Notes on Estimation of an Elliptically-symmetric Log-concave Density

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1 Introduction

1.1 Notation

For a vector $x \in \mathbb{R}^p$, $\|x\|, \|x\|_2$ both denotes the l_2 norm. For a matrix A, $\|A\|_2$ denotes the operator norm. We represent positive definiteness of a matrix A as $A \succ 0$, and semidefiniteness as $A \succeq 0$. Given a vector x and a matrix $A \succ 0$, $\|x\|_A = \sqrt{x^\top A^{-1} x}$ denotes the Mahalanobis distance.

1.2 Elliptical Density

A p-dimensional random vector has the Elliptical density if the pdf is of the form

$$f(x\,;\,\mu, \Sigma) = |\Sigma|^{-1/2} g_p\big(\|x-\mu\|_\Sigma\big)$$

where Ω is a positive definite matrix and $\|x\|_\Sigma$ is the Mahalanobis distance, $\|x\|_\Sigma^2 = x^\top \Sigma^{-1} x$.

where $g_p : \mathbb{R}^+ \to \mathbb{R}^+$ is a generator function with the property

$$\int_{\mathbb{R}^p} g_p(\|x\|_2) dx = 1$$

1.3 Identifiability

There is one degree of non-identifiability. Let $a > 0$, let $\Sigma' = \frac{\Sigma}{a}$, then we have that

$$\begin{aligned} f(x\,;\,\mu, \Sigma) &= \left|\frac{\Sigma}{a}\right|^{-1/2} a^{-p/2} g_p\left(\sqrt{\frac{1}{a}}\sqrt{x^\top \left(\frac{\Sigma}{a}\right)^{-1} x}\right) \\ &= |\Sigma'|^{-1/2} g'_p(\sqrt{x^\top \Sigma'^{-1} x}) \end{aligned}$$

where $g'_p(r) = a^{-p/2} g_p(r/\sqrt{a})$. It is easy to check that $\int_{\mathbb{R}^p} g'_p(\|x\|_2)dx = 1$. Thus, without loss of generality, we may assume that $\|\Sigma\|_2 = 1$.

To prove identifiability, we note the following lemma:

Lemma 1. *Suppose $A, B \succ 0$. Let $a, b > 0$, the sets $\{x : x^\top A x = a\}$ and $\{x : x^\top B x = b\}$ are equal iff $(bA)/a = B$.*

Proof. Let $S = \{x : x^\top A x = a\}$. We have that for any $x \in S$, $x^\top((bA/a) - B)x = 0$. Since S contains p independent vectors, namely the elementary basis appropriately scaled, we have that $(bA/a) - B = 0$. $\square$

Now suppose (Σ, g_p) and (Σ', g'_p) induce the same density f. We have then that $g_p(\sqrt{x^\top \Sigma^{-1} x}) = c g'_p(\sqrt{x^\top \Sigma'^{-1} x}) = f(x)$ for some $c > 0$.

[TODO:finish, intuition: we look at the level sets of g_p and g'_p, i.e., $g_p^{-1}(\{a\})$ for some $a > 0$. If the level sets are singletons, this is easy. If the level sets are bounded, this is easy too. If the level sets are unbounded, what to do?

1.4 Characterizations

Let X follow a centered elliptical distribution. Then, we have that

$$X = \Sigma^{1/2} \Phi Z$$

where Φ is random vector from $\mathbb{S}^{p-1}$ and Z is a non-negative random variable that follows the density

$$f_Z(r) = c_p r^{p-1} g_p(r)$$

$c_p = 2\frac{\pi^{p/2}}{\Gamma(p/2)}$.

2 Log-Concavity

A related lemma in [TODO:cite Bhattacharyya] states that f is unimodal iff g_p is non-increasing.

Lemma 2. *f is log-concave iff g_p is log-concave and non-increasing.*

Proof. Without loss of generality, suppose that $\mu = 0$.

Suppose g_p is log-concave and non-increasing. Then, we have that

$$\begin{aligned}
\log f(\lambda x + (1-\lambda) y) &= (-1/2) \log |\Sigma| + \log g_p(\|\lambda x + (1-\lambda) y\|_\Sigma) \\
&\geq (-1/2) \log |\Sigma| + \log g_p(\lambda \|x\|_\Sigma + (1-\lambda)\|y\|_\Sigma) \\
&\geq (-1/2) \log |\Sigma| + \lambda \log g_p(\|x\|_\Sigma) + (1-\lambda) \log g_p(\|y\|_\Sigma) \\
&= \lambda \log f(x) + (1-\lambda) \log f(y)
\end{aligned}$$

The first inequality follows because $\|\lambda x + (1-\lambda) y\|_\Sigma \leq \lambda \|x\|_\Sigma + (1-\lambda)\|y\|_\Sigma$; since $\|\cdot\|_\Sigma$ is a norm, it is convex. The first inequality follows also because $\log g_p$ is a non-increasing function. The second inequality follows from the log-concavity of g_p.

Now we turn to the converse. If g_p is increasing at any point, then it is clear that f is no longer unimodal and hence not log-concave. If g_p is not log-concave, then for some $t, s \in \mathbb{R}^+$, $\log g_p(\lambda t + (1-\lambda)s) < \lambda \log g_p(t) + (1-\lambda)\log g_p(s)$. Let $z \in \mathbb{R}^p$ satisfy $\|z\|_\Sigma = 1$, then

$$\begin{aligned}\log f(\lambda t z + (1-\lambda)sz) &= \log g_p(\lambda t + (1-\lambda)s) \\ &< \lambda \log g_p(t) + (1-\lambda)\log g_p(s) \\ &= \lambda \log f(tz) + (1-\lambda)\log f(sz)\end{aligned}$$

and thus $\log f$ is concave either.

□

2.1 Projection Operator

Define $\mathcal{F} = \{\phi(\mathbb{R}^+, \mathbb{R}^+) : \phi \text{ is concave, decreasing}\}$. We describe projection onto the class of p-variate densities of the form $f(x) = |\Sigma|^{-1/2}\exp(\phi(\|x\|_\Sigma))$ where ϕ is such that $\exp(\phi(r))r^{p-1}c_p$ is a density over $[0, \infty)$.

First, we fix $\Sigma \succ 0$. We can without loss of generality assume that $\|\Sigma\|_2 = 1$ as we have discussed in the section on identifiability.

Definition Let $\Sigma \succ 0$ be fixed. For a probability measure P over $\mathbb{R}^p$ and a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, we define

$$L_\Sigma(\phi, P) = \int \phi(\|x\|_\Sigma)dP - \int_0^\infty \exp(\phi(r))r^{p-1}c_p dr$$

The projection of P is $\phi^* \in \mathcal{F}$ such that

$$L_\Sigma(\phi^*, P) = \sup_{\phi \in \mathcal{F}} L_\Sigma(\phi, P)$$

Note 1:

First, we check that if ϕ^* exists, then $\exp(\phi^*(r))r^{p-1}c_p$ is indeed a density. To see this, note that

$$\partial_c L_\Sigma(\phi^* + c, P) = 1 - e^c \int_0^\infty \exp(\phi^*(r))r^{p-1}c_p dr$$

By definition of ϕ^*, $c = 0$ implies that $\partial_c L_\Sigma(\phi^* + c, P) = 0$, implying further that $\exp(\phi^*(r))r^{p-1}c_p$ is indeed a valid density.

Note 2:

It is clear that $L_\Sigma(\phi, P)$ is concave in ϕ.

Proposition 3. *Let $\Sigma \succ 0$ be fixed. Define $\mathcal{P}_d = \{P : \int \|x\| dP < \infty, \, P(\{0\}) < 1\}$. Then, we have,*

1. *If $\int \|x\| dP = \infty$, then $L_\Sigma(\phi, P) = -\infty$ for all $\phi \in \mathcal{F}$.*
2. *If $P(\{0\}) = 1$, then $\sup_{\phi \in \mathcal{F}} L_\Sigma(\phi, P) = \infty$*
3. *If $P(\{0\}) < 1$ and $\int \|x\| dP < \infty$, then $\sup_{\phi \in \mathcal{F}} L_\Sigma(\phi, P) < \infty$ and there exists a maximizer $\phi^* \in \mathcal{F}$ which achieves this value.*

Proof. Suppose that $\int \|x\| dP = \infty$. Then $\int \|x\|_\Sigma dP \geq \int \frac{\|x\|_2}{\|\Sigma\|_2} dP = \infty$.

First, suppose that $\lim_{r \to \infty} \phi(r) = c > -\infty$. Then $L(\phi, P) \leq \phi(0) - \int_0^\infty r^{p-1} e^c c_p dr = -\infty$. Thus, we may consider only ϕ such that $\lim_{r \to \infty} \phi(r) = -\infty$. For any such ϕ, there exists $a, b > 0$ such that $\phi(r) \leq a - b|r|$.

Hence, $L(\phi, P) \leq \int \phi dP \leq \int a - b\|x\|_\Sigma dP \leq -\infty$. This proves claim 1.

For claim 2, suppose $P(\{0\}) = 1$. Let $\phi_n(r) = n - e^n r$. Then, we have that

$$\begin{aligned} L(\phi_n, P) &= n - \int e^n \exp(-e^n r) r^{p-1} c_p dr \\ &= n - e^n \int e^{-s} \left(\frac{s}{e^n}\right)^{p-1} \frac{ds}{e^n} \\ &= n - (e^n)^{1-p} \Gamma(p) \end{aligned}$$

Thus, we have that $\lim_{n \to \infty} L(\phi_n, P) = \infty$. This proves the second claim.

Onto the third claim. We will first prove that if $P(\{0\}) < 1$ and $\int \|x\| dP < \infty$, then $-\infty < \sup_{\phi \in \mathcal{F}} L_\Sigma(\phi, P) < \infty$. Then we will prove that the maximizer exists.

By plugging in $\phi(r) = -r$, we have that $L_\Sigma(\phi, P) = -\int \|x\|_\Sigma dP - \int e^{-r} r^{p-1} c_p dr = -\int \|x\|_\Sigma dP - c_p \Gamma(p/2)$. Since $\int \|x\|_\Sigma dP \leq \int \|x\| \|\Sigma^{-1}\|_2 dP < \infty$, we have shown that $L_\Sigma > -\infty$ for some ϕ.

Define $b^* = \inf\{b : P(B_\Sigma(0;b)) \geq \frac{P(\{0\})}{2} + \frac{1}{2}\}$. $b^* > 0$ since $P(\{0\}) < 1$. Let $b = b^*/2$, $c = P(B_\Sigma(0;b))$, then we have that $0 < c < 1$.

Suppose $\phi(0) = M$ and $\phi(b) = M'$, because ϕ is non-increasing, $M = \sup_r \phi(r)$ and $M' = \inf_{r\in[0,b]} \phi(r) = \sup_{r\in[b,\infty)} \phi(r)$.

$$\int \phi dP \leq \int_{B_\Sigma(0;b)} \phi dP + \int_{B_\Sigma(0;b)^c} \phi dP \leq Mc + M'(1-c) = (M - M')c + M'$$

Then, we have that

$$\begin{aligned}
L_\Sigma(\phi, P) &= \int \phi dP - \int e^{\phi(r)} r^{p-1} c_p dr \\
&\leq \int \phi dP - \int_0^b e^{\phi(r)} r^{p-1} c_p dr \\
&\leq (M - M')c + M' - \int_0^b \exp(M - \frac{r}{b}(M - M')) r^{p-1} c_p dr \\
&\leq \Delta(c-1) + M - e^M \int_0^b \exp(-\frac{r}{b}\Delta) r^{p-1} c_p dr
\end{aligned}$$

where we have used the notation $\Delta = M - M'$.

First, let us suppose that $\Delta(1-c) \leq 2M$. Then, we have that

$$\begin{aligned}
L_\Sigma(\phi, P) &\leq M - e^M \int_0^b \exp(-\frac{r}{b}\frac{2M}{1-c}) r^{p-1} c_p dr \\
&\leq M - e^M \int_0^{2M/(1-c)} e^{-s} s^{p-1} c_p ds \left(\frac{b}{2M/(1-c)}\right)^p
\end{aligned}$$

Which is bounded since the RHS goes to $-\infty$ as M goes to ∞.

Now, let us suppose that $\Delta(1-c) > 2M$, then we have that

$$L_\Sigma(\phi, P) \leq -M$$

Thus, we see that $L_\Sigma(\phi, P)$ is bounded and, furthermore, there exists a constant M^* such that $\sup\{L(\phi, P) : \phi \in \mathcal{F}\} = \sup\{L(\phi, P) : \phi \in \mathcal{F}, \|\phi\|_\infty \leq M^*\}$.

Let $r^* = \sup\{r : P(B_\Sigma(0;r)) < 1\}$, Then

$\square$

Suppose $P(H) = 1$ for some hyperplane, then let $\Sigma_n \to A$ where H is the nullspace of A.

Suppose interior(csupp(P)) is non-empty, then its Lebesgue measure is some $c > 0$. The Lebesgue measure of $B_\Sigma(0;b)$ is at $c_p \frac{b^p}{p} |\Sigma|^{1/2}$. We assume that $\|\Sigma\|_2 = 1$.

3 Algorithm

Let $X_1, ..., X_n \sim P$ be the samples and let μ be zero and Σ be fixed. The log-likelihood is

$$l(g_p\,;\, X_1, ..., X_n) = \sum_{i=1}^{n} \log g_p(\|X_i\|_\Sigma)$$

where g_p is decreasing, log-concave, and satisfies $\int_0^\infty g_p(r) r^{p-1} c_p dr = 1$. If we reparametrize the problem by writing $\phi(r) = \log g_p(r)$, $Y_i = \|X_i\|_\Sigma$ and also put the integral constraint in the Lagrangian form, we get an equivalent optimization

$$\max_{\phi \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \phi(Y_i) - \int_0^\infty \exp(\phi(r)) r^{p-1} c_p dr$$

where $\mathcal{F} = \{\phi\ :\ \ \phi \text{ decreasing and concave }\}$. We use the notation from the previous section and denote $L_\Sigma(\phi, P_n) = \frac{1}{n}\sum_{i=1}^n \phi(Y_i) - \int_0^\infty \exp(\phi(r)) r^{p-1} c_p dr$.

Lemma 4. *Let $\phi \in \mathcal{F}$ and let $\bar{\phi}$ be the piecewise linear function with the property that $\bar{\phi}(Y_i) = \phi(Y_i)$ for all $i = 1, ..., n$ and $\bar{\phi}(0) = \phi(0)$. Then, we have that $\bar{\phi} \in \mathcal{F}$ and*

$$L_\Sigma(\bar{\phi}, P_n) \geq L_\Sigma(\phi, P_n)$$

The implication is that we need only optimize over piecewise linear functions whose knots are placed at $\{Y_1, ..., Y_n\} \cup \{0\}$.

Proof. It is clear that $\bar{\phi} \in \mathcal{F}$ and that $\phi \geq \bar{\phi}$.

Therefore,

$$\sum_{i=1}^{n} \phi(Y_i) = \sum_{i=1}^{n} \bar{\phi}(Y_i)$$
$$\int_0^\infty \exp(\phi(r)) r^{p-1} c_p dr \geq \int_0^\infty \exp(\bar{\phi}(r)) r^{p-1} c_p dr$$

□

3.1 Piecewise Linear Parametrization

Let $\bar{\mathcal{F}} = \{\phi\ :\ \phi \text{ is p.w. linear, decreasing, concave}\}$.

Given samples $Y_1, ..., Y_n \in \mathbb{R}^+$, any $\phi \in \bar{\mathcal{F}}$ can be parametrized by a vector $(\phi_1, ..., \phi_n)$,

$$\phi(r) = \sum_{i=1}^{n-1} \left[\left(\frac{Y_{i+1} - r}{Y_{i+1} - Y_i} \right) \phi_i + \left(\frac{r - Y_i}{Y_{i+1} - Y_i} \right) \phi_{i+1} \right] \mathbf{1}_{r \in [Y_i, Y_{i+1}]} + \phi_1 \mathbf{1}_{r \in [0, Y_1]}$$

Thus, we can write the full optimization as

$$\max_{\phi_1,\dots,\phi_n} \frac{1}{n}\sum_{i=1}^{n}\phi_i - \sum_{i=1}^{n-1}\int_{Y_i}^{Y_{i+1}} \exp\left(\frac{Y_{i+1}-r}{Y_{i+1}-Y_i}\phi_i + \frac{r-Y_i}{Y_{i+1}-Y_i}\phi_{i+1}\right) r^{p-1}c_p dr - \int_0^{Y_1} \exp(\phi_1) r^{p-1} c_p dr$$

$$\text{subject to } \frac{\phi_{i+1}-\phi_i}{Y_{i+1}-Y_i} \geq \frac{\phi_{i+2}-\phi_{i+1}}{Y_{i+2}-Y_{i+1}} \quad \text{for all } i = 1,..,n-2$$

$$\frac{\phi_2-\phi_1}{Y_2-Y_1} \leq 0$$

3.1.1 Derivatives

Define the F function as the objective

$$F(\phi_1,\dots,\phi_n) = \frac{1}{n}\sum_{i=1}^{n}\phi_i - \sum_{i=1}^{n-1}\int_{Y_i}^{Y_{i+1}} \exp\left(\frac{Y_{i+1}-r}{Y_{i+1}-Y_i}\phi_i + \frac{r-Y_i}{Y_{i+1}-Y_i}\phi_{i+1}\right) r^{p-1}c_p dr - \int_0^{Y_1} \exp(\phi_1) r^{p-1} c_p dr$$

We will rewrite F to facilitate the differentiation.

$$F(\phi) = \frac{1}{n}\mathbf{1}^\top\phi - \sum_{i=1}^{n-1}\int_{Y_i}^{Y_{i+1}} \exp(a_i(r)^\top\phi) r^{p-1}c_p dr - \int_0^{Y_1} \exp(\phi_1) r^{p-1} c_p dr$$

where $a_i(r) \in \mathbb{R}^n$ is the following form: $(0,\dots,0,\frac{Y_{i+1}-r}{Y_{i+1}-Y_i},\frac{r-Y_i}{Y_{i+1}-Y_i},0,\dots,0)$ where the two non-zero coordinates are $i, i+1$. Then, we have that

$$\nabla F = \frac{1}{n}\mathbf{1} - \sum_{i=1}^{n-1}\int_{Y_i}^{Y_{i+1}} a_i(r)\exp(a_i(r)^\top\phi) r^{p-1}c_p dr - \int_0^{Y_1} e_1 \exp(\phi_1) r^{p-1} c_p dr$$

$$H\,F = -\sum_{i=1}^{n-1}\int_{Y_i}^{Y_{i+1}} a_i(r)a_i(r)^\top\exp(a_i(r)^\top\phi) r^{p-1}c_p dr - \int_0^{Y_1} e_1 e_1^\top \exp(\phi_1) r^{p-1} c_p dr$$

For a given ϕ, these can be evaluated by numerical integration.

3.1.2 Active Set

Let us express the constraints as $v_i^\top \phi \leq 0$ for $i = 1, ..., n-1$, where

$$\begin{aligned}
v_1 &= (-\frac{1}{Y_2 - Y_1}, \frac{1}{Y_2 - Y_1}, 0, ..., 0) \\
v_2 &= (\frac{1}{Y_2 - Y_1}, -\frac{1}{Y_3 - Y_2} - \frac{1}{Y_2 - Y_1}, \frac{1}{Y_3 - Y_2}, 0, ...0) \\
&... \\
v_i &= (0, ..., 0, \frac{1}{Y_i - Y_{i-1}}, -\frac{1}{Y_{i+1} - Y_i} - \frac{1}{Y_i - Y_{i-1}}, \frac{1}{Y_{i+1} - Y_i}, 0, ...0) \\
&...
\end{aligned}$$

Define the active set $A \subset \{1, ..., n-1\}$ as $A = \{i\,;\, v_i^\top \phi = 0\}$. Define $I = \{1, ..., n-1\} - A$.

Proposition 5. *Define $V \in \mathbb{R}^{n \times n}$ such that the i-th row $V_i = v_i$ for $i = 1, ..., n-1$ and the n-th row $V_n = \mathbf{1}_n$.*

Then, define $B^\top = -V^{-1}$ and let b_i be the i-th row of B. We have that $b_i^\top v_i = -1$ and $b_i^\top v_j = 0$ for $j \neq i$.

The proof follows from the observation that V is invertible and that $B^\top V = -I$.

3.1.3 Optimization over an Active Set

In this section, we solve

$$\begin{aligned}
&\min_\phi F(\phi) \\
&\text{s.t. } v_i^\top \phi = 0 \qquad \text{for all } i \in A
\end{aligned}$$

Given the active set A, let us define $I = \{1, ..., n\} - A$. We index the elements of I by $i_1, ..., i_T$ where T denotes the cardinality of I. By definition, $n \in I$ always.

Proposition 6. *The subspace $\{\phi : v_A^\top \phi = 0\}$ is equal to the subspace of ϕ where*

1. $\phi_I \in \mathbb{R}^T$
2. *For $j \in A$ where $j < i_1$, $\phi_j = \phi_{i_1}$*
3. *For $j \in A$ where $j > i_1$, $\phi_j = \frac{Y_{i_{t+1}} - Y_j}{Y_{i_{t+1}} - Y_{i_t}} \phi_{i_t} + \frac{Y_j - Y_{i_t}}{Y_{i_{t+1}} - Y_{i_t}} \phi_{i_{t+1}}$, $i_t < j < i_{t+1}$*

Given the proposition, we can solve the optimization over an active set with an unconstrained optimization.

$$\min_{\phi_I} \frac{1}{n} \left(i_1 \phi_{i_1} + \sum_{t=1}^{T-1} \sum_{j=i_t+1}^{i_{t+1}} \frac{Y_{i_{t+1}} - Y_j}{Y_{i_{t+1}} - Y_{i_t}} \phi_{i_t} + \frac{Y_j - Y_{i_t}}{Y_{i_{t+1}} - Y_{i_t}} \phi_{i_{t+1}} \right)$$
$$+ \sum_{t=1}^{T-1} \int_{Y_{i_t}}^{Y_{i_{t+1}}} \exp\left(\frac{Y_{i_{t+1}} - r}{Y_{i_{t+1}} - Y_{i_t}} \phi_i + \frac{r - Y_{i_t}}{Y_{i_{t+1}} - Y_{i_t}} \phi_{i_{t+1}} \right) r^{p-1} c_p dr - \int_0^{Y_{i_1}} \exp(\phi_{i_1}) r^{p-1} c_p dr$$

We let $F(\phi_I)$ denote the objective function. Again, we can simplify the notation with vector representation.

$$F(\phi_I) = \frac{1}{n} w^\top \phi_I + \sum_{t=1}^{T} \int_{Y_{i_t}}^{Y_{i_{t+1}}} \exp(a_t(r)^\top \phi_I) r^{p-1} c_p dr - \int_0^{Y_{i_1}} \exp(\phi_{i_1}) r^{p-1} c_p dr$$

where $w \in \mathbb{R}^T$ is of the form $w_1 = i_1 + \sum_{j=i_1+1}^{i_2} \frac{Y_{i_2} - Y_j}{Y_{i_2} - Y_{i_1}}$ and $w_t = \sum_{j=i_{t-1}+1}^{i_t} \frac{Y_j - Y_{i_{t-1}}}{Y_{i_t} - Y_{i_{t-1}}} + \sum_{j=i_t+1}^{i_{t+1}} \frac{Y_{i_{t+1}} - Y_j}{Y_{i_{t+1}} - Y_{i_t}}$.

And, $a_t(r) \in \mathbb{R}^T$ is of the form $(0, ..., 0, \frac{Y_{i_{t+1}} - r}{Y_{i_{t+1}} - Y_{i_t}}, \frac{r - Y_{i_t}}{Y_{i_{t+1}} - Y_{i_t}}, 0, ..., 0)$ where the two non-zero coordinates are $t, t+1$.

4 Moment Properties

As described in previous sections, if $X \in \mathbb{R}^p$ follows a spherically-symmetric log-concave density, then the norm of X follows a density of the form $e^{\phi(r)} r^{p-1}$ where ϕ is decreasing and concave.

We define the following:

$$\Phi \equiv \{\phi : \mathbb{R}^+ \to \mathbb{R} : \text{ decreasing, concave}\} \tag{1}$$
$$\mathcal{H} \equiv \left\{ h \, : \, h(r) = r^{p-1} e^{\phi(r)}, \, \phi \in \Phi, \, \int h(r) dr = 1, \, \sigma_h = 1 \right\} \tag{2}$$

For technical reasons made clear later, we will also consider densities of the form $e^{\phi(r)} r^{p-1}$ where $\phi(r) = 0$ for $r \leq a_0$ for some a_0 and ϕ is decreasing and concave on $[a_0, \infty)$. Accordingly, we define the following:

$$\Phi_{a_0} \equiv \{\phi : [a_0, \infty) \to \mathbb{R} : \text{ decreasing, concave}\} \tag{3}$$
$$\mathcal{H}_{a_0} \equiv \left\{ h \, : \, h(r) = r^{p-1} e^{\phi(r)}, \, \phi \in \Phi_{a_0}, \, \int h(r) dr = 1, \, \sigma_h = 1 \right\} \tag{4}$$

It is clear that $\mathcal{H}_{a_0} = \mathcal{H}$ for $a_0 = 0$.

4.1 Bounds on First Moment

Proposition 7. *For any $a_0 \geq 0$, for any $h \in \mathcal{H}_{a_0}$, we have that $a_0 \leq \mu_h \leq a_0 + c_1 p$ where c_1 is some absolute constant.*

Proof. Let $h \in \mathcal{H}_{a_0}$. It is clear that $\mu_h \geq a_0$ since h is zero on $(-\infty, a_0)$.

We prove that $\mu_h \leq a_0 + c_1 p$ by proving the contrapositive, that if $\mu_h \geq a_0 + c_1 p$ for some absolute constant c_1, then $\sigma_h > 1$.

By lemma **??**, if $\mu_h \geq a_0 + c_1 p$, then $\|h\|_\infty \leq \frac{16}{c_1}$. Therefore, $\sigma_h \geq \frac{Cc_1}{16}$ by lemma 22, where C is some universal constant. Taking $c_1 > 16/C$ yields that $\sigma_h > 1$. The proof is thus complete. $\square$

Define

$$\mathcal{H}^{\mu}_{a_0} \equiv \left\{ h \,:\, h(r) = r^{p-1} e^{\phi(r)},\, \phi \in \Phi_{a_0},\, \int h(r)dr = 1,\, \mu_h \geq a_0 + \mu \right\}$$

Lemma 8. *For any $a_0 \geq 0$ and any $r \in [a_0, \infty)$, we have that*

$$\sup_{h \in \mathcal{H}^{\mu}_{a_0}} h(r) \leq \min\left\{ 16\frac{p}{\mu}, \frac{p}{r - a_0} \right\}$$

Proof. For any $h \in \mathcal{H}^{\mu}_{a_0}$, we define $\tilde{h}(r)$ as the location shift $h(r - a_0)$. Note that $\tilde{h}(r)$ has a density of the form $e^{\phi(r)}(r + a_0)^{p+1}$ where $\phi \in \Phi$, i.e., it is decreasing and concave on $[0, \infty)$. We will prove the lemma by proving that $\sup_{h \in \mathcal{H}^{\mu}_{a_0}} \tilde{h}(r) \leq \min\left\{ 16\frac{p}{\mu}, \frac{p}{r} \right\}$.

Fix $r_0 \in [0, \infty)$. We first prove that $\sup_{h \in \mathcal{H}^{\mu}_{a_0}} \tilde{h}(r_0) \leq \frac{p}{r_0}$.

Consider an indicator ϕ and denote the resulting density $\tilde{h}_{r_0}(r) = \alpha \mathbb{1}_{r \in [0, r_0]}(r + a_0)^{p-1}$ where $\alpha^{-1} = \int_0^{r_0} (r + a_0)^{p-1} dr = \frac{(r_0 + a_0)^p}{p} - \frac{a_0^p}{p}$.

Let $\tilde{h}(r) = e^{\phi(r)}(r + a_0)^{p-1}$. Then, we have that

$$\begin{aligned}
\int_0^{r_0} \alpha (r + a_0)^{p-1} dr = 1 &\geq \int_0^{r_0} e^{\phi(r)} (r + a_0)^{p-1} dr \\
&\geq \int_0^{r_0} e^{\phi(r_0)} (r + a_0)^{p-1} dr
\end{aligned}$$

Thus, $e^{\phi(r_0)} \leq \alpha$ and $\tilde{h}(r_0) \leq \tilde{h}_{r_0}(r_0) = \alpha (r_0 + a_0)^{p-1}$.

Since

$$
\begin{aligned}
&\alpha(a_0+r_0)^{p-1}\frac{(a_0+r_0)}{p}-\alpha\frac{a_0^p}{p}=1\\
&(\Rightarrow)\quad \alpha(a_0+r_0)^{p-1}(a_0+r_0)-\alpha(r_0+a_0)^{p-1}a_0\le p\\
&(\Rightarrow)\quad \alpha(a_0+r_0)^{p-1}\le\frac{p}{r_0}
\end{aligned}
$$

Thus, we have shown that $\tilde{h}(r_0)\le\frac{p}{r_0}$ as desired.

Now we move onto the second bound in the inequality. Note that if $r_0\ge\mu/8$, then $\frac{p}{r_0}\le 8\frac{p}{\mu}$ and the second bound is proven.

Therefore, let us fix $r_0\in[0,\mu/8)$. We will prove that $\sup_{h\in\mathcal{H}^{\mu}_{a_0}}\tilde{h}(r_0)\le\frac{8p}{\mu-4r_0}\le 16\frac{p}{\mu}$.

To this end, fix $M\ge\log 16$, and $m\in(-\infty,M-2]$. Suppose that $h\in\mathcal{H}^{\mu}_{a_0}$ satisfies $\log\tilde{h}(r_0)\ge M$ for some $r_0\in[0,p^{1/2}]$. Note that $\log\tilde{h}$ itself is a concave function on $\mathbb{R}^+$ and so $\tilde{h}$ is a log-concave density.

For $t\in[m,M]$, let $D_t:=\{r\in[0,\infty):\log\tilde{h}(r)\ge t\}$. First note that for any $t\in[m,M]$ and $r\in D_m$, we have

$$\log\tilde{h}\bigg(\frac{t-m}{M-m}r_0+\frac{M-t}{M-m}r\bigg)\ge\frac{(t-m)M}{M-m}+\frac{(M-t)m}{M-m}=t.$$

Hence, writing μ for Lebesgue measure on $\mathbb{R}$,

$$\mu(D_t)\ge\mu\bigg(\frac{t-m}{M-m}r_0+\frac{M-t}{M-m}D_m\bigg)=\frac{M-t}{M-m}\mu(D_m).$$

Using Fubini's theorem, we can now compute

$$
\begin{aligned}
1&\ge\int_{D_m}\tilde{h}(r)-e^m\,dr\ge\int_{D_m}\int_m^M e^s\mathbb{1}_{\{\log\tilde{h}(r)\ge s\}}\,ds\,dr\\
&=\int_m^M e^s\mu(D_s)\,ds\ge\frac{\mu(D_m)}{M-m}\int_m^M(M-s)e^s\,ds=\frac{\mu(D_m)e^M}{M-m}\int_0^{M-m}te^{-t}\,dt\\
&\ge\frac{\mu(D_m)e^M}{2(M-m)}.
\end{aligned}
$$

Since D_m is an interval containing r_0, we conclude that $\log\tilde{h}(r)\le m$ whenever $|r-r_0|\ge 2(M-m)e^{-M}$. Thus

$$\log\tilde{h}(r)\le M-\frac{|r-r_0|e^M}{2}$$

for $|r-r_0|\ge 4e^{-M}$. First, suppose that $r_0-4e^{-M}>0$ and using the bound $\tilde{h}(r)\le p/r$, it now follows that

$$\mu \le \int_0^\infty r h(r)\, dr \le \int_0^{r_0-4e^{-M}} r \exp\left\{ M - \frac{(r_0-r)e^M}{2} \right\} dr + \int_{r_0-4e^{-M}}^{r_0+4e^{-M}} r \frac{p}{r}\, dr$$
$$+ \int_{r_0+4e^{-M}}^\infty r \exp\left\{ M - \frac{(r-r_0)e^M}{2} \right\} dr$$
$$\le 2\int_2^\infty \left(r_0 - \frac{2s}{e^M} \right) e^{-s}\, ds + 8e^{-M}p + 2\int_2^\infty \left(r_0 + \frac{2s}{e^M} \right) e^{-s}\, ds$$
$$= 4r_0 \int_2^\infty e^{-s} ds + 8e^{-M}p$$
$$\le 4r_0 + 8e^{-M}p$$

Thus yielding $\mu \le 4r_0 + 8e^{-M}p$. Since $4r_0 \le \mu/2$ by assumption, we have that $e^M \le \frac{8p}{\mu - 4r_0} \le 16\frac{p}{\mu}$.

Now, suppose $r_0 - 4e^{-M} \le 0$. Then, similarly, we have that

$$\mu \le \int_0^\infty r h(r)\, dr \le \int_0^{r_0+4e^{-M}} r \frac{p}{r}\, dr$$
$$+ \int_{r_0+4e^{-M}}^\infty r \exp\left\{ M - \frac{(r-r_0)e^M}{2} \right\} dr$$
$$\le 8e^{-M}p + 2\int_2^\infty \left(r_0 + \frac{2s}{e^M} \right) e^{-s}\, ds$$
$$= 4r_0 + 2e^{-M} + 8e^{-M}p$$
$$\le 4r_0 + 2e^{-M}(4p+1)$$

Similar reasoning yields the desired upper bound on e^M.

□

5 Envelope bounds

We define the following:

$$\Phi \equiv \{\phi : \mathbb{R}^+ \to \mathbb{R} : \text{ decreasing, concave}\} \tag{5}$$
$$\mathcal{G} \equiv \{e^\phi : \phi \in \Phi\} \tag{6}$$
$$\mathcal{H} \equiv \left\{ h(r) : h(r) = c_p r^{p-1} e^{\phi(r)},\ \phi \in \Phi,\ \int h(r)dr = 1,\ \int r^2 h(r) dr = p \right\} \tag{7}$$

Thus $\mathcal{H}$ consists of densities of random variables $\|X\|$, where X has a spherically symmetric, log-concave density on $\mathbb{R}^p$, and $\mathbb{E}(\|X\|^2) = p$.

The following result provides crude upper bounds for $\mathcal{H}$.

Lemma 9. *For all $r \in [0,\infty)$, we have*

$$\sup_{h\in\mathcal{H}} h(r) \leq \begin{cases} \min(\sqrt{2}, 1/r) & \text{if } p = 1 \\ \min\Bigl\{\frac{(p+1)^{p/2}}{(p-1)!} r^{p-1}\,, 24r\,, \frac{p}{r}\Bigr\} & \text{if } p \geq 2. \end{cases} \tag{8}$$

Remark: The only difference between the cases $p = 1$ and $p \geq 2$ is that the bound $\sup_{h\in\mathcal{H}} h(r) \leq 24r$ does not hold when $p = 1$. The bounds $\frac{(p+1)^{p/2}}{(p-1)!} r^{p-1}$ and p/r are sharp when $r = 0$ and $r = (p+2)^{1/2}$ respectively. The first of these facts is trivial unless $p = 1$, but in that case one can observe that if we define $h : [0,\infty) \to [0,\infty)$ by $h(r) := \sqrt{2}e^{-\sqrt{2}r}$ then $h \in \mathcal{H}$ and $h(0) = \sqrt{2}$. The second fact follows because if we define $h : [0,\infty) \to [0,\infty)$ by $h(r) := \frac{p}{(p+2)^{p/2}} r^{p-1} \mathbb{1}_{\{r\in[0,(p+2)^{1/2}]\}}$, then $h \in \mathcal{H}$ and $h(\sqrt{p+2}) = p/(p+2)^{1/2}$.

Remark for us: The second bound in (8) seems to be unnecessary.

Proof. For the first bound in (8) (treating the cases $p = 1$ and $p \geq 2$ simultaneously), for $r \in [0,\infty)$, let

$$g_0^*(r) := \frac{(p+1)^{p/2}}{c_p(p-1)!} e^{-(p+1)^{1/2} r},$$

so $g_0^* \in \mathcal{G}$, and let $h_0^*(r) := c_p r^{p-1} g_0^*(r)$. Then h_0^* is the $\Gamma(p, (p+1)^{1/2})$ density, so $h_0^* \in \mathcal{H}$. Suppose for a contradiction that $g \in \mathcal{G}$ satisfies the conditions the function $h : [0,\infty) \to [0,\infty)$ given by $h(r) := c_p r^{p-1} g(r)$ belongs to $\mathcal{H}$, and $g(0) > g_0^*(0)$. Then since $\log g_0^*$ is an affine function and h is a log-concave density, there exists $r_0 \in (0,\infty)$ such that $g(r) > g_0^*(r)$ for $r < r_0$ and $g(r) < g_0^*(r)$ for $r > r_0$. But then $h <_{\mathrm{st}} h^*$, so $c_p \int_0^\infty r^{p+1} g(r)\,dr < p$, which establishes our desired contradiction. But since every $\phi \in \Phi$ is decreasing, it follows that $r \mapsto \sup_{g\in\mathcal{G}} g(r)$ is decreasing, so

$$\sup_{h\in\mathcal{H}} h(r) = c_p \sup_{g\in\mathcal{G}} r^{p-1} g(r) \leq c_p r^{p-1} \sup_{g\in\mathcal{G}} g(0) = c_p r^{p-1} g_0^*(0) = \frac{(p+1)^{p/2}}{(p-1)!} r^{p-1}.$$

Next we establish the third bound in (8), again treating $p = 1$ and $p \geq 2$ simultaneously. For $a \in (0,\infty)$ and $r \in (0,\infty)$, consider the function

$$g_a(r) := \frac{p}{c_p a^p} \mathbb{1}_{\{r\in[0,a]\}}.$$

Then $g_a \in \mathcal{G}$ and $c_p \int_0^\infty r^{p-1} g_a(r)\,dr = 1$. Thus if $g \in \mathcal{G}$ satisfies $g(a) > g_a(a)$, then $g(r) > g_a(r)$ for all $r \in [0,a]$ and $g(r) \geq g_a(r)$ for all $r \in [0,\infty)$. But then $c_p \int_0^\infty r^{p-1} g(r)\,dr > 1$, so the function $h : [0,\infty) \to [0,\infty)$ given by $h(r) := c_p r^{p-1} g(r)$ does not belong to $\mathcal{H}$. We deduce that for every $r \in (0,\infty)$,

$$\sup_{h\in\mathcal{H}} h(r) \leq c_p r^{p-1} g_r(r) = \frac{p}{r}.$$

Finally, we prove the second bound in (8) in the case $p \geq 2$. To this end, fix $M \geq \log 16$, and $m \in (-\infty, M-2]$. Suppose that $h \in \mathcal{H}$ satisfies $\log h(r_0) \geq M$ for some $r_0 \in (1/4, p^{1/2}]$, and for $t \in [m, M]$, let $D_t := \{r \in [0,\infty) : \log h(r) \geq t\}$. First note that for any $t \in [m, M]$ and $r \in D_m$, we have

$$\log h\biggl(\frac{t-m}{M-m}r_0 + \frac{M-t}{M-m}r\biggr) \geq \frac{(t-m)M}{M-m} + \frac{(M-t)m}{M-m} = t.$$

Hence, writing μ for Lebesgue measure on $\mathbb{R}$,

$$\mu(D_t) \geq \mu\biggl(\frac{t-m}{M-m}r_0 + \frac{M-t}{M-m}D_m\biggr) = \frac{M-t}{M-m}\mu(D_m).$$

Using Fubini's theorem, we can now compute

$$\begin{aligned}
1 &\geq \int_{D_m} h(r) - e^m \, dr \geq \int_{D_m}\int_m^M e^s \mathbb{1}_{\{\log h(r) \geq s\}} \, ds \, dr \\
&= \int_m^M e^s \mu(D_s) \, ds \geq \frac{\mu(D_m)}{M-m}\int_m^M (M-s)e^s \, ds = \frac{\mu(D_m)e^M}{M-m}\int_0^{M-m} te^{-t} \, dt \\
&\geq \frac{\mu(D_m)e^M}{2(M-m)}.
\end{aligned}$$

Since D_m is an interval containing r_0, we conclude that $\log h(r) \leq m$ whenever $|r - r_0| \geq 2(M-m)e^{-M}$. Thus

$$\log h(r) \leq M - \frac{|r-r_0|e^M}{2}$$

for $|r - r_0| \geq 4e^{-M}$. Noting that $r_0 - 4e^{-M} > 0$ and using the bound $h(r) \leq p/r$, it now follows that

$$\begin{aligned}
p = \int_0^\infty r^2 h(r) \, dr &\leq \int_0^{r_0 - 4e^{-M}} r^2 \exp\biggl\{M - \frac{(r_0 - r)e^M}{2}\biggr\} \, dr + p\int_{r_0-4e^{-M}}^{r_0+4e^{-M}} r \, dr \\
&\qquad + \int_{r_0+4e^{-M}}^\infty r^2 \exp\biggl\{M - \frac{(r-r_0)e^M}{2}\biggr\} \, dr \\
&\leq 2\int_2^\infty \biggl(r_0 - \frac{2s}{e^M}\biggr)^2 e^{-s} \, ds + 8e^{-M}r_0 p + 2\int_2^\infty \biggl(r_0 + \frac{2s}{e^M}\biggr)^2 e^{-s} \, ds \\
&= 4e^{-2}r_0^2 + 32e^{-2M} + 8e^{-M}r_0 p \leq p\biggl(\frac{2}{3} + 8e^{-M}r_0\biggr).
\end{aligned}$$

We deduce that $e^{-M}r_0 \geq 1/24$, so $h(r) \leq \min(16, 24r)$ for $r \in (1/4, p^{1/2}]$. But our first bound in (8) is at most $5r$ for $r \leq 1$ and $p \geq 2$, and the conclusion follows. □

Corollary 10. *Let $Z \sim h \in \mathcal{H}$. Then there exists a universal constant $c_0 > 0$ such that* $\mathrm{Var}(Z) \geq c_0 p^{-1}$.

Remark: Define $h : [0,\infty) \to [0,\infty)$ by $h(r) := \frac{p}{(p+2)^{p/2}} r^{p-1} \mathbb{1}_{\{r \in [0,(p+2)^{1/2}]\}}$. Then it can be shown that $h \in \mathcal{H}$, and if $Z \sim h$, then $\mathrm{Var}(Z) = p/(p+1)^2$. Thus the bound given in Corollary 10 is sharp in terms of its dependence on p.

Proof. From the first bound in Lemma 9, we have

$$\sup_{h \in \mathcal{H}} \sup_{r \in [0,\infty)} h(r) \leq \sqrt{2}$$

for $r \leq p^{1/2}/e$. Write $\mu := \mathbb{E}(Z)$ and $\sigma^2 := \mathrm{Var}(Z)$. By Lovász and Vempala [2007, Theorem 5.14(d)], we have

$$\frac{1}{128\sigma} \leq h(\mu) \leq \sup_{h \in \mathcal{H}} \sup_{r \in [0,\infty)} h(r) \leq e p^{1/2}.$$

The result follows. □

An upper bound on the variance of $Z \sim h \in \mathcal{H}$ is readiy available.

Lemma 11. *Bobkov [2003, Lemma 1] Suppose h is a density of the form $r^{p-1} g(r) c_p$ for some log-concave function $g(r)$, suppose $Z \sim h$, then,*

$$\mathrm{Var}(Z) \leq \frac{1}{p} (\mathbb{E}Z)^2$$

Under our constraint on $h \in \mathcal{H}$, we have that $(\mathbb{E}Z)^2 \leq \mathbb{E}[Z^2] = p$. This gives us the following corollary:

Corollary 12. *Let $h \in \mathcal{H}$ and suppose $Z \sim h$. Then,*

$$\mathrm{Var}(Z) \leq 1$$

The upper bound is also tight. If we let $g(r) = e^{-ar} c$ where $a = \sqrt{\frac{(p+2)(p+1)}{p}}$ and c be chosen such that $cc_p = \frac{a^p}{\Gamma(p)}$, then we have that the mean is $\sqrt{\frac{p^3}{(p+2)(p+1)}}$ and the variance is $\frac{p^2}{(p+2)(p+1)}$. Thus, the variance of our chosen $g(r)$ gets arbitrarily close to 1 for increasing p.

We need one more ingredient before we can state our envelope bound.

Lemma 13. *Let $\mathcal{F}^{\mu,\sigma^2} = \{f \text{ log-concave density} : \mu_f = \mu, \sigma_f^2 = \sigma^2\}$. Then, there exists universal constants A, B such that*

$$\sup_{f \in \mathcal{F}^{\mu,\sigma^2}} f(x) \leq \frac{A}{\sigma} \exp\left(-B \frac{|x-\mu|}{\sigma}\right)$$

Proof. This follows directly from Kim and Samworth [2016, Theorem 2] by specializing to $d = 1$ and performing a change of variables. □

The following theorem gives an envelope bound for the density class $\mathcal{H}$.

Theorem 14. *For any absolute constant $c_1, c_2 > 0$, there exists constants C_1, C_2 such that*

$$\sup_{f\in\mathcal{H}} f(x) \leq \begin{cases} \frac{A'}{c_0}\sqrt{p} & \left|x-\sqrt{p}\right| \leq \frac{c_0}{B\sqrt{p}} \\ \frac{A'}{eB}\frac{1}{|x-\sqrt{p}|} & \frac{c_0}{B\sqrt{p}} \leq \left|x-\sqrt{p}\right| \leq \frac{1}{B} \\ A'e^{-B|x-\sqrt{p}|} & \frac{1}{B} \leq \left|x-\sqrt{p}\right| \end{cases}$$

Proof. Define $\mathcal{H}_\sigma = \{f \in \mathcal{H} : \sigma_f = \sigma\}$ as the sub-class of $\mathcal{H}$ in which the densities have standard deviation σ. It is clear that $\mathcal{H}_\sigma = \emptyset$ for $\sigma \notin [\frac{c_0}{\sqrt{p}}, 1]$ by our upper and lower bounds on the variance of densities in $\mathcal{H}$ (Corollary 10, 12).

First, we observe

$$\sup_{f\in\mathcal{H}} f(x) = \sup_{\sigma\in[\frac{c_0}{\sqrt{p}},1]} \sup_{f\in\mathcal{H}_\sigma} f(x)$$

And, by Lemma 13 and by the fact that $\mu = \sqrt{\mathbb{E}Z^2 - \sigma^2} = \sqrt{p-\sigma^2}$,

$$\begin{aligned} \sup_{f\in\mathcal{H}_\sigma} f(x) &\leq \frac{A}{\sigma}\exp\left(-B\frac{|x-\sqrt{p-\sigma^2}|}{\sigma}\right) \\ &\leq \frac{A}{\sigma}\exp\left(-B\frac{|x-\sqrt{p}| - (\sqrt{p}-\sqrt{p-\sigma^2})}{\sigma}\right) \\ &\leq \frac{A}{\sigma}\exp\left(-B\frac{|x-\sqrt{p}|}{\sigma} + B\frac{\frac{\sigma^2}{2\sqrt{p-\sigma^2}}}{\sigma}\right) \\ &= \frac{A}{\sigma}\exp\left(-B\frac{|x-\sqrt{p}|}{\sigma} + B\frac{\sigma}{2\sqrt{p-\sigma^2}}\right) \\ &\leq \frac{A}{\sigma}\exp\left(-B\frac{|x-\sqrt{p}|}{\sigma} + B\frac{1}{2\sqrt{p-1}}\right) \\ &\leq \frac{A'}{\sigma}\exp\left(-B\frac{|x-\sqrt{p}|}{\sigma}\right) \end{aligned}$$

The second inequality follows from triangle inequality. The third inequality follows because $\sqrt{p} - \sqrt{p-\sigma^2} \leq \frac{d\sqrt{p-\sigma^2}}{dp}\sigma^2$ by concavity of the square root function. The fourth inequality follows because $\sigma \leq 1$. And, in the last inequality, we define A' to be a constant since $B\frac{1}{2\sqrt{p-1}}$ is bounded by a constant for all p.

Let $H_\sigma(x) = \frac{A'}{\sigma}\exp\left(-B\frac{|x-\sqrt{p}|}{\sigma}\right)$.

If we write $\nu = \frac{1}{\sigma}$, then $\log H_\sigma(x)$ is a concave function of ν. We solve for the optimal and get that $\sigma = \frac{1}{\sqrt{p}}$ if $B|x-\sqrt{p}| \leq \sqrt{c_0}\sqrt{p}$, $\sigma = B|x-\sqrt{p}|$ if $\frac{c_0}{\sqrt{p}} \leq B|x-\sqrt{p}| \leq 1$, and $\sigma = 1$ if $1 \leq B|x-\sqrt{p}|$.

Thus, if $B|x-\sqrt{p}| \leq \frac{c_0}{\sqrt{p}}$, we have that $\sup_{\sigma \in [c_0/\sqrt{p}, 1]} H_\sigma(x) \leq A' \frac{\sqrt{p}}{c_0}$. If $\frac{c_0}{\sqrt{p}} \leq B|x-\sqrt{p}| \leq 1$, we have that $\sup_{\sigma \in [c_0/\sqrt{p}, 1]} H_\sigma(x) \leq \frac{A'}{eB|x-\sqrt{p}|}$. If $1 \leq B|x-\sqrt{p}|$, then $\sup_{\sigma \in [c_0/\sqrt{p}, 1]} H_\sigma(x) \leq A' \exp(-B|x-\sqrt{p}|)$.

□

6 Rate of Convergence

Let f_0 be a spherically symmetric log-concave density and suppose that $X_1, ..., X_n \sim f_0$. Let $\hat{f}$ be the spherically-symmetric log-concave (SSLC) MLE. Since MLE is scale-invariant, we assume without loss of generality that $\mathrm{Var}_{f_0}(\|X\|) = 1$.

We aim to show the following:

$$\mathbb{E} d_H(f_0, \hat{f}_n) \lesssim n^{-4/5}$$

Let h_0 be the density of $\|X\|$, then we know that $h_0(r) = c_p r^{p-1} e^{\phi(r)}$ where ϕ is decreasing and concave. Likewise, we define $\hat{h}_n = c_p r^{p-1} e^{\hat{\phi}(r)}$ where $\hat{f}_n(x) = e^{\hat{\phi}(\|x\|)}$. Let $Z_1, ..., Z_n \sim h_0$ and let $\mathbb{H}_n$ denote their empirical distribution. It is straightforward to see that

$$\hat{\phi} = \operatorname*{argmax}_{\phi \in \Phi, \int e^{\phi} r^{p-1} c_p dr = 1} \int \phi(r) d\mathbb{H}_n \tag{9}$$

By a change of variable, it suffices to prove that

$$\mathbb{E} d_H(h_0, \hat{h}_n) \lesssim n^{-4/5}$$

Let Φ be defined as 5 and let

$$\mathcal{H} = \left\{ h \,:\, h(r) = c_p r^{p-1} e^{\phi(r)},\, \phi \in \Phi,\, \int h(r) dr = 1 \right\} \tag{10}$$

6.1 Moment Characterizations of h_0

We first bound the moments of h_0 and $\hat{h}_n$.

Proposition 15. *Suppose $h_0 \in \mathcal{H}$ and that $Var_{h_0} = 1$, then we have that*

$$\sqrt{p} \leq \mathbb{E}_{h_0} \leq c_0 p$$

for some absolute constant c_0.

It is possible to derive a tight version of this proposition with more work.

Proof. By lemma 11, we have that $\mathbb{E}_{h_0} \geq \sqrt{p}\sqrt{\mathrm{Var}_{h_0}} \geq \sqrt{p}$.

Let $\kappa > 0$ be such that $\tilde{h}_0 = \frac{1}{\kappa} h_0\left(\frac{r}{\kappa}\right)$ has second moment p, that is, $\int \tilde{h}_0(r) r^2 dr = p$. By corollary 10, we have that $\mathrm{Var}_{\tilde{h}_0} \geq c_0 p^{-1}$. It is also clear that $\mathbb{E}_{\tilde{h}_0} \leq \sqrt{p}$.

Since $\mathrm{Var}_{h_0} = 1$, it must be that $\kappa \geq \sqrt{c_0 p^{-1}}$ and therefore,

$$\mathbb{E}_{h_0} = \frac{1}{\kappa}\mathbb{E}_{\tilde{h}_0} \leq \sqrt{\frac{p}{c_0}}\sqrt{p} = \frac{p}{\sqrt{c_0}}$$

□

Corollary 16. *Suppose $h_0 \in \mathcal{H}$ and has variance 1. Let $Z_1, ..., Z_n \sim h_0$, then we have that, with probability at least $1 - \frac{1}{n}$,*

$$\bar{Z} \leq c_0 p$$

for an absolute constant c_0.

This corollary is an easy consequence of Proposition 15 and Chebyshev inequality.

6.2 Differential Entropy of h_0

Proposition 17. *Suppose $Z_1, ..., Z_n \sim h_0$ where $h_0(r) = \exp^{\phi_0(r)} r^{p-1} c_p$ has variance 1 and $\phi_0 \in \Phi$.*

Then, we have that, with probability at least $1 - 2e^{-(1/16)\sqrt{n}}$:

$$\int \log h_0(r) dQ_n \geq -2 - \log 2$$

Proof. By [Bobkov and Madiman, 2011, Theorem 1.1], we have that, with probability at least $1 - 2e^{-1/16\sqrt{n}}$,

$$\left| \int h_0(r) \log h_0(r) dr - \frac{1}{n}\sum_{i=1}^{n} \log h_0(Z_i) \right| \leq 1$$

By lemma 18, we have that $\int h_0(r) \log h_0(r) dr \geq -1 - \log 2$. Therefore, we have that, with probability at least $1 - 2e^{-1/16\sqrt{n}}$:

$$\begin{aligned} \int \log h_0(r) dQ_n &\geq \int h_0(r) \log h_0(r) dr - 1 \\ &\geq -2 - \log 2 \end{aligned}$$

□

Lemma 18. *For any univariate log-concave density $f(x)$ with variance 1, we have that*

$$-1 - \log 2 \leq \int f(x) \log f(x) dx \leq 8 \log 2$$

Proof. Since entropy is variant with respect to location shift, let us suppose without loss of generality that $f(x)$ has mean 0.

By [Lovász and Vempala, 2007, Theorem 5.14], $f(x) \leq 2^8$.

Therefore,

$$\int f(x) \log f(x) dx \leq 8 \log 2$$

Let $g(x) = \frac{1}{2} e^{-|x|}$, then $g(x)$ is a density and

$$\begin{aligned}
\int f(x) \log f(x) dx &\geq \int f(x) \log g(x) dx \\
&= \int f(x)(-|x|) - \log 2 dx \\
&= -\int f(x)|x| dx - \log 2 \\
&\geq -\sqrt{\int f(x) x^2 dx} - \log 2 \\
&= -1 - \log 2
\end{aligned}$$

□

6.3 Moment Preservation of MLE

Let $Z_1, ..., Z_n$ be samples with empirical distribution Q_n and assume the following conditions hold on the Z_i's.

A1 $|\bar{Z} - \mathbb{E}_{h_0}| \leq 1$

A2 There exists two absolute constants δ_c, ϵ_c such that for all interval A of length at most δ_c, we have that $Q_n(A) \leq 1 - \epsilon_c$

A3 There is an absolute constant c_{ent} such that, for some $\phi \in \Phi$ satisfying $\int e^{\phi} r^{p-1} c_p = 1$, we have

$$\int \phi(r) + (p-1) \log r + \log c_p dQ_n \geq c_{ent}$$

Proposition 19. *Suppose conditions A1-3 hold true. Let $\hat{h}_n = e^{\hat{\phi}(r)} r^{p-1} c_p$ where $\hat{\phi}$ is defined in equation 9. Then, we have that,*

$$|\mathbb{E}_{\hat{h}_n} - \mathbb{E}_{h_0}| \leq c_\mu$$

for some absolute constant c_μ and that

$$\frac{1}{c_\sigma} \leq Var_{\hat{h}_n} \leq c_\sigma$$

for absolute constants $c_\sigma > 1$.

Proof. By proposition 15, we have that $\mathbb{E}_{h_0} \leq c_0 p$. Therefore, by condition A1, we also have that $\bar{Z} \leq \mathbb{E}_{h_0} + 1 \leq (c_0 + 1)p$.

Combine this bound on $\bar{Z}$ with proposition 20 and we have that $|\mathbb{E}_{\hat{h}_n} - \bar{Z}| \leq (c_0 + 1)$.

Through condition A1 again, we have that $|\mathbb{E}_{\hat{h}_n} - \mathbb{E}_{h_0}| \leq c_0 + 2$ as desired.

The variance bound is an easy consequences of Proposition 21. $\square$

Proposition 20. *Let Q_n be an empirical distribution with samples $Z_1, ..., Z_n$. Let*

$$\phi^* = \underset{\phi \in \Phi}{\operatorname{argmax}} \int \phi dQ_n(r) - \int_0^\infty r^{p-1} e^{\phi(r)} c_p dr.$$

Then, $h^*(r) = r^{p-1} e^{\phi^*(r)} c_p$ is a density and we have that

$$\bar{Z}\left(1 - \frac{1}{p+1}\right) \leq \mathbb{E}[Z_{\phi^*}] \leq \bar{Z}$$

Proof. For a real number x, we define $x_+ = \max(x, 0)$ and $x_- = \min(x, 0)$.

Let $r_0 = \sup_r \phi^*(r) = \phi^*(0)$. We know that $r_0 = Z_i$ for some i because Q_n is an empirical distribution. We also know that the right derivative of ϕ^* at r_0 is strictly less than zero. Therefore,

$$\mathbb{E}[(Z_{\phi^*} - r_0)_+] = \frac{1}{n} \sum_{i=1}^n (Z_i - r_0)_+$$

$$\begin{aligned}
\mathbb{E}Z_{\phi^*} &= \mathbb{E}[(Z_{\phi^*} - r_0)_+] + \mathbb{E}[(Z_{\phi^*} - r_0)_-] + r_0 \\
&= \frac{1}{n} \sum_{i=1}^n (Z_i - r_0)_+ + \mathbb{E}[(Z_{\phi^*} - r_0)_-] + r_0 \\
&= \frac{1}{n} \sum_{i=1}^n (Z_i - r_0) - \frac{1}{n} \sum_{i=1}^n (Z_i - r_0)_- + \mathbb{E}[(Z_{\phi^*} - r_0)_-] + r_0 \\
&= \frac{1}{n} \sum_{i=1}^n Z_i + \frac{1}{n} \sum_{i=1}^n (r_0 - Z_i)_+ - \mathbb{E}[(r_0 - Z_{\phi^*})_+]
\end{aligned}$$

Now, because $\phi^*(r) = \phi^*(0)$ for all $r \leq r_0$, we have that

$$\begin{aligned}\mathbb{E}[(r_0 - Z_{\phi^*})_+] &= \int_0^{r_0} (r_0 - r) r^{p-1} e^{\phi^*(0)} c_p dr \\ &= \frac{r_0^{p+1}}{p} e^{\phi^*(0)} c_p - \frac{r_0^{p+1}}{p+1} e^{\phi^*(0)} c_p \\ &= e^{\phi^*(0)} c_p r_0^{p+1} \left(\frac{1}{p} - \frac{1}{p+1} \right)\end{aligned}$$

Since $\int_0^{r_0} r^{p-1} e^{\phi^*(0)} c_p dr = \frac{r_0^p}{p} e^{\phi^*(0)} c_p \leq 1$, we have that

$$\mathbb{E}[(r_0 - Z_{\phi^*})_+] \leq \frac{r_0}{p+1}$$

Therefore,

$$\mathbb{E} Z_{\phi^*} \geq \bar{Z} + \frac{1}{n} \sum_{i=1}^n (r_0 - Z_i)_+ - \frac{r_0}{p+1}$$

We will finish the proof by showing that $\frac{r_0}{p+1} - \frac{1}{n} \sum_{i=1}^n (r_0 - Z_i)_+ \leq \frac{\bar{Z}}{p+1}$.

To see this, note that $\frac{1}{n} \sum_{i=1}^n (r_0 - Z_i)_+ = r_0 - \frac{1}{n} \sum_{i=1}^n \tilde{Z}_i$ where $\tilde{Z}_i = Z_i$ if $Z_i \leq r_0$ and $\tilde{Z}_i = r_0$ if $Z_i > r_0$. It is clear that $r_0 - \frac{1}{n} \sum_{i=1}^n \tilde{Z}_i \geq 0$ and that $\frac{1}{n} \sum_{i=1}^n \tilde{Z}_i \leq \bar{Z}$.

$$\begin{aligned}\frac{r_0}{p+1} - \frac{1}{n} \sum_{i=1}^n (r_0 - Z_i)_+ &= \frac{r_0 - \frac{1}{n} \sum_{i=1}^n \tilde{Z}_i}{p+1} + \frac{\frac{1}{n} \sum_{i=1}^n \tilde{Z}_i}{p+1} - (r_0 - \frac{1}{n} \sum_{i=1}^n \tilde{Z}_i) \\ &\leq (r_0 - \frac{1}{n} \sum_{i=1}^n \tilde{Z}_i)(\frac{1}{p+1} - 1) + \frac{\bar{Z}}{p+1} \\ &\leq \frac{\bar{Z}}{p+1}\end{aligned}$$

□

Proposition 21. *Suppose A2 and A3 hold, then there exists absolute constants $c_\sigma, C_\sigma > 0$ such that*

$$c_\sigma \leq Var_{\hat{h}_n} \leq C_\sigma$$

Furthermore, we have that

$$\sup_r \log \hat{h}_n(r) \leq \max \left\{ -2 \log \left(\frac{\epsilon_c \delta_c}{10} \right), -\frac{c_{ent}}{\epsilon_c} \right\}$$

where $\epsilon_c, \delta_c, c_{ent}$ are defined in condition A2, A3.

Proof. Let $\hat{h}(r) = e^{\hat{\phi}(r)} r^{p-1} c_p$. We prove the proposition by proving that there exists absolute constants M_h, m_h such that $\hat{h}(r) \leq M_h$ for all $r \in [0, \infty]$ and that $\hat{h}(r) \geq m_h$ for some r. The proposition follows then from lemma 22.

By A3 and the fact that $\hat{\phi}$ is the MLE, we have that $\int \Big(\hat{\phi}(r) + (p-1)\log r + \log c_p\Big) dQ_n \geq c_{ent}$. On the other hand, we have that

$$\int \Big(\hat{\phi}(r) + (p-1)\log r + \log c_p\Big) dQ_n < \int \|\hat{h}\|_\infty dQ_n = \|\hat{h}\|_\infty$$

Thus, $\hat{h}(r) \geq c_{ent}$ for some r and we can set $m_h = c_{ent}$.

Now, let $M = \sup_r \log \hat{h}(r)$ and let $D_t = \{t : \log \hat{h}(r) \geq t\}$.

By [Dümbgen et al., 2011, Lemma 4.1], we have that $\lambda(D_{-\frac{M}{\epsilon_c}}) \leq 5(1+\frac{1}{\epsilon_c}) M e^{-M}$ where $\lambda(\cdot)$ denotes the Lebesgue measure.

Suppose now for contradiction that $M > (-2\log\big(\frac{\epsilon_c \delta_c}{10}\big)) \vee -\frac{c_{ent}}{\epsilon_c}$.

Then, we have that

$$\begin{aligned}
\lambda(D_{-\frac{M}{\epsilon_c}}) &< 5(1+\frac{1}{\epsilon_c}) e^{-M/2} = 5(1+\frac{1}{\epsilon_c}) \frac{\epsilon_c \delta_c}{10} \\
&\leq (1+\epsilon_c)\frac{1}{2}\delta_c \leq \delta_c
\end{aligned}$$

Therefore, by A2, we have that $Q_n(D_{-\frac{M}{\epsilon_c}}) \leq 1 - \epsilon_c$.

$$\begin{aligned}
\int \log \hat{h}(r) dQ_n &\leq -\frac{M}{\epsilon_c}(1 - Q_n(D_{-\frac{M}{\epsilon_c}})) + M Q_n(D_{-\frac{M}{\epsilon_c}}) \\
&\leq -\frac{M}{\epsilon_c} + M(1+\frac{1}{\epsilon_c}) Q_n(D_{-\frac{M}{\epsilon_c}}) \\
&\leq -\frac{M}{\epsilon_c} + M(1+\frac{1}{\epsilon_c})(1-\epsilon_c) \\
&\leq -\epsilon_c M < c_{ent}
\end{aligned}$$

Therefore, we have that $M \leq (-2\log\big(\frac{\epsilon_c \delta_c}{10}\big)) \vee -\frac{c_{ent}}{\epsilon_c}$. We can then set $M_h = \exp\Big((-2\log\big(\frac{\epsilon_c \delta_c}{10}\big)) \vee -\frac{c_{ent}}{\epsilon_c}\Big)$.

□

The following lemma is useful; it characterizes the variance of a log-concave density in terms of the density value.

Lemma 22. *Let h be a univariate log-concave density. Let C, C' be two universal constants.*

1. *If $h(r) \leq M$ for all r, then $\sigma_h \geq \frac{C}{M}$.*

2. If $h(r) \geq M'$ for some r, then $\sigma_h \leq \frac{C'}{M'}$

Proof. Let h be an arbitrary log-concave density in one variable. Suppose first that $h(r) \leq M$ for all r.

Define $f(r) = \sigma_h h(\sigma_h r)$, then f is a log-concave density with unit variance. Hence,

$$f(0) = \sigma_h h(0) \leq \sigma_h M$$

which gives us $\sigma_h geq \frac{f(0)}{M}$.

Now suppose that $h(r) \geq M'$ for some r'. Then

$$f(\frac{r'}{\sigma_h}) = \sigma_h h(r') \geq \sigma_h M'$$

which gives us $\sigma_h \leq \frac{\|f\|_\infty}{M'}$. □

6.4 Proof of Rate

Proposition 23.

$$\mathbb{E} d_H(h_0, \hat{h}_n) \leq \frac{C}{n^{4/5}}$$

Proof. Since both d_H and the SSLC-MLE are scale-invariant, we can without loss of generality assume that $\mathrm{Var}_{h_0} = 1$.

We first verify that conditions A1, A2, A3 hold with high probability. Condition A1 holds with probability at least $1 - \frac{1}{n}$ by Chebyshev inequality.

Now we move on to the second condition. Note that because h_0 has unit variance, $\|h_0\|_\infty \leq 2^8$ ([Lovász and Vempala, 2007, Theorem 5.14]). Let H_0 denote the CDF and h_0 and let $(a, b]$ be an interval of length at most $\frac{1}{2^{10}}$,

$$\begin{aligned} Q_n(b) - Q_n(a) &= Q_n(b) - H_0(b) + H_0(b) - H_0(a) + (H_0(a) - Q_n(a)) \\ &\leq 2\|Q_n - H_0\|_\infty + \|h_0\|_\infty (b - a) \\ &\leq 2\|Q_n - H_0\|_\infty + \frac{1}{4} \end{aligned}$$

By DKW inequality, $\|Q_n - H_0\| \leq \sqrt{\frac{1}{2n} \log 2n}$ with probability at least $1 - \frac{1}{n}$. Therefore, for any $n \geq 8$, $\|Q_n - H_0\| \leq \frac{1}{4}$ with probability at least $1 - \frac{1}{n}$. This proves condition A2 with $\epsilon_c = \frac{1}{4}$ and $\delta_c = \frac{1}{2^{10}}$.

Condition A3 holds with probability at least $1 - 2e^{-\frac{1}{16}\sqrt{n}}$ by proposition 17. It is clear that, for large enough n, that this probability is lower bounded by $1 - \frac{1}{n}$.

Since A1-A3 hold simultaneously with probability at least $1 - \frac{3}{n}$, we have that, with at least that probability, $\hat{h}_n \in \mathcal{H}(h_0, c_\mu, c_\sigma)$ with some absolute constant c_μ, c_σ by proposition 19.

By lemma 25, we have that

$$\int_0^\delta H_{[]}^{1/2}(\epsilon, \mathcal{H}(h_0, c_\mu, c_\sigma), d_H) d\epsilon \leq C\delta^{3/4}$$

for some universal constant C.

Define $\Psi(\delta) = C\delta^{3/4}$ and we have that $\frac{\Psi(\delta)}{\delta^2}$ is non-increasing.

By [Kim et al., 2016, Theorem 10], we then have that

$$P(d_X^2(\hat{h}_n, h_0) > \delta^2) \leq \exp\left(-\frac{n\delta^2}{C^2}\right)$$

for all $\delta \geq \delta_* \equiv C' n^{-2/5}$ for some universal constant C'.

Therefore,

$$\begin{aligned}
\mathbb{E} d_H^2(\hat{h}_n, h_0) \leq & \mathbb{E} d_X^2(\hat{h}_n, h_0) \\
& \leq \int_0^{16 \log n} P\Big(d_X^2(\hat{h}_n, h_0) \geq t \cap \hat{h}_n \in \mathcal{H}(h_0, c_\mu, c_\sigma)\Big) dt + \\
& \quad 16 \log n P(\hat{h}_n \notin \mathcal{H}(h_0, c_\mu, c_\sigma)) + \int_{16 \log n}^\infty P\Big(d_X^2(\hat{h}_n, h_0) \geq t\Big) dt \\
& \leq \delta_*^2 + \int_{\delta_*^2}^{16 \log n} \exp\left(-\frac{nt^2}{C^2}\right) dt + \frac{24 \log n}{n} + \int_{16 \log n}^\infty P\Big(\max_{i=1,\dots,n} \log \frac{\hat{h}_n(X_i)}{h_0(X_i)} \geq t\Big) dt
\end{aligned}$$

By lemma 26 and 27, we have that

$$\begin{aligned}
\int_{16 \log n}^\infty P\Big(\max_{i=1,\dots,n} \log \frac{\hat{h}_n(X_i)}{h_0(X_i)} \geq t\Big) dt &= \int_{16}^\infty P\Big(\max_{i=1,\dots,n} \log \frac{\hat{h}_n(X_i)}{h_0(X_i)} \geq s \log n\Big) ds \\
&\leq \int_{16}^\infty C n^{-s} + 2n^{-s\sqrt{n}/48} + n^{-\frac{s}{4}n + n} ds \\
&\leq \frac{C}{n}
\end{aligned}$$

Therefore, we have that $\mathbb{E} d_H^2(\hat{h}_n, h_0) \leq \frac{C}{n^{4/5}}$. This concludes the proof. $\square$

$$\mathcal{H}(h_0, c_\mu, c_\sigma) = \left\{ h, \ : \ \int h(r) dr = 1, \ h \text{ log-concave}, \ |\mathbb{E}_h - \mathbb{E}_{h_0}| \leq c_\mu, \ \frac{1}{c_\sigma} \leq \mathrm{Var}_h \leq c_\sigma \right\} \tag{11}$$

Lemma 24.

$$\sup_{h \in \mathcal{H}(h_0, c_\mu, c_\sigma)} h(r) \leq \begin{cases} C_0 c_\sigma & \text{if } r \in [\mu_0 - c_\mu, \mu_0 + c_\mu] \\ C_0 c_\sigma \exp(-\frac{a_0}{c_\sigma}|r - (\mu_0 - c_\mu)|) & \text{if } r < \mu_0 - c_\mu \\ C_0 c_\sigma \exp(-\frac{a_0}{c_\sigma}|r - (\mu_0 + c_\mu)|) & \text{if } r > \mu_0 - c_\mu \end{cases}$$

where C_0, a_0 are absolute constants.

Proof. Let $\mathcal{H}_{\mu,\sigma}$ be the set of univariate log-concave densities with mean μ and variance σ^2. By Kim and Samworth [2016, Theorem 2], there exists a_0, C_0 such that $\sup_{h \in \mathcal{H}_{\mu,\sigma}} h(x) \leq \frac{C_0}{\sigma} \exp\left(-a_0 \left|\frac{x-\mu}{\sigma}\right|\right)$.

Therefore,

$$\begin{aligned} \sup_{h \in \mathcal{H}(h_0, c_\mu, c_\sigma)} h(r) &\leq \sup_{\mu \in [\mu_0 - c_\mu,\, \mu_0 + c_\mu]} \sup_{\sigma \in [\frac{1}{c_\sigma}, c_\sigma]} \sup_{h \in \mathcal{H}_{\mu,\sigma}} h(r) \\ &\leq \sup_{\mu \in [\mu_0 - c_\mu,\, \mu_0 + c_\mu]} \sup_{\sigma \in [\frac{1}{c_\sigma}, c_\sigma]} \frac{C_0}{\sigma} \exp\left(-a_0 \left|\frac{x-\mu}{\sigma}\right|\right) \end{aligned}$$

The result follows readily. □

We can derive the bracketing entropy from the envelope bound.

Lemma 25.

$$H_{[]}(\epsilon, \mathcal{H}(h_0, c_\mu, c_\sigma), d_H) \leq \frac{C}{\epsilon^{1/2}}$$

where C depends only on c_μ, c_σ.

Proof. Let $\epsilon > 0$ be fixed. Suppose $\epsilon \leq C_0 c_\sigma$.

We construct the bracketing by dividing the region $\mathbb{R}^+$ into segments.

Let $J = \lfloor \mu_0 - c_\mu \rfloor$.

$$\begin{aligned} S_0 &= [\mu_0 - c_\mu,\, \mu_0 + c_\mu] \\ S_j^L &= [(\mu_0 - c_\mu) - j, (\mu_0 - c_\mu) - (j-1)] \\ S_j^R &= [(\mu_0 + c_\mu) + (j-1), (\mu_0 + c_\mu) + j] \end{aligned}$$

where S_j^R is defined for $j = 1, ..., \infty$ and S_j^L is defined for $j = 1, ..., J+1$ where $S_{J+1}^L = [0, (\mu_0 - c_\mu) - J]$.

Let $\mathcal{F}([a,b], -\infty, B)$ be the set of log-concave functions f such that f is supported on $[a,b]$ and that $\log f(x) \leq B$. On $S_j^R \cup S_j^L$, $h(r) \leq e^{B_j} \equiv C_0 c_\sigma e^{-\frac{a_0}{c_\sigma}(j-1)}$ for any $h \in \mathcal{H}(h_0, c_\mu, c_\sigma)$. Therefore, any $h \in \mathcal{H}(h_0, c_\mu, c_\sigma)$, when restricted to S_j^R, lies in $\mathcal{F}(S_j^R, -\infty, B_j)$; likewise for S_j^L.

Then, we have that, for any $\epsilon_0, ..., \epsilon_j, ...$ that satisfy $\epsilon_0^2 + \sum_{j=0}^{\infty} 2\epsilon_j^2 = \epsilon^2$,

$$\begin{aligned}H_{[]}(\epsilon, \mathcal{H}(h_0, c_\mu, c_\sigma), d_H) \leq & H_{[]}(\epsilon_0, \mathcal{F}(S_0, -\infty, B_0), d_H, S_0) + \\ & \sum_{j=1}^{\infty} H_{[]}(\epsilon_j, \mathcal{F}(S_j^R, -\infty, B_j), d_H, S_j^R) + \\ & \sum_{j=1}^{J+1} H_{[]}(\epsilon_j, \mathcal{F}(S_j^L, -\infty, B_j), d_H, S_j^L)\end{aligned}$$

Define $\epsilon_j^2 = c' \exp\left(-\frac{a_0}{c_\sigma}\frac{1}{2}(j-1)\right)\epsilon^2$ and $\epsilon_0^2 = c'\epsilon^2$ where c' ensures that $c' + 2c' \sum_{j=1}^{\infty} \exp\left(-\frac{a_0}{c_\sigma}\frac{1}{2}(j-1)\right)$ sums to 1.

By Corollary 29, for $j = 1, ..., \infty$,

$$\begin{aligned}H_{[]}(\epsilon_j, \mathcal{F}(S_j^R, -\infty, B_j), d_H, S_j^R) &\leq C(2C_0 c_\sigma)^{1/4} e^{-\frac{a_0}{c_\sigma 4}(j-1)} \frac{1}{\epsilon_j^{1/2}} \\ &\leq C(2(1/c')C_0 c_\sigma)^{1/4} e^{-\frac{a_0}{c_\sigma 8}(j-1)} \frac{1}{\epsilon^{1/2}}\end{aligned}$$

And likewise for S_j^L.

$$\begin{aligned}H_{[]}(\epsilon_j, \mathcal{F}(S_0, -\infty, B_0), d_H, S_0) &\leq C(2C_0 c_\sigma)^{1/4} \frac{1}{\epsilon_0^{1/2}} \\ &\leq C(2(1/c')C_0 c_\sigma)^{1/4} \frac{1}{\epsilon^{1/2}}\end{aligned}$$

The result directly follows. ☐

The following lemma controls the probability that $\sup_r \log \hat{h}_n(r) \geq t \log n$ for $t \geq 8$, where h_0 has unit variance and $\hat{h}_n$ is the MLE.

Lemma 26. *Let $Z_1, ..., Z_n$ be samples from $h_0 \in \mathcal{H}$ (10). Suppose that h_0 has unit variance and suppose $n \geq n_0$ for some absolute constant n_0 and let $t \geq 8$.*

Then, with probability at least $1 - 2n^{-t\sqrt{n}/48} - n^{-\frac{t}{4}n+n}$, we have that condition A2, A3 holds with $\epsilon_c = 1/2$, $\delta_c = 20n^{-t/2}$ and $c_{ent} \geq -\frac{1}{2}t\log n$.

Furthermore,

$$\sup_r \log \hat{h}_n(r) \leq t \log n$$

Proof. Let Q_n be the empirical distribution of the samples $Z_1, ..., Z_n$.

Let E_{A2}^c be the event that there exists an interval A of length at most $20n^{-t/2}$ such that $Q_n(A) > 1/2$. E_{A2} is then the event that condition A2 holds with $\delta_c = n^{-t/2}$ and $\epsilon_c = 1/2$.

E_{A2}^c occurs only if at least $n/2 + 1$ sample points fall inside an interval of length $n^{-t/2}$.

This occurs with probability at most

$$
\begin{aligned}
(20Cn^{-t/2})^{n/2}\binom{n}{n/2} &\overset{(a)}{\leq} n^{-\frac{t}{2}(n/2)+n} \\
&\leq n^{-\frac{t}{4}n+n}
\end{aligned}
$$

where $C = \sup\{\|h\|_\infty : h \text{ log-concave density, unit variance}\}$ is an absolute constant. (a) follows by assuming that $n \geq 20C$.

Let E_{A3}^c be event that $\int \log h_0 dQ_n \leq -(t/2)\log n$.

$$
\begin{aligned}
&\int \log h_0 dQ_n \leq -(t/2)\log n \\
(\Rightarrow)\ &\int h_0(r)\log h_0(r)dr - \int \log h_0 dQ_n \geq \int h_0(r)\log h_0(r)dr + (t/2)\log n \\
(\overset{(a)}{\Rightarrow})\ &\left|\int h_0(r)\log h_0(r)dr - \int \log h_0 dQ_n\right| \geq (t/2)\log n - (1+\log 2) \overset{(b)}{(}\ t/3)\log n
\end{aligned}
$$

where (a) follows from lemma 18 and (b) follows from the assumption that $n > 2$.

By [Bobkov and Madiman, 2011, Theorem 1.1], we have that, with probability at least $1 - 2e^{-1/16s\sqrt{n}}$,

$$
\left|\int h_0(r)\log h_0(r)dr - \frac{1}{n}\sum_{i=1}^{n}\log h_0(Z_i)\right| \leq s
$$

By letting $s = (t/3)\log n$, we have that E_{A3} holds with probability at least $1 - 2e^{-1/48t\sqrt{n}\log n} = 1 - 2n^{-t\sqrt{n}/48}$.

By union bound, we then have that E_{A2} and E_{A3} occur simultaneously with probability at least

$$
1 - 2n^{-t\sqrt{n}/48} - n^{-\frac{t}{4}n+n}
$$

By proposition 21, we have that

$$
\begin{aligned}
\sup_r \log \hat{h}_n(r) &\leq \max\left\{-2\log\left(\frac{\epsilon_c\delta_c}{10}\right), -\frac{c_{ent}}{\epsilon_c}\right\} \\
&\leq \max\left\{-2\log\left(\frac{\delta_c}{20}\right), -2c_{ent}\right\} \\
&\leq t\log n
\end{aligned}
$$

□

The next lemma is from Kim et al. [2016] and it controls the probability that $\min_i \log h_0(Z_i) \leq -t\log n$ for $t > 8$. It is a counterpart to lemma 26.

Lemma 27. *Let h_0 be a log-concave density with unit variance and suppose $Z_1, ..., Z_n \sim h_0$. Then, for any $t \geq 4$, with probability at least $1 - \frac{C}{n^t}$,*

$$\sup_{r \in [Z_{(1)}, Z_{(n)}]} \log h_0(r) \geq t \log n$$

Proof. Proof in Kim et al. [2016]. [TODO:give more detail].

□

6.5 Bracketing Entropy

We start with a proposition from Kim et al. [2016]. Let $\mathcal{F}([a,b], -\infty, B)$ be the set of log-concave functions f such that f is supported on $[a,b]$ and that $\log f(x) \leq B$.

Proposition 28. *(Kim et al. [2016, Proposition 14])*
There exists a universal constant $C > 0$ such that

$$H_{[]}(\epsilon, \mathcal{F}([a,b], -\infty, B), d_H, [a,b]) \leq C(1 + B^{1/2}) \frac{e^{B/4}(b-a)^{1/4}}{\epsilon^{1/2}}$$

We can slightly improve the result through a scaling argument.

Corollary 29. *There exists a universal constant $C > 0$ such that*

$$H_{[]}(\epsilon, \mathcal{F}([a,b], -\infty, B), d_H, [a,b]) \leq C \frac{e^{B/4}(b-a)^{1/4}}{\epsilon^{1/2}}$$

Proof. Let $\sigma > 0$. For $f \in \mathcal{F}([a,b], -\infty, B)$, define $f_\sigma(x) = \frac{1}{\sigma} f(\frac{x}{\sigma})$ and define $\mathcal{F}_\sigma([a,b], -\infty, B) = \{f_\sigma \,:\, f \in \mathcal{F}([a,b], -\infty, B)\}$.

Since the Hellinger distance d_H is affine invariant, we have that

$$H_{[]}(\epsilon, \mathcal{F}([a,b], -\infty, B), d_H, [a,b]) = H_{[]}(\epsilon, \mathcal{F}_\sigma([a,b], -\infty, B), d_H, [a,b])$$

However, we also know that $\mathcal{F}_\sigma([a,b], -\infty, B) \subset \mathcal{F}([\sigma a, \sigma b], -\infty, B + \log \sigma)$. Thus,

$$\begin{aligned} H_{[]}(\epsilon, \mathcal{F}([a,b], -\infty, B), d_H, [a,b]) &\leq H_{[]}(\epsilon, \mathcal{F}([\sigma a, \sigma b], -\infty, B + \log \sigma), d_H, [a,b]) \\ &\leq C(1 + (B + \log \sigma)^{1/2}) \frac{e^{B/4}(b-a)^{1/4}}{\epsilon^{1/2}} \end{aligned}$$

Since this holds true for all $\sigma > 0$, the corollary follows.

□

7 Adaptation

7.1 Adaptive Rate of Convergence

Proposition 30. *Let S be some subset of $\mathbb{R}^+$. Suppose that $Z_1, ..., Z_n \sim h_0$ for some $h_0 = e^{\phi_0(r)} r^{p-1}$ where ϕ_0 is concave and decreasing on S.*

Let $\hat{h}_n$ be the SSLC-MLE estimated from $Z_1, ..., Z_n$.

[TODO:finish]

7.2 Local Bracketing

Define

$$\mathcal{H}^1 = \left\{ h \,:\, h(r) = e^{\phi(r)} r^{p-1},\, \int h(r) dr = 1,\, \phi \text{ affine} \right\}$$

Define also the following class of spherically symmetric log-concave densities.

$$\begin{aligned}
&\mathcal{H}(h_0, \delta, c_\mu, c_\sigma) && (12)\\
&= \left\{ h \,:\, h(r) = e^{\phi(r)} r^{p-1},\, \int \left(\sqrt{h_0} - \sqrt{h} \right)^2 dr \le \delta^2, \right. && (13)\\
&\left. \frac{1}{c_\sigma} \le \frac{\sigma_h}{\sigma_0} \le c_\sigma,\, |\mu_h - \mu_0| \le c_\mu \right\} && (14)
\end{aligned}$$

Proposition 31. *Let $h_0 = e^{\phi_0(r)} r^{p-1}$ where ϕ_0 is concave and decreasing and where the support of h_0 is any subset of $\mathbb{R}^+$. Let c_μ, c_σ be absolute constants. Let $\nu = \inf\{d_H(h_0, h_1) : h_1 \in \mathcal{H}^1\}$. Then, there exists an absolute constant κ such that, for all $\delta + \nu < \kappa$,*

we have that,

$$H_{[]}(\epsilon, \mathcal{H}(h_0, \delta, c_\mu, c_\sigma), d_H) \le C \left(\frac{\delta + \nu}{\epsilon} \right)^{1/2} \log^2 \frac{1}{\delta}$$

Proof. Suppose $h \in \mathcal{H}(h_0, \delta, c_\mu, c_\sigma)$. By triangle inequality, $d_H(h_1, h) \le d_H(h_0, h) + d_H(h_0, h_1) \le \delta + \nu$.

Furthermore, by lemma 32, we have that there exists constants c'_μ, c'_σ such that $\frac{1}{c'_\sigma} \le \frac{\sigma_{h_1}}{\sigma_{h_0}} \le c'_\sigma$ and $|\mu_{h_1} - \mu_{h_0}| \le c'_\mu$.

Therefore, $\frac{1}{c_\sigma c'_\sigma} \le \frac{\sigma_h}{\sigma_{h_1}} \le c_\sigma c'_\sigma$ and $|\mu_h - \mu_{h_1}| \le c'_\mu + c_\mu$.

We have then shown that

$$\mathcal{H}(h_0, \delta, c_\mu, c_\sigma) \subset \mathcal{H}(h_1, \delta + \nu, c_\mu + c'_\mu, c_\sigma c'_\sigma)$$

Since $h_1(r) = r^{p-1} e^{\phi_1(r)}$ where ϕ_1 is affine, $\log h - \log h_1$ is concave for any $h \in \mathcal{H}(h_1, \delta + \nu, c_\mu + c'_\mu, c_\sigma c'_\sigma)$.

Therefore,

$$\mathcal{H}(h_1, \delta + \nu, c_\mu + c'_\mu, c_\sigma c'_\sigma) \subset \mathcal{F}(h_1, \delta + \nu, c_\mu + c'_\mu, c_\sigma c'_\sigma)$$

where the RHS is defined in equation 17. The result then follow from corollary 34.

□

Lemma 32. *Let p, q be two log-concave densities. There are absolute constants C, c'_μ, c'_σ such that for all $\delta < C$, if $\int(\sqrt{p} - \sqrt{q})^2 dr \leq \delta^2$, then we have that*

$$\frac{1}{c'_\sigma} \leq \frac{\sigma_p}{\sigma_q} \leq c'_\sigma \qquad |\mu_p - \mu_q| \leq c'_\mu$$

Proof. [TODO:finish]

□

7.3 A General Result on Local Brackets

We prove a general result on the local bracketing entropy of log-concave densities.

Let $h_0(r) = e^{\phi_0(r)}$ be a log-concave density with zero mean and unit variance. We make no assumption on the support of h_0. We define

$$\tilde{\mathcal{F}}(h_0, \delta, c_1, a_1) = \left\{ e^\phi \,:\, \phi - \phi_0 \text{ concave}, \int e^{\phi_0(r)} \left(e^{\frac{\phi(r) - \phi_0(r)}{2}} - 1 \right)^2 dr \leq \delta^2,\, e^{\phi(r)} \vee e^{\phi_0(r)} \leq c_1 e^{-a_1|r|} \right\} \tag{15}$$

$\tilde{\mathcal{F}}(h_0, \delta, c_1, a_1)$ is the class of densities δ-close to h_0 in the Hellinger sense and also bounded by an envelope of the form $c_1 e^{-a_1|r|}$.

Proposition 33. *Suppose that $\delta^2 \leq 2^{-18}$. Let $c_1 > 0, a_1 > 0$ be absolute constants. The bracketing entropy of $\tilde{\mathcal{F}}(h_0, \delta, c_1, a_1)$ can be bounded as*

$$H_{[]}(\epsilon, \tilde{\mathcal{F}}(h_0, \delta, c_1, a_1), d_H) \lesssim \frac{\delta^{1/2}}{\epsilon^{1/2}} \log^2 \frac{1}{\delta}$$

When it is clear that c_1, a_1 are absolute constants, we write $\tilde{\mathcal{F}}(h_0, \delta)$ in place of $\tilde{\mathcal{F}}(h_0, \delta, c_1, a_1)$.

We can get an immediate corollary from this proposition. Let $h_0 = e^{\phi_0}$ be an arbitrary log-concave density.

Let us define the moment constrained function class

$$\mathcal{F}(h_0, \delta, c_\mu, c_\sigma) \tag{16}$$

$$= \left\{ e^\phi \,:\, \phi - \phi_0 \text{ concave}, \int \left(\sqrt{h_0} - \sqrt{h} \right)^2 dr \leq \delta^2,\, \frac{1}{c_\sigma} \leq \frac{\sigma_h}{\sigma_0} \leq c_\sigma, |\mu_h - \mu_0| \leq c_\mu \right\} \tag{17}$$

where μ_0, σ_0 denote the mean and standard deviation of h_0 respectively.

Corollary 34. *Let $\delta^2 \leq 2^{-18}$ and let c_μ, c_σ be absolute constants. We have then that*

$$H_{[]}(\epsilon, \mathcal{F}(h_0, \delta, c_\mu, c_\sigma), d_H) \lesssim \frac{\delta^{1/2}}{\epsilon^{1/2}} \log^2 \frac{1}{\delta}$$

Proof. Since d_H is location and scale invariant, we can without loss of generality assume that $\mu_0 = 0$ and $\sigma_0 = 1$.

If $h \in \mathcal{F}(h_0, \delta, c_\mu, c_\sigma)$, then $|\mu_h| \leq c_\mu$ and $\frac{1}{c_\sigma} \leq \sigma_h \leq c_\sigma$. There exists c_1, a_1, dependent only on c_μ, c_σ, such that, for all $r \in \mathbb{R}$,

$$\sup \left\{ h(r) \,:\, h \in \mathcal{H},\, |\mu_h| \leq c_\mu, \frac{1}{c_\sigma} \leq \sigma_h \leq c_\sigma \right\} \leq c_1 e^{-a_1 |r|}$$

Since c_μ, c_σ are taken to be absolute constants, $\mathcal{F}(h_0, \delta, c_\mu, c_\sigma) \subset \tilde{\mathcal{F}}(h_0, \delta, c_1, a_1)$ for some absolute constant c_1, a_1. An application of proposition 33 immediately yields the corollary. □

Now we turn to the proof of the main proposition.

Proof. (of proposition 33)

In this proof, we let C be a generic absolute constant whose value may vary from instance to instance.

Define $a_L = \inf\{r \,:\, e^{\phi_0(r)} \geq \delta^2\}$ and $a_R = \sup\{r \,:\, e^{\phi_0(r)} \geq \delta^2\}$.

First, we will bracket the region $[\infty, a_L] \cup [a_R, \infty]$. Recall that we have the envelope $e^{\phi_0(r)} \leq c_1 e^{-a_1 |r|}$.

For $k = 1, ..., \infty$, define $S_k = [a_L + k, a_L + (k-1)]$ and $S'_K = [a_R + (k-1), a_R]$. Define the set $\mathcal{K} = \{k \,:\, c_1 e^{-a_1(k - a_L)} \wedge c_1 e^{-a_1(k + a_R)} \geq \delta^4\}$.

Define $\epsilon_k^2 = \frac{\epsilon^2}{8|\mathcal{K}|}$ if $k \in \mathcal{K}$. Note that $\sum_{k \in \mathcal{K}} 2\epsilon_k^2 = \frac{\epsilon^2}{4}$. It is clear that $|\mathcal{K}| \leq C \log \frac{1}{\delta}$.

Otherwise, define $\epsilon_k^2 = C\epsilon^2 (e^{-a_1(k-a_L)/8} \wedge e^{-a_1(k+a_R)/8})$ where C is a constant chosen such that $\sum_{k=1}^{\infty} 2\epsilon_k^2 = \frac{\epsilon^2}{2}$.

From proposition , we have that, for $k \in \mathcal{K}$:

$$H_{[]}(\epsilon_k, \tilde{\mathcal{F}}(f_0, \delta), d_H, S_k) \leq C \frac{\delta^{1/2}}{\epsilon_k^{1/2}} \leq C \frac{\delta^{1/2}}{\epsilon^{1/2}} \log^{1/4} \frac{1}{\delta}$$
$$H_{[]}(\epsilon_k, \tilde{\mathcal{F}}(f_0, \delta), d_H, S'_k) \leq C \frac{\delta^{1/2}}{\epsilon_k^{1/2}} \leq C \frac{\delta^{1/2}}{\epsilon^{1/2}} \log^{1/4} \frac{1}{\delta}$$

For $k \notin \mathcal{K}$:

$$H_{[]}(\epsilon_k, \tilde{\mathcal{F}}(f_0, \delta), d_H, S_k) \leq C\frac{c_1 e^{-a_1(k-a_L)/4}}{\epsilon_k^{1/2}} \leq C\frac{c_1 e^{-a_1(k-a_L)/8}}{\epsilon^{1/2}}$$
$$H_{[]}(\epsilon_k, \tilde{\mathcal{F}}(f_0, \delta), d_H, S'_k) \leq C\frac{c_1 e^{-a_1(k+a_R)/4}}{\epsilon_k^{1/2}} \leq C\frac{c_1 e^{-a_1(k+a_R)/8}}{\epsilon^{1/2}}$$

and therefore,

$$H_{[]}(\frac{\epsilon}{\sqrt{2}}, \tilde{\mathcal{F}}(f_0, \delta), d_H, [-\infty, a_L] \cup [a_R, \infty]) \leq \sum_{k=1}^{\infty} H_{[]}(\epsilon_k, \tilde{\mathcal{F}}(f_0, \delta), d_H, S_k) + H_{[]}(\epsilon_k, \tilde{\mathcal{F}}(f_0, \delta), d_H, S'_k) \tag{18}$$

$$\leq C\frac{\delta^{1/2}}{\epsilon^{1/2}} \log^{5/4} \frac{1}{\delta} \tag{19}$$

Next, we bracket the region $[a_L, a_R]$. Recall from lemma 37 that $a_L \leq -1/9$ and $a_R \geq 1/9$.

We divide the region into segments $[a_L, s_1], [s_1, s_2], ..., [s'_2, s'_1], [s'_1, a_R]$ according to the following algorithm:

1. Select s_1 as the smallest real number such that $(s_1 - a_L) \cdot \sup_{t \in [a_L, s_1]} e^{\phi_0(t)} \geq C\delta^2 \log \frac{1}{\delta}$ where C is some constant specified later.

2. For $j > 1$. Choose s_j as the smallest real number such that either

   [a] $\int_{s_{j-1}}^{s_j} e^{\phi_0(t)} dt = 2 \int_{-\infty}^{s_{j-1}} e^{\phi_0(t)} dt$ or

   [b] $s_j \in [-1/16, 0]$.

   If $s_j \in [-1/16, 0]$, then stop.

On the right side, we do the same:

1. Select s'_1 as the largest real number such that $(a_R - s'_1) \cdot \sup_{t \in [s'_1, a_R]} e^{\phi_0(t)} \geq C\delta^2 \log \frac{1}{\delta}$ where C is some constant specified later.

2. For $j > 1$. Choose s'_j as the largest real number such that either

   [a] $\int_{s'_{j-1}}^{s'_j} e^{\phi_0(t)} dt = 2 \int_{-\infty}^{s'_{j-1}} e^{\phi_0(t)} dt$ or

   [b] $s'_j \in [0, 1/16]$.

   If $s'_j \in [0, 1/16]$, then stop.

We first focus on the left side segmentation. We make the following five claims:

(1) $s_1 < -1/16$.

(2) $\int_{-\infty}^{s_1} e^{\phi_0(t)}dt \geq \delta^2$

(3) For each $j > 1$, $\int_{s_j}^{\infty} e^{\phi_0(t)}dt > 2^{-10}$.

(4) For each $j > 1$ where $s_j < -1/16$, $(s_j - s_{j-1}) \cdot \sup_{t\in[s_{j-1},s_j]} e^{\phi_0(t)} \leq C \log \frac{1}{\delta} \cdot \int_{-\infty}^{s_{j-1}} e^{\phi_0(t)}dt$.

(5) If $s_j \in [-1/16, 0]$, $\int_{-\infty}^{s_{j-1}} e^{\phi_0(t)}dt \geq 2^{-14}$.

We also claim that analogous statements hold for the right side segmentation.

(1) $s'_1 > 1/16$.

(2) $\int_{s'_1}^{\infty} e^{\phi_0(t)}dt \geq \delta^2$.

(3) For each $j > 1$, $\int_{-\infty}^{s'_j} e^{\phi_0(t)}dt > 2^{-10}$.

(4) For each $j > 1$ where $s'_j > 1/16$, $(s'_{j-1} - s'_j) \cdot \sup_{t\in[s'_{j-1},s'_j]} e^{\phi_0(t)} \leq C \log \frac{1}{\delta} \cdot \int_{s'_{j-1}}^{\infty} e^{\phi_0(t)}dt$.

(5) If $s'_j \in [0, 1/16]$, $\int_{s'_{j-1}}^{\infty} e^{\phi_0(t)}dt \geq 2^{-14}$.

Before proving these claims, let us see how these claims yield a bracketing on $[a_L, a_R]$.

Claim (1) and (2) and the fact that $\int_{-\infty}^{s_j} e^{\phi_0(t)}dt$ triples with every iteration show that the algorithm will terminate in at most $C \log \frac{1}{\delta}$ iterations, generating $C \log \frac{1}{\delta}$ segments.

Define $\tilde{\epsilon}$ such that $\tilde{\epsilon}^2 = \frac{\epsilon^2}{4L}$ where L is the total number of segments generated by the algorithm. $L \leq C \log \frac{1}{\delta}$ as previously discussed.

On $[a_L, s_1]$, it is clear that

$$\begin{aligned}
H_{[]}(\tilde{\epsilon}, \tilde{\mathcal{F}}(f_0,\delta), d_H, [a_L, s_1]) &\leq C(s_1 - a_L)^{1/4} \left(\sup_{\phi\in\tilde{\mathcal{F}}(f_0,\delta)} \sup_{t\in[a_L,s_1]} e^{\phi(t)} \right)^{1/4} \frac{1}{\tilde{\epsilon}^{1/2}} \\
&\leq C(s_1 - a_L)^{1/4} \left(\sup_{t\in[a_L,s_1]} e^{\phi_0(t)} \right)^{1/4} \left(\sup_{\phi\in\tilde{\mathcal{F}}(f_0,\delta)} \sup_{t\in[a_L,s_1]} e^{\phi(t)-\phi_0(t)} \right)^{1/4} \frac{1}{\tilde{\epsilon}^{1/2}} \\
&\leq C \frac{\delta^{1/2}}{\tilde{\epsilon}^{1/2}} \log^{1/4} \frac{1}{\delta} \cdot \left(\sup_{\phi\in\tilde{\mathcal{F}}(f_0,\delta)} \sup_{t\in[a_L,s_1]} e^{\phi(t)-\phi_0(t)} \right)^{1/4} \\
&\overset{(a)}{\leq} C \frac{\delta^{1/2}}{\epsilon^{1/2}} \log^{1/2} \frac{1}{\delta}
\end{aligned}$$

where (a) follows because $\sup_{\phi\in\tilde{\mathcal{F}}(f_0,\delta)} \sup_{t\in[a_L,s_1]} e^{\phi(t)-\phi_0(t)} \leq e^{C\delta^2 \log \frac{1}{\delta}} \leq C$ by lemma 36 and also because $\tilde{\epsilon}^2 \geq \frac{\epsilon^2}{C \log \frac{1}{\delta}}$.

Suppose now that $j > 1$. On $[s_{j-1}, s_j]$, we have that, by proposition 40 and lemma 36 and lemma 35 (note that lemma 35 applies because $\int_{-\infty}^{s_{j-1}} e^{\phi_0(t)}dt \geq \delta^2$ by claim (2) and $\int_{s_j}^{\infty} e^{\phi_0(t)}dt \geq 2^{-10} \geq \delta^2$ by claim (3))

$$\begin{aligned}
H_{[]}(\tilde{\epsilon}, \tilde{\mathcal{F}}(f_0,\delta), d_H, [s_{j-1}, s_j]) \leq & C\delta^{1/2}\left(\log\frac{1}{\delta} + \frac{1}{\sqrt{\int_{-\infty}^{s_{j-1}} e^{\phi_0(t)}dt} \wedge \sqrt{\int_{s_j}^{\infty} e^{\phi_0(t)}dt}}\right)^{1/2} \cdot \\
& (s_j - s_{j-1})^{1/4}\left(\sup_{\phi\in\tilde{\mathcal{F}}(f_0,\delta)} \sup_{t\in[a_L,s_1]} e^{\phi(t)}\right)^{1/4} \frac{1}{\tilde{\epsilon}^{1/2}} \\
\leq & C\delta^{1/2}\left(\log\frac{1}{\delta} + \frac{1}{\sqrt{\int_{-\infty}^{s_{j-1}} e^{\phi_0(t)}dt} \wedge \sqrt{\int_{s_j}^{\infty} e^{\phi_0(t)}dt}}\right)^{1/2} \cdot \\
& (s_j - s_{j-1})^{1/4}\left(\sup_{t\in[s_{j-1},s_j]} e^{\phi_0(t)}\right)^{1/4}\left(\sup_{\phi\in\tilde{\mathcal{F}}(f_0,\delta)} \sup_{t\in[s_{j-1},s_j]} e^{\phi(t)-\phi_0(t)}\right)^{1/4} \frac{1}{\tilde{\epsilon}^{1/2}} \\
\stackrel{(a)}{\leq} & C\delta^{1/2}\log^{3/4}\frac{1}{\delta} \cdot \frac{1}{\tilde{\epsilon}^{1/2}} \\
\leq & C\frac{\delta^{1/2}}{\epsilon^{1/2}}\log\frac{1}{\delta}
\end{aligned}$$

(a) is difficult to deduce and requires further explanation. Let us first suppose that $\log\frac{1}{\delta} \geq \frac{1}{\sqrt{\int_{-\infty}^{s_{j-1}} e^{\phi_0(t)}dt} \wedge \sqrt{\int_{s_j}^{\infty} e^{\phi_0(t)}dt}}$. Then, (a) follows from the fact that $|s_j - s_{j-1}| \leq C\log\frac{1}{\delta}$ and $e^{\phi_0(t)} \leq c_1$ for all t.

Now let us suppose that $\log\frac{1}{\delta}$ is smaller. By claim (3), $\int_{s_j}^{\infty} e^{\phi_0(t)}dt > 2^{-10}$. If $\int_{-\infty}^{s_{j-1}} e^{\phi_0(t)}dt$ is larger, then the (a) again follows. So, let us suppose that $\int_{-\infty}^{s_{j-1}} e^{\phi_0(t)}dt$ is smaller than 2^{-10}.

If j is the last iteration, then $\int_{-\infty}^{s_{j-1}} e^{\phi_0(t)}dt \geq 2^{-14}$ by claim (5) and (a) follows from the bound on $(s_j - s_{j-1})\sup_{t\in[s_{j-1},s_j]} e^{\phi_0(t)}$ again.

If j is not the last iteration, then $\frac{(s_j - s_{j-1})\sup_{t\in[s_{j-1},s_j]} e^{\phi_0(t)}}{\int_{-\infty}^{s_{j-1}} e^{\phi_0(t)}dt} \leq C\log\frac{1}{2}$ by claim (4) and then (a) again follows.

Therefore, (a) follows in all cases. Thus, we have a bracketing for $[a_L, s_J]$ where J is the terminating iteration for the algorithm on the left side.

$$\begin{aligned}
H_{[]}(\epsilon, \tilde{\mathcal{F}}(f_0,\delta), d_H, [a_L, s_J]) &\leq \sum_{j=1}^{J} H_{[]}(\tilde{\epsilon}, \tilde{\mathcal{F}}(f_0,\delta), d_H, [s_{j-1}, s_j]) \\
&\leq C\frac{\delta^{1/2}}{\epsilon^{1/2}}\log^2\frac{1}{\delta}
\end{aligned}$$

Using an identical argument, we can get the same bracketing for $[s'_{J'}, a_R]$ where J' is the last iteration of the algorithm on the right side.

Thus, all that is left is to bracket $[s_J, s'_{J'}$. Since $s_J \geq -1/16$ and $s'_{J'} \leq 1/16$, and that $\int_{-\infty}^{s_J} e^{\phi_0(t)}dt \geq 2^{-10}$ and $\int_{s'_{J'}}^{\infty} e^{\phi_0(t)}dt \geq 2^{-10}$ by claim (5). We have that

$$H_{[]}(\epsilon, \tilde{\mathcal{F}}(f_0, \delta), d_H, [s_J, s'_{J'}]) \leq C\frac{\delta^{1/2}}{\epsilon^{1/2}} \log \frac{1}{\delta}$$

Now, we just have to prove claims (1)-(5).

For claim (1), we first observe that $(s - a_L) \sup_{t \in [a_L, s]} e^{\phi_0(t)}$ increases monotonically as s increases. Now, by Lovász and Vempala [2007], we have $e^{\phi_0(t)} \geq 2^{-8}$ for all $t \in [-1/9, 1/9]$ and that $a_L < -1/9$. Therefore, for any $s \geq -1/16$, we have that $(s - a_L) \sup_{t \in [a_L, s]} e^{\phi_0(t)} \geq (1/9 - 1/16)2^{-8} \geq 2^{-13} > \delta^2$. Hence, $s_1 < -1/16$.

On to claim (2). $\int_{-\infty}^{s_1} e^{\phi_0(t)}dt \geq \int_{a_L}^{s_1} e^{\phi_0(t)}dt$. On $[a_L, s_1]$, ϕ_0 is between $-2\log\frac{1}{\delta}$ and $\log c_1$. Therefore,

$\int_{a_L}^{s_1} e^{\phi_0(t)}dt$ is, by lemma 38, at least $|s_1 - a_L| \sup_{t \in [a_L, s_1]} e^{\phi_0(t)} \frac{C}{\log\frac{1}{\delta}} \geq C\frac{\delta^2}{\log\frac{1}{\delta}}$.

For claim (3), note that $\int_{s_j}^{\infty} e^{\phi_0(t)}dt \geq \int_0^{\infty} e^{\phi_0(t)}dt \geq 2^{-8}(1/9 + 1/16) \geq 2^{-10}$.

On to claim (4). Because j is not the terminating iteration, it must be that $2\int_{-\infty}^{s_{j-1}} e^{\phi_0(t)}dt = \int_{s_{j-1}}^{s_j} e^{\phi_0(t)}dt$. Hence, we have that

$$\frac{(s_j - s_{j-1}) \cdot \sup_{t \in [s_{j-1}, s_j]} e^{\phi_0(t)}}{\int_{-\infty}^{s_{j-1}} e^{\phi_0(t)}dt} \leq \frac{(s_j - s_{j-1}) \cdot \sup_{t \in [s_{j-1}, s_j]} e^{\phi_0(t)}}{2\int_{s_{j-1}}^{s_j} e^{\phi_0(t)}dt} \overset{(a)}{\leq} C\log\frac{1}{\delta}$$

where (a) follows from lemma 38.

For claim (5), observe that because j is the terminating iteration, we have that $\int_{-\infty}^{s_{j-1}} e^{\phi_0(t)}dt \geq \frac{1}{2}\int_{s_{j-1}}^{s_j} e^{\phi_0(t)}dt \geq 2^{-9}(1/9 - 1/16) \geq 2^{-14}$.

□

7.3.1 Bound Lemmas

Lemma 35. *Let $e^{\phi} \in \tilde{\mathcal{F}}(\phi_0, \delta)$. Let r be arbitrary.*

Suppose that $\int_{-\infty}^{r} e^{\phi_0(t)}dt \wedge \int_r^{\infty} e^{\phi_0(t)}dt \geq \delta^2$, then, we have that,

$$\frac{\phi(r) - \phi_0(r)}{2} \geq -\frac{1}{2}\frac{\delta}{\sqrt{\int_{-\infty}^{r} e^{\phi_0(t)}dt \wedge \int_r^{\infty} e^{\phi_0(t)}dt}}$$

Proof. Let $\psi(r) \equiv \frac{\phi(r) - \phi_0(r)}{2}$. Suppose that $\psi(r) < 0$ or there is nothing to prove.

Because ψ is concave, $\psi < \psi(r)$ either for $t \in [-\infty, r)$ or for $t \in (r, \infty]$. Suppose it is the former without loss of generality.

$$
\begin{aligned}
&\int e^{\phi_0(t)}(e^{\psi(t)}-1)^2 dt \le \delta^2 \\
(\Rightarrow)\quad &\int_{-\infty}^{r} e^{\phi_0(t)}(e^{\psi(r)}-1)^2 dt \le \delta^2 \\
(\Rightarrow)\quad &(e^{\psi(r)}-1)^2 \le \frac{\delta^2}{\int_{-\infty}^{r} e^{\phi_0(t)} dt} \\
(\Rightarrow)\quad &(1-e^{\psi(r)}) \le \frac{\delta^2}{\int_{-\infty}^{r} e^{\phi_0(t)} dt} \\
(\Rightarrow)\quad &\psi(r) \ge \log\left(1 - \frac{\delta^2}{\int_{-\infty}^{r} e^{\phi_0(t)} dt}\right) \\
(\Rightarrow)\quad &\psi(r) \ge \frac{1}{2}\frac{\delta^2}{\int_{-\infty}^{r} e^{\phi_0(t)} dt}
\end{aligned}
$$

where the last inequality follows from the assumption that $\int_{-\infty}^{r} e^{\phi_0(t)} dt \ge \delta^2$.

The identical argument applies if $\psi < \psi(r)$ for $t \in (r, \infty]$. The lemma follows easily.

□

Lemma 36. *Let $e^{\phi} \in \tilde{\mathcal{F}}(\phi_0, \delta)$. Define $a_L = \inf\{r : e^{\phi_0(r)} \ge \delta^2\}$ and $a_R = \sup\{r : e^{\phi_0(r)} \ge \delta^2\}$. Suppose that δ is small enough such that $\frac{2^{20} c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta} \ge 2$ and that $\delta^2 < 2^{-18}$.*

Then, we have that, for any $r \in (a_L, a_R)$:

$$
\frac{\phi(r) - \phi_0(r)}{2} \le C\delta \log \frac{c}{\delta}
$$

where C, c are absolute constants.

Proof. As a shorthand, let us write $\psi(r) \equiv \frac{\phi(r)-\phi_0(r)}{2}$. Suppose $\psi(r) > 0$ or there is nothing to prove.

Define $s_L = \inf\left\{ t \in [a_L, r) \,:\, \psi(t) > \frac{\psi(r)}{\frac{2^{20} c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta}} \right\}$.

Define also $s_R = \sup\left\{ t \in (r, s_R] \,:\, \psi(t) > \frac{\psi(r)}{\frac{2^{20} c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta}} \right\}$. Note that $\psi(s_L), \psi(s_R) \ge \frac{\psi(r)}{\frac{2^{20} c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta}}$.

Then, we have that

$$\int_{s_L}^{s_R} e^{\phi_0(t)} \left(e^{\psi(t)} - 1\right)^2 dt \le \delta^2$$
$$(\Rightarrow) \quad \int_{s_L}^{s_R} e^{\phi_0(t)} \psi(t)^2 dt \le \delta^2$$
$$(\Rightarrow) \quad \int_{s_L}^{s_R} e^{\phi_0(t)} \left(\frac{\psi(r)}{\frac{2^{20}c_1}{a_1}\log\frac{\sqrt{c_1}}{\delta}}\right)^2 dt \le \delta^2$$
$$(\Rightarrow) \quad \psi(r) \le \frac{\delta}{\sqrt{\int_{s_L}^{s_R} e^{\phi_0(t)} dt}} \frac{2^{20}c_1}{a_1} \log\frac{\sqrt{c_1}}{\delta} \tag{20}$$

Now, define Define $s_L' = \inf\left\{t \in [a_L, r) \,:\, \psi(t) > -\frac{\psi(r)}{\frac{2^{20}c_1}{a_1}\log\frac{\sqrt{c_1}}{\delta}}\right\}$.
Define also $s_R' = \sup\left\{t \in (r, s_R] \,:\, \psi(t) > -\frac{\psi(r)}{\frac{2^{20}c_1}{a_1}\log\frac{\sqrt{c_1}}{\delta}}\right\}$.
On $[a_L, s_L')$ and $(s_R', a_R]$, $\psi \le -\frac{\psi(r)}{\frac{2^{20}c_1}{a_1}\log\frac{\sqrt{c_1}}{\delta}}$. Therefore,

$$\int_{a_L}^{s_L'} e^{\phi_0(t)} \left(e^{-\frac{\psi(r)}{\frac{2^{20}c_1}{a_1}\log\frac{\sqrt{c_1}}{\delta}}} - 1\right)^2 dt \le \delta^2$$
$$\left(e^{-\frac{\psi(r)}{\frac{2^{20}c_1}{a_1}\log\frac{\sqrt{c_1}}{\delta}}} - 1\right)^2 \int_{a_L}^{s_L'} e^{\phi_0(t)} dt \le \delta^2$$
$$\left(e^{-\frac{\psi(r)}{\frac{2^{20}c_1}{a_1}\log\frac{\sqrt{c_1}}{\delta}}} - 1\right)^2 \le \frac{\delta^2}{\int_{a_L}^{s_L'} e^{\phi_0(t)} dt}$$
$$1 - e^{-\frac{\psi(r)}{\frac{2^{20}c_1}{a_1}\log\frac{\sqrt{c_1}}{\delta}}} \le \frac{\delta}{\sqrt{\int_{a_L}^{s_L'} e^{\phi_0(t)} dt}}$$
$$\frac{\psi(r)}{\frac{2^{20}c_1}{a_1}\log\frac{\sqrt{c_1}}{\delta}} \le -\log\left(\frac{\delta}{\sqrt{\int_{a_L}^{s_L'} e^{\phi_0(t)} dt}}\right)$$

Therefore, we have that

$$\psi(r) \le -\frac{2^{20}c_1}{a_1}\log\frac{\sqrt{c_1}}{\delta} \cdot \log\left(1 - \frac{\delta}{\sqrt{\int_{a_L}^{s_L'} e^{\phi_0(t)} dt}}\right) \tag{21}$$

and, by an identical argument, we have that,

$$\psi(r) \leq -\frac{2^{20}c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta} \cdot \log \left(1 - \frac{\delta}{\sqrt{\int_{s'_R}^{a_R} e^{\phi_0(t)} dt}} \right) \tag{22}$$

Define $T_1 = \int_{s_L}^{s_R} e^{\phi_0(t)} dt$, $T_2 = \int_{s'_L}^{s_L} e^{\phi_0(t)} dt + \int_{s'_R}^{s_R} e^{\phi_0(t)} dt$, and $T_3 = \int_{a_L}^{s'_L} e^{\phi_0(t)} dt + \int_{s'_R}^{a_R} e^{\phi_0(t)} dt$. Collecting inequalities 20, 21, 22 and using the T_1, T_2, T_3 definitions, we have that

$$\psi(r) \leq \frac{2^{20}c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta} \cdot \frac{\delta}{T_1^{1/2}}$$

$$\psi(r) \leq -\frac{2^{20}c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta} \cdot \log \left(1 - \frac{\delta}{T_3^{1/2}} \right)$$

Note that in the second inequality, if $T_3^{1/2} \geq 2^{-16}$, then $\frac{\delta}{T_3^{1/2}} \leq 1/2$ and the second inequality becomes $\psi(r) \leq \frac{2^{20}c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta} \cdot \frac{\delta}{T_3^{1/2}}$.

We have that $T_1 + T_2 + T_3 \geq 2^{-15}$ by lemma 37. We claim that $T_2 < 2^{-16}$. Then, either $T_1 \geq 2^{-16}$ or $T_3 \geq 2^{-16}$ and we prove the lemma with $C = 2^{36} \frac{c_1}{a_1}$ and $c = \sqrt{c_1}$.

Let us then prove the claim. If $|s_L - s'_L| + |s_R - s'_R| = 0$, then there is nothing to prove so let us assume that $|s_L - s'_L| + |s_R - s'_R| > 0$.

First note that, we have, by concavity of ψ,

$$\begin{aligned} |s'_R - s'_L| &\leq |s_R - s_L| \frac{\psi(r) + \psi(r) \frac{1}{\frac{2^{20}c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta}}}{\psi(r) - \psi(r) \frac{1}{\frac{2^{20}c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta}}} \\ &\leq |s_R - s_L| \frac{1 + \frac{1}{\frac{2^{20}c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta}}}{1 - \frac{1}{\frac{2^{20}c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta}}} \\ &\leq |s_R - s_L| \left(1 + \frac{4}{\frac{2^{20}c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta}} \right) \end{aligned}$$

where the last inequality uses the condition that $\frac{2^{20}c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta} \geq 2$.

$$|s'_R - s'_L| \le |s_R - s_L| \left(1 + \frac{4}{\frac{2^{20}c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta}}\right)$$

$$|s_R - s_L| + |s'_L - s_L| + |s'_R - s_R| \le |s_R - s_L| \left(1 + \frac{4}{\frac{2^{20}c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta}}\right)$$

$$|s'_L - s_L| + |s'_R - s_R| \le |s_R - s_L| \frac{4}{\frac{2^{20}c_1}{a_1} \log \frac{\sqrt{c_1}}{\delta}}$$

$$|s_R - s_L| \ge (|s'_L - s_L| + |s'_R - s_R|) \frac{2^{20}c_1}{4a_1} \log \frac{\sqrt{c_1}}{\delta}$$

Suppose for the sake of contradiction that $T_2 \ge 2^{-16}$. Then, because $e^{\phi_0(r)} \le c_1$, we have that

$$c_1 |s_L - s'_L| + c_1 |s_R - s'_R| \ge 2^{-16}$$

$$\begin{aligned} |s_L - s_R| &\ge \frac{1}{c_1} 2^{-16} \frac{2^{20}c_1}{4a_1} \log \frac{\sqrt{c_1}}{\delta} \\ &\ge \frac{4}{a_1} \log \frac{\sqrt{c_1}}{\delta} \end{aligned}$$

Now, $e^{\phi_0(r)} \le c_1 e^{-a_1|r|}$ by assumption. Plugging in $\delta^2 = c_1 e^{-a_1|r|}$ yields that $|a_L|, |a_R| \le \frac{2}{a_1} \log \frac{\sqrt{c_1}}{\delta}$ and therefore, $|a_L - a_R| \le frac4a_1 \log \frac{\sqrt{c_1}}{\delta}$ Since $|s_L - s_R| \ge |a_L - a_R|$, it must be that $|s_L - s'_L| + |s_R - s'_R| = 0$, which is a contradiction. This proves the claim and hence the lemma.

□

7.3.2 Technical Lemmas

Lemma 37. *Let f be a log-concave density with mean zero and unit variance, let $a_L = \inf\{r : f(r) \ge \delta^2\}$ and $a_R = \inf\{r : f(r) \ge \delta^2\}$. If $\delta^2 \le 2^{-10}$, then we have that*

$$\int_{a_L}^{a_R} f(r) dr \ge 2^{-15}$$

Proof. By Lovász and Vempala [2007, Theorem 5.14(d)], we have that $2^4 \le f(0) \ge 2^{-7}$ and that $f(r) \ge 2^{-8}$ for all $r \in [-1/9, 1/9]$. By log-concavity, we have then that $a_L \le -1/9$ and $a_R \ge 1/9$.

Define $\alpha = \log f(0)$ and $\beta = \log f(a_L)$. It is clear that $4\log 2 \le \alpha \ge -7\log 2$ and $\beta \ge -10\log 2$.

$$\begin{aligned}\int_{a_L}^{0} f(r)dr &\ge |a_L| \int_0^1 e^{(1-\lambda)\alpha+\lambda\beta} d\lambda \\ &\ge |a_L| \int_0^1 e^{\alpha} e^{-\lambda(\alpha-\beta)} d\lambda \\ &\ge |a_L| e^{\alpha} \frac{1}{\alpha-\beta}(1-e^{-(\alpha-\beta)}) \\ &\ge (1/9)2^{-7}\frac{1}{\alpha-\beta}(1-e^{-(\alpha-\beta)})\end{aligned}$$

If $e^{-(\alpha-\beta)} \ge 1/2$, then $\frac{1}{\alpha-\beta}(1-e^{-(\alpha-\beta)}) \ge 1/2$. If not, then, by our bound on α and β, we have $\frac{1}{\alpha-\beta}(1-e^{-(\alpha-\beta)}) \le \frac{1}{14\log 2}\frac{1}{2}$. Either way, we have that $\int_{a_L}^{0} f(r)dr \ge 2^{-15}$.

The same bound holds for $\int_0^{a_R} f(r)dr$. The lemma follows thus.

□

Lemma 38. *Let e^{ϕ_0} be a log-concave density. Let $[a,b]$ be an interval and suppose that $e^{\phi_0(t)}$ achieves its maximum in $[a,b]$ at t^*. Then, we have that*

$$\begin{aligned}\int_a^b e^{\phi_0(t)}dt \ge &|a-t^*|e^{\phi_0(t^*)} \cdot \frac{1}{\phi_0(t^*)-\phi_0(a)}\left\{1-e^{-(\phi_0(t^*)-\phi_0(a))}\right\} \\ &+ |b-t^*|e^{\phi_0(t^*)} \cdot \frac{1}{\phi_0(t^*)-\phi_0(b)}\left\{1-e^{-(\phi_0(t^*)-\phi_0(b))}\right\}\end{aligned}$$

And, if $\phi_0(t^*)-\phi_0(b), \phi_0(t^*)-\phi_0(a) \le \tau$ for some upper bound $\tau > 1$, then

$$\int_a^b e^{\phi_0(t)}dt \ge |a-b|e^{\phi_0(t^*)}\frac{1}{4\tau}$$

Proof. On $[a,t^*]$, $\phi_0(t) \ge \lambda\phi_0(a) + (1-\lambda)\phi_0(t^*)$ where $\lambda = \frac{t^*-t}{t^*-a}$.

Therefore,

$$\begin{aligned}\int_a^{t^*} e^{\phi_0(t)}dt &\ge (t^*-a)\int_0^1 e^{\lambda\phi_0(a)+(1-\lambda)\phi_0(t^*)}d\lambda \\ &\ge |a-t^*|e^{\phi_0(t^*)} \cdot \frac{1}{\phi_0(t^*)-\phi_0(a)}\left\{1-e^{-(\phi_0(t^*)-\phi_0(a))}\right\}\end{aligned}$$

Similar argument applies for $\int_{t^*}^b e^{\phi_0(t)}dt$.

For the second part of the lemma, note that if $\phi_0(t^*)-\phi_0(a) \le 1$, then $\frac{1-e^{-(\phi_0(t^*)-\phi_0(b))}}{\phi_0(t^*)-\phi_0(b)} \ge e^{-1}$. If $\phi_0(t^*)-\phi_0(a) \ge 1$, then $\frac{1-e^{-(\phi_0(t^*)-\phi_0(b))}}{\phi_0(t^*)-\phi_0(b)} \ge e^{-1}\frac{1}{\phi_0(t^*)-\phi_0(b)} \ge e^{-1}\frac{1}{\tau}$.

□

7.3.3 Bracketing Lemmas

The following two propositions are from Kim et al. [2016].

Proposition 39. *There exists a universal constant $C > 0$ such that*

$$H_{[]}(\epsilon, \tilde{\mathcal{F}}([a,b], -\infty, B), d_H, [a,b]) \leq C \frac{e^{B/4}(b-a)^{1/4}}{\epsilon^{1/2}}$$

Proposition 40. *There exists a universal constant $C > 0$ such that*

$$H_{[]}(\epsilon, \tilde{\mathcal{F}}([a,b], B_2, B_1), d_H, [a,b]) \leq C(B_1 - B_2)^{1/2} \frac{e^{B_1/4}(b-a)^{1/4}}{\epsilon^{1/2}}$$

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