepsilon^{2^{k_0-3}/3^{k_0-1}} \le \varepsilon^{2^{l-3}/3^{l-1}} \le \varepsilon^{2^{k_0-3}/3^{k_0}}$ and $\varepsilon^{2^{k_0-3}/3^{k_0-1}} \le \varepsilon^{2^{l-3}/3^{l}} \le \varepsilon^{2^{k_0-3}/3^{k_0}}$. Equivalently: $(2/3)^{k_0} \ge (2/3)^l \ge (2/3)^{k_0}/3$, and $(2/3)^{k_0} \ge (2/3)^l/3 \ge (2/3)^{k_0}/3$, i.e., $k_0 \le l \le k_0 + \ln(3)/\ln(3/2)$, and $-\ln(3)/\ln(3/2) + k_0 \le l \le k_0$, that is, $-2 + k_0 \le l \le k_0 + 2$ (see Fig. 3). Hence $\sum_{k=1}^{\infty}(\varepsilon^{2^{k-3}/3^k} - \varepsilon^{2^{k-3}/3^{k-1}}) \le 5 \times$ (the length of [0,1]) $= 5$ $(0 < \varepsilon < 1)$.

Summing up, the following estimate holds for the number $\mathbb{N}_{\Lambda_{L_1[\varepsilon,1-\varepsilon]}}(\varepsilon)$ of elements in a minimal ε-net of $\Lambda_{L_1[\varepsilon,1-\varepsilon]}$: $\mathbb{N}_{\Lambda_{L_1[\varepsilon,1-\varepsilon]}}(\varepsilon) \le c_1 2^{\frac{c_2}{\sqrt{\varepsilon}}}$.

Given two functions $f, g \in \Lambda_{L_1}$, the trivial inequality $\|f - g\|_{L_1[0,1]} \le 4\varepsilon + \|f - g\|_{L_1[\varepsilon,1-\varepsilon]}$ implies that an $\varepsilon/5$-net of $\Lambda_{L_1[\varepsilon,1-\varepsilon]}$ will be an ε-net of Λ_{L_1}. In view of this, $\mathbb{N}_{\Lambda_{L_1}(\varepsilon)} \le c_1 2^{\frac{c_2}{\sqrt{\varepsilon}}}$, where $\mathbb{N}_{\Lambda_{L_1}(\varepsilon)}$ is the number of elements in a minimal ε-net of Λ_{L_1}. *In fact, the elements of our ε-net are piecewise convex functions with possible discontinuities at the points $\varepsilon^{(2/3)^k}$, $k=0,1,\dots$, but trivially, by using the triangle inequality we can construct an ε-net of Λ_{L_1} consisting of elements from Λ_{L_1} and having at most $c_1 2^{\frac{c_2}{\sqrt{\varepsilon}}}$ elements. The same holds in the general case $1 \le p < \infty$ given below.*

Estimate from above for an arbitrary p $(1 \le p < \infty)$. The proof is based on a multistep approximation procedure similar to that in the case $p=1$ but more involved technically. Let Λ_{L_p} be the compact metric spaces of convex functions defined on the interval $[0,1]$, uniformly bounded with a uniform bound 1 and equipped with $L_p[0,1]$-metric. Similar to the proof of $p=1$, given $p\in[1,\infty)$ and $0<\varepsilon<1/2^{1+1/p}$, consider

$$\bigcup_{k:k\ge 1;\ \varepsilon^{p[(p+1)/(p+2)]^k}\le 1/2^{p+1}} \left[\varepsilon^{p[(p+1)/(p+2)]^{k-1}}, \varepsilon^{p[(p+1)/(p+2)]^k}\right] \subset \left[\varepsilon^p, 1/2^{p+1}\right],$$

and start from ε^p towards the right endpoint 1 of the interval $[\varepsilon^p,1]$ *a multistep approximation procedure* consisting of a finite number (depending on p and ε) of approximation steps on the local compact metric spaces $\Lambda_{L_p[\varepsilon^{p[(p+1)/(p+2)]^{k-1}}, \varepsilon^{p[(p+1)/(p+2)]^k}]}$ for $k=1,2,\dots;$ $\varepsilon^{p[(p+1)/(p+2)]^k}\le 1/2^{p+1}$.

Step k of the approximation procedure for an arbitrary p $(1\le p<\infty)$ $k=1,2,3,\dots$: Consider the interval $[\varepsilon^{p[(p+1)/(p+2)]^{k-1}}, \varepsilon^{p[(p+1)/(p+2)]^k}]$ and the corresponding compact metric space $\Lambda_{L_p[\varepsilon^{p[(p+1)/(p+2)]^{k-1}}, \varepsilon^{p[(p+1)/(p+2)]^k}]}$. Obviously, this is a family of uniformly Lipschitz functions with a uniform Lipschitz constant $1/\varepsilon^{p[(p+1)/(p+2)]^{k-1}}$ and uniformly bounded by 1.

Let $f,g\in\Lambda_{L_p[\varepsilon^{p[(p+1)/(p+2)]^{k-1}}, \varepsilon^{p[(p+1)/(p+2)]^k}]}$. By using the substitution $x=\varepsilon^{p^2\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}v$ we obtain

$$\begin{aligned}
&\int_{\varepsilon^{p\left(\frac{p+1}{p+2}\right)^{k-1}}}^{\varepsilon^{p\left(\frac{p+1}{p+2}\right)^{k}}} \left|f(x)-g(x)\right|^p dx \\
&= \int_{\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}}^{\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}}} \varepsilon^{p^2\frac{(p+1)^{k-2}}{(p+2)^{k-1}}} \left|f\left(\varepsilon^{p^2\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}v\right)-g\left(\varepsilon^{p^2\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}v\right)\right|^p dv \\
&= \int_{\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}}^{\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}}} \left|\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}} f\left(\varepsilon^{p^2\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}v\right)-\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}} g\left(\varepsilon^{p^2\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}v\right)\right|^p dv \\
&\le \varepsilon^p\left(\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}}-\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}\right)
\end{aligned}$$

if $\|\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}} f(\varepsilon^{p^2\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}v)-\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}} g(\varepsilon^{p^2\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}v)\|_{C[\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}},\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}}]}\le\varepsilon$. For $f\in\Lambda_{L_p[\varepsilon^{p[(p+1)/(p+2)]^{k-1}}, \varepsilon^{p[(p+1)/(p+2)]^k}]}$ the normalized functions $\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}\times$

$f(\varepsilon^{p^2\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}v)$ defined for $v \in [\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}, \varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}}]$ are uniformly Lipschitz with a Lipschitz constant $M = 1$ and uniformly bounded with an upper bound $B = \varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}$. By Lemma 3.1 the number of elements in a minimal ε-net for the metric space $\Lambda_{C[\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}, \varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}}]}$ of uniformly Lipschitz functions with a uniform Lipschitz constant 1 and uniformly bounded with a uniform bound $\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}$ is bounded from above as follows:

$$\begin{aligned}
&\mathbb{N}_{\Lambda_{C[\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}, \varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}}]}}(\varepsilon) \\
&\quad \le c_1 2^{\frac{c_2}{\sqrt{\varepsilon}}[(\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}} - \varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}})^2 + (\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}})^2]^{1/4}} \\
&\quad \le c_1 2^{\frac{c_2}{\sqrt{\varepsilon}}(\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}} - \varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}})^{1/2}}, \quad (c_1 > 0,\ c_2 > 0)
\end{aligned}$$

if $\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}} - \varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}} \ge \varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}$ which is equivalent to $\frac{1}{2^{p+1}} \ge \varepsilon^{p(\frac{p+1}{p+2})^k}$.

We follow the above approximation procedure for $k = 1, 2, 3, 4, \ldots, k^*$ until the first $k = k^*(\varepsilon, p) = k^*$ such that $\frac{2^{p+1}-1}{4^{p+1}} \le \varepsilon^{p(\frac{p+1}{p+2})^{k^*}} \le \frac{1}{2^{p+1}}$. Similar to the case $p = 1$, $k^* = O(\ln\ln(1/\varepsilon))$, $\varepsilon \to 0_+$. Next we perform the same procedure starting from $1 - \varepsilon^p$ towards 0 (the left endpoint of the interval [0, 1]) up to the approximation step for which the left endpoint of the corresponding interval is in $[1 - \frac{1}{2^{p+1}}, 1 - \frac{2^{p+1}-1}{4^{p+1}}]$. By Theorem 2.1, the local metric space $\Lambda_{L_p[\frac{2^{p+1}-1}{4^{p+1}}, 1-\frac{2^{p+1}-1}{4^{p+1}}]}$ of functions, uniformly Lipschitz with a uniform Lipschitz constant $M = \frac{4^{p+1}}{2^{p+1}-1}$ and uniformly bounded with a uniform bound 1 has a minimal $L_p[\frac{2^{p+1}-1}{4^{p+1}}, 1 - \frac{2^{p+1}-1}{4^{p+1}}](\varepsilon)$-net containing not more than $c_1 2^{c_2(p)/\sqrt{\varepsilon}}$ elements.

Summing up, a minimal $\varepsilon[1 + 2\sum_{k=1}^{k^*}(\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}} - \varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}})]^{1/p}$*-net of* $\Lambda_{L_p[\varepsilon^p, 1-\varepsilon^p]}$ *contains at most* $\tilde{c}_1(p)\, 2^{\frac{\tilde{c}_2(p)}{\sqrt{\varepsilon}}[1+2\sum_{k=1}^{k^*}(\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}} - \varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}})^{1/2}}$ *elements* (*functions*).

Furthermore, $(\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}} - \varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}})^{1/2} + (\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}})^{1/2} \le \sqrt{2}(\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}})^{1/2}$, and therefore, $(\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}} - \varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}})^{1/2} \le (\sqrt{2} + 1)(\varepsilon^{p\frac{(p+1)^{k-2}}{2(p+2)^{k}}} - \varepsilon^{p\frac{(p+1)^{k-2}}{2(p+2)^{k-1}}})$ if $\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}} > 2\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}$ which is equivalent to $\frac{1}{2^{p+1}} \ge \varepsilon^{p(\frac{p+1}{p+2})^k}$.

The final step of the proof for an estimate from above in the general case of p, $1 \le p < \infty$, is to show that the series

$$\sum_{k=1}^{\infty}\bigl(\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}} - \varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}\bigr) \quad \text{and} \quad \sum_{k=1}^{\infty}\bigl(\varepsilon^{p\frac{(p+1)^{k-2}}{2(p+2)^{k}}} - \varepsilon^{p\frac{(p+1)^{k-2}}{2(p+2)^{k-1}}}\bigr)$$

are bounded from above for each $\varepsilon < 1$ by a constant depending only on p. Similar to the proof in the particular case $p = 1$, in the interval sequence

$$\Big\{\Big[\varepsilon^{p\frac{(p+1)^{k-2}}{2(p+2)^{k-1}}}, \varepsilon^{p\frac{(p+1)^{k-2}}{2(p+2)^{k}}}\Big]\Big\}_{k=1}^{\infty}$$

each fixed interval has at most $2\lfloor \ln(p+2)/\ln[(p+2)/(p+1)]\rfloor$ overlapping with other intervals of the corresponding sequence, where $\lfloor a\rfloor$ is the integer part of a. Therefore,

$$\begin{aligned}
&\max\left[\sum_{k=1}^{\infty}\Big(\varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k}}} - \varepsilon^{p\frac{(p+1)^{k-2}}{(p+2)^{k-1}}}\Big), \sum_{k=1}^{\infty}\Big(\varepsilon^{p\frac{(p+1)^{k-2}}{2(p+2)^{k}}} - \varepsilon^{p\frac{(p+1)^{k-2}}{2(p+2)^{k-1}}}\Big)\right]\\
&\quad \le 2\Big[\Big(2\Big\lfloor \ln(p+2)/\ln\big((p+2)/(p+1)\big)\Big\rfloor + 1\Big)\times\big(\text{the length of } [0,1]\big)\Big]\\
&\quad = 4\Big\lfloor \ln(p+2)/\ln\big((p+2)/(p+1)\big)\Big\rfloor + 2.
\end{aligned}$$

Hence for the number $\mathbb{N}_{\Lambda_{L_p[\varepsilon^p,1-\varepsilon^p]}}(\varepsilon)$ of elements in a minimal ε-net of $\Lambda_{L_p[\varepsilon^p,1-\varepsilon^p]}$, the following estimate holds: $\mathbb{N}_{\Lambda_{L_p[\varepsilon^p,1-\varepsilon^p]}}(\varepsilon) \le c_1(p)\, 2^{\frac{c_2(p)}{\sqrt{\varepsilon}}}$. Given two functions $f, g \in \Lambda_{L_p}$, Minkowski's inequality implies: $\|f-g\|_{L_p[0,1]} \le \|f-g\|_{L_p[0,\varepsilon^p]} + \|f-g\|_{L_p[\varepsilon^p,1-\varepsilon^p]} + \|f-g\|_{L_p[1-\varepsilon^p,1]} \le 4\varepsilon + \|f-g\|_{L_p[\varepsilon^p,1-\varepsilon^p]}$. From here an $\varepsilon/5$-net of $\Lambda_{L_p[\varepsilon^p,1-\varepsilon^p]}$ will be an ε-net of Λ_{L_p}.

We conclude that a minimal ε-net of Λ_{L_p} contains at most $c_1(p)\, 2^{\frac{c_2(p)}{\sqrt{\varepsilon}}}$ elements (functions), i.e., $\mathbb{N}_{\Lambda_{L_p}(\varepsilon)} \le c_1(p)\, 2^{\frac{c_2(p)}{\sqrt{\varepsilon}}}$. Hence, $H_{\Lambda_{L_p}}(\varepsilon) \le c(p)\, \varepsilon^{-1/2}$.

Estimate from below Let $\mathfrak{M}_{1,\rho} \subset \mathfrak{M}_{2,\rho}$ be two metric spaces endowed with the same metric ρ. Obviously, a maximal ε-distant set of $\mathfrak{M}_{2,\rho}$ contains not less elements than a maximal ε-distant set of $\mathfrak{M}_{1,\rho}$. In view of this the estimate from below is a corollary of Theorem 2.1. The proof of Theorem 3.1 is completed. □

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