Miscellaneous Notes for Elliptical Log-Concave Density

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Define $K_n := K \backslash (1 - \frac{1}{n})K$. Let $Z \sim \text{Unif}(K)$, then we have that

$$Z = Z_n W + Z_n'(1 - W)$$

where $W := \mathbb{I}\{Z \in K_n\}$, $Z_n \sim \text{Unif}(K_n)$, $Z_n' \sim \text{Unif}((1 - \frac{1}{n})K)$, and W, Z_n, Z_n' are independent. If $W = 1$, then $Z/\|Z\|_K = Z_n/\|Z_n\|_K$ and if $W = 0$, then $Z/\|Z\|_K = Z_n'/\|Z_n'\|_K$. Thus,

$$\frac{Z}{\|Z\|_K} = \frac{Z_n}{\|Z_n\|_K} W + \frac{Z_n'}{\|Z_n'\|_K}(1 - W).$$

Note that $\frac{Z_n'}{\|Z_n'\|_K}$ is identically distributed as $\frac{Z}{\|Z\|_K}$ since Z_n' is identically distributed as $(1 - \frac{1}{n})Z$. Then,

$$\phi_{\frac{Z}{\|Z\|_K}}(t) = \lambda \phi_{\frac{Z}{\|Z\|_K}}(t) + (1 - \lambda)\phi_{\frac{Z_n}{\|Z_n\|_K}}(t),$$

where $\lambda = \mathbb{P}(W = 1)$. Thus, we have that $\frac{Z_n}{\|Z_n\|_K}$ is identically distributed as $\frac{Z}{\|Z\|_K}$.

Since $\|Z_n - \frac{Z_n}{\|Z_n\|_K}\| \leq \frac{1}{\|Z_n\|_K} - 1 \leq \frac{1}{1 - \frac{1}{n}} - 1 \to 0$, we have that Z_n converges weakly to $\frac{Z}{\|Z\|_K}$.

1 Envelope Search

Problem: Let $h(r) = r^{p-1} g(r) c_p$ be a density where $g(r)$ is log-concave and decreasing. Suppose also that the second moment is p.

$$\int r^{p-1} g(r) c_p dr = 1$$

$$\int r^2 r^{p-1} g(r) c_p dr = p$$

Let $\mathcal{H}$ be the set of all such densities. Then, we want to have an exponentially decaying envelope

$$\sup_{h \in \mathcal{H}} h(r) \leq \exp(-a_p r + b_p)$$

for scalar a_p, b_p dependent on p.

1.1 Thoughts and Examples

One possibly useful fact. If $f(x)$ is an isotropic log-concave density, then there exists absolute constants a, b such that $f(x) \leq \exp(-ax + b)$.

Therefore, if $f(\frac{x}{\sigma})\frac{1}{\sigma}$ has variance σ^2 and can be bounded by $\exp(-\frac{a}{\sigma}x + b - \log \sigma)$.

One example to keep in mind is if $g(r) = M > 0$ is uniform on $[0, r_0]$ and 0 elsewhere. It is easy to solve for r_0:

$$\int_0^{r_0} r^{p-1} M c_p dr = \frac{r_0^p}{p} M c_p = 1$$

$$\int_0^{r_0} r^{p+1} M c_p dr = \frac{r_0^{p+2}}{p+2} M c_p = p$$

$$\frac{r_0^{p+2}}{p+2} M c_p = \frac{r_0^p}{p} M c_p \frac{p}{p+2} r_0^2 = \frac{p}{p+2} r_0^2 = p$$

Therefore, $r_0 = \sqrt{p+2}$. This density has vanishing variance and its maximum value explodes. The maximum value is $h(r_0) = r_0^{p-1} M c_p = \frac{p}{r_0} \frac{r_0^{p-1}}{p} M c_p = \frac{p}{\sqrt{p+2}}$.

To compute the variance, we first find the mean.

$$\int_0^{r_0} r^p M c_p dr = \frac{r_0^{p+1}}{p+1} M c_p = \frac{p}{p+1} \sqrt{p+2}$$

$$\text{variance: } \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = p^2 - \left(\frac{p}{p+1}\right)^2 (p+2) = \frac{p}{(p+2)^2}$$

Two points bound. Let $M = \log g(0)$. Let $r_0 > 0$ and let $M' = \log g(r_0)$. Define $\Delta = M - M' \geq 0$.

Then, we have the following upper and lower bound on g:

$$\log g(r) \geq \left\{ \begin{array}{cc} M - \Delta \frac{r}{r_0} & r \leq r_0 \\ -\infty & r \geq r_0 \end{array} \right\}$$

$$\log g(r) \leq \left\{ \begin{array}{cc} M & r \leq r_0 \\ M - \Delta \frac{r}{r_0} & r \geq r_0 \end{array} \right\}$$

So then, we have that

$$\begin{aligned} 1 &= \int_0^\infty r^{p-1} g(r) c_p dr \\ &\geq \int_0^{r_0} r^{p-1} \exp\left(M - \Delta \frac{r}{r_0}\right) c_p dr \end{aligned}$$

1.2 Change of Variables

Recall that the density we are interested in is:

$$h(r) = r^{p-1} g(r) c_p$$

Satisfying the two conditions that

$$\int_0^\infty r^{p-1} g(r) c_p dr = 1$$
$$\int_0^\infty r^{p+1} g(r) c_p dr = p$$

Let us perform a change of variables: $s = \frac{r}{\sqrt{p}}$ and thus $r = s\sqrt{p}$.
Then, the two integral equations become:

$$c_p \sqrt{p}^p \int_0^\infty s^{p-1} g(\sqrt{p}(s)) ds = 1$$
$$c_p \sqrt{p}^{p+2} \int_0^\infty s^{p+1} g(\sqrt{p}(s)) ds = p$$

With some cancelation and with the replacement of $\widetilde{g}(s) = c_p \sqrt{p}^p g(\sqrt{p}(s))$, we have that

$$\int_0^\infty s^{p-1} \widetilde{g}(s) ds = 1$$
$$\int_0^\infty s^{p+1} \widetilde{g}(s) ds = 1$$

Note that $\widetilde{g}(s)$ is log-concave and decreasing.

An Observation

Let r_0 be arbitrary. Then we have that

$$\int_0^{r_0} s^{p-1}(1 - s^2) \widetilde{g}(s) ds + \int_{r_0}^\infty s^{p-1}(1 - s^2) \widetilde{g}(s) ds = 0$$

If $r_0 \leq 1$, then the first term is positive, which implies that the second term is negative. If $r_0 \geq 1$, then the second term is negative, which implies that the first term is positive.

Thus, for any r_0, we have that the first term is positive and the second term is negative.

1.3 Hinge Example

The analysis of this example provides some useful calculations.

Let $g(r)$ be of the form:

$$g(r) = \begin{cases} e^{m_0} & r \leq r_0\sqrt{p} \\ e^{m_0 - a(r - r_0\sqrt{p})} & r \geq r_0\sqrt{p} \end{cases}$$

g is thus parametrized by three parameters: m_0, a, r_0.

We want $g(r)$ to satisfy two integral conditions:

$$\int_0^\infty r^{p-1} g(r) c_p dr = 1$$
$$\int_0^\infty r^{p+1} g(r) c_p dr = p$$

The first integral equation breaks down into two halves:

$$\int_0^{r_0\sqrt{p}} r^{p-1} e^{m_0} c_p dr + \int_{r_0\sqrt{p}}^\infty r^{p-1} e^{m_0} e^{-a(r - r_0\sqrt{p})} c_p dr = 1$$

We apply a change of variables: $s = \frac{r}{\sqrt{p}}$ and $r = s\sqrt{p}$.

$$e^{m_0} c_p \sqrt{p}^p \left\{ \int_0^{r_0} s^{p-1} ds + \int_{r_0}^\infty s^{p-1} e^{-a\sqrt{p}(s - r_0)} ds \right\} = 1$$

Likewise, we have that second equation as well:

$$e^{m_0} c_p \sqrt{p}^{p+2} \left\{ \int_0^{r_0} s^{p+1} ds + \int_{r_0}^\infty s^{p+1} e^{-a\sqrt{p}(s - r_0)} ds \right\} = p$$

We will simplify by letting $\bar{a} = a\sqrt{p}$. Then, we have:

$$e^{m_0} c_p \sqrt{p}^p \left\{ \int_0^{r_0} s^{p-1} ds + \int_{r_0}^\infty s^{p-1} e^{-\bar{a}(s - r_0)} ds \right\} = 1$$
$$e^{m_0} c_p \sqrt{p}^p \left\{ \int_0^{r_0} s^{p+1} ds + \int_{r_0}^\infty s^{p+1} e^{-\bar{a}(s - r_0)} ds \right\} = 1$$

Setting the two equation equal to each other:

$$\int_0^{r_0} s^{p-1} ds - \int_0^{r_0} s^{p+1} ds + \int_{r_0}^\infty s^{p-1} e^{-\bar{a}(s - r_0)} ds - \int_{r_0}^\infty s^{p+1} e^{-\bar{a}(s - r_0)} ds = 0$$
$$\left(\frac{r_0^p}{p} - \frac{r_0^{p+2}}{p+2} \right) + \int_{r_0}^\infty s^{p-1} e^{-\bar{a}(s - r_0)} ds - \int_{r_0}^\infty s^{p+1} e^{-\bar{a}(s - r_0)} ds = 0$$

We know that for all plausible r_0, it must be that $\frac{r_0^P}{p} - \frac{r_0^{p+2}}{p+2} \geq 0$. Thus, to solve for the maximum value of r_0, we set $\frac{r_0^p}{p} = \frac{r_0^{p+2}}{p+2}$, yielding $\sqrt{1 + \frac{2}{p}}$.

Now we turn our attention to $\bar{a}$. At $r_0 = 1$, we have that

$$\int_1^\infty s^{p-1}(1-s^2)e^{-\bar{a}(s-1)} = \frac{1}{p+2} - \frac{1}{p}$$
$$\int_1^\infty s^{p-1}(1-s^2)e^{-\bar{a}s}ds = e^{-\bar{a}}\left(\frac{1}{p+2} - \frac{1}{p}\right)$$
$$\int_1^\infty s^{p-1}e^{-\bar{a}s} - s^{p+1}e^{-\bar{a}s}ds = e^{-\bar{a}}\left(\frac{1}{p+2} - \frac{1}{p}\right)$$