CHAPTER 5

The Theory of the Simplex Method

Chapter 4 introduced the basic mechanics of the simplex method. Now we shall delve a little more deeply into this algorithm by examining some of its underlying theory. The first section further develops the general geometric and algebraic properties that form the foundation of the simplex method. We then describe the *matrix form* of the simplex method, which streamlines the procedure considerably for computer implementation. Next we use this matrix form to present a fundamental insight about a property of the simplex method that enables us to deduce how changes that are made in the original model get carried along to the final simplex tableau. This insight will provide the key to the important topics of Chap. 6 (duality theory and sensitivity analysis). The chapter then concludes by presenting the *revised simplex method*, which further streamlines the matrix form of the simplex method. Commercial computer codes of the simplex method normally are based on the revised simplex method.

5.1 FOUNDATIONS OF THE SIMPLEX METHOD

Section 4.1 introduced *corner-point feasible (CPF) solutions* and the key role they play in the simplex method. These geometric concepts were related to the algebra of the simplex method in Secs. 4.2 and 4.3. However, all this was done in the context of the Wyndor Glass Co. problem, which has only *two decision variables* and so has a straightforward geometric interpretation. How do these concepts generalize to higher dimensions when we deal with larger problems? We address this question in this section.

We begin by introducing some basic terminology for any linear programming problem with n decision variables. While we are doing this, you may find it helpful to refer to Fig. 5.1 (which repeats Fig. 4.1) to interpret these definitions in two dimensions ($n = 2$).

Terminology

It may seem intuitively clear that optimal solutions for any linear programming problem must lie on the boundary of the feasible region, and in fact this is a general property. Because boundary is a geometric concept, our initial definitions clarify how the boundary of the feasible region is identified algebraically.

> The **constraint boundary equation** for any constraint is obtained by replacing its $\leq$, $=$, or $\geq$ sign with an $=$ sign.

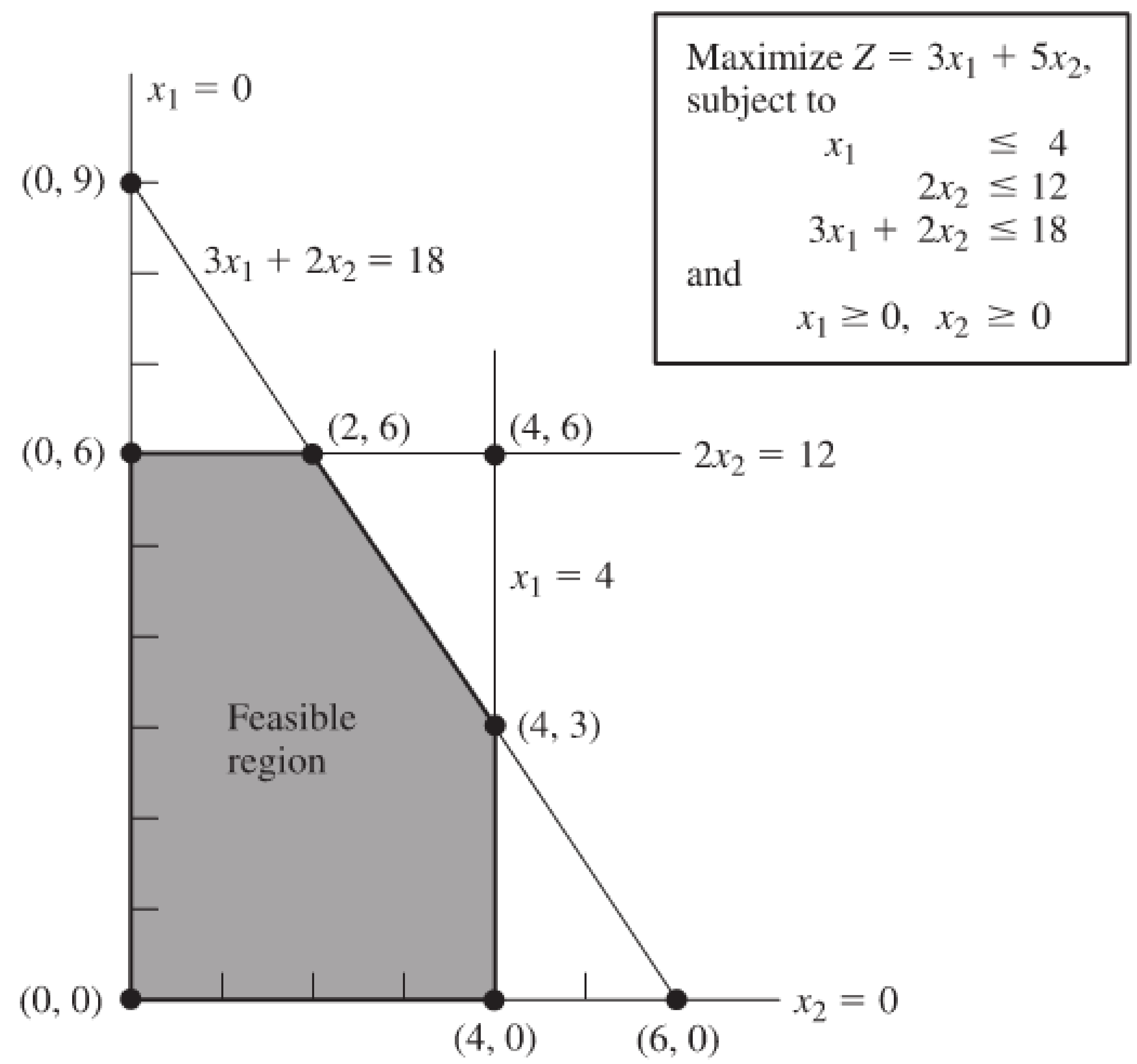


FIGURE 5.1 Constraint boundaries, constraint boundary equations, and corner-point solutions for the Wyndor Glass Co. problem.

Consequently, the form of a constraint boundary equation is $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$ for functional constraints and $x_j = 0$ for nonnegativity constraints. Each such equation defines a "flat" geometric shape (called a **hyperplane**) in n-dimensional space, analogous to the line in two-dimensional space and the plane in three-dimensional space. This hyperplane forms the **constraint boundary** for the corresponding constraint. When the constraint has either a $\le$ or a $\ge$ sign, this *constraint boundary* separates the points that satisfy the constraint (all the points on one side up to and including the constraint boundary) from the points that violate the constraint (all those on the other side of the constraint boundary). When the constraint has an $=$ sign, only the points on the constraint boundary satisfy the constraint.

For example, the Wyndor Glass Co. problem has five constraints (three functional constraints and two nonnegativity constraints), so it has the five *constraint boundary equations* shown in Fig. 5.1. Because $n = 2$, the hyperplanes defined by these constraint boundary equations are simply lines. Therefore, the constraint boundaries for the five constraints are the five lines shown in Fig. 5.1.

> The **boundary** of the feasible region contains just those feasible solutions that satisfy one or more of the constraint boundary equations.

Geometrically, any point on the boundary of the feasible region lies on one or more of the hyperplanes defined by the respective constraint boundary equations. Thus, in Fig. 5.1, the boundary consists of the five darker line segments.

Next, we give a general definition of *CPF solution* in n-dimensional space.

> A **corner-point feasible (CPF) solution** is a feasible solution that does not lie on *any* line segment[1] connecting two *other* feasible solutions.

As this definition implies, a feasible solution that *does* lie on a line segment connecting two other feasible solutions is *not* a CPF solution. To illustrate when $n = 2$, consider Fig. 5.1.

[1]An algebraic expression for a line segment is given in Appendix 2.

The point (2, 3) is *not* a CPF solution, because it lies on various such line segments, e.g., the line segment connecting (0, 3) and (4, 3). Similarly, (0, 3) is *not* a CPF solution, because it lies on the line segment connecting (0, 0) and (0, 6). However, (0, 0) *is* a CPF solution, because it is impossible to find two *other* feasible solutions that lie on completely opposite sides of (0, 0). (Try it.)

When the number of decision variables n is greater than 2 or 3, this definition for *CPF solution* is not a very convenient one for identifying such solutions. Therefore, it will prove most helpful to interpret these solutions algebraically. For the Wyndor Glass Co. example, each CPF solution in Fig. 5.1 lies at the intersection of two ($n = 2$) constraint lines; i.e., it is the *simultaneous solution* of a system of two constraint boundary equations. This situation is summarized in Table 5.1, where **defining equations** refer to the constraint boundary equations that yield (define) the indicated CPF solution.

> For any linear programming problem with n decision variables, each CPF solution lies at the intersection of n constraint boundaries; i.e., it is the *simultaneous solution* of a system of n constraint boundary equations.

However, this is not to say that *every* set of n constraint boundary equations chosen from the $n + m$ constraints (n nonnegativity and m functional constraints) yields a CPF solution. In particular, the simultaneous solution of such a system of equations might violate one or more of the other m constraints not chosen, in which case it is a corner-point *infeasible* solution. The example has three such solutions, as summarized in Table 5.2. (Check to see why they are infeasible.)

TABLE 5.1 Defining equations for each CPF solution for the Wyndor Glass Co. problem

CPF Solution	Defining Equations
(0, 0)	$x_1 = 0$ $x_2 = 0$
(0, 6)	$x_1 = 0$ $2x_2 = 12$
(2, 6)	$2x_2 = 12$ $3x_1 + 2x_2 = 18$
(4, 3)	$3x_1 + 2x_2 = 18$ $x_1 = 4$
(4, 0)	$x_1 = 4$ $x_2 = 0$

TABLE 5.2 Defining equations for each corner-point infeasible solution for the Wyndor Glass Co. problem

Corner-Point Infeasible Solution	Defining Equations
(0, 9)	$x_1 = 0$ $3x_1 + 2x_2 = 18$
(4, 6)	$2x_2 = 12$ $x_1 = 4$
(6, 0)	$3x_1 + 2x_2 = 18$ $x_2 = 0$

Furthermore, a system of n constraint boundary equations might have no solution at all. This occurs twice in the example, with the pairs of equations (1) $x_1 = 0$ and $x_1 = 4$ and (2) $x_2 = 0$ and $2x_2 = 12$. Such systems are of no interest to us.

The final possibility (which never occurs in the example) is that a system of n constraint boundary equations has multiple solutions because of redundant equations. You need not be concerned with this case either, because the simplex method circumvents its difficulties.

We also should mention that it is possible for more than one system of n constraint boundary equations to yield the same CP solution. For example, if the $x_1 \leq 4$ constraint in the Wyndor Glass Co. problem were to be replaced by $x_1 \leq 2$, note in Fig. 5.1 how the CPF solution (2, 6) can be derived from any one of three pairs of constraint boundary equations. (This is an example of the *degeneracy* discussed in a different context in Sec. 4.5.)

To summarize for the example, with five constraints and two variables, there are 10 pairs of constraint boundary equations. Five of these pairs became defining equations for CPF solutions (Table 5.1), three became defining equations for corner-point infeasible solutions (Table 5.2), and each of the final two pairs had no solution.

Adjacent CPF Solutions

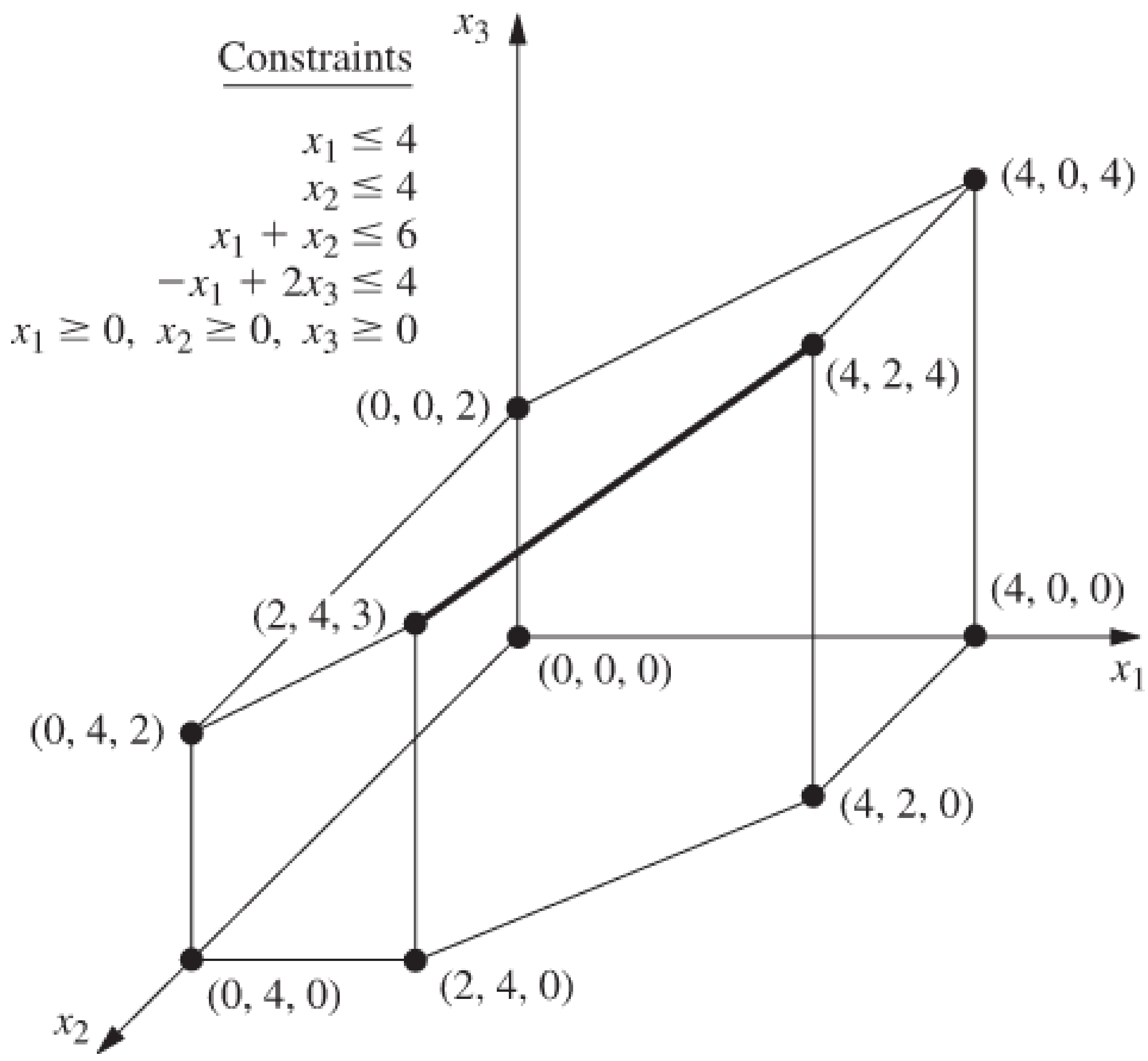
Section 4.1 introduced adjacent CPF solutions and their role in solving linear programming problems. We now elaborate.

Recall from Chap. 4 that (when we ignore slack, surplus, and artificial variables) each iteration of the simplex method moves from the current CPF solution to an *adjacent* one. What is the *path* followed in this process? What really is meant by *adjacent* CPF solution? First we address these questions from a geometric viewpoint, and then we turn to algebraic interpretations.

These questions are easy to answer when $n = 2$. In this case, the *boundary* of the feasible region consists of several connected *line segments* forming a *polygon,* as shown in Fig. 5.1 by the five darker line segments. These line segments are the *edges* of the feasible region. Emanating from each CPF solution are *two* such edges leading to an adjacent CPF solution at the other end. (Note in Fig. 5.1 how each CPF solution has two adjacent ones.) The path followed in an iteration is to move along one of these edges from one end to the other. In Fig. 5.1, the first iteration involves moving along the edge from (0, 0) to (0, 6), and then the next iteration moves along the edge from (0, 6) to (2, 6). As Table 5.1 illustrates, each of these moves to an adjacent CPF solution involves just one change in the set of defining equations (constraint boundaries on which the solution lies).

When $n = 3$, the answers are slightly more complicated. To help you visualize what is going on, Fig. 5.2 shows a three-dimensional drawing of a typical feasible region when $n = 3$, where the dots are the CPF solutions. This feasible region is a *polyhedron* rather than the polygon we had with $n = 2$ (Fig. 5.1), because the constraint boundaries now are *planes* rather than lines. The faces of the polyhedron form the *boundary* of the feasible region, where each face is the portion of a constraint boundary that satisfies the other constraints as well. Note that each CPF solution lies at the intersection of three constraint boundaries (sometimes including some of the $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ constraint boundaries for the nonnegativity constraints), and the solution also satisfies the other constraints. Such intersections that do not satisfy one or more of the other constraints yield corner-point *infeasible* solutions instead.

The darker line segment in Fig. 5.2 depicts the path of the simplex method on a typical iteration. The point (2, 4, 3) is the *current* CPF solution to begin the iteration, and the point (4, 2, 4) will be the new CPF solution at the end of the iteration. The point (2, 4, 3) lies at the intersection of the $x_2 = 4$, $x_1 + x_2 = 6$, and $-x_1 + 2x_3 = 4$ constraint boundaries, so these three equations are the *defining equations* for this CPF solution. If the $x_2 = 4$ defining equation were removed, the intersection of the other two constraint



■ **FIGURE 5.2**
Feasible region and CPF solutions for a three-variable linear programming problem.

boundaries (planes) would form a line. One segment of this line, shown as the dark line segment from (2, 4, 3) to (4, 2, 4) in Fig. 5.2, lies on the boundary of the feasible region, whereas the rest of the line is infeasible. This line segment is an edge of the feasible region, and its endpoints (2, 4, 3) and (4, 2, 4) are adjacent CPF solutions.

For $n = 3$, all the *edges* of the feasible region are formed in this way as the feasible segment of the line lying at the intersection of two constraint boundaries, and the two endpoints of an edge are *adjacent* CPF solutions. In Fig. 5.2 there are 15 edges of the feasible region, and so there are 15 pairs of adjacent CPF solutions. For the current CPF solution (2, 4, 3), there are three ways to remove one of its three defining equations to obtain an intersection of the other two constraint boundaries, so there are three edges emanating from (2, 4, 3). These edges lead to (4, 2, 4), (0, 4, 2), and (2, 4, 0), so these are the CPF solutions that are adjacent to (2, 4, 3).

For the next iteration, the simplex method chooses one of these three edges, say, the darker line segment in Fig. 5.2, and then moves along this edge away from (2, 4, 3) until it reaches the first new constraint boundary, $x_1 = 4$, at its other endpoint. [We cannot continue farther along this line to the next constraint boundary, $x_2 = 0$, because this leads to a corner-point infeasible solution—(6, 0, 5).] The intersection of this first new constraint boundary with the two constraint boundaries forming the edge yields the *new* CPF solution (4, 2, 4).

When $n > 3$, these same concepts generalize to higher dimensions, except the constraint boundaries now are *hyperplanes* instead of planes. Let us summarize.

> Consider any linear programming problem with n decision variables and a bounded feasible region. A CPF solution lies at the intersection of n constraint boundaries (and satisfies the other constraints as well). An **edge** of the feasible region is a feasible line segment that lies at the intersection of $n - 1$ constraint boundaries, where each endpoint lies on one additional constraint boundary (so that these endpoints are CPF solutions). Two CPF solutions are **adjacent** if the line segment connecting them is an edge of the feasible region. Emanating from each CPF solution are n such edges, each one leading to one of the n adjacent CPF solutions. Each iteration of the simplex method moves from the current CPF solution to an adjacent one by moving along one of these n edges.

When you shift from a geometric viewpoint to an algebraic one, *intersection of constraint boundaries* changes to *simultaneous solution of constraint boundary equations.* The n constraint boundary equations yielding (defining) a CPF solution are its defining equations, where deleting one of these equations yields a line whose feasible segment is an edge of the feasible region.

We next analyze some key properties of CPF solutions and then describe the implications of all these concepts for interpreting the simplex method. However, while the summary on the previous page is fresh in your mind, let us give you a preview of its implications. When the simplex method chooses an entering basic variable, the geometric interpretation is that it is choosing one of the edges emanating from the current CPF solution to move along. Increasing this variable from zero (and simultaneously changing the values of the other basic variables accordingly) corresponds to moving along this edge. Having one of the basic variables (the leaving basic variable) decrease so far that it reaches zero corresponds to reaching the first new constraint boundary at the other end of this edge of the feasible region.

Properties of CPF Solutions

We now focus on three key properties of CPF solutions that hold for *any* linear programming problem that has feasible solutions and a bounded feasible region.

Property 1: (a) If there is exactly one optimal solution, then it must be a CPF solution. (b) If there are multiple optimal solutions (and a bounded feasible region), then at least two must be adjacent CPF solutions.

Property 1 is a rather intuitive one from a geometric viewpoint. First consider Case (a), which is illustrated by the Wyndor Glass Co. problem (see Fig. 5.1) where the one optimal solution (2, 6) is indeed a CPF solution. Note that there is nothing special about this example that led to this result. For any problem having just one optimal solution, it always is possible to keep raising the objective function line (hyperplane) until it just touches one point (the optimal solution) at a corner of the feasible region.

We now give an algebraic proof for this case.

Proof of Case (*a*) of Property 1: We set up a *proof by contradiction* by assuming that there is exactly one optimal solution and that it is *not* a CPF solution. We then show below that this assumption leads to a contradiction and so cannot be true. (The solution assumed to be optimal will be denoted by $\mathbf{x}^*$, and its objective function value by Z^*.)

Recall the definition of *CPF solution* (a feasible solution that does not lie on any line segment connecting two other feasible solutions). Since we have assumed that the optimal solution $\mathbf{x}^*$ is not a CPF solution, this implies that there must be two other feasible solutions such that the line segment connecting them contains the optimal solution. Let the vectors $\mathbf{x}'$ and $\mathbf{x}''$ denote these two other feasible solutions, and let Z_1 and Z_2 denote their respective objective function values. Like each other point on the line segment connecting $\mathbf{x}'$ and $\mathbf{x}''$,

$$\mathbf{x}^* = \alpha\mathbf{x}'' + (1 - \alpha)\mathbf{x}'$$

for some value of α such that $0 < \alpha < 1$. Thus, since the coefficients of the variables are identical for Z^*, Z_1, and Z_2, it follows that

$$Z^* = \alpha Z_2 + (1 - \alpha)Z_1.$$

Since the weights α and $1 - \alpha$ add to 1, the only possibilities for how Z^*, Z_1, and Z_2 compare are (1) $Z^* = Z_1 = Z_2$, (2) $Z_1 < Z^* < Z_2$, and (3) $Z_1 > Z^* > Z_2$. The first possibility implies that $\mathbf{x}'$ and $\mathbf{x}''$ also are optimal, which contradicts the assumption that there is exactly one optimal solution. Both the latter possibilities contradict the assumption that $\mathbf{x}^*$ (not a CPF solution) is optimal. The resulting conclusion is that it is impossible to have a single optimal solution that is not a CPF solution.

Now consider Case (*b*), which was demonstrated in Sec. 3.2 under the definition of *optimal solution* by changing the objective function in the example to $Z = 3x_1 + 2x_2$ (see Fig. 3.5 in Sec. 3.2). What then happens when you are solving graphically is that the objective function line keeps getting raised until it contains the line segment connecting the two CPF solutions (2, 6) and (4, 3). The same thing would happen in higher dimensions except that an objective function *hyperplane* would keep getting raised until it contained the line segment(s) connecting two (or more) adjacent CPF solutions. As a consequence, *all* optimal solutions can be obtained as weighted averages of optimal CPF solutions. (This situation is described further in Probs. 4.5-5 and 4.5-6.)

The real significance of Property 1 is that it greatly simplifies the search for an optimal solution because now only CPF solutions need to be considered. The magnitude of this simplification is emphasized in Property 2.

Property 2: There are only a *finite* number of CPF solutions.

This property certainly holds in Figs. 5.1 and 5.2, where there are just 5 and 10 CPF solutions, respectively. To see why the number is finite in general, recall that each CPF solution is the simultaneous solution of a system of n out of the $m + n$ constraint boundary equations. The number of different combinations of $m + n$ equations taken n at a time is

$$\binom{m+n}{n} = \frac{(m+n)!}{m!n!},$$

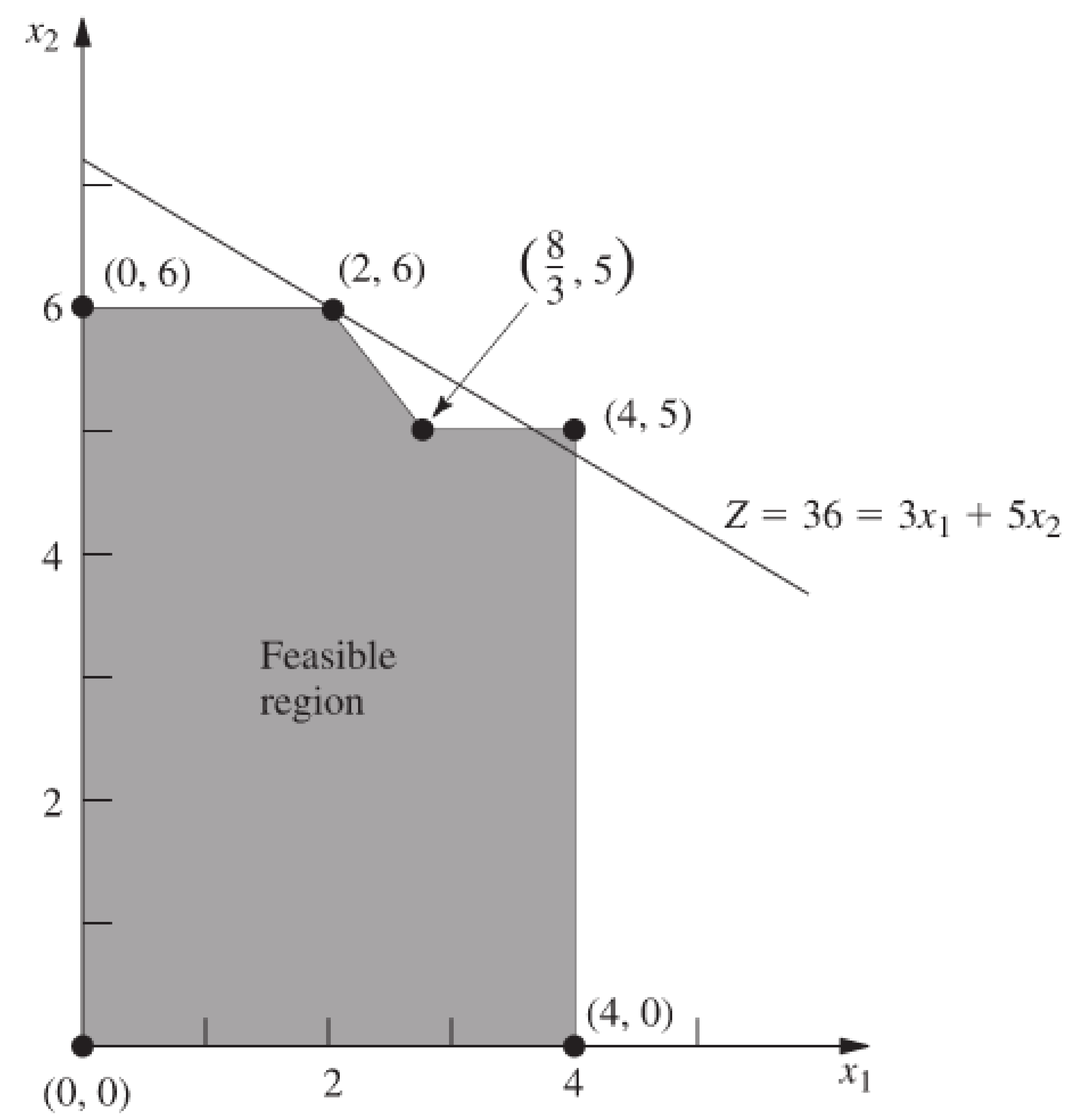
which is a finite number. This number, in turn, in an *upper bound* on the number of CPF solutions. In Fig. 5.1, $m = 3$ and $n = 2$, so there are 10 different systems of two equations, but only half of them yield CPF solutions. In Fig. 5.2, $m = 4$ and $n = 3$, which gives 35 different systems of three equations, but only 10 yield CPF solutions.

Property 2 suggests that, in principle, an optimal solution can be obtained by exhaustive enumeration; i.e., find and compare all the finite number of CPF solutions. Unfortunately, there are finite numbers, and then there are finite numbers that (for all practical purposes) might as well be infinite. For example, a rather small linear programming problem with only $m = 50$ and $n = 50$ would have $100!/(50!)^2 \approx 10^{29}$ systems of equations to be solved! By contrast, the simplex method would need to examine only approximately 100 CPF solutions for a problem of this size. This tremendous savings can be obtained because of the optimality test given in Sec. 4.1 and restated here as Property 3.

Property 3: If a CPF solution has no *adjacent* CPF solutions that are *better* (as measured by Z), then there are no *better* CPF solutions anywhere. Therefore, such a CPF solution is guaranteed to be an *optimal* solution (by Property 1), assuming only that the problem possesses at least one optimal solution (guaranteed if the problem possesses feasible solutions and a bounded feasible region).

To illustrate Property 3, consider Fig. 5.1 for the Wyndor Glass Co. example. For the CPF solution (2, 6), its adjacent CPF solutions are (0, 6) and (4, 3), and neither has a better value of Z than (2, 6) does. This outcome implies that none of the other CPF solutions—(0, 0) and (4, 0)—can be better than (2, 6), so (2, 6) must be optimal.

By contrast, Fig. 5.3 shows a feasible region that can *never* occur for a linear programming problem (since the continuation of the constraint boundary lines that pass



■ **FIGURE 5.3**
Modification of the Wyndor Glass Co. problem that violates both linear programming and Property 3 for CPF solutions in linear programming.

through $(\frac{8}{3}, 5)$ would chop off part of this region) but that does violate Property 3. The problem shown is identical to the Wyndor Glass Co. example (including the same objective function) *except* for the enlargement of the feasible region to the right of $(\frac{8}{3}, 5)$. Consequently, the adjacent CPF solutions for (2, 6) now are (0, 6) and $(\frac{8}{3}, 5)$, and again neither is better than (2, 6). However, another CPF solution (4, 5) now is better than (2, 6), thereby violating Property 3. The reason is that the boundary of the feasible region goes down from (2, 6) to $(\frac{8}{3}, 5)$ and then "bends outward" to (4, 5), beyond the objective function line passing through (2, 6).

The key point is that the kind of situation illustrated in Fig. 5.3 can never occur in linear programming. The feasible region in Fig. 5.3 implies that the $2x_2 \leq 12$ and $3x_1 + 2x_2 \leq 18$ constraints apply for $0 \leq x_1 \leq \frac{8}{3}$. However, under the condition that $\frac{8}{3} \leq x_1 \leq 4$, the $3x_1 + 2x_2 \leq 18$ constraint is dropped and replaced by $x_2 \leq 5$. Such "conditional constraints" just are not allowed in linear programming.

The basic reason that Property 3 holds for any linear programming problem is that the feasible region always has the property of being a *convex set*[2], as defined in Appendix 2 and illustrated in several figures there. For two-variable linear programming problems, this convex property means that the *angle* inside the feasible region at *every* CPF solution is less than 180°. This property is illustrated in Fig. 5.1, where the angles at (0, 0), (0, 6), and (4, 0) are 90° and those at (2, 6) and (4, 3) are between 90° and 180°. By contrast, the feasible region in Fig. 5.3 is *not* a convex set, because the angle at $(\frac{8}{3}, 5)$ is more than 180°. This is the kind of "bending outward" at an angle greater than 180° that can never occur in linear programming. In higher dimensions, the same intuitive notion of "never bending outward" (a basic property of a convex set) continues to apply.

[2]If you already are familiar with convex sets, note that the set of solutions that satisfy any linear programming constraint (whether it be an inequality or equality constraint) is a convex set. For any linear programming problem, its feasible region is the *intersection* of the sets of solutions that satisfy its individual constraints. Since the intersection of convex sets is a convex set, this feasible region necessarily is a convex set.

To clarify the significance of a convex feasible region, consider the objective functic hyperplane that passes through a CPF solution that has no adjacent CPF solutions that a better. [In the original Wyndor Glass Co. example, this hyperplane is the objective functic line passing through (2, 6).] All these adjacent solutions [(0, 6) and (4, 3) in the exampl must lie either on the hyperplane or on the unfavorable side (as measured by Z) of tl hyperplane. The feasible region being convex means that its boundary cannot "bend ou ward" beyond an adjacent CPF solution to give another CPF solution that lies on the f vorable side of the hyperplane. So Property 3 holds.

Extensions to the Augmented Form of the Problem

For any linear programming problem in our standard form (including functional constrain in $\le$ form), the appearance of the functional constraints after slack variables are intr duced is as follows:

$$
\begin{array}{lllll}
(1) & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & + \; x_{n+1} & & = b_1 \\
(2) & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & & + \; x_{n+2} & = b_2 \\
\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & \\
(m) & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & & + \; x_{n+m} & = b_m,
\end{array}
$$

where $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ are the slack variables. For other linear programming prol lems, Sec. 4.6 described how essentially this same appearance (proper form from Gaussi elimination) can be obtained by introducing artificial variables, etc. Thus, the original s lutions $(x_1, x_2, \ldots, x_n)$ now are augmented by the corresponding values of the slack artificial variables $(x_{n+1}, x_{n+2}, \ldots, x_{n+m})$ and perhaps some surplus variables as well. Th augmentation led in Sec. 4.2 to defining **basic solutions** as *augmented corner-point solutio* and **basic feasible solutions (BF solutions)** as *augmented CPF solutions*. Consequently, tl preceding three properties of CPF solutions also hold for BF solutions.

Now let us clarify the algebraic relationships between basic solutions and corner-poi solutions. Recall that each corner-point solution is the simultaneous solution of a syste of n constraint boundary equations, which we called its *defining equations*. The key que tion is: How do we tell whether a particular constraint boundary equation is one of tl defining equations when the problem is in augmented form? The answer, fortunately, a simple one. Each constraint has an **indicating variable** that completely indicates (t whether its value is zero) whether that constraint's boundary equation is satisfied by tl current solution. A summary appears in Table 5.3. For the type of constraint in each ro

TABLE 5.3 Indicating variables for constraint boundary equations*

Type of Constraint	Form of Constraint	Constraint in Augmented Form	Constraint Boundary Equation	Indicating Variable
Nonnegativity	$x_j \ge 0$	$x_j \ge 0$	$x_j = 0$	x_j
Functional ($\le$)	$\sum_{j=1}^{n} a_{ij}x_j \le b_i$	$\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i$	$\sum_{j=1}^{n} a_{ij}x_j = b_i$	x_{n+i}
Functional ($=$)	$\sum_{j=1}^{n} a_{ij}x_j = b_i$	$\sum_{j=1}^{n} a_{ij}x_j + \bar{x}_{n+i} = b_i$	$\sum_{j=1}^{n} a_{ij}x_j = b_i$	$\bar{x}_{n+i}$
Functional ($\ge$)	$\sum_{j=1}^{n} a_{ij}x_j \ge b_i$	$\sum_{j=1}^{n} a_{ij}x_j + \bar{x}_{n+i} - x_{s_i} = b_i$	$\sum_{j=1}^{n} a_{ij}x_j = b_i$	$\bar{x}_{n+i} - x_{s_i}$

*Indicating variable $= 0 \Rightarrow$ constraint boundary equation satisfied;
indicating variable $\ne 0 \Rightarrow$ constraint boundary equation violated.

of the table, note that the corresponding constraint boundary equation (fourth column) is satisfied if and only if this constraint's indicating variable (fifth column) equals zero. In the last row (functional constraint in $\geq$ form), the indicating variable $\bar{x}_{n+i} - x_{s_i}$ actually is the difference between the artificial variable $\bar{x}_{n+i}$ and the surplus variable x_{s_i}.

Thus, whenever a constraint boundary equation is one of the defining equations for a corner-point solution, its indicating variable has a value of zero in the augmented form of the problem. Each such indicating variable is called a *nonbasic variable* for the corresponding basic solution. The resulting conclusions and terminology (already introduced in Sec. 4.2) are summarized next.

> Each **basic solution** has m **basic variables,** and the rest of the variables are **nonbasic variables** set equal to zero. (The number of nonbasic variables equals n plus the number of surplus variables.) The values of the **basic variables** are given by the simultaneous solution of the system of m equations for the problem in augmented form (after the nonbasic variables are set to zero). This basic solution is the augmented corner-point solution whose n defining equations are those indicated by the nonbasic variables. In particular, whenever an indicating variable in the fifth column of Table 5.3 is a nonbasic variable, the constraint boundary equation in the fourth column is a defining equation for the corner-point solution. (For functional constraints in $\geq$ form, at least one of the two supplementary variables $\bar{x}_{n+i}$ and x_{s_i} always is a nonbasic variable, but the constraint boundary equation becomes a defining equation only if *both* of these variables are nonbasic variables.)

Now consider the basic *feasible* solutions. Note that the only requirements for a solution to be feasible in the augmented form of the problem are that it satisfy the system of equations and that *all* the variables be *nonnegative.*

> A **BF solution** is a basic solution where all m basic variables are nonnegative (≥ 0).
> A BF solution is said to be **degenerate** if any of these m variables equals zero.

Thus, it is possible for a variable to be zero and still not be a nonbasic variable for the current BF solution. (This case corresponds to a CPF solution that satisfies another constraint boundary equation in addition to its n defining equations.) Therefore, it is necessary to keep track of which is the current set of nonbasic variables (or the current set of basic variables) rather than to rely upon their zero values.

We noted earlier that not every system of n constraint boundary equations yields a corner-point solution, because the system may have no solution or it may have multiple solutions. For analogous reasons, not every set of n nonbasic variables yields a basic solution. However, these cases are avoided by the simplex method.

To illustrate these definitions, consider the Wyndor Glass Co. example once more. Its constraint boundary equations and indicating variables are shown in Table 5.4.

TABLE 5.4 Indicating variables for the constraint boundary equations of the Wyndor Glass Co. problem*

Constraint	Constraint in Augmented Form	Constraint Boundary Equation	Indicating Variable
$x_1 \geq 0$	$x_1 \geq 0$	$x_1 = 0$	x_1
$x_2 \geq 0$	$x_2 \geq 0$	$x_2 = 0$	x_2
$x_1 \leq 4$	(1) $x_1 + x_3 = 4$	$x_1 = 4$	x_3
$2x_2 \leq 12$	(2) $2x_2 + x_4 = 12$	$2x_2 = 12$	x_4
$3x_1 + 2x_2 \leq 18$	(3) $3x_1 + 2x_2 + x_5 = 18$	$3x_1 + 2x_2 = 18$	x_5

*Indicating variable $= 0 \Rightarrow$ constraint boundary equation satisfied;
indicating variable $\neq 0 \Rightarrow$ constraint boundary equation violated.

Augmenting each of the CPF solutions (see Table 5.1) yields the BF solutions listed in Table 5.5. This table places adjacent BF solutions next to each other, except for the pair consisting of the first and last solutions listed. Notice that in each case the nonbasic variables necessarily are the indicating variables for the defining equations. Thus, adjacent BF solutions differ by having just one different nonbasic variable. Also notice that each BF solution is the simultaneous solution of the system of equations for the problem in augmented form (see Table 5.4) when the nonbasic variables are set equal to zero.

Similarly, the three corner-point *infeasible* solutions (see Table 5.2) yield the three basic *infeasible* solutions shown in Table 5.6.

The other two sets of nonbasic variables, (1) x_1 and x_3 and (2) x_2 and x_4, do not yield a basic solution, because setting either pair of variables equal to zero leads to having no solution for the system of Eqs. (1) to (3) given in Table 5.4. This conclusion parallels the observation we made early in this section that the corresponding sets of constraint boundary equations do not yield a solution.

The *simplex method* starts at a BF solution and then iteratively moves to a better adjacent BF solution until an optimal solution is reached. At each iteration, how is the adjacent BF solution reached?

For the original form of the problem, recall that an adjacent CPF solution is reached from the current one by (1) deleting one constraint boundary (defining equation) from the set of n constraint boundaries defining the current solution, (2) moving away from the current solution in the feasible direction along the intersection of the remaining $n - 1$ constraint boundaries (an edge of the feasible region), and (3) stopping when the *first* new constraint boundary (defining equation) is reached.

TABLE 5.5 BF solutions for the Wyndor Glass Co. problem

CPF Solution	Defining Equations	BF Solution	Nonbasic Variables
(0, 0)	$x_1 = 0$ $x_2 = 0$	(0, 0, 4, 12, 18)	x_1 x_2
(0, 6)	$x_1 = 0$ $2x_2 = 12$	(0, 6, 4, 0, 6)	x_1 x_4
(2, 6)	$2x_2 = 12$ $3x_1 + 2x_2 = 18$	(2, 6, 2, 0, 0)	x_4 x_5
(4, 3)	$3x_1 + 2x_2 = 18$ $x_1 = 4$	(4, 3, 0, 6, 0)	x_5 x_3
(4, 0)	$x_1 = 4$ $x_2 = 0$	(4, 0, 0, 12, 6)	x_3 x_2

TABLE 5.6 Basic infeasible solutions for the Wyndor Glass Co. problem

Corner-Point Infeasible Solution	Defining Equations	Basic Infeasible Solution	Nonbasic Variables
(0, 9)	$x_1 = 0$ $3x_1 + 2x_2 = 18$	(0, 9, 4, −6, 0)	x_1 x_5
(4, 6)	$2x_2 = 12$ $x_1 = 4$	(4, 6, 0, 0, −6)	x_4 x_3
(6, 0)	$3x_1 + 2x_2 = 18$ $x_2 = 0$	(6, 0, −2, 12, 0)	x_5 x_2

TABLE 5.7 Sequence of solutions obtained by the simplex method for the Wyndor Glass Co. problem

Iteration	CPF Solution	Defining Equations	BF Solution	Nonbasic Variables	Functional Constraints in Augmented Form
0	(0, 0)	$x_1 = 0$ $x_2 = 0$	(0, 0, 4, 12, 18)	$x_1 = 0$ $x_2 = 0$	$x_1 + \mathbf{x_3} = 4$ $2x_2 + \mathbf{x_4} = 12$ $3x_1 + 2x_2 + \mathbf{x_5} = 18$
1	(0, 6)	$x_1 = 0$ $2x_2 = 12$	(0, 6, 4, 0, 6)	$x_1 = 0$ $x_4 = 0$	$x_1 + \mathbf{x_3} = 4$ $2\mathbf{x_2} + x_4 = 12$ $3x_1 + 2\mathbf{x_2} + \mathbf{x_5} = 18$
2	(2, 6)	$2x_2 = 12$ $3x_1 + 2x_2 = 18$	(2, 6, 2, 0, 0)	$x_4 = 0$ $x_5 = 0$	$\mathbf{x_1} + \mathbf{x_3} = 4$ $2\mathbf{x_2} + x_4 = 12$ $3\mathbf{x_1} + 2\mathbf{x_2} + x_5 = 18$

Equivalently, in our new terminology, the simplex method reaches an adjacent BF solution from the current one by (1) deleting one variable (the entering basic variable) from the set of n nonbasic variables defining the current solution, (2) moving away from the current solution by *increasing* this one variable from zero (and adjusting the other basic variables to still satisfy the system of equations) while keeping the remaining $n - 1$ nonbasic variables at zero, and (3) stopping when the *first* of the basic variables (the leaving basic variable) reaches a value of zero (its constraint boundary). With either interpretation, the choice among the n alternatives in step 1 is made by selecting the one that would give the best rate of improvement in Z (per unit increase in the entering basic variable) during step 2.

Table 5.7 illustrates the close correspondence between these geometric and algebraic interpretations of the simplex method. Using the results already presented in Secs. 4.3 and 4.4, the fourth column summarizes the sequence of BF solutions found for the Wyndor Glass Co. problem, and the second column shows the corresponding CPF solutions. In the third column, note how each iteration results in deleting one constraint boundary (defining equation) and substituting a new one to obtain the new CPF solution. Similarly, note in the fifth column how each iteration results in deleting one nonbasic variable and substituting a new one to obtain the new BF solution. Furthermore, the nonbasic variables being deleted and added are the indicating variables for the defining equations being deleted and added in the third column. The last column displays the initial system of equations [excluding Eq. (0)] for the augmented form of the problem, with the current basic variables shown in bold type. In each case, note how setting the nonbasic variables equal to zero and then solving this system of equations for the basic variables must yield the same solution for (x_1, x_2) as the corresponding pair of defining equations in the third column.

The Worked Examples section of the book's website provides **another example** of developing the type of information given in Table 5.7 for a minimization problem.

5.2 THE SIMPLEX METHOD IN MATRIX FORM

Chapter 4 describes the simplex method in both an algebraic form and a tabular form. Further insight into the theory and power of the simplex method can be obtained by examining its *matrix* form. We begin by introducing matrix notation to represent linear programming problems. (See Appendix 4 for a review of matrices.).

To help you distinguish between matrices, vectors, and scalars, we consistently use **BOLDFACE CAPITAL** letters to represent matrices, **boldface lowercase** letters to represent vectors, and *italicized* letters in ordinary print to represent scalars. We also use a boldface zero (**0**) to denote a *null vector* (a vector whose elements all are zero) in either column or row form (which one should be clear from the context), whereas a zero in ordinary print (0) continues to represent the number zero.

Using matrices, our standard form for the general linear programming model given in Sec. 3.2 becomes

$$\boxed{\begin{array}{l} \text{Maximize} \qquad Z = \mathbf{cx}, \\ \text{subject to} \\ \mathbf{Ax} \le \mathbf{b} \qquad \text{and} \qquad \mathbf{x} \ge \mathbf{0}, \end{array}}$$

where **c** is the row vector

$$\mathbf{c} = [c_1, c_2, \ldots, c_n],$$

x, b, and **0** are the column vectors such that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and **A** is the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

To obtain the *augmented form* of the problem, introduce the column vector of slack variables

$$\mathbf{x}_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

so that the constraints become

$$[\mathbf{A}, \mathbf{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b} \qquad \text{and} \qquad \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} \ge \mathbf{0},$$

where **I** is the $m \times m$ identity matrix, and the null vector **0** now has $n + m$ elements. (We comment at the end of the section about how to deal with problems that are not in our standard form.)

Solving for a Basic Feasible Solution

Recall that the general approach of the simplex method is to obtain a sequence of *improving BF solutions* until an optimal solution is reached. One of the key features of the matrix form of the simplex method involves the way in which it solves for each new

BF solution after identifying its basic and nonbasic variables. Given these variables, the resulting basic solution is the solution of the m equations

$$[\mathbf{A}, \mathbf{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b},$$

in which the n *nonbasic variables* from the $n + m$ elements of

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix}$$

are set equal to zero. Eliminating these n variables by equating them to zero leaves a set of m equations in m unknowns (the *basic variables*). This set of equations can be denoted by

$$\mathbf{B}\mathbf{x}_B = \mathbf{b},$$

where the **vector of basic variables**

$$\mathbf{x}_B = \begin{bmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bm} \end{bmatrix}$$

is obtained by eliminating the nonbasic variables from

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix},$$

and the **basis matrix**

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1m} \\ B_{21} & B_{22} & \dots & B_{2m} \\ \dots & \dots & \dots & \dots \\ B_{m1} & B_{m2} & \dots & B_{mm} \end{bmatrix}$$

is obtained by eliminating the columns corresponding to coefficients of nonbasic variables from $[\mathbf{A}, \mathbf{I}]$. (In addition, the elements of $\mathbf{x}_B$ and, therefore, the columns of $\mathbf{B}$ may be placed in a different order when the simplex method is executed.)

The simplex method introduces only basic variables such that $\mathbf{B}$ is *nonsingular,* so that $\mathbf{B}^{-1}$ always will exist. Therefore, to solve $\mathbf{B}\mathbf{x}_B = \mathbf{b}$, both sides are premultiplied by $\mathbf{B}^{-1}$:

$$\mathbf{B}^{-1}\mathbf{B}\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}.$$

Since $\mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$, the desired solution for the basic variables is

$$\boxed{\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}.}$$

Let $\mathbf{c}_B$ be the vector whose elements are the objective function coefficients (including zeros for slack variables) for the corresponding elements of $\mathbf{x}_B$. The value of the objective function for this basic solution is then

$$\boxed{Z = \mathbf{c}_B\mathbf{x}_B = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}.}$$

Example. To illustrate this method of solving for a BF solution, consider again the Wyndor Glass Co. problem presented in Sec. 3.1 and solved by the original simplex method in Table 4.8. In this case,

$$\mathbf{c} = [3, 5], \qquad [\mathbf{A}, \mathbf{I}] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{x}_s = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

Referring to Table 4.8, we see that the sequence of BF solutions obtained by the simplex method is the following:

Iteration 0

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{B}^{-1}, \qquad \text{so} \qquad \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix},$$

$$\mathbf{c}_B = [0, 0, 0], \qquad \text{so} \qquad Z = [0, 0, 0] \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 0.$$

Iteration 1

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \qquad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

so

$$\begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix},$$

$$\mathbf{c}_B = [0, 5, 0], \qquad \text{so} \qquad Z = [0, 5, 0] \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} = 30.$$

Iteration 2

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}, \qquad \mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

so

$$\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix},$$

$$\mathbf{c}_B = [0, 5, 3], \qquad \text{so} \qquad Z = [0, 5, 3] \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = 36.$$

Matrix Form of the Current Set of Equations

The last preliminary before we summarize the matrix form of the simplex method is to show the matrix form of the set of equations appearing in the simplex tableau for any iteration of the original simplex method.

For the *original* set of equations, the matrix form is

$$\boxed{\begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}.}$$

This set of equations also is exhibited in the first simplex tableau of Table 5.8.

The algebraic operations performed by the simplex method (multiply an equation by a constant and add a multiple of one equation to another equation) are expressed in matrix form by premultiplying both sides of the original set of equations by the appropriate matrix. This matrix would have the same elements as the identity matrix, *except* that each multiple for an algebraic operation would go into the spot needed to have the matrix multiplication perform this operation. Even after a series of algebraic operations over several iterations, we still can deduce what this matrix must be (symbolically) for the entire series by using what we already know about the right-hand sides of the new set of equations. In particular, after any iteration, $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ and $Z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$, so the right-hand sides of the new set of equations have become

$$\begin{bmatrix} Z \\ \mathbf{x}_B \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}.$$

Because we perform the same series of algebraic operations on *both* sides of the original set of equations, we use this same matrix that premultiplies the original right-hand side to premultiply the original left-hand side. Consequently, since

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix},$$

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Coefficient of: Original Variables	Coefficient of: Slack Variables	Right Side
0	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 $\mathbf{0}$	$-\mathbf{c}$ $\mathbf{A}$	$\mathbf{0}$ $\mathbf{I}$	0 $\mathbf{b}$
		⋮	⋮	⋮	⋮	⋮
Any	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 $\mathbf{0}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$ $\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{c}_B\mathbf{B}^{-1}$ $\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ $\mathbf{B}^{-1}\mathbf{b}$

the desired matrix form of the *set of equations after any iteration* is

$$\boxed{\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}.}$$

The second simplex tableau of Table 5.8 also exhibits this same set of equations.

Example. To illustrate this matrix form for the current set of equations, we will show how it yields the final set of equations resulting from iteration 2 for the Wyndor Glass Co. problem. Using the $\mathbf{B}^{-1}$ and $\mathbf{c}_B$ given for iteration 2 at the end of the preceding subsection, we have

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 3] \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = [0, \tfrac{3}{2}, 1],$$

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [0, 5, 3] \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} - [3, 5] = [0, 0].$$

Also, by using the values of $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ and $Z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ calculated at the end of the preceding subsection, these results give the following set of equations:

$$\left[\begin{array}{c|cc|ccc} 1 & 0 & 0 & 0 & \frac{3}{2} & 1 \\ \hline 0 & 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \begin{bmatrix} Z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 36 \\ 2 \\ 6 \\ 2 \end{bmatrix},$$

as shown in the final simplex tableau in Table 4.8.

The matrix form of the set of equations after any iteration (as shown in the box just before the above example) provides the key to the execution of the matrix form of the simplex method. The matrix expressions shown in these equations (or in the bottom part of Table 5.8) provide a direct way of calculating all the numbers that would appear in the current set of equations (for the algebraic form of the simplex method) or in the current simplex tableau (for the tableau form of the simplex method). The three forms of the simplex method make exactly the same decisions (entering basic variable, leaving basic variable, etc.) step after step and iteration after iteration. The only difference between these forms is in the methods used to calculate the numbers needed to make those decisions. As summarized below, the matrix form provides a convenient and compact way of calculating these numbers without carrying along a series of systems of equations or a series of simplex tableaux.

Summary of the Matrix Form of the Simplex Method

1. *Initialization*: Introduce slack variables, etc., to obtain the initial basic variables, as described in Chap. 4. This yields the initial $\mathbf{x}_B$, $\mathbf{c}_B$, $\mathbf{B}$, and $\mathbf{B}^{-1}$ (where $\mathbf{B} = \mathbf{I} = \mathbf{B}^{-1}$ under our current assumption that the problem being solved fits our standard form). Then go to the optimality test.

2. *Iteration*:

Step 1. Determine the entering basic variable: Refer to the coefficients of the *nonbasic* variables in Eq. (0) that were obtained in the preceding application of the optimality test below. Then (just as described in Sec. 4.4), select the variable with the *negative coefficient* having the largest absolute value as the entering basic variable.

Step 2. Determine the leaving basic variable: Use the matrix expressions, $\mathbf{B}^{-1}\mathbf{A}$ (for the coefficients of the original variables) and $\mathbf{B}^{-1}$ (for the coefficients of the slack variables), to calculate the coefficients of the entering basic variable in every equation except Eq. (0). Also use the preceding calculation of $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ (see Step 3) to identify the right-hand sides of these equations. Then (just as described in Sec. 4.4), use the *minimum ratio test* to select the leaving basic variable.

Step 3. Determine the new BF solution: Update the basis matrix $\mathbf{B}$ by replacing the column for the leaving basic variable by the corresponding column in $[\mathbf{A}, \mathbf{I}]$ for the entering basic variable. Also make the corresponding replacements in $\mathbf{x}_B$ and $\mathbf{c}_B$. Then derive $\mathbf{B}^{-1}$ (as illustrated in Appendix 4) and set $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$.

3. *Optimality test*: Use the matrix expressions, $\mathbf{c}_B \mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$ (for the coefficients of the original variables) and $\mathbf{c}_B \mathbf{B}^{-1}$ (for the coefficients of the slack variables), to calculate the coefficients of the nonbasic variables in Eq. (0). The current BF solution is optimal if and only if all of these coefficients are nonnegative. If it is optimal, stop. Otherwise, go to an iteration to obtain the next BF solution.

Example. We already have performed some of the above matrix calculations for the Wyndor Glass Co. problem earlier in this section. We now will put all the pieces together in applying the full simplex method in matrix form to this problem. As a starting point, recall that

$$\mathbf{c} = [3, 5], \qquad [\mathbf{A}, \mathbf{I}] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}.$$

Initialization

The initial basic variables are the slack variables, so (as already noted for Iteration 0 for the first example in this section)

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad \mathbf{c}_B = [0, 0, 0], \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \mathbf{B}^{-1}.$$

Optimality test
The coefficients of the nonbasic variables (x_1 and x_2) are

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [0, 0] - [3, 5] = [-3, -5]$$

so these negative coefficients indicate that the initial BF solution ($\mathbf{x}_B = \mathbf{b}$) is not optimal.

Iteration 1
Since -5 is larger in absolute value than -3, the entering basic variable is x_2. Performing only the relevant portion of a matrix multiplication, the coefficients of x_2 in every equation except Eq. (0) are

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} — & 0 \\ — & 2 \\ — & 2 \end{bmatrix}$$

and the right-hand side of these equations are given by the value of $\mathbf{x}_B$ shown in the initialization step. Therefore, the minimum ratio test indicates that the leaving basic variable is x_4 since $12/2 < 18/2$. Iteration 1 for the first example in this section already shows the resulting updated $\mathbf{B}$, $\mathbf{x}_B$, $\mathbf{c}_B$, and $\mathbf{B}^{-1}$, namely,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}, \mathbf{c}_B = [0, 5, 0],$$

so x_2 has replaced x_4 in $\mathbf{x}_B$, in providing an element of $\mathbf{c}_B$ from [3, 5, 0, 0, 0], and in providing a column from **[A, I]** in **B.**

Optimality test
The nonbasic variables now are x_1 and x_4 , and their coefficients in Eq. (0) are

For x_1: $$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [0, 5, 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} - [3, 5] = [-3, —, —]$$

For x_4: $$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} = [—, 5/2, —]$$

Since x_1 has a negative coefficient, the current BF is not optimal, so we go on to the next iteration.

Iteration 2:
Since x_1 is the one nonbasic variable with a negative coefficient in Eq. (0), it now becomes the entering basic variable. Its coefficients in the other equations are

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & — \\ 0 & — \\ 3 & — \end{bmatrix}$$

Also using $\mathbf{x}_B$ obtained at the end of the preceding iteration, the minimum ratio test indicates that x_5 is the leaving basic variable since $6/3 < 4/1$. Iteration 2 for the first example in this section already shows the resulting updated $\mathbf{B}$, $\mathbf{B}^{-1}$, $\mathbf{x}_B$, and $\mathbf{c}_B$, namely,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}, \mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}, \mathbf{c}_B = [0, 5, 3],$$

so x_1 has replaced x_5 in $\mathbf{x}_B$, in providing an element of $\mathbf{c}_B$ from [3, 5, 0, 0, 0], and in providing a column from **[A,I]** in **B**.

Optimality test
The nonbasic variables now are x_4 and x_5. Using the calculations already shown for the second example in this section, their coefficients in Eq. (0) are 3/2 and 1, respectively. Since neither of these coefficients are negative, the current BF solution ($x_1 = 2$, $x_2 = 6$, $x_3 = 2$, $x_4 = 0$, $x_5 = 0$) is optimal and the procedure terminates.

Final Observations

The above example illustrates that the matrix form of the simplex method uses just a few matrix expressions to perform all the needed calculations. These matrix expressions are summarized in the bottom part of Table 5.8. A fundamental insight from this table is that it is only necessary to know the current $\mathbf{B}^{-1}$ and $\mathbf{c}_B\mathbf{B}^{-1}$, which appear in the slack variables portion of the current simplex tableau, in order to calculate all the other numbers in this tableau in terms of the original parameters (**A, b,** and **c**) of the model being solved. When dealing with the *final* simplex tableau, this insight proves to be a particularly valuable one, as will be described in the next section.

A drawback of the matrix form of the simplex method as it has been outlined in this section is that it is necessary to derive $\mathbf{B}^{-1}$, the inverse of the updated basis matrix, at the end of each iteration. Although routines are available for inverting small square (nonsingular) matrices (and this can even be done readily by hand for 2 x 2 or perhaps 3 x 3 matrices), the time required to invert matrices grows very rapidly with the size of the matrices. Fortunately, there is a much more efficient procedure available for updating $\mathbf{B}^{-1}$ from one iteration to the next rather than inverting the new basis matrix from scratch. When this procedure is incorporated into the matrix form of the simplex method, this improved version of the matrix form is conventionally called the **revised simplex method.** This is the version of the simplex method (along with further improvements) that normally is used in commercial software for linear programming. We will describe the procedure for updating $\mathbf{B}^{-1}$ in Sec. 5.4.

The Worked Examples section of the book's website gives **another example** of applying the matrix form of the simplex method. This example also incorporates the efficient procedure for updating $\mathbf{B}^{-1}$ at each iteration instead of inverting the updated basis matrix from scratch, so the full-fledged revised simplex method is applied.

Finally, we should remind you that the description of the matrix form of the simplex method throughout this section has assumed that the problem being solved fits *our standard form* for the general linear programming model given in Sec. 3.2. However, the modifications for other forms of the model are relatively straightforward. The initialization step would be conducted just as was described in Sec. 4.6 for either the algebraic form or tabular form of the simplex method. When this step involves introducing artificial variables to obtain an initial BF solution (and thereby to obtain an *identity matrix as the initial basis matrix*), these variables are included among the m elements of $\mathbf{x}_s$.

5.3 A FUNDAMENTAL INSIGHT

We shall now focus on a property of the simplex method (in any form) that has been revealed by the matrix form of the simplex method in Sec. 5.2. This fundamental insight provides the key to both duality theory and sensitivity analysis (Chap. 6), two very important parts of linear programming.

We shall first describe this insight when the problem being solved fits *our standard form* for linear programming models (Sec. 3.2) and then discuss how to adapt to other forms later. The insight is based directly on Table 5.8 in Sec. 5.2, as described below.

> **The insight provided by Table 5.8:** Using matrix notation, Table 5.8 gives the rows of the *initial* simplex tableau as $[-\mathbf{c}, \mathbf{0}, 0]$ for row 0 and $[\mathbf{A}, \mathbf{I}, \mathbf{b}]$ for the rest of the rows. After any iteration, the coefficients of the slack variables in the current simplex tableau become $\mathbf{c}_B\mathbf{B}^{-1}$ for row 0 and $\mathbf{B}^{-1}$ for the rest of the rows, where $\mathbf{B}$ is the current basis matrix. Examining the rest of the current simplex tableau, the insight is that these coefficients of the slack variables immediately reveal how the *entire* rows of the current simplex tableau have been obtained from the rows in the *initial* simplex tableau. In particular, after any iteration,
>
> $$\text{Row } 0 = [-\mathbf{c}, \mathbf{0}, 0] + \mathbf{c}_B\mathbf{B}^{-1}[\mathbf{A}, \mathbf{I}, \mathbf{b}]$$
> $$\text{Rows } 1 \text{ to } m = \mathbf{B}^{-1}[\mathbf{A}, \mathbf{I}, \mathbf{b}]$$

We shall describe the applications of this insight at the end of this section. These applications are particularly important only when we are dealing with the *final* simplex tableau after the optimal solution has been obtained. Therefore, we will focus hereafter on discussing the "fundamental insight" just in terms of the optimal solution.

To distinguish between the matrix notation used after *any* iteration ($\mathbf{B}^{-1}$, etc.) and the corresponding notation after just the *last* iteration, we now introduce the following notation for the latter case.

When $\mathbf{B}$ is the basis matrix for the *optimal solution* found by the simplex method, let

$\mathbf{S}^* = \mathbf{B}^{-1}$ = coefficients of the *slack* variables in rows 1 to m

$\mathbf{A}^* = \mathbf{B}^{-1}\mathbf{A}$ = coefficients of the *original* variables in rows 1 to m

$\mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1}$ = coefficients of the *slack* variables in row 0

$\mathbf{z}^* = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}$, so $\mathbf{z}^* - \mathbf{c}$ = coefficients of the *original* variables in row 0

$Z^* = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ = optimal value of the objective function

$\mathbf{b}^* = \mathbf{B}^{-1}\mathbf{b}$ = optimal right-hand sides of rows 1 to m

The bottom half of Table 5.9 shows where each of these symbols fits in the final simplex tableau. To illustrate all the notation, the top half of Table 5.9 includes the initial tableau for the Wyndor Glass Co. problem and the bottom half includes the final tableau for this problem.

Referring to this again, suppose now that you are given the initial tableau, $\mathbf{t}$ and $\mathbf{T}$, and just $\mathbf{y}^*$ and $\mathbf{S}^*$ from the final tableau. How can this information alone be used to calculate the rest of the final tableau? The answer is provided by the fundamental insight summarized below.

Fundamental Insight

(1) $\mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T} = [\mathbf{y}^*\mathbf{A} - \mathbf{c} \mid \mathbf{y}^* \mid \mathbf{y}^*\mathbf{b}]$.

(2) $\mathbf{T}^* = \mathbf{S}^*\mathbf{T} = [\mathbf{S}^*\mathbf{A} \mid \mathbf{S}^* \mid \mathbf{S}^*\mathbf{b}]$.

TABLE 5.9 General notation for initial and final simplex tableaux in matrix form, illustrated by the Wyndor Glass Co. problem

Initial Tableau	
Row 0:	$\mathbf{t} = [-3, -5 \mid 0, 0, 0 \mid 0] = [-\mathbf{c} \mid \mathbf{0} \mid 0]$.
Other rows:	$\mathbf{T} = \left[\begin{array}{cc\|ccc\|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}]$.
Combined:	$\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c\|c\|c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right]$.
Final Tableau	
Row 0:	$\mathbf{t}^* = [0, 0 \mid 0, \frac{3}{2}, 1 \mid 36] = [\mathbf{z}^* - \mathbf{c} \mid \mathbf{y}^* \mid Z^*]$.
Other rows:	$\mathbf{T}^* = \left[\begin{array}{cc\|ccc\|c} 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 2 \end{array}\right] = [\mathbf{A}^* \mid \mathbf{S}^* \mid \mathbf{b}^*]$.
Combined:	$\begin{bmatrix} \mathbf{t}^* \\ \mathbf{T}^* \end{bmatrix} = \left[\begin{array}{c\|c\|c} \mathbf{z}^* - \mathbf{c} & \mathbf{y}^* & Z^* \\ \mathbf{A}^* & \mathbf{S}^* & \mathbf{b}^* \end{array}\right]$.

Thus, by knowing the parameters of the model in the initial tableau (**c**, **A**, and **b**) and *only* the coefficients of the slack variables in the final tableau ($\mathbf{y}^*$ and $\mathbf{S}^*$), these equations enable calculating *all* the other numbers in the final tableau.

Now let us summarize the mathematical logic behind the two equations for the fundamental insight. To derive Eq. (2), recall that the entire sequence of algebraic operations performed by the simplex method (excluding those involving row 0) is equivalent to premultiplying **T** by some matrix, call it **M.** Therefore,

$$\mathbf{T}^* = \mathbf{MT},$$

but now we need to identify **M.** By writing out the component parts of **T** and $\mathbf{T}^*$, this equation becomes

$$\begin{aligned}[\mathbf{A}^* \mid \mathbf{S}^* \mid \mathbf{b}^*] &= \mathbf{M}\,[\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}] \\ &= [\mathbf{MA} \mid \mathbf{M} \mid \mathbf{Mb}].\end{aligned}$$

Because the middle (or any other) component of these equal matrices must be the same, it follows that $\mathbf{M} = \mathbf{S}^*$, so Eq. (2) is a valid equation.

Equation (1) is derived in a similar fashion by noting that the entire sequence of algebraic operations involving row 0 amounts to adding some linear combination of the rows in **T** to **t,** which is equivalent to adding to **t** some *vector* times **T.** Denoting this vector by **v,** we thereby have

$$\mathbf{t}^* = \mathbf{t} + \mathbf{vT},$$

but **v** still needs to be identified. Writing out the component parts of **t** and $\mathbf{t}^*$ yields

$$\begin{aligned}[\mathbf{z}^* - \mathbf{c} \mid \mathbf{y}^* \mid Z^*] &= [-\mathbf{c} \mid \mathbf{0} \mid 0] + \mathbf{v}\,[\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}] \\ &= [-\mathbf{c} + \mathbf{vA} \mid \mathbf{v} \mid \mathbf{vb}].\end{aligned}$$

Equating the middle component of these equal vectors gives $\mathbf{v} = \mathbf{y}^*$, which validates Eq. (1).

Adapting to Other Model Forms

Thus far, the fundamental insight has been described under the assumption that the original model is in our standard form, described in Sec. 3.2. However, the above mathematical logic now reveals just what adjustments are needed for other forms of the original model. The key is the identity matrix $\mathbf{I}$ in the initial tableau, which turns into $\mathbf{S}^*$ in the final tableau. If some artificial variables must be introduced into the initial tableau to serve as initial basic variables, then it is the set of columns (appropriately ordered) for *all* the initial basic variables (both slack and artificial) that forms $\mathbf{I}$ in this tableau. (The columns for any surplus variables are extraneous.) The *same* columns in the final tableau provide $\mathbf{S}^*$ for the $\mathbf{T}^* = \mathbf{S}^*\mathbf{T}$ equation and $\mathbf{y}^*$ for the $\mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T}$ equation. If M's were introduced into the preliminary row 0 as coefficients for artificial variables, then the $\mathbf{t}$ for the $\mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T}$ equation is the row 0 for the initial tableau after these nonzero coefficients for basic variables are algebraically eliminated. (Alternatively, the preliminary row 0 can be used for $\mathbf{t}$, but then these M's must be subtracted from the final row 0 to give $\mathbf{y}^*$.) (See Prob. 5.3-9.)

Applications

The fundamental insight has a variety of important applications in linear programming. One of these applications involves the revised simplex method, which is based mainly on the matrix form of the simplex method presented in Sec. 5.2. As described in this preceding section (see Table 5.8), this method used $\mathbf{B}^{-1}$ and the initial tableau to calculate all the relevant numbers in the current tableau for *every* iteration. It goes even further than the fundamental insight by using $\mathbf{B}^{-1}$ to calculate $\mathbf{y}^*$ itself as $\mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1}$.

Another application involves the interpretation of the *shadow prices* $(y_1^*, y_2^*, \dots, y_m^*)$ described in Sec. 4.7. The fundamental insight reveals that Z^* (the value of Z for the optimal solution) is

$$Z^* = \mathbf{y}^*\mathbf{b} = \sum_{i=1}^{m} y_i^* b_i,$$

so, e.g.,

$$Z^* = 0b_1 + \frac{3}{2}b_2 + b_3$$

for the Wyndor Glass Co. problem. This equation immediately yields the interpretation for the y_i^* values given in Sec. 4.7.

Another group of extremely important applications involves various *postoptimality tasks* (reoptimization technique, sensitivity analysis, parametric linear programming—described in Sec. 4.7) that investigate the effect of making one or more changes in the original model. In particular, suppose that the simplex method already has been applied to obtain an optimal solution (as well as $\mathbf{y}^*$ and $\mathbf{S}^*$) for the original model, and then these changes are made. If exactly the same sequence of algebraic operations were to be applied to the revised initial tableau, what would be the resulting changes in the final tableau? Because $\mathbf{y}^*$ and $\mathbf{S}^*$ don't change, the fundamental insight reveals the answer immediately.

One particularly common type of postoptimality analysis involves investigating possible changes in $\mathbf{b}$. The elements of $\mathbf{b}$ often represent managerial decisions about the amounts of various resources being made available to the activities under consideration in the linear programming model. Therefore, after the optimal solution has been obtained by the simplex method, management often wants to explore what would happen if some of these managerial decisions on resource allocations were to be changed in various ways. By using the formulas,

$$\begin{aligned}\mathbf{x}_B &= \mathbf{S}^*\mathbf{b}\\ \mathbf{Z}^* &= \mathbf{y}^*\mathbf{b},\end{aligned}$$

you can see exactly how the optimal BF solution changes (or whether it becomes infeasible because of negative variables), as well as how the optimal value of the objective function changes, as a function of **b**. You do *not* have to reapply the simplex method over and over for each new **b**, because the coefficients of the slack variables tell all!

For example, consider the change from $b_2 = 12$ to $b_2 = 13$ as illustrated in Fig. 4.8 for the Wyndor Glass Co. problem. It is not necessary to *solve* for the new optimal solution $(x_1, x_2) = (\frac{5}{3}, \frac{13}{2})$ because the values of the basic variables in the final tableau ($\mathbf{b}^*$) are immediately revealed by the fundamental insight:

$$\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \mathbf{b}^* = \mathbf{S}^*\mathbf{b} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 13 \\ 18 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{13}{2} \\ \frac{5}{3} \end{bmatrix}.$$

There is an even easier way to make this calculation. Since the only change is in the *second* component of **b** ($\Delta b_2 = 1$), which gets premultiplied by only the *second* column of $\mathbf{S}^*$, the *change* in $\mathbf{b}^*$ can be calculated as simply

$$\Delta\mathbf{b}^* = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ -\frac{1}{3} \end{bmatrix} \Delta b_2 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ -\frac{1}{3} \end{bmatrix},$$

so the original values of the basic variables in the final tableau ($x_3 = 2$, $x_2 = 6$, $x_1 = 2$) now become

$$\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{7}{13} \\ \frac{13}{2} \\ \frac{5}{3} \end{bmatrix}.$$

(If any of these new values were *negative,* and thus infeasible, then the reoptimization technique described in Sec. 4.7 would be applied, starting from this revised final tableau.) Applying *incremental analysis* to the preceding equation for Z^* also immediately yields

$$\Delta Z^* = \frac{3}{2}\Delta b_2 = \frac{3}{2}.$$

The fundamental insight can be applied to investigating other kinds of changes in the original model in a very similar fashion; it is the crux of the sensitivity analysis procedure described in the latter part of Chap. 6.

You also will see in the next chapter that the fundamental insight plays a key role in the very useful duality theory for linear programming.

5.4 THE REVISED SIMPLEX METHOD

The revised simplex method is based directly on the matrix form of the simplex method presented in Sec. 5.2. However, as mentioned at the end of that section, the difference is that the revised simplex method incorporates a key improvement into the matrix form. Instead of needing to invert the new basis matrix **B** after each iteration, which is computationally expensive for large matrices, the revised simplex method uses a much more efficient procedure that simply updates $\mathbf{B}^{-1}$ from one iteration to the next. We focus on describing and illustrating this procedure in this section.

This procedure is based on two properties of the simplex method. One is described in *the insight provided by Table 5.8* at the beginning of Sec. 5.3. In particular, after any iteration, the coefficients of the *slack variables* for all the rows except row 0 in the current simplex tableau become $\mathbf{B}^{-1}$, where $\mathbf{B}$ is the current basis matrix. This property always holds as long as the problem being solved fits *our standard form* described in Sec. 3.2 for linear programming models. (For nonstandard forms where artificial variables need to be introduced, the only difference is that it is the set of appropriately ordered columns that form an identity matrix $\mathbf{I}$ below row 0 in the initial simplex tableau that then provides $\mathbf{B}^{-1}$ in any subsequent tableau.)

The other relevant property of the simplex method is that step 3 of an iteration changes the numbers in the simplex tableau, including the numbers giving $\mathbf{B}^{-1}$, only by performing the elementary algebraic operations (such as dividing an equation by a constant or subtracting a multiple of some equation from another equation) that are needed to restore proper form from Gaussian elimination. Therefore, all that is needed to update $\mathbf{B}^{-1}$ from one iteration to the next is to obtain the new $\mathbf{B}^{-1}$ (denote it by $\mathbf{B}_{\text{new}}^{-1}$) from the old $\mathbf{B}^{-1}$ (denote it by $\mathbf{B}_{\text{old}}^{-1}$) by performing the usual algebraic operations on $\mathbf{B}_{\text{old}}^{-1}$ that the algebraic form of the simplex method would perform on the entire system of equations (except Eq. (0)) for this iteration. Thus, given the choice of the entering basic variable and leaving basic variable from steps 1 and 2 of an iteration, the procedure is to apply step 3 of an iteration (as described in Secs. 4.3 and 4.4) to the $\mathbf{B}^{-1}$ portion of the current simplex tableau or system of equations.

To describe this procedure formally, let

x_k = entering basic variable,

a'_{ik} = coefficient of x_k in current Eq. (i), for $i = 1, 2, \ldots, m$ (identified in step 2 of an iteration),

r = number of equation containing the leaving basic variable.

Recall that the new set of equations [excluding Eq. (0)] can be obtained from the preceding set by subtracting a'_{ik}/a'_{rk} times Eq. (r) from Eq. (i), for all $i = 1, 2, \ldots, m$ except $i = r$, and then dividing Eq. (r) by a'_{rk}. Therefore, the element in row i and column j of $\mathbf{B}_{\text{new}}^{-1}$ is

$$(\mathbf{B}_{\text{new}}^{-1})_{ij} = \begin{cases} (\mathbf{B}_{\text{old}}^{-1})_{ij} - \dfrac{a'_{ik}}{a'_{rk}}(\mathbf{B}_{\text{old}}^{-1})_{rj} & \text{if } i \neq r, \\ \dfrac{1}{a'_{rk}}(\mathbf{B}_{\text{old}}^{-1})_{rj} & \text{if } i = r. \end{cases}$$

These formulas are expressed in matrix notation as

$$\mathbf{B}_{\text{new}}^{-1} = \mathbf{E}\mathbf{B}_{\text{old}}^{-1},$$

where matrix $\mathbf{E}$ is an identity matrix except that its rth column is replaced by the vector

$$\boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{bmatrix}, \qquad \text{where} \qquad \eta_i = \begin{cases} -\dfrac{a'_{ik}}{a'_{rk}} & \text{if } i \neq r, \\ \dfrac{1}{a'_{rk}} & \text{if } i = r. \end{cases}$$

Thus, $\mathbf{E} = [\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_{r-1}, \boldsymbol{\eta}, \mathbf{U}_{r+1}, \ldots, \mathbf{U}_m]$, where the m elements of each of the $\mathbf{U}_i$ column vectors are 0 except for a 1 in the ith position.

Example. We shall illustrate this procedure by applying it to the Wyndor Glass Co. problem. We already have applied the matrix form of the simplex method to this same problem in Sec. 5.2, so we will refer to the results obtained there for each iteration (the entering basic variable, leaving basic variable, etc.) for the information needed to apply the procedure.

Iteration 1

We found in Sec. 5.2 that the initial $\mathbf{B}^{-1} = \mathbf{I}$, the entering basic variable is x_2 (so $k = 2$), the coefficients of x_2 in Eqs. 1, 2, and 3 are $a_{12} = 0$, $a_{22} = 2$, and $a_{32} = 2$, the leaving basic variable is x_4, and the number of the equation containing x_4 is $r = 2$. To obtain the new $\mathbf{B}^{-1}$,

$$\boldsymbol{\eta} = \begin{bmatrix} -\dfrac{a_{12}}{a_{22}} \\[2ex] \dfrac{1}{a_{22}} \\[2ex] -\dfrac{a_{32}}{a_{22}} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{1}{2} \\ -1 \end{bmatrix},$$

so

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Iteration 2

We found in Sec. 5.2 for this iteration that the entering basic variable is x_1 (so $k = 1$), the coefficients of x_1 in the current Eqs. 1, 2, and 3 are $a'_{11} = 1$, $a'_{21} = 0$, and $a'_{31} = 3$, the leaving basic variable is x_5, and the number of the equation containing x_5 is $r = 3$. These results yield

$$\boldsymbol{\eta} = \begin{bmatrix} -\dfrac{a'_{11}}{a'_{31}} \\[2ex] -\dfrac{a'_{21}}{a'_{31}} \\[2ex] \dfrac{1}{a'_{31}} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{3} \\ 0 \\ \dfrac{1}{3} \end{bmatrix}.$$

Therefore, the new $\mathbf{B}^{-1}$ is

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

No more iterations are needed at this point, so this example is finished.

Since the revised simplex method consists of combining this procedure for updating $\mathbf{B}^{-1}$ at each iteration with the rest of the matrix form of the simplex method presented in Sec. 5.2, combining this example with the one in Sec. 5.2 applying the matrix form to the same problem provides a complete example of applying the revised simplex method. As mentioned at the end of Sec. 5.2, the Worked Examples section of the book's website also gives **another example** of applying the revised simplex method.

Let us conclude this section by summarizing the advantages of the revised simplex method over the algebraic or tabular form of the simplex method. One advantage is that the number of arithmetic computations may be reduced. This is especially true when the **A** matrix contains a large number of zero elements (which is usually the case for the large problems arising in practice). The amount of information that must be stored at each iteration is less, sometimes considerably so. The revised simplex method also permits the control of the rounding errors inevitably generated by computers. This control can be exercised by periodically obtaining the current $\mathbf{B}^{-1}$ by directly inverting **B.** Furthermore, some of the postoptimality analysis problems discussed in Sec. 4.7 and the end of Sec.5.3 can be handled more conveniently with the revised simplex method. For all these reasons, the revised simplex method is usually the preferable form of the simplex method for computer execution.

5.5 CONCLUSIONS

Although the simplex method is an algebraic procedure, it is based on some fairly simple geometric concepts. These concepts enable one to use the algorithm to examine only a relatively small number of BF solutions before reaching and identifying an optimal solution.

Chapter 4 describes how *elementary algebraic operations* are used to execute the *algebraic form* of the simplex method, and then how the *tableau form* of the simplex method uses the equivalent *elementary row operations* in the same way. Studying the simplex method in these forms is a good way of getting started in learning its basic concepts. However, these forms of the simplex method do not provide the most efficient form for execution on a computer. *Matrix operations* are a faster way of combining and executing elementary algebraic operations or row operations. Therefore, the *matrix form* of the simplex method provides an effective way of adapting the simplex method for computer implementation. The *revised simplex method* provides a further improvement for computer implementation by combining the matrix form of the simplex method with an efficient procedure for updating the inverse of the current basis matrix from iteration to iteration.

The final simplex tableau includes complete information on how it can be algebraically reconstructed directly from the initial simplex tableau. This fundamental insight has some very important applications, especially for postoptimality analysis.

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3. Luenberger, D., and Y. Ye: *Linear and Nonlinear Programming*, 3rd ed., Springer, New York, 2008.
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■ LEARNING AIDS FOR THIS CHAPTER ON OUR WEBSITE (www.mhhe.com/hillier)

Worked Examples:

Examples for Chapter 5

A Demonstration Example in OR Tutor:

Fundamental Insight

Interactive Procedures in IOR Tutorial:

Interactive Graphical Method
Enter or Revise a General Linear Programming Model
Set Up for the Simplex Method—Interactive Only
Solve Interactively by the Simplex Method

Automatic Procedures in IOR Tutorial:

Solve Automatically by the Simplex Method
Graphical Method and Sensitivity Analysis

Files (Chapter 3) for Solving the Wyndor Example:

Excel Files
LINGO/LINDO File
MPL/CPLEX File

Glossary for Chapter 5

See Appendix 1 for documentation of the software.

■ PROBLEMS

The symbols to the left of some of the problems (or their parts) have the following meaning:

D: The demonstration example listed above may be helpful.
I: You can check some of your work by using procedures listed above.

An asterisk on the problem number indicates that at least a partial answer is given in the back of the book.

5.1-1.* Consider the following problem.

Maximize $Z = 3x_1 + 2x_2$,

subject to

$$2x_1 + x_2 \le 6$$
$$x_1 + 2x_2 \le 6$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

I **(a)** Solve this problem graphically. Identify the CPF solutions by circling them on the graph.
(b) Identify all the sets of two defining equations for this problem. For each set, solve (if a solution exists) for the corresponding corner-point solution, and classify it as a CPF solution or corner-point infeasible solution.
(c) Introduce slack variables in order to write the functional constraints in augmented form. Use these slack variables to identify the basic solution that corresponds to each corner-point solution found in part (*b*).
(d) Do the following for *each* set of two defining equations from part (*b*): Identify the indicating variable for each defining equation. Display the set of equations from part (*c*) *after* deleting these two indicating (nonbasic) variables. Then use the latter set of equations to solve for the two remaining variables (the basic variables). Compare the resulting basic solution to the corresponding basic solution obtained in part (*c*).
(e) Without executing the simplex method, use its geometric interpretation (and the objective function) to identify the path

(sequence of CPF solutions) it would follow to reach the optimal solution. For each of these CPF solutions in turn, identify the following decisions being made for the next iteration: (*i*) which defining equation is being deleted and which is being added; (*ii*) which indicating variable is being deleted (the entering basic variable) and which is being added (the leaving basic variable).

5.1-2. Repeat Prob. 5.1-1 for the model in Prob. 3.1-6.

5.1-3. Consider the following problem.

Maximize $Z = 5x_1 + 8x_2$,

subject to

$$
\begin{aligned}
4x_1 + 2x_2 &\le 80\\
-3x_1 + x_2 &\le 4\\
-x_1 + 2x_2 &\le 20\\
4x_1 - x_2 &\le 40
\end{aligned}
$$

and

$x_1 \ge 0, \quad x_2 \ge 0.$

I **(a)** Solve this problem graphically. Identify the CPF solutions by circling them on the graph.

(b) Develop a table giving each of the CPF solutions and the corresponding defining equations, BF solution, and nonbasic variables. Calculate Z for each of these solutions, and use just this information to identify the optimal solution.

(c) Develop the corresponding table for the corner-point infeasible solutions, etc. Also identify the sets of defining equations and nonbasic variables that do not yield a solution.

5.1-4. Consider the following problem.

Maximize $Z = 2x_1 - x_2 + x_3$,

subject to

$$
\begin{aligned}
3x_1 + x_2 + x_3 &\le 60\\
x_1 - x_2 + 2x_3 &\le 10\\
x_1 + x_2 - x_3 &\le 20
\end{aligned}
$$

and

$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$

After slack variables are introduced and then one complete iteration of the simplex method is performed, the following simplex tableau is obtained.

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
1	Z	(0)	1	0	−1	3	0	2	0	20
	x_4	(1)	0	0	4	−5	1	−3	0	30
	x_1	(2)	0	1	−1	2	0	1	0	10
	x_6	(3)	0	0	2	−3	0	−1	1	10

(a) Identify the CPF solution obtained at iteration 1.

(b) Identify the constraint boundary equations that define this CPF solution.

5.1-5. Consider the three-variable linear programming problem shown in Fig. 5.2.

(a) Construct a table like Table 5.1, giving the set of defining equations for each CPF solution.

(b) What are the defining equations for the corner-point infeasible solution (6, 0, 5)?

(c) Identify one of the systems of three constraint boundary equations that yields neither a CPF solution nor a corner-point infeasible solution. Explain why this occurs for this system.

5.1-6. Consider the following problem.

Minimize $Z = 8x_1 + 5x_2$,

subject to

$$
\begin{aligned}
-3x_1 + 2x_2 &\le 30\\
2x_1 + x_2 &\ge 50\\
x_1 + x_2 &\ge 30
\end{aligned}
$$

and

$x_1 \ge 0, \quad x_2 \ge 0.$

(a) Identify the 10 sets of defining equations for this problem. For each one, solve (if a solution exists) for the corresponding corner-point solution, and classify it as a CPF solution or a corner-point infeasible solution.

(b) For each corner-point solution, give the corresponding basic solution and its set of nonbasic variables.

5.1-7. Reconsider the model in Prob. 3.1-5.

(a) Identify the 15 sets of defining equations for this problem. For each one, solve (if a solution exists) for the corresponding corner-point solution, and classify it as a CPF solution or a corner-point infeasible solution.

(b) For each corner-point solution, give the corresponding basic solution and its set of nonbasic variables.

5.1-8. Each of the following statements is true under most circumstances, but not always. In each case, indicate when the statement will not be true and why.

(a) The best CPF solution is an optimal solution.

(b) An optimal solution is a CPF solution.

(c) A CPF solution is the only optimal solution if none of its adjacent CPF solutions are better (as measured by the value of the objective function).

5.1-9. Consider the original form (before augmenting) of a linear programming problem with n decision variables (each with a nonnegativity constraint) and m functional constraints. Label each of the following statements as true or false, and then justify your answer with specific references (including page citations) to material in the chapter.

(a) If a feasible solution is optimal, it must be a CPF solution.

(b) The number of CPF solutions is at least

$$\frac{(m+n)!}{m!n!}.$$

(c) If a CPF solution has adjacent CPF solutions that are better (as measured by Z), then one of these adjacent CPF solutions must be an optimal solution.

5.1-10. Label each of the following statements about linear programming problems as true or false, and then justify your answer.
(a) If a feasible solution is optimal but not a CPF solution, then infinitely many optimal solutions exist.
(b) If the value of the objective function is equal at two different feasible points $\mathbf{x}^*$ and $\mathbf{x}^{**}$, then all points on the line segment connecting $\mathbf{x}^*$ and $\mathbf{x}^{**}$ are feasible and Z has the same value at all those points.
(c) If the problem has n variables (before augmenting), then the simultaneous solution of any set of n constraint boundary equations is a CPF solution.

5.1-11. Consider the augmented form of linear programming problems that have feasible solutions and a bounded feasible region. Label each of the following statements as true or false, and then justify your answer by referring to specific statements (with page citations) in the chapter.
(a) There must be at least one optimal solution.
(b) An optimal solution must be a BF solution.
(c) The number of BF solutions is finite.

5.1-12.* Reconsider the model in Prob. 4.6-9. Now you are given the information that the basic variables in the optimal solution are x_2 and x_3. Use this information to identify a system of three constraint boundary equations whose simultaneous solution must be this optimal solution. Then solve this system of equations to obtain this solution.

5.1-13. Reconsider Prob. 4.3-6. Now use the given information and the theory of the simplex method to identify a system of three constraint boundary equations (in x_1, x_2, x_3) whose simultaneous solution must be the optimal solution, without applying the simplex method. Solve this system of equations to find the optimal solution.

5.1-14. Consider the following problem.

$$\text{Maximize} \quad Z = 2x_1 + 2x_2 + 3x_3,$$

subject to

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &\leq 4 \\ x_1 + x_2 + x_3 &\leq 3 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Let x_4 and x_5 be the slack variables for the respective functional constraints. Starting with these two variables as the basic variables for the initial BF solution, you now are given the information that the simplex method proceeds as follows to obtain the optimal solution in two iterations: (1) In iteration 1, the entering basic variable is x_3 and the leaving basic variable is x_4; (2) in iteration 2, the entering basic variable is x_2 and the leaving basic variable is x_5.
(a) Develop a three-dimensional drawing of the feasible region for this problem, and show the path followed by the simplex method.
(b) Give a geometric interpretation of why the simplex method followed this path.
(c) For each of the two edges of the feasible region traversed by the simplex method, give the equation of each of the two constraint boundaries on which it lies, and then give the equation of the additional constraint boundary at each endpoint.
(d) Identify the set of defining equations for each of the three CPF solutions (including the initial one) obtained by the simplex method. Use the defining equations to solve for these solutions.
(e) For each CPF solution obtained in part (d), give the corresponding BF solution and its set of nonbasic variables. Explain how these nonbasic variables identify the defining equations obtained in part (d).

5.1-15. Consider the following problem.

$$\text{Maximize} \quad Z = 3x_1 + 4x_2 + 2x_3,$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 20 \\ x_1 + 2x_2 + x_3 &\leq 30 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Let x_4 and x_5 be the slack variables for the respective functional constraints. Starting with these two variables as the basic variables for the initial BF solution, you now are given the information that the simplex method proceeds as follows to obtain the optimal solution in two iterations: (1) In iteration 1, the entering basic variable is x_2 and the leaving basic variable is x_5; (2) in iteration 2, the entering basic variable is x_1 and the leaving basic variable is x_4.

Follow the instructions of Prob. 5.1-14 for this situation.

5.1-16. By inspecting Fig. 5.2, explain why Property 1b for CPF solutions holds for this problem if it has the following objective function.
(a) Maximize $Z = x_3$.
(b) Maximize $Z = -x_1 + 2x_3$.

5.1-17. Consider the three-variable linear programming problem shown in Fig. 5.2.
(a) Explain in geometric terms why the set of solutions satisfying any individual constraint is a convex set, as defined in Appendix 2.

(b) Use the conclusion in part (*a*) to explain why the entire feasible region (the set of solutions that simultaneously satisfies every constraint) is a convex set.

5.1-18. Suppose that the three-variable linear programming problem given in Fig. 5.2 has the objective function

$$\text{Maximize} \quad Z = 3x_1 + 4x_2 + 3x_3.$$

Without using the algebra of the simplex method, apply just its geometric reasoning (including choosing the edge giving the maximum rate of increase of Z) to determine and explain the path it would follow in Fig. 5.2 from the origin to the optimal solution.

5.1-19. Consider the three-variable linear programming problem shown in Fig. 5.2.

(a) Construct a table like Table 5.4, giving the indicating variable for each constraint boundary equation and original constraint.

(b) For the CPF solution (2, 4, 3) and its three adjacent CPF solutions (4, 2, 4), (0, 4, 2), and (2, 4, 0), construct a table like Table 5.5, showing the corresponding defining equations, BF solution, and nonbasic variables.

(c) Use the sets of defining equations from part (*b*) to demonstrate that (4, 2, 4), (0, 4, 2), and (2, 4, 0) are indeed adjacent to (2, 4, 3), but that none of these three CPF solutions are adjacent to each other. Then use the sets of nonbasic variables from part (*b*) to demonstrate the same thing.

5.1-20. The formula for the line passing through (2, 4, 3) and (4, 2, 4) in Fig. 5.2 can be written as

$$(2, 4, 3) + \alpha[(4, 2, 4) - (2, 4, 3)] = (2, 4, 3) + \alpha(2, -2, 1),$$

where $0 \le \alpha \le 1$ for just the line segment between these points. After augmenting with the slack variables x_4, x_5, x_6, x_7 for the respective functional constraints, this formula becomes

$$(2, 4, 3, 2, 0, 0, 0) + \alpha(2, -2, 1, -2, 2, 0, 0).$$

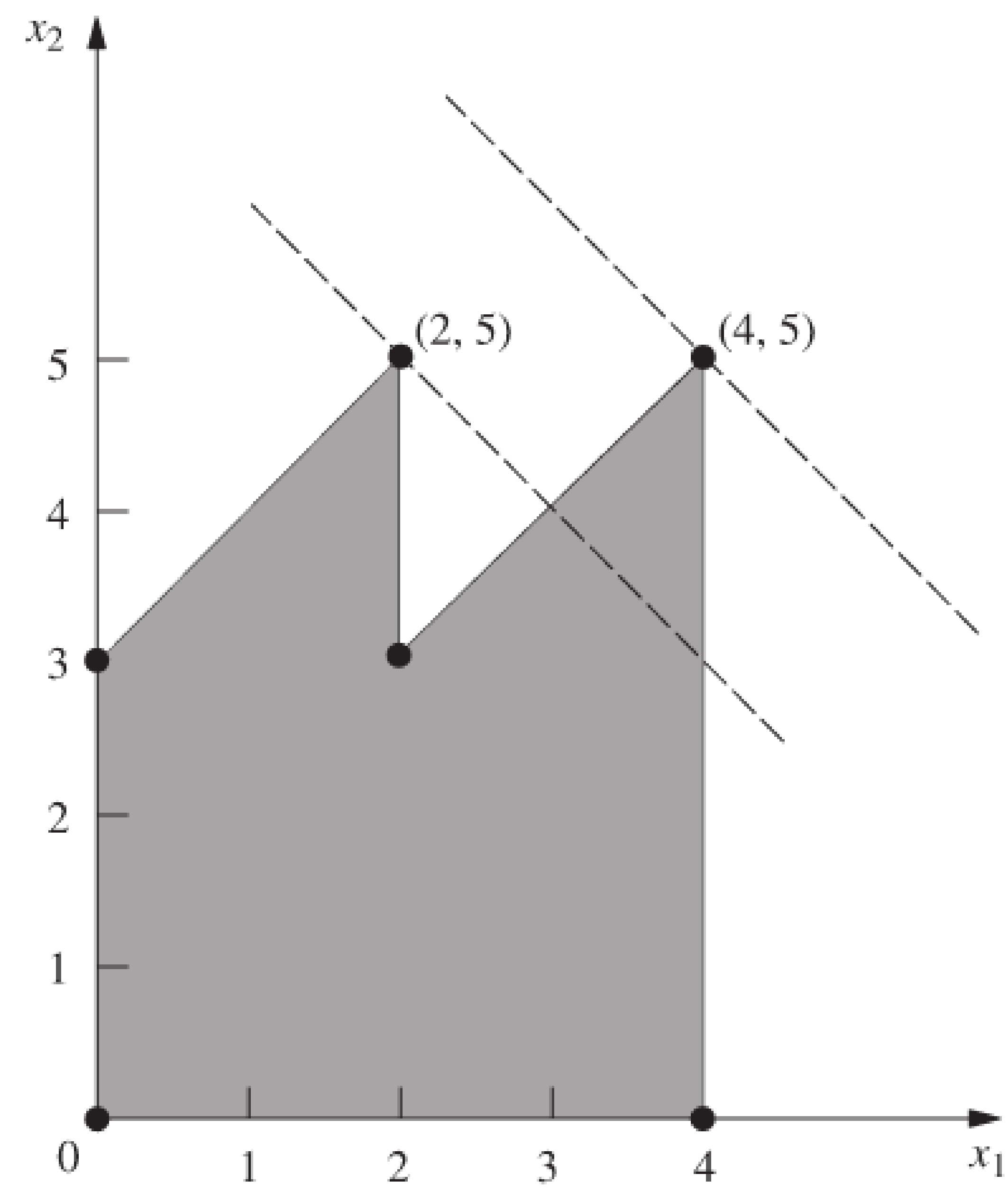
Use this formula directly to answer each of the following questions, and thereby relate the algebra and geometry of the simplex method as it goes through one iteration in moving from (2, 4, 3) to (4, 2, 4). (You are given the information that it is moving along this line segment.)

(a) What is the entering basic variable?

(b) What is the leaving basic variable?

(c) What is the new BF solution?

5.1-21. Consider a two-variable mathematical programming problem that has the feasible region shown on the graph, where the six dots correspond to CPF solutions. The problem has a linear objective function, and the two dashed lines are objective function lines passing through the optimal solution (4, 5) and the second-best CPF solution (2, 5). Note that the nonoptimal solution (2, 5) is better than both of its adjacent CPF solutions, which violates Property 3 in Sec. 5.1 for CPF solutions in linear programming. Demonstrate that this problem *cannot* be a linear programming problem by constructing the feasible region that would result if the six line segments on the boundary were constraint boundaries for linear programming constraints.



5.2-1. Consider the following problem.

$$\text{Maximize} \quad Z = 8x_1 + 4x_2 + 6x_3 + 3x_4 + 9x_5,$$

subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 3x_4 \phantom{{}+ x_5} &\le 180 \quad \text{(resource 1)} \\ 4x_1 + 3x_2 + 2x_3 + x_4 + x_5 &\le 270 \quad \text{(resource 2)} \\ x_1 + 3x_2 \phantom{{}+ 2x_3} + x_4 + 3x_5 &\le 180 \quad \text{(resource 3)} \end{aligned}$$

and

$$x_j \ge 0, \qquad j = 1, \ldots, 5.$$

You are given the facts that the basic variables in the optimal solution are x_3, x_1, and x_5 and that

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}^{-1} = \frac{1}{27}\begin{bmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{bmatrix}.$$

(a) Use the given information to identify the optimal solution.

(b) Use the given information to identify the shadow prices for the three resources.

I **5.2-2.*** Work through the matrix form of the simplex method step by step to solve the following problem.

$$\text{Maximize} \quad Z = 5x_1 + 8x_2 + 7x_3 + 4x_4 + 6x_5,$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 &\le 20 \\ 3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 &\le 30 \end{aligned}$$

and

$$x_j \geq 0, \qquad j = 1, 2, 3, 4, 5.$$

5.2-3. Reconsider Prob. 5.1-1. For the sequence of CPF solutions identified in part (*e*), construct the basis matrix **B** for each of the corresponding BF solutions. For each one, invert **B** manually, use this $\mathbf{B}^{-1}$ to calculate the current solution, and then perform the next iteration (or demonstrate that the current solution is optimal).

I **5.2-4.** Work through the matrix form of the simplex method step by step to solve the model given in Prob. 4.1-5.

I **5.2-5.** Work through the matrix form of the simplex method step by step to solve the model given in Prob. 4.7-6.

D **5.3-1.*** Consider the following problem.

$$\text{Maximize} \quad Z = x_1 - x_2 + 2x_3,$$

subject to

$$\begin{aligned} 2x_1 - 2x_2 + 3x_3 &\leq 5 \\ x_1 + x_2 - x_3 &\leq 3 \\ x_1 - x_2 + x_3 &\leq 2 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

Let x_4, x_5, and x_6 denote the slack variables for the respective constraints. After you apply the simplex method, a portion of the final simplex tableau is as follows:

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1				1	1	0	
x_2	(1)	0				1	3	0	
x_6	(2)	0				0	1	1	
x_3	(3)	0				1	2	0	

(a) Use the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the final simplex tableau. Show your calculations.
(b) Identify the defining equations of the CPF solution corresponding to the optimal BF solution in the final simplex tableau.

D **5.3-2.** Consider the following problem.

$$\text{Maximize} \quad Z = 4x_1 + 3x_2 + x_3 + 2x_4,$$

subject to

$$\begin{aligned} 4x_1 + 2x_2 + x_3 + x_4 &\leq 5 \\ 3x_1 + x_2 + 2x_3 + x_4 &\leq 4 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0, \qquad x_4 \geq 0.$$

Let x_5 and x_6 denote the slack variables for the respective constraints. After you apply the simplex method, a portion of the final simplex tableau is as follows:

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1					1	1	
x_2	(1)	0					1	−1	
x_4	(2)	0					−1	2	

(a) Use the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the final simplex tableau. Show your calculations.
(b) Identify the defining equations of the CPF solution corresponding to the optimal BF solution in the final simplex tableau.

D **5.3-3.** Consider the following problem.

$$\text{Maximize} \quad Z = 6x_1 + x_2 + 2x_3,$$

subject to

$$\begin{aligned} 2x_1 + 2x_2 + \tfrac{1}{2}x_3 &\leq 2 \\ -4x_1 - 2x_2 - \tfrac{3}{2}x_3 &\leq 3 \\ x_1 + 2x_2 + \tfrac{1}{2}x_3 &\leq 1 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

Let x_4, x_5, and x_6 denote the slack variables for the respective constraints. After you apply the simplex method, a portion of the final simplex tableau is as follows:

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1				2	0	2	
x_5	(1)	0				1	1	2	
x_3	(2)	0				−2	0	4	
x_1	(3)	0				1	0	−1	

Use the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the final simplex tableau. Show your calculations.

D **5.3-4.** Consider the following problem.

Maximize $\quad Z = 20x_1 + 6x_2 + 8x_3,$

subject to

$$\begin{aligned} 8x_1 + 2x_2 + 3x_3 &\le 200 \\ 4x_1 + 3x_2 \quad\quad\; &\le 100 \\ 2x_1 \quad\quad + x_3 &\le 50 \\ x_3 &\le 20 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

Let x_4, x_5, x_6, and x_7 denote the slack variables for the first through fourth constraints, respectively. Suppose that after some number of iterations of the simplex method, a portion of the current simplex tableau is as follows:

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Right Side
Z	(0)	1				$\frac{9}{4}$	$\frac{1}{2}$	0	0	
x_1	(1)	0				$\frac{3}{16}$	$-\frac{1}{8}$	0	0	
x_2	(2)	0				$-\frac{1}{4}$	$\frac{1}{2}$	0	0	
x_6	(3)	0				$-\frac{3}{8}$	$\frac{1}{4}$	1	0	
x_7	(4)	0				0	0	0	1	

(a) Use the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the current simplex tableau. Show your calculations.
(b) Indicate which of these missing numbers would be generated by the matrix form of the simplex method to perform the next iteration.
(c) Identify the defining equations of the CPF solution corresponding to the BF solution in the current simplex tableau.

D **5.3-5.** Consider the following problem.

Maximize $\quad Z = c_1x_1 + c_2x_2 + c_3x_3,$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le b \\ 2x_1 + x_2 + 3x_3 &\le 2b \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

Note that values have not been assigned to the coefficients in the objective function (c_1, c_2, c_3), and that the only specification for the right-hand side of the functional constraints is that the second one $(2b)$ be twice as large as the first (b).

Now suppose that your boss has inserted her best estimate of the values of c_1, c_2, c_3, and b without informing you and then has run the simplex method. You are given the resulting final simplex tableau below (where x_4 and x_5 are the slack variables for the respective functional constraints), but you are unable to read the value of Z^*.

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	$\frac{7}{10}$	0	0	$\frac{3}{5}$	$\frac{4}{5}$	Z^*
x_2	(1)	0	$\frac{1}{5}$	1	0	$\frac{3}{5}$	$-\frac{1}{5}$	1
x_3	(2)	0	$\frac{3}{5}$	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	3

(a) Use the fundamental insight presented in Sec. 5.3 to identify the value of (c_1, c_2, c_3) that was used.
(b) Use the fundamental insight presented in Sec. 5.3 to identify the value of b that was used.
(c) Calculate the value of Z^* in two ways, where one way uses your results from part (a) and the other way uses your result from part (b). Show your two methods for finding Z^*.

5.3-6. For iteration 2 of the example in Sec. 5.3, the following expression was shown:

$$\text{Final row } 0 = [-3, \quad -5 \mid 0, \quad 0, \quad 0 \mid 0]$$

$$+ \left[0, \quad \tfrac{3}{2}, \quad 1\right] \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right].$$

Derive this expression by combining the algebraic operations (in matrix form) for iterations 1 and 2 that affect row 0.

5.3-7. Most of the description of the fundamental insight presented in Sec. 5.3 assumes that the problem is in our standard form. Now consider each of the following other forms, where the additional adjustments in the initialization step are those presented in Sec. 4.6, including the use of artificial variables and the Big M method where appropriate. Describe the resulting adjustments in the fundamental insight.
(a) Equality constraints
(b) Functional constraints in $\ge$ form
(c) Negative right-hand sides
(d) Variables allowed to be negative (with no lower bound)

5.3-8. Reconsider the model in Prob. 4.6-5. Use artificial variables and the Big M method to construct the complete first simplex tableau for the simplex method, and then identify the columns that will contain $\mathbf{S}^*$ for applying the fundamental insight in the final tableau. Explain why these are the appropriate columns.

5.3-9. Consider the following problem.

Minimize $\quad Z = 2x_1 + 3x_2 + 2x_3,$

subject to

$$x_1 + 4x_2 + 2x_3 \geq 8$$
$$3x_1 + 2x_2 + 2x_3 \geq 6$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Let x_4 and x_6 be the surplus variables for the first and second constraints, respectively. Let $\bar{x}_5$ and $\bar{x}_7$ be the corresponding artificial variables. After you make the adjustments described in Sec. 4.6 for this model form when using the Big M method, the initial simplex tableau ready to apply the simplex method is as follows:

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	x_6	$\bar{x}_7$	Right Side
Z	(0)	-1	$-4M + 2$	$-6M + 3$	$-2M + 2$	M	0	M	0	$-14M$
$\bar{x}_5$	(1)	0	1	4	2	-1	1	0	0	8
$\bar{x}_7$	(2)	0	3	2	0	0	0	-1	1	6

After you apply the simplex method, a portion of the final simplex tableau is as follows:

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	x_6	$\bar{x}_7$	Right Side
Z	(0)	-1					$M - 0.5$		$M - 0.5$	
x_2	(1)	0					0.3		-0.1	
x_1	(2)	0					-0.2		0.4	

(a) Based on the above tableaux, use the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the final simplex tableau. Show your calculations.

(b) Examine the mathematical logic presented in Sec. 5.3 to validate the fundamental insight (see the $\mathbf{T}^* = \mathbf{MT}$ and $\mathbf{t}^* = \mathbf{t} + \mathbf{vT}$ equations and the subsequent derivations of $\mathbf{M}$ and $\mathbf{v}$). This logic assumes that the original model fits our standard form, whereas the current problem does not fit this form. Show how, with minor adjustments, this same logic applies to the current problem when $\mathbf{t}$ is row 0 and $\mathbf{T}$ is rows 1 and 2 in the initial simplex tableau given above. Derive $\mathbf{M}$ and $\mathbf{v}$ for this problem.

(c) When you apply the $\mathbf{t}^* = \mathbf{t} + \mathbf{vT}$ equation, another option is to use $\mathbf{t} = [2, 3, 2, 0, M, 0, M, 0]$, which is the *preliminary* row 0 before the algebraic elimination of the nonzero coefficients of the initial basic variables $\bar{x}_5$ and $\bar{x}_7$. Repeat part (*b*) for this equation with this new $\mathbf{t}$. After you derive the new $\mathbf{v}$, show that this equation yields the same final row 0 for this problem as the equation derived in part (*b*).

(d) Identify the defining equations of the CPF solution corresponding to the optimal BF solution in the final simplex tableau.

5.3-10. Consider the following problem.

Maximize $\quad Z = 3x_1 + 7x_2 + 2x_3,$

subject to

$$-2x_1 + 2x_2 + x_3 \leq 10$$
$$3x_1 + x_2 - x_3 \leq 20$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

You are given the fact that the basic variables in the optimal solution are x_1 and x_3.

(a) Introduce slack variables, and then use the given information to find the optimal solution directly by Gaussian elimination.

(b) Extend the work in part (*a*) to find the shadow prices.

(c) Use the given information to identify the defining equations of the optimal CPF solution, and then solve these equations to obtain the optimal solution.

(d) Construct the basis matrix $\mathbf{B}$ for the optimal BF solution, invert $\mathbf{B}$ manually, and then use this $\mathbf{B}^{-1}$ to solve for the optimal solution and the shadow prices $\mathbf{y}^*$. Then apply the optimality test for the matrix form of the simplex method to verify that this solution is optimal.

(e) Given $\mathbf{B}^{-1}$ and $\mathbf{y}^*$ from part (*d*), use the fundamental insight presented in Sec. 5.3 to construct the complete final simplex tableau.

5.4-1. Consider the model given in Prob. 5.2-2. Let x_6 and x_7 be the slack variables for the first and second constraints, respectively. You are given the information that x_2 is the entering basic variable and x_7 is the leaving basic variable for the first iteration of the simplex method and then x_4 is the entering basic variable and x_6 is the leaving basic variable for the second (final) iteration. Use the procedure presented in Sec. 5.4 for updating $\mathbf{B}^{-1}$ from one iteration to the next to find $\mathbf{B}^{-1}$ after the first iteration and then after the second iteration.

I **5.4-2.*** Work through the revised simplex method step by step to solve the model given in Prob. 4.3-4.

I **5.4-3.** Work through the revised simplex method step by step to solve the model given in Prob. 4.7-5.

I **5.4-4.** Work through the revised simplex method step by step to solve the model given in Prob. 3.1-6.

CHAPTER 6

Duality Theory and Sensitivity Analysis

One of the most important discoveries in the early development of linear programming was the concept of duality and its many important ramifications. This discovery revealed that every linear programming problem has associated with it another linear programming problem called the **dual.** The relationships between the dual problem and the original problem (called the **primal**) prove to be extremely useful in a variety of ways. For example, you soon will see that the shadow prices described in Sec. 4.7 actually are provided by the optimal solution for the dual problem. We shall describe many other valuable applications of duality theory in this chapter as well.

One of the key uses of duality theory lies in the interpretation and implementation of *sensitivity analysis.* As we already mentioned in Secs. 2.3, 3.3, and 4.7, sensitivity analysis is a very important part of almost every linear programming study. Because most of the parameter values used in the original model are just *estimates* of future conditions, the effect on the optimal solution if other conditions prevail instead needs to be investigated. Furthermore, certain parameter values (such as resource amounts) may represent *managerial decisions,* in which case the choice of the parameter values may be the main issue to be studied, which can be done through sensitivity analysis.

For greater clarity, the first three sections discuss duality theory under the assumption that the *primal* linear programming problem is in *our standard form* (but with no restriction that the b_i values need to be positive). Other forms are then discussed in Sec. 6.4. We begin the chapter by introducing the essence of duality theory and its applications. We then describe the economic interpretation of the dual problem (Sec. 6.2) and delve deeper into the relationships between the primal and dual problems (Sec. 6.3). Section 6.5 focuses on the role of duality theory in sensitivity analysis. The basic procedure for sensitivity analysis (which is based on the fundamental insight of Sec. 5.3) is summarized in Sec. 6.6 and illustrated in Sec. 6.7. Section 6.8 focuses on how to use spreadsheets to perform sensitivity analysis in a straightforward way. (If you don't have much time to devote to this chapter, it is feasible to read only Sec. 6.8 by itself to obtain a relatively brief introduction to sensitivity analysis.)

6.1 THE ESSENCE OF DUALITY THEORY

Given our standard form for the *primal problem* at the left (perhaps after conversion from another form), its *dual problem* has the form shown to the right.

Primal Problem	*Dual Problem*
Maximize $Z = \sum_{j=1}^{n} c_j x_j$,	Minimize $W = \sum_{i=1}^{m} b_i y_i$,
subject to	subject to
$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, for $i = 1, 2, \ldots, m$	$\sum_{i=1}^{m} a_{ij} y_i \geq c_j$, for $j = 1, 2, \ldots, n$
and	and
$x_j \geq 0$, for $j = 1, 2, \ldots, n$.	$y_i \geq 0$, for $i = 1, 2, \ldots, m$.

Thus, with the primal problem in *maximization* form, the dual problem is in *minimization* form instead. Furthermore, the dual problem uses exactly the same *parameters* as the primal problem, but in different locations, as summarized below.

1. The coefficients in the objective function of the primal problem are the *right-hand sides* of the functional constraints in the dual problem.
2. The right-hand sides of the functional constraints in the primal problem are the coefficients in the objective function of the dual problem.
3. The coefficients of a variable in the functional constraints of the primal problem are the coefficients in a functional constraint of the dual problem.

To highlight the comparison, now look at these same two problems in matrix notation (as introduced at the beginning of Sec. 5.2), where $\mathbf{c}$ and $\mathbf{y} = [y_1, y_2, \ldots, y_m]$ are row vectors but $\mathbf{b}$ and $\mathbf{x}$ are column vectors.

Primal Problem	*Dual Problem*
Maximize $Z = \mathbf{cx}$,	Minimize $W = \mathbf{yb}$,
subject to	subject to
$\mathbf{Ax} \leq \mathbf{b}$	$\mathbf{yA} \geq \mathbf{c}$
and	and
$\mathbf{x} \geq \mathbf{0}$.	$\mathbf{y} \geq \mathbf{0}$.

To illustrate, the primal and dual problems for the Wyndor Glass Co. example of Sec. 3.1 are shown in Table 6.1 in both algebraic and matrix form.

The **primal-dual table** for linear programming (Table 6.2) also helps to highlight the correspondence between the two problems. It shows all the linear programming parameters (the a_{ij}, b_i, and c_j) and how they are used to construct the two problems. All the headings for the primal problem are horizontal, whereas the headings for the dual problem are read by turning the book sideways. For the primal problem, each *column* (except the Right Side column) gives the coefficients of a single variable in the respective constraints and then in the objective function, whereas each *row* (except the bottom one) gives the parameters for a single contraint. For the dual problem, each *row* (except the Right Side row) gives the coefficients of a single variable in the respective constraints and then in the objective function, whereas each *column* (except the rightmost one) gives

TABLE 6.1 Primal and dual problems for the Wyndor Glass Co. example

Primal Problem in Algebraic Form

$$
\begin{aligned}
&\text{Maximize} \quad Z = 3x_1 + 5x_2,\\
&\text{subject to}\\
&\qquad x_1 \phantom{{}+2x_2} \le 4\\
&\qquad \phantom{3x_1+{}} 2x_2 \le 12\\
&\qquad 3x_1 + 2x_2 \le 18\\
&\text{and} \quad x_1 \ge 0, \quad x_2 \ge 0.
\end{aligned}
$$

Dual Problem in Algebraic Form

$$
\begin{aligned}
&\text{Minimize} \quad W = 4y_1 + 12y_2 + 18y_3,\\
&\text{subject to}\\
&\qquad y_1 \phantom{{}+2y_2} + 3y_3 \ge 3\\
&\qquad 2y_2 + 2y_3 \ge 5\\
&\text{and}\\
&\qquad y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.
\end{aligned}
$$

Primal Problem in Matrix Form

$$
\text{Maximize} \quad Z = [3, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},
$$

subject to

$$
\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}
$$

and

$$
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

Dual Problem in Matrix Form

$$
\text{Minimize} \quad W = [y_1, y_2, y_3]\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}
$$

subject to

$$
[y_1, y_2, y_3]\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} \ge [3, 5]
$$

and

$$
[y_1, y_2, y_3] \ge [0, 0, 0].
$$

TABLE 6.2 Primal-dual table for linear programming, illustrated by the Wyndor Glass Co. example

(a) General Case

			Primal Problem					
			Coefficient of:				**Right Side**	
			x_1	x_2	$\cdots$	x_n		
Dual Problem	**Coefficient of:**	y_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$\le b_1$	Coefficients for Objective Function (Minimize)
		y_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$\le b_2$	
		$\vdots$					$\vdots$	
		y_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$\le b_m$	
		Right Side	VI c_1	VI c_2	$\cdots$ $\cdots$	VI c_n		
			Coefficients for Objective Function (Maximize)					

(b) Wyndor Glass Co. Example

	x_1	x_2	
y_1	1	0	≤ 4
y_2	0	2	≤ 12
y_3	3	2	≤ 18
	VI 3	VI 5	

the parameters for a single constraint. In addition, the Right Side column gives the right-hand sides for the primal problem and the objective function coefficients for the dual problem, whereas the bottom row gives the objective function coefficients for the primal problem and the right-hand sides for the dual problem.

Consequently, we now have the following general relationships between the primal and dual problems.

1. The parameters for a (functional) *constraint* in either problem are the coefficients of a *variable* in the other problem.
2. The coefficients in the *objective function* of either problem are the *right-hand sides* for the other problem.

Thus, there is a direct correspondence between these entities in the two problems, as summarized in Table 6.3. These correspondences are a key to some of the applications of duality theory, including sensitivity analysis.

The Worked Examples section of the book's website provides **another example** of using the primal-dual table to construct the dual problem for a linear programming model.

Origin of the Dual Problem

Duality theory is based directly on the *fundamental insight* (particularly with regard to row 0) presented in Sec. 5.3. To see why, we continue to use the notation introduced in Table 5.9 for row 0 of the *final* tableau, except for replacing Z^* by W^* and dropping the asterisks from $\mathbf{z}^*$ and $\mathbf{y}^*$ when referring to *any* tableau. Thus, at *any* given iteration of the simplex method for the primal problem, the current numbers in row 0 are denoted as shown in the (partial) tableau given in Table 6.4. For the coefficients of $x_1, x_2, \ldots, x_n$, recall that $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ denotes the vector that the simplex method added to the vector of *initial* coefficients, $-\mathbf{c}$, in the process of reaching the current tableau. (Do not confuse $\mathbf{z}$ with the value of the objective function Z.) Similarly, since the *initial* coefficients of $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ in row 0 all are 0, $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ denotes the vector that the simplex method has added to these coefficients. Also recall [see Eq. (1) in the statement of the fundamental insight in Sec. 5.3] that the fundamental insight led to the following relationships between these quantities and the parameters of the original model:

$$W = \mathbf{yb} = \sum_{i=1}^{m} b_i y_i,$$

$$\mathbf{z} = \mathbf{yA}, \qquad \text{so} \qquad z_j = \sum_{i=1}^{m} a_{ij} y_i, \qquad \text{for } j = 1, 2, \ldots, n.$$

TABLE 6.3 Correspondence between entities in primal and dual problems

One Problem	Other Problem
Constraint i ⟷	Variable i
Objective function ⟷	Right-hand sides

TABLE 6.4 Notation for entries in row 0 of a simplex tableau

			Coefficient of:									
Iteration	**Basic Variable**	**Eq.**	Z	x_1	x_2	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_{n+m}	**Right Side**
Any	Z	(0)	1	$z_1 - c_1$	$z_2 - c_2$	$\cdots$	$z_n - c_n$	y_1	y_2	$\cdots$	y_m	W

To illustrate these relationships with the Wyndor example, the first equation gives $W = 4y_1 + 12y_2 + 18y_3$, which is just the objective function for the dual problem shown in the upper right-hand box of Table 6.1. The second set of equations give $z_1 = y_1 + 3y_3$ and $z_2 = 2y_2 + 2y_3$, which are the left-hand sides of the functional constraints for this dual problem. Thus, by subtracting the right-hand sides of these $\geq$ constraints ($c_1 = 3$ and $c_2 = 5$), $(z_1 - c_1)$ and $(z_2 - c_2)$ can be interpreted as being the *surplus variables* for these functional constraints.

The remaining key is to express what the simplex method tries to accomplish (according to the optimality test) in terms of these symbols. Specifically, it seeks a set of basic variables, and the corresponding BF solution, such that *all* coefficients in row 0 are *nonnegative.* It then stops with this optimal solution. Using the notation in Table 6.4, this goal is expressed symbolically as follows:

Condition for Optimality:

$$z_j - c_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n,$$
$$y_i \geq 0 \qquad \text{for } i = 1, 2, \ldots, m.$$

After we substitute the preceding expression for z_j, the condition for optimality says that the simplex method can be interpreted as seeking values for $y_1, y_2, \ldots, y_m$ such that

$$W = \sum_{i=1}^{m} b_i y_i,$$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \qquad \text{for } j = 1, 2, \ldots, n$$

and

$$y_i \geq 0, \qquad \text{for } i = 1, 2, \ldots, m.$$

But, except for lacking an objective for W, this problem is precisely the *dual problem*! To complete the formulation, let us now explore what the missing objective should be.

Since W is just the current value of Z, and since the objective for the primal problem is to maximize Z, a natural first reaction is that W should be maximized also. However, this is not correct for the following rather subtle reason: The only *feasible* solutions for this new problem are those that satisfy the condition for *optimality* for the primal problem. Therefore, it is *only* the optimal solution for the primal problem that corresponds to a feasible solution for this new problem. As a consequence, the optimal value of Z in the primal problem is the *minimum* feasible value of W in the new problem, so W should be minimized. (The full justification for this conclusion is provided by the relationships we develop in Sec. 6.3.) Adding this objective of minimizing W gives the *complete* dual problem.

Consequently, the dual problem may be viewed as a restatement in linear programming terms of the *goal* of the simplex method, namely, to reach a solution for the primal problem that *satisfies the optimality test. Before* this goal has been reached, the corresponding $\mathbf{y}$ in row 0 (coefficients of slack variables) of the current tableau must be *infeasible* for the *dual problem.* However, *after* the goal is reached, the corresponding $\mathbf{y}$ must be an *optimal solution* (labeled $\mathbf{y}^*$) for the *dual problem,* because it is a feasible solution that attains the minimum feasible value of W. This optimal solution $(y_1^*, y_2^*, \ldots, y_m^*)$ provides for the primal problem the shadow prices that were described in Sec. 4.7. Furthermore, this optimal W is just the optimal value of Z, so the *optimal objective function values are equal* for the two problems. This fact also implies that $\mathbf{cx} \leq \mathbf{yb}$ for any $\mathbf{x}$ and $\mathbf{y}$ that are *feasible* for the primal and dual problems, respectively.

To illustrate, the left-hand side of Table 6.5 shows row 0 for the respective iterations when the simplex method is applied to the Wyndor Glass Co. example. In each case, row 0 is partitioned into three parts: the coefficients of the decision variables (x_1, x_2), the coefficients of the slack variables (x_3, x_4, x_5), and the right-hand side (value of Z). Since the coefficients of the slack variables give the corresponding values of the dual variables (y_1, y_2, y_3), each row 0 identifies a corresponding solution for the dual problem, as shown in the y_1, y_2, and y_3 columns of Table 6.5. To interpret the next two columns, recall that $(z_1 - c_1)$ and $(z_2 - c_2)$ are the surplus variables for the functional constraints in the dual problem, so the full dual problem after augmenting with these surplus variables is

$$\text{Minimize} \qquad W = 4y_1 + 12y_2 + 18y_3,$$

subject to

$$\begin{aligned} y_1 \qquad + 3y_3 - (z_1 - c_1) &= 3 \\ 2y_2 + 2y_3 - (z_2 - c_2) &= 5 \end{aligned}$$

and

$$y_1 \ge 0, \qquad y_2 \ge 0, \qquad y_3 \ge 0.$$

Therefore, by using the numbers in the y_1, y_2, and y_3 columns, the values of these surplus variables can be calculated as

$$\begin{aligned} z_1 - c_1 &= y_1 + 3y_3 - 3, \\ z_2 - c_2 &= 2y_2 + 2y_3 - 5. \end{aligned}$$

Thus, a negative value for either surplus variable indicates that the corresponding constraint is violated. Also included in the rightmost column of the table is the calculated value of the dual objective function $W = 4y_1 + 12y_2 + 18y_3$.

As displayed in Table 6.4, *all* these quantities to the right of row 0 in Table 6.5 already are identified by row 0 without requiring any new calculations. In particular, note in Table 6.5 how *each* number obtained for the dual problem already appears in row 0 in the spot indicated by Table 6.4.

For the initial row 0, Table 6.5 shows that the corresponding dual solution $(y_1, y_2, y_3) = (0, 0, 0)$ is infeasible because both surplus variables are negative. The first iteration succeeds in eliminating one of these negative values, but not the other. After two iterations, the optimality test is satisfied for the primal problem because all the dual variables and surplus variables are nonnegative. This dual solution $(y_1^*, y_2^*, y_3^*) = (0, \frac{3}{2}, 1)$ is optimal (as could be verified by applying the simplex method directly to the dual problem), so the optimal value of Z and W is $Z^* = 36 = W^*$.

TABLE 6.5 Row 0 and corresponding dual solution for each iteration for the Wyndor Glass Co. example

	Primal Problem	Dual Problem					
Iteration	**Row 0**	y_1	y_2	y_3	$z_1 - c_1$	$z_2 - c_2$	W
0	$[-3, \; -5 \;\vdots\; 0, \; 0, \; 0 \;\vdots\; 0]$	0	0	0	−3	−5	0
1	$[-3, \; 0 \;\vdots\; 0, \; \frac{5}{2}, \; 0 \;\vdots\; 30]$	0	$\frac{5}{2}$	0	−3	0	30
2	$[\;0, \; 0 \;\vdots\; 0, \; \frac{3}{2}, \; 1 \;\vdots\; 36]$	0	$\frac{3}{2}$	1	0	0	36

Summary of Primal-Dual Relationships

Now let us summarize the newly discovered key relationships between the primal and dual problems.

Weak duality property: If **x** is a feasible solution for the primal problem and **y** is a feasible solution for the dual problem, then

$$\mathbf{cx} \le \mathbf{yb}.$$

For example, for the Wyndor Glass Co. problem, one feasible solution is $x_1 = 3$, $x_2 = 3$, which yields $Z = \mathbf{cx} = 24$, and one feasible solution for the dual problem is $y_1 = 1$, $y_2 = 1$, $y_3 = 2$, which yields a larger objective function value $W = \mathbf{yb} = 52$. These are just sample feasible solutions for the two problems. For *any* such pair of feasible solutions, this inequality must hold because the *maximum* feasible value of $Z = \mathbf{cx}$ (36) *equals* the *minimum* feasible value of the dual objective function $W = \mathbf{yb}$, which is our next property.

Strong duality property: If $\mathbf{x}^*$ is an optimal solution for the primal problem and $\mathbf{y}^*$ is an optimal solution for the dual problem, then

$$\mathbf{cx}^* = \mathbf{y}^*\mathbf{b}.$$

Thus, these two properties imply that $\mathbf{cx} < \mathbf{yb}$ for feasible solutions if one or both of them are *not optimal* for their respective problems, whereas equality holds when both are optimal.

The *weak duality property* describes the relationship between any pair of solutions for the primal and dual problems where *both* solutions are *feasible* for their respective problems. At each iteration, the simplex method finds a specific pair of solutions for the two problems, where the primal solution is feasible but the dual solution is *not feasible* (except at the final iteration). Our next property describes this situation and the relationship between this pair of solutions.

Complementary solutions property: At each iteration, the simplex method simultaneously identifies a CPF solution **x** for the primal problem and a **complementary solution y** for the dual problem (found in row 0, the coefficients of the slack variables), where

$$\mathbf{cx} = \mathbf{yb}.$$

If **x** is *not optimal* for the primal problem, then **y** is *not feasible* for the dual problem.

To illustrate, after one iteration for the Wyndor Glass Co. problem, $x_1 = 0$, $x_2 = 6$, and $y_1 = 0$, $y_2 = \frac{5}{2}$, $y_3 = 0$, with $\mathbf{cx} = 30 = \mathbf{yb}$. This **x** is feasible for the primal problem, but this **y** is not feasible for the dual problem (since it violates the constraint, $y_1 + 3y_3 \ge 3$).

The complementary solutions property also holds at the final iteration of the simplex method, where an optimal solution is found for the primal problem. However, more can be said about the complementary solution **y** in this case, as presented in the next property.

Complementary optimal solutions property: At the final iteration, the simplex method simultaneously identifies an optimal solution $\mathbf{x}^*$ for the primal problem and a **complementary optimal solution** $\mathbf{y}^*$ for the dual problem (found in row 0, the coefficients of the slack variables), where

$$\mathbf{cx}^* = \mathbf{y}^*\mathbf{b}.$$

The y_i^* are the shadow prices for the primal problem.

For the example, the final iteration yields $x_1^* = 2$, $x_2^* = 6$, and $y_1^* = 0$, $y_2^* = \frac{3}{2}$, $y_3^* = 1$, with $\mathbf{cx}^* = 36 = \mathbf{y}^*\mathbf{b}$.

We shall take a closer look at some of these properties in Sec. 6.3. There you will see that the complementary solutions property can be extended considerably further. In particular, after slack and surplus variables are introduced to augment the respective problems, every *basic* solution in the primal problem has a complementary *basic* solution in the dual problem. We already have noted that the simplex method identifies the values of the surplus variables for the dual problem as $z_j - c_j$ in Table 6.4. This result then leads to an additional *complementary slackness property* that relates the basic variables in one problem to the nonbasic variables in the other (Tables 6.7 and 6.8), but more about that later.

In Sec. 6.4, after describing how to construct the dual problem when the primal problem is *not* in our standard form, we discuss another very useful property, which is summarized as follows:

Symmetry property: For *any* primal problem and its dual problem, all relationships between them must be *symmetric* because the dual of this dual problem is this primal problem.

Therefore, all the preceding properties hold regardless of which of the two problems is labeled as the primal problem. (The direction of the inequality for the weak duality property does require that the primal problem be expressed or reexpressed in maximization form and the dual problem in minimization form.) Consequently, the simplex method can be applied to either problem, and it simultaneously will identify complementary solutions (ultimately a complementary optimal solution) for the other problem.

So far, we have focused on the relationships between *feasible* or *optimal* solutions in the primal problem and corresponding solutions in the dual problem. However, it is possible that the primal (or dual) problem either has *no feasible solutions* or has feasible solutions but *no optimal solution* (because the objective function is unbounded). Our final property summarizes the primal-dual relationships under all these possibilities.

Duality theorem: The following are the only possible relationships between the primal and dual problems.

1. If one problem has *feasible solutions* and a *bounded* objective function (and so has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable.
2. If one problem has *feasible solutions* and an *unbounded* objective function (and so *no optimal solution*), then the other problem has *no feasible solutions.*
3. If one problem has *no feasible solutions,* then the other problem has either *no feasible solutions* or an *unbounded* objective function.

Applications

As we have just implied, one important application of duality theory is that the *dual* problem can be solved directly by the simplex method in order to identify an optimal solution for the primal problem. We discussed in Sec. 4.8 that the number of functional constraints affects the computational effort of the simplex method far more than the number of variables does. If $m > n,$ so that the dual problem has fewer functional constraints (n) than the primal problem (m), then applying the simplex method directly to the dual problem instead of the primal problem probably will achieve a substantial reduction in computational effort.

The *weak* and *strong duality properties* describe key relationships between the primal and dual problems. One useful application is for evaluating a proposed solution for the primal problem. For example, suppose that **x** is a feasible solution that has been proposed for implementation and that a feasible solution **y** has been found by inspection for the dual problem such that $\mathbf{cx} = \mathbf{yb}$. In this case, $\mathbf{x}$ must be *optimal* without the simplex method even being applied! Even if $\mathbf{cx} < \mathbf{yb}$, then $\mathbf{yb}$ still provides an upper bound on the optimal value of Z, so if $\mathbf{yb} - \mathbf{cx}$ is small, intangible factors favoring $\mathbf{x}$ may lead to its selection without further ado.

One of the key applications of the complementary solutions property is its use in the dual simplex method presented in Sec. 7.1. This algorithm operates on the primal problem exactly as if the simplex method were being applied simultaneously to the dual problem, which can be done because of this property. Because the roles of row 0 and the right side in the simplex tableau have been reversed, the dual simplex method requires that row 0 *begin and remain nonnegative* while the right side *begins* with some *negative* values (subsequent iterations strive to reach a nonnegative right side). Consequently, this algorithm occasionally is used because it is more convenient to set up the initial tableau in this form than in the form required by the simplex method. Furthermore, it frequently is used for reoptimization (discussed in Sec. 4.7), because changes in the original model lead to the revised final tableau fitting this form. This situation is common for certain types of sensitivity analysis, as you will see later in the chapter.

In general terms, duality theory plays a central role in sensitivity analysis. This role is the topic of Sec. 6.5.

Another important application is its use in the economic interpretation of the dual problem and the resulting insights for analyzing the primal problem. You already have seen one example when we discussed shadow prices in Sec. 4.7. Section 6.2 describes how this interpretation extends to the entire dual problem and then to the simplex method.

6.2 ECONOMIC INTERPRETATION OF DUALITY

The economic interpretation of duality is based directly upon the typical interpretation for the primal problem (linear programming problem in our standard form) presented in Sec. 3.2. To refresh your memory, we have summarized this interpretation of the primal problem in Table 6.6.

Interpretation of the Dual Problem

To see how this interpretation of the primal problem leads to an economic interpretation for the dual problem,[1] note in Table 6.4 that W is the value of Z (total profit) at the current iteration. Because

$$W = b_1y_1 + b_2y_2 + \cdots + b_my_m,$$

TABLE 6.6 Economic interpretation of the primal problem

Quantity	Interpretation
x_j	Level of activity j $(j = 1, 2, \ldots, n)$
c_j	Unit profit from activity j
Z	Total profit from all activities
b_i	Amount of resource i available $(i = 1, 2, \ldots, m)$
a_{ij}	Amount of resource i consumed by each unit of activity j

[1]Actually, several slightly different interpretations have been proposed. The one presented here seems to us to be the most useful because it also directly interprets what the simplex method does in the primal problem.

each $b_i y_i$ can thereby be interpreted as the current *contribution to profit* by having b_i units of resource i available for the primal problem. Thus,

> The dual variable y_i is interpreted as the contribution to profit per unit of resource i ($i = 1, 2, \ldots, m$), when the current set of basic variables is used to obtain the primal solution.

In other words, the y_i values (or y_i^* values in the optimal solution) are just the **shadow prices** discussed in Sec. 4.7.

For example, when iteration 2 of the simplex method finds the optimal solution for the Wyndor problem, it also finds the optimal values of the dual variables (as shown in the bottom row of Table 6.5) to be $y_1^* = 0$, $y_2^* = \frac{3}{2}$, and $y_3^* = 1$. These are precisely the shadow prices found in Sec. 4.7 for this problem through graphical analysis. Recall that the resources for the Wyndor problem are the production capacities of the three plants being made available to the two new products under consideration, so that b_i is the number of hours of production time per week being made available in Plant i for these new products, where $i = 1, 2, 3$. As discussed in Sec. 4.7, the shadow prices indicate that individually increasing any b_i by 1 would increase the optimal value of the objective function (total weekly profit in units of thousands of dollars) by y_i^*. Thus, y_i^* can be interpreted as the contribution to profit per unit of resource i when using the optimal solution.

This interpretation of the dual variables leads to our interpretation of the overall dual problem. Specifically, since each unit of activity j in the primal problem consumes a_{ij} units of resource i,

> $\Sigma_{i=1}^{m} a_{ij} y_i$ is interpreted as the current contribution to profit of the mix of resources that would be consumed if 1 unit of activity j were used ($j = 1, 2, \ldots, n$).

For the Wyndor problem, 1 unit of activity j corresponds to producing 1 batch of product j per week, where $j = 1, 2$. The mix of resources consumed by producing 1 batch of product 1 is 1 hour of production time in Plant 1 and 3 hours in Plant 3. The corresponding mix per batch of product 2 is 2 hours each in Plants 2 and 3. Thus, $y_1 + 3y_3$ and $2y_2 + 2y_3$ are interpreted as the current contributions to profit (in thousands of dollars per week) of these respective mixes of resources per batch produced per week of the respective products.

For each activity j, this same mix of resources (and more) probably can be used in other ways as well, but no alternative use should be considered if it is less profitable than 1 unit of activity j. Since c_j is interpreted as the unit profit from activity j, each functional constraint in the dual problem is interpreted as follows:

> $\Sigma_{i=1}^{m} a_{ij} y_i \geq c_j$ says that the actual contribution to profit of the above mix of resources must be at least as much as if they were used by 1 unit of activity j; otherwise, we would not be making the best possible use of these resources.

For the Wyndor problem, the unit profits (in thousands of dollars per week) are $c_1 = 3$ and $c_2 = 5$, so the dual functional constraints with this interpretation are $y_1 + 3y_3 \geq 3$ and $2y_2 + 2y_3 \geq 5$. Similarly, the interpretation of the nonnegativity constraints is the following:

> $y_i \geq 0$ says that the contribution to profit of resource i ($i = 1, 2, \ldots, m$) must be nonnegative: otherwise, it would be better not to use this resource at all.

The objective

$$\text{Minimize} \qquad W = \sum_{i=1}^{m} b_i y_i$$

can be viewed as minimizing the total implicit value of the resources consumed by the activities. For the Wyndor problem, the total implicit value (in thousands of dollars per week) of the resources consumed by the two products is $W = 4y_1 + 12y_2 + 18y_3$.

This interpretation can be sharpened somewhat by differentiating between basic and nonbasic variables in the primal problem for any given BF solution $(x_1, x_2, \ldots, x_{n+m})$. Recall that the *basic* variables (the only variables whose values can be nonzero) *always* have a coefficient of *zero* in row 0. Therefore, referring again to Table 6.4 and the accompanying equation for z_j, we see that

$$\sum_{i=1}^{m} a_{ij} y_i = c_j, \qquad \text{if } x_j > 0 \qquad (j = 1, 2, \ldots, n),$$

$$y_i = 0, \qquad \text{if } x_{n+i} > 0 \qquad (i = 1, 2, \ldots, m).$$

(This is one version of the complementary slackness property discussed in Sec. 6.3.) The economic interpretation of the first statement is that whenever an activity j operates at a strictly positive level ($x_j > 0$), the marginal value of the resources it consumes *must equal* (as opposed to exceeding) the unit profit from this activity. The second statement implies that the marginal value of resource i is *zero* ($y_i = 0$) whenever the supply of this resource is not exhausted by the activities ($x_{n+i} > 0$). In economic terminology, such a resource is a "free good"; the price of goods that are oversupplied must drop to zero by the law of supply and demand. This fact is what justifies interpreting the objective for the dual problem as minimizing the total implicit value of the resources *consumed,* rather than the resources *allocated.*

To illustrate these two statements, consider the optimal BF solution (2, 6, 2, 0, 0) for the Wyndor problem. The basic variables are x_1, x_2, and x_3, so their coefficients in row 0 are zero, as shown in the bottom row of Table 6.5. This bottom row also gives the corresponding dual solution: $y_1^* = 0$, $y_2^* = \frac{3}{2}$, $y_3^* = 1$, with surplus variables $(z_1^* - c_1) = 0$ and $(z_2^* - c_2) = 0$. Since $x_1 > 0$ and $x_2 > 0$, both these surplus variables and direct calculations indicate that $y_1^* + 3y_3^* = c_1 = 3$ and $2y_2^* + 2y_3^* = c_2 = 5$. Therefore, the value of the resources consumed per batch of the respective products produced does indeed equal the respective unit profits. The slack variable for the constraint on the amount of Plant 1 capacity used is $x_3 > 0$, so the marginal value of adding any Plant 1 capacity would be zero ($y_1^* = 0$).

Interpretation of the Simplex Method

The interpretation of the dual problem also provides an economic interpretation of what the simplex method does in the primal problem. The *goal* of the simplex method is to find how to use the available resources in the most profitable feasible way. To attain this goal, we must reach a BF solution that satisfies all the *requirements* on profitable use of the resources (the constraints of the dual problem). These requirements comprise the *condition for optimality* for the algorithm. For any given BF solution, the requirements (dual constraints) associated with the basic variables are automatically satisfied (with equality). However, those associated with nonbasic variables may or may not be satisfied.

In particular, if an original variable x_j is nonbasic so that activity j is not used, then the current contribution to profit of the resources that would be required to undertake each unit of activity j

$$\sum_{i=1}^{m} a_{ij} y_i$$

may be smaller than, larger than, or equal to the unit profit c_j obtainable from the activity. If it is smaller, so that $z_j - c_j < 0$ in row 0 of the simplex tableau, then these resources can be used more profitably by initiating this activity. If it is larger ($z_j - c_j > 0$), then these resources already are being assigned elsewhere in a more profitable way, so they should not be diverted to activity j. If $z_j - c_j = 0$, there would be no change in profitability by initiating activity j.

Similarly, if a slack variable x_{n+i} is nonbasic so that the total allocation b_i of resource i is being used, then y_i is the current contribution to profit of this resource on a marginal basis. Hence, if $y_i < 0$, profit can be increased by cutting back on the use of this resource (i.e., increasing x_{n+i}). If $y_i > 0$, it is worthwhile to continue fully using this resource, whereas this decision does not affect profitability if $y_i = 0$.

Therefore, what the simplex method does is to examine all the nonbasic variables in the current BF solution to see which ones can provide a *more profitable use of the resources* by being increased. If *none* can, so that no feasible shifts or reductions in the current proposed use of the resources can increase profit, then the current solution must be optimal. If one or more can, the simplex method selects the variable that, if increased by 1, would *improve the profitability* of the use of the resources the most. It then actually increases this variable (the entering basic variable) as much as it can until the marginal values of the resources change. This increase results in a new BF solution with a new row 0 (dual solution), and the whole process is repeated.

The economic interpretation of the dual problem considerably expands our ability to analyze the primal problem. However, you already have seen in Sec. 6.1 that this interpretation is just one ramification of the relationships between the two problems. In Sec 6.3, we delve into these relationships more deeply.

6.3 PRIMAL–DUAL RELATIONSHIPS

Because the dual problem is a linear programming problem, it also has corner-point solutions. Furthermore, by using the augmented form of the problem, we can express these corner-point solutions as basic solutions. Because the functional constraints have the $\geq$ form, this augmented form is obtained by *subtracting* the surplus (rather than adding the slack) from the left-hand side of each constraint j ($j = 1, 2, \ldots, n$).[2] This surplus is

$$z_j - c_j = \sum_{i=1}^{m} a_{ij}y_i - c_j, \qquad \text{for } j = 1, 2, \ldots, n.$$

Thus, $z_j - c_j$ plays the role of the *surplus variable* for constraint j (or its slack variable if the constraint is multiplied through by -1). Therefore, augmenting each corner-point solution $(y_1, y_2, \ldots, y_m)$ yields a basic solution $(y_1, y_2, \ldots, y_m, z_1 - c_1, z_2 - c_2, \ldots, z_n - c_n)$ by using this expression for $z_j - c_j$. Since the augmented form of the dual problem has n functional constraints and $n + m$ variables, each basic solution has n basic variables and m nonbasic variables. (Note how m and n reverse their previous roles here because, as Table 6.3 indicates, dual constraints correspond to primal variables and dual variables correspond to primal constraints.)

[2]You might wonder why we do not also introduce *artificial variables* into these constraints as discussed in Sec. 4.6. The reason is that these variables have no purpose other than to change the feasible region temporarily as a convenience in starting the simplex method. We are not interested now in applying the simplex method to the dual problem, and we do not want to change its feasible region.

Complementary Basic Solutions

One of the important relationships between the primal and dual problems is a direct correspondence between their basic solutions. The key to this correspondence is row 0 of the simplex tableau for the primal basic solution, such as shown in Table 6.4 or 6.5. Such a row 0 can be obtained for *any* primal basic solution, feasible or not, by using the formulas given in the bottom part of Table 5.8.

Note again in Tables 6.4 and 6.5 how a complete solution for the dual problem (including the surplus variables) can be read directly from row 0. Thus, because of its coefficient in row 0, each variable in the primal problem has an associated variable in the dual problem, as summarized in Table 6.7, first for any problem and then for the Wyndor problem.

A key insight here is that the dual solution read from row 0 must also be a basic solution! The reason is that the m basic variables for the primal problem are required to have a coefficient of zero in row 0, which thereby requires the m associated dual variables to be zero, i.e., nonbasic variables for the dual problem. The values of the remaining n (basic) variables then will be the simultaneous solution to the system of equations given at the beginning of this section. In matrix form, this system of equations is $\mathbf{z} - \mathbf{c} = \mathbf{yA} - \mathbf{c}$, and the fundamental insight of Sec. 5.3 actually identifies its solution for $\mathbf{z} - \mathbf{c}$ and $\mathbf{y}$ as being the corresponding entries in row 0.

Because of the symmetry property quoted in Sec. 6.1 (and the direct association between variables shown in Table 6.7), the correspondence between basic solutions in the primal and dual problems is a symmetric one. Furthermore, a pair of complementary basic solutions has the same objective function value, shown as W in Table 6.4.

Let us now summarize our conclusions about the correspondence between primal and dual basic solutions, where the first property extends the complementary solutions property of Sec. 6.1 to the augmented forms of the two problems and then to any basic solution (feasible or not) in the primal problem.

> **Complementary basic solutions property:** Each *basic* solution in the *primal problem* has a **complementary basic solution** in the *dual problem,* where their respective objective function values (Z and W) are equal. Given row 0 of the simplex tableau for the primal basic solution, the complementary dual basic solution $(\mathbf{y}, \mathbf{z} - \mathbf{c})$ is found as shown in Table 6.4.

The next property shows how to identify the basic and nonbasic variables in this complementary basic solution.

> **Complementary slackness property:** Given the association between variables in Table 6.7, the variables in the primal basic solution and the complementary dual basic solution satisfy the **complementary slackness** relationship shown in Table 6.8. Furthermore, this relationship is a symmetric one, so that these two basic solutions are complementary to each other.

TABLE 6.7 Association between variables in primal and dual problems

	Primal Variable	Associated Dual Variable
Any problem	(Decision variable) x_j (Slack variable) x_{n+i}	$z_j - c_j$ (surplus variable) $j = 1, 2, \ldots, n$ y_i (decision variable) $i = 1, 2, \ldots, m$
Wyndor problem	Decision variables: x_1 x_2 Slack variables: x_3 x_4 x_5	$z_1 - c_1$ (surplus variables) $z_2 - c_2$ y_1 (decision variables) y_2 y_3

TABLE 6.8 Complementary slackness relationship for complementary basic solutions

Primal Variable	Associated Dual Variable	
Basic	Nonbasic	(*m* variables)
Nonbasic	Basic	(*n* variables)

TABLE 6.9 Complementary basic solutions for the Wyndor Glass Co. example

	Primal Problem			Dual Problem	
No.	Basic Solution	Feasible?	$Z = W$	Feasible?	Basic Solution
1	(0, 0, 4, 12, 18)	Yes	0	No	(0, 0, 0, −3, −5)
2	(4, 0, 0, 12, 6)	Yes	12	No	(3, 0, 0, 0, −5)
3	(6, 0, −2, 12, 0)	No	18	No	(0, 0, 1, 0, −3)
4	(4, 3, 0, 6, 0)	Yes	27	No	$\left(-\frac{9}{2}, 0, \frac{5}{2}, 0, 0\right)$
5	(0, 6, 4, 0, 6)	Yes	30	No	$\left(0, \frac{5}{2}, 0, -3, 0\right)$
6	(2, 6, 2, 0, 0)	Yes	36	Yes	$\left(0, \frac{3}{2}, 1, 0, 0\right)$
7	(4, 6, 0, 0, −6)	No	42	Yes	$\left(3, \frac{5}{2}, 0, 0, 0\right)$
8	(0, 9, 4, −6, 0)	No	45	Yes	$\left(0, 0, \frac{5}{2}, \frac{9}{2}, 0\right)$

The reason for using the name *complementary slackness* for this latter property is that it says (in part) that for each pair of associated variables, if one of them has *slack* in its nonnegativity constraint (a basic variable > 0), then the other one must have *no slack* (a nonbasic variable $= 0$). We mentioned in Sec. 6.2 that this property has a useful economic interpretation for linear programming problems.

Example. To illustrate these two properties, again consider the Wyndor Glass Co. problem of Sec. 3.1. All eight of its basic solutions (five feasible and three infeasible) are shown in Table 6.9. Thus, its dual problem (see Table 6.1) also must have eight basic solutions, each complementary to one of these primal solutions, as shown in Table 6.9.

The three BF solutions obtained by the simplex method for the primal problem are the first, fifth, and sixth primal solutions shown in Table 6.9. You already saw in Table 6.5 how the complementary basic solutions for the dual problem can be read directly from row 0, starting with the coefficients of the slack variables and then the original variables. The other dual basic solutions also could be identified in this way by constructing row 0 for each of the other primal basic solutions, using the formulas given in the bottom part of Table 5.8.

Alternatively, for each primal basic solution, the complementary slackness property can be used to identify the basic and nonbasic variables for the complementary dual basic solution, so that the system of equations given at the beginning of the section can be solved directly to obtain this complementary solution. For example, consider the next-to-last primal basic solution in Table 6.9, (4, 6, 0, 0, −6). Note that x_1, x_2, and x_5 are *basic variables,* since these variables are not equal to 0. Table 6.7 indicates that the associated dual variables are $(z_1 - c_1)$, $(z_2 - c_2)$, and y_3. Table 6.8 specifies that these associated dual variables are *nonbasic variables* in the complementary basic solution, so

$$z_1 - c_1 = 0, \qquad z_2 - c_2 = 0, \qquad y_3 = 0.$$

Consequently, the augmented form of the functional constraints in the dual problem,

$$\begin{aligned} y_1 \quad + 3y_3 - (z_1 - c_1) &= 3 \\ 2y_2 + 2y_3 - (z_2 - c_2) &= 5, \end{aligned}$$

reduce to

$$\begin{aligned} y_1 \quad + 0 - 0 &= 3 \\ 2y_2 + 0 - 0 &= 5, \end{aligned}$$

so that $y_1 = 3$ and $y_2 = \frac{5}{2}$. Combining these values with the values of 0 for the nonbasic variables gives the basic solution $(3, \frac{5}{2}, 0, 0, 0)$, shown in the rightmost column and next-to-last row of Table 6.9. Note that this dual solution is feasible for the dual problem because all five variables satisfy the nonnegativity constraints.

Finally, notice that Table 6.9 demonstrates that $(0, \frac{3}{2}, 1, 0, 0)$ is the optimal solution for the dual problem, because it is the basic *feasible* solution with minimal W (36).

Relationships between Complementary Basic Solutions

We now turn our attention to the relationships between complementary basic solutions, beginning with their *feasibility* relationships. The middle columns in Table 6.9 provide some valuable clues. For the pairs of complementary solutions, notice how the yes or no answers on feasibility also satisfy a complementary relationship in most cases. In particular, with one exception, whenever one solution is feasible, the other is not. (It also is possible for *neither* solution to be feasible, as happened with the third pair.) The one exception is the sixth pair, where the primal solution is known to be optimal. The explanation is suggested by the $Z = W$ column. Because the sixth dual solution also is optimal (by the complementary optimal solutions property), with $W = 36$, the first five dual solutions *cannot be feasible* because $W < 36$ (remember that the dual problem objective is to *minimize* W). By the same token, the last two primal solutions cannot be feasible because $Z > 36$.

This explanation is further supported by the strong duality property that optimal primal and dual solutions have $Z = W$.

Next, let us state the *extension* of the complementary optimal solutions property of Sec. 6.1 for the augmented forms of the two problems.

> **Complementary optimal basic solutions property:** An *optimal* basic solution in the *primal problem* has a **complementary optimal basic solution** in the dual problem, where their respective objective function values (Z and W) are equal. Given row 0 of the simplex tableau for the optimal primal solution, the complementary optimal dual solution $(\mathbf{y}^*, \mathbf{z}^* - \mathbf{c})$ is found as shown in Table 6.4.

To review the reasoning behind this property, note that the dual solution $(\mathbf{y}^*, \mathbf{z}^* - \mathbf{c})$ must be feasible for the dual problem because the condition for optimality for the primal problem requires that *all* these dual variables (including surplus variables) be *nonnegative.* Since this solution is *feasible,* it must be *optimal* for the dual problem by the weak duality property (since $W = Z$, so $\mathbf{y}^*\mathbf{b} = \mathbf{c}\mathbf{x}^*$ where $\mathbf{x}^*$ is optimal for the primal problem).

Basic solutions can be classified according to whether they satisfy each of two conditions. One is the *condition for feasibility,* namely, whether *all* the variables (including slack variables) in the augmented solution are *nonnegative.* The other is the *condition for optimality,* namely, whether *all* the coefficients in row 0 (i.e., all the variables in the complementary basic solution) are *nonnegative.* Our names for the different types of basic solutions are summarized in Table 6.10. For example, in Table 6.9, primal basic

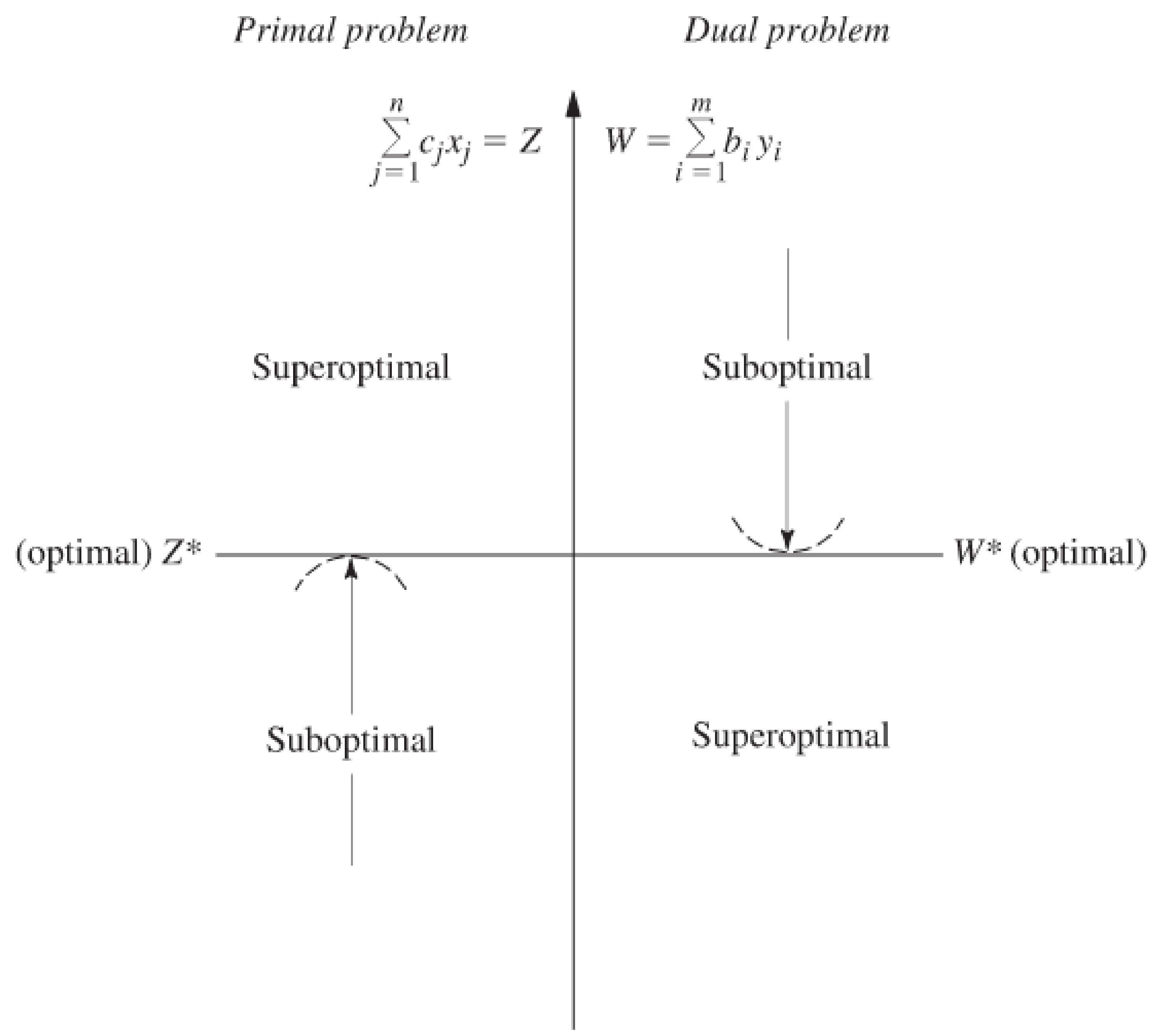


FIGURE 6.1 Range of possible values of $Z = W$ for certain types of complementary basic solutions.

TABLE 6.10 Classification of basic solutions

		Satisfies Condition for Optimality?	
		Yes	**No**
Feasible?	**Yes**	Optimal	Suboptimal
	No	Superoptimal	Neither feasible nor superoptimal

TABLE 6.11 Relationships between complementary basic solutions

Primal Basic Solution	Complementary Dual Basic Solution	Both Basic Solutions: Primal Feasible?	Dual Feasible?
Suboptimal	Superoptimal	Yes	No
Optimal	Optimal	Yes	Yes
Superoptimal	Suboptimal	No	Yes
Neither feasible nor superoptimal	Neither feasible nor superoptimal	No	No

solutions 1, 2, 4, and 5 are suboptimal, 6 is optimal, 7 and 8 are superoptimal, and 3 is neither feasible nor superoptimal.

Given these definitions, the general relationships between complementary basic solutions are summarized in Table 6.11. The resulting range of possible (common) values for the objective functions ($Z = W$) for the first three pairs given in Table 6.11 (the last pair can have any value) is shown in Fig. 6.1. Thus, while the simplex method is dealing directly with suboptimal basic solutions and working toward optimality in the primal problem, it is simultaneously dealing indirectly with complementary superoptimal solutions and working toward feasibility in the dual problem. Conversely, it sometimes is more convenient (or necessary) to work directly with superoptimal basic solutions and to move toward feasibility in the primal problem, which is the purpose of the dual simplex method described in Sec. 7.1.

The third and fourth columns of Table 6.11 introduce two other common terms that are used to describe a pair of complementary basic solutions. The two solutions are said to be **primal feasible** if the primal basic solution is feasible, whereas they are called **dual feasible** if the complementary dual basic solution is feasible for the dual problem. Using this terminology, the simplex method deals with primal feasible solutions and strives toward achieving dual feasibility as well. When this is achieved, the two complementary basic solutions are optimal for their respective problems.

These relationships prove very useful, particularly in sensitivity analysis, as you will see later in the chapter.

6.4 ADAPTING TO OTHER PRIMAL FORMS

Thus far it has been assumed that the model for the primal problem is in our standard form. However, we indicated at the beginning of the chapter that any linear programming problem, whether in our standard form or not, possesses a dual problem. Therefore, this section focuses on how the dual problem changes for other primal forms.

Each nonstandard form was discussed in Sec. 4.6, and we pointed out how it is possible to convert each one to an equivalent standard form if so desired. These conversions are summarized in Table 6.12. Hence, you always have the option of converting any model to our standard form and *then* constructing its dual problem in the usual way. To illustrate, we do this for our standard dual problem (it must have a dual also) in Table 6.13. Note that what we end up with is just our standard primal problem! Since any pair of primal and dual problems can be converted to these forms, this fact implies that the dual of the dual problem always is the primal problem. Therefore, for any primal problem and its dual problem, all relationships between them must be symmetric. This is just the symmetry property already stated in Sec. 6.1 (without proof), but now Table 6.13 demonstrates why it holds.

One consequence of the symmetry property is that all the statements made earlier in the chapter about the relationships of the dual problem to the primal problem also hold in reverse.

TABLE 6.12 Conversions to standard form for linear programming models

Nonstandard Form	Equivalent Standard Form
Minimize Z	Maximize $(-Z)$
$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$	$-\sum_{j=1}^{n} a_{ij}x_j \leq -b_i$
$\sum_{j=1}^{n} a_{ij}x_j = b_i$	$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ and $-\sum_{j=1}^{n} a_{ij}x_j \leq -b_i$
x_j unconstrained in sign	$x_j^+ - x_j^-$, $x_j^+ \geq 0$, $x_j^- \geq 0$

TABLE 6.13 Constructing the dual of the dual problem

Dual Problem		*Converted to Standard Form*
Minimize $W = \mathbf{yb}$, subject to $\mathbf{yA} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}.$	$\longrightarrow$	Maximize $(-W) = -\mathbf{yb}$, subject to $-\mathbf{yA} \leq -\mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}.$
		$\downarrow$
Converted to Standard Form		*Its Dual Problem*
Maximize $Z = \mathbf{cx}$, subject to $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}.$	$\longleftarrow$	Minimize $(-Z) = -\mathbf{cx}$, subject to $-\mathbf{Ax} \geq -\mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}.$

Another consequence is that it is immaterial which problem is called the primal and which is called the dual. In practice, you might see a linear programming problem fitting our standard form being referred to as the dual problem. The convention is that the model formulated to fit the actual problem is called the primal problem, regardless of its form.

Our illustration of how to construct the dual problem for a nonstandard primal problem did not involve either equality constraints or variables unconstrained in sign. Actually, for these two forms, a shortcut is available. It is possible to show (see Probs. 6.4-7 and 6.4-2*a*) that an *equality constraint* in the primal problem should be treated just like a $\leq$ constraint in constructing the dual problem except that the nonnegativity constraint for the corresponding dual variable should be *deleted* (i.e., this variable is unconstrained in sign). By the symmetry property, deleting a nonnegativity constraint in the primal problem affects the dual problem only by changing the corresponding inequality constraint to an equality constraint.

Another shortcut involves functional constraints in $\geq$ form for a maximization problem. The straightforward (but longer) approach would begin by converting each such constraint to $\leq$ form

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \longrightarrow -\sum_{j=1}^{n} a_{ij}x_j \leq -b_i.$$

Constructing the dual problem in the usual way then gives $-a_{ij}$ as the coefficient of y_i in functional constraint j (which has $\geq$ form) and a coefficient of $-b_i$ in the objective function (which is to be minimized), where y_i also has a nonnegativity constraint $y_i \geq 0$. Now suppose we define a new variable $y_i' = -y_i$. The changes caused by expressing the dual problem in terms of y_i' instead of y_i are that (1) the coefficients of the variable become a_{ij} for functional constraint j and b_i for the objective function and (2) the constraint on the variable becomes $y_i' \leq 0$ (a *nonpositivity constraint*). The shortcut is to use y_i' instead of y_i as a dual variable so that the parameters in the original constraint (a_{ij} and b_i) immediately become the coefficients of this variable in the dual problem.

Here is a useful mnemonic device for remembering what the forms of dual constraints should be. With a maximization problem, it might seem *sensible* for a functional constraint to be in $\leq$ form, slightly *odd* to be in $=$ form, and somewhat *bizarre* to be in $\geq$ form. Similarly, for a minimization problem, it might seem *sensible* to be in $\geq$ form, slightly *odd* to be in $=$ form, and somewhat *bizarre* to be in $\leq$ form. For the constraint on an individual variable in either kind of problem, it might seem *sensible* to have a nonnegativity constraint, somewhat *odd* to have no constraint (so the variable is unconstrained in sign), and quite *bizarre* for the variable to be restricted to be *less* than or equal to zero. Now recall the correspondence between entities in the primal and dual problems indicated in Table 6.3; namely, functional constraint i in one problem corresponds to variable i in the other problem, and vice versa. The **sensible-odd-bizarre method,** or **SOB method** for short, says that the form of a functional constraint or the constraint on a variable in the dual problem should be sensible, odd, or bizarre, depending on whether the form for the corresponding entity in the primal problem is sensible, odd, or bizarre. Here is a summary.

The SOB Method for Determining the Form of Constraints in the Dual.[3]

1. Formulate the primal problem in either maximization form or minimization form, and then the dual problem automatically will be in the other form.
2. Label the different forms of functional constraints and of constraints on individual variables in the primal problem as being *sensible, odd,* or *bizarre* according to Table 6.14. The labeling of the functional constraints depends on whether the problem is a *maximization* problem (use the second column) or a *minimization* problem (use the third column).
3. For each constraint on an *individual variable* in the dual problem, use the form that has the same label as for the functional constraint in the primal problem that corresponds to this dual variable (as indicated by Table 6.3).
4. For each *functional constraint* in the dual problem, use the form that has the same label as for the constraint on the corresponding individual variable in the primal problem (as indicated by Table 6.3).

The arrows between the second and third columns of Table 6.14 spell out the correspondence between the forms of constraints in the primal and dual. Note that the correspondence always is between a functional constraint in one problem and a constraint on an individual variable in the other problem. Since the primal problem can be either a maximization or minimization problem, where the dual then will be of the opposite type, the second column of the table gives the form for whichever is the maximization problem and the third column gives the form for the other problem (a minimization problem).

To illustrate, consider the radiation therapy example presented at the beginning of Sec. 3.4. To show the conversion in both directions in Table 6.14, we begin with the maximization form of this model as the primal problem, before using the (original) minimization form.

The primal problem in maximization form is shown on the left side of Table 6.15. By using the second column of Table 6.14 to represent this problem, the arrows in this table indicate the form of the dual problem in the third column. These same arrows are used in Table 6.15 to show the resulting dual problem. (Because of these arrows, we have placed the functional constraints last in the dual problem rather than in their usual top position.)

[3]This particular mnemonic device (and a related one) for remembering what the forms of the dual constraints should be has been suggested by Arthur T. Benjamin, a mathematics professor at Harvey Mudd College. An interesting and wonderfully bizarre fact about Professor Benjamin himself is that he is one of the world's great human calculators who can perform such feats as quickly multiplying six-digit numbers in his head. For a further discussion and derivation of the SOB method, see A. T. Benjamin: "Sensible Rules for Remembering Duals — The S-O-B Method," *SIAM Review*, **37**(1): 85–87, 1995.

TABLE 6.14 Corresponding primal-dual forms

Label	Primal Problem (or Dual Problem)	Dual Problem (or Primal Problem)
	Maximize Z (or W)	Minimize W (or Z)
	Constraint i:	Variable y_i (or x_i):
Sensible	$\le$ form ⟷	$y_i \ge 0$
Odd	$=$ form ⟷	Unconstrained
Bizarre	$\ge$ form ⟷	$y_i' \le 0$
	Variable x_j (or y_j):	Constraint j:
Sensible	$x_j \ge 0$ ⟷	$\ge$ form
Odd	Unconstrained ⟷	$=$ form
Bizarre	$x_j' \le 0$ ⟷	$\le$ form

TABLE 6.15 One primal-dual form for the radiation therapy example

Primal Problem			Dual Problem	
Maximize $-Z = -0.4x_1 - 0.5x_2$,			Minimize $W = 2.7y_1 + 6y_2 + 6y_3'$,	
subject to			subject to	
(S)	$0.3x_1 + 0.1x_2 \le 2.7$	⟷	$y_1 \ge 0$	(S)
(O)	$0.5x_1 + 0.5x_2 = 6$	⟷	y_2 unconstrained in sign	(O)
(B)	$0.6x_1 + 0.4x_2 \ge 6$	⟷	$y_3' \le 0$	(B)
and			and	
(S)	$x_1 \ge 0$	⟷	$0.3y_1 + 0.5y_2 + 0.6y_3' \ge -0.4$	(S)
(S)	$x_2 \ge 0$	⟷	$0.1y_1 + 0.5y_2 + 0.4y_3' \ge -0.5$	(S)

Beside each constraint in both problems, we have inserted (in parentheses) an S, O, or B to label the form as sensible, odd, or bizarre. As prescribed by the SOB method, the label for each dual constraint always is the same as for the corresponding primal constraint.

However, there was no need (other than for illustrative purposes) to convert the primal problem to maximization form. Using the original minimization form, the equivalent primal problem is shown on the left side of Table 6.16. Now we use the *third column* of Table 6.14 to represent this primal problem, where the arrows indicate the form of the dual problem in the *second column*. These same arrows in Table 6.16 show the resulting dual problem on the right side. Again, the labels on the constraints show the application of the SOB method.

Just as the primal problems in Tables 6.15 and 6.16 are equivalent, the two dual problems also are completely equivalent. The key to recognizing this equivalency lies in the fact that the variables in each version of the dual problem are the negative of those in the other version ($y_1' = -y_1$, $y_2' = -y_2$, $y_3 = -y_3'$). Therefore, for each version, if the variables in the other version are used instead, and if both the objective function and the constraints are multiplied through by -1, then the other version is obtained. (Problem 6.4-5 asks you to verify this.)

If you would like to see **another example** of using the SOB method to construct a dual problem, one is given in the Worked Examples section of the book's website.

If the simplex method is to be applied to either a primal or a dual problem that has any variables constrained to be *nonpositive* (for example, $y_3' \le 0$ in the dual problem of Table 6.15), this variable may be replaced by its *nonnegative* counterpart (for example, $y_3 = -y_3'$).

TABLE 6.16 The other primal-dual form for the radiation therapy example

Primal Problem		Dual Problem	
Minimize $Z = 0.4x_1 + 0.5x_2$,		Maximize $W = 2.7y_1' + 6y_2' + 6y_3$,	
subject to		subject to	
(B)	$0.3x_1 + 0.1x_2 \leq 2.7$ $\longleftrightarrow$	$y_1' \leq 0$	(B)
(O)	$0.5x_1 + 0.5x_2 = 6$ $\longleftrightarrow$	y_2' unconstrained in sign	(O)
(S)	$0.6x_1 + 0.4x_2 \geq 6$ $\longleftrightarrow$	$y_3 \geq 0$	(S)
and		and	
(S)	$x_1 \geq 0$ $\longleftrightarrow$	$0.3y_1' + 0.5y_2' + 0.6y_3 \leq 0.4$	(S)
(S)	$x_2 \geq 0$ $\longleftrightarrow$	$0.1y_1' + 0.5y_2' + 0.4y_3 \leq 0.6$	(S)

When artificial variables are used to help the simplex method solve a primal problem, the duality interpretation of row 0 of the simplex tableau is the following: Since artificial variables play the role of slack variables, their coefficients in row 0 now provide the values of the corresponding dual variables in the complementary basic solution for the dual problem. Since artificial variables are used to replace the real problem with a more convenient artificial problem, this dual problem actually is the dual of the artificial problem. However, after all the artificial variables become nonbasic, we are back to the real primal and dual problems. With the two-phase method, the artificial variables would need to be retained in phase 2 in order to read off the complete dual solution from row 0. With the Big M method, since M has been added initially to the coefficient of each artificial variable in row 0, the current value of each corresponding dual variable is the current coefficient of this artificial variable *minus* M.

For example, look at row 0 in the final simplex tableau for the radiation therapy example, given at the bottom of Table 4.12. After M is subtracted from the coefficients of the artificial variables $\bar{x}_4$ and $\bar{x}_6$, the optimal solution for the corresponding dual problem given in Table 6.15 is read from the coefficients of x_3, $\bar{x}_4$, and $\bar{x}_6$ as $(y_1, y_2, y_3') = (0.5, -1.1, 0)$. As usual, the surplus variables for the two functional constraints are read from the coefficients of x_1 and x_2 as $z_1 - c_1 = 0$ and $z_2 - c_2 = 0$.

6.5 THE ROLE OF DUALITY THEORY IN SENSITIVITY ANALYSIS

As described further in the next three sections, sensitivity analysis basically involves investigating the effect on the optimal solution of making changes in the values of the model parameters a_{ij}, b_i, and c_j. However, changing parameter values in the primal problem also changes the corresponding values in the dual problem. Therefore, you have your choice of which problem to use to investigate each change. Because of the primal-dual relationships presented in Secs. 6.1 and 6.3 (especially the complementary basic solutions property), it is easy to move back and forth between the two problems as desired. In some cases, it is more convenient to analyze the dual problem directly in order to determine the complementary effect on the primal problem. We begin by considering two such cases.

Changes in the Coefficients of a Nonbasic Variable

Suppose that the changes made in the original model occur in the coefficients of a variable that was nonbasic in the original optimal solution. What is the effect of these changes on this solution? Is it still feasible? Is it still optimal?

Because the variable involved is nonbasic (value of zero), changing its coefficients cannot affect the feasibility of the solution. Therefore, the open question in this case is whether it is still optimal. As Tables 6.10 and 6.11 indicate, an equivalent question is whether the complementary basic solution for the dual problem is still feasible after these changes are made. Since these changes affect the dual problem by changing only one constraint, this question can be answered simply by checking whether this complementary basic solution still satisfies this revised constraint.

We shall illustrate this case in the corresponding subsection of Sec. 6.7 after developing a relevant example. The Worked Examples section of the book's website also gives **another example** for both this case and the next one.

Introduction of a New Variable

As indicated in Table 6.6, the decision variables in the model typically represent the levels of the various activities under consideration. In some situations, these activities were selected from a larger group of *possible* activities, where the remaining activities were not included in the original model because they seemed less attractive. Or perhaps these other activities did not come to light until after the original model was formulated and solved. Either way, the key question is whether any of these previously unconsidered activities are sufficiently worthwhile to warrant initiation. In other words, would adding any of these activities to the model change the original optimal solution?

Adding another activity amounts to introducing a new variable, with the appropriate coefficients in the functional constraints and objective function, into the model. The only resulting change in the dual problem is to add a *new constraint* (see Table 6.3).

After these changes are made, would the original optimal solution, along with the new variable equal to zero (nonbasic), still be optimal for the primal problem? As for the preceding case, an equivalent question is whether the complementary basic solution for the dual problem is still feasible. And, as before, this question can be answered simply by checking whether this complementary basic solution satisfies one constraint, which in this case is the new constraint for the dual problem.

To illustrate, suppose for the Wyndor Glass Co. problem of Sec. 3.1 that a possible third new product now is being considered for inclusion in the product line. Letting x_{new} represent the production rate for this product, we show the resulting revised model as follows:

$$\text{Maximize} \quad Z = 3x_1 + 5x_2 + 4x_{\text{new}},$$

subject to

$$\begin{aligned} x_1 \qquad\quad + 2x_{\text{new}} &\le 4 \\ 2x_2 + 3x_{\text{new}} &\le 12 \\ 3x_1 + 2x_2 + x_{\text{new}} &\le 18 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_{\text{new}} \ge 0.$$

After we introduced slack variables, the original optimal solution for this problem without x_{new} (given by Table 4.8) was $(x_1, x_2, x_3, x_4, x_5) = (2, 6, 2, 0, 0)$. Is this solution, along with $x_{\text{new}} = 0$, still optimal?

To answer this question, we need to check the complementary basic solution for the dual problem. As indicated by the *complementary optimal basic solutions property* in Sec. 6.3, this solution is given in row 0 of the *final* simplex tableau for the primal problem, using the locations shown in Table 6.4 and illustrated in Table 6.5. Therefore, as given in both the bottom row of Table 6.5 and the sixth row of Table 6.9, the solution is

$$(y_1, y_2, y_3, z_1 - c_1, z_2 - c_2) = \left(0, \frac{3}{2}, 1, 0, 0\right).$$

(Alternatively, this complementary basic solution can be derived in the way that was illustrated in Sec. 6.3 for the complementary basic solution in the next-to-last row of Table 6.9.)

Since this solution was optimal for the original dual problem, it certainly satisfies the original dual constraints shown in Table 6.1. But does it satisfy this new dual constraint?

$$2y_1 + 3y_2 + y_3 \geq 4$$

Plugging in this solution, we see that

$$2(0) + 3\left(\frac{3}{2}\right) + (1) \geq 4$$

is satisfied, so this dual solution is still feasible (and thus still optimal). Consequently, the original primal solution (2, 6, 2, 0, 0), along with $x_{\text{new}} = 0$, is still optimal, so this third possible new product should *not* be added to the product line.

This approach also makes it very easy to conduct sensitivity analysis on the coefficients of the new variable added to the primal problem. By simply checking the new dual constraint, you can immediately see how far any of these parameter values can be changed before they affect the feasibility of the dual solution and so the optimality of the primal solution.

Other Applications

Already we have discussed two other key applications of duality theory to sensitivity analysis, namely, *shadow prices* and the *dual simplex method*. As described in Secs. 4.7 and 6.2, the optimal dual solution $(y_1^*, y_2^*, \ldots, y_m^*)$ provides the shadow prices for the respective resources that indicate how Z would change if (small) changes were made in the b_i (the resource amounts). The resulting analysis will be illustrated in some detail in Sec. 6.7.

In more general terms, the economic interpretation of the dual problem and of the simplex method presented in Sec. 6.2 provides some useful insights for sensitivity analysis.

When we investigate the effect of changing the b_i or the a_{ij} values (for basic variables), the original optimal solution may become a *superoptimal* basic solution (as defined in Table 6.10) instead. If we then want to *reoptimize* to identify the new optimal solution, the dual simplex method (discussed at the end of Secs. 6.1 and 6.3) should be applied, starting from this basic solution. (This important variant of the simplex method will be described in Sec. 7.1.)

We mentioned in Sec. 6.1 that sometimes it is more efficient to solve the dual problem directly by the simplex method in order to identify an optimal solution for the primal problem. When the solution has been found in this way, sensitivity analysis for the primal problem then is conducted by applying the procedure described in the next two sections directly to the dual problem and then inferring the complementary effects on the primal problem (e.g., see Table 6.11). This approach to sensitivity analysis is relatively straightforward because of the close primal-dual relationships described in Secs. 6.1 and 6.3. (See Prob. 6.6-3.)

6.6 THE ESSENCE OF SENSITIVITY ANALYSIS

The work of the operations research team usually is not even nearly done when the simplex method has been successfully applied to identify an optimal solution for the model. As we pointed out at the end of Sec. 3.3, one assumption of linear programming is that all the parameters of the model (a_{ij}, b_i, and c_j) are *known constants.* Actually, the parameter values used in the model normally are just *estimates* based on a *prediction of future conditions.* The data obtained to develop these estimates often are rather crude or nonexistent, so that the parameters in the original formulation may represent little more than quick rules of thumb provided by busy line personnel. The data may even represent deliberate overestimates or underestimates to protect the interests of the estimators.

Thus, the successful manager and operations research staff will maintain a healthy skepticism about the original numbers coming out of the computer and will view them in many cases as only a starting point for further analysis of the problem. An "optimal" solution is optimal only with respect to the specific model being used to represent the real problem, and such a solution becomes a reliable guide for action only after it has been verified as performing well for other reasonable representations of the problem. Furthermore, the model parameters (particularly b_i) sometimes are set as a result of managerial policy decisions (e.g., the amount of certain resources to be made available to the activities), and these decisions should be reviewed after their potential consequences are recognized.

For these reasons it is important to perform **sensitivity analysis** to investigate the effect on the optimal solution provided by the simplex method if the parameters take on other possible values. Usually there will be some parameters that can be assigned any reasonable value without the optimality of this solution being affected. However, there may also be parameters with likely alternative values that would yield a new optimal solution. This situation is particularly serious if the original solution would then have a substantially inferior value of the objective function, or perhaps even be infeasible!

Therefore, one main purpose of sensitivity analysis is to identify the **sensitive parameters** (i.e., the parameters whose values cannot be changed without changing the optimal solution). For coefficients in the objective function that are not categorized as sensitive, it is also very helpful to determine the *range of values* of the coefficient over which the optimal solution will remain unchanged. (We call this range of values the *allowable range for that coefficient.*) In some cases, changing the right-hand side of a functional constraint can affect the *feasibility* of the optimal BF solution. For such parameters, it is useful to determine the range of values over which the optimal BF solution (with adjusted values for the basic variables) will remain feasible. (We call this range of values the *allowable range* for the right-hand side involved.) This range of values also is the range over which the current *shadow price* for the corresponding constraint remains valid. In the next section, we will describe the specific procedures for obtaining this kind of information.

Such information is invaluable in two ways. First, it identifies the more important parameters, so that special care can be taken to estimate them closely and to select a solution that performs well for most of their likely values. Second, it identifies the parameters that will need to be monitored particularly closely as the study is implemented. If it is discovered that the true value of a parameter lies outside its allowable range, this immediately signals a need to change the solution.

For small problems, it would be straightforward to check the effect of a variety of changes in parameter values simply by reapplying the simplex method each time to see if the optimal solution changes. This is particularly convenient when using a spreadsheet formulation. Once the Solver has been set up to obtain an optimal solution, all you have to do is make any desired change on the spreadsheet and then click on the Solve button again.

However, for larger problems of the size typically encountered in practice, sensitivity analysis would require an exorbitant computational effort if it were necessary to reapply the simplex method from the beginning to investigate each new change in a parameter value. Fortunately, the fundamental insight discussed in Sec. 5.3 virtually eliminates computational effort. The basic idea is that the fundamental insight *immediately* reveals just how any changes in the original model would change the numbers in the final simplex tableau (assuming that the *same* sequence of algebraic operations originally performed by the simplex method were to be *duplicated*). Therefore, after making a few simple calculations to revise this tableau, we can check easily whether the original optimal BF solution is now nonoptimal (or infeasible). If so, this solution would be used as the initial basic solution to restart the simplex method (or dual simplex method) to find the new optimal solution, if desired. If the changes in the model are not major, only a very few iterations should be required to reach the new optimal solution from this "advanced" initial basic solution.

To describe this procedure more specifically, consider the following situation. The simplex method already has been used to obtain an optimal solution for a linear programming model with specified values for the b_i, c_j, and a_{ij} parameters. To initiate sensitivity analysis, at least one of the parameters is changed. After the changes are made, let $\bar{b}_i$, $\bar{c}_j$, and $\bar{a}_{ij}$ denote the values of the various parameters. Thus, in matrix notation,

$$\mathbf{b} \rightarrow \bar{\mathbf{b}}, \qquad \mathbf{c} \rightarrow \bar{\mathbf{c}}, \qquad \mathbf{A} \rightarrow \bar{\mathbf{A}},$$

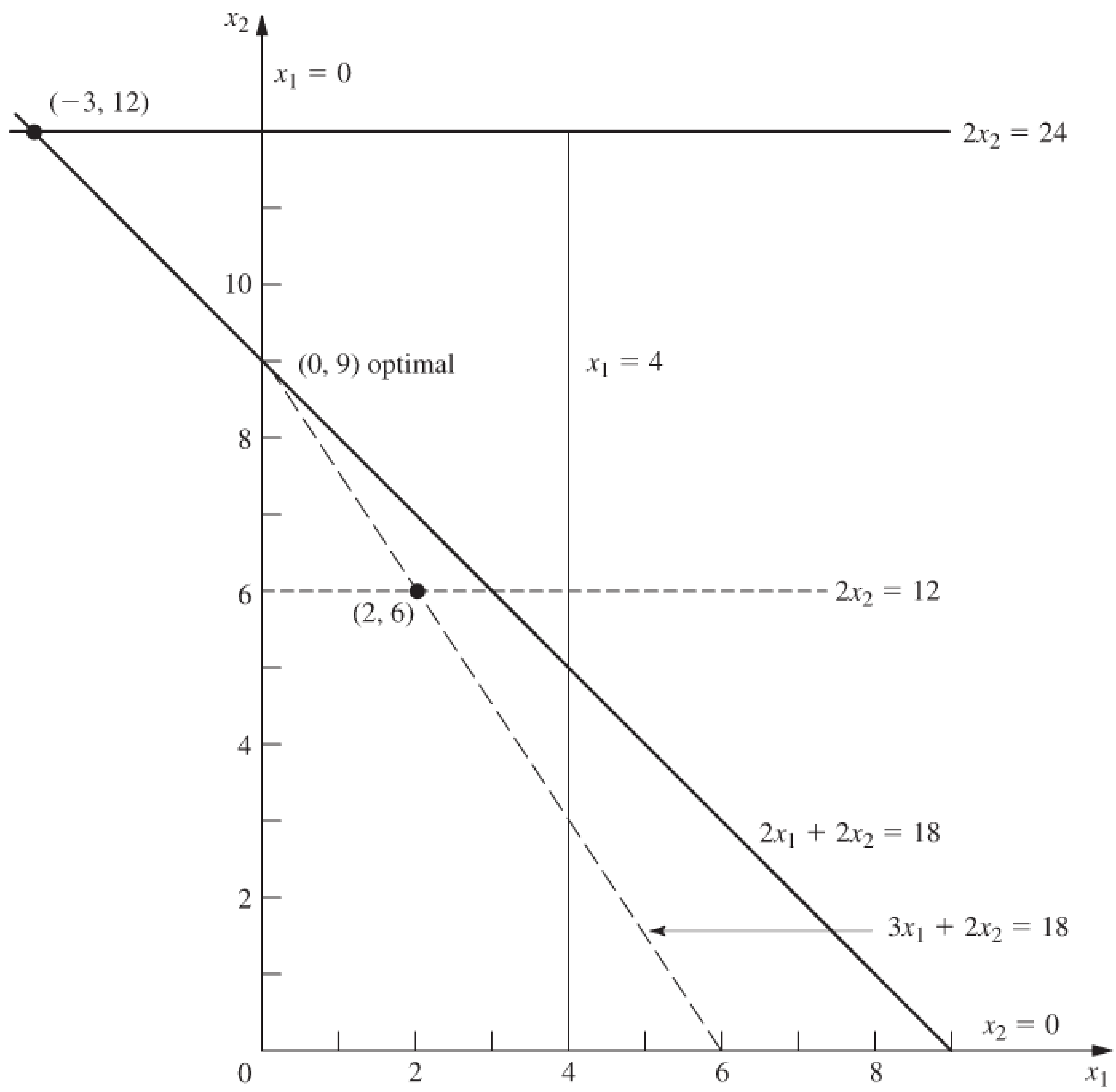
for the revised model.

The first step is to revise the final simplex tableau to reflect these changes. In particular, we want to find the revised final tableau that would result if *exactly* the same algebraic operations (including the same multiples of rows being added to or subtracted from other rows) that led from the initial tableau to the final tableau were repeated when starting from the new initial tableau. (This isn't necessarily the same as reapplying the simplex method since the changes in the initial tableau might cause the simplex method to change some of the algebraic operations being used.) Continuing to use the notation presented in Table 5.9, as well as the accompanying formulas for the fundamental insight [(1) $\mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T}$ and (2) $\mathbf{T}^* = \mathbf{S}^*\mathbf{T}$], the revised final tableau is calculated from $\mathbf{y}^*$ and $\mathbf{S}^*$ (which have not changed) and the new initial tableau, as shown in Table 6.17. Note that $\mathbf{y}^*$ and $\mathbf{S}^*$ together are the coefficients of the *slack variables* in the final simplex tableau, where the vector $\mathbf{y}^*$ (the dual variables) equals these coefficients in row 0 and the matrix $\mathbf{S}^*$ gives these coefficients in the other rows of the tableau. Thus, simply by using $\mathbf{y}^*$, $\mathbf{S}^*$, and the revised numbers in the *initial* tableau, Table 6.17 reveals how the revised numbers in the rest of the *final* tableau are calculated immediately without having to repeat any algebraic operations.

Example (Variation 1 of the Wyndor Model). To illustrate, suppose that the first revision in the model for the Wyndor Glass Co. problem of Sec. 3.1 is the one shown in Table 6.18.

TABLE 6.17 Revised final simplex tableau resulting from changes in original model

		Coefficient of:			
	Eq.	Z	**Original Variables**	**Slack Variables**	**Right Side**
New initial tableau	(0)	1	$-\bar{\mathbf{c}}$	$\mathbf{0}$	0
	$(1, 2, \ldots, m)$	$\mathbf{0}$	$\bar{\mathbf{A}}$	$\mathbf{I}$	$\bar{\mathbf{b}}$
Revised final tableau	(0)	1	$\mathbf{z}^* - \bar{\mathbf{c}} = \mathbf{y}^*\bar{\mathbf{A}} - \bar{\mathbf{c}}$	$\mathbf{y}^*$	$Z^* = \mathbf{y}^*\bar{\mathbf{b}}$
	$(1, 2, \ldots, m)$	$\mathbf{0}$	$\mathbf{A}^* = \mathbf{S}^*\bar{\mathbf{A}}$	$\mathbf{S}^*$	$\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}}$



■ **FIGURE 6.2**
Shift of the final corner-point solution from (2, 6) to (−3, 12) for Variation 1 of the Wyndor Glass Co. model where $c_1 = 3 \to 4$, $a_{31} = 3 \to 2$, and $b_2 = 12 \to 24$.

■ **TABLE 6.18 The original model and the first revised model (variation 1) for conducting sensitivity analysis on the Wyndor Glass Co. model**

Original Model

Maximize $\quad Z = [3, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$

subject to

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

and

$$\mathbf{x} \ge \mathbf{0}.$$

Revised Model

Maximize $\quad Z = [4, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$

subject to

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$$

and

$$\mathbf{x} \ge \mathbf{0}.$$

Thus, the changes from the original model are $c_1 = 3 \to 4$, $a_{31} = 3 \to 2$, and $b_2 = 12 \to 24$. Figure 6.2 shows the graphical effect of these changes. For the original model, the simplex method already has identified the optimal CPF solution as (2, 6), lying at the intersection of the two constraint boundaries, shown as dashed lines $2x_2 = 12$ and $3x_1 + 2x_2 = 18$. Now the revision of the model has shifted both of these constraint boundaries as shown by the dark lines $2x_2 = 24$ and $2x_1 + 2x_2 = 18$. Consequently, the previous

CPF solution (2, 6) now shifts to the new intersection (−3, 12), which is a corner-point *infeasible* solution for the revised model. The procedure described in the preceding paragraphs finds this shift *algebraically* (in augmented form). Furthermore, it does so in a manner that is very efficient even for huge problems where graphical analysis is impossible.

To carry out this procedure, we begin by displaying the parameters of the revised model in matrix form:

$$\bar{\mathbf{c}} = [4, 5], \qquad \bar{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix}, \qquad \bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}.$$

The resulting new initial simplex tableau is shown at the top of Table 6.19. Below this tableau is the original final tableau (as first given in Table 4.8). We have drawn dark boxes around the portions of this final tableau that the changes in the model definitely *do not change,* namely, the coefficients of the slack variables in both row 0 ($\mathbf{y}^*$) and the rest of the rows ($\mathbf{S}^*$). Thus,

$$\mathbf{y}^* = [0, \tfrac{3}{2}, 1], \qquad \mathbf{S}^* = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

These coefficients of the slack variables necessarily are unchanged with the same algebraic operations originally performed by the simplex method because the coefficients of these same variables in the initial tableau are unchanged.

TABLE 6.19 Obtaining the revised final simplex tableau for Variation 1 of the Wyndor Glass Co. model

	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
New initial tableau	Z	(0)	1	−4	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	24
	x_5	(3)	0	2	2	0	0	1	18
Final tableau for original model	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
Revised final tableau	Z	(0)	1	−2	0	0	$\frac{3}{2}$	1	54
	x_3	(1)	0	$\frac{1}{3}$	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	6
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	12
	x_1	(3)	0	$\frac{2}{3}$	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	−2

However, because other portions of the initial tableau have changed, there will be changes in the rest of the final tableau as well. Using the formulas in Table 6.17, we calculate the revised numbers in the rest of the final tableau as follows:

$$\mathbf{z}^* - \bar{\mathbf{c}} = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} - [4, 5] = [-2, 0], \qquad Z^* = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = 54,$$

$$\mathbf{A}^* = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{bmatrix},$$

$$\mathbf{b}^* = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}.$$

The resulting revised final tableau is shown at the bottom of Table 6.19.

Actually, we can substantially streamline these calculations for obtaining the revised final tableau. Because none of the coefficients of x_2 changed in the original model (tableau), none of them can change in the final tableau, so we can delete their calculation. Several other original parameters (a_{11}, a_{21}, b_1, b_3) also were not changed, so another shortcut is to calculate only the *incremental changes* in the final tableau in terms of the incremental changes in the initial tableau, ignoring those terms in the vector or matrix multiplication that involve zero change in the initial tableau. In particular, the only incremental changes in the initial tableau are $\Delta c_1 = 1$, $\Delta a_{31} = -1$, and $\Delta b_2 = 12$, so these are the only terms that need be considered. This streamlined approach is shown below, where a zero or dash appears in each spot where no calculation is needed.

$$\Delta(\mathbf{z}^* - \mathbf{c}) = \mathbf{y}^*\,\Delta\mathbf{A} - \Delta\mathbf{c} = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 0 & — \\ 0 & — \\ -1 & — \end{bmatrix} - [1, —] = [-2, —].$$

$$\Delta Z^* = \mathbf{y}^*\,\Delta\mathbf{b} = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} = 18.$$

$$\Delta\mathbf{A}^* = \mathbf{S}^*\,\Delta\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 & — \\ 0 & — \\ -1 & — \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & — \\ 0 & — \\ -\frac{1}{3} & — \end{bmatrix}.$$

$$\Delta\mathbf{b}^* = \mathbf{S}^*\,\Delta\mathbf{b} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -4 \end{bmatrix}.$$

Adding these increments to the original quantities in the final tableau (middle of Table 6.19) then yields the revised final tableau (bottom of Table 6.19).

This *incremental analysis* also provides a useful general insight, namely, that changes in the final tableau must be *proportional* to each change in the initial tableau. We illustrate in the next section how this property enables us to use linear interpolation or extrapolation to determine the range of values for a given parameter over which the final basic solution remains both feasible and optimal.

After obtaining the revised final simplex tableau, we next convert the tableau to proper form from Gaussian elimination (as needed). In particular, the basic variable for row i must have a coefficient of 1 in that row and a coefficient of 0 in every other row (including row 0) for the tableau to be in the proper form for identifying and evaluating the current basic solution. Therefore, if the changes have violated this requirement (which can occur only if the original constraint coefficients of a basic variable have been changed), further changes must be made to restore this form. This restoration is done by using Gaussian elimination, i.e., by successively applying step 3 of an iteration for the simplex method (see Chap. 4) as if each violating basic variable were an entering basic variable. Note that these algebraic operations may also cause further changes in the *right side* column, so that the current basic solution can be read from this column only when the proper form from Gaussian elimination has been fully restored.

For the example, the revised final simplex tableau shown in the top half of Table 6.20 is not in proper form from Gaussian elimination because of the column for the basic variable x_1. Specifically, the coefficient of x_1 in *its* row (row 3) is $\frac{2}{3}$ instead of 1, and it has nonzero coefficients (-2 and $\frac{1}{3}$) in rows 0 and 1. To restore proper form, row 3 is multiplied by $\frac{3}{2}$; then 2 times this new row 3 is added to row 0 and $\frac{1}{3}$ times new row 3 is subtracted from row 1. This yields the proper form from Gaussian elimination shown in the bottom half of Table 6.20, which now can be used to identify the new values for the current (previously optimal) basic solution:

$$(x_1, x_2, x_3, x_4, x_5) = (-3, 12, 7, 0, 0).$$

Because x_1 is negative, this basic solution no longer is feasible. However, it is *superoptimal* (as defined in Table 6.10), and so *dual feasible,* because *all* the coefficients in row 0 still are *nonnegative.* Therefore, the dual simplex method can be used to reoptimize (if desired), by starting from this basic solution. (The sensitivity analysis procedure in IOR Tutorial includes this option.) Referring to Fig. 6.2 (and ignoring slack variables), the dual simplex method

TABLE 6.20 Converting the revised final simplex tableau to proper form from Gaussian elimination for Variation 1 of the Wyndor Glass Co. model

	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Revised final tableau	Z	(0)	1	-2	0	0	$\frac{3}{2}$	1	54
	x_3	(1)	0	$\frac{1}{3}$	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	6
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	12
	x_1	(3)	0	$\frac{2}{3}$	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	-2
Converted to proper form	Z	(0)	1	0	0	0	$\frac{1}{2}$	2	48
	x_3	(1)	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	7
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	12
	x_1	(3)	0	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	-3

uses just one iteration to move from the corner-point solution $(-3, 12)$ to the optimal CPF solution $(0, 9)$. (It is often useful in sensitivity analysis to identify the solutions that are optimal for some set of likely values of the model parameters and then to determine which of these solutions most *consistently* performs well for the various likely parameter values.)

If the basic solution $(-3, 12, 7, 0, 0)$ had been *neither* primal feasible nor dual feasible (i.e., if the tableau had negative entries in *both* the *right side* column and row 0), artificial variables could have been introduced to convert the tableau to the proper form for an initial simplex tableau.[4]

The General Procedure. When one is testing to see how *sensitive* the original optimal solution is to the various parameters of the model, the common approach is to check each parameter (or at least c_j and b_i) individually. In addition to finding allowable ranges as described in the next section, this check might include changing the value of the parameter from its initial estimate to other possibilities in the *range of likely values* (including the endpoints of this range). Then some combinations of simultaneous changes of parameter values (such as changing an entire functional constraint) may be investigated. *Each* time one (or more) of the parameters is changed, the procedure described and illustrated here would be applied. Let us now summarize this procedure.

Summary of Procedure for Sensitivity Analysis

1. *Revision of model:* Make the desired change or changes in the model to be investigated next.
2. *Revision of final tableau:* Use the fundamental insight (as summarized by the formulas on the bottom of Table 6.17) to determine the resulting changes in the final simplex tableau. (See Table 6.19 for an illustration.)
3. *Conversion to proper form from Gaussian elimination:* Convert this tableau to the proper form for identifying and evaluating the current basic solution by applying (as necessary) Gaussian elimination. (See Table 6.20 for an illustration.)
4. *Feasibility test:* Test this solution for feasibility by checking whether all its basic variable values in the right-side column of the tableau still are nonnegative.
5. *Optimality test:* Test this solution for optimality (if feasible) by checking whether all its nonbasic variable coefficients in row 0 of the tableau still are nonnegative.
6. *Reoptimization:* If this solution fails either test, the new optimal solution can be obtained (if desired) by using the current tableau as the initial simplex tableau (and making any necessary conversions) for the simplex method or dual simplex method.

The interactive routine entitled *sensitivity analysis* in IOR Tutorial will enable you to efficiently practice applying this procedure. In addition, a demonstration in OR Tutor (also entitled *sensitivity analysis*) provides you with **another example**.

For problems with only two decision variables, graphical analysis provides an alternative to the above algebraic procedure for performing sensitivity analysis. IOR Tutorial includes a procedure called *Graphical Method and Sensitivity Analysis* for performing such graphical analysis efficiently.

In the next section, we shall discuss and illustrate the application of the above algebraic procedure to each of the major categories of revisions in the original model. We also will use graphical analysis to illuminate what is being accomplished algebraically. This discussion will involve, in part, expanding upon the example introduced in this section for investigating changes in the Wyndor Glass Co. model. In fact, we shall begin by *individually* checking each of the preceding changes. At the same time, we shall integrate some of the applications of duality theory to sensitivity analysis discussed in Sec. 6.5.

[4]There also exists a primal-dual algorithm that can be directly applied to such a simplex tableau without any conversion.

6.7 APPLYING SENSITIVITY ANALYSIS

Sensitivity analysis often begins with the investigation of changes in the values of b_i, the amount of resource i ($i = 1, 2, \ldots, m$) being made available for the activities under consideration. The reason is that there generally is more flexibility in setting and adjusting these values than there is for the other parameters of the model. As already discussed in Secs. 4.7 and 6.2, the economic interpretation of the dual variables (the y_i) as shadow prices is extremely useful for deciding which changes should be considered.

Case 1—Changes in b_i

Suppose that the only changes in the current model are that one or more of the b_i parameters ($i = 1, 2, \ldots, m$) has been changed. In this case, the *only* resulting changes in the final simplex tableau are in the *right-side* column. Consequently, the tableau still will be in proper form from Gaussian elimination and all the nonbasic variable coefficients in row 0 still will be nonnegative. Therefore, both the *conversion to proper form from Gaussian elimination* and the *optimality test* steps of the general procedure can be skipped. After revising the right-side column of the tableau, the only question will be whether all the basic variable values in this column still are nonnegative (the feasibility test).

As shown in Table 6.17, when the vector of the b_i values is changed from $\mathbf{b}$ to $\overline{\mathbf{b}}$, the formulas for calculating the new *right-side* column in the final tableau are

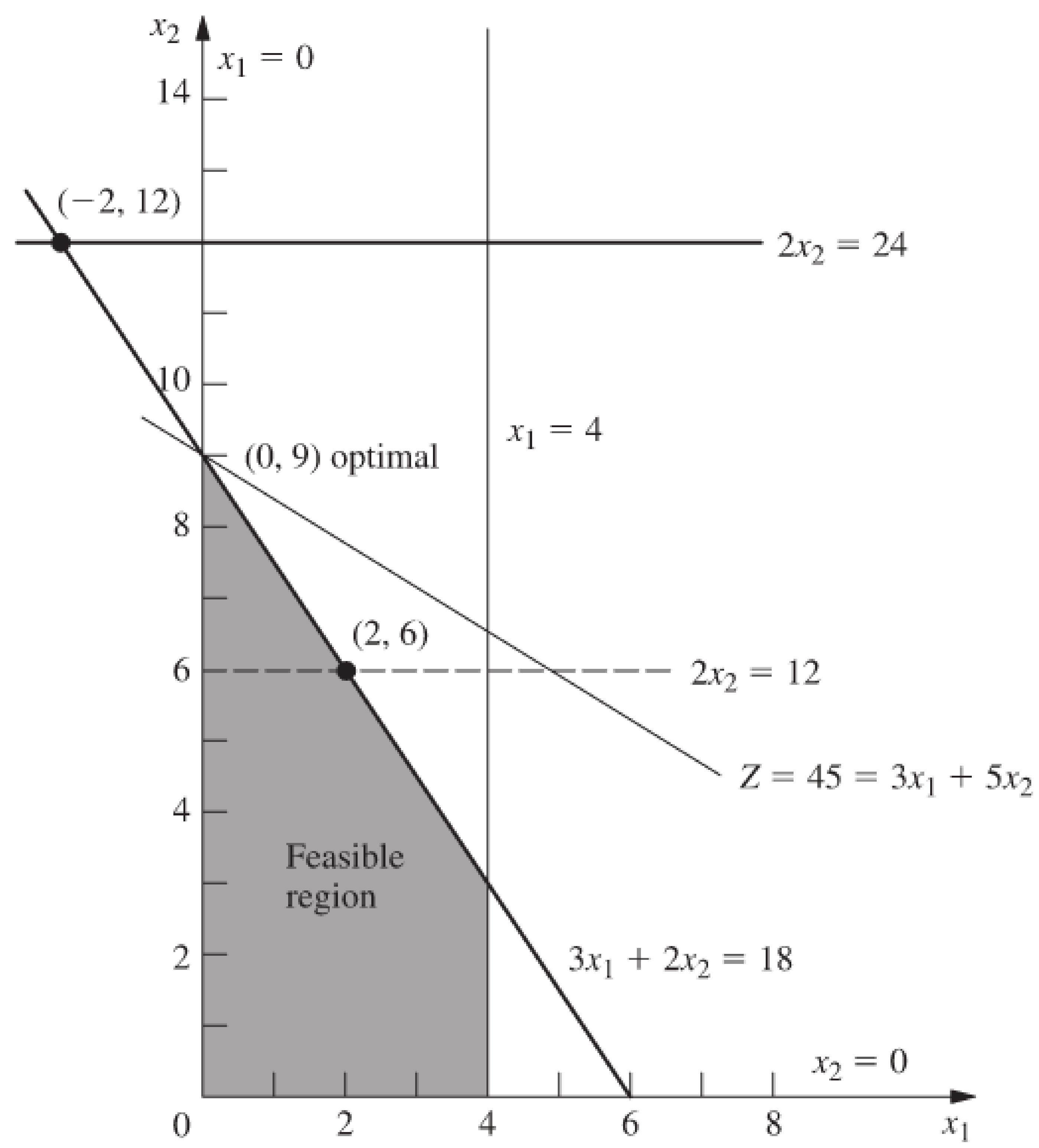
Right side of final row 0: $\quad Z^* = \mathbf{y}^*\overline{\mathbf{b}}$,

Right side of final rows $1, 2, \ldots, m$: $\quad \mathbf{b}^* = \mathbf{S}^*\overline{\mathbf{b}}$.

(See the bottom of Table 6.17 for the location of the unchanged vector $\mathbf{y}^*$ and matrix $\mathbf{S}^*$ in the final tableau.) The first equation has a natural economic interpretation that relates to the economic interpretation of the dual variables presented at the beginning of Sec. 6.2. The vector $\mathbf{y}^*$ gives the optimal values of the dual variables, where these values are interpreted as the *shadow prices* of the respective resources. In particular, when $\mathbf{Z}^*$ represents the profit from using the optimal primal solution $\mathbf{x}^*$ and each b_i represents the amount of resource i being made available, $\mathbf{y}_i^*$ indicates how much the profit could be increased per unit increase in b_i (for small increases in b_i).

Example (Variation 2 of the Wyndor Model). Sensitivity analysis is begun for the original Wyndor Glass Co. problem of Sec. 3.1 by examining the optimal values of the y_i dual variables ($y_1^* = 0$, $y_2^* = \frac{3}{2}$, $y_3^* = 1$). These *shadow prices* give the marginal value of each resource i (the available production capacity of Plant i) for the activities (two new products) under consideration, where marginal value is expressed in the units of Z (thousands of dollars of profit per week). As discussed in Sec. 4.7 (see Fig. 4.8), the total profit from these activities can be increased \$1,500 per week (y_2^* times \$1,000 per week) for each additional unit of resource 2 (hour of production time per week in Plant 2) that is made available. This increase in profit holds for relatively small changes that do not affect the feasibility of the current basic solution (and so do not affect the y_i^* values).

Consequently, the OR team has investigated the marginal profitability from the other current uses of this resource to determine if any are less than \$1,500 per week. This investigation reveals that one old product is far less profitable. The production rate for this product already has been reduced to the minimum amount that would justify its marketing expenses. However, it can be discontinued altogether, which would provide an additional 12 units of resource 2 for the new products. Thus, the next step is to determine the profit that could be obtained from the new products if this shift were made. This shift changes b_2 from 12 to 24 in the linear programming model. Figure 6.3 shows the graphical effect of this change, including the shift in the final corner-point solution from (2, 6) to (−2, 12).



■ **FIGURE 6.3**
Feasible region for Variation 2 of the Wyndor Glass Co. model where $b_2 = 12 \rightarrow 24$.

(Note that this figure differs from Fig. 6.2, which depicts Variation 1 of the Wyndor model, because the constraint $3x_1 + 2x_2 \leq 18$ has not been changed here.)

Thus, for Variation 2 of the Wyndor model, the only revision in the original model is the following change in the vector of the b_i values:

$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} \longrightarrow \bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}.$$

so only b_2 has a new value.

Analysis of Variation 2. When the fundamental insight (Table 6.17) is applied, the effect of this change in b_2 on the original final simplex tableau (middle of Table 6.19) is that the entries in the right-side column change to the following values:

$$Z^* = \mathbf{y}^*\bar{\mathbf{b}} = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = 54,$$

$$\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}, \qquad \text{so} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}.$$

Equivalently, because the only change in the original model is $\Delta b_2 = 24 - 12 = 12$, incremental analysis can be used to calculate these same values more quickly. Incremental analysis involves calculating just the *increments* in the tableau values caused by the change (or changes) in the original model, and then adding these increments to the original values. In this case, the increments in Z^* and $\mathbf{b}^*$ are

$$\Delta Z^* = \mathbf{y}^* \Delta \mathbf{b} = \mathbf{y}^* \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = \mathbf{y}^* \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix},$$

$$\Delta \mathbf{b}^* = \mathbf{S}^* \, \Delta \mathbf{b} = \mathbf{S}^* \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = \mathbf{S}^* \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}.$$

Therefore, using the second component of $\mathbf{y}^*$ and the second column of $\mathbf{S}^*$, the only calculations needed are

$$\Delta Z^* = \frac{3}{2}(12) = 18, \qquad \text{so } Z^* = 36 + 18 = 54,$$

$$\Delta b_1^* = \frac{1}{3}(12) = 4, \qquad \text{so } b_1^* = 2 + 4 = 6,$$

$$\Delta b_2^* = \frac{1}{2}(12) = 6, \qquad \text{so } b_2^* = 6 + 6 = 12,$$

$$\Delta b_3^* = -\frac{1}{3}(12) = -4, \qquad \text{so } b_3^* = 2 - 4 = -2,$$

where the original values of these quantities are obtained from the right-side column in the original final tableau (middle of Table 6.19). The resulting revised final tableau corresponds completely to this original final tableau except for replacing the right-side column with these new values.

Therefore, the current (previously optimal) basic solution has become

$$(x_1, x_2, x_3, x_4, x_5) = (-2, 12, 6, 0, 0),$$

which fails the feasibility test because of the negative value. The dual simplex method now can be applied, starting with this revised simplex tableau, to find the new optimal solution. This method leads in just one iteration to the new final simplex tableau shown in Table 6.21. (Alternatively, the simplex method could be applied from the beginning, which also would lead to this final tableau in just one iteration in this case.) This tableau indicates that the new optimal solution is

$$(x_1, x_2, x_3, x_4, x_5) = (0, 9, 4, 6, 0),$$

with $Z = 45$, thereby providing an increase in profit from the new products of 9 units ($9,000 per week) over the previous $Z = 36$. The fact that $x_4 = 6$ indicates that 6 of the 12 additional units of resource 2 are unused by this solution.

TABLE 6.21 Data for Variation 2 of the Wyndor Glass Co. model

Model Parameters

$c_1 = 3,$	$c_2 = 5$	$(n = 2)$
$a_{11} = 1,$	$a_{12} = 0,$	$b_1 = 4$
$a_{21} = 0,$	$a_{22} = 2,$	$b_2 = 24$
$a_{31} = 3,$	$a_{32} = 2,$	$b_3 = 18$

Final Simplex Tableau after Reoptimization

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	45
x_3	(1)	0	1	0	1	0	0	4
x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
x_4	(3)	0	−3	0	0	1	−1	6

An Application Vignette

The **Pacific Lumber Company (PALCO)** is a large timber-holding company with headquarters in Scotia, California. The company has over 200,000 acres of highly productive forest lands that support five mills located in Humboldt County in northern California. The lands include some of the most spectacular redwood groves in the world that have been given or sold at low cost to be preserved as parks. PALCO manages the remaining lands intensively for sustained timber production, subject to strong forest practice laws. Since PALCO's forests are home to many species of wildlife, including endangered species such as spotted owls and marbled murrelets, the provisions of the federal Endangered Species Act also need to be carefully observed.

To obtain a sustained yield plan for the entire landholding, PALCO management contracted with a team of OR consultants to develop a 120-year, 12-period, long-term forest ecosystem management plan. The OR team performed this task by formulating and applying a linear programming model to optimize the company's overall timberland operations and profitability after satisfying the various constraints. The model was a huge one with approximately 8,500 functional constraints and 353,000 decision variables.

A major challenge in applying the linear programming model was the many uncertainties in estimating what the parameters of the model should be. The major factors causing these uncertainties were the continuing fluctuations in market supply and demand, logging costs, and environmental regulations. Therefore, the OR team made extensive use of *detailed sensitivity analysis*. The resulting sustained yield plan *increased the company's present net worth by over* **\$398 million** while also generating a better mix of wildlife habitat acres.

Source: L. R. Fletcher, H. Alden, S. P. Holmen, D. P. Angelis, and M. J. Etzenhouser: "Long-Term Forest Ecosystem Planning at Pacific Lumber," *Interfaces*, **29**(1): 90–112, Jan–Feb. 1999. (A link to this article is provided on our website, www.mhhe.com/hillier.)

Based on the results with $b_2 = 24$, the relatively unprofitable old product will be discontinued and the unused 6 units of resource 2 will be saved for some future use. Since y_3^* still is positive, a similar study is made of the possibility of changing the allocation of resource 3, but the resulting decision is to retain the current allocation. Therefore, the current linear programming model at this point (Variation 2) has the parameter values and optimal solution shown in Table 6.21. This model will be used as the starting point for investigating other types of changes in the model later in this section. However, before turning to these other cases, let us take a broader look at the current case.

The Allowable Range for a Right-Hand Side. Although $\Delta b_2 = 12$ proved to be too large an increase in b_2 to retain feasibility (and so optimality) with the basic solution where x_1, x_2, and x_3 are the basic variables (middle of Table 6.19), the above incremental analysis shows immediately just how large an increase is feasible. In particular, note that

$$b_1^* = 2 + \frac{1}{3}\Delta b_2,$$
$$b_2^* = 6 + \frac{1}{2}\Delta b_2,$$
$$b_3^* = 2 - \frac{1}{3}\Delta b_2,$$

where these three quantities are the values of x_3, x_2, and x_1, respectively, for this basic solution. The solution remains feasible, and so optimal, as long as all three quantities remain nonnegative.

$$2 + \frac{1}{3}\Delta b_2 \ge 0 \quad \Rightarrow \quad \frac{1}{3}\Delta b_2 \ge -2 \quad \Rightarrow \quad \Delta b_2 \ge -6,$$
$$6 + \frac{1}{2}\Delta b_2 \ge 0 \quad \Rightarrow \quad \frac{1}{2}\Delta b_2 \ge -6 \quad \Rightarrow \quad \Delta b_2 \ge -12,$$
$$2 - \frac{1}{3}\Delta b_2 \ge 0 \quad \Rightarrow \quad 2 \ge \frac{1}{3}\Delta b_2 \quad \Rightarrow \quad \Delta b_2 \le 6.$$

Therefore, since $b_2 = 12 + \Delta b_2$, the solution remains feasible only if

$$-6 \le \Delta b_2 \le 6, \qquad \text{that is,} \qquad 6 \le b_2 \le 18.$$

(Verify this graphically in Fig. 6.3.) As introduced in Sec. 4.7, this range of values for b_2 is referred to as its *allowable range.*

> For any b_i, recall from Sec. 4.7 that its **allowable range** is the range of values over which the current optimal BF solution[5] (with adjusted values for the basic variables) remains feasible. Thus, the *shadow price* for b_i remains valid for evaluating the effect on Z of changing b_i only as long as b_i remains within this allowable range. (It is assumed that the change in this one b_i value is the only change in the model.) The adjusted values for the basic variables are obtained from the formula $\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}}$. The calculation of the allowable range then is based on finding the range of values of b_i such that $\mathbf{b}^* \ge \mathbf{0}$.

Many linear programming software packages use this same technique for automatically generating the allowable range for each b_i. (A similar technique, discussed under Cases 2*a* and 3, also is used to generate an *allowable range* for each c_j.) In Chap. 4, we showed the corresponding output for the Excel Solver and LINDO in Figs. 4.10 and A4.2, respectively. Table 6.22 summarizes this same output with respect to the b_i for the original Wyndor Glass Co. model. For example, both the *allowable increase* and *allowable decrease* for b_2 are 6, that is, $-6 \le \Delta b_2 \le 6$. The analysis in the preceding paragraph shows how these quantities were calculated.

Analyzing Simultaneous Changes in Right-Hand Sides. When multiple b_i values are changed simultaneously, the formula $\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}}$ can again be used to see how the right-hand sides change in the final tableau. If all these right-hand sides still are nonnegative, the feasibility test will indicate that the revised solution provided by this tableau still is feasible. Since row 0 has not changed, being feasible implies that this solution also is optimal.

Although this approach works fine for checking the effect of a *specific* set of changes in the b_i, it does not give much insight into how far the b_i can be simultaneously changed from their original values before the revised solution will no longer be feasible. As part of postoptimality analysis, the management of an organization often is interested in investigating the effect of various changes in policy decisions (e.g., the amounts of resources being made available to the activities under consideration) that determine the right-hand sides. Rather than considering just one specific set of changes, management may want to explore *directions* of changes where some right-hand sides increase while others decrease. Shadow

TABLE 6.22 Typical software output for sensitivity analysis of the right-hand sides for the original Wyndor Glass Co. model

Constraint	Shadow Price	Current RHS	Allowable Increase	Allowable Decrease
Plant 1	0	4	∞	2
Plant 2	1.5	12	6	6
Plant 3	1	18	6	6

[5]When there is more than one optimal BF solution for the current model (before changing b_i), we are referring here to the one obtained by the simplex method.

prices are invaluable for this kind of exploration. However, shadow prices remain valid for evaluating the effect of such changes on Z only within certain ranges of changes. For each b_i, the *allowable range* gives this range if *none* of the other b_i are changing at the same time. What do these *allowable ranges* become when some of the b_i are changing simultaneously?

A partial answer to this question is provided by the following 100 percent rule, which combines the *allowable changes* (increase or decrease) for the individual b_i that are given by the last two columns of a table like Table 6.22.

The 100 Percent Rule for Simultaneous Changes in Right-Hand Sides: The shadow prices remain valid for predicting the effect of simultaneously changing the right-hand sides of some of the functional constraints as long as the changes are not too large. To check whether the changes are small enough, calculate for each change the percentage of the allowable change (increase or decrease) for that right-hand side to remain within its allowable range. If the *sum* of the percentage changes does *not* exceed 100 percent, the shadow prices definitely will still be valid. (If the sum *does* exceed 100 percent, then we cannot be sure.)

Example (Variation 3 of the Wyndor Model). To illustrate this rule, consider *Variation* 3 of the Wyndor Glass Co. model, which revises the original model by changing the right-hand side vector as follows:

$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} \rightarrow \overline{\mathbf{b}} = \begin{bmatrix} 4 \\ 15 \\ 15 \end{bmatrix}.$$

The calculations for the 100 percent rule in this case are

$$b_2\text{: } 12 \rightarrow 15. \quad \text{Percentage of allowable increase} = 100\left(\frac{15-12}{6}\right) = 50\%$$

$$b_3\text{: } 18 \rightarrow 15. \quad \text{Percentage of allowable decrease} = 100\left(\frac{18-15}{6}\right) = \underline{50\%}$$

$$\text{Sum} = 100\%$$

Since the sum of 100 percent barely does *not* exceed 100 percent, the shadow prices definitely are valid for predicting the effect of these changes on Z. In particular, since the shadow prices of b_2 and b_3 are 1.5 and 1, respectively, the resulting change in Z would be

$$\Delta Z = 1.5(3) + 1(-3) = 1.5,$$

so Z^* would increase from 36 to 37.5.

Figure 6.4 shows the feasible region for this revised model. (The dashed lines show the original locations of the revised constraint boundary lines.) The optimal solution now is the CPF solution (0, 7.5), which gives

$$Z = 3x_1 + 5x_2 = 0 + 5(7.5) = 37.5,$$

just as predicted by the shadow prices. However, note what would happen if either b_2 were further increased above 15 or b_3 were further decreased below 15, so that the sum of the percentages of allowable changes would exceed 100 percent. This would cause the previously optimal corner-point solution to slide to the left of the x_2 axis ($x_1 < 0$), so this *infeasible* solution would no longer be optimal. Consequently, the old shadow prices would no longer be valid for predicting the new value of Z^*.

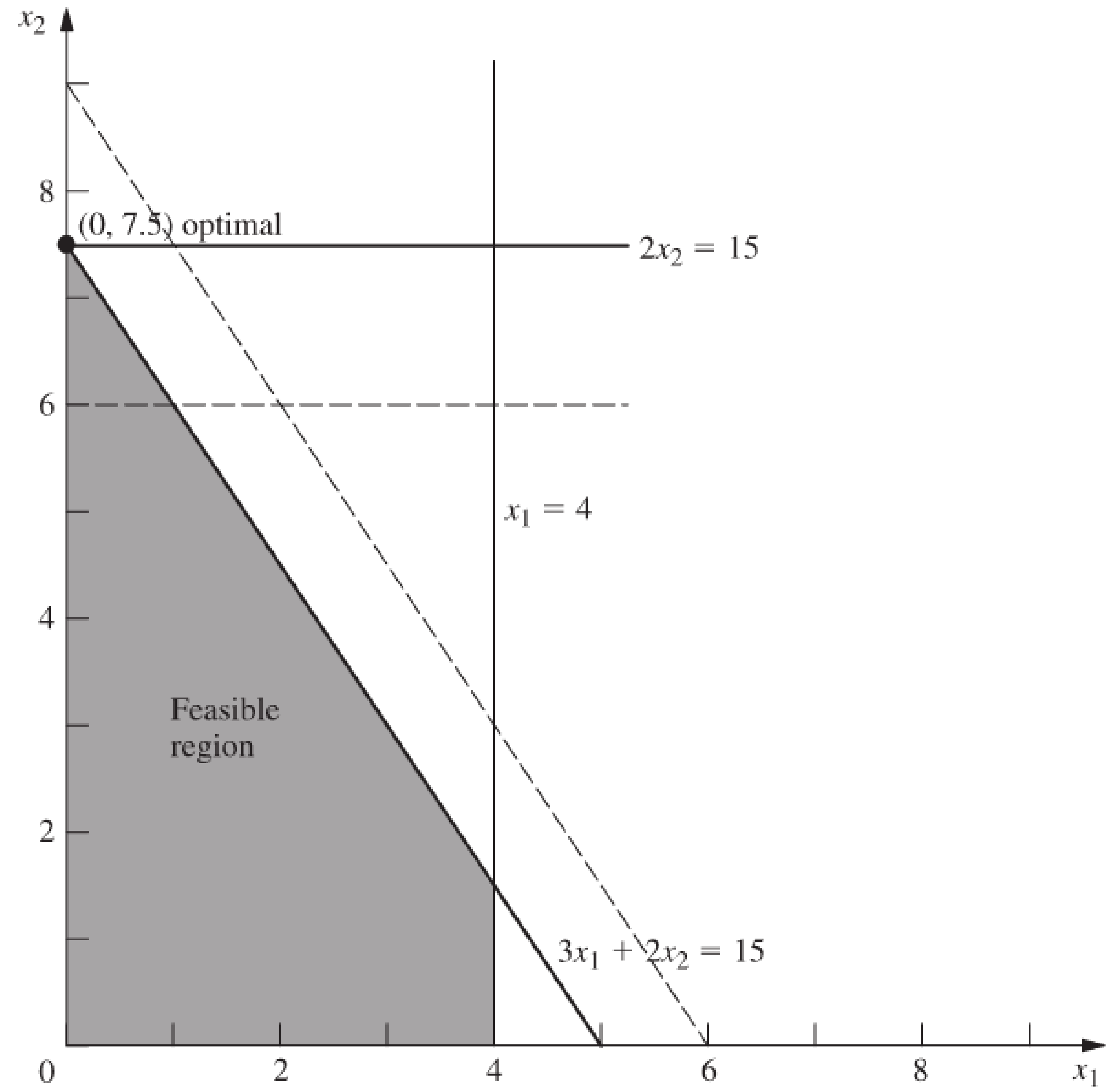


FIGURE 6.4
Feasible region for Variation 3 of the Wyndor Glass Co. model where $b_2 = 12 \to 15$ and $b_3 = 18 \to 15$.

Case 2*a*—Changes in the Coefficients of a Nonbasic Variable

Consider a particular variable x_j (fixed j) that is a nonbasic variable in the optimal solution shown by the final simplex tableau. In Case 2*a,* the only change in the current model is that one or more of the coefficients of this variable—c_j, a_{1j}, a_{2j}, . . . , a_{mj}—have been changed. Thus, letting $\overline{c}_j$ and $\overline{a}_{ij}$ denote the new values of these parameters, with $\overline{\mathbf{A}}_j$ (column j of matrix $\overline{\mathbf{A}}$) as the vector containing the $\overline{a}_{ij}$, we have

$$c_j \longrightarrow \overline{c}_j, \qquad \mathbf{A}_j \longrightarrow \overline{\mathbf{A}}_j$$

for the revised model.

As described at the beginning of Sec. 6.5, duality theory provides a very convenient way of checking these changes. In particular, if the *complementary* basic solution $\mathbf{y}^*$ in the dual problem still satisfies the single dual constraint that has changed, then the original optimal solution in the primal problem *remains optimal* as is. Conversely, if $\mathbf{y}^*$ violates this dual constraint, then this primal solution is *no longer optimal.*

If the optimal solution has changed and you wish to find the new one, you can do so rather easily. Simply apply the fundamental insight to revise the x_j column (the only one that has changed) in the final simplex tableau. Specifically, the formulas in Table 6.17 reduce to the following:

Coefficient of x_j in final row 0: $\quad z_j^* - \overline{c}_j = \mathbf{y}^*\overline{\mathbf{A}}_j - \overline{c}_j,$

Coefficient of x_j in final rows 1 to m: $\quad \mathbf{A}_j^* = \mathbf{S}^*\overline{\mathbf{A}}_j.$

With the current basic solution no longer optimal, the new value of $z_j^* - c_j$ now will be the one negative coefficient in row 0, so restart the simplex method with x_j as the initial entering basic variable.

Note that this procedure is a streamlined version of the general procedure summarized at the end of Sec. 6.6. Steps 3 and 4 (conversion to proper form from Gaussian elimination and the feasibility test) have been deleted as irrelevant, because the only column being changed in the revision of the final tableau (before reoptimization) is for the nonbasic variable x_j. Step 5 (optimality test) has been replaced by a quicker test of optimality to be performed right after step 1 (revision of model). It is only if this test reveals that the optimal solution has changed, and you wish to find the new one, that steps 2 and 6 (revision of final tableau and reoptimization) are needed.

Example (Variation 4 of the Wyndor Model). Since x_1 is nonbasic in the current optimal solution (see Table 6.21) for Variation 2 of the Wyndor Glass Co. model, the next step in its sensitivity analysis is to check whether any reasonable changes in the estimates of the coefficients of x_1 could still make it advisable to introduce product 1. The set of changes that goes as far as realistically possible to make product 1 more attractive would be to reset $c_1 = 4$ and $a_{31} = 2$. Rather than exploring each of these changes independently (as is often done in sensitivity analysis), we will consider them together. Thus, the changes under consideration are

$$c_1 = 3 \longrightarrow \bar{c}_1 = 4, \qquad \mathbf{A}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \longrightarrow \bar{\mathbf{A}}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

These two changes in Variation 2 give us *Variation* 4 of the Wyndor model. Variation 4 actually is equivalent to Variation 1 considered in Sec. 6.6 and depicted in Fig. 6.2, since Variation 1 combined these two changes with the change in the original Wyndor model ($b_2 = 12 \rightarrow 24$) that gave Variation 2. However, the key difference from the treatment of Variation 1 in Sec. 6.6 is that the analysis of Variation 4 treats Variation 2 as being the original model, so our starting point is the final simplex tableau given in Table 6.21 where x_1 now is a nonbasic variable.

The change in a_{31} revises the feasible region from that shown in Fig. 6.3 to the corresponding region in Fig. 6.5. The change in c_1 revises the objective function from $Z = 3x_1 + 5x_2$ to $Z = 4x_1 + 5x_2$. Figure 6.5 shows that the optimal objective function line $Z = 45 = 4x_1 + 5x_2$ still passes through the current optimal solution (0, 9), so this solution remains optimal after these changes in a_{31} and c_1.

To use duality theory to draw this same conclusion, observe that the changes in c_1 and a_{31} lead to a single revised constraint for the dual problem, namely, the constraint that $a_{11}y_1 + a_{21}y_2 + a_{31}y_3 \geq c_1$. Both this revised constraint and the current $\mathbf{y}^*$ (coefficients of the slack variables in row 0 of Table 6.21) are shown below.

$$y_1^* = 0, \qquad y_2^* = 0, \qquad y_3^* = \frac{5}{2},$$

$$y_1 + 3y_3 \geq 3 \longrightarrow y_1 + 2y_3 \geq 4,$$

$$0 + 2\left(\frac{5}{2}\right) \geq 4.$$

Since $\mathbf{y}^*$ *still* satisfies the revised constraint, the current primal solution (Table 6.21) is still optimal.

Because this solution is still optimal, there is no need to revise the x_j column in the final tableau (step 2). Nevertheless, we do so below for illustrative purposes.

$$z_1^* - \bar{c}_1 = \mathbf{y}^*\bar{\mathbf{A}}_1 - c_1 = [0, 0, \tfrac{5}{2}] \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 4 = 1.$$

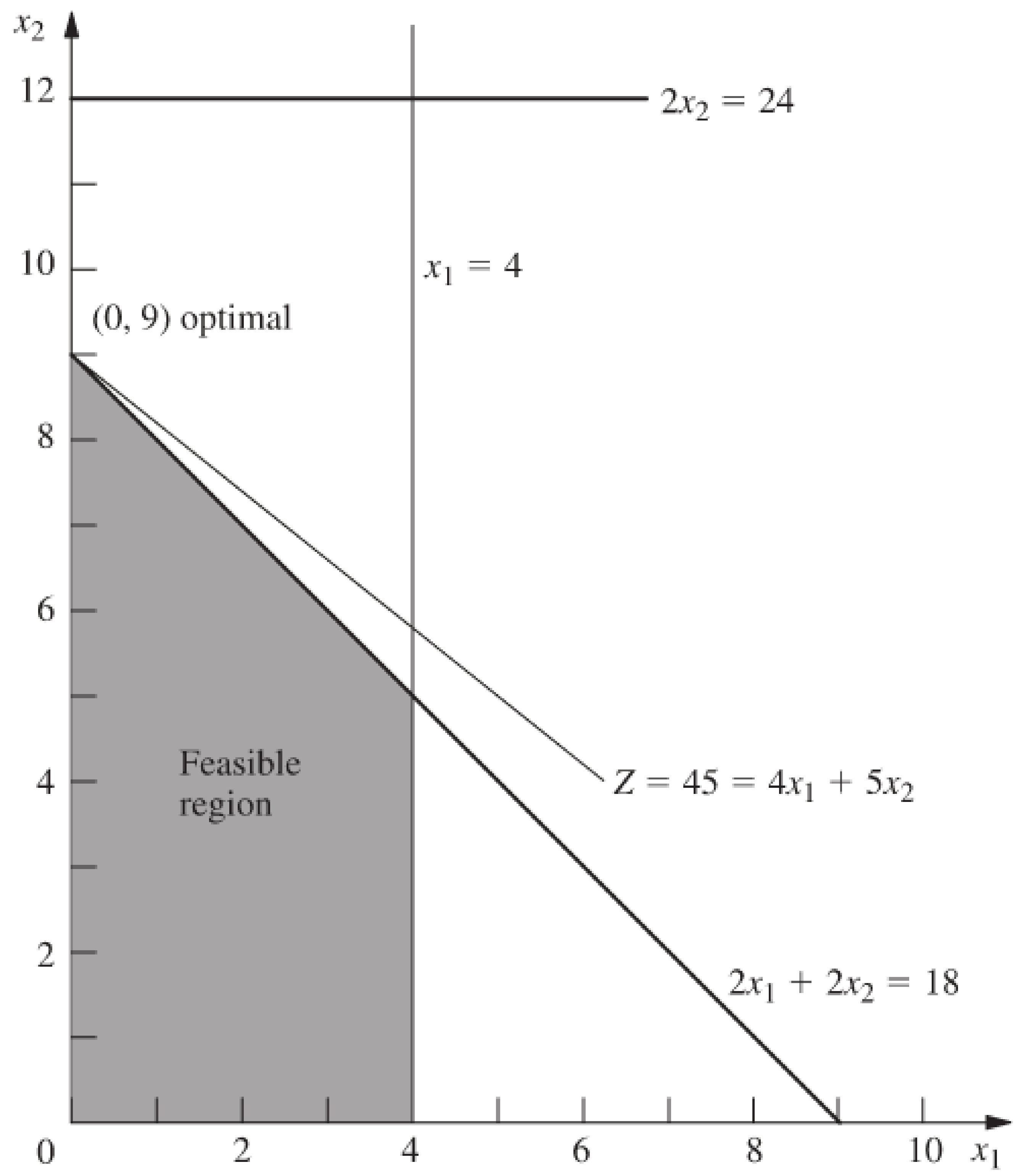


FIGURE 6.5
Feasible region for Variation 4 of the Wyndor model where Variation 2 (Fig. 6.3) has been revised so $a_{31} = 3 \rightarrow 2$ and $c_1 = 3 \rightarrow 4$.

$$\mathbf{A}_1^* = \mathbf{S}^*\overline{\mathbf{A}}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

The fact that $z_1^* - \bar{c}_1 \geq 0$ again confirms the optimality of the current solution. Since $z_1^* - c_1$ is the surplus variable for the revised constraint in the dual problem, this way of testing for optimality is equivalent to the one used above.

This completes the analysis of the effect of changing the current model (Variation 2) to Variation 4. Because any larger changes in the original estimates of the coefficients of x_1 would be unrealistic, the OR team concludes that these coefficients are *insensitive* parameters in the current model. Therefore, they will be kept fixed at their best estimates shown in Table 6.21—$c_1 = 3$ and $a_{31} = 3$—for the remainder of the sensitivity analysis.

The Allowable Range for an Objective Function Coefficient of a Nonbasic Variable. We have just described and illustrated how to analyze *simultaneous* changes in the coefficients of a nonbasic variable x_j. It is common practice in sensitivity analysis to also focus on the effect of changing just *one* parameter, c_j. As introduced in Sec. 4.7, this involves streamlining the above approach to find the *allowable range* for c_j.

> For any c_j, recall from Sec. 4.7 that its **allowable range** is the range of values over which the current optimal solution (as obtained by the simplex method for the current model before c_j is changed) remains optimal. (It is assumed that the change in this one c_j is the only change in the current model.) When x_j is a

nonbasic variable for this solution, the solution remains optimal as long as $z_j^* - c_j \geq 0$, where $z_j^* = \mathbf{y}^*\mathbf{A}_j$ is a constant unaffected by any change in the value of c_j. Therefore, the allowable range for c_j can be calculated as $c_j \leq \mathbf{y}^*\mathbf{A}_j$.

For example, consider the current model (Variation 2) for the Wyndor Glass Co. problem summarized on the left side of Table 6.21, where the current optimal solution (with $c_1 = 3$) is given on the right side. When considering only the decision variables, x_1 and x_2, this optimal solution is $(x_1, x_2) = (0, 9)$, as displayed in Fig. 6.3. When just c_1 is changed, this solution remains optimal as long as

$$c_1 \leq \mathbf{y}^*\mathbf{A}_1 = [0, 0, \tfrac{5}{2}]\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = 7\tfrac{1}{2},$$

so $c_1 \leq 7\frac{1}{2}$ is the allowable range.

An alternative to performing this vector multiplication is to note in Table 6.21 that $z_1^* - c_1 = \frac{9}{2}$ (the coefficient of x_1 in row 0) when $c_1 = 3$, so $z_1^* = 3 + \frac{9}{2} = 7\frac{1}{2}$. Since $z_1^* = \mathbf{y}^*\mathbf{A}_1$, this immediately yields the same allowable range.

Figure 6.3 provides graphical insight into why $c_1 \leq 7\frac{1}{2}$ is the allowable range. At $c_1 = 7\frac{1}{2}$, the objective function becomes $Z = 7.5x_1 + 5x_2 = 2.5(3x_1 + 2x_2)$, so the optimal objective line will lie on top of the constraint boundary line $3x_1 + 2x_2 = 18$ shown in the figure. Thus, at this endpoint of the allowable range, we have multiple optimal solutions consisting of the line segment between (0, 9) and (4, 3). If c_1 were to be increased any further ($c_1 > 7\frac{1}{2}$), only (4, 3) would be optimal. Consequently, we need $c_1 \leq 7\frac{1}{2}$ for (0, 9) to remain optimal.

IOR Tutorial includes a procedure called *Graphical Method and Sensitivity Analysis* that enables you to perform this kind of graphical analysis very efficiently.

For any nonbasic decision variable x_j, the value of $z_j^* - c_j$ sometimes is referred to as the **reduced cost** for x_j, because it is the minimum amount by which the unit *cost* of activity j would have to be *reduced* to make it worthwhile to undertake activity j (increase x_j from zero). Interpreting c_j as the unit profit of activity j (so reducing the unit cost increases c_j by the same amount), the value of $z_j^* - c_j$ thereby is the maximum allowable increase in c_j to keep the current BF solution optimal.

The sensitivity analysis information generated by linear programming software packages normally includes both the reduced cost and the allowable range for each coefficient in the objective function (along with the types of information displayed in Table 6.22). This was illustrated in Fig. 4.10 for the Excel Solver and in Figs. A4.1 and A4.2 for LINGO and LINDO. Table 6.23 displays this information in a typical form for our current model (Variation 2 of the Wyndor Glass Co. model). The last three columns are used to calculate the allowable range for each coefficient, so these allowable ranges are

$$c_1 \leq 3 + 4.5 = 7.5,$$
$$c_2 \geq 5 - 3 = 2.$$

TABLE 6.23 Typical software output for sensitivity analysis of the objective function coefficients for Variation 2 of the Wyndor Glass Co. model

Variable	Value	Reduced Cost	Current Coefficient	Allowable Increase	Allowable Decrease
x_1	0	4.5	3	4.5	∞
x_2	9	0	5	∞	3

As was discussed in Sec. 4.7, if any of the allowable increases or decreases had turned out to be zero, this would have been a signpost that the optimal solution given in the table is only one of multiple optimal solutions. In this case, changing the corresponding coefficient a tiny amount beyond the zero allowed and re-solving would provide another optimal CPF solution for the original model.

Thus far, we have described how to calculate the type of information in Table 6.23 for only nonbasic variables. For a basic variable like x_2, the reduced cost automatically is 0. We will discuss how to obtain the allowable range for c_j when x_j is a basic variable under Case 3.

Analyzing Simultaneous Changes in Objective Function Coefficients. Regardless of whether x_j is a basic or nonbasic variable, the allowable range for c_j is valid only if this objective function coefficient is the only one being changed. However, when simultaneous changes are made in the coefficients of the objective function, a 100 percent rule is available for checking whether the original solution must still be optimal. Much like the 100 percent rule for simultaneous changes in right-hand sides, this 100 percent rule combines the *allowable changes* (increase or decrease) for the individual c_j that are given by the last two columns of a table like Table 6.23, as described below.

> **The 100 Percent Rule for Simultaneous Changes in Objective Function Coefficients:** If simultaneous changes are made in the coefficients of the objective function, calculate for each change the percentage of the allowable change (increase or decrease) for that coefficient to remain within its allowable range. If the *sum* of the percentage changes does *not* exceed 100 percent, the original optimal solution definitely will still be optimal. (If the sum *does* exceed 100 percent, then we cannot be sure.)

Using Table 6.23 (and referring to Fig. 6.3 for visualization), this 100 percent rule says that (0, 9) will remain optimal for Variation 2 of the Wyndor Glass Co. model even if we simultaneously increase c_1 from 3 and decrease c_2 from 5 as long as these changes are not too large. For example, if c_1 is increased by 1.5 ($33\frac{1}{3}$ percent of the allowable change), then c_2 can be decreased by as much as 2 ($66\frac{2}{3}$ percent of the allowable change). Similarly, if c_1 is increased by 3 ($66\frac{2}{3}$ percent of the allowable change), then c_2 can only be decreased by as much as 1 ($33\frac{1}{3}$ percent of the allowable change). These maximum changes revise the objective function to either $Z = 4.5x_1 + 3x_2$ or $Z = 6x_1 + 4x_2$, which causes the optimal objective function line in Fig. 6.3 to rotate clockwise until it coincides with the constraint boundary equation $3x_1 + 2x_2 = 18$.

In general, when objective function coefficients change in the *same* direction, it is possible for the percentages of allowable changes to sum to more than 100 percent without changing the optimal solution. We will give an example at the end of the discussion of Case 3.

Case 2*b*—Introduction of a New Variable

After solving for the optimal solution, we may discover that the linear programming formulation did not consider all the attractive alternative activities. Considering a new activity requires introducing a new variable with the appropriate coefficients into the objective function and constraints of the current model—which is Case 2*b*.

The convenient way to deal with this case is to treat it just as if it were Case 2*a*! This is done by pretending that the new variable x_j actually was in the original model with all its coefficients equal to zero (so that they still are zero in the final simplex tableau) and that x_j is a nonbasic variable in the current BF solution. Therefore, if we change these zero coefficients to their actual values for the new variable, the procedure (including any reoptimization) does indeed become identical to that for Case 2*a*.

In particular, all you have to do to check whether the current solution still is optimal is to check whether the complementary basic solution $\mathbf{y}^*$ satisfies the one new dual constraint that corresponds to the new variable in the primal problem. We already have described this approach and then illustrated it for the Wyndor Glass Co. problem in Sec. 6.5.

Case 3—Changes in the Coefficients of a Basic Variable

Now suppose that the variable x_j (fixed j) under consideration is a *basic* variable in the optimal solution shown by the final simplex tableau. Case 3 assumes that the only changes in the current model are made to the coefficients of this variable.

Case 3 differs from Case 2*a* because of the requirement that a simplex tableau be in proper form from Gaussian elimination. This requirement allows the column for a nonbasic variable to be anything, so it does not affect Case 2*a*. However, for Case 3, the basic variable x_j must have a coefficient of 1 in its row of the simplex tableau and a coefficient of 0 in every other row (including row 0). Therefore, after the changes in the x_j column of the final simplex tableau have been calculated,[6] it probably will be necessary to apply Gaussian elimination to restore this form, as illustrated in Table 6.20. In turn, this step probably will change the value of the current basic solution and may make it either infeasible or nonoptimal (so reoptimization may be needed). Consequently, all the steps of the overall procedure summarized at the end of Sec. 6.6 are required for Case 3.

Before Gaussian elimination is applied, the formulas for revising the x_j column are the same as for Case 2*a*, as summarized below.

Coefficient of x_j in final row 0: $\quad z_j^* - \bar{c}_j = \mathbf{y}^*\bar{\mathbf{A}}_j - \bar{c}_j.$

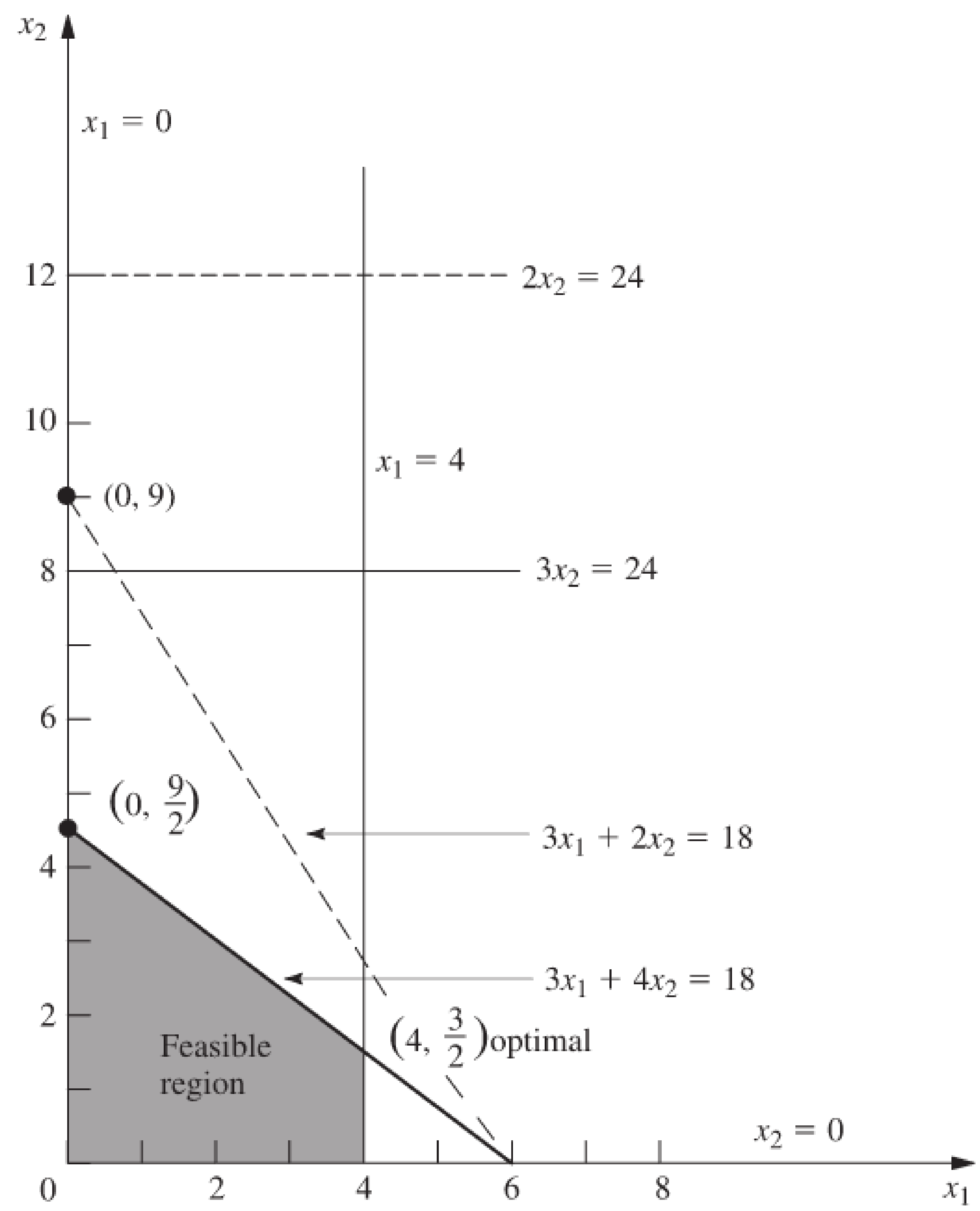
Coefficient of x_j in final rows 1 to m: $\quad \mathbf{A}_j^* = \mathbf{S}^*\bar{\mathbf{A}}_j.$

Example (Variation 5 of the Wyndor Model). Because x_2 is a basic variable in Table 6.21 for Variation 2 of the Wyndor Glass Co. model, sensitivity analysis of its coefficients fits Case 3. Given the current optimal solution ($x_1 = 0$, $x_2 = 9$), product 2 is the *only* new product that should be introduced, and its production rate should be relatively large. Therefore, the key question now is whether the initial estimates that led to the coefficients of x_2 in the current model (Variation 2) could have *overestimated* the attractiveness of product 2 so much as to invalidate this conclusion. This question can be tested by checking the *most pessimistic* set of reasonable estimates for these coefficients, which turns out to be $c_2 = 3$, $a_{22} = 3$, and $a_{32} = 4$. Consequently, the changes to be investigated (Variation 5 of the Wyndor model) are

$$c_2 = 5 \longrightarrow \bar{c}_2 = 3, \qquad \mathbf{A}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \longrightarrow \bar{\mathbf{A}}_2 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}.$$

The graphical effect of these changes is that the feasible region changes from the one shown in Fig. 6.3 to the one in Fig. 6.6. The optimal solution in Fig. 6.3 is $(x_1, x_2) = (0, 9)$, which is the corner-point solution lying at the intersection of the $x_1 = 0$ and $3x_1 + 2x_2 = 18$ constraint boundaries. With the revision of the constraints, the corresponding corner-point solution in Fig. 6.6 is $(0, \frac{9}{2})$. However, this solution no longer is optimal, because the revised objective function of $Z = 3x_1 + 3x_2$ now yields a new optimal solution of $(x_1, x_2) = (4, \frac{3}{2})$.

[6]For the relatively sophisticated reader, we should point out a possible pitfall for Case 3 that would be discovered at this point. Specifically, the changes in the initial tableau can destroy the linear independence of the columns of coefficients of basic variables. This event occurs only if the unit coefficient of the basic variable x_j in the final tableau has been changed to zero at this point, in which case more extensive simplex method calculations must be used for Case 3.



■ **FIGURE 6.6**
Feasible region for Variation 5 of the Wyndor model where Variation 2 (Fig. 6.3) has been revised so $c_2 = 5 \to 3$, $a_{22} = 2 \to 3$, and $a_{32} = 2 \to 4$.

Analysis of Variation 5. Now let us see how we draw these same conclusions algebraically. Because the only changes in the model are in the coefficients of x_2, the *only* resulting changes in the final simplex tableau (Table 6.21) are in the x_2 column. Therefore, the above formulas are used to recompute just this column.

$$z_2 - \bar{c}_2 = \mathbf{y}^*\bar{\mathbf{A}}_2 - \bar{c}_2 = [0, 0, \tfrac{5}{2}] \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - 3 = 7.$$

$$\mathbf{A}_2^* = \mathbf{S}^*\bar{\mathbf{A}}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$

(Equivalently, incremental analysis with $\Delta c_2 = -2$, $\Delta a_{22} = 1$, and $\Delta a_{32} = 2$ can be used in the same way to obtain this column.)

The resulting revised final tableau is shown at the top of Table 6.24. Note that the new coefficients of the basic variable x_2 do not have the required values, so the conversion to proper form from Gaussian elimination must be applied next. This step involves dividing row 2 by 2, subtracting 7 times the new row 2 from row 0, and adding the new row 2 to row 3.

The resulting second tableau in Table 6.24 gives the new value of the current basic solution, namely, $x_3 = 4$, $x_2 = \frac{9}{2}$, $x_4 = \frac{21}{2}$ ($x_1 = 0$, $x_5 = 0$). Since all these variables are

TABLE 6.24 Sensitivity analysis procedure applied to Variation 5 of the Wyndor Glass Co. model

	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
Revised final tableau	Z	(0)	1	$\frac{9}{2}$	7	0	0	$\frac{5}{2}$	45
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	$\frac{3}{2}$	2	0	0	$\frac{1}{2}$	9
	x_4	(3)	0	-3	-1	0	1	-1	6
Converted to proper form	Z	(0)	1	$-\frac{3}{4}$	0	0	0	$\frac{3}{4}$	$\frac{27}{2}$
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	$\frac{3}{4}$	1	0	0	$\frac{1}{4}$	$\frac{9}{2}$
	x_4	(3)	0	$-\frac{9}{4}$	0	0	1	$-\frac{3}{4}$	$\frac{21}{2}$
New final tableau after reoptimization (only one iteration of the simplex method needed in this case)	Z	(0)	1	0	0	$\frac{3}{4}$	0	$\frac{3}{4}$	$\frac{33}{2}$
	x_1	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	$-\frac{3}{4}$	0	$\frac{1}{4}$	$\frac{3}{2}$
	x_4	(3)	0	0	0	$\frac{9}{4}$	1	$-\frac{3}{4}$	$\frac{39}{2}$

nonnegative, the solution is still feasible. However, because of the negative coefficient of x_1 in row 0, we know that it is no longer optimal. Therefore, the simplex method would be applied to this tableau, with this solution as the initial BF solution, to find the new optimal solution. The initial entering basic variable is x_1, with x_3 as the leaving basic variable. Just one iteration is needed in this case to reach the new optimal solution $x_1 = 4$, $x_2 = \frac{3}{2}$, $x_4 = \frac{39}{2}$ $(x_3 = 0, x_5 = 0)$, as shown in the last tableau of Table 6.24.

All this analysis suggests that c_2, a_{22}, and a_{32} are relatively sensitive parameters. However, additional data for estimating them more closely can be obtained only by conducting a pilot run. Therefore, the OR team recommends that production of product 2 be initiated immediately on a small scale ($x_2 = \frac{3}{2}$) and that this experience be used to guide the decision on whether the remaining production capacity should be allocated to product 2 or product 1.

The Allowable Range for an Objective Function Coefficient of a Basic Variable. For Case 2*a*, we described how to find the allowable range for any c_j such that x_j is a nonbasic variable for the current optimal solution (before c_j is changed). When x_j is a basic variable instead, the procedure is somewhat more involved because of the need to convert to proper form from Gaussian elimination before testing for optimality.

To illustrate the procedure, consider Variation 5 of the Wyndor Glass Co. model (with $c_2 = 3$, $a_{22} = 3$, $a_{23} = 4$) that is graphed in Fig. 6.6 and solved in Table 6.24. Since x_2 is a basic variable for the optimal solution (with $c_2 = 3$) given at the bottom of this table, the steps needed to find the allowable range for c_2 are the following:

1. Since x_2 is a basic variable, note that its coefficient in the new final row 0 (see the bottom tableau in Table 6.24) is automatically $z_2^* - c_2 = 0$ before c_2 is changed from its current value of 3.

2. Now increment $c_2 = 3$ by Δc_2 (so $c_2 = 3 + \Delta c_2$). This changes the coefficient noted in step 1 to $z_2^* - c_2 = -\Delta c_2$, which changes row 0 to

$$\text{Row } 0 = \left[0,\ -\Delta c_2,\ \frac{3}{4},\ 0,\ \frac{3}{4}\ \vdots\ \frac{33}{2}\right].$$

3. With this coefficient now not zero, we must perform elementary row operations to restore proper form from Gaussian elimination. In particular, add to row 0 the product, Δc_2 times row 2, to obtain the new row 0, as shown below.

$$\begin{array}{rl} & \left[0,\ -\Delta c_2,\ \frac{3}{4},\quad 0,\ \frac{3}{4}\qquad\qquad \vdots\ \frac{33}{2}\right] \\ + & \left[0,\quad \Delta c_2,\ -\frac{3}{4}\Delta c_2,\ 0,\ \frac{1}{4}\Delta c_2\qquad \vdots\ \frac{3}{2}\Delta c_2\right] \\ \hline \text{New row } 0 = & \left[0,\ \ 0,\ \frac{3}{4} - \frac{3}{4}\Delta c_2,\ 0,\ \frac{3}{4} + \frac{1}{4}\Delta c_2\ \vdots\ \frac{33}{2} + \frac{3}{2}\Delta c_2\right] \end{array}$$

4. Using this new row 0, solve for the range of values of Δc_2 that keeps the coefficients of the nonbasic variables (x_3 and x_5) nonnegative.

$$\frac{3}{4} - \frac{3}{4}\Delta c_2 \geq 0 \quad \Rightarrow \quad \frac{3}{4} \geq \frac{3}{4}\Delta c_2 \quad \Rightarrow \quad \Delta c_2 \leq 1.$$

$$\frac{3}{4} + \frac{1}{4}\Delta c_2 \geq 0 \quad \Rightarrow \quad \frac{1}{4}\Delta c_2 \geq -\frac{3}{4} \quad \Rightarrow \quad \Delta c_2 \geq -3.$$

Thus, the range of values is $-3 \leq \Delta c_2 \leq 1$.

5. Since $c_2 = 3 + \Delta c_2$, add 3 to this range of values, which yields

$$0 \leq c_2 \leq 4$$

as the allowable range for c_2.

With just two decision variables, this allowable range can be verified graphically by using Fig. 6.6 with an objective function of $Z = 3x_1 + c_2x_2$. With the current value of $c_2 = 3$, the optimal solution is $(4, \frac{3}{2})$. When c_2 is increased, this solution remains optimal only for $c_2 \leq 4$. For $c_2 \geq 4$, $(0, \frac{9}{2})$ becomes optimal (with a tie at $c_2 = 4$), because of the constraint boundary $3x_1 + 4x_2 = 18$. When c_2 is decreased instead, $(4, \frac{3}{2})$ remains optimal only for $c_2 \geq 0$. For $c_2 \leq 0$, $(4, 0)$ becomes optimal because of the constraint boundary $x_1 = 4$.

In a similar manner, the allowable range for c_1 (with c_2 fixed at 3) can be derived either algebraically or graphically to be $c_1 \geq \frac{9}{4}$. (Problem 6.7-10 asks you to verify this both ways.)

Thus, the *allowable decrease* for c_1 from its current value of 3 is only $\frac{3}{4}$. However, it is possible to decrease c_1 by a larger amount without changing the optimal solution if c_2 also decreases sufficiently. For example, suppose that *both* c_1 and c_2 are decreased by 1 from their current value of 3, so that the objective function changes from $Z = 3x_1 + 3x_2$ to $Z = 2x_1 + 2x_2$. According to the 100 percent rule for simultaneous changes in objective function coefficients, the percentages of allowable changes are $133\frac{1}{3}$ percent and $33\frac{1}{3}$ percent, respectively, which sum to far over 100 percent. However, the slope of the objective function line has not changed at all, so $(4, \frac{3}{2})$ still is optimal.

Case 4—Introduction of a New Constraint

In this case, a new constraint must be introduced to the model after it has already been solved. This case may occur because the constraint was overlooked initially or because new considerations have arisen since the model was formulated. Another possibility is that the constraint was deleted purposely to decrease computational effort because it appeared to be less restrictive than other constraints already in the model, but now this impression needs to be checked with the optimal solution actually obtained.

To see if the current optimal solution would be affected by a new constraint, all you have to do is to check directly whether the optimal solution satisfies the constraint. If it does, then it would still be the *best feasible solution* (i.e., the optimal solution), even if the constraint were added to the model. The reason is that a new constraint can only eliminate some previously feasible solutions without adding any new ones.

If the new constraint does eliminate the current optimal solution, and if you want to find the new solution, then introduce this constraint into the final simplex tableau (as an additional row) *just* as if this were the initial tableau, where the usual additional variable (slack variable or artificial variable) is designated to be the basic variable for this new row. Because the new row probably will have *nonzero* coefficients for some of the other basic variables, the conversion to proper form from Gaussian elimination is applied next, and then the reoptimization step is applied in the usual way.

Just as for some of the preceding cases, this procedure for Case 4 is a streamlined version of the general procedure summarized at the end of Sec. 6.6. The only question to be addressed for this case is whether the previously optimal solution still is *feasible,* so step 5 (optimality test) has been deleted. Step 4 (feasibility test) has been replaced by a much quicker test of feasibility (does the previously optimal solution satisfy the new constraint?) to be performed right after step 1 (revision of model). It is only if this test provides a negative answer, and you wish to reoptimize, that steps 2, 3, and 6 are used (revision of final tableau, conversion to proper form from Gaussian elimination, and reoptimization).

Example (Variation 6 of the Wyndor Model). To illustrate this case, we consider Variation 6 of the Wyndor Glass Co. model, which simply introduces the new constraint

$$2x_1 + 3x_2 \leq 24$$

into the Variation 2 model given in Table 6.21. The graphical effect is shown in Fig. 6.7. The previous optimal solution (0, 9) violates the new constraint, so the optimal solution changes to (0, 8).

To analyze this example algebraically, note that (0, 9) yields $2x_1 + 3x_2 = 27 > 24$, so this previous optimal solution is no longer feasible. To find the new optimal solution, add the new constraint to the current final simplex tableau as just described, with the slack variable x_6 as its initial basic variable. This step yields the first tableau shown in Table 6.25. The conversion to proper form from Gaussian elimination then requires subtracting from the new row the product, 3 times row 2, which identifies the current basic solution $x_3 = 4$, $x_2 = 9$, $x_4 = 6$, $x_6 = -3$ ($x_1 = 0$, $x_5 = 0$), as shown in the second tableau. Applying the dual simplex method (described in Sec. 7.1) to this tableau then leads in just one iteration (more are sometimes needed) to the new optimal solution in the last tableau of Table 6.25.

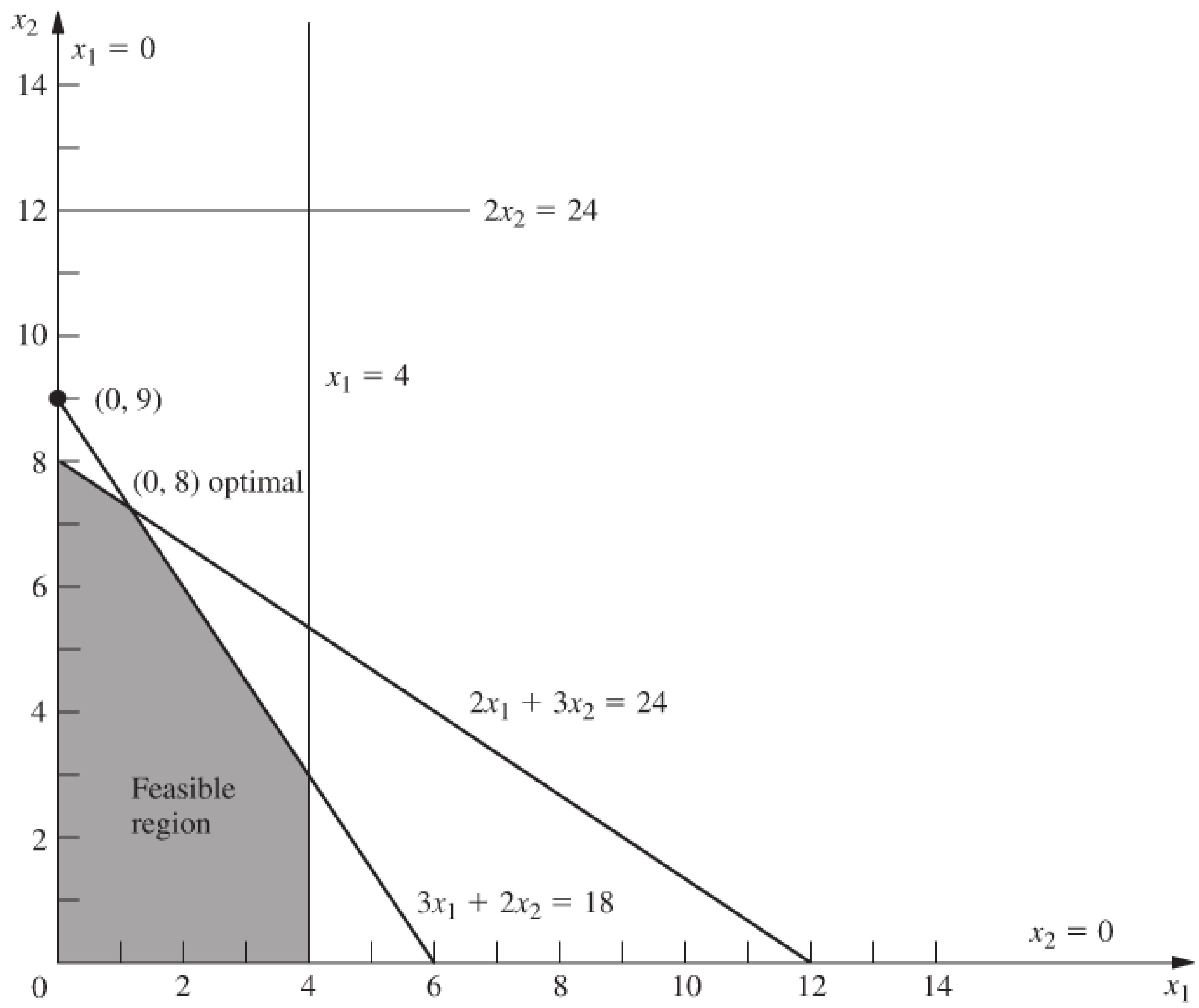
Systematic Sensitivity Analysis—Parametric Programming

So far we have described how to test specific changes in the model parameters. Another common approach to sensitivity analysis is to vary one or more parameters continuously over some interval(s) to see when the optimal solution changes.

For example, with Variation 2 of the Wyndor Glass Co. model, rather than beginning by testing the specific change from $b_2 = 12$ to $\bar{b}_2 = 24$, we might instead set

$$\bar{b}_2 = 12 + \theta$$

and then vary θ continuously from 0 to 12 (the maximum value of interest). The geometric interpretation in Fig. 6.3 is that the $2x_2 = 12$ constraint line is being shifted upward to



■ **FIGURE 6.7**
Feasible region for Variation 6 of the Wyndor model where Variation 2 (Fig. 6.3) has been revised by adding the new constraint, $2x_1 + 3x_2 \leq 24$.

$2x_2 = 12 + \theta$, with θ being increased from 0 to 12. The result is that the original optimal CPF solution (2, 6) shifts up the $3x_1 + 2x_2 = 18$ constraint line toward $(-2, 12)$. This corner-point solution remains optimal as long as it is still feasible ($x_1 \geq 0$), after which (0, 9) becomes the optimal solution.

The algebraic calculations of the effect of having $\Delta b_2 = \theta$ are directly analogous to those for the Case 1 example where $\Delta b_2 = 12$. In particular, we use the expressions for Z^* and $\mathbf{b}^*$ given for Case 1,

$$Z^* = \mathbf{y}^*\bar{\mathbf{b}}$$
$$\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}}$$

where $\bar{\mathbf{b}}$ now is

$$\bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 12+\theta \\ 18 \end{bmatrix}$$

and where $\mathbf{y}^*$ and $\mathbf{S}^*$ are given in the boxes in the middle tableau in Table 6.19. These equations indicate that the optimal solution is

$$\begin{aligned} Z^* &= 36 + \frac{3}{2}\theta \\ x_3 &= 2 + \frac{1}{3}\theta \\ x_2 &= 6 + \frac{1}{2}\theta \\ x_1 &= 2 - \frac{1}{3}\theta \end{aligned} \qquad (x_4 = 0,\ x_5 = 0)$$

TABLE 6.25 Sensitivity analysis procedure applied to Variation 6 of the Wyndor Glass Co. model

	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Revised final tableau	Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	0	45
	x_3	(1)	0	1	0	1	0	0	0	4
	x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	0	9
	x_4	(3)	0	-3	0	0	1	-1	0	6
	x_6	New	0	2	3	0	0	0	1	24
Converted to proper form	Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	0	45
	x_3	(1)	0	1	0	1	0	0	0	4
	x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	0	9
	x_4	(3)	0	-3	0	0	1	-1	0	6
	x_6	New	0	$-\frac{5}{2}$	0	0	0	$-\frac{3}{2}$	1	-3
New final tableau after reoptimization (only one iteration of dual simplex method needed in this case)	Z	(0)	1	$\frac{1}{3}$	0	0	0	0	$\frac{5}{3}$	40
	x_3	(1)	0	1	0	1	0	0	0	4
	x_2	(2)	0	$\frac{2}{3}$	1	0	0	0	$\frac{1}{3}$	8
	x_4	(3)	0	$-\frac{4}{3}$	0	0	1	0	$-\frac{2}{3}$	8
	x_5	New	0	$\frac{5}{3}$	0	0	0	1	$-\frac{2}{3}$	2

for θ small enough that this solution still is feasible, i.e., for $\theta \leq 6$. For $\theta > 6$, the dual simplex method (described in Sec. 7.1) yields the tableau shown in Table 6.21 except for the value of x_4. Thus, $Z = 45$, $x_3 = 4$, $x_2 = 9$ (along with $x_1 = 0$, $x_5 = 0$), and the expression for $\mathbf{b}^*$ yields

$$x_4 = b_3^* = 0(4) + 1(12 + \theta) - 1(18) = -6 + \theta.$$

This information can then be used (along with other data not incorporated into the model on the effect of increasing b_2) to decide whether to retain the original optimal solution and, if not, how much to increase b_2.

In a similar way, we can investigate the effect on the optimal solution of varying several parameters simultaneously. When we vary just the b_i parameters, we express the new value $\bar{b}_i$ in terms of the original value b_i as follows:

$$\bar{b}_i = b_i + \alpha_i\theta, \qquad \text{for } i = 1, 2, \ldots, m,$$

where the α_i values are input constants specifying the desired rate of increase (positive or negative) of the corresponding right-hand side as θ is increased.

For example, suppose that it is possible to shift some of the production of a current Wyndor Glass Co. product from Plant 2 to Plant 3, thereby increasing b_2 by decreasing b_3. Also suppose that b_3 decreases twice as fast as b_2 increases. Then

$$\bar{b}_2 = 12 + \theta$$

$$\bar{b}_3 = 18 - 2\theta,$$

where the (nonnegative) value of θ measures the amount of production shifted. (Thus, $\alpha_1 = 0$, $\alpha_2 = 1$, and $\alpha_3 = -2$ in this case.) In Fig. 6.3, the geometric interpretation is that as θ is increased from 0, the $2x_2 = 12$ constraint line is being pushed up to $2x_2 = 12 + \theta$ (ignore the $2x_2 = 24$ line) and simultaneously the $3x_1 + 2x_2 = 18$ constraint line is being pushed down to $3x_1 + 2x_2 = 18 - 2\theta$. The original optimal CPF solution (2, 6) lies at the intersection of the $2x_2 = 12$ and $3x_1 + 2x_2 = 18$ lines, so shifting these lines causes this corner-point solution to shift. However, with the objective function of $Z = 3x_1 + 5x_2$, this corner-point solution will remain optimal as long as it is still feasible ($x_1 \geq 0$).

An algebraic investigation of simultaneously changing b_2 and b_3 in this way again involves using the formulas for Case 1 (treating θ as representing an unknown number) to calculate the resulting changes in the final tableau (middle of Table 6.19), namely,

$$Z^* = \mathbf{y}^*\overline{\mathbf{b}} = [0, \tfrac{3}{2}, 1]\begin{bmatrix} 4 \\ 12 + \theta \\ 18 - 2\theta \end{bmatrix} = 36 - \tfrac{1}{2}\theta,$$

$$\mathbf{b}^* = \mathbf{S}^*\overline{\mathbf{b}} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 4 \\ 12 + \theta \\ 18 - 2\theta \end{bmatrix} = \begin{bmatrix} 2 + \theta \\ 6 + \frac{1}{2}\theta \\ 2 - \theta \end{bmatrix}.$$

Therefore, the optimal solution becomes

$$\begin{aligned} Z^* &= 36 - \frac{1}{2}\theta \\ x_3 &= 2 + \theta \qquad\qquad (x_4 = 0, \qquad x_5 = 0) \\ x_2 &= 6 + \frac{1}{2}\theta \\ x_1 &= 2 - \theta \end{aligned}$$

for θ small enough that this solution still is feasible, i.e., for $\theta \leq 2$. (Check this conclusion in Fig. 6.3.) However, the fact that Z decreases as θ increases from 0 indicates that the best choice for θ is $\theta = 0$, so none of the possible shifting of production should be done.

The approach to varying several c_j parameters simultaneously is similar. In this case, we express the new value $\overline{c}_j$ in terms of the original value of c_j as

$$\overline{c}_j = c_j + \alpha_j\theta, \qquad \text{for } j = 1, 2, \ldots, n,$$

where the α_j are input constants specifying the desired rate of increase (positive or negative) of c_j as θ is increased.

To illustrate this case, reconsider the sensitivity analysis of c_1 and c_2 for the Wyndor Glass Co. problem that was performed earlier in this section. Starting with Variation 2 of the Wyndor model presented in Table 6.21 and Fig. 6.3, we separately considered the effect of changing c_1 from 3 to 4 (its most optimistic estimate) and c_2 from 5 to 3 (its most pessimistic estimate). Now we can simultaneously consider both changes, as well as various intermediate cases with smaller changes, by setting

$$\overline{c}_1 = 3 + \theta \qquad \text{and} \qquad \overline{c}_2 = 5 - 2\theta,$$

where the value of θ measures the *fraction* of the maximum possible change that is made. The result is to replace the original objective function $Z = 3x_1 + 5x_2$ by a *function* of θ

$$Z(\theta) = (3 + \theta)x_1 + (5 - 2\theta)x_2,$$

so the optimization now can be performed for any desired (fixed) value of θ between 0 and 1. By checking the effect as θ increases from 0 to 1, we can determine just when and how the optimal solution changes as the error in the original estimates of these parameters increases.

Considering these changes simultaneously is especially appropriate if there are factors that cause the parameters to change together. Are the two products competitive in some sense, so that a larger-than-expected unit profit for one implies a smaller-than-expected unit profit for the other? Are they both affected by some exogenous factor, such as the advertising emphasis of a competitor? Is it possible to simultaneously change both unit profits through appropriate shifting of personnel and equipment?

In the feasible region shown in Fig. 6.3, the geometric interpretation of changing the objective function from $Z = 3x_1 + 5x_2$ to $Z(\theta) = (3 + \theta)x_1 + (5 - 2\theta)x_2$ is that we are changing the *slope* of the original objective function line ($Z = 45 = 3x_1 + 5x_2$) that passes through the optimal solution (0, 9). If θ is increased enough, this slope will change sufficiently that the optimal solution will switch from (0, 9) to another CPF solution (4, 3). (Check graphically whether this occurs for $\theta \leq 1$.)

The algebraic procedure for dealing simultaneously with these two changes ($\Delta c_1 = \theta$ and $\Delta c_2 = -2\theta$) is shown in Table 6.26. Although the changes now are expressed in terms of θ rather than specific numerical amounts, θ is treated just as an unknown number. The table displays just the relevant rows of the tableaux involved (row 0 and the row for the basic variable x_2). The first tableau shown is just the final tableau for the current version of the model (before c_1 and c_2 are changed) as given in Table 6.21. Refer to the formulas in Table 6.17. The only changes in the *revised* final tableau shown next are that Δc_1 and Δc_2 are subtracted from the row 0 coefficients of x_1 and x_2, respectively. To convert this tableau to proper form from Gaussian elimination, we subtract 2θ times row 2 from row 0, which yields the last tableau shown. The expressions in terms of θ for the coefficients of nonbasic variables x_1 and x_5 in row 0 of this tableau show that the current BF solution remains optimal for $\theta \leq \frac{9}{8}$. Because $\theta = 1$ is the maximum realistic value of θ, this indicates that c_1 and c_2 together are insensitive parameters with respect to the Variation 2 model in Table 6.21. There is no need to try to estimate these parameters more closely unless other parameters change (as occurred for Variation 5 of the Wyndor model).

TABLE 6.26 Dealing with $\Delta c_1 = \theta$ and $\Delta c_2 = -2\theta$ for Variation 2 of the Wyndor model as given in Table 6.21

	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Final tableau	Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	45
	x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
Revised final tableau when $\Delta c_1 = \theta$ and $\Delta c_2 = -2\theta$	$Z(\theta)$	(0)	1	$\frac{9}{2} - \theta$	2θ	0	0	$\frac{5}{2}$	45
	x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
Converted to proper form	$Z(\theta)$	(0)	1	$\frac{9}{2} - 4\theta$	0	0	0	$\frac{5}{2} - \theta$	$45 - 18\theta$
	x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9

As we discussed in Sec. 4.7, this way of continuously varying several parameters simultaneously is referred to as *parametric linear programming.* Section 7.2 presents the complete parametric linear programming procedure (including identifying new optimal solutions for larger values of θ) when just the c_j parameters are being varied and then when just the b_i parameters are being varied. Some linear programming software packages also include routines for varying just the coefficients of a single variable or just the parameters of a single constraint. In addition to the other applications discussed in Sec. 4.7, these procedures provide a convenient way of conducting sensitivity analysis systematically.

6.8 PERFORMING SENSITIVITY ANALYSIS ON A SPREADSHEET[7]

With the help of the Excel Solver, spreadsheets provide an alternative, relatively straightforward way of performing much of the sensitivity analysis described in Secs. 6.5–6.7. The spreadsheet approach is basically the same for each of the cases considered in Sec. 6.7 for the types of changes made in the original model. Therefore, we will focus on only the effect of changes in the coefficients of the variables in the objective function (Cases 2*a* and 3 in Sec. 6.7). We will illustrate this effect by making changes in the *original* Wyndor model formulated in Sec. 3.1, where the coefficients of x_1 (number of batches of the new door produced per week) and x_2 (number of batches of the new window produced per week) in the objective function are

$c_1 = 3 =$ profit (in thousands of dollars) per batch of the new type of door,
$c_2 = 5 =$ profit (in thousands of dollars) per batch of the new type of window.

For your convenience, the spreadsheet formulation of this model (Fig. 3.22) is repeated here as Fig. 6.8. Note that the cells containing the quantities to be changed are ProfitPerBatch (C4:D4). Since the profits in these cells are expressed in dollars, whereas c_1 and c_2 are in units of thousands of dollars, we hereafter will discuss the sensitivity analysis in terms of the changes in the profits shown in these cells instead of changes in c_1 and c_2. To this end, we will denote these profits by

$P_D =$ profit per batch of doors currently entered in cell C4,
$P_W =$ profit per batch of windows currently entered in cell D4.

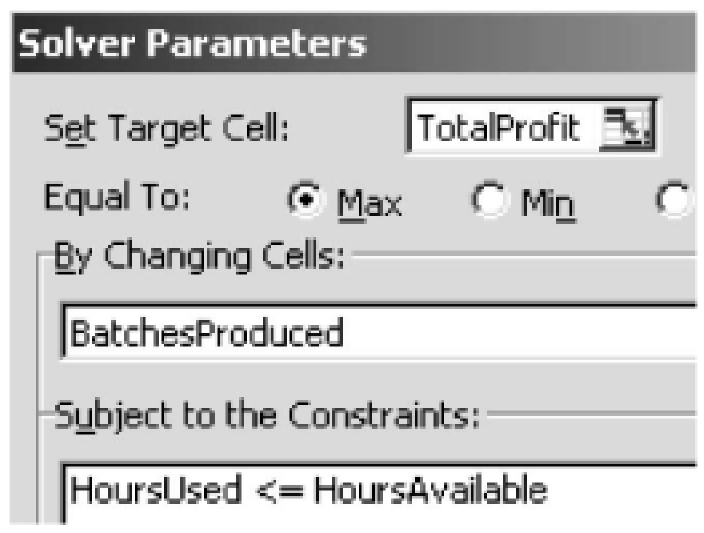
Spreadsheets actually provide three methods of performing sensitivity analysis. One is to check the effect of an individual change in the model by simply making the change on the spreadsheet and re-solving. A second is to systematically generate a table on a single spreadsheet that shows the effect of a series of changes in one or two parameters of the model. A third is to obtain and apply Excel's sensitivity report. We describe each of these methods in turn below.

Checking Individual Changes in the Model

One of the great strengths of a spreadsheet is the ease with which it can be used interactively to perform various kinds of sensitivity analysis. Once the Solver has been set up to obtain an optimal solution, you can immediately find out what would happen if one of the parameters of the model were changed to some other value. All you have to do is make this change on the spreadsheet and then click on the Solve button again.

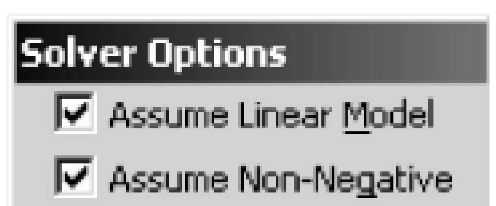
[7]We have written this section in a way that can be understood without first reading any of the preceding sections in this chapter. However, Sec. 4.7 is important background for the latter part of this section.

	A	B	C	D	E	F	G
1	Wyndor Glass Co. Product-Mix Problem						
2							
3			Doors	Windows			
4		Profit Per Batch	$3,000	$5,000			
5					Hours		Hours
6			Hours Used Per Batch Produced		Used		Available
7		Plant 1	1	0	2	<=	4
8		Plant 2	0	2	12	<=	12
9		Plant 3	3	2	18	<=	18
10							
11			Doors	Windows			Total Profit
12		Batches Produced	2	6			$36,000



	E
5	Hours
6	Used
7	=SUMPRODUCT(C7:D7,BatchesProduced)
8	=SUMPRODUCT(C8:D8,BatchesProduced)
9	=SUMPRODUCT(C9:D9,BatchesProduced)

	G
11	Total Profit
12	=SUMPRODUCT(ProfitPerBatch,BatchesProduced)



Range Name	Cells
BatchesProduced	C12:D12
HoursAvailable	G7:G9
HoursUsed	E7:E9
HoursUsedPerBatchProduced	C7:D9
ProfitPerBatch	C4:D4
TotalProfit	G12

■ **FIGURE 6.8**
The spreadsheet model and the optimal solution obtained for the original Wyndor problem before performing sensitivity analysis.

	A	B	C	D	E	F	G
1	Wyndor Glass Co. Product-Mix Problem						
2							
3			Doors	Windows			
4		Profit Per Batch	$2,000	$5,000			
5					Hours		Hours
6			Hours Used Per Batch Produced		Used		Available
7		Plant 1	1	0	2	<=	4
8		Plant 2	0	2	12	<=	12
9		Plant 3	3	2	18	<=	18
10							
11			Doors	Windows			Total Profit
12		Batches Produced	2	6			$34,000

■ **FIGURE 6.9**
The revised Wyndor problem where the estimate of the profit per batch of doors has been decreased from $P_D = \$3,000$ to $P_D = \$2,000$, but no change occurs in the optimal solution for the product mix.

To illustrate, suppose that Wyndor management is quite uncertain about what the profit per batch of doors (P_D) will turn out to be. Although the figure of $3,000 given in Fig. 6.8 is considered to be a reasonable initial estimate, management feels that the true profit could end up deviating substantially from this figure in either direction. However, the range between $P_D = \$2,000$ and $P_D = \$5,000$ is considered fairly likely.

Figure 6.9 shows what would happen if the profit per batch of doors were to drop from $P_D = \$3,000$ to $P_D = \$2,000$. Comparing with Fig. 6.8, there is no change at all in

	A	B	C	D	E	F	G
1	Wyndor Glass Co. Product-Mix Problem						
2							
3			Doors	Windows			
4		Profit Per Batch	$5,000	$5,000			
5					Hours		Hours
6			Hours Used Per Batch Produced		Used		Available
7		Plant 1	1	0	2	<=	4
8		Plant 2	0	2	12	<=	12
9		Plant 3	3	2	18	<=	18
10							
11			Doors	Windows			Total Profit
12		Batches Produced	2	6			$40,000

■ **FIGURE 6.10**
The revised Wyndor problem where the estimate of the profit per batch of doors has been increased from $P_D = \$3,000$ to $P_D = \$5,000$, but no change occurs in the optimal solution for the product mix.

the optimal solution for the product mix. In fact, the *only* changes in the new spreadsheet are the new value of P_D in cell C4 and a decrease of $2,000 in the total profit shown in cell G12 (because each of the two batches of doors produced per week provides $1,000 less profit). Because the optimal solution does not change, we now know that the original estimate of $P_D = \$3,000$ can be considerably *too high* without invalidating the model's optimal solution.

But what happens if this estimate is *too low* instead? Figure 6.10 shows what would happen if P_D were increased to $P_D = \$5,000$. Again, there is no change in the optimal solution. Therefore, we now know that the range of values of P_D over which the current optimal solution remains optimal (i.e., the *allowable range* discussed in Sec. 6.7) includes the range from $2,000 to $5,000 and may extend further.

Because the original value of $P_D = \$3,000$ can be changed considerably in either direction without changing the optimal solution, P_D is a relatively insensitive parameter. It is not necessary to pin down this estimate with great accuracy in order to have confidence that the model is providing the correct optimal solution.

This may be all the information that is needed about P_D. However, if there is a good possibility that the true value of P_D will turn out to be even outside this broad range from $2,000 to $5,000, further investigation would be desirable. How much higher or lower can P_D be before the optimal solution would change?

Figure 6.11 demonstrates that the optimal solution would indeed change if P_D is increased all the way up to $P_D = \$10,000$. Thus, we now know that this change occurs somewhere between $5,000 and $10,000 during the process of increasing P_D.

	A	B	C	D	E	F	G
1	Wyndor Glass Co. Product-Mix Problem						
2							
3			Doors	Windows			
4		Profit Per Batch	$10,000	$5,000			
5					Hours		Hours
6			Hours Used Per Batch Produced		Used		Available
7		Plant 1	1	0	4	<=	4
8		Plant 2	0	2	6	<=	12
9		Plant 3	3	2	18	<=	18
10							
11			Doors	Windows			Total Profit
12		Batches Produced	4	3			$55,000

■ **FIGURE 6.11**
The revised Wyndor problem where the estimate of the profit per batch of doors has been increased from $P_D = \$3,000$ to $P_D = \$10,000$, which results in a change in the optimal solution for the product mix.

Using the Solver Table to Do Sensitivity Analysis Systematically

To pin down just when the optimal solution will change, we could continue selecting new values of P_D at random. However, a better approach is to systematically consider a range of values of P_D. An Excel add-in developed by Professor Mark Hillier, called the *Solver Table,* is designed to perform just this sort of analysis. It is available to you in your OR Courseware on the book's website. To install it, you need simply to open the Solver Table file in OR Courseware.

The Solver Table is used to show the results in the changing cells and/or certain output cells for various trial values in a data cell. For each trial value in the data cell, Solver is called on to re-solve the problem. Therefore, the Solver Table (or any comparable Excel add-in) provides a systematic way of performing sensitivity analysis and then displaying the results to managers and others who are not familiar with the more technical aspects of sensitivity analysis.

To use the Solver Table, first expand the original spreadsheet (Fig. 6.8) to make a table with headings as shown in Fig. 6.12. In the first column of the table (cells B19:B28), list the trial values for the data cell (the profit per batch of doors), except leave the first row (cell B18) blank. The headings of the next columns specify which output will be evaluated. For each of these columns, use the first row of the table (cells C18:E18) to write an equation that sets the value in each of these cells equal to the relevant changing cell or output cell. In this case, the cells of interest are DoorBatchesProduced (C12), WindowBatchesProduced (D12), and TotalProfit (G12), so the equations for C18:E18 are those shown just below the spreadsheet in Fig. 6.12.

Next, select the entire table by clicking and dragging from cells B18 through E28, and then choose Solver Table from the Add-Ins tab (for Excel 2007) or Tools menu (for earlier versions of Excel), after having installed this Excel add-in provided in your OR Courseware. In the Solver Table dialogue box (as shown at the bottom of Fig. 6.12), indicate the column input cell (C4), which refers to the data cell that is being changed in the first column of the table. Nothing is entered for the row input cell because no row is being used to list the trial values of a data cell in this case.

The Solver Table shown in Fig. 6.13 is then generated automatically by clicking on the OK button. For each trial value listed in the first column of the table for the data cell of interest, Excel re-solves the problem using Solver and then fills in the corresponding values in the other columns of the tables. (The numbers in the first row of the table come from the original solution in the spreadsheet before the original value in the data cell was changed.)

The table reveals that the optimal solution remains the same all the way from $P_D = \$1{,}000$ (and perhaps lower) to $P_D = \$7{,}000$, but that a change occurs somewhere between \$7,000 and \$8,000. We next could systematically consider values of P_D between \$7,000 and \$8,000 to determine more closely where the optimal solution changes. However, this is not necessary since, as discussed a little later, a shortcut is to use the Excel sensitivity report to determine exactly where the optimal solution changes.

Thus far, we have illustrated how to systematically investigate the effect of changing only P_D (cell C4 in Fig. 6.8). The approach is the same for P_W (cell D4). In fact, the Solver Table can be used in this way to investigate the effect of changing *any* single data cell in the model, including any cell in HoursAvailable (G7:G9) or HoursUsedPerBatchProduced (C7:D9).

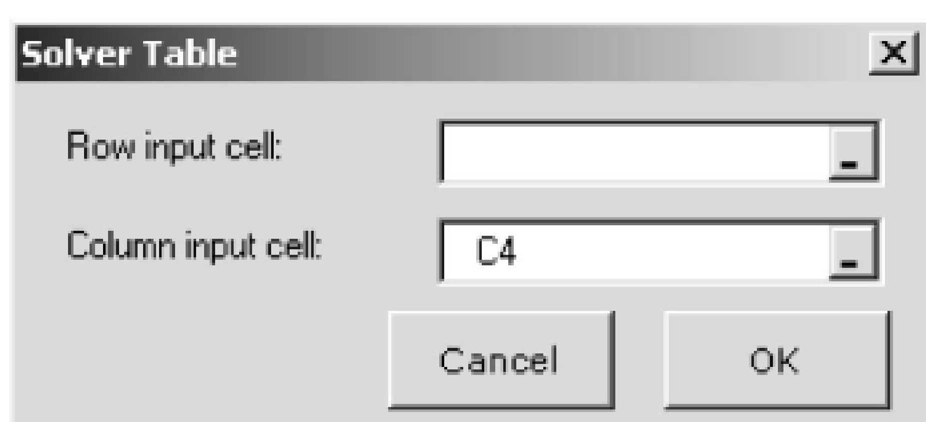
We next will illustrate how to investigate simultaneous changes in two data cells with a spreadsheet, first by itself and then with the help of the Solver Table.

Checking Two-Way Changes in the Model

When using the original estimates for P_D (\$3,000) and P_W (\$5,000), the optimal solution indicated by the model (Fig. 6.8) is heavily weighted toward producing the windows (6 batches per week) rather than the doors (only 2 batches per week). Suppose that

	A	B	C	D	E	F	G
1	Wyndor Glass Co. Product-Mix Problem						
2							
3			Doors	Windows			
4		Profit Per Batch	$3,000	$5,000			
5					Hours		Hours
6			Hours Used Per Batch Produced		Used		Available
7		Plant 1	1	0	2	<=	4
8		Plant 2	0	2	12	<=	12
9		Plant 3	3	2	18	<=	18
10							
11			Doors	Windows			Total Profit
12		Batches Produced	2	6			$36,000
13							
14							
15							
16		Profit Per Batch	Optimal Batches Produced		Total		
17		for Doors	Doors	Windows	Profit		Select these cells (B18:E28), before choosing the Solver Table.
18			2	6	$36,000		
19		$1,000					
20		$2,000					
21		$3,000					
22		$4,000					
23		$5,000					
24		$6,000					
25		$7,000					
26		$8,000					
27		$9,000					
28		$10,000					

	C	D	E
16	Optimal Batches Produced		Total
17	Doors	Windows	Profit
18	=DoorBatchesProduced	=WindowBatchesProduced	=TotalProfit



Range Name	Cells
DoorBatchesProduced	C12
TotalProfit	G12
WIndowBatchesProduced	D12

■ **FIGURE 6.12**
Expansion of the spreadsheet in Fig. 6.8 to prepare for using the Solver Table to show the effect of systematically varying the estimate of the profit per batch of doors in the Wyndor problem.

Wyndor management is concerned about this imbalance and feels that the problem may be that the estimate for P_D is too low and the estimate for P_W is too high. This raises the question: If the estimates are indeed off in these directions, would this lead to a more balanced product mix being the most profitable one? (Keep in mind that it is the *ratio* of P_D to P_W that is relevant for determining the optimal product mix, so having their estimates be off in the *same* direction with little change in this ratio is unlikely to change the optimal product mix).

This question can be answered in a matter of seconds simply by substituting new estimates of the profits per batch in the original spreadsheet in Fig. 6.8 and clicking on the Solve button. Figure 6.14 shows that new estimates of $4,500 for doors and $4,000 for windows causes no change at all in the solution for the optimal product mix. (The total profit does change, but this occurs only because of the changes in the profits per batch.) Would even larger changes in the estimates of profits per batch finally lead to a change

	A	B	C	D	E	F	G
1	Wyndor Glass Co. Product-Mix Problem						
2							
3			Doors	Windows			
4		Profit Per Batch	$3,000	$5,000			
5					Hours		Hours
6			Hours Used Per Batch Produced		Used		Available
7		Plant 1	1	0	2	<=	4
8		Plant 2	0	2	12	<=	12
9		Plant 3	3	2	18	<=	18
10							
11			Doors	Windows			Total Profit
12		Batches Produced	2	6			$36,000
13							
14							
15							
16		Profit Per Batch	Optimal Batches Produced		Total		
17		for Doors	Doors	Windows	Profit		
18			2	6	$36,000		
19		$1,000	2	6	$32,000		
20		$2,000	2	6	$34,000		
21		$3,000	2	6	$36,000		
22		$4,000	2	6	$38,000		
23		$5,000	2	6	$40,000		
24		$6,000	2	6	$42,000		
25		$7,000	2	6	$44,000		
26		$8,000	4	3	$47,000		
27		$9,000	4	3	$51,000		
28		$10,000	4	3	$55,000		

■ **FIGURE 6.13**
An application of the Solver Table that shows the effect of systematically varying the estimate of the profit per batch for doors in the Wyndor problem.

in the optimal product mix? Figure 6.15 shows that this does happen, yielding a relatively balanced product mix of $(x_1, x_2) = (4, 3)$, when estimates of $6,000 for doors and $3,000 for windows are used.

Figures 6.14 and 6.15 don't reveal where the optimal product mix changes as the profit estimates increase from $4,500 to $6,000 for doors and decrease from $4,000 to $3,000 for windows. We next describe how the Solver Table can systematically help to pin this down better.

	A	B	C	D	E	F	G
1	Wyndor Glass Co. Product-Mix Problem						
2							
3			Doors	Windows			
4		Profit Per Batch	$4,500	$4,000			
5					Hours		Hours
6			Hours Used Per Batch Produced		Used		Available
7		Plant 1	1	0	2	<=	4
8		Plant 2	0	2	12	<=	12
9		Plant 3	3	2	18	<=	18
10							
11			Doors	Windows			Total Profit
12		Batches Produced	2	6			$33,000

■ **FIGURE 6.14**
The revised Wyndor problem where the estimates of the profits per batch of doors and windows have been changed to $P_D = \$4,500$ and $P_W = \$4,000$, respectively, but no change occurs in the optimal product mix.

	A	B	C	D	E	F	G
1	Wyndor Glass Co. Product-Mix Problem						
2							
3			Doors	Windows			
4		Profit Per Batch	$6,000	$3,000			
5					Hours		Hours
6			Hours Used Per Batch Produced		Used		Available
7		Plant 1	1	0	4	<=	4
8		Plant 2	0	2	6	<=	12
9		Plant 3	3	2	18	<=	18
10							
11			Doors	Windows			Total Profit
12		Batches Produced	4	3			$33,000

■ **FIGURE 6.15**
The revised Wyndor problem where the estimates of the profits per batch of doors and windows have been changed to \$6,000 and \$3,000, respectively, which results in a change in the optimal product mix.

Using the Solver Table for Two-Way Sensitivity Analysis

A two-way version of the Solver Table provides a way of systematically investigating the effect if the estimates entered into *two* data cells are inaccurate simultaneously. (However, two is the maximum number of data cells that can be considered simultaneously by the Solver Table.) In this case, the Solver Table shows the results in a single output cell for various trial values in the two data cells.

To illustrate this approach, we again will investigate the effect of increasing P_D and decreasing P_W simultaneously. Before considering the effect on the optimal product mix, we will look at the effect on the total profit. To do this, the Solver Table will be used to show how TotalProfit (G12) in Fig. 6.8 varies over a range of trial values in the two data cells, ProfitPerBatch (C4:D4). For each pair of trial values in these data cells, Solver will be called on to re-solve the problem.

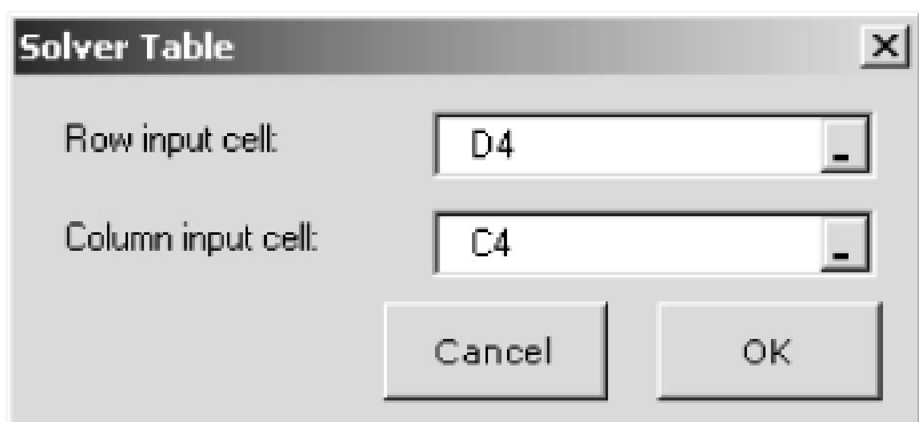
To create a two-way Solver Table for the Wyndor problem, expand the original spreadsheet (Fig. 6.8) to make a table with column and row headings as shown in rows 16–21 of the spreadsheet in Fig. 6.16. In the upper left-hand corner of the table (C17), write an equation (=TotalProfit) that refers to the target cell. In the first column of the table (column C, below the equation in cell C17), insert various trial values for the first data cell of interest (the profit per batch of the doors). In the first row of the table (row 17, to the right of the equation in cell C17), insert various trial values for the second data cell of interest (the profit per batch of the windows).

Next, select the entire table (C17:H21) and choose Solver Table from the Add-Ins tab (for Excel 2007) or Tools menu (for earlier versions of Excel), after having installed this Excel add-in provided in your OR Courseware. In the Solver Table dialogue box (shown at the bottom of Fig. 6.16), indicate which data cells are being changed simultaneously. The column input cell C4 refers to the data cell whose various trial values are listed in the first column of the table (C18:C21), while the row input cell D4 refers to the data cell whose various trial values are listed in the first row of the table (D17:H17).

The Solver Table shown in Fig. 6.17 is then generated automatically by clicking on the OK button. For each pair of trial values for the two data cells, Excel re-solves the problem using Solver and then fills in the total profit in the corresponding spot in the table. (The number in C17 comes from the target cell in the original spreadsheet before the original values in the two data cells are changed.)

Unlike a one-way Solver Table that can show the results of *multiple* changing cells and/or output cells for various trial values of a single data cell, a two-way Solver Table is limited to showing the results in a *single* cell for each pair of trial values in the two data cells of interest.

	A	B	C	D	E	F	G	H	I
1	Wyndor Glass Co. Product-Mix Problem								
2									
3			Doors	Windows					
4		Profit Per Batch	$3,000	$5,000					
5					Hours		Hours		
6			Hours Used Per Batch Produced		Used		Available		
7		Plant 1	1	0	2	<=	4		
8		Plant 2	0	2	12	<=	12		
9		Plant 3	3	2	18	<=	18		Select these cells (C17:H21), before choosing the Solver Table.
10									
11			Doors	Windows			Total Profit		
12		Batches Produced	2	6			$36,000		
13									
14									
15									
16		Total Profit			Profit Per Batch for Windows				
17			$36,000	$1,000	$2,000	$3,000	$4,000	$5,000	
18			$3,000						
19		Profit Per Batch	$4,000						
20		for Doors	$5,000						
21			$6,000						



	C
17	=TotalProfit

Range Name	Cell
TotalProfit	G12

FIGURE 6.16
Expansion of the spreadsheet in Fig. 6.8 to prepare for using a two-dimensional Solver Table to show the effect on total profits of systematically varying the estimates of the profits per batch of doors and windows for the Wyndor problem.

FIGURE 6.17
A two-dimensional application of the Solver Table that shows the effect on the optimal total profit of systematically varying the estimates of the profits per batch of doors and windows for the Wyndor problem.

	B	C	D	E	F	G	H
16	Total Profit			Profit Per Batch for Windows			
17		$36,000	$1,000	$2,000	$3,000	$4,000	$5,000
18		$3,000	$15,000	$18,000	$24,000	$30,000	$36,000
19	Profit Per Batch	$4,000	$19,000	$22,000	$26,000	$32,000	$38,000
20	for Doors	$5,000	$23,000	$26,000	$29,000	$34,000	$40,000
21		$6,000	$27,000	$30,000	$33,000	$36,000	$42,000

However, there is a trick using the & symbol that enables Solver Table to show the results from multiple changing cells and/or output cells within a single cell of the table. We utilize this trick in the Solver Table shown in Fig. 6.18 to show the results for *both* changing cells, DoorBatchesProduced (C12) and WindowBatchesProduced (D12), for each pair of trial values for ProfitPerBatch (C4:D4). The key formula is in cell C25:

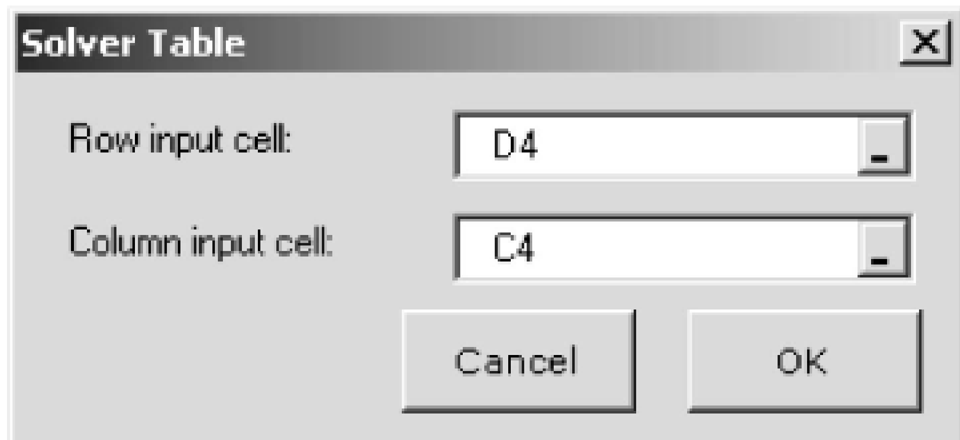
C25 = "("& DoorBatchesProduced &", "& WindowBatchesProduced &")"

The & character tells Excel to concatenate, so the result will be a left parenthesis, followed by the value in DoorBatchesProduced (C12), then a comma and the contents in Window-BatchesProduced (D12), and finally a right parenthesis. If DoorBatchesProduced = 2 and WindowBatchesProduced = 6, the result is (2, 6). Thus, the results from *both* changing cells are displayed within a *single* cell of the table.

After the usual preliminaries in entering the information shown in rows 24–25 and columns B-C of Fig. 6.18, along with the formula in C25, clicking on the OK button

	B	C	D	E	F	G	H
24	Batches Produced (Doors, Windows)			Profit Per Batch for Windows			
25		(2,6)	$1,000	$2,000	$3,000	$4,000	$5,000
26		$3,000	(4,3)	(4,3)	(2,6)	(2,6)	(2,6)
27	Profit Per Batch	$4,000	(4,3)	(4,3)	(2,6)	(2,6)	(2,6)
28	for Doors	$5,000	(4,3)	(4,3)	(4,3)	(2,6)	(2,6)
29		$6,000	(4,3)	(4,3)	(4,3)	(4,3)	(4,3)

	C
25	="(" & DoorBatchesProduced & "," & WindowBatchesProduced & ")"



Range Name	Cells
DoorBatchesProduced	C12
WindowBatchesProduced	D12

FIGURE 6.18
A two-dimensional application of the Solver Table that shows the effect on the optimal product mix of systematically varying the estimates of the profits per batch of doors and windows for the Wyndor problem.

automatically generates the entire Solver Table. Cells D26:H29 show the optimal solution for the various combinations of trial values for the profits per batch of the doors and windows. The upper right-hand corner (cell H26) of this Solver Table gives the optimal solution of $(x_1, x_2) = (2, 6)$ when using the original profit estimates of \$3,000 per batch of doors and \$5,000 per batch of windows. Moving down from this cell corresponds to increasing this estimate for doors while moving to the left amounts to decreasing the estimate for windows. (The cells when moving up or to the right of H26 are not shown because these changes would only increase the attractiveness of $(x_1, x_2) = (2, 6)$ as the optimal solution.) Note that $(x_1, x_2) = (2, 6)$ continues to be the optimal solution for all the cells near H26. This indicates that the original estimates of profit per batch would need to be very inaccurate indeed before the optimal product mix would change.

Using the Sensitivity Report to Perform Sensitivity Analysis

You now have seen how some sensitivity analysis can be performed readily on a spreadsheet either by interactively making changes in data cells and re-solving or by using the Solver Table to generate similar information systematically. However, there is a shortcut. Some of the same information (and more) can be obtained more quickly and precisely by simply using the sensitivity report provided by the Excel Solver. (Essentially the same sensitivity report is a standard part of the output available from other linear programming software packages as well, including MPL/CPLEX, LINDO, and LINGO.)

Section 4.7 already has discussed the sensitivity report and how it is used to perform sensitivity analysis. Figure 4.10 in that section shows the sensitivity report for the Wyndor problem. Part of this report is shown here in Fig. 6.19. Rather than repeating Sec. 4.7, we will focus here on illustrating how the sensitivity report can efficiently address the specific questions raised in the preceding subsections for the Wyndor problem.

The question considered in the first two subsections was how far the initial estimate of \$3,000 for P_D could be off before the current optimal solution, $(x_1, x_2) = (2, 6)$, would change. Figures 6.10 and 6.11 showed that the optimal solution would not change until

FIGURE 6.19
Part of the sensitivity report generated by the Excel Solver for the original Wyndor problem (Fig. 6.8), where the last three columns identify the allowable ranges for the profits per batch of doors and windows.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$12	DoorBatchesProduced	2	0	3000	4500	3000
$D$12	WindowBatchesProduced	6	0	5000	1E+30	3000

P_D is raised to somewhere between \$5,000 and \$10,000. Figure 6.13 then narrowed down the gap for where the optimal solution changes to somewhere between \$7,000 and \$8,000. This figure also showed that if the initial estimate of \$3,000 for P_D is too high rather than too low, P_D would need to be dropped to somewhere below \$1,000 before the optimal solution would change.

Now look at how the portion of the sensitivity report in Figure 6.19 addresses this same question. The DoorBatchesProduced row in this report provides the following information (without the dollar signs) about P_D.

Current value of P_D:	3,000.	
Allowable increase in P_D:	4,500.	So $P_D \leq 3{,}000 + 4{,}500 = 7{,}500$
Allowable decrease in P_D:	3,000.	So $P_D \geq 3{,}000 - 3{,}000 = 0$.
Allowable range for P_D:		$0 \leq P_D \leq 7{,}500$.

Therefore, if P_D is changed from its current value (without making any other change in the model), the current solution $(x_1, x_2) = (2, 6)$ will remain optimal so long as the new value of P_D is within this *allowable range,* $0 \leq P_D \leq \$7{,}500$.

Figure 6.20 provides graphical insight into this allowable range. For the original value of $P_D = 3{,}000$, the solid line in the figure shows the slope of the objective function line passing through (2, 6). At the lower end of the allowable range, where $P_D = 0$, the objective function line that passes through (2, 6) now is line B in the figure, so every point on the line segment between (0, 6) and (2, 6) is an optimal solution. For any value of $P_D < 0$, the objective function line will have rotated even further so that (0, 6) becomes the only optimal solution. At the upper end of the allowable range, when $P_D = 7{,}500$, the objective function line that passes through (2, 6) becomes line C, so every point on the line segment between (2, 6) and (4, 3) becomes an optimal solution. For any value of $P_D > 7{,}500$, the objective function line is even steeper than line C, so (4, 3) becomes the only optimal solution. Consequently, the original optimal solution, $(x_1, x_2) = (2, 6)$ remains optimal only as long as $0 \leq P_D \leq \$7{,}500$.

The procedure called *Graphical Method and Sensitivity Analysis* in IOR Tutorial is designed to help you perform this kind of graphical analysis. After you enter the model for the original Wyndor problem, the module provides you with the graph shown in Fig. 6.20 (without the dashed lines). You then can simply drag one end of the objective line up or down to see how far you can increase or decrease P_D before $(x_1, x_2) = (2, 6)$ will no longer be optimal.

Conclusion: The allowable range for P_D is $0 \leq P_D \leq \$7{,}500$, because $(x_1, x_2) = (2, 6)$ remains optimal over this range but not beyond. (When $P_D = 0$ or $P_D = \$7{,}500$, there are multiple optimal solutions, but $(x_1, x_2) = (2, 6)$ still is one of them.) With the range this wide around the original estimate of \$3,000

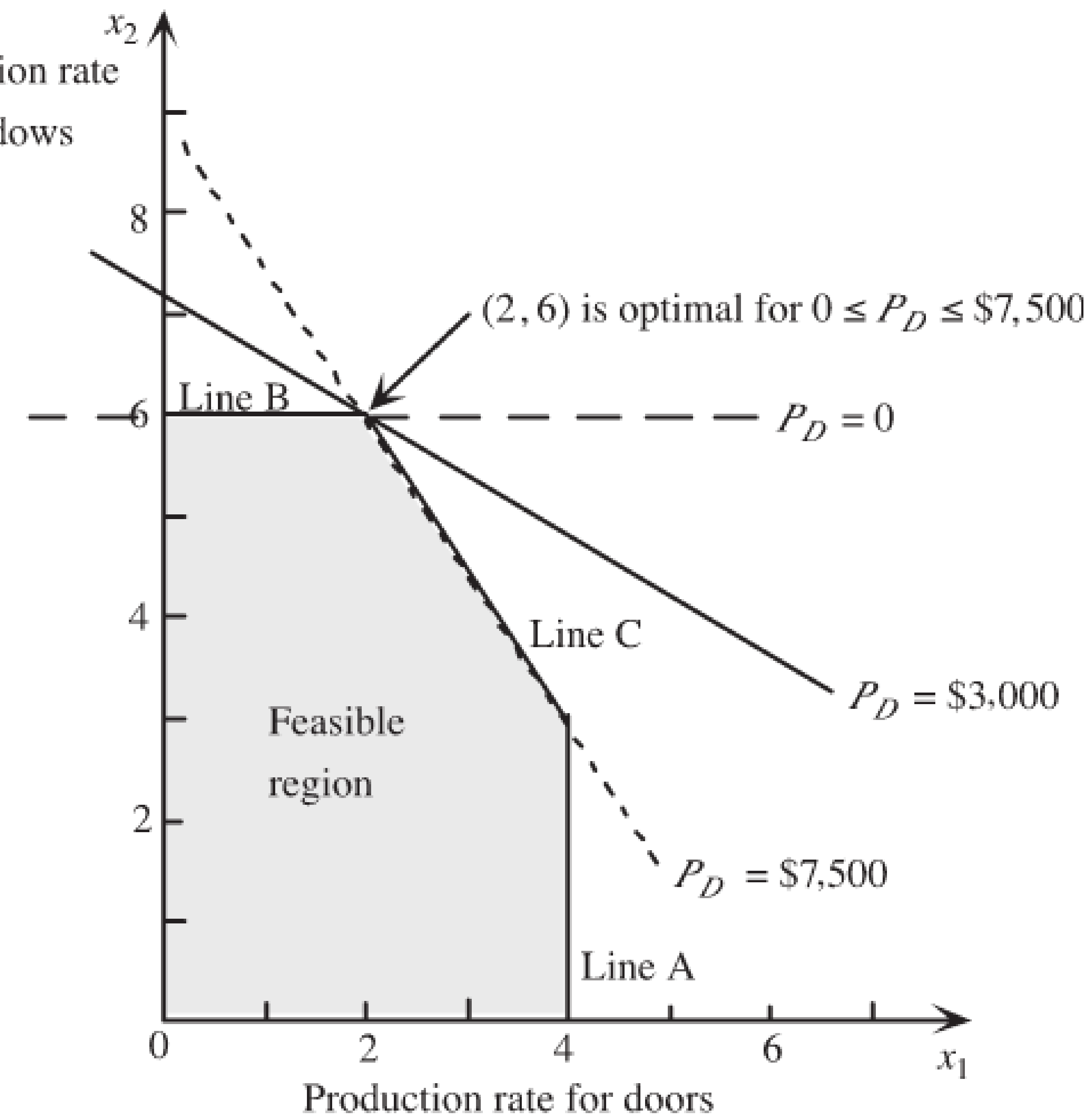


FIGURE 6.20
The two dashed lines that pass through solid constraint boundary lines are the objective function lines when P_D (the profit per batch of doors) is at an endpoint of its allowable range, $0 \le P_D \le \$7,500$, since either line or any objective function line in between still yields $(x_1, x_2) = (2, 6)$ as an optimal solution for the Wyndor problem.

(P_D = \$3,000) for the profit per batch of doors, we can be quite confident of obtaining the correct optimal solution for the true profit.

Now let us turn to the question considered in the preceding two subsections. What would happen if the estimate of P_D (\$3,000) were too low and the estimate of P_W (\$5,000) were too high simultaneously? Specifically, how far can the estimates be off in these directions before the current optimal solution, $(x_1, x_2) = (2, 6)$, would change?

Figure 6.14 showed that if P_D were increased by \$1,500 (from \$3,000 to \$4,500) and P_W were decreased by \$1,000 (from \$5,000 to \$4,000), the optimal solution would remain the same. Figure 6.15 then indicated that doubling these changes would result in a change in the optimal solution. However, it is unclear where the change in the optimal solution occurs. Figure 6.18 provided further information, but not a definitive answer to this question.

Fortunately, additional information can be gleaned from the sensitivity report (Fig. 6.19) by using its allowable increases and allowable decreases in P_D and P_W. The key is to apply the following rule (as first stated in Sec. 6.7):

> **The 100 Percent Rule for Simultaneous Changes in Objective Function Coefficients:** If simultaneous changes are made in the coefficients of the objective function, calculate for each change the percentage of the allowable change (increase or decrease) for that coefficient to remain within its allowable range. If the *sum* of the percentage changes does *not* exceed 100 percent, the original optimal solution definitely will still be optimal. (If the sum *does* exceed 100 percent, then we cannot be sure.)

This rule does not spell out what happens if the sum of the percentage changes *does* exceed 100 percent. The consequence depends on the directions of the changes in the coefficients. Remember that it is the *ratios* of the coefficients that are relevant for determining the optimal solution, so the original optimal solution might indeed remain optimal even when the sum of the percentage changes greatly exceeds 100 percent if the changes in the coefficients are in the same direction. Thus, exceeding 100 percent may or may not change the optimal solution, but so long as 100 percent is not exceeded, the original optimal solution *definitely* will still be optimal.

Keep in mind that we can safely use the entire allowable increase or decrease in a single objective function coefficient only if none of the other coefficients have changed at all. With simultaneous changes in the coefficients, we focus on the *percentage* of the allowable increase or decrease that is being used for each coefficient.

To illustrate, consider the Wyndor problem again, along with the information provided by the sensitivity report in Fig. 6.19. Suppose now that the estimate of P_D has increased from \$3,000 to \$4,500 while the estimate of P_W has decreased from \$5,000 to \$4,000. The calculations for the 100 percent rule now are

P_D: \$3,000 $\rightarrow$ \$4,500.

$$\text{Percentage of allowable increase} = 100\left(\frac{4{,}500 - 3{,}000}{4{,}500}\right)\% = 33\frac{1}{3}\%$$

P_W: \$5,000 $\rightarrow$ \$4,000.

$$\text{Percentage of allowable decrease} = 100\left(\frac{5{,}000 - 4{,}000}{3{,}000}\right)\% = 33\frac{1}{3}\%$$

$$\text{Sum} = 66\frac{2}{3}\%.$$

Since the sum of the percentages does not exceed 100 percent, the original optimal solution $(x_1, x_2) = (2, 6)$ definitely is still optimal, just as we found earlier in Fig. 6.14.

Now suppose that the estimate of P_D has increased from \$3,000 to \$6,000 while the estimate P_W has decreased from \$5,000 to \$3,000. The calculations for the 100 percent rule now are

P_D: \$3,000 $\rightarrow$ \$6,000.

$$\text{Percentage of allowable increase} = 100\left(\frac{6{,}000 - 3{,}000}{4{,}500}\right)\% = 66\frac{2}{3}\%$$

P_W: \$5,000 $\rightarrow$ \$3,000.

$$\text{Percentage of allowable decrease} = 100\left(\frac{5{,}000 - 3{,}000}{3{,}000}\right)\% = 66\frac{2}{3}\%$$

$$\text{Sum} = 133\frac{1}{3}\%.$$

Since the sum of the percentages now exceeds 100 percent, the 100 percent rule says that we can no longer guarantee that $(x_1, x_2) = (2, 6)$ is still optimal. In fact, we found earlier in both Figs. 6.15 and 6.18 that the optimal solution has changed to $(x_1, x_2) = (4, 3)$.

These results suggest how to find just where the optimal solution changes while P_D is being increased and P_W is being decreased by these relative amounts. Since 100 percent is midway between $66\frac{2}{3}$ percent and $133\frac{1}{3}$ percent, the sum of the percentage changes will equal 100 percent when the values of P_D and P_W are midway between their values in the above cases. In particular, $P_D = \$5{,}250$ is midway between \$4,500 and \$6,000 and

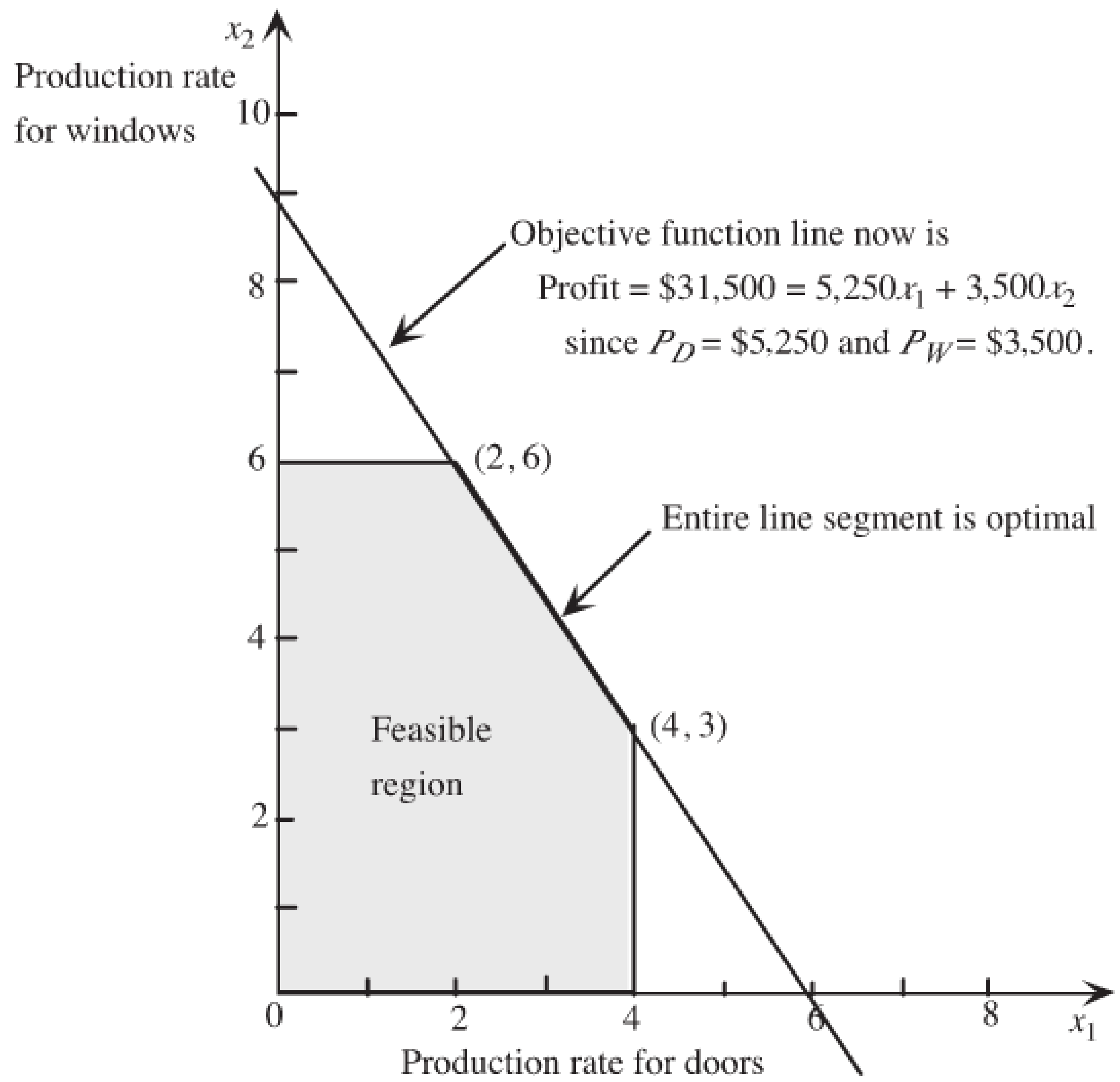


FIGURE 6.21
When the estimates of the profits per batch of doors and windows change to P_D = \$5,250 and P_W = \$3,500, which lies at the edge of what is allowed by the 100 percent rule, the graphical method shows that $(x_1, x_2) = (2, 6)$ still is an optimal solution, but now every other point on the line segment between this solution and (4, 3) also is optimal.

P_W = \$3,500 is midway between \$4,000 and \$3,000. The corresponding calculations for the 100 percent rule are

P_D: \$3,000 $\rightarrow$ \$5,250.

$$\text{Percentage of allowable increase} = 100\left(\frac{5{,}250 - 3{,}000}{4{,}500}\right)\% = 50\%$$

P_W: \$5,000 $\rightarrow$ \$3,500.

$$\text{Percentage of allowable decrease} = 100\left(\frac{5{,}000 - 3{,}500}{3{,}000}\right)\% = 50\%$$

$$\text{Sum} = 100\%.$$

Although the sum of the percentages equals 100 percent, the fact that it does not *exceed* 100 percent guarantees that $(x_1, x_2) = (2, 6)$ is still optimal. Figure 6.21 shows graphically that *both* (2, 6) and (4, 3) are now optimal, as well as all the points on the line segment connecting these two points. However. If P_D and P_W were to be changed any further from their original values (so that the sum of the percentages exceeds 100 percent), the objective function line would be rotated so far toward the vertical that $(x_1, x_2) = (4, 3)$ would become the only optimal solution.

At the same time, keep in mind that having the sum of the percentages of allowable changes exceed 100 percent does not automatically mean that the optimal solution will change. For example, suppose that the estimates of both unit profits are halved. The resulting calculations for the 100 percent rule are

P_D: \$3,000 $\rightarrow$ \$1,500.

$$\text{Percentage of allowable decrease} = 100\left(\frac{3{,}000 - 1{,}500}{3{,}000}\right)\% = \quad 50\%$$

P_W: \$5,000 $\rightarrow$ \$2,500.

$$\text{Percentage of allowable decrease} = 100\left(\frac{5{,}000 - 2{,}500}{3{,}000}\right)\% = \quad 83\frac{1}{3}\%$$

$$\text{Sum} = 133\frac{1}{3}\%.$$

Even though this sum exceeds 100 percent, Fig. 6.22 shows that the original optimal solution is still optimal. In fact, the objective function line has the same slope as the original objective function line (the solid line in Fig. 6.20). This happens whenever *proportional changes* are made to all the profit estimates, which will automatically lead to the same optimal solution.

Other Types of Sensitivity Analysis

This section has focused on how to use a spreadsheet to investigate the effect of changes in only the coefficients of the variables in the objective function. One often is interested in investigating the effect of changes in the right-hand sides of the functional constraints

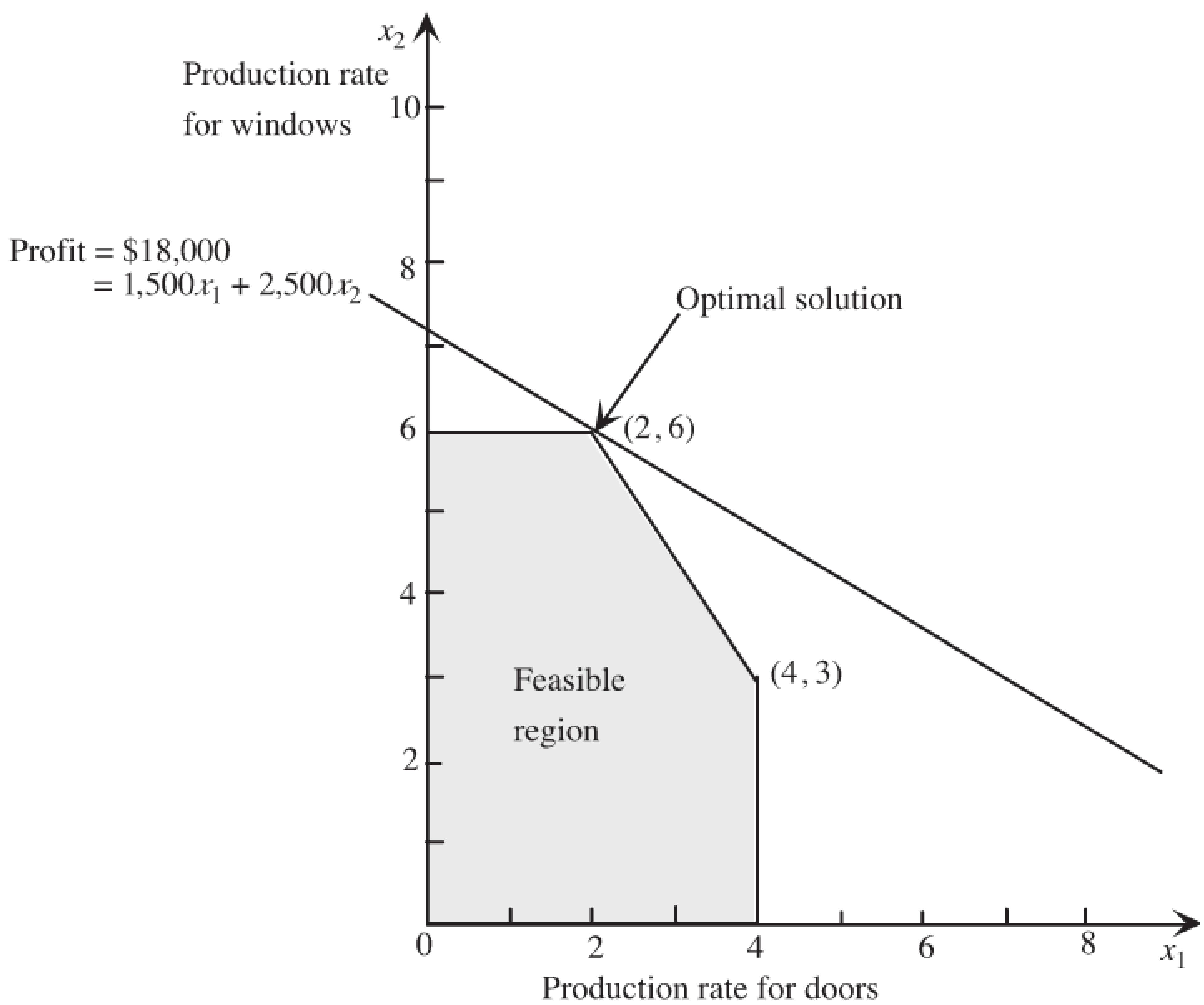


FIGURE 6.22 When the estimates of the profits per batch of doors and windows change to P_D = \$1,500 and P_W = \$2,500 (half of their original values), the graphical method shows that the optimal solution still is $(x_1, x_2) = (2, 6)$, even though the 100 percent rule says that the optimal solution might change.

as well. Occasionally you might even want to check whether the optimal solution would change if changes need to be made in some coefficients in the functional constraints.

The spreadsheet approach for investigating these other kinds of changes in the model is virtually the same as for the coefficients in the objective function. Once again, you can try out any changes in the data cells by simply making these changes on the spreadsheet and using the Excel Solver to re-solve the model. And once again, you can systematically check the effect of a series of changes in any one or two data cells by using the Solver Table. As already described in Sec. 4.7, the sensitivity report generated by the Excel Solver (or any other linear programming software package) also provides some valuable information, including the shadow prices, regarding the effect of changing the right-hand side of any single functional constraint. When changing a number of right-hand sides simultaneously, there also is a "100 percent rule" for this case that is analogous to the 100 percent rule for simultaneous changes in objective function constraints. (See the Case 1 portion of Sec. 6.7 for details about how to investigate the effect of changes in right-hand sides, including the application of the 100 percent rule for simultaneous changes in right-hand sides.)

The Worked Examples section of the book's website includes **another example** of using a spreadsheet to investigate the effect of changing individual right-hand sides.

6.9 CONCLUSIONS

Every linear programming problem has associated with it a dual linear programming problem. There are a number of very useful relationships between the original (primal) problem and its dual problem that enhance our ability to analyze the primal problem. For example, the economic interpretation of the dual problem gives shadow prices that measure the marginal value of the resources in the primal problem and provides an interpretation of the simplex method. Because the simplex method can be applied directly to either problem in order to solve both of them simultaneously, considerable computational effort sometimes can be saved by dealing directly with the dual problem. Duality theory, including the dual simplex method (Sec. 7.1) for working with superoptimal basic solutions, also plays a major role in sensitivity analysis.

The values used for the parameters of a linear programming model generally are just estimates. Therefore, sensitivity analysis needs to be performed to investigate what happens if these estimates are wrong. The fundamental insight of Sec. 5.3 provides the key to performing this investigation efficiently. The general objectives are to identify the sensitive parameters that affect the optimal solution, to try to estimate these sensitive parameters more closely, and then to select a solution that remains good over the range of likely values of the sensitive parameters. This analysis is a very important part of most linear programming studies.

With the help of the Excel Solver, spreadsheets also provide some useful methods of performing sensitivity analysis. One method is to repeatedly enter changes in one or more parameters of the model into the spreadsheet and then click on the Solve button to see immediately if the optimal solution changes. A second is to use the Solver Table to systematically check on the effect of making a series of changes in one or two parameters of the model. A third is to use the sensitivity report provided by the Excel Solver to identify the allowable range for the coefficients in the objective function, the shadow prices for the functional constraints, and the allowable range for each right-hand side over which its shadow price remains valid. (Other software that applies the simplex method—including MPL/CPLEX, LINDO, and LINGO—also provides such a sensitivity report upon request.)

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Worked Examples:

Examples for Chapter 6

A Demonstration Example in OR Tutor:

Sensitivity Analysis

Interactive Procedures in IOR Tutorial:

Interactive Graphical Method
Enter or Revise a General Linear Programming Model
Solve Interactively by the Simplex Method
Sensitivity Analysis

Automatic Procedures in IOR Tutorial:

Solve Automatically by the Simplex Method
Graphical Method and Sensitivity Analysis

Excel Add-Ins:

Premium Solver for Education
Solver Table

Files (Chapter 3) for Solving the Wyndor Example:

Excel Files
LINGO/LINDO File
MPL/CPLEX File

Glossary for Chapter 6

See Appendix 1 for documentation of the software.

PROBLEMS

The symbols to the left of some of the problems (or their parts) have the following meaning:

D: The demonstration example just listed may be helpful.
I: We suggest that you use the corresponding interactive procedure just listed (the printout records your work).
C: Use the computer with any of the software options available to you (or as instructed by your instructor) to solve the problem automatically.
E*: Use Excel.

An asterisk on the problem number indicates that at least a partial answer is given in the back of the book.

6.1-1.* Construct the dual problem for each of the following linear programming models fitting our standard form.
(a) Model in Prob. 3.1-6
(b) Model in Prob. 4.7-5

6.1-2. Consider the linear programming model in Prob. 4.5-4.
(a) Construct the primal-dual table and the dual problem for this model.
(b) What does the fact that Z is unbounded for this model imply about its dual problem?

6.1-3. For each of the following linear programming models, give your recommendation on which is the more efficient way (probably) to obtain an optimal solution: by applying the simplex method directly to this primal problem or by applying the simplex method directly to the dual problem instead. Explain.

(a) Maximize $Z = 10x_1 - 4x_2 + 7x_3$,

subject to

$$\begin{aligned} 3x_1 - x_2 + 2x_3 &\le 25 \\ x_1 - 2x_2 + 3x_3 &\le 25 \\ 5x_1 + x_2 + 2x_3 &\le 40 \\ x_1 + x_2 + x_3 &\le 90 \\ 2x_1 - x_2 + x_3 &\le 20 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

(b) Maximize $Z = 2x_1 + 5x_2 + 3x_3 + 4x_4 + x_5$,

subject to

$$\begin{aligned} x_1 + 3x_2 + 2x_3 + 3x_4 + x_5 &\le 6 \\ 4x_1 + 6x_2 + 5x_3 + 7x_4 + x_5 &\le 15 \end{aligned}$$

and

$$x_j \ge 0, \quad \text{for } j = 1, 2, 3, 4, 5.$$

6.1-4. Consider the following problem.

Maximize $Z = -x_1 - 2x_2 - x_3$,

subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 &\le 12 \\ x_1 + x_2 - x_3 &\le 1 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

(a) Construct the dual problem.
(b) Use duality theory to show that the optimal solution for the primal problem has $Z \le 0$.

6.1-5. Consider the following problem.

Maximize $Z = 5x_1 + 4x_2 + 3x_3$,

subject to

$$\begin{aligned} x_1 + x_3 &\le 15 \quad \text{(resource 1)} \\ x_2 + 2x_3 &\le 25 \quad \text{(resource 2)} \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

(a) Construct the dual problem for this primal problem.
I **(b)** Solve the dual problem graphically. Use this solution to identify the shadow prices for the resources in the primal problem.
C **(c)** Confirm your results from part (*b*) by solving the primal problem automatically by the simplex method and then identifying the shadow prices.

6.1-6. Follow the instructions of Prob. 6.1-5 for the following problem.

Maximize $Z = x_1 - 3x_2 + 2x_3$,

subject to

$$\begin{aligned} 2x_1 + 2x_2 - 2x_3 &\le 6 \quad \text{(resource 1)} \\ -x_2 + 2x_3 &\le 4 \quad \text{(resource 2)} \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

6.1-7. Consider the following problem.

Maximize $Z = 2x_1 + 3x_2$,

subject to

$$\begin{aligned} 4x_1 + x_2 &\le 20 \\ -x_1 + x_2 &\le 10 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

I **(a)** Demonstrate graphically that this problem has no feasible solutions.
(b) Construct the dual problem.

I **(c)** Demonstrate graphically that the dual problem has an unbounded objective function.

I **6.1-8.** Construct and graph a primal problem with two decision variables and two functional constraints that has feasible solutions and an unbounded objective function. Then construct the dual problem and demonstrate graphically that it has no feasible solutions.

I **6.1-9.** Construct a pair of primal and dual problems, each with two decision variables and two functional constraints, such that both problems have no feasible solutions. Demonstrate this property graphically.

6.1-10. Construct a pair of primal and dual problems, each with two decision variables and two functional constraints, such that the primal problem has no feasible solutions and the dual problem has an unbounded objective function.

6.1-11. Use the weak duality property to prove that if both the primal and the dual problem have feasible solutions, then both must have an optimal solution.

6.1-12. Consider the primal and dual problems in our standard form presented in matrix notation at the beginning of Sec. 6.1. Use only this definition of the dual problem for a primal problem in this form to prove each of the following results.
(a) The weak duality property presented in Sec. 6.1.
(b) If the primal problem has an unbounded feasible region that permits increasing Z indefinitely, then the dual problem has no feasible solutions.

6.1-13. Consider the primal and dual problems in our standard form presented in matrix notation at the beginning of Sec. 6.1. Let $\mathbf{y}^*$ denote the optimal solution for this dual problem. Suppose that $\mathbf{b}$ is then replaced by $\bar{\mathbf{b}}$. Let $\bar{\mathbf{x}}$ denote the optimal solution for the new primal problem. Prove that

$$\mathbf{c}\bar{\mathbf{x}} \leq \mathbf{y}^*\bar{\mathbf{b}}.$$

6.1-14. For any linear programming problem in our standard form and its dual problem, label each of the following statements as true or false and then justify your answer.
(a) The sum of the number of functional constraints and the number of variables (before augmenting) is the same for both the primal and the dual problems.
(b) At each iteration, the simplex method simultaneously identifies a CPF solution for the primal problem and a CPF solution for the dual problem such that their objective function values are the same.
(c) If the primal problem has an unbounded objective function, then the optimal value of the objective function for the dual problem must be zero.

6.2-1. Consider the simplex tableaux for the Wyndor Glass Co. problem given in Table 4.8. For each tableau, give the economic interpretation of the following items:
(a) Each of the coefficients of the slack variables (x_3, x_4, x_5) in row 0
(b) Each of the coefficients of the decision variables (x_1, x_2) in row 0
(c) The resulting choice for the entering basic variable (or the decision to stop after the final tableau)

6.3-1.* Consider the following problem.

$$\text{Maximize} \quad Z = 6x_1 + 8x_2,$$

subject to

$$5x_1 + 2x_2 \leq 20$$
$$x_1 + 2x_2 \leq 10$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

(a) Construct the dual problem for this primal problem.
(b) Solve both the primal problem and the dual problem graphically. Identify the CPF solutions and corner-point infeasible solutions for both problems. Calculate the objective function values for all these solutions.
(c) Use the information obtained in part (*b*) to construct a table listing the complementary basic solutions for these problems. (Use the same column headings as for Table 6.9.)
I **(d)** Work through the simplex method step by step to solve the primal problem. After each iteration (including iteration 0), identify the BF solution for this problem and the complementary basic solution for the dual problem. Also identify the corresponding corner-point solutions.

6.3-2. Consider the model with two functional constraints and two variables given in Prob. 4.1-5. Follow the instructions of Prob. 6.3-1 for this model.

6.3-3. Consider the primal and dual problems for the Wyndor Glass Co. example given in Table 6.1. Using Tables 5.5, 5.6, 6.8, and 6.9, construct a new table showing the eight sets of nonbasic variables for the primal problem in column 1, the corresponding sets of associated variables for the dual problem in column 2, and the set of nonbasic variables for each complementary basic solution in the dual problem in column 3. Explain why this table demonstrates the complementary slackness property for this example.

6.3-4. Suppose that a primal problem has a *degenerate* BF solution (one or more basic variables equal to zero) as its optimal solution. What does this degeneracy imply about the dual problem? Why? Is the converse also true?

6.3-5. Consider the following problem.

$$\text{Maximize} \quad Z = 3x_1 - 8x_2,$$

subject to

$$x_1 - 2x_2 \leq 10$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

(a) Construct the dual problem, and then find its optimal solution by inspection.
(b) Use the complementary slackness property and the optimal

solution for the dual problem to find the optimal solution for the primal problem.

(c) Suppose that c_1, the coefficient of x_1 in the primal objective function, actually can have any value in the model. For what values of c_1 does the dual problem have no feasible solutions? For these values, what does duality theory then imply about the primal problem?

6.3-6. Consider the following problem.

$$\text{Maximize} \quad Z = 2x_1 + 7x_2 + 4x_3,$$

subject to

$$x_1 + 2x_2 + x_3 \le 10$$
$$3x_1 + 3x_2 + 2x_3 \le 10$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

(a) Construct the dual problem for this primal problem.

(b) Use the dual problem to demonstrate that the optimal value of Z for the primal problem cannot exceed 25.

(c) It has been conjectured that x_2 and x_3 should be the basic variables for the optimal solution of the primal problem. Directly derive this basic solution (and Z) by using Gaussian elimination. Simultaneously derive and identify the complementary basic solution for the dual problem by using Eq. (0) for the primal problem. Then draw your conclusions about whether these two basic solutions are optimal for their respective problems.

I **(d)** Solve the dual problem graphically. Use this solution to identify the basic variables and the nonbasic variables for the optimal solution of the primal problem. Directly derive this solution, using Gaussian elimination.

6.3-7.* Reconsider the model of Prob. 6.1-3*b*.

(a) Construct its dual problem.

I **(b)** Solve this dual problem graphically.

(c) Use the result from part (*b*) to identify the nonbasic variables and basic variables for the optimal BF solution for the primal problem.

(d) Use the results from part (*c*) to obtain the optimal solution for the primal problem directly by using Gaussian elimination to solve for its basic variables, starting from the initial system of equations [excluding Eq. (0)] constructed for the simplex method and setting the nonbasic variables to zero.

(e) Use the results from part (*c*) to identify the defining equations (see Sec. 5.1) for the optimal CPF solution for the primal problem, and then use these equations to find this solution.

6.3-8. Consider the model given in Prob. 5.3-10.

(a) Construct the dual problem.

(b) Use the given information about the basic variables in the optimal primal solution to identify the nonbasic variables and basic variables for the optimal dual solution.

(c) Use the results from part (*b*) to identify the defining equations (see Sec. 5.1) for the optimal CPF solution for the dual problem, and then use these equations to find this solution.

I **(d)** Solve the dual problem graphically to verify your results from part (*c*).

6.3-9. Consider the model given in Prob. 3.1-5.

(a) Construct the dual problem for this model.

(b) Use the fact that $(x_1, x_2) = (13, 5)$ is optimal for the primal problem to identify the nonbasic variables and basic variables for the optimal BF solution for the dual problem.

(c) Identify this optimal solution for the dual problem by directly deriving Eq. (0) corresponding to the optimal primal solution identified in part (*b*). Derive this equation by using Gaussian elimination.

(d) Use the results from part (*b*) to identify the defining equations (see Sec. 5.1) for the optimal CPF solution for the dual problem. Verify your optimal dual solution from part (*c*) by checking to see that it satisfies this system of equations.

6.3-10. Suppose that you also want information about the dual problem when you apply the matrix form of the simplex method (see Sec. 5.2) to the primal problem in our standard form.

(a) How would you identify the optimal solution for the dual problem?

(b) After obtaining the BF solution at each iteration, how would you identify the complementary basic solution in the dual problem?

6.4-1. Consider the following problem.

$$\text{Maximize} \quad Z = 5x_1 + 4x_2,$$

subject to

$$2x_1 + 3x_2 \ge 10$$
$$x_1 + 2x_2 = 20$$

and

$$x_2 \ge 0 \quad (x_1 \text{ unconstrained in sign}).$$

(a) Use the SOB method to construct the dual problem.

(b) Use Table 6.12 to convert the primal problem to our standard form given at the beginning of Sec. 6.1, and construct the corresponding dual problem. Then show that this dual problem is equivalent to the one obtained in part (*a*).

6.4-2. Consider the primal and dual problems in our standard form presented in matrix notation at the beginning of Sec. 6.1. Use only this definition of the dual problem for a primal problem in this form to prove each of the following results.

(a) If the functional constraints for the primal problem $\mathbf{Ax} \le \mathbf{b}$ are changed to $\mathbf{Ax} = \mathbf{b}$, the only resulting change in the dual problem is to *delete* the nonnegativity constraints, $\mathbf{y} \ge \mathbf{0}$. (*Hint:* The constraints $\mathbf{Ax} = \mathbf{b}$ are equivalent to the set of constraints $\mathbf{Ax} \le \mathbf{b}$ *and* $\mathbf{Ax} \ge \mathbf{b}$.)

(b) If the functional constraints for the primal problem $\mathbf{Ax} \le \mathbf{b}$ are changed to $\mathbf{Ax} \ge \mathbf{b}$, the only resulting change in the dual problem is that the nonnegativity constraints $\mathbf{y} \ge \mathbf{0}$ are replaced by nonpositivity constraints $\mathbf{y} \le \mathbf{0}$, where the current dual variables are interpreted as the negative of the original dual variables. (*Hint:* The constraints $\mathbf{Ax} \ge \mathbf{b}$ are equivalent to $-\mathbf{Ax} \le -\mathbf{b}$.)

(c) If the nonnegativity constraints for the primal problem $\mathbf{x} \geq \mathbf{0}$ are deleted, the only resulting change in the dual problem is to replace the functional constraints $\mathbf{yA} \geq \mathbf{c}$ by $\mathbf{yA} = \mathbf{c}$. (*Hint:* A variable unconstrained in sign can be replaced by the difference of two nonnegative variables.)

6.4-3.* Construct the dual problem for the linear programming problem given in Prob. 4.6-3.

6.4-4. Consider the following problem.

Minimize $\quad Z = 5x_1 + 10x_2,$

subject to

$$\begin{aligned} -4x_1 + 2x_2 &\geq 4 \\ 5x_1 - 10x_2 &\geq 10 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

(a) Construct the dual problem.

I **(b)** Use graphical analysis of the dual problem to determine whether the primal problem has feasible solutions and, if so, whether its objective function is bounded.

6.4-5. Consider the two versions of the dual problem for the radiation therapy example that are given in Tables 6.15 and 6.16. Review in Sec. 6.4 the general discussion of why these two versions are completely equivalent. Then fill in the details to verify this equivalency by proceeding step by step to convert the version in Table 6.15 to equivalent forms until the version in Table 6.16 is obtained.

6.4-6. For each of the following linear programming models, use the SOB method to construct its dual problem.

(a) Model in Prob. 4.6-7

(b) Model in Prob. 4.6-16

6.4-7. Consider the model with equality constraints given in Prob. 4.6-2.

(a) Construct its dual problem.

(b) Demonstrate that the answer in part (*a*) is correct (i.e., equality constraints yield dual variables without nonnegativity constraints) by first converting the primal problem to our standard form (see Table 6.12), then constructing its dual problem, and next converting this dual problem to the form obtained in part (*a*).

6.4-8.* Consider the model without nonnegativity constraints given in Prob. 4.6-14.

(a) Construct its dual problem.

(b) Demonstrate that the answer in part (*a*) is correct (i.e., variables without nonnegativity constraints yield equality constraints in the dual problem) by first converting the primal problem to our standard form (see Table 6.12), then constructing its dual problem, and finally converting this dual problem to the form obtained in part (*a*).

6.4-9. Consider the dual problem for the Wyndor Glass Co. example given in Table 6.1. Demonstrate that *its* dual problem is the primal problem given in Table 6.1 by going through the conversion steps given in Table 6.13.

6.4-10. Consider the following problem.

Minimize $\quad Z = -5x_1 - 15x_2,$

subject to

$$\begin{aligned} 2x_1 - 4x_2 &\leq 8 \\ -3x_1 + 3x_2 &\leq 24 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

I **(a)** Demonstrate graphically that this problem has an unbounded objective function.

(b) Construct the dual problem.

I **(c)** Demonstrate graphically that the dual problem has no feasible solutions.

6.5-1. Consider the model of Prob. 6.7-2. Use duality theory directly to determine whether the current basic solution remains optimal after each of the following independent changes.

(a) The change in part (*e*) of Prob. 6.7-2

(b) The change in part (*g*) of Prob. 6.7-2

6.5-2. Consider the model of Prob. 6.7-4. Use duality theory directly to determine whether the current basic solution remains optimal after each of the following independent changes.

(a) The change in part (*b*) of Prob. 6.7-4

(b) The change in part (*d*) of Prob. 6.7-4

6.5-3. Reconsider part (*d*) of Prob. 6.7-6. Use duality theory directly to determine whether the original optimal solution is still optimal.

6.6-1.* Consider the following problem.

Maximize $\quad Z = 3x_1 + x_2 + 4x_3,$

subject to

$$\begin{aligned} 6x_1 + 3x_2 + 5x_3 &\leq 25 \\ 3x_1 + 4x_2 + 5x_3 &\leq 20 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

The corresponding final set of equations yielding the optimal solution is

$$\begin{aligned} &(0) \quad Z + 2x_2 + \frac{1}{5}x_4 + \frac{3}{5}x_5 = 17 \\ &(1) \quad x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_4 - \frac{1}{3}x_5 = \frac{5}{3} \\ &(2) \quad x_2 + x_3 - \frac{1}{5}x_4 + \frac{2}{5}x_5 = 3. \end{aligned}$$

(a) Identify the optimal solution from this set of equations.

(b) Construct the dual problem.

I **(c)** Identify the optimal solution for the dual problem from the final set of equations. Verify this solution by solving the dual problem graphically.

(d) Suppose that the original problem is changed to

$$\text{Maximize} \quad Z = 3x_1 + 3x_2 + 4x_3,$$

subject to

$$6x_1 + 2x_2 + 5x_3 \leq 25$$
$$3x_1 + 3x_2 + 5x_3 \leq 20$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Use duality theory to determine whether the previous optimal solution is still optimal.

(e) Use the fundamental insight presented in Sec. 5.3 to identify the new coefficients of x_2 in the final set of equations after it has been adjusted for the changes in the original problem given in part (*d*).

(f) Now suppose that the only change in the original problem is that a new variable x_{new} has been introduced into the model as follows:

$$\text{Maximize} \quad Z = 3x_1 + x_2 + 4x_3 + 2x_{\text{new}},$$

subject to

$$6x_1 + 3x_2 + 5x_3 + 3x_{\text{new}} \leq 25$$
$$3x_1 + 4x_2 + 5x_3 + 2x_{\text{new}} \leq 20$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_{\text{new}} \geq 0.$$

Use duality theory to determine whether the previous optimal solution, along with $x_{\text{new}} = 0$, is still optimal.

(g) Use the fundamental insight presented in Sec. 5.3 to identify the coefficients of x_{new} as a nonbasic variable in the final set of equations resulting from the introduction of x_{new} into the original model as shown in part (*f*).

D,I **6.6-2.** Reconsider the model of Prob. 6.6-1. You are now to conduct sensitivity analysis by *independently* investigating each of the following six changes in the original model. For each change, use the sensitivity analysis procedure to revise the given final set of equations (in tableau form) and convert it to proper form from Gaussian elimination. Then test this solution for feasibility and for optimality. (Do not reoptimize.)

(a) Change the right-hand side of constraint 1 to $b_1 = 10$.
(b) Change the right-hand side of constraint 2 to $b_2 = 10$.
(c) Change the coefficient of x_2 in the objective function to $c_2 = 3$.
(d) Change the coefficient of x_3 in the objective function to $c_3 = 2$.
(e) Change the coefficient of x_2 in constraint 2 to $a_{22} = 2$.
(f) Change the coefficient of x_1 in constraint 1 to $a_{11} = 8$.

D,I **6.6-3.** Consider the following problem.

$$\text{Minimize} \quad W = 5y_1 + 4y_2,$$

subject to

$$4y_1 + 3y_2 \geq 4$$
$$2y_1 + y_2 \geq 3$$
$$y_1 + 2y_2 \geq 1$$
$$y_1 + y_2 \geq 2$$

and

$$y_1 \geq 0, \quad y_2 \geq 0.$$

Because this primal problem has more functional constraints than variables, suppose that the simplex method has been applied directly to its dual problem. If we let x_5 and x_6 denote the slack variables for this dual problem, the resulting final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1	3	0	2	0	1	1	9
x_2	(1)	0	1	1	−1	0	1	−1	1
x_4	(2)	0	2	0	3	1	−1	2	3

For each of the following independent changes in the original primal model, you now are to conduct sensitivity analysis by directly investigating the effect on the dual problem and then inferring the complementary effect on the primal problem. For each change, apply the procedure for sensitivity analysis summarized at the end of Sec. 6.6 to the dual problem (do *not* reoptimize), and then give your conclusions as to whether the current basic solution for the primal problem still is feasible and whether it still is optimal. Then check your conclusions by a direct graphical analysis of the primal problem.

(a) Change the objective function to $W = 3y_1 + 5y_2$.
(b) Change the right-hand sides of the functional constraints to 3, 5, 2, and 3, respectively.
(c) Change the first constraint to $2y_1 + 4y_2 \geq 7$.
(d) Change the second constraint to $5y_1 + 2y_2 \geq 10$.

6.7-1. Read the referenced article that fully describes the OR study summarized in the application vignette presented in Sec. 6.7. Briefly describe how sensitivity analysis was applied in this study. Then list the various financial and nonfinancial benefits that resulted from the study.

D,I **6.7-2.*** Consider the following problem.

$$\text{Maximize} \quad Z = -5x_1 + 5x_2 + 13x_3,$$

subject to

$$-x_1 + x_2 + 3x_3 \leq 20$$
$$12x_1 + 4x_2 + 10x_3 \leq 90$$

and

$$x_j \geq 0 \quad (j = 1, 2, 3).$$

If we let x_4 and x_5 be the slack variables for the respective constraints, the simplex method yields the following final set of equations:

$$\begin{array}{llr}
(0) & Z \qquad\qquad + 2x_3 + 5x_4 \qquad & = 100 \\
(1) & \quad -x_1 + x_2 + 3x_3 + \;\; x_4 \qquad & = \;\; 20 \\
(2) & \quad 16x_1 \qquad - 2x_3 - 4x_4 + x_5 & = \;\; 10.
\end{array}$$

Now you are to conduct sensitivity analysis by *independently* investigating each of the following nine changes in the original model. For each change, use the sensitivity analysis procedure to revise this set of equations (in tableau form) and convert it to proper form from Gaussian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. (Do not reoptimize.)

(a) Change the right-hand side of constraint 1 to

$$b_1 = 30.$$

(b) Change the right-hand side of constraint 2 to

$$b_2 = 70.$$

(c) Change the right-hand sides to

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 100 \end{bmatrix}.$$

(d) Change the coefficient of x_3 in the objective function to

$$c_3 = 8.$$

(e) Change the coefficients of x_1 to

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}.$$

(f) Change the coefficients of x_2 to

$$\begin{bmatrix} c_2 \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}.$$

(g) Introduce a new variable x_6 with coefficients

$$\begin{bmatrix} c_6 \\ a_{16} \\ a_{26} \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 5 \end{bmatrix}.$$

(h) Introduce a new constraint $2x_1 + 3x_2 + 5x_3 \leq 50$. (Denote its slack variable by x_6.)

(i) Change constraint 2 to

$$10x_1 + 5x_2 + 10x_3 \leq 100.$$

6.7-3.* Reconsider the model of Prob. 6.7-2. Suppose that we now want to apply parametric linear programming analysis to this problem. Specifically, the right-hand sides of the functional constraints are changed to

$20 + 2\theta$ (for constraint 1)

and

$90 - \theta$ (for constraint 2),

where θ can be assigned any positive or negative values.

Express the basic solution (and Z) corresponding to the original optimal solution as a function of θ. Determine the lower and upper bounds on θ before this solution would become infeasible.

D,I **6.7-4.** Consider the following problem.

$$\text{Maximize} \quad Z = 2x_1 + 7x_2 - 3x_3,$$

subject to

$$\begin{aligned} x_1 + 3x_2 + 4x_3 &\leq 30 \\ x_1 + 4x_2 - \;\; x_3 &\leq 10 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

By letting x_4 and x_5 be the slack variables for the respective constraints, the simplex method yields the following final set of equations:

$$\begin{array}{llr}
(0) & Z \;+ \quad x_2 + \; x_3 \qquad\quad + 2x_5 & = 20 \\
(1) & \qquad - \; x_2 + 5x_3 + x_4 - \;\; x_5 & = 20 \\
(2) & \quad x_1 + 4x_2 - \; x_3 \qquad\quad + \;\; x_5 & = 10.
\end{array}$$

Now you are to conduct sensitivity analysis by *independently* investigating each of the following seven changes in the original model. For each change, use the sensitivity analysis procedure to revise this set of equations (in tableau form) and convert it to proper form from Gaussian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. If either test fails, reoptimize to find a new optimal solution.

(a) Change the right-hand sides to

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \end{bmatrix}.$$

(b) Change the coefficients of x_3 to

$$\begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}.$$

(c) Change the coefficients of x_1 to

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}.$$

(d) Introduce a new variable x_6 with coefficients

$$\begin{bmatrix} c_6 \\ a_{16} \\ a_{26} \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}.$$

(e) Change the objective function to $Z = x_1 + 5x_2 - 2x_3$.

(f) Introduce a new constraint $3x_1 + 2x_2 + 3x_3 \leq 25$.
(g) Change constraint 2 to $x_1 + 2x_2 + 2x_3 \leq 35$.

6.7-5. Reconsider the model of Prob. 6.7-4. Suppose that we now want to apply parametric linear programming analysis to this problem. Specifically, the right-hand sides of the functional constraints are changed to

$$30 + 3\theta \qquad \text{(for constraint 1)}$$

and

$$10 - \theta \qquad \text{(for constraint 2)},$$

where θ can be assigned any positive or negative values.

Express the basic solution (and Z) corresponding to the original optimal solution as a function of θ. Determine the lower and upper bounds on θ before this solution would become infeasible.

D,I **6.7-6.** Consider the following problem.

$$\text{Maximize} \quad Z = 2x_1 - x_2 + x_3,$$

subject to

$$\begin{aligned} 3x_1 - 2x_2 + 2x_3 &\leq 15 \\ -x_1 + x_2 + x_3 &\leq 3 \\ x_1 - x_2 + x_3 &\leq 4 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

If we let x_4, x_5, and x_6 be the slack variables for the respective constraints, the simplex method yields the following final set of equations:

$$\begin{array}{llr} (0) & Z \qquad + 2x_3 + x_4 + x_5 & = 18 \\ (1) & x_2 + 5x_3 + x_4 + 3x_5 & = 24 \\ (2) & 2x_3 + x_5 + x_6 & = 7 \\ (3) & x_1 + 4x_3 + x_4 + 2x_5 & = 21. \end{array}$$

Now you are to conduct sensitivity analysis by *independently* investigating each of the following eight changes in the original model. For each change, use the sensitivity analysis procedure to revise this set of equations (in tableau form) and convert it to proper form from Gaussian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. If either test fails, reoptimize to find a new optimal solution.

(a) Change the right-hand sides to

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 2 \end{bmatrix}.$$

(b) Change the coefficient of x_3 in the objective function to $c_3 = 2$.
(c) Change the coefficient of x_1 in the objective function to $c_1 = 3$.
(d) Change the coefficients of x_3 to

$$\begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}.$$

(e) Change the coefficients of x_1 and x_2 to

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_2 \\ a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 3 \\ 2 \end{bmatrix},$$

respectively.
(f) Change the objective function to $Z = 5x_1 + x_2 + 3x_3$.
(g) Change constraint 1 to $2x_1 - x_2 + 4x_3 \leq 12$.
(h) Introduce a new constraint $2x_1 + x_2 + 3x_3 \leq 60$.

C **6.7-7** Consider the Distribution Unlimited Co. problem presented in Sec. 3.4 and summarized in Fig. 3.13.

Although Fig. 3.13 gives estimated unit costs for shipping through the various shipping lanes, there actually is some uncertainty about what these unit costs will turn out to be. Therefore, before adopting the optimal solution given at the end of Sec. 3.4, management wants additional information about the effect of inaccuracies in estimating these unit costs.

Use a computer package based on the simplex method to generate sensitivity analysis information preparatory to addressing the following questions.

(a) Which of the unit shipping costs given in Fig. 3.13 has the smallest margin for error without invalidating the optimal solution given in Sec. 3.4? Where should the greatest effort be placed in estimating the unit shipping costs?
(b) What is the allowable range for each of the unit shipping costs?
(c) How should these allowable ranges be interpreted to management?
(d) If the estimates change for more than one of the unit shipping costs, how can you use the generated sensitivity analysis information to determine whether the optimal solution might change?

6.7-8. Consider the following problem.

$$\text{Maximize} \quad Z = c_1x_1 + c_2x_2,$$

subject to

$$\begin{aligned} 2x_1 - x_2 &\leq b_1 \\ x_1 - x_2 &\leq b_2 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

Let x_3 and x_4 denote the slack variables for the respective functional constraints. When $c_1 = 3$, $c_2 = -2$, $b_1 = 30$, and $b_2 = 10$, the simplex method yields the following final simplex tableau.

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	Right Side
Z	(0)	1	0	0	1	1	40
x_2	(1)	0	0	1	1	−2	10
x_1	(2)	0	1	0	1	−1	20

I **(a)** Use graphical analysis to determine the allowable range for c_1 and c_2.
(b) Use algebraic analysis to derive and verify your answers in part *(a)*.
I **(c)** Use graphical analysis to determine the allowable range for b_1 and b_2.
(d) Use algebraic analysis to derive and verify your answers in part *(c)*
C **(e)** Use a software package based on the simplex method to find these allowable ranges.

I **6.7-9.** Consider Variation 5 of the Wyndor Glass Co. model (see Fig. 6.6 and Table 6.24), where the changes in the parameter values given in Table 6.21 are $\bar{c}_2 = 3$, $\bar{a}_{22} = 3$, and $\bar{a}_{32} = 4$. Use the formula $\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}}$ to find the allowable range for each b_i. Then interpret each allowable range graphically.

I **6.7-10.** Consider Variation 5 of the Wyndor Glass Co. model (see Fig. 6.6 and Table 6.24), where the changes in the parameter values given in Table 6.21 are $\bar{c}_2 = 3$, $\bar{a}_{22} = 3$, and $\bar{a}_{32} = 4$. Verify both algebraically and graphically that the allowable range for c_1 is $c_1 \geq \frac{9}{4}$.

6.7-11. For the problem given in Table 6.21, find the allowable range for c_2. Show your work algebraically, using the tableau given in Table 6.21. Then justify your answer from a geometric viewpoint, referring to Fig. 6.3.

6.7-12.* For the original Wyndor Glass Co. problem, use the last tableau in Table 4.8 to do the following.
(a) Find the allowable range for each b_i.
(b) Find the allowable range for c_1 and c_2.
C **(c)** Use a software package based on the simplex method to find these allowable ranges.

6.7-13. For Variation 6 of the Wyndor Glass Co. model presented in Sec. 6.7, use the last tableau in Table 6.25 to do the following.
(a) Find the allowable range for each b_i.
(b) Find the allowable range for c_1 and c_2.
C **(c)** Use a software package based on the simplex method to find these allowable ranges.

6.7-14. Consider Variation 5 of the Wyndor Glass Co. model presented in Sec. 6.7, where $\bar{c}_2 = 3$, $\bar{a}_{22} = 3$, $\bar{a}_{32} = 4$, and where the other parameters are given in Table 6.21. Starting from the resulting final tableau given at the bottom of Table 6.24, construct a table like Table 6.26 to perform parametric linear programming analysis, where

$$c_1 = 3 + \theta \quad \text{and} \quad c_2 = 3 + 2\theta.$$

How far can θ be increased above 0 before the current basic solution is no longer optimal?

6.7-15. Reconsider the model of Prob. 6.7-6. Suppose that you now have the option of making trade-offs in the profitability of the first two activities, whereby the objective function coefficient of x_1 can be increased by any amount by simultaneously decreasing the objective function coefficient of x_2 by the same amount. Thus, the alternative choices of the objective function are

$$Z(\theta) = (2 + \theta)x_1 - (1 + \theta)x_2 + x_3,$$

where any nonnegative value of θ can be chosen.

Construct a table like Table 6.26 to perform parametric linear programming analysis on this problem. Determine the upper bound on θ before the original optimal solution would become nonoptimal. Then determine the best choice of θ over this range.

6.7-16. Consider the following parametric linear programming problem.

$$\text{Maximize} \quad Z(\theta) = (10 - 4\theta)x_1 + (4 - \theta)x_2 + (7 + \theta)x_3,$$

subject to

$$\begin{aligned} 3x_1 + x_2 + 2x_3 &\leq 7 \qquad \text{(resource 1)},\\ 2x_1 + x_2 + 3x_3 &\leq 5 \qquad \text{(resource 2)}, \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0,$$

where θ can be assigned any positive or negative values. Let x_4 and x_5 be the slack variables for the respective constraints. After we apply the simplex method with $\theta = 0$, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	0	0	3	2	2	24
x_1	(1)	0	1	0	−1	1	−1	2
x_2	(2)	0	0	1	5	−2	3	1

(a) Determine the range of values of θ over which the above BF solution will remain optimal. Then find the best choice of θ within this range.
(b) Given that θ is within the range of values found in part *(a)*, find the allowable range for b_1 (the available amount of resource 1). Then do the same for b_2 (the available amount of resource 2).
(c) Given that θ is within the range of values found in part *(a)*, identify the shadow prices (as a function of θ) for the two resources. Use this information to determine how the optimal value of the objective function would change (as a function of θ) if the available amount of resource 1 were decreased by 1 and the available amount of resource 2 simultaneously were increased by 1.
(d) Construct the dual of this parametric linear programming problem. Set $\theta = 0$ and solve this dual problem graphically to find the corresponding shadow prices for the two resources of the primal problem. Then find these shadow prices as a function of θ [within the range of values found in part *(a)*] by algebraically solving for this same optimal CPF solution for the dual problem as a function of θ.

6.7-17. Consider the following parametric linear programming problem.

$$\text{Maximize} \quad Z(\theta) = 2x_1 + 4x_2 + 5x_3,$$

subject to

$$x_1 + 3x_2 + 2x_3 \le 5 + \theta$$
$$x_1 + 2x_2 + 3x_3 \le 6 + 2\theta$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0,$$

where θ can be assigned any positive or negative values. Let x_4 and x_5 be the slack variables for the respective functional constraints. After we apply the simplex method with $\theta = 0$, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	0	0	1	0	1	1	11
x_1	(1)	1	1	5	0	3	−2	3
x_3	(2)	2	0	−1	1	−1	1	1

(a) Express the BF solution (and Z) given in this tableau as a function of θ. Determine the lower and upper bounds on θ before this optimal BF solution would become infeasible. Then determine the best choice of θ between these bounds.

(b) Given that θ is between the bounds found in part (*a*), determine the allowable range for c_1 (the coefficient of x_1 in the objective function).

6.7-18. Consider the following problem.

Maximize $\quad Z = 10x_1 + 4x_2,$

subject to

$$3x_1 + x_2 \le 30$$
$$2x_1 + x_2 \le 25$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

Let x_3 and x_4 denote the slack variables for the respective functional constraints. After we apply the simplex method, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	Right Side
Z	(0)	1	0	0	2	2	110
x_2	(1)	0	0	1	−2	3	15
x_1	(2)	0	1	0	1	−1	5

Now suppose that both of the following changes are made simultaneously in the original model:

1. The first constraint is changed to $4x_1 + x_2 \le 40$.
2. Parametric programming is introduced to change the objective function to the alternative choices of

$$Z(\theta) = (10 - 2\theta)x_1 + (4 + \theta)x_2,$$

where any nonnegative value of θ can be chosen.

(a) Construct the resulting revised final tableau (as a function of θ), and then convert this tableau to proper form from Gaussian elimination. Use this tableau to identify the new optimal solution that applies for either $\theta = 0$ or sufficiently small values of θ.

(b) What is the upper bound on θ before this optimal solution would become nonoptimal?

(c) Over the range of θ from zero to this upper bound, which choice of θ gives the largest value of the objective function?

6.7-19. Consider the following problem.

Maximize $\quad Z = 9x_1 + 8x_2 + 5x_3,$

subject to

$$2x_1 + 3x_2 + x_3 \le 4$$
$$5x_1 + 4x_2 + 3x_3 \le 11$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Let x_4 and x_5 denote the slack variables for the respective functional constraints. After we apply the simplex method, the final simplex tableau is

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	0	2	0	2	1	19
x_1	(1)	0	1	5	0	3	−1	1
x_3	(2)	0	0	−7	1	−5	2	2

D,I **(a)** Suppose that a new technology has become available for conducting the first activity considered in this problem. If the new technology were adopted to replace the existing one, the coefficients of x_1 in the model would change

$$\text{from} \quad \begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 5 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 18 \\ 3 \\ 6 \end{bmatrix}.$$

Use the sensitivity analysis procedure to investigate the potential effect and desirability of adopting the new technology. Specifically, assuming it were adopted, construct the resulting revised final tableau, convert this tableau to proper form from Gaussian elimination, and then reoptimize (if necessary) to find the new optimal solution.

(b) Now suppose that you have the option of mixing the old and new technologies for conducting the first activity. Let θ denote the fraction of the technology used that is from the new technology, so $0 \le \theta \le 1$. Given θ, the coefficients of x_1 in the model become

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 9 + 9\theta \\ 2 + \theta \\ 5 + \theta \end{bmatrix}.$$

Construct the resulting revised final tableau (as a function of θ), and convert this tableau to proper form from Gaussian elimination. Use this tableau to identify the current basic solution as a function of θ. Over the allowable values of $0 \le \theta \le 1$, give the range of values of θ for which this solution is both feasible and optimal. What is the best choice of θ within this range?

6.7-20. Consider the following problem.

Maximize $\quad Z = 3x_1 + 5x_2 + 2x_3,$

subject to

$$\begin{aligned} -2x_1 + 2x_2 + x_3 &\le 5 \\ 3x_1 + x_2 - x_3 &\le 10 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Let x_4 and x_5 be the slack variables for the respective functional constraints. After we apply the simplex method, the final simplex tableau is

Basic Variable	Eq.	Coefficient of:						Right Side
		Z	x_1	x_2	x_3	x_4	x_5	
Z	(0)	1	0	20	0	9	7	115
x_1	(1)	0	1	3	0	1	1	15
x_3	(2)	0	0	8	1	3	2	35

Parametric linear programming analysis now is to be applied simultaneously to the objective function and right-hand sides, where the model in terms of the new parameter is the following:

Maximize $\quad Z(\theta) = (3 + 2\theta)x_1 + (5 + \theta)x_2 + (2 - \theta)x_3,$

subject to

$$\begin{aligned} -2x_1 + 2x_2 + x_3 &\le 5 + 6\theta \\ 3x_1 + x_2 - x_3 &\le 10 - 8\theta \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Construct the resulting revised final tableau (as a function of θ), and convert this tableau to proper form from Gaussian elimination. Use this tableau to identify the current basic solution as a function of θ. For $\theta \ge 0$, give the range of values of θ for which this solution is both feasible and optimal. What is the best choice of θ within this range?

6.7-21. Consider the Wyndor Glass Co. problem described in Sec. 3.1. Suppose that, in addition to considering the introduction of two new products, management now is considering changing the production rate of a certain old product that is still profitable. Refer to Table 3.1. The number of production hours per week used per unit production rate of this old product is 1, 4, and 3 for Plants 1, 2, and 3, respectively. Therefore, if we let θ denote the *change* (positive or negative) in the production rate of this old product, the right-hand sides of the three functional constraints in Sec. 3.1 become $4 - \theta$, $12 - 4\theta$, and $18 - 3\theta$, respectively. Thus, choosing a negative value of θ would free additional capacity for producing more of the two new products, whereas a positive value would have the opposite effect.

(a) Use a parametric linear programming formulation to determine the effect of different choices of θ on the optimal solution for the product mix of the two new products given in the final tableau of Table 4.8. In particular, use the fundamental insight of Sec. 5.3 to obtain expressions for Z and the basic variables x_3, x_2, and x_1 in terms of θ, assuming that θ is sufficiently close to zero that this "final" basic solution still is feasible and thus optimal for the given value of θ.

(b) Now consider the broader question of the choice of θ along with the product mix for the two new products. What is the breakeven unit profit for the old product (in comparison with the two new products) below which its production rate should be decreased ($\theta < 0$) in favor of the new products and above which its production rate should be increased ($\theta > 0$)?

(c) If the unit profit is above this breakeven point, how much can the old product's production rate be increased before the final BF solution would become infeasible?

(d) If the unit profit is below this breakeven point, how much can the old product's production rate be decreased (assuming its previous rate was larger than this decrease) before the final BF solution would become infeasible?

6.7-22. Consider the following problem.

Maximize $\quad Z = 2x_1 - x_2 + 3x_3,$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - 2x_2 + x_3 &\ge 1 \\ 2x_2 + x_3 &\le 2 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Suppose that the Big M method (see Sec. 4.6) is used to obtain the initial (artificial) BF solution. Let $\bar{x}_4$ be the artificial slack variable for the first constraint, x_5 the surplus variable for the second constraint, $\bar{x}_6$ the artificial variable for the second constraint, and x_7 the slack variable for the third constraint. The corresponding final set of equations yielding the optimal solution is

$$\begin{aligned}
&(0) \quad Z + 5x_2 + (M + 2)\bar{x}_4 + M\bar{x}_6 + x_7 = 8 \\
&(1) \quad x_1 - x_2 + \bar{x}_4 - x_7 = 1 \\
&(2) \quad 2x_2 + x_3 + x_7 = 2 \\
&(3) \quad 3x_2 + \bar{x}_4 + x_5 - \bar{x}_6 = 2.
\end{aligned}$$

Suppose that the original objective function is changed to $Z = 2x_1 + 3x_2 + 4x_3$ and that the original third constraint is changed to $2x_2 + x_3 \leq 1$. Use the sensitivity analysis procedure to revise the final set of equations (in tableau form) and convert it to proper form from Gaussian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. (Do not reoptimize.)

6.8-1. Consider the following problem.

Maximize $\quad Z = 2x_1 + 5x_2,$

subject to

$$x_1 + 2x_2 \leq 10 \text{ (resource 1)}$$
$$x_1 + 3x_2 \leq 12 \text{ (resource 2)}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0,$$

where Z measures the profit in dollars from the two activities.

While doing sensitivity analysis, you learn that the estimates of the unit profits are accurate only to within ± 50 percent. In other words, the ranges of *likely values* for these unit profits are \$1 to \$3 for activity 1 and \$2.50 to \$7.50 for activity 2.

E* **(a)** Formulate a spreadsheet model for this problem based on the original estimates of the unit profits. Then use the Solver to find an optimal solution and to generate the sensitivity report.

E* **(b)** Use the spreadsheet and Solver to check whether this optimal solution remains optimal if the unit profit for activity 1 changes from \$2 to \$1. From \$2 to \$3.

E* **(c)** Also check whether the optimal solution remains optimal if the unit profit for activity 1 still is \$2 but the unit profit for activity 2 changes from \$5 to \$2.50. From \$5 to \$7.50.

E* **(d)** Use the Solver Table to systematically generate the optimal solution and total profit as the unit profit of activity 1 increases in 20¢ increments from \$1 to \$3 (without changing the unit profit of activity 2). Then do the same as the unit profit of activity 2 increases in 50¢ increments from \$2.50 to \$7.50 (without changing the unit profit of activity 1). Use these results to estimate the allowable range for the unit profit of each activity.

I **(e)** Use the Graphical Method and Sensitivity Analysis procedure in IOR Tutorial to estimate the allowable range for the unit profit of each activity.

E* **(f)** Use the sensitivity report provided by the Excel Solver to find the allowable range for the unit profit of each activity. Then use these ranges to check your results in parts (*b–e*).

E* **(g)** Use a two-way Solver Table to systematically generate the optimal solution as the unit profits of the two activities are changed simultaneously as described in part (*d*).

I **(h)** Use the Graphical Method and Sensitivity Analysis procedure in IOR Tutorial to interpret the results in part (*g*) graphically.

E* **6.8-2.** Reconsider the model given in Prob. 6.8-1. While doing sensitivity analysis, you learn that the estimates of the right-hand sides of the two functional constraints are accurate only to within ± 50 percent. In other words, the ranges of *likely values* for these parameters are 5 to 15 for the first right-hand side and 6 to 18 for the second right-hand side.

(a) After solving the original spreadsheet model, determine the shadow price for the first functional constraint by increasing its right-hand side by 1 and solving again.

(b) Use the Solver Table to generate the optimal solution and total profit as the right-hand side of the first functional constraint is incremented by 1 from 5 to 15. Use this table to estimate the allowable range for this right-hand side, i.e., the range over which the shadow price obtained in part (*a*) is valid.

(c) Repeat part (*a*) for the second functional constraint.

(d) Repeat part (*b*) for the second functional constraint where its right-hand side is incremented by 1 from 6 to 18.

(e) Use the Solver's sensitivity report to determine the shadow price for each functional constraint and the allowable range for the right-hand side of each of these constraints.

6.8-3. Consider the following problem.

Maximize $\quad Z = x_1 + 2x_2,$

subject to

$$x_1 + 3x_2 \leq 8 \text{ (resource 1)}$$
$$x_1 + x_2 \leq 4 \text{ (resource 2)}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0,$$

where Z measures the profit in dollars from the two activities and the right-hand sides are the number of units available of the respective resources.

I **(a)** Use the graphical method to solve this model.

I **(b)** Use graphical analysis to determine the shadow price for each of these resources by solving again after increasing the amount of the resource available by 1.

E* **(c)** Use the spreadsheet model and the Solver instead to do parts (*a*) and (*b*).

E* **(d)** For each resource in turn, use the Solver Table to systematically generate the optimal solution and the total profit when the only change is that the amount of that resource available increases in increments of 1 from 4 less than the original value up to 6 more than the current value. Use these results to estimate the allowable range for the amount available of each resource.

(e) Use the Solver's sensitivity report to obtain the shadow prices. Also use this report to find the range for the amount of each resource available over which the corresponding shadow price remains valid.

(f) Describe why these shadow prices are useful when management has the flexibility to change the amounts of the resources being made available.

6.8-4.* One of the products of the G.A. Tanner Company is a special kind of toy that provides an estimated unit profit of \$3. Because of a large demand for this toy, management would like to increase its production rate from the current level of 1,000 per day.

However, a limited supply of two subassemblies (A and B) from vendors makes this difficult. Each toy requires two subassemblies of type A, but the vendor providing these subassemblies would only be able to increase its supply rate from the current 2,000 per day to a maximum of 3,000 per day. Each toy requires only one subassembly of type B, but the vendor providing these subassemblies would be unable to increase its supply rate above the current level of 1,000 per day. Because no other vendors currently are available to provide these subassemblies, management is considering initiating a new production process internally that would simultaneously produce an equal number of subassemblies of the two types to supplement the supply from the two vendors. It is estimated that the company's cost for producing one subassembly of each type would be \$2.50 more than the cost of purchasing these subassemblies from the two vendors. Management wants to determine both the production rate of the toy and the production rate of each pair of subassemblies (one A and one B) that would maximize the total profit.

The following table summarizes the data for the problem.

	Resource Usage per Unit of Each Activity		
	Activity		
Resource	**Produce Toys**	**Produce Subassemblies**	**Amount of Resource Available**
Subassembly A	2	–1	3,000
Subassembly B	1	–1	1,000
Unit profit	\$3	–\$2.50	

E* **(a)** Formulate and solve a spreadsheet model for this problem.

E* **(b)** Since the stated unit profits for the two activities are only estimates, management wants to know how much each of these estimates can be off before the optimal solution would change. Begin exploring this question for the first activity (producing toys) by using the spreadsheet and Solver to manually generate a table that gives the optimal solution and total profit as the unit profit for this activity increases in 50¢ increments from \$2 to \$4. What conclusion can be drawn about how much the estimate of this unit profit can differ in each direction from its original value of \$3 before the optimal solution would change?

E* **(c)** Repeat part (*b*) for the second activity (producing subassemblies) by generating a table as the unit profit for this activity increases in 50¢ increments from –\$3.50 to –\$1.50 (with the unit profit for the first activity fixed at \$3).

E* **(d)** Use the Solver Table to systematically generate all the data requested in parts (*b*) and (*c*), except use 25¢ increments instead of 50¢ increments. Use these data to refine your conclusions in parts (*b*) and (*c*).

I **(e)** Use the Graphical Method and Sensitivity Analysis procedure in IOR Tutorial to determine how much the unit profit of each activity can change in either direction (without changing the unit profit of the other activity) before the optimal solution would change. Use this information to specify the allowable range for the unit profit of each activity.

E* **(f)** Use Excel's sensitivity report to find the allowable range for the unit profit of each activity.

E* **(g)** Use a two-way Solver Table to systematically generate the optimal solution as the unit profits of the two activities are changed simultaneously as described in parts (*b*) and (*c*).

(h) Use the information provided by Excel's sensitivity report to describe how far the unit profits of the two activities can change simultaneously before the optimal solution might change.

E* **6.8-5.** Reconsider Prob. 6.8-4. After further negotiations with each vendor, management of the G.A. Tanner Co. has learned that either of them would be willing to consider increasing their supply of their respective subassemblies over the previously stated maxima (3,000 subassemblies of type A per day and 1,000 of type B per day) if the company would pay a small premium over the regular price for the extra subassemblies. The size of the premium for each type of subassembly remains to be negotiated. The demand for the toy being produced is sufficiently high so that 2,500 per day could be sold if the supply of subassemblies could be increased enough to support this production rate. Assume that the original estimates of unit profits given in Prob. 6.8-4 are accurate.

(a) Formulate and solve a spreadsheet model for this problem with the original maximum supply levels and the additional constraint that no more than 2,500 toys should be produced per day.

(b) Without considering the premium, use the spreadsheet and Solver to determine the shadow price for the subassembly A constraint by solving the model again after increasing the maximum supply by 1. Use this shadow price to determine the maximum premium that the company should be willing to pay for each subassembly of this type.

(c) Repeat part (*b*) for the subassembly B constraint.

(d) Estimate how much the maximum supply of subassemblies of type A could be increased before the shadow price (and the corresponding premium) found in part (*b*) would no longer be valid by using the Solver Table to generate the optimal solution and total profit (excluding the premium) as the maximum supply increases in increments of 100 from 3,000 to 4,000.

(e) Repeat part (*d*) for subassemblies of type B by using the Solver Table as the maximum supply increases in increments of 100 from 1,000 to 2,000.

(f) Use the Solver's sensitivity report to determine the shadow price for each of the subassembly constraints and the allowable range for the right-hand side of each of these constraints.

E* **6.8-6.*** Consider the Union Airways problem presented in Sec. 3.4, including the data given in Table 3.19. The Excel files for Chap. 3 include a spreadsheet that shows the formulation and optimal solution for this problem. You are to use this spreadsheet and the Excel Solver to do parts (*a*) to (*g*) below.

Management is about to begin negotiations on a new contract with the union that represents the company's customer service agents. This might result in some small changes in the daily costs per agent given in Table 3.19 for the various shifts. Several possible changes listed below are being considered separately. In each case, management would like to know whether the change might result in the solution in the spreadsheet no longer being optimal. Answer this question in parts (*a*) to (*e*) by using the spreadsheet and Solver directly. If the optimal solution changes, record the new solution.

(a) The daily cost per agent for Shift 2 changes from \$160 to \$165.
(b) The daily cost per agent for Shift 4 changes from \$180 to \$170.
(c) The changes in parts (*a*) and (*b*) both occur.
(d) The daily cost per agent increases by \$4 for shifts 2, 4, and 5, but decreases by \$4 for shifts 1 and 3.
(e) The daily cost per agent increases by 2 percent for each shift.
(f) Use the Solver to generate the sensitivity report for this problem. Suppose that the above changes are being considered later without having the spreadsheet model immediately available on a computer. Show in each case how the sensitivity report can be used to check whether the original optimal solution must still be optimal.
(g) For each of the five shifts in turn, use the Solver Table to systematically generate the optimal solution and total cost when the only change is that the daily cost per agent on that shift increases in \$3 increments from \$15 less than the current cost up to \$15 more than the current cost.

E* **6.8-7.** Reconsider the Union Airways problem and its spreadsheet model that was dealt with in Prob. 6.8-6.

Management now is considering increasing the level of service provided to customers by increasing one or more of the numbers in the rightmost column of Table 3.19 for the minimum number of agents needed in the various time periods. To guide them in making this decision, they would like to know what impact this change would have on total cost.

Use the Excel Solver to generate the sensitivity report in preparation for addressing the following questions.

(a) Which of the numbers in the rightmost column of Table 3.19 can be increased without increasing total cost? In each case, indicate how much it can be increased (if it is the only one being changed) without increasing total cost.
(b) For each of the other numbers, how much would the total cost increase per increase of 1 in the number? For each answer, indicate how much the number can be increased (if it is the only one being changed) before the answer is no longer valid.
(c) Do your answers in part (*b*) definitely remain valid if all the numbers considered in part (*b*) are simultaneously increased by one?
(d) Do your answers in part (*b*) definitely remain valid if all 10 numbers are simultaneously increased by one?
(e) How far can all 10 numbers be simultaneously increased by the same amount before your answers in part (*b*) may no longer be valid?

6.8-8. David, LaDeana, and Lydia are the sole partners and workers in a company which produces fine clocks. David and LaDeana each are available to work a maximum of 40 hours per week at the company, while Lydia is available to work a maximum of 20 hours per week.

The company makes two different types of clocks: a grandfather clock and a wall clock. To make a clock, David (a mechanical engineer) assembles the inside mechanical parts of the clock while LaDeana (a woodworker) produces the handcarved wood casings. Lydia is responsible for taking orders and shipping the clocks. The amount of time required for each of these tasks is shown below.

	Time Required	
Task	**Grandfather Clock**	**Wall Clock**
Assemble clock mechanism	6 hours	4 hours
Carve wood casing	8 hours	4 hours
Shipping	3 hours	3 hours

Each grandfather clock built and shipped yields a profit of \$300, while each wall clock yields a profit of \$200.

The three partners now want to determine how many clocks of each type should be produced per week to maximize the total profit.

(a) Formulate a linear programming model in algebraic form for this problem.
I **(b)** Use the Graphical Method and Sensitivity Analysis procedure in IOR Tutorial to solve the model. Then use this procedure to check if the optimal solution would change if the unit profit for grandfather clocks is changed from \$300 to \$375 (with no other changes in the model). Then check if the optimal solution would change if, in addition to this change in the unit profit for grandfather clocks, the estimated unit profit for wall clocks also changes from \$200 to \$175.
E* **(c)** Formulate and solve this model on a spreadsheet.
E* **(d)** Use the Excel Solver to check the effect of the changes specified in part (*b*).
E* **(e)** Use the Solver Table to systematically generate the optimal solution and total profit as the unit profit for grandfather clocks is increased in \$20 increments from \$150 to \$450 (with no change in the unit profit for wall clocks). Then do the same as the unit profit for wall clocks is increased in \$20 increments from \$50 to \$350 (with no change in the unit profit for grandfather clocks). Use this information to estimate the allowable range for the unit profit of each type of clock.
E* **(f)** Use a two-way Solver Table to systematically generate the optimal solution as the unit profits for the two types of clocks are changed simultaneously as specified in part (*e*), except use \$50 increments instead of \$20 increments.
E* **(g)** For each of the three partners in turn, use the Excel Solver to determine the effect on the optimal solution and the total profit if that partner alone were to increase the maximum number of hours available to work per week by 5 hours.
E* **(h)** Use the Solver Table to systematically generate the optimal solution and the total profit when the only change is that David's maximum number of hours available to work per week changes to each of the following values: 35, 37, 39, 41, 43, 45. Then do the same when the only change is that LaDeana's number changes in the same way. Then do the

same when the only change is that Lydia's number changes to each of the following values: 15, 17, 19, 21, 23, 25.

E* **(i)** Generate the Excel sensitivity report and use it to determine the allowable range for the unit profit for each type of clock and the allowable range for the maximum number of hours each partner is available to work per week.

(j) To increase the total profit, the three partners have agreed that one of them will slightly increase the maximum number of hours available to work per week. The choice of which one will be based on which one would increase the total profit the most. Use the sensitivity report to make this choice. (Assume no change in the original estimates of the unit profits.)

(k) Explain why one of the shadow prices is equal to zero.

(l) Can the shadow prices in the sensitivity report be validly used to determine the effect if Lydia were to change her maximum number of hours available to work per week from 20 to 25? If so, what would be the increase in the total profit?

(m) Repeat part (*l*) if, in addition to the change for Lydia, David also were to change his maximum number of hours available to work per week from 40 to 35.

I **(n)** Use graphical analysis to verify your answer in part (*m*).

CASES

CASE 6.1 Controlling Air Pollution

Refer to Sec. 3.4 (subsection entitled "Controlling Air Pollution") for the Nori & Leets Co. problem. After the OR team obtained an optimal solution, we mentioned that the team then conducted sensitivity analysis. We now continue this story by having you retrace the steps taken by the OR team, after we provide some additional background.

The values of the various parameters in the original formulation of the model are given in Tables 3.12, 3.13, and 3.14. Since the company does not have much prior experience with the pollution abatement methods under consideration, the cost estimates given in Table 3.14 are fairly rough, and each one could easily be off by as much as 10 percent in either direction. There also is some uncertainty about the parameter values given in Table 3.13, but less so than for Table 3.14. By contrast, the values in Table 3.12 are policy standards, and so are prescribed constants.

However, there still is considerable debate about where to set these policy standards on the required reductions in the emission rates of the various pollutants. The numbers in Table 3.12 actually are preliminary values tentatively agreed upon before learning what the total cost would be to meet these standards. Both the city and company officials agree that the final decision on these policy standards should be based on the *trade-off* between costs and benefits. With this in mind, the city has concluded that each 10 percent increase in the policy standards over the current values (all the numbers in Table 3.12) would be worth \$3.5 million to the city. Therefore, the city has agreed to reduce the company's tax payments to the city by \$3.5 million for *each* 10 percent reduction in the policy standards (up to 50 percent) that is accepted by the company.

Finally, there has been some debate about the *relative* values of the policy standards for the three pollutants. As indicated in Table 3.12, the required reduction for particulates now is less than half of that for either sulfur oxides or hydrocarbons. Some have argued for decreasing this disparity. Others contend that an even greater disparity is justified because sulfur oxides and hydrocarbons cause considerably more damage than particulates. Agreement has been reached that this issue will be reexamined after information is obtained about which trade-offs in policy standards (increasing one while decreasing another) are available without increasing the total cost.

(a) Use any available linear programming software to solve the model for this problem as formulated in Sec. 3.4. In addition to the optimal solution, obtain the additional output provided for performing postoptimality analysis (e.g., the Sensitivity Report when using Excel). This output provides the basis for the following steps.

(b) Ignoring the constraints with no uncertainty about their parameter values (namely, $x_j \leq 1$ for $j = 1, 2, \ldots, 6$), identify the parameters of the model that should be classified as *sensitive parameters*. (*Hint:* See the subsection "Sensitivity Analysis" in Sec. 4.7.) Make a resulting recommendation about which parameters should be estimated more closely, if possible.

(c) Analyze the effect of an inaccuracy in estimating each cost parameter given in Table 3.14. If the true value is 10 percent *less* than the estimated value, would this alter the optimal solution? Would it change if the true value were 10 percent *more* than the estimated value? Make a resulting recommendation about where to focus further work in estimating the cost parameters more closely.

(d) Consider the case where your model has been converted to maximization form before applying the simplex method. Use Table 6.14 to construct the corresponding dual problem, and use the output from applying the simplex method to the primal problem to identify an optimal solution for this dual problem. If the primal problem had been left in minimization form, how would this affect the form of the dual problem and the sign of the optimal dual variables?

(e) For each pollutant, use your results from part (*d*) to specify the rate at which the total cost of an optimal solution would change with any small change in the required reduction in the annual emission rate of the pollutant. Also specify how much this required reduction can be changed (up or down) without affecting the rate of change in the total cost.

(f) For each unit change in the policy standard for particulates given in Table 3.12, determine the change in the opposite direction for sulfur oxides that would keep the total cost of an optimal solution unchanged. Repeat this for hydrocarbons instead of sulfur oxides. Then do it for a simultaneous and equal change for both sulfur oxides and hydrocarbons in the opposite direction from particulates.

(g) Letting θ denote the percentage increase in all the policy standards given in Table 3.12, formulate the problem of analyzing the effect of simultaneous proportional increases in these standards as a parametric linear programming problem. Then use your results from part (*e*) to determine the rate at which the total cost of an optimal solution would increase with a small increase in θ from zero.

(h) Use the simplex method to find an optimal solution for the parametric linear programming problem formulated in part (*g*) for each $\theta = 10, 20, 30, 40, 50$. Considering the tax incentive offered by the city, use these results to determine which value of θ (including the option of $\theta = 0$) should be chosen to minimize the company's total cost of both pollution abatement and taxes.

(i) For the value of θ chosen in part (*h*), repeat parts (*e*) and (*f*) so that the decision makers can make a final decision on the *relative* values of the policy standards for the three pollutants.

PREVIEWS OF ADDED CASES ON OUR WEBSITE (www.mhhe.com/hillier)

CASE 6.2 Farm Management

The Ploughman family has owned and operated a 640-acre farm for several generations. The family now needs to make a decision about the mix of livestock and crops for the coming year. By assuming that normal weather conditions will prevail next year, a linear programming model can be formulated and solved to guide this decision. However, adverse weather conditions would harm the crops and greatly reduce the resulting value. Therefore, considerable postoptimality analysis is needed to explore the effect of several possible scenarios for the weather next year and the implications for the family's decision.

CASE 6.3 Assigning Students to Schools, Revisited

This case is a continuation of Case 4.3, which involved the Springfield School Board assigning students from six residential areas to the city's three remaining middle schools. After solving a linear programming model for the problem with any software package, that package's sensitivity analysis report now needs to be used for two purposes. One is to check on the effect of an increase in certain bussing costs because of ongoing road construction in one of the residential areas. The other is to explore the advisability of adding portable classrooms to increase the capacity of one or more of the middle schools for a few years.

CASE 6.4 Writing a Nontechnical Memo

After setting goals for how much the sales of three products should increase as a result of an upcoming advertising campaign, the management of the Profit & Gambit Co. now wants to explore the trade-off between advertising cost and increased sales. Your first task is to perform the associated sensitivity analysis. Your main task then is to write a nontechnical memo to Profit & Gambit management presenting your results in the language of management.

CHAPTER 7

Other Algorithms for Linear Programming

The key to the extremely widespread use of linear programming is the availability of an exceptionally efficient algorithm—the simplex method—that will routinely solve the large-size problems that typically arise in practice. However, the simplex method is only part of the arsenal of algorithms regularly used by linear programming practitioners. We now turn to these other algorithms.

This chapter begins with three algorithms that are, in fact, *variants* of the simplex method. In particular, the next three sections introduce the *dual simplex method* (a modification particularly useful for sensitivity analysis), *parametric linear programming* (an extension for systematic sensitivity analysis), and the *upper bound technique* (a streamlined version of the simplex method for dealing with variables having upper bounds). We will not go into the kind of detail with these algorithms that we did with the simplex method in Chaps. 4 and 5. The goal instead will be to briefly introduce their main ideas.

Section 4.9 introduced another algorithmic approach to linear programming—a type of algorithm that moves through the interior of the feasible region. We describe this *interior-point approach* further in Sec. 7.4.

A supplement to this chapter on the book's website also introduces *linear goal programming*. In this case, rather than having a *single objective* (maximize or minimize Z) as for linear programming, the problem instead has *several goals* toward which we must strive simultaneously. Certain formulation techniques enable converting a linear goal programming problem back into a linear programming problem so that solution procedures based on the simplex method can still be used. The supplement describes these techniques and procedures.

7.1 THE DUAL SIMPLEX METHOD

The *dual simplex method* is based on the duality theory presented in the first part of Chap. 6. To describe the basic idea behind this method, it is helpful to use some terminology introduced in Tables 6.10 and 6.11 of Sec. 6.3 for describing any pair of complementary basic solutions in the primal and dual problems. In particular, recall that both solutions are said to be *primal feasible* if the primal basic solution is feasible, whereas they are called *dual feasible* if the complementary dual basic solution is feasible for the dual problem. Also recall (as indicated on the right side of Table 6.11) that each complementary basic solution is optimal for its problem only if it is *both* primal feasible and dual feasible.

The dual simplex method can be thought of as the *mirror image* of the simplex method. The simplex method deals directly with basic solutions in the primal problem that are *primal feasible* but not dual feasible. It then moves toward an optimal solution by striving to achieve dual feasibility as well (the optimality test for the simplex method). By contrast, the dual simplex method deals with basic solutions in the primal problem that are *dual feasible* but not primal feasible. It then moves toward an optimal solution by striving to achieve primal feasibility as well.

Furthermore, the dual simplex method deals with a problem as if the simplex method were being applied simultaneously to its dual problem. If we make their *initial* basic solutions *complementary,* the two methods move in complete sequence, obtaining *complementary* basic solutions with each iteration.

The dual simplex method is very useful in certain special types of situations. Ordinarily it is easier to find an initial basic solution that is feasible than one that is dual feasible. However, it is occasionally necessary to introduce many *artificial* variables to construct an initial BF solution artificially. In such cases it may be easier to begin with a dual feasible basic solution and use the dual simplex method. Furthermore, fewer iterations may be required when it is not necessary to drive many artificial variables to zero.

When dealing with a problem whose initial basic solutions (without artificial variables) are *neither* primal feasible nor dual feasible, it also is possible to combine the ideas of the simplex method and dual simplex method into a *primal-dual algorithm* that strives toward both primal feasibility and dual feasibility.

As we mentioned several times in Chap. 6 as well as in Sec. 4.7, another important primary application of the dual simplex method is its use in conjunction with sensitivity analysis. Suppose that an optimal solution has been obtained by the simplex method but that it becomes necessary (or of interest for sensitivity analysis) to make minor changes in the model. If the formerly optimal basic solution is *no longer primal feasible* (but still satisfies the optimality test), you can immediately apply the dual simplex method by starting with this *dual feasible* basic solution. (We will illustrate this at the end of this section.) Applying the dual simplex method in this way usually leads to the new optimal solution much more quickly than would solving the new problem from the beginning with the simplex method.

The dual simplex method also can be useful in solving certain huge linear programming problems from scratch because it is such an efficient algorithm. Computational experience with the most powerful versions of CPLEX indicates that the dual simplex method often is more efficient than the simplex method for solving particularly massive problems encountered in practice.

The rules for the dual simplex method are very similar to those for the simplex method. In fact, once the methods are started, the only difference between them is in the criteria used for selecting the entering and leaving basic variables and for stopping the algorithm.

To start the dual simplex method (for a maximization problem), we must have all the coefficients in Eq. (0) *nonnegative* (so that the basic solution is dual feasible). The basic solutions will be infeasible (except for the last one) only because some of the variables are negative. The method continues to decrease the value of the objective function, always retaining *nonnegative coefficients* in Eq. (0), until all the *variables* are nonnegative. Such a basic solution is feasible (it satisfies all the equations) and is, therefore, optimal by the simplex method criterion of nonnegative coefficients in Eq. (0).

The details of the dual simplex method are summarized next.

Summary of the Dual Simplex Method

1. *Initialization:* After converting any functional constraints in $\geq$ form to $\leq$ form (by multiplying through both sides by -1), introduce slack variables as needed to construct a set of equations describing the problem. Find a basic solution such that the coefficients in Eq. (0) are zero for basic variables and nonnegative for nonbasic variables (so the solution is optimal if it is feasible). Go to the feasibility test.
2. *Feasibility test:* Check to see whether all the basic variables are *nonnegative.* If they are, then this solution is feasible, and therefore optimal, so stop. Otherwise, go to an iteration.
3. *Iteration:*

   Step 1 Determine the *leaving basic variable:* Select the *negative* basic variable that has the largest absolute value.

   Step 2 Determine the *entering basic variable:* Select the nonbasic variable whose coefficient in Eq. (0) reaches zero first as an increasing multiple of the equation containing the leaving basic variable is added to Eq. (0). This selection is made by checking the nonbasic variables with *negative coefficients* in that equation (the one containing the leaving basic variable) and selecting the one with the smallest absolute value of the ratio of the Eq. (0) coefficient to the coefficient in that equation.

   Step 3 Determine the *new basic solution:* Starting from the current set of equations, solve for the basic variables in terms of the nonbasic variables by Gaussian elimination. When we set the nonbasic variables equal to zero, each basic variable (and Z) equals the new right-hand side of the one equation in which it appears (with a coefficient of $+1$). Return to the feasibility test.

To fully understand the dual simplex method, you must realize that the method proceeds just as if the *simplex method* were being applied to the complementary basic solutions in the *dual problem.* (In fact, this interpretation was the motivation for constructing the method as it is.) Step 1 of an iteration, determining the leaving basic variable, is equivalent to determining the entering basic variable in the dual problem. The negative variable with the largest absolute value corresponds to the negative coefficient with the largest absolute value in Eq. (0) of the dual problem (see Table 6.3). Step 2, determining the entering basic variable, is equivalent to determining the leaving basic variable in the dual problem. The coefficient in Eq. (0) that reaches zero first corresponds to the variable in the dual problem that reaches zero first. The two criteria for stopping the algorithm are also complementary.

An Example

We shall now illustrate the dual simplex method by applying it to the *dual problem* for the Wyndor Glass Co. (see Table 6.1). Normally this method is applied directly to the problem of concern (a primal problem). However, we have chosen this problem because you have already seen the simplex method applied to *its* dual problem (namely, the primal problem[1]) in Table 4.8 so you can compare the two. To facilitate the comparison, we shall continue to denote the decision variables in the problem being solved by y_i rather than x_j.

In *maximization* form, the problem to be solved is

$$\text{Maximize} \quad Z = -4y_1 - 12y_2 - 18y_3,$$

subject to

$$\begin{aligned} y_1 \qquad + 3y_3 &\geq 3 \\ 2y_2 + 2y_3 &\geq 5 \end{aligned}$$

[1]Recall that the symmetry property in Sec. 6.1 points out that the dual of a dual problem is the original primal problem.

TABLE 7.1 Dual simplex method applied to the Wyndor Glass Co. dual problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	y_1	y_2	y_3	y_4	y_5	Right Side
0	Z	(0)	1	4	12	18	0	0	0
	y_4	(1)	0	−1	0	−3	1	0	−3
	y_5	(2)	0	0	−2	−2	0	1	−5
1	Z	(0)	1	4	0	6	0	6	−30
	y_4	(1)	0	−1	0	−3	1	0	−3
	y_2	(2)	0	0	1	1	0	$-\frac{1}{2}$	$\frac{5}{2}$
2	Z	(0)	1	2	0	0	2	6	−36
	y_3	(1)	0	$\frac{1}{3}$	0	1	$-\frac{1}{3}$	0	1
	y_2	(2)	0	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{3}{2}$

and

$$y_1 \geq 0, \qquad y_2 \geq 0, \qquad y_3 \geq 0.$$

Since negative right-hand sides are now allowed, we do not need to introduce artificial variables to be the initial basic variables. Instead, we simply convert the functional constraints to $\leq$ form and introduce slack variables to play this role. The resulting initial set of equations is that shown for iteration 0 in Table 7.1. Notice that all the coefficients in Eq. (0) are nonnegative, so the solution is optimal if it is feasible.

The initial basic solution is $y_1 = 0$, $y_2 = 0$, $y_3 = 0$, $y_4 = -3$, $y_5 = -5$, with $Z = 0$, which is not feasible because of the negative values. The leaving basic variable is y_5 ($5 > 3$), and the entering basic variable is y_2 ($12/2 < 18/2$), which leads to the second set of equations, labeled as iteration 1 in Table 7.1. The corresponding basic solution is $y_1 = 0$, $y_2 = \frac{5}{2}$, $y_3 = 0$, $y_4 = -3$, $y_5 = 0$, with $Z = -30$, which is not feasible.

The next leaving basic variable is y_4, and the entering basic variable is y_3 ($6/3 < 4/1$), which leads to the final set of equations in Table 7.1. The corresponding basic solution is $y_1 = 0$, $y_2 = \frac{3}{2}$, $y_3 = 1$, $y_4 = 0$, $y_5 = 0$, with $Z = -36$, which is feasible and therefore optimal.

Notice that the optimal solution for the dual of this problem[2] is $x_1^* = 2$, $x_2^* = 6$, $x_3^* = 2$, $x_4^* = 0$, $x_5^* = 0$, as was obtained in Table 4.8 by the simplex method. We suggest that you now trace through Tables 7.1 and 4.8 simultaneously and compare the complementary steps for the two mirror-image methods.

As mentioned earlier, an important primary application of the dual simplex method is that it frequently can be used to quickly re-solve a problem when sensitivity analysis results in making small changes in the original model. In particular, if the formerly optimal basic solution is no longer primal feasible (one or more right-hand sides now are negative) but still satisfies the optimality test (no negative coefficients in Row 0), you can immediately apply the dual simplex method by starting with this dual feasible basic solution. For example, this situation arises when a new constraint that violates the formerly optimal solution is added to the original model. To illustrate, suppose that the problem solved in Table 7.1 originally did not include its first functional constraint ($y_1 + 3y_3 \geq 3$).

[2]The *complementary optimal basic solutions property* presented in Sec. 6.3 indicates how to read the optimal solution for the dual problem from row 0 of the final simplex tableau for the primal problem. This same conclusion holds regardless of whether the simplex method or the dual simplex method is used to obtain the final tableau.

After deleting Row 1, the iteration 1 tableau in Table 7.1 shows that the resulting optimal solution is $y_1 = 0$, $y_2 = \frac{5}{2}$, $y_3 = 0$, $y_5 = 0$, with $Z = -30$. Now suppose that sensitivity analysis leads to adding the originally omitted constraint, $y_1 + 3y_3 \geq 3$, which is violated by the original optimal solution since both $y_1 = 0$ and $y_3 = 0$. To find the new optimal solution, this constraint (including its slack variable y_4) now would be added as Row 1 of the middle tableau in Table 7.1. Regardless of whether this tableau had been obtained by applying the simplex method or the dual simplex method to obtain the original optimal solution (perhaps after many iterations), applying the dual simplex method to this tableau leads to the new optimal solution in just one iteration.

If you would like to see **another example** of applying the dual simplex method, one is provided in the Worked Examples section of the book's website.

7.2 PARAMETRIC LINEAR PROGRAMMING

At the end of Sec. 6.7 we described *parametric linear programming* and its use for conducting sensitivity analysis systematically by gradually changing various model parameters simultaneously. We shall now present the algorithmic procedure, first for the case where the c_j parameters are being changed and then where the b_i parameters are varied.

Systematic Changes in the c_j Parameters

For the case where the c_j parameters are being changed, the *objective function* of the ordinary linear programming model

$$Z = \sum_{j=1}^{n} c_j x_j$$

is replaced by

$$Z(\theta) = \sum_{j=1}^{n} (c_j + \alpha_j \theta) x_j,$$

where the α_j are given input constants representing the *relative* rates at which the coefficients are to be changed. Therefore, gradually increasing θ from zero changes the coefficients at these relative rates.

The values assigned to the α_j may represent interesting simultaneous changes of the c_j for systematic sensitivity analysis of the effect of increasing the magnitude of these changes. They may also be based on how the coefficients (e.g., unit profits) would change together with respect to some factor measured by θ. This factor might be uncontrollable, e.g., the state of the economy. However, it may also be under the control of the decision maker, e.g., the amount of personnel and equipment to shift from some of the activities to others.

For any given value of θ, the optimal solution of the corresponding linear programming problem can be obtained by the simplex method. This solution may have been obtained already for the original problem where $\theta = 0$. However, the objective is to *find the optimal solution* of the modified linear programming problem [maximize $Z(\theta)$ subject to the original constraints] *as a function of* θ. Therefore, in the solution procedure you need to be able to determine when and how the optimal solution changes (if it does) as θ increases from zero to any specified positive number.

Figure 7.1 illustrates how $Z^*(\theta)$, the objective function value for the optimal solution (given θ), changes as θ increases. In fact, $Z^*(\theta)$ always has this *piecewise linear* and *convex*[3] form (see Prob. 7.2-7). The corresponding optimal solution changes (as θ increases) *just*

[3]See Appendix 2 for a definition and discussion of convex functions.

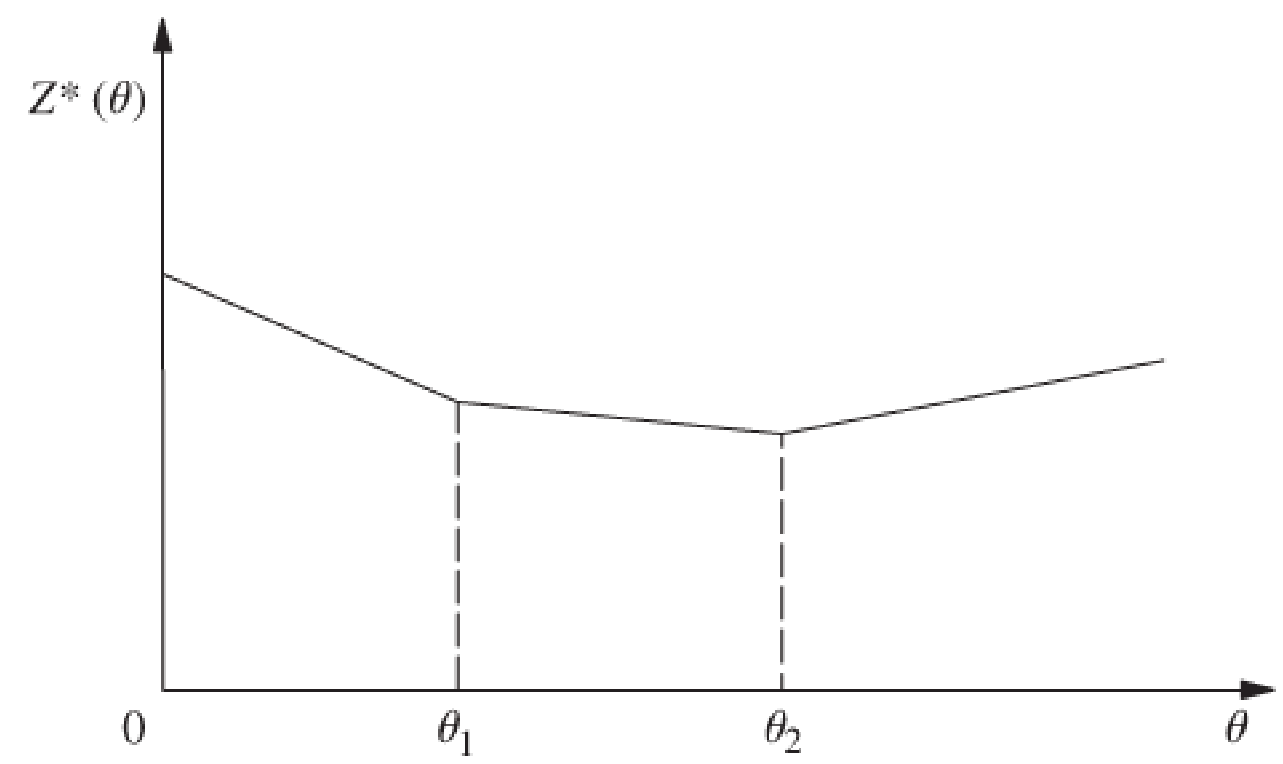


FIGURE 7.1 The objective function value for an optimal solution as a function of θ for parametric linear programming with systematic changes in the c_j parameters.

at the values of θ where the slope of the $Z^*(\theta)$ function changes. Thus, Fig. 7.1 depicts a problem where three different solutions are optimal for different values of θ, the first for $0 \le \theta \le \theta_1$, the second for $\theta_1 \le \theta \le \theta_2$, and the third for $\theta \ge \theta_2$. Because the value of each x_j remains the same within each of these intervals for θ, the value of $Z^*(\theta)$ varies with θ only because the *coefficients* of the x_j are changing as a linear function of θ. The solution procedure is based directly upon the sensitivity analysis procedure for investigating changes in the c_j parameters (Cases 2*a* and 3, Sec. 6.7). As described in the last subsection of Sec. 6.7, the only basic difference with parametric linear programming is that the changes now are expressed in terms of θ rather than as specific numbers.

Example. To illustrate the solution procedure, suppose that $\alpha_1 = 2$ and $\alpha_2 = -1$ for the original Wyndor Glass Co. problem presented in Sec. 3.1, so that

$$Z(\theta) = (3 + 2\theta)x_1 + (5 - \theta)x_2.$$

Beginning with the final simplex tableau for $\theta = 0$ (Table 4.8), we see that its Eq. (0)

$$(0) \qquad Z + \frac{3}{2}x_4 + x_5 = 36$$

would first have these changes from the original ($\theta = 0$) coefficients added into it on the left-hand side:

$$(0) \qquad Z - 2\theta x_1 + \theta x_2 + \frac{3}{2}x_4 + x_5 = 36.$$

Because both x_1 and x_2 are basic variables [appearing in Eqs. (3) and (2), respectively], they both need to be eliminated algebraically from Eq. (0):

$$\begin{array}{lr} & Z - 2\theta x_1 + \theta x_2 + \frac{3}{2}x_4 + x_5 = 36 \\ & +\ 2\theta \text{ times Eq. (3)} \\ & -\ \ \theta \text{ times Eq. (2)} \\ \hline (0) & Z + \left(\frac{3}{2} - \frac{7}{6}\theta\right)x_4 + \left(1 + \frac{2}{3}\theta\right)x_5 = 36 - 2\theta. \end{array}$$

The optimality test says that the current BF solution will remain optimal as long as these coefficients of the nonbasic variables remain nonnegative:

$$\frac{3}{2} - \frac{7}{6}\theta \ge 0, \qquad \text{for } 0 \le \theta \le \frac{9}{7},$$

$$1 + \frac{2}{3}\theta \ge 0, \qquad \text{for all } \theta \ge 0.$$

TABLE 7.2 The c_j parametric linear programming procedure applied to the Wyndor Glass Co. example

Range of θ	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side	Optimal Solution
	$Z(\theta)$	(0)	1	0	0	0	$\frac{9-7\theta}{6}$	$\frac{3+2\theta}{3}$	$36-2\theta$	$x_4=0$
										$x_5=0$
$0 \le \theta \le \frac{9}{7}$	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	$x_3=2$
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6	$x_2=6$
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	$x_1=2$
	$Z(\theta)$	(0)	1	0	0	$\frac{-9+7\theta}{2}$	0	$\frac{5-\theta}{2}$	$27+5\theta$	$x_3=0$
										$x_5=0$
$\frac{9}{7} \le \theta \le 5$	x_4	(1)	0	0	0	3	1	-1	6	$x_4=6$
	x_2	(2)	0	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3	$x_2=3$
	x_1	(3)	0	1	0	1	0	0	4	$x_1=4$
	$Z(\theta)$	(0)	1	0	$-5+\theta$	$3+2\theta$	0	0	$12+8\theta$	$x_2=0$
										$x_3=0$
$\theta \ge 5$	x_4	(1)	0	0	2	0	1	0	12	$x_4=12$
	x_5	(2)	0	0	2	-3	0	1	6	$x_5=6$
	x_1	(3)	0	1	0	1	0	0	4	$x_1=4$

Therefore, after θ is increased past $\theta = \frac{9}{7}$, x_4 would need to be the entering basic variable for another iteration of the simplex method to find the new optimal solution. Then θ would be increased further until another coefficient goes negative, and so on until θ has been increased as far as desired.

This entire procedure is now summarized, and the example is completed in Table 7.2.

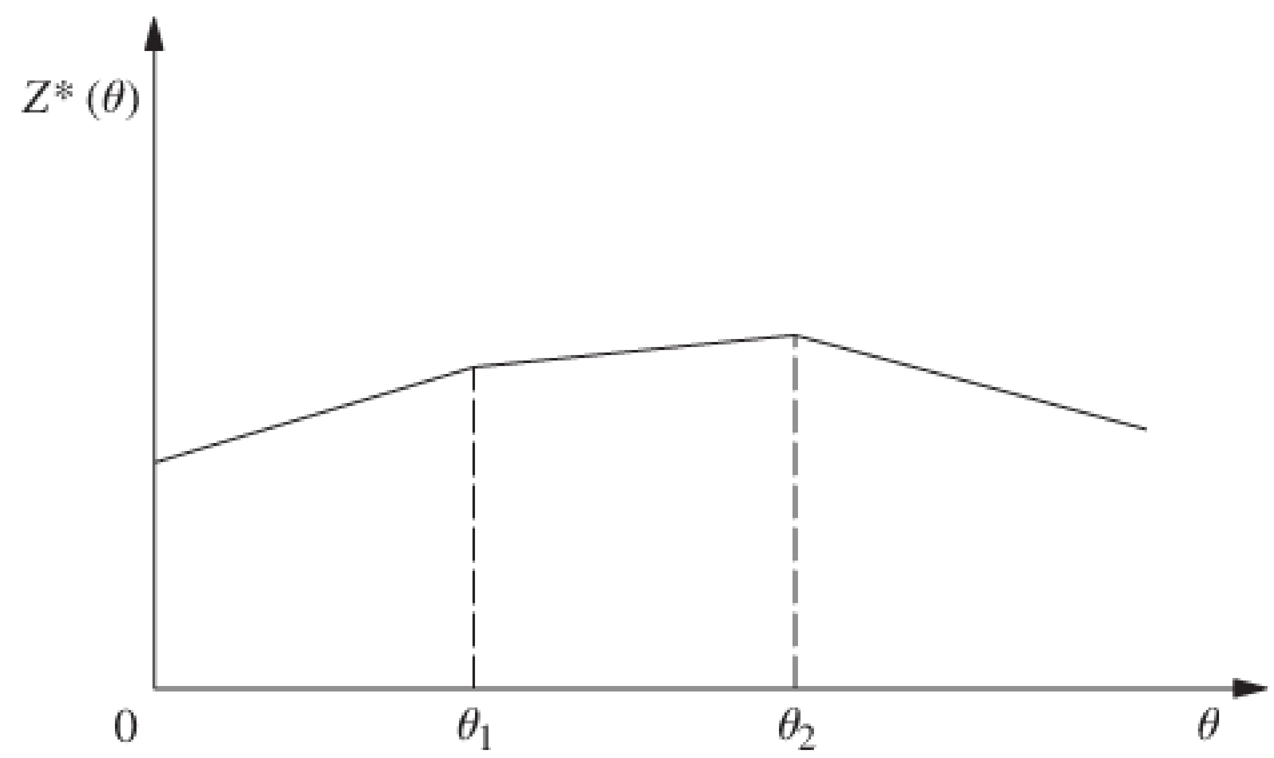
Summary of the Parametric Linear Programming Procedure for Systematic Changes in the c_j Parameters

1. Solve the problem with $\theta = 0$ by the simplex method.
2. Use the sensitivity analysis procedure (Cases 2*a* and 3, Sec. 6.7) to introduce the $\Delta c_j = \alpha_j \theta$ changes into Eq. (0).
3. Increase θ until one of the nonbasic variables has its coefficient in Eq. (0) go negative (or until θ has been increased as far as desired).
4. Use this variable as the entering basic variable for an iteration of the simplex method to find the new optimal solution. Return to step 3.

Systematic Changes in the b_i Parameters

For the case where the b_i parameters change systematically, the one modification made in the original linear programming model is that b_i is replaced by $b_i + \alpha_i\theta$, for $i = 1, 2, \ldots, m$, where the α_i are given input constants. Thus, the problem becomes

$$\text{Maximize} \qquad Z(\theta) = \sum_{j=1}^{n} c_j x_j,$$



■ **FIGURE 7.2**
The objective function value for an optimal solution as a function of θ for parametric linear programming with systematic changes in the b_i parameters.

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i + \alpha_i\theta \qquad \text{for } i = 1, 2, \ldots, m$$

and

$$x_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n.$$

The goal is to identify the optimal solution as a function of θ.

With this formulation, the corresponding objective function value $Z^*(\theta)$ always has the *piecewise linear* and *concave*[4] form shown in Fig. 7.2. (See Prob. 7.2-8.) The set of basic variables in the optimal solution still changes (as θ increases) *only* where the slope of $Z^*(\theta)$ changes. However, in contrast to the preceding case, the values of these variables now change as a (linear) function of θ between the slope changes. The reason is that increasing θ changes the right-hand sides in the initial set of equations, which then causes changes in the right-hand sides in the final set of equations, i.e., in the values of the final set of basic variables. Figure 7.2 depicts a problem with three sets of basic variables that are optimal for different values of θ, the first for $0 \leq \theta \leq \theta_1$, the second for $\theta_1 \leq \theta \leq \theta_2$, and the third for $\theta \geq \theta_2$. Within each of these intervals of θ, the value of $Z^*(\theta)$ varies with θ despite the fixed coefficients c_j because the x_j values are changing.

The following solution procedure summary is very similar to that just presented for systematic changes in the c_j parameters. The reason is that changing the b_i values is equivalent to changing the coefficients in the objective function of the *dual* model. Therefore, the procedure for the primal problem is exactly *complementary* to applying simultaneously the procedure for systematic changes in the c_j parameters to the *dual* problem. Consequently, the *dual simplex method* (see Sec. 7.1) now would be used to obtain each new optimal solution, and the applicable sensitivity analysis case (see Sec. 6.7) now is Case 1, but these differences are the only major differences.

Summary of the Parametric Linear Programming Procedure for Systematic Changes in the b_i Parameters

1. Solve the problem with $\theta = 0$ by the simplex method.
2. Use the sensitivity analysis procedure (Case 1, Sec. 6.7) to introduce the $\Delta b_i = \alpha_i\theta$ changes to the *right side* column.

[4]See Appendix 2 for a definition and discussion of concave functions.

3. Increase θ until one of the basic variables has its value in the *right side* column go negative (or until θ has been increased as far as desired).
4. Use this variable as the leaving basic variable for an iteration of the dual simplex method to find the new optimal solution. Return to step 3.

Example. To illustrate this procedure in a way that demonstrates its *duality* relationship with the procedure for systematic changes in the c_j parameters, we now apply it to the dual problem for the Wyndor Glass Co. (see Table 6.1). In particular, suppose that $\alpha_1 = 2$ and $\alpha_2 = -1$ so that the functional constraints become

$$\begin{array}{lcl} y_1 \quad + 3y_3 \ge 3 + 2\theta & \text{or} & -y_1 \quad - 3y_3 \le -3 - 2\theta \\ 2y_2 + 2y_3 \ge 5 - \theta & \text{or} & -2y_2 - 2y_3 \le -5 + \theta. \end{array}$$

Thus, the dual of *this* problem is just the example considered in Table 7.2.

This problem with $\theta = 0$ has already been solved in Table 7.1, so we begin with the final simplex tableau given there. Using the sensitivity analysis procedure for Case 1, Sec. 6.7, we find that the entries in the *right side* column of the tableau change to the values given below.

$$Z^* = \mathbf{y}^*\bar{\mathbf{b}} = [2, 6]\begin{bmatrix} -3 - 2\theta \\ -5 + \theta \end{bmatrix} = -36 + 2\theta,$$

$$\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}} = \begin{bmatrix} -\frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -3 - 2\theta \\ -5 + \theta \end{bmatrix} = \begin{bmatrix} 1 + \frac{2\theta}{3} \\ \frac{3}{2} - \frac{7\theta}{6} \end{bmatrix}.$$

Therefore, the two basic variables in this tableau

$$y_3 = \frac{3 + 2\theta}{3} \qquad \text{and} \qquad y_2 = \frac{9 - 7\theta}{6}$$

remain nonnegative for $0 \le \theta \le \frac{9}{7}$. Increasing θ past $\theta = \frac{9}{7}$ requires making y_2 a leaving basic variable for another iteration of the dual simplex method, and so on, as summarized in Table 7.3.

TABLE 7.3 The b_i parametric linear programming procedure applied to the dual of the Wyndor Glass Co. example

Range of θ	Basic Variable	Eq.	Coefficient of: Z	y_1	y_2	y_3	y_4	y_5	Right Side	Optimal Solution
	$Z(\theta)$	(0)	1	2	0	0	2	6	$-36 + 2\theta$	$y_1 = y_4 = y_5 = 0$
$0 \le \theta \le \frac{9}{7}$	y_3	(1)	0	$\frac{1}{3}$	0	1	$-\frac{1}{3}$	0	$\frac{3 + 2\theta}{3}$	$y_3 = \frac{3 + 2\theta}{3}$
	y_2	(2)	0	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{9 - 7\theta}{6}$	$y_2 = \frac{9 - 7\theta}{6}$
	$Z(\theta)$	(0)	1	0	6	0	4	3	$-27 - 5\theta$	$y_2 = y_4 = y_5 = 0$
$\frac{9}{7} \le \theta \le 5$	y_3	(1)	0	0	1	1	0	$-\frac{1}{2}$	$\frac{5 - \theta}{2}$	$y_3 = \frac{5 - \theta}{2}$
	y_1	(2)	0	1	-3	0	-1	$\frac{3}{2}$	$\frac{-9 + 7\theta}{2}$	$y_1 = \frac{-9 + 7\theta}{2}$
	$Z(\theta)$	(0)	1	0	12	6	4	0	$-12 - 8\theta$	$y_2 = y_3 = y_4 = 0$
$\theta \ge 5$	y_5	(1)	0	0	-2	-2	0	1	$-5 + \theta$	$y_5 = -5 + \theta$
	y_1	(2)	0	1	0	3	-1	0	$3 + 2\theta$	$y_1 = 3 + 2\theta$

We suggest that you now trace through Tables 7.2 and 7.3 simultaneously to note the duality relationship between the two procedures.

The Worked Examples section of the book's website includes **another example** of the procedure for systematic changes in the b_i parameters.

7.3 THE UPPER BOUND TECHNIQUE

It is fairly common in linear programming problems for some of or all the *individual* x_j variables to have *upper bound constraints*

$$x_j \leq u_j,$$

where u_j is a positive constant representing the maximum *feasible* value of x_j. We pointed out in Sec. 4.8 that the most important determinant of computation time for the simplex method is the *number of functional constraints,* whereas the number of *nonnegativity* constraints is relatively unimportant. Therefore, having a large number of upper bound constraints among the functional constraints greatly increases the computational effort required.

The *upper bound technique* avoids this increased effort by removing the upper bound constraints from the functional constraints and treating them separately, essentially like nonnegativity constraints.[5] Removing the upper bound constraints in this way causes no problems as long as none of the variables gets increased over its upper bound. The only time the simplex method increases some of the variables is when the entering basic variable is increased to obtain a new BF solution. Therefore, the upper bound technique simply applies the simplex method in the usual way to the *remainder* of the problem (i.e., without the upper bound constraints) but with the one additional restriction that each new BF solution must satisfy the upper bound constraints in addition to the usual lower bound (nonnegativity) constraints.

To implement this idea, note that a decision variable x_j with an upper bound constraint $x_j \leq u_j$ can always be replaced by

$$x_j = u_j - y_j,$$

where y_j would then be the decision variable. In other words, you have a choice between letting the decision variable be the *amount above zero* (x_j) or the *amount below* u_j ($y_j = u_j - x_j$). (We shall refer to x_j and y_j as *complementary* decision variables.) Because

$$0 \leq x_j \leq u_j$$

it also follows that

$$0 \leq y_j \leq u_j.$$

Thus, at any point during the simplex method, you can either

1. Use x_j, where $0 \leq x_j \leq u_j$, or
2. Replace x_j by $u_j - y_j$, where $0 \leq y_j \leq u_j$.

The upper bound technique uses the following rule to make this choice:

Rule: Begin with choice 1.
Whenever $x_j = 0$, use choice 1, so x_j is *nonbasic*.

[5]The upper bound technique assumes that the variables have the usual nonnegativity constraints in addition to the upper bound constraints. If a variable has a lower bound other than 0, say, $x_j \geq L_j$, then this constraint can be converted into a nonnegativity constraint by making the change of variables, $x_j' = x_j - L_j$, so $x_j' \geq 0$.

Whenever $x_j = u_j$, use choice 2, so $y_j = 0$ is *nonbasic*.
Switch choices only when the other extreme value of x_j is reached.

Therefore, whenever a basic variable reaches its upper bound, you should switch choices and use its complementary decision variable as the new nonbasic variable (the leaving basic variable) for identifying the new BF solution. Thus, the one substantive modification being made in the simplex method is in the rule for selecting the leaving basic variable.

Recall that the simplex method selects as the leaving basic variable the one that would be the first to become infeasible by going negative as the entering basic variable is increased. The modification now made is to select instead the variable that would be the first to become infeasible *in any way,* either by going negative or by going over the upper bound, as the entering basic variable is increased. (Notice that one possibility is that the entering basic variable may become infeasible first by going over its upper bound, so that its complementary decision variable becomes the leaving basic variable.) If the leaving basic variable reaches zero, then proceed as usual with the simplex method. However, if it reaches its upper bound instead, then switch choices and make its complementary decision variable the leaving basic variable.

An Example

To illustrate the upper bound technique, consider this problem:

$$\text{Maximize} \qquad Z = 2x_1 + x_2 + 2x_3,$$

subject to

$$\begin{aligned} 4x_1 + x_2 &= 12 \\ -2x_1 + x_3 &= 4 \end{aligned}$$

and

$$0 \le x_1 \le 4, \qquad 0 \le x_2 \le 15, \qquad 0 \le x_3 \le 6.$$

Thus, all three variables have upper bound constraints ($u_1 = 4$, $u_2 = 15$, $u_3 = 6$).

The two equality constraints are already in proper form from Gaussian elimination for identifying the initial BF solution ($x_1 = 0$, $x_2 = 12$, $x_3 = 4$), and none of the variables in this solution exceeds its upper bound, so x_2 and x_3 can be used as the initial basic variables without artificial variables being introduced. However, these variables then need to be eliminated algebraically from the objective function to obtain the initial Eq. (0), as follows:

$$\begin{array}{lllr} & Z & -\ 2x_1 - x_2 - 2x_3 & = \ \ 0 \\ & & +\ (4x_1 + x_2 & = 12) \\ & & +\ 2(-\ 2x_1 \qquad\ \ + \ x_3 & = \ \ 4) \\ \hline (0) & Z & -\ 2x_1 & = 20. \end{array}$$

To start the first iteration, this initial Eq. (0) indicates that the initial *entering* basic variable is x_1. Since the upper bound constraints are not to be included, the entire initial set of equations and the corresponding calculations for selecting the leaving basic variables are those shown in Table 7.4. The second column shows how much the entering basic variable x_1 can be *increased* from zero before some basic variable (including x_1) becomes infeasible. The maximum value given next to Eq. (0) is just the upper bound constraint for x_1. For Eq. (1), since the coefficient of x_1 is *positive, increasing* x_1 to 3 decreases the basic variable in this equation (x_2) from 12 to its *lower* bound of *zero*. For Eq. (2), since the coefficient of x_1 is *negative, increasing* x_1 to 1 *increases* the basic variable in this equation (x_3) from 4 to its *upper bound* of 6.

TABLE 7.4 Equations and calculations for the initial leaving basic variable in the example for the upper bound technique

Initial Set of Equations	Maximum Feasible Value of x_1
(0) $Z - 2x_1 = 20$	$x_1 \le 4$ (since $u_1 = 4$)
(1) $4x_1 + x_2 = 12$	$x_1 \le \frac{12}{4} = 3$
(2) $-2x_1 + x_3 = 4$	$x_1 \le \frac{6-4}{2} = 1 \leftarrow$ minimum (because $u_3 = 6$)

Because Eq. (2) has the *smallest* maximum feasible value of x_1 in Table 7.4, the basic variable in this equation (x_3) provides the *leaving* basic variable. However, because x_3 reached its *upper* bound, replace x_3 by $6 - y_3$, so that $y_3 = 0$ becomes the new nonbasic variable for the next BF solution and x_1 becomes the new basic variable in Eq. (2). This replacement leads to the following changes in this equation:

$$
\begin{aligned}
(2) \qquad & -2x_1 + x_3 = 4 \\
\rightarrow & -2x_1 + 6 - y_3 = 4 \\
\rightarrow & -2x_1 - y_3 = -2 \\
\rightarrow & \;\; x_1 + \frac{1}{2}y_3 = 1
\end{aligned}
$$

Therefore, after we eliminate x_1 algebraically from the other equations, the *second* complete set of equations becomes

$$
\begin{aligned}
(0) \quad & Z + y_3 = 22 \\
(1) \quad & x_2 - 2y_3 = 8 \\
(2) \quad & x_1 + \frac{1}{2}y_3 = 1.
\end{aligned}
$$

The resulting BF solution is $x_1 = 1$, $x_2 = 8$, $y_3 = 0$. By the optimality test, it also is an optimal solution, so $x_1 = 1$, $x_2 = 8$, $x_3 = 6 - y_3 = 6$ is the desired solution for the original problem.

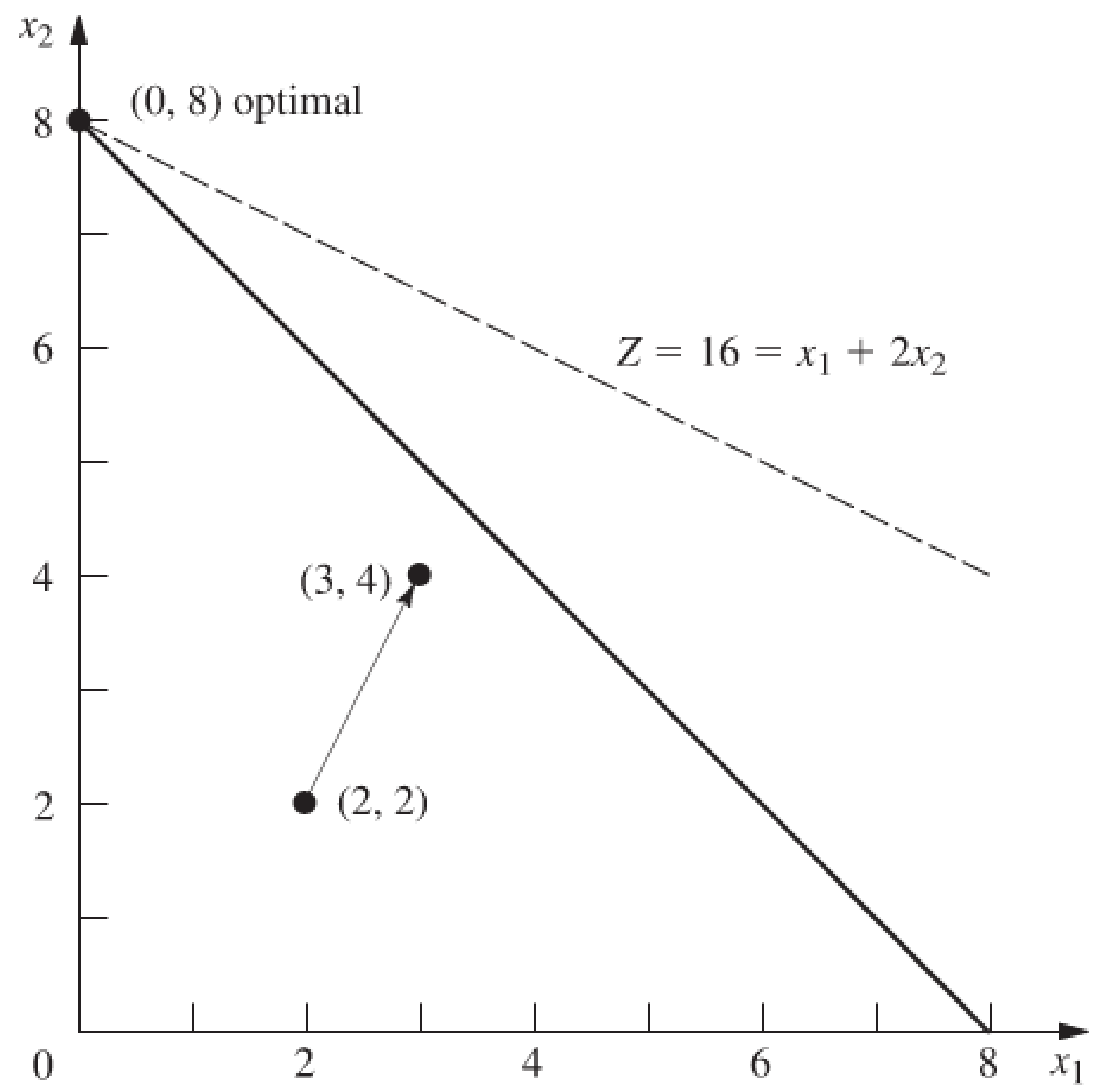
If you would like to see **another example** of the upper bound technique, the Worked Examples section of the book's website includes one.

7.4 AN INTERIOR-POINT ALGORITHM

In Sec. 4.9 we discussed a dramatic development in linear programming that occurred in 1984, namely, the invention by Narendra Karmarkar of AT&T Bell Laboratories of a powerful algorithm for solving huge linear programming problems with an approach very different from the simplex method. We now introduce the nature of Karmarkar's approach by describing a relatively elementary variant (the "affine" or "affine-scaling" variant) of his algorithm.[6] (Your IOR Tutorial also includes this variant under the title, *Solve Automatically by the Interior-Point Algorithm.*)

Throughout this section we shall focus on Karmarkar's main ideas on an intuitive level while avoiding mathematical details. In particular, we shall bypass certain details

[6]The basic approach for this variant actually was proposed in 1967 by a Russian mathematician I. I. Dikin and then rediscovered soon after the appearance of Karmarkar's work by a number of researchers, including E. R. Barnes, T. M. Cavalier, and A. L. Soyster. Also see R. J. Vanderbei, M. S. Meketon, and B. A. Freedman, "A Modification of Karmarkar's Linear Programming Algorithm," *Algorithmica,* **1**(4) (Special Issue on New Approaches to Linear Programming): 395–407, 1986.



■ **FIGURE 7.3**
Example for the interior-point algorithm.

that are needed for the full implementation of the algorithm (e.g., how to find an initial feasible trial solution) but are not central to a basic conceptual understanding. The ideas to be described can be summarized as follows:

Concept 1: Shoot through the *interior* of the feasible region toward an optimal solution.
Concept 2: Move in a direction that improves the objective function value at the fastest possible rate.
Concept 3: Transform the feasible region to place the current trial solution near its center, thereby enabling a large improvement when concept 2 is implemented.

To illustrate these ideas throughout the section, we shall use the following example:

$$\text{Maximize} \qquad Z = x_1 + 2x_2,$$

subject to

$$x_1 + x_2 \leq 8$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

This problem is depicted graphically in Fig. 7.3, where the optimal solution is seen to be $(x_1, x_2) = (0, 8)$ with $Z = 16$. (We will describe the significance of the arrow in the figure shortly.)

You will see that our interior-point algorithm requires a considerable amount of work to solve this tiny example. The reason is that the algorithm is designed to solve *huge* problems efficiently, but is much less efficient than the simplex method (or the graphical method in this case) for small problems. However, having an example with only two variables will allow us to depict graphically what the algorithm is doing.

The Relevance of the Gradient for Concepts 1 and 2

The algorithm begins with an initial trial solution that (like all subsequent trial solutions) lies in the *interior* of the feasible region, i.e., *inside the boundary* of the feasible region. Thus, for the example, the solution must not lie on any of the three lines ($x_1 = 0$, $x_2 = 0$,

$x_1 + x_2 = 8$) that form the boundary of this region in Fig. 7.3. (A trial solution that lies on the boundary cannot be used because this would lead to the undefined mathematical operation of division by zero at one point in the algorithm.) We have arbitrarily chosen $(x_1, x_2) = (2, 2)$ to be the initial trial solution.

To begin implementing concepts 1 and 2, note in Fig. 7.3 that the direction of movement from (2, 2) that increases Z at the fastest possible rate is *perpendicular* to (and toward) the objective function line $Z = 16 = x_1 + 2x_2$. We have shown this direction by the arrow from (2, 2) to (3, 4). Using vector addition, we have

$$(3, 4) = (2, 2) + (1, 2),$$

where the vector (1, 2) is the **gradient** of the objective function. (We will discuss gradients further in Sec. 12.5 in the broader context of *nonlinear programming,* where algorithms similar to Karmarkar's have long been used.) The components of (1, 2) are just the coefficients in the objective function. Thus, with one subsequent modification, the gradient (1, 2) defines the ideal direction to which to move, where the question of the *distance to move* will be considered later.

The algorithm actually operates on linear programming problems after they have been rewritten in augmented form. Letting x_3 be the slack variable for the functional constraint of the example, we see that this form is

$$\text{Maximize} \qquad Z = x_1 + 2x_2,$$

subject to

$$x_1 + x_2 + x_3 = 8$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

In matrix notation (slightly different from Chap. 5 because the slack variable now is incorporated into the notation), the augmented form can be written in general as

$$\text{Maximize} \qquad Z = \mathbf{c}^T\mathbf{x},$$

subject to

$$\mathbf{Ax} = \mathbf{b}$$

and

$$\mathbf{x} \geq \mathbf{0},$$

where

$$\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathbf{A} = [1, \quad 1, \quad 1], \qquad \mathbf{b} = [8], \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for the example. Note that $\mathbf{c}^T = [1, 2, 0]$ now is the gradient of the objective function.

The augmented form of the example is depicted graphically in Fig. 7.4. The feasible region now consists of the triangle with vertices (8, 0, 0), (0, 8, 0), and (0, 0, 8). Points in the interior of this feasible region are those where $x_1 > 0$, $x_2 > 0$, and $x_3 > 0$. Each of these three $x_j > 0$ conditions has the effect of forcing (x_1, x_2) away from one of the three lines forming the boundary of the feasible region in Fig. 7.3.

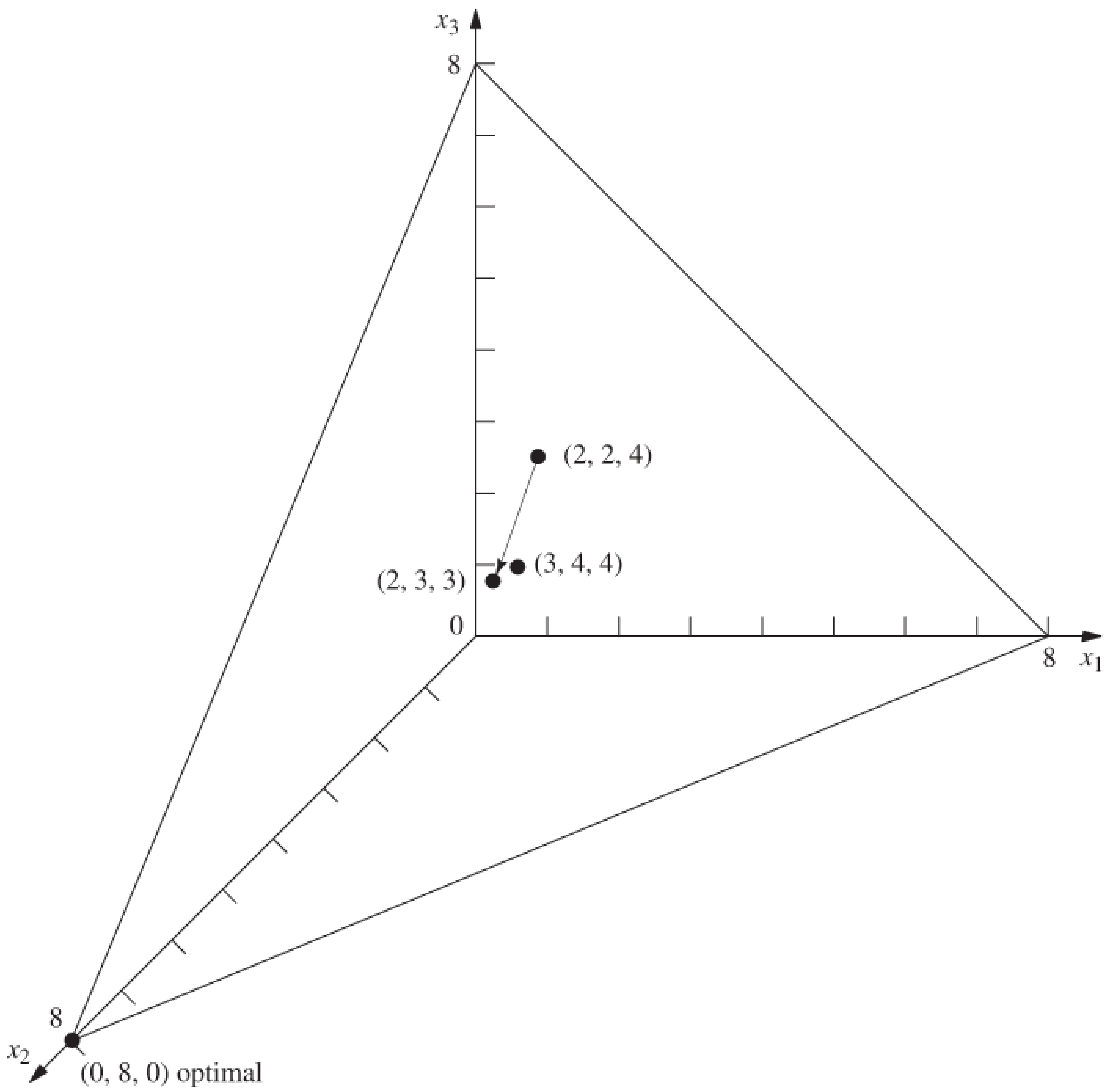


FIGURE 7.4
Example in augmented form for the interior-point algorithm.

Using the Projected Gradient to Implement Concepts 1 and 2

In augmented form, the initial trial solution for the example is $(x_1, x_2, x_3) = (2, 2, 4)$. Adding the gradient (1, 2, 0) leads to

$$(3, 4, 4) = (2, 2, 4) + (1, 2, 0).$$

However, now there is a complication. The algorithm cannot move from (2, 2, 4) to (3, 4, 4), because (3, 4, 4) is infeasible! When $x_1 = 3$ and $x_2 = 4$, then $x_3 = 8 - x_1 - x_2 = 1$ instead of 4. The point (3, 4, 4) lies on the near side as you look down on the feasible triangle in Fig. 7.4. Therefore, to remain feasible, the algorithm (indirectly) *projects* the point (3, 4, 4) down onto the feasible triangle by dropping a line that is *perpendicular* to this triangle. A vector from (0, 0, 0) to (1, 1, 1) is perpendicular to this triangle, so the perpendicular line through (3, 4, 4) is given by the equation

$$(x_1, x_2, x_3) = (3, 4, 4) - \theta(1, 1, 1),$$

where θ is a scalar. Since the triangle satisfies the equation $x_1 + x_2 + x_3 = 8$, this perpendicular line intersects the triangle at (2, 3, 3). Because

$$(2, 3, 3) = (2, 2, 4) + (0, 1, -1),$$

the **projected gradient** of the objective function (the gradient projected onto the feasible region) is (0, 1, −1). It is this projected gradient that defines the direction of movement from (2, 2, 4) for the algorithm, as shown by the arrow in Fig. 7.4.

A formula is available for computing the projected gradient directly. By defining the *projection matrix* **P** as

$$\mathbf{P} = \mathbf{I} - \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A},$$

the *projected gradient* (in column form) is

$$\mathbf{c}_p = \mathbf{P}\mathbf{c}.$$

Thus, for the example,

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \left([1 \quad 1 \quad 1] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)^{-1} [1 \quad 1 \quad 1]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [1 \quad 1 \quad 1]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix},$$

so

$$\mathbf{c}_p = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Moving from (2, 2, 4) in the direction of the projected gradient (0, 1, −1) involves increasing α from zero in the formula

$$\mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} + 4\alpha\mathbf{c}_p = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} + 4\alpha \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

where the coefficient 4 is used simply to give an upper bound of 1 for α to maintain feasibility (all $x_j \geq 0$). Note that increasing α to $\alpha = 1$ would cause x_3 to decrease to $x_3 = 4 + 4(1)(-1) = 0$, where $\alpha > 1$ yields $x_3 < 0$. Thus, α measures the fraction used of the distance that could be moved before the feasible region is left.

How large should α be made for moving to the next trial solution? Because the increase in Z is proportional to α, a value close to the upper bound of 1 is good for giving a relatively large step toward optimality on the current iteration. However, the problem with a value too close to 1 is that the next trial solution then is jammed against a constraint boundary, thereby making it difficult to take large improving steps during subsequent iterations. Therefore, it is very helpful for trial solutions to be near the center of the feasible region (or at least near the center of the portion of the feasible region in the vicinity of an optimal solution), and not too close to any constraint boundary. With this in mind, Karmarkar has stated for his algorithm that a value as large as $\alpha = 0.25$ should be "safe." In practice, much larger values (for example, $\alpha = 0.9$) sometimes are used. For the purposes of this example (and the problems at the end of the chapter), we have chosen $\alpha = 0.5$. (Your IOR Tutorial uses $\alpha = 0.5$ as the default value, but also has $\alpha = 0.9$ available.)

A Centering Scheme for Implementing Concept 3

We now have just one more step to complete the description of the algorithm, namely, a special scheme for transforming the feasible region to place the current trial solution near its center. We have just described the benefit of having the trial solution near the center, but another important benefit of this centering scheme is that it keeps turning the direction of the projected gradient to point more nearly toward an optimal solution as the algorithm converges toward this solution.

The basic idea of the centering scheme is straightforward—simply change the scale (units) for each of the variables so that the trial solution becomes equidistant from the constraint boundaries in the new coordinate system. (Karmarkar's original algorithm uses a more sophisticated centering scheme.)

For the example, there are three constraint boundaries in Fig. 7.3, each one corresponding to a zero value for one of the three variables of the problem in augmented form, namely, $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$. In Fig. 7.4, see how these three constraint boundaries intersect the $\mathbf{Ax} = \mathbf{b}$ ($x_1 + x_2 + x_3 = 8$) plane to form the boundary of the feasible region. The initial trial solution is $(x_1, x_2, x_3) = (2, 2, 4)$, so this solution is 2 units away from the $x_1 = 0$ and $x_2 = 0$ constraint boundaries and 4 units away from the $x_3 = 0$ constraint boundary, when the units of the respective variables are used. However, whatever these units are in each case, they are quite arbitrary and can be changed as desired without changing the problem. Therefore, let us rescale the variables as follows:

$$\tilde{x}_1 = \frac{x_1}{2}, \qquad \tilde{x}_2 = \frac{x_2}{2}, \qquad \tilde{x}_3 = \frac{x_3}{4}$$

in order to make the current trial solution of $(x_1, x_2, x_3) = (2, 2, 4)$ become

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (1, 1, 1).$$

In these new coordinates (substituting $2\tilde{x}_1$ for x_1, $2\tilde{x}_2$ for x_2, and $4\tilde{x}_3$ for x_3), the problem becomes

$$\text{Maximize} \qquad Z = 2\tilde{x}_1 + 4\tilde{x}_2,$$

subject to

$$2\tilde{x}_1 + 2\tilde{x}_2 + 4\tilde{x}_3 = 8$$

and

$$\tilde{x}_1 \geq 0, \qquad \tilde{x}_2 \geq 0, \qquad \tilde{x}_3 \geq 0,$$

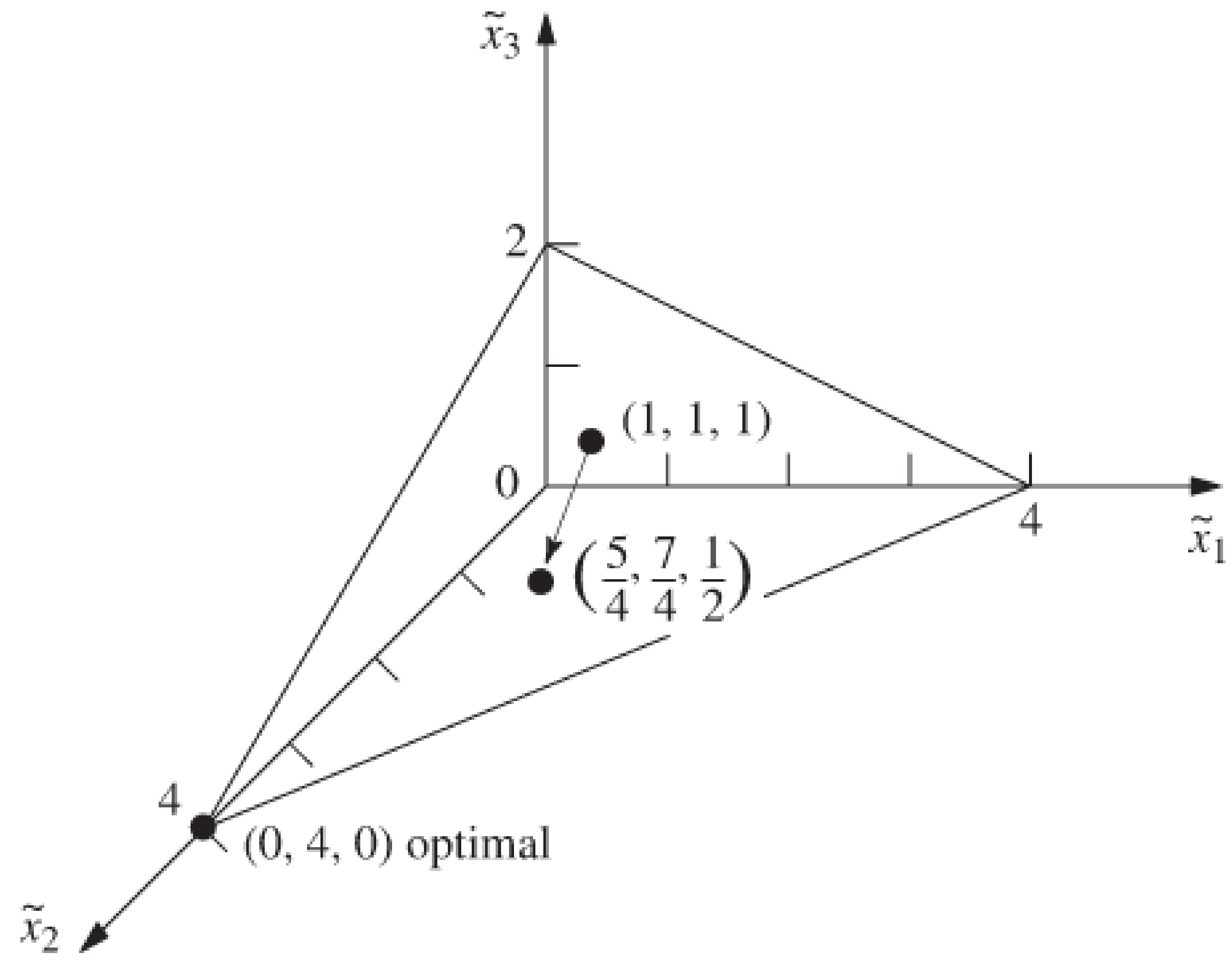
as depicted graphically in Fig. 7.5.

Note that the trial solution (1, 1, 1) in Fig. 7.5 is equidistant from the three constraint boundaries $\tilde{x}_1 = 0$, $\tilde{x}_2 = 0$, $\tilde{x}_3 = 0$. For each subsequent iteration as well, the problem is rescaled again to achieve this same property, so that the current trial solution always is (1, 1, 1) in the current coordinates.

Summary and Illustration of the Algorithm

Now let us summarize and illustrate the algorithm by going through the first iteration for the example, then giving a summary of the general procedure, and finally applying this summary to a second iteration.

Iteration 1. Given the initial trial solution $(x_1, x_2, x_3) = (2, 2, 4)$, let $\mathbf{D}$ be the corresponding *diagonal matrix* such that $\mathbf{x} = \mathbf{D}\tilde{\mathbf{x}}$, so that



■ **FIGURE 7.5**
Example after rescaling for iteration 1.

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

The rescaled variables then are the components of

$$\tilde{\mathbf{x}} = \mathbf{D}^{-1}\mathbf{x} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{x_1}{2} \\ \frac{x_2}{2} \\ \frac{x_3}{4} \end{bmatrix}.$$

In these new coordinates, **A** and **c** have become

$$\tilde{\mathbf{A}} = \mathbf{AD} = [1 \quad 1 \quad 1] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = [2 \quad 2 \quad 4],$$

$$\tilde{\mathbf{c}} = \mathbf{Dc} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}.$$

Therefore, the projection matrix is

$$\mathbf{P} = \mathbf{I} - \tilde{\mathbf{A}}^T(\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T)^{-1}\tilde{\mathbf{A}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \left([2 \quad 2 \quad 4] \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \right)^{-1} [2 \quad 2 \quad 4]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{24}\begin{bmatrix} 4 & 4 & 8 \\ 4 & 4 & 8 \\ 8 & 8 & 16 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

so that the projected gradient is

$$\mathbf{c}_p = \mathbf{P}\tilde{\mathbf{c}} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}.$$

Define v as the *absolute value* of the *negative* component of $\mathbf{c}_p$ having the *largest* absolute value, so that $v = |-2| = 2$ in this case. Consequently, in the current coordinates, the algorithm now moves from the current trial solution $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (1, 1, 1)$ to the next trial solution

$$\tilde{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\alpha}{v}\mathbf{c}_p = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{0.5}{2}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ \frac{7}{4} \\ \frac{1}{2} \end{bmatrix},$$

as shown in Fig. 7.5. (The definition of v has been chosen to make the smallest component of $\tilde{\mathbf{x}}$ equal to zero when $\alpha = 1$ in this equation for the next trial solution.) In the original coordinates, this solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{D}\tilde{\mathbf{x}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{5}{4} \\ \frac{7}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{7}{2} \\ 2 \end{bmatrix}.$$

This completes the iteration, and this new solution will be used to start the next iteration.

These steps can be summarized as follows for any iteration.

Summary of the Interior-Point Algorithm

1. Given the current trial solution $(x_1, x_2, \ldots, x_n)$, set

$$\mathbf{D} = \begin{bmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & x_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & x_n \end{bmatrix}$$

2. Calculate $\tilde{\mathbf{A}} = \mathbf{AD}$ and $\tilde{\mathbf{c}} = \mathbf{Dc}$.
3. Calculate $\mathbf{P} = \mathbf{I} - \tilde{\mathbf{A}}^T(\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T)^{-1}\tilde{\mathbf{A}}$ and $\mathbf{c}_p = \mathbf{P}\tilde{\mathbf{c}}$.
4. Identify the negative component of $\mathbf{c}_p$ having the largest absolute value, and set v equal to this absolute value. Then calculate

$$\tilde{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \frac{\alpha}{v}\mathbf{c}_p,$$

where α is a selected constant between 0 and 1 (for example, $\alpha = 0.5$).

5. Calculate $\mathbf{x} = \mathbf{D}\tilde{\mathbf{x}}$ as the trial solution for the next iteration (step 1). (If this trial solution is virtually unchanged from the preceding one, then the algorithm has virtually converged to an optimal solution, so stop.)

Now let us apply this summary to iteration 2 for the example.

Iteration 2

Step 1:
Given the current trial solution $(x_1, x_2, x_3) = (\frac{5}{2}, \frac{7}{2}, 2)$, set

$$\mathbf{D} = \begin{bmatrix} \frac{5}{2} & 0 & 0 \\ 0 & \frac{7}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(Note that the rescaled variables are

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \mathbf{D}^{-1}\mathbf{x} = \begin{bmatrix} \frac{2}{5} & 0 & 0 \\ 0 & \frac{2}{7} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{5}x_1 \\ \frac{2}{7}x_2 \\ \frac{1}{2}x_3 \end{bmatrix},$$

so that the BF solutions in these new coordinates are

$$\tilde{\mathbf{x}} = \mathbf{D}^{-1}\begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{16}{5} \\ 0 \\ 0 \end{bmatrix}, \qquad \tilde{\mathbf{x}} = \mathbf{D}^{-1}\begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{16}{7} \\ 0 \end{bmatrix},$$

and

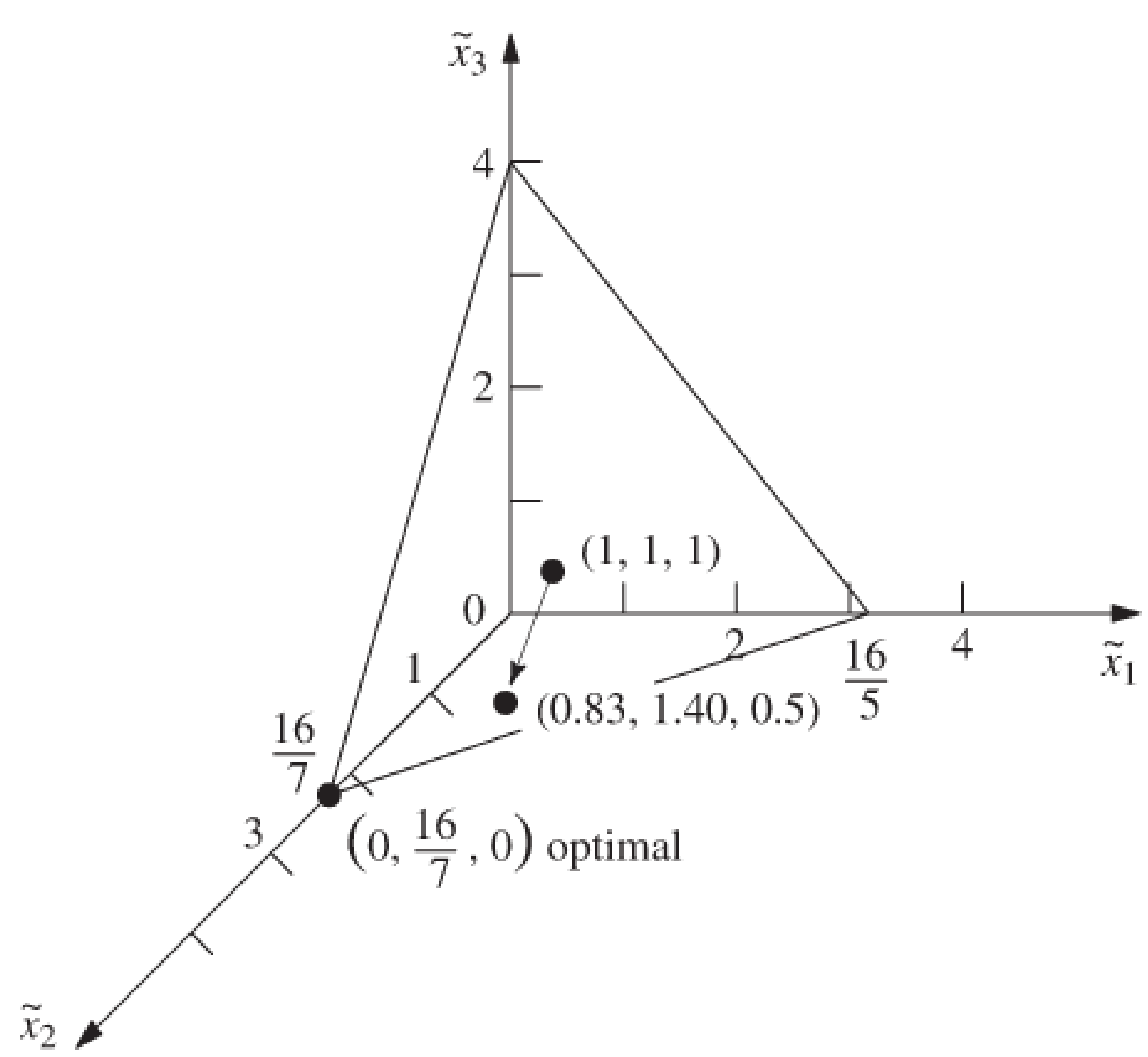
$$\tilde{\mathbf{x}} = \mathbf{D}^{-1}\begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix},$$

as depicted in Fig. 7.6.)

Step 2:

$$\tilde{\mathbf{A}} = \mathbf{A}\mathbf{D} = [\tfrac{5}{2}, \tfrac{7}{2}, 2] \qquad \text{and} \qquad \tilde{\mathbf{c}} = \mathbf{D}\mathbf{c} = \begin{bmatrix} \frac{5}{2} \\ 7 \\ 0 \end{bmatrix}.$$

FIGURE 7.6
Example after rescaling for iteration 2.



Step 3:

$$\mathbf{P} = \begin{bmatrix} \frac{13}{18} & -\frac{7}{18} & -\frac{2}{9} \\ -\frac{7}{18} & \frac{41}{90} & -\frac{14}{45} \\ -\frac{2}{9} & -\frac{14}{45} & \frac{37}{45} \end{bmatrix} \quad \text{and} \quad \mathbf{c}_p = \begin{bmatrix} -\frac{11}{12} \\ \frac{133}{60} \\ -\frac{41}{15} \end{bmatrix}.$$

Step 4:

$\left|-\frac{41}{15}\right| > \left|-\frac{11}{12}\right|$, so $v = \frac{41}{15}$ and

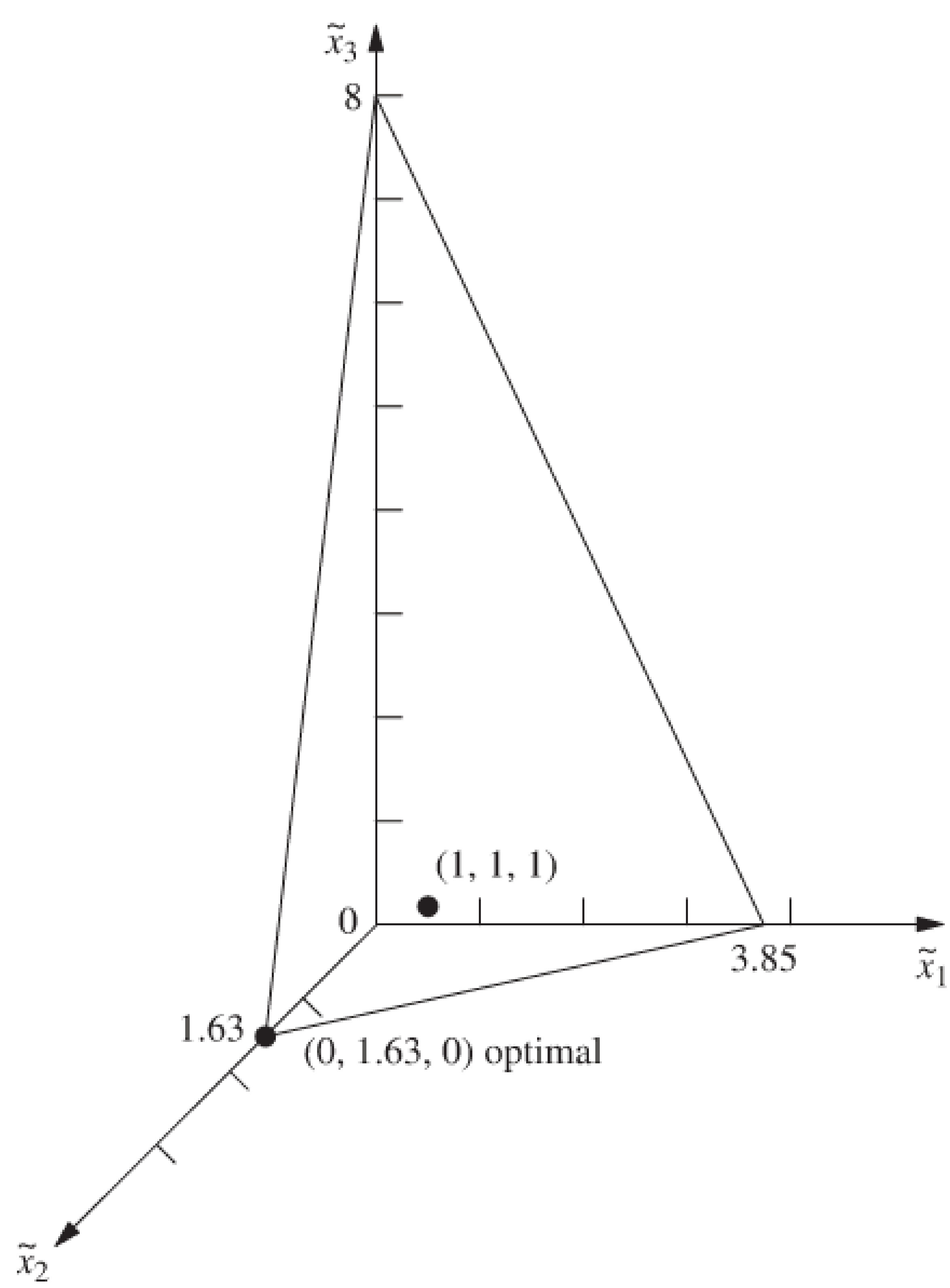
$$\tilde{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{0.5}{\frac{41}{15}} \begin{bmatrix} -\frac{11}{12} \\ \frac{133}{60} \\ -\frac{41}{15} \end{bmatrix} = \begin{bmatrix} \frac{273}{328} \\ \frac{461}{328} \\ \frac{1}{2} \end{bmatrix} \approx \begin{bmatrix} 0.83 \\ 1.40 \\ 0.50 \end{bmatrix}.$$

Step 5:

$$\mathbf{x} = \mathbf{D}\tilde{\mathbf{x}} = \begin{bmatrix} \frac{1365}{656} \\ \frac{3227}{656} \\ 1 \end{bmatrix} \approx \begin{bmatrix} 2.08 \\ 4.92 \\ 1.00 \end{bmatrix}$$

is the trial solution for iteration 3.

Since there is little to be learned by repeating these calculations for additional iterations, we shall stop here. However, we do show in Fig. 7.7 the reconfigured feasible region after rescaling based on the trial solution just obtained for iteration 3. As always,



■ **FIGURE 7.7**
Example after rescaling for iteration 3.

the rescaling has placed the trial solution at $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (1, 1, 1)$, equidistant from the $\tilde{x}_1 = 0$, $\tilde{x}_2 = 0$, and $\tilde{x}_3 = 0$ constraint boundaries. Note in Figs. 7.5, 7.6, and 7.7 how the sequence of iterations and rescaling have the effect of "sliding" the optimal solution toward (1, 1, 1) while the other BF solutions tend to slide away. Eventually, after enough iterations, the optimal solution will lie very near $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (0, 1, 0)$ after rescaling, while the other two BF solutions will be *very* far from the origin on the $\tilde{x}_1$ and $\tilde{x}_3$ axes. Step 5 of that iteration then will yield a solution in the original coordinates very near the optimal solution of $(x_1, x_2, x_3) = (0, 8, 0)$.

Figure 7.8 shows the progress of the algorithm in the original $x_1 = x_2$ coordinate system before the problem is augmented. The three points—$(x_1, x_2) = (2, 2)$, $(2.5, 3.5)$, and $(2.08, 4.92)$—are the trial solutions for initiating iterations 1, 2, and 3, respectively. We then have drawn a smooth curve through and beyond these points to show the trajectory of the algorithm in subsequent iterations as it approaches $(x_1, x_2) = (0, 8)$.

The functional constraint for this particular example happened to be an inequality constraint. However, equality constraints cause no difficulty for the algorithm, since it deals with the constraints only after any necessary augmenting has been done to convert them to equality form ($\mathbf{Ax} = \mathbf{b}$) anyway. To illustrate, suppose that the only change in the example is that the constraint $x_1 + x_2 \leq 8$ is changed to $x_1 + x_2 = 8$. Thus, the feasible region in Fig. 7.3 changes to just the line segment between (8, 0) and (0, 8). Given an initial feasible trial solution in the interior ($x_1 > 0$ and $x_2 > 0$) of this line segment—say, $(x_1, x_2) = (4, 4)$—the algorithm can proceed just as presented in the five-step summary with just the two variables and $\mathbf{A} = [1, 1]$. For each iteration, the projected gradient points along this line segment in the direction of (0, 8). With $\alpha = \frac{1}{2}$, iteration 1 leads from (4, 4) to (2, 6), iteration 2 leads from (2, 6) to (1, 7), etc. (Problem 7.4-3 asks you to verify these results.)

Although either version of the example has only one functional constraint, having more than one leads to just one change in the procedure as already illustrated (other than more extensive calculations). Having a single functional constraint in the example meant that $\mathbf{A}$

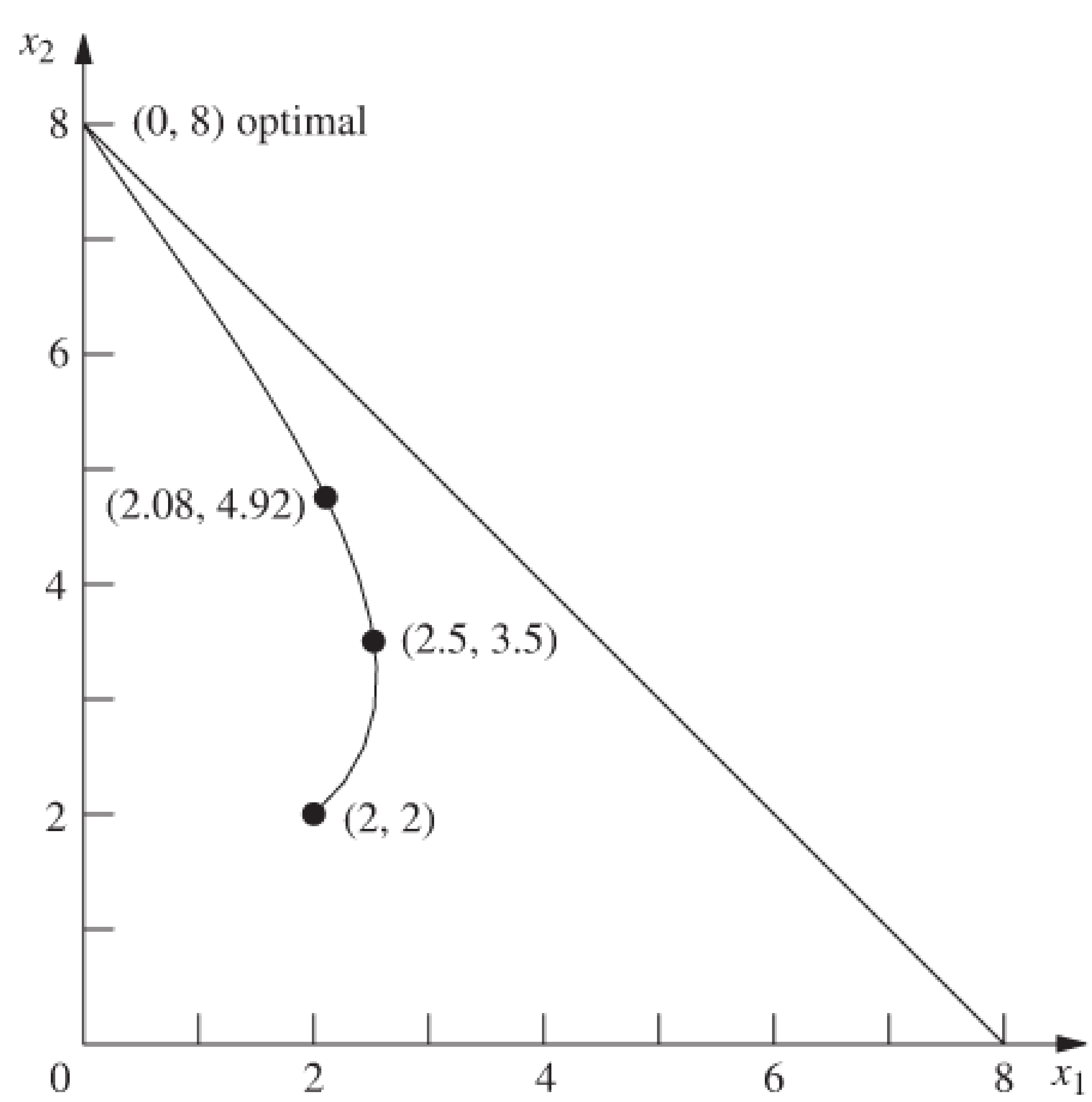


FIGURE 7.8
Trajectory of the interior-point algorithm for the example in the original x_1-x_2 coordinate system.

had only a single row, so the $(\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T)^{-1}$ term in step 3 only involved taking the reciprocal of the number obtained from the vector product $\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T$. Multiple functional constraints mean that $\mathbf{A}$ has multiple rows, so then the $(\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T)^{-1}$ term involves finding the *inverse* of the matrix obtained from the matrix product $\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T$.

To conclude, we need to add a comment to place the algorithm into better perspective. For our extremely small example, the algorithm requires relatively extensive calculations and then, after many iterations, obtains only an approximation of the optimal solution. By contrast, the graphical procedure of Sec. 3.1 finds the optimal solution in Fig. 7.3 immediately, and the simplex method requires only one quick iteration. However, do not let this contrast fool you into downgrading the efficiency of the interior-point algorithm. This algorithm is designed for dealing with *big* problems that may have many thousands of functional constraints. The simplex method typically requires thousands of iterations on such problems. By "shooting" through the interior of the feasible region, the interior-point algorithm tends to require a substantially smaller number of iterations (although with considerably more work per iteration). This sometimes enables an interior-point algorithm to efficiently solve huge linear programming problems that might even be beyond the reach of either the simplex method or the dual simplex method. Therefore, interior-point algorithms similar to the one presented here should play an important role in the future of linear programming.

See Sec. 4.9 for a comparison of the interior-point approach with the simplex method. Section 4.9 also discusses the complementary roles of the interior-point approach and the simplex method, including how they can even be combined into a hybrid algorithm.

Finally, we should emphasize that this section has provided only a conceptual introduction to the interior-point approach to linear programming by describing an elementary variant of Karmakar's path-breaking 1984 algorithm. Over the many subsequent years, a number of top-notch researchers have developed many key advances in the interior-point approach. Further coverage of this advanced topic is beyond the scope of this book. However, the interested reader can find many details in the selected references listed at the end of this chapter.

7.5 CONCLUSIONS

The *dual simplex method* and *parametric linear programming* are especially valuable for postoptimality analysis, although they also can be very useful in other contexts.

The *upper bound technique* provides a way of streamlining the simplex method for the common situation in which many or all of the variables have explicit upper bounds. It can greatly reduce the computational effort for large problems.

Mathematical-programming computer packages usually include all three of these procedures, and they are widely used. Because their basic structure is based largely upon the simplex method as presented in Chap. 4, they retain the exceptional computational efficiency to handle very large problems of the sizes described in Sec. 4.8.

Various other special-purpose algorithms also have been developed to exploit the special structure of particular types of linear programming problems (such as those to be discussed in Chaps. 8 and 9). Much research is currently being done in this area.

Karmarkar's interior-point algorithm initiated another key line of research into how to solve linear programming problems. Variants of this algorithm now provide a powerful approach for efficiently solving some very large problems.

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Worked Examples:

Examples for Chapter 7

Interactive Procedures in IOR Tutorial:

Enter or Revise a General Linear Programming Model
Set Up for the Simplex Method—Interactive Only
Solve Interactively by the Simplex Method
Interactive Graphical Method

Automatic Procedures in IOR Tutorial:

Solve Automatically by the Simplex Method
Solve Automatically by the Interior-Point Algorithm
Graphical Method and Sensitivity Analysis

An Excel Add-In:

Premium Solver for Education

"Ch. 7—Other Algorithms for LP" Files for Solving the Examples:

Excel Files
LINGO/LINDO File
MPL/CPLEX File

Glossary for Chapter 7

Supplement to This Chapter:

Linear Goal Programming and Its Solution Procedures (includes two accompanying cases: A Cure for Cuba and Airport Security)

See Appendix 1 for documentation of the software.

PROBLEMS

The symbols to the left of some of the problems (or their parts) have the following meaning:

I: We suggest that you use one of the procedures in IOR Tutorial (the print-out records your work). For parametric linear programming, this only applies to $\theta = 0$, after which you should proceed manually.

C: Use the computer to solve the problem by using the automatic procedure for the interior-point algorithm in IOR Tutorial.

An asterisk on the problem number indicates that at least a partial answer is given in the back of the book.

7.1-1. Consider the following problem.

$$\text{Maximize} \quad Z = -x_1 - 2x_2,$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\le 40 \\ x_2 &\ge 15 \\ -2x_1 + x_2 &\le 10 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0.$$

I **(a)** Solve this problem graphically.

(b) Use the *dual simplex method* manually to solve this problem.

(c) Trace graphically the path taken by the dual simplex method.

7.1-2.* Use the *dual simplex method* manually to solve the following problem.

$$\text{Minimize} \quad Z = 5x_1 + 2x_2 + 4x_3,$$

subject to

$$\begin{aligned} 3x_1 + x_2 + 2x_3 &\ge 4 \\ 6x_1 + 3x_2 + 5x_3 &\ge 10 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

7.1-3. Use the *dual simplex method* manually to solve the following problem.

$$\text{Minimize} \quad Z = 7x_1 + 2x_2 + 5x_3 + 4x_4,$$

subject to

$$\begin{aligned} 2x_1 + 4x_2 + 7x_3 + x_4 &\ge 5 \\ 8x_1 + 4x_2 + 6x_3 + 4x_4 &\ge 8 \\ 3x_1 + 8x_2 + x_3 + 4x_4 &\ge 4 \end{aligned}$$

and

$$x_j \ge 0, \qquad \text{for } j = 1, 2, 3, 4.$$

7.1-4. Consider the following problem.

$$\text{Maximize} \quad Z = 5x_1 + 10x_2,$$

subject to

$$\begin{aligned} 3x_1 + x_2 &\le 40 \\ x_1 + x_2 &\le 20 \\ 5x_1 + 3x_2 &\le 90 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0.$$

I **(a)** Solve by the *original simplex method* (in tabular form). Identify the *complementary* basic solution for the dual problem obtained at each iteration.

(b) Solve the *dual* of this problem manually by the *dual simplex method.* Compare the resulting sequence of basic solutions with the complementary basic solutions obtained in part (*a*).

7.1-5. Consider the example for case 1 of sensitivity analysis given in Sec. 6.7, where the initial simplex tableau of Table 4.8 is modified by changing b_2 from 12 to 24, thereby changing the respective entries in the right-side column of the *final* simplex tableau to 54, 6, 12, and −2. Starting from this revised final simplex tableau, use the *dual simplex method* to obtain the new optimal solution shown in Table 6.21. Show your work.

7.1-6.* Consider part (*a*) of Prob. 6.7-2. Use the *dual simplex method* manually to reoptimize, starting from the revised final tableau.

7.2-1.* Consider the following problem.

$$\text{Maximize} \quad Z = 8x_1 + 24x_2,$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\le 10 \\ 2x_1 + x_2 &\le 10 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0.$$

Suppose that Z represents profit and that it is possible to modify the objective function somewhat by an appropriate shifting of key personnel between the two activities. In particular, suppose that the unit profit of activity 1 can be increased above 8 (to a maximum of 18) at the expense of decreasing the unit profit of activity 2 below 24 by twice the amount. Thus, Z can actually be represented as

$$Z(\theta) = (8 + \theta)x_1 + (24 - 2\theta)x_2,$$

where θ is also a decision variable such that $0 \le \theta \le 10$.

I **(a)** Solve the original form of this problem graphically. Then extend this graphical procedure to solve the parametric extension of the problem; i.e., find the optimal solution and the optimal value of $Z(\theta)$ as a function of θ, for $0 \le \theta \le 10$.

I **(b)** Find an optimal solution for the original form of the problem by the simplex method. Then use *parametric linear programming* to find an optimal solution and the optimal value of $Z(\theta)$ as a function of θ, for $0 \le \theta \le 10$. Plot $Z(\theta)$.

(c) Determine the optimal value of θ. Then indicate how this optimal value could have been identified directly by solving only two ordinary linear programming problems. (*Hint:* A convex function achieves its maximum at an endpoint.)

I **7.2-2.** Use *parametric linear programming* to find the optimal solution for the following problem as a function of θ, for $0 \le \theta \le 20$.

$$\text{Maximize} \quad Z(\theta) = (20 + 4\theta)x_1 + (30 - 3\theta)x_2 + 5x_3,$$

subject to

$$\begin{aligned} 3x_1 + 3x_2 + x_3 &\le 10 \\ 8x_1 + 6x_2 + 4x_3 &\le 25 \\ 6x_1 + x_2 + x_3 &\le 15 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

I **7.2-3.** Consider the following problem.

$$\text{Maximize} \quad Z(\theta) = (10 - \theta)x_1 + (12 + \theta)x_2 + (7 + 2\theta)x_3,$$

subject to

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &\le 30 \\ x_1 + x_2 + x_3 &\le 20 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

(a) Use *parametric linear programming* to find an optimal solution for this problem as a function of θ, for $\theta \ge 0$.
(b) Construct the dual model for this problem. Then find an optimal solution for this dual problem as a function of θ, for $\theta \ge 0$, by the method described in the latter part of Sec. 7.2. Indicate graphically what this algebraic procedure is doing. Compare the basic solutions obtained with the complementary basic solutions obtained in part (*a*).

I **7.2-4.*** Use the *parametric linear programming* procedure for making systematic changes in the b_i parameters to find an optimal solution for the following problem as a function of θ, for $0 \le \theta \le 25$.

$$\text{Maximize} \quad Z(\theta) = 2x_1 + x_2,$$

subject to

$$\begin{aligned} x_1 &\le 10 + 2\theta \\ x_1 + x_2 &\le 25 - \theta \\ x_2 &\le 10 + 2\theta \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

Indicate graphically what this algebraic procedure is doing.

I **7.2-5.** Use *parametric linear programming* to find an optimal solution for the following problem as a function of θ, for $0 \le \theta \le 30$.

$$\text{Maximize} \quad Z(\theta) = 5x_1 + 42x_2 + 28x_3 + 49x_4,$$

subject to

$$\begin{aligned} 3x_1 - 2x_2 + x_3 + 3x_4 &\le 135 - 2\theta \\ 2x_1 + 4x_2 - x_3 + 2x_4 &\le 78 - \theta \\ x_1 + 2x_2 + x_3 + 2x_4 &\le 30 + \theta \end{aligned}$$

and

$$x_j \ge 0, \qquad \text{for } j = 1, 2, 3, 4.$$

Then identify the value of θ that gives the largest optimal value of $Z(\theta)$.

7.2-6. Consider Prob. 6.7-3. Use *parametric linear programming* to find an optimal solution as a function of θ for $-20 \le \theta \le 0$. (*Hint:* Substitute $-\theta'$ for θ, and then increase θ' from zero.)

7.2-7. Consider the $Z^*(\theta)$ function shown in Fig. 7.1 for *parametric linear programming* with systematic changes in the c_j parameters.
(a) Explain why this function is piecewise linear.
(b) Show that this function must be convex.

7.2-8. Consider the $Z^*(\theta)$ function shown in Fig. 7.2 for *parametric linear programming* with systematic changes in the b_i parameters.
(a) Explain why this function is piecewise linear.
(b) Show that this function must be concave.

7.2-9. Let

$$Z^* = \max\left\{\sum_{j=1}^{n} c_j x_j\right\},$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i, \qquad \text{for } i = 1, 2, \ldots, m,$$

and

$$x_j \ge 0, \qquad \text{for } j = 1, 2, \ldots, n$$

(where the a_{ij}, b_i, and c_j are fixed constants), and let $(y_1^*, y_2^*, \ldots, y_m^*)$ be the corresponding optimal dual solution. Then let

$$Z^{**} = \max\left\{\sum_{j=1}^{n} c_j x_j\right\},$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i + k_i, \qquad \text{for } i = 1, 2, \ldots, m,$$

and

$$x_j \ge 0, \qquad \text{for } j = 1, 2, \ldots, n,$$

where $k_1, k_2, \ldots, k_m$ are given constants. Show that

$$Z^{**} \le Z^* + \sum_{i=1}^{m} k_i y_i^*.$$

7.3-1. Consider the following problem.

$$\text{Maximize} \quad Z = 2x_1 + 3x_2,$$

subject to

$$\begin{aligned} 3x_1 - 9x_2 &\le 20 \\ 3x_1 &\le 40 \\ 9x_2 &\le 40 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0.$$

I **(a)** Solve this problem graphically.
(b) Use the *upper bound technique* manually to solve this problem.
(c) Trace graphically the path taken by the upper bound technique.

7.3-2.* Use the *upper bound technique* manually to solve the following problem.

$$\text{Maximize} \qquad Z = x_1 + 3x_2 - 2x_3,$$

subject to

$$\begin{aligned} x_2 - 2x_3 &\le 1 \\ 2x_1 + x_2 + 2x_3 &\le 8 \\ x_1 &\le 1 \\ x_2 &\le 3 \\ x_3 &\le 2 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

7.3-3. Use the *upper bound technique* manually to solve the following problem.

$$\text{Maximize} \qquad Z = 2x_1 + 3x_2 - 2x_3 + 5x_4,$$

subject to

$$\begin{aligned} 2x_1 + 2x_2 + x_3 + 2x_4 &\le 5 \\ x_1 + 2x_2 - 3x_3 + 4x_4 &\le 5 \end{aligned}$$

and

$$0 \le x_j \le 1, \qquad \text{for } j = 1, 2, 3, 4.$$

7.3-4. Use the *upper bound technique* manually to solve the following problem.

$$\text{Maximize} \qquad Z = 2x_1 + 5x_2 + 3x_3 + 4x_4 + x_5,$$

subject to

$$\begin{aligned} x_1 + 3x_2 + 2x_3 + 3x_4 + x_5 &\le 6 \\ 4x_1 + 6x_2 + 5x_3 + 7x_4 + x_5 &\le 15 \end{aligned}$$

and

$$0 \le x_j \le 1, \qquad \text{for } j = 1, 2, 3, 4, 5.$$

7.3-5. Simultaneously use the *upper bound technique* and the *dual simplex method* manually to solve the following problem.

$$\text{Minimize} \qquad Z = 3x_1 + 4x_2 + 2x_3,$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\ge 15 \\ x_2 + x_3 &\ge 10 \end{aligned}$$

and

$$0 \le x_1 \le 25, \qquad 0 \le x_2 \le 5, \qquad 0 \le x_3 \le 15.$$

C **7.4-1.** Reconsider the example used to illustrate the interior-point algorithm in Sec. 7.4. Suppose that $(x_1, x_2) = (1, 3)$ were used instead as the initial feasible trial solution. Perform two iterations manually, starting from this solution. Then use the automatic procedure in your IOR Tutorial to check your work.

7.4-2. Consider the following problem.

$$\text{Maximize} \qquad Z = 3x_1 + x_2,$$

subject to

$$x_1 + x_2 \le 4$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0.$$

I **(a)** Solve this problem graphically. Also identify all CPF solutions.
C **(b)** Starting from the initial trial solution $(x_1, x_2) = (1, 1)$, perform four iterations of the interior-point algorithm presented in Sec. 7.4 manually. Then use the automatic procedure in your IOR Tutorial to check your work.
(c) Draw figures corresponding to Figs. 7.4, 7.5, 7.6, 7.7, and 7.8 for this problem. In each case, identify the basic (or corner-point) feasible solutions in the current coordinate system. (Trial solutions can be used to determine projected gradients.)

7.4-3. Consider the following problem.

$$\text{Maximize} \qquad Z = x_1 + 2x_2,$$

subject to

$$x_1 + x_2 = 8$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0.$$

C **(a)** Near the end of Sec. 7.4, there is a discussion of what the interior-point algorithm does on this problem when starting from the initial feasible trial solution $(x_1, x_2) = (4, 4)$. Verify the results presented there by performing two iterations manually. Then use the automatic procedure in your IOR Tutorial to check your work.
(b) Use these results to predict what subsequent trial solutions would be if additional iterations were to be performed.
(c) Suppose that the stopping rule adopted for the algorithm in this application is that the algorithm stops when two successive trial solutions differ by no more than 0.01 in any component. Use your predictions from part (*b*) to predict the final trial solution and the total number of iterations required to get there. How close would this solution be to the optimal solution $(x_1, x_2) = (0, 8)$?

7.4-4. Consider the following problem.

Maximize $Z = 3x_1 + x_2$,

subject to

$$3x_1 + 2x_2 \le 45$$
$$6x_1 + x_2 \le 45$$

and

$$x_1 \ge 0, \quad x_2 \ge 0.$$

I **(a)** Solve the problem graphically.
(b) Find the *gradient* of the objective function in the original x_1-x_2 coordinate system. If you move from the origin in the direction of the gradient until you reach the boundary of the feasible region, where does it lead relative to the optimal solution?
C **(c)** Starting from the initial trial solution $(x_1, x_2) = (1, 1)$, use your IOR Tutorial to perform 10 iterations of the interior-point algorithm presented in Sec. 7.4.
C **(d)** Repeat part (*c*) with $\alpha = 0.9$.

7.4-5. Consider the following problem.

Maximize $Z = 2x_1 + 5x_2 + 7x_3$,

subject to

$$x_1 + 2x_2 + 3x_3 = 6$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

I **(a)** Graph the feasible region.
(b) Find the *gradient* of the objective function, and then find the *projected gradient* onto the feasible region.
(c) Starting from the initial trial solution $(x_1, x_2, x_3) = (1, 1, 1)$, perform two iterations of the interior-point algorithm presented in Sec. 7.4 manually.
C **(d)** Starting from this same initial trial solution, use your IOR Tutorial to perform 10 iterations of this algorithm.

C **7.4-6.** Starting from the initial trial solution $(x_1, x_2) = (2, 2)$, use your IOR Tutorial to apply 15 iterations of the interior-point algorithm presented in Sec. 7.4 to the Wyndor Glass Co. problem presented in Sec. 3.1. Also draw a figure like Fig. 7.8 to show the trajectory of the algorithm in the original x_1-x_2 coordinate system.

CHAPTER 8

The Transportation and Assignment Problems

Chapter 3 emphasized the wide applicability of linear programming. We continue to broaden our horizons in this chapter by discussing two particularly important (and related) types of linear programming problems. One type, called the *transportation problem,* received this name because many of its applications involve determining how to optimally transport goods. However, some of its important applications (e.g., production scheduling) actually have nothing to do with transportation.

The second type, called the *assignment problem,* involves such applications as assigning people to tasks. Although its applications appear to be quite different from those for the transportation problem, we shall see that the assignment problem can be viewed as a special type of transportation problem.

The next chapter will introduce additional special types of linear programming problems involving *networks,* including the *minimum cost flow problem* (Sec. 9.6). There we shall see that both the transportation and assignment problems actually are special cases of the minimum cost flow problem. We introduce the network representation of the transportation and assignment problems in this chapter.

Applications of the transportation and assignment problems tend to require a very large number of constraints and variables, so a straightforward computer application of the simplex method may require an exorbitant computational effort. Fortunately, a key characteristic of these problems is that most of the a_{ij} coefficients in the constraints are zeros, and the relatively few nonzero coefficients appear in a distinctive pattern. As a result, it has been possible to develop special *streamlined* algorithms that achieve dramatic computational savings by exploiting this special structure of the problem. Therefore, it is important to become sufficiently familiar with these special types of problems that you can recognize them when they arise and apply the proper computational procedure.

To describe special structures, we shall introduce the table (matrix) of constraint coefficients shown in Table 8.1, where a_{ij} is the coefficient of the jth variable in the ith functional constraint. Later, portions of the table containing only coefficients equal to zero will be indicated by leaving them blank, whereas blocks containing nonzero coefficients will be shaded.

After presenting a prototype example for the transportation problem, we describe the special structure in its model and give additional examples of its applications. Section 8.2 presents the *transportation simplex method,* a special streamlined version of the simplex

TABLE 8.1 Table of constraint coefficients for linear programming

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

method for efficiently solving transportation problems. (You will see in Sec. 9.7 that this algorithm is related to the *network simplex method,* another streamlined version of the simplex method for efficiently solving any minimum cost flow problem, including both transportation and assignment problems.) Section 8.3 focuses on the assignment problem. Section 8.4 then presents a specialized algorithm, called the *Hungarian algorithm,* for solving only assignment problems very efficiently.

The book's website also provides a supplement to this chapter. It is a complete case study (including the analysis) that illustrates how a corporate decision regarding where to locate a new facility (an oil refinery in this case) may require solving many transportation problems. (One of the cases for this chapter asks you to continue the analysis for an extension of this case study.)

8.1 THE TRANSPORTATION PROBLEM

Prototype Example

One of the main products of the P & T COMPANY is canned peas. The peas are prepared at three canneries (near Bellingham, Washington; Eugene, Oregon; and Albert Lea, Minnesota) and then shipped by truck to four distributing warehouses in the western United States (Sacramento, California; Salt Lake City, Utah; Rapid City, South Dakota; and Albuquerque, New Mexico), as shown in Fig. 8.1. Because the shipping costs are a major expense, management is initiating a study to reduce them as much as possible. For the upcoming season, an estimate has been made of the output from each cannery, and each warehouse has been allocated a certain amount from the total supply of peas. This information (in units of truckloads), along with the shipping cost per truckload for each cannery-warehouse combination, is given in Table 8.2. Thus, there are a total of 300 truckloads to be shipped. The problem now is to determine which plan for assigning these shipments to the various cannery-warehouse combinations would *minimize the total shipping cost.*

By ignoring the geographical layout of the canneries and warehouses, we can provide a *network representation* of this problem in a simple way by lining up all the canneries in one column on the left and all the warehouses in one column on the right. This representation is shown in Fig. 8.2. The arrows show the possible routes for the truckloads, where the number next to each arrow is the shipping cost per truckload for that route. A square bracket next to each location gives the number of truckloads to be shipped *out* of that location (so that the allocation into each warehouse is given as a negative number).

The problem depicted in Fig. 8.2 is actually a linear programming problem of the *transportation problem type*. To formulate the model, let Z denote total shipping cost, and let x_{ij} ($i = 1, 2, 3; j = 1, 2, 3, 4$) be the number of truckloads to be shipped from cannery

An Application Vignette

Procter & Gamble (P & G) makes and markets over 300 brands of consumer goods worldwide. The company has grown continuously over its long history tracing back to the 1830s. To maintain and accelerate that growth, a major OR study was undertaken to strengthen P & G's global effectiveness. Prior to the study, the company's supply chain consisted of hundreds of suppliers, over 50 product categories, over 60 plants, 15 distribution centers, and over 1,000 customer zones. However, as the company moved toward global brands, management realized that it needed to consolidate plants to reduce manufacturing expenses, improve speed to market, and reduce capital investment. Therefore, the study focused on redesigning the company's production and distribution system for its North American operations. The result was a reduction in the number of North American plants by almost 20 percent, *saving over* **$200 million** in pretax costs *per year*.

A major part of the study revolved around *formulating and solving transportation problems* for individual product categories. For each option regarding the plants to keep open, and so forth, solving the corresponding transportation problem for a product category showed what the distribution cost would be for shipping the product category from those plants to the distribution centers and customer zones.

Source: J. D. Camm, T. E. Chorman, F. A. Dill, J. R. Evans, D. J. Sweeney, and G. W. Wegryn: "Blending OR/MS, Judgment, and GIS: Restructuring P & G's Supply Chain," *Interfaces,* **27**(1): 128–142, Jan.–Feb. 1997. (A link to this article is provided on our website, www.mhhe.com/hillier.)

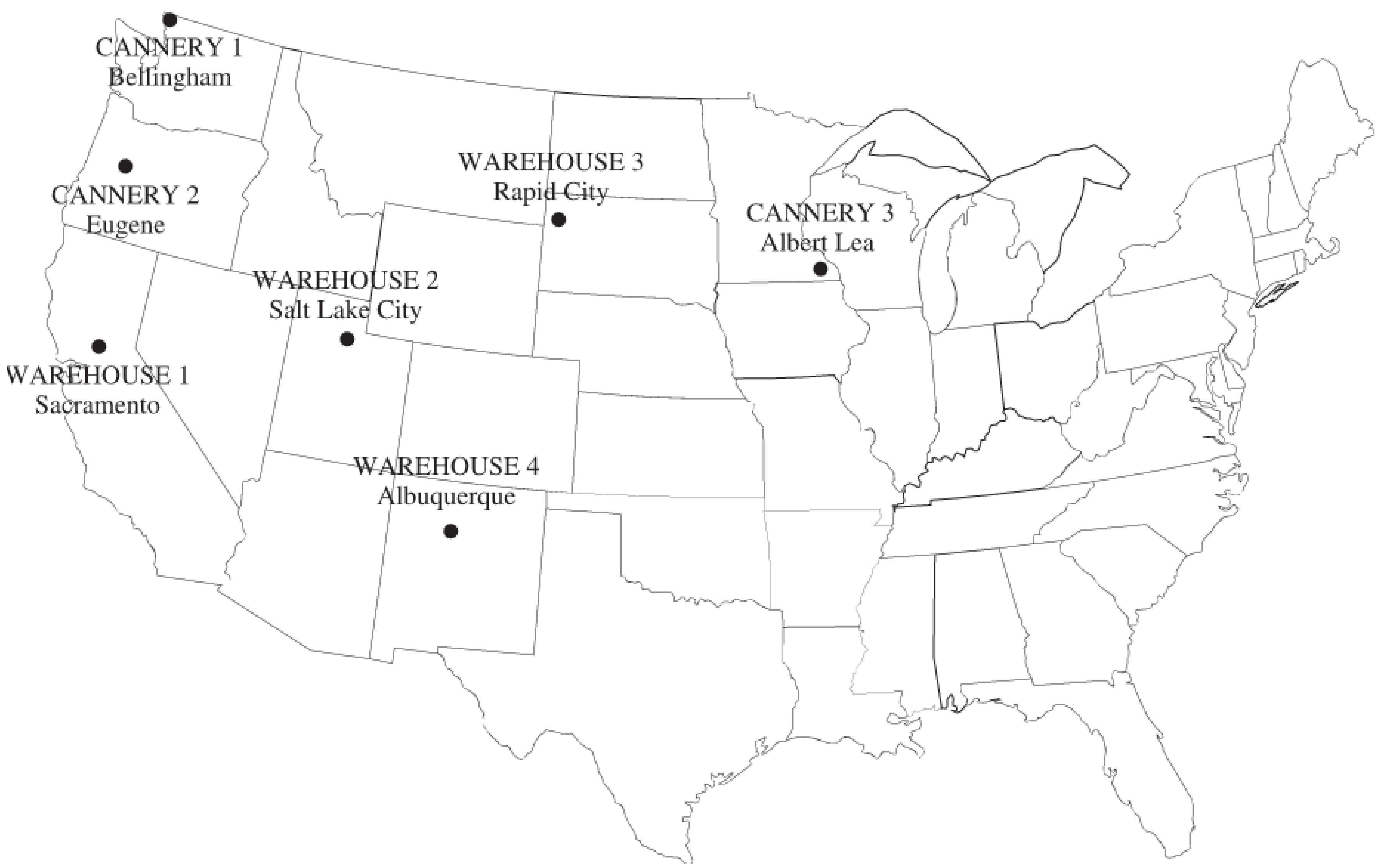


FIGURE 8.1
Location of canneries and warehouses for the P & T Co. problem.

TABLE 8.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
	1	464	513	654	867	75
Cannery	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

i to warehouse j. Thus, the objective is to choose the values of these 12 decision variables (the x_{ij}) so as to

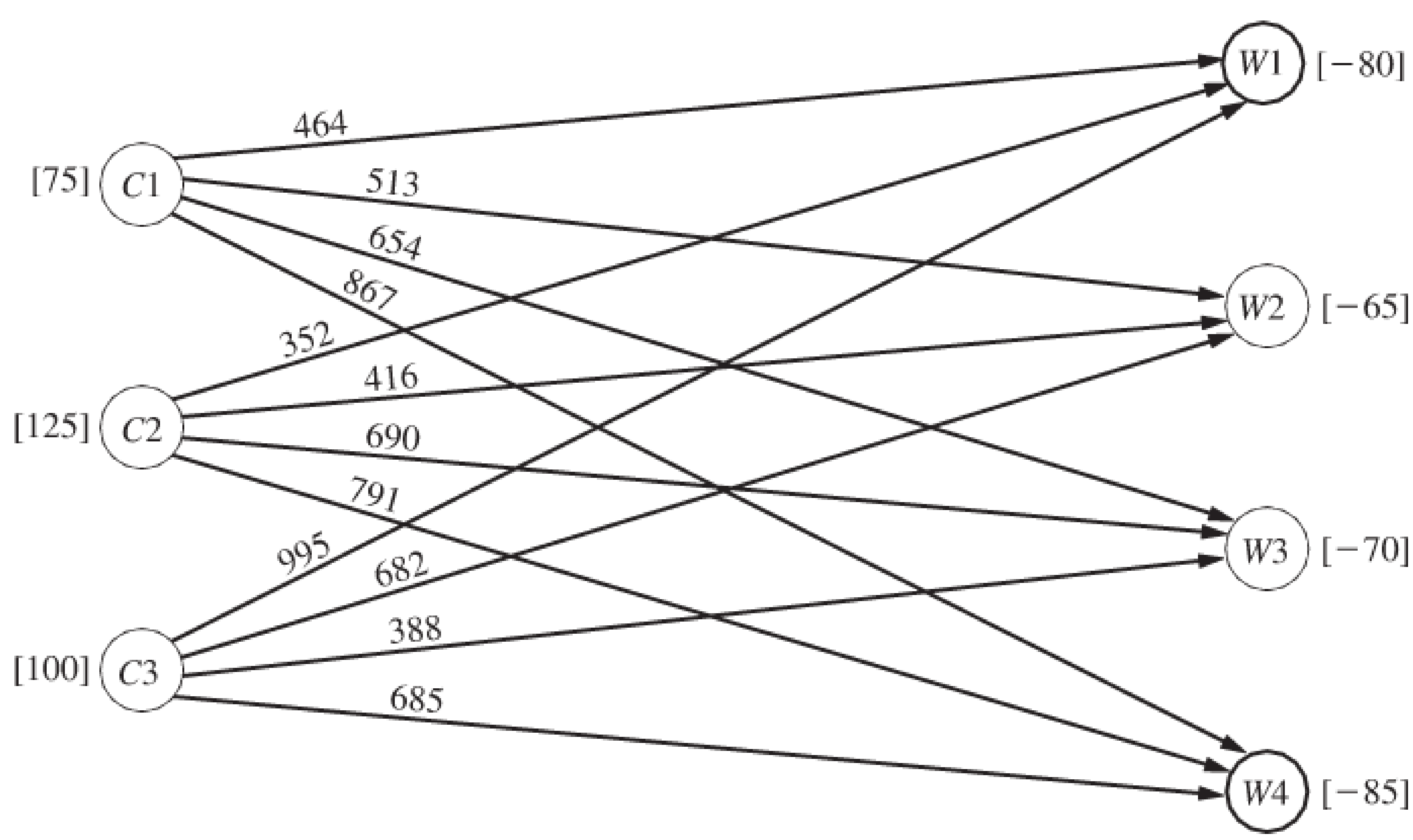
$$\begin{aligned}\text{Minimize} \quad Z = {} & 464x_{11} + 513x_{12} + 654x_{13} + 867x_{14} + 352x_{21} + 416x_{22} \\ & + 690x_{23} + 791x_{24} + 995x_{31} + 682x_{32} + 388x_{33} + 685x_{34},\end{aligned}$$

subject to the constraints

$$\begin{array}{llllllllllllr}
x_{11} + x_{12} + x_{13} + x_{14} & & & & & & & & & & & & = 75 \\
& x_{21} + x_{22} + x_{23} + x_{24} & & & & & & & & & & & = 125 \\
& & x_{31} + x_{32} + x_{33} + x_{34} & & & & & & & & & & = 100 \\
x_{11} & + x_{21} & + x_{31} & & & & & & & & & & = 80 \\
x_{12} & + x_{22} & + x_{32} & & & & & & & & & & = 65 \\
x_{13} & + x_{23} & + x_{33} & & & & & & & & & & = 70 \\
x_{14} & + x_{24} & + x_{34} & & & & & & & & & & = 85
\end{array}$$

and

$$x_{ij} \geq 0 \qquad (i = 1, 2, 3;\ j = 1, 2, 3, 4).$$



■ **FIGURE 8.2**
Network representation of the P & T Co. problem.

TABLE 8.3 Constraint coefficients for P & T Co.

Coefficient of:

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccc} x_{11} & x_{12} & x_{13} & x_{14} & x_{21} & x_{22} & x_{23} & x_{24} & x_{31} & x_{32} & x_{33} & x_{34} \end{array} \\
\mathbf{A} = & \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 \\
1 & & & & 1 & & & & 1 & & & \\
 & 1 & & & & 1 & & & & 1 & & \\
 & & 1 & & & & 1 & & & & 1 & \\
 & & & 1 & & & & 1 & & & & 1
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{Cannery constraints} \\
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \text{Warehouse constraints}
\end{array}
\end{array}
$$

Table 8.3 shows the constraint coefficients. As you will see later in this section, it is the special structure in the pattern of these coefficients that distinguishes this problem as a transportation problem, not its context. However, we first will describe the various other characteristics of the transportation problem model.

The Transportation Problem Model

To describe the general model for the transportation problem, we need to use terms that are considerably less specific than those for the components of the prototype example. In particular, the general transportation problem is concerned (literally or figuratively) with distributing *any* commodity from *any* group of supply centers, called **sources**, to *any* group of receiving centers, called **destinations,** in such a way as to minimize the total distribution cost. The correspondence in terminology between the prototype example and the general problem is summarized in Table 8.4.

As indicated by the fourth and fifth rows of the table, each source has a certain **supply** of units to distribute to the destinations, and each destination has a certain **demand** for units to be received from the sources. The model for a transportation problem makes the following assumption about these supplies and demands.

> **The requirements assumption:** Each source has a fixed *supply* of units, where this entire supply must be distributed to the destinations. (We let s_i denote the number of units being supplied by source i, for $i = 1, 2, \dots, m$.) Similarly, each destination has a fixed *demand* for units, where this entire demand must be received from the sources. (We let d_j denote the number of units being received by destination j, for $j = 1, 2, \dots, n$.)

This assumption holds for the P & T Co. problem since each cannery (source) has a fixed output and each warehouse (destination) has a fixed allocation.

TABLE 8.4 Terminology for the transportation problem

Prototype Example	General Problem
Truckloads of canned peas	Units of a commodity
Three canneries	m sources
Four warehouses	n destinations
Output from cannery i	Supply s_i from source i
Allocation to warehouse j	Demand d_j at destination j
Shipping cost per truckload from cannery i to warehouse j	Cost c_{ij} per unit distributed from source i to destination j

This assumption that there is no leeway in the amounts to be sent or received means that there needs to be a balance between the total supply from all sources and the total demand at all destinations.

The feasible solutions property: A transportation problem will have feasible solutions if and only if

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

Fortunately, these sums are equal for the P & T Co. since Table 8.2 indicates that the supplies (outputs) sum to 300 truckloads and so do the demands (allocations).

In some real problems, the supplies actually represent *maximum* amounts (rather than fixed amounts) to be distributed. Similarly, in other cases, the demands represent maximum amounts (rather than fixed amounts) to be received. Such problems do not quite fit the model for a transportation problem because they violate the *requirements assumption.* However, it is possible to *reformulate* the problem so that they then fit this model by introducing a *dummy destination* or a *dummy source* to take up the slack between the actual amounts and maximum amounts being distributed. We will illustrate how this is done with two examples at the end of this section.

The last row of Table 8.4 refers to a cost per unit distributed. This reference to a *unit cost* implies the following basic assumption for any transportation problem.

The cost assumption: The cost of distributing units from any particular source to any particular destination is *directly proportional* to the number of units distributed. Therefore, this cost is just the *unit cost* of distribution *times* the *number of units distributed.* (We let c_{ij} denote this unit cost for source i and destination j.)

This assumption holds for the P & T Co. problem since the cost of shipping peas from any cannery to any warehouse is directly proportional to the number of truckloads being shipped.

The only data needed for a transportation problem model are the supplies, demands, and unit costs. These are the *parameters of the model.* All these parameters can be summarized conveniently in a single *parameter table* as shown in Table 8.5.

The model: Any problem (whether involving transportation or not) fits the model for a transportation problem if it can be described completely in terms of a *parameter table* like Table 8.5 and it satisfies both the *requirements assumption* and the *cost assumption.* The objective is to minimize the total cost of distributing the units. All the parameters of the model are included in this parameter table.

TABLE 8.5 Parameter table for the transportation problem

		Cost per Unit Distributed			
		Destination			
	1	**2**	$\cdots$	***n***	**Supply**
Source 1	c_{11}	c_{12}	$\cdots$	c_{1n}	s_1
2	c_{21}	c_{22}	$\cdots$	c_{2n}	s_2
$\vdots$					$\vdots$
m	c_{m1}	c_{m2}	$\cdots$	c_{mn}	s_m
Demand	d_1	d_2	$\cdots$	d_n	

Therefore, formulating a problem as a transportation problem only requires filling out a parameter table in the format of Table 8.5. (The parameter table for the P & T Co. problem is shown in Table 8.2.) Alternatively, the same information can be provided by using the network representation of the problem shown in Fig. 8.3 (as was done in Fig. 8.2 for the P & T Co. problem). Some problems that have nothing to do with transportation also can be formulated as a transportation problem in either of these two ways. The Worked Examples section of the book's website includes **another example** of such a problem.

Since a transportation problem can be formulated simply by either filling out a parameter table or drawing its network representation, it is not necessary to write out a formal mathematical model for the problem. However, we will go ahead and show you this model once for the general transportation problem just to emphasize that it is indeed a special type of linear programming problem.

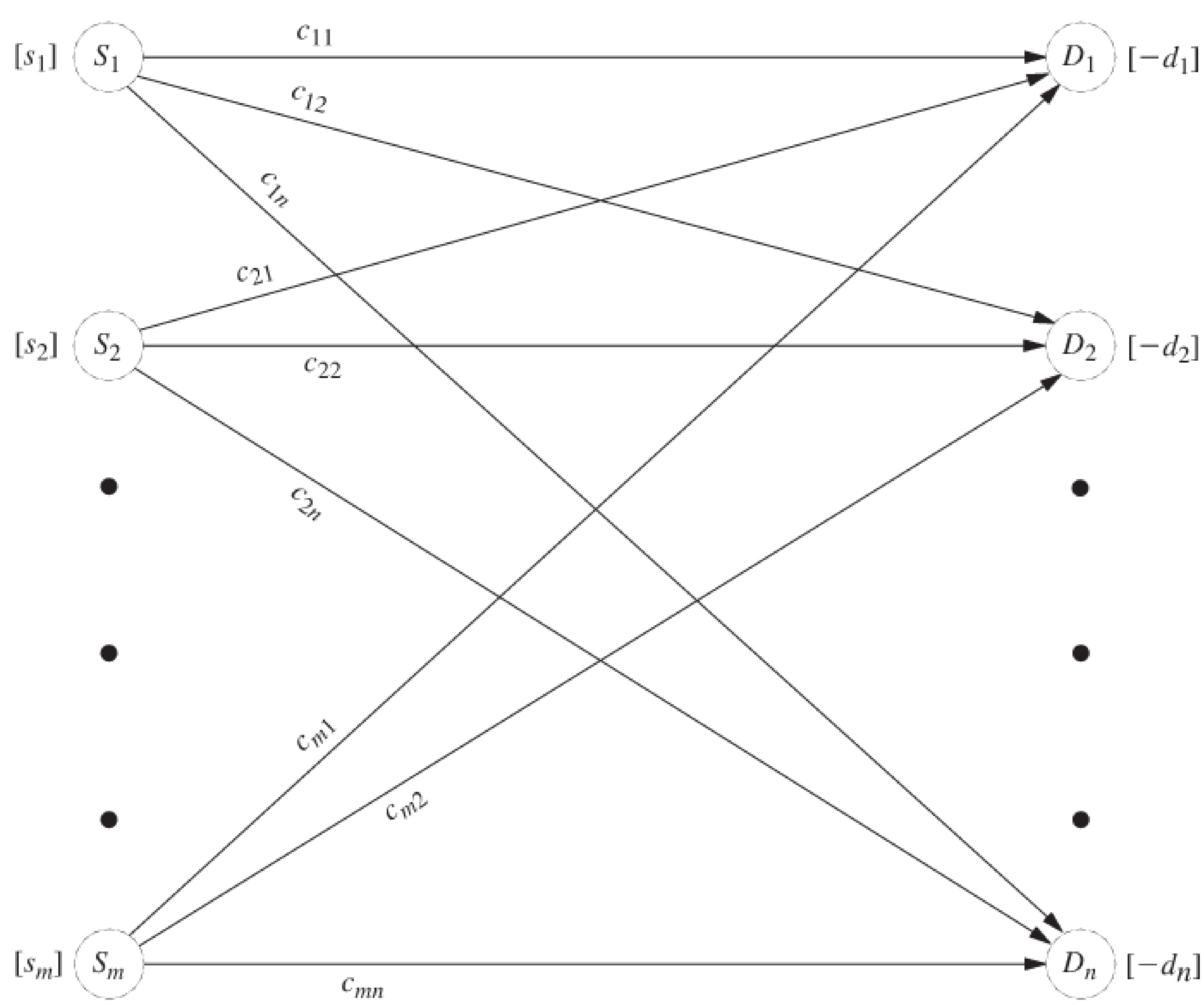
Letting Z be the total distribution cost and x_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) be the number of units to be distributed from source i to destination j, the linear programming formulation of this problem is

$$\text{Minimize} \qquad Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij},$$

subject to

$$\sum_{j=1}^{n} x_{ij} = s_i \qquad \text{for } i = 1, 2, \ldots, m,$$

■ **FIGURE 8.3**
Network representation of the transportation problem.



$$\sum_{i=1}^{m} x_{ij} = d_j \qquad \text{for } j = 1, 2, \ldots, n,$$

and

$$x_{ij} \geq 0, \qquad \text{for all } i \text{ and } j.$$

Note that the resulting table of constraint coefficients has the special structure shown in Table 8.6. *Any* linear programming problem that fits this special formulation is of the transportation problem type, regardless of its physical context. In fact, there have been numerous applications unrelated to transportation that have been fitted to this special structure, as we shall illustrate in the next example later in this section. (The assignment problem described in Sec. 8.3 is an additional example.) This is one of the reasons why the transportation problem is considered such an important special type of linear programming problem.

For many applications, the supply and demand quantities in the model (the s_i and d_j) have integer values, and implementation will require that the distribution quantities (the x_{ij}) also have integer values. Fortunately, because of the special structure shown in Table 8.6, all such problems have the following property.

> **Integer solutions property:** For transportation problems where every s_i and d_j have an integer value, all the basic variables (allocations) in *every* basic feasible (BF) solution (including an optimal one) also have *integer* values.

The solution procedure described in Sec. 8.2 deals only with BF solutions, so it automatically will obtain an *integer* optimal solution for this case. (You will be able to see why this solution procedure actually gives a proof of the integer solutions property after you learn the procedure; Prob. 8.2-20 guides you through the reasoning involved.) Therefore, it is unnecessary to add a constraint to the model that the x_{ij} must have integer values.

As with other linear programming problems, the usual software options (Excel, LINGO/LINDO, MPL/CPLEX) are available to you for setting up and solving transportation problems (and assignment problems), as demonstrated in the files for this chapter in your OR Courseware. However, because the Excel approach now is somewhat different from what you have seen previously, we next describe this approach.

Using Excel to Formulate and Solve Transportation Problems

As described in Sec. 3.6, the process of using a spreadsheet to formulate a linear programming model for a problem begins by developing answers to three questions. What are the *decisions* to be made? What are the *constraints* on these decisions? What is the *overall measure of performance* for these decisions? Since a transportation problem is a special type of

TABLE 8.6 Constraint coefficients for the transportation problem

Coefficient of:

$$
\begin{array}{cc}
& \begin{array}{ccccccccccccc} x_{11} & x_{12} & \cdots & x_{1n} & x_{21} & x_{22} & \cdots & x_{2n} & \cdots & x_{m1} & x_{m2} & \cdots & x_{mn}\end{array} \\
\mathbf{A} = & \left[\begin{array}{ccccccccccccc} 1 & 1 & \cdots & 1 & & & & & & & & & \\ & & & & 1 & 1 & \cdots & 1 & & & & & \\ & & & & & & & & \ddots & & & & \\ & & & & & & & & & 1 & 1 & \cdots & 1 \\ 1 & & & & 1 & & & & & 1 & & & \\ & 1 & & & & 1 & & & \cdots & & 1 & & \\ & & \ddots & & & & \ddots & & & & & \ddots & \\ & & & 1 & & & & 1 & & & & & 1 \end{array}\right] \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \text{Supply constraints} \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \text{Demand constraints} \end{array}
\end{array}
$$

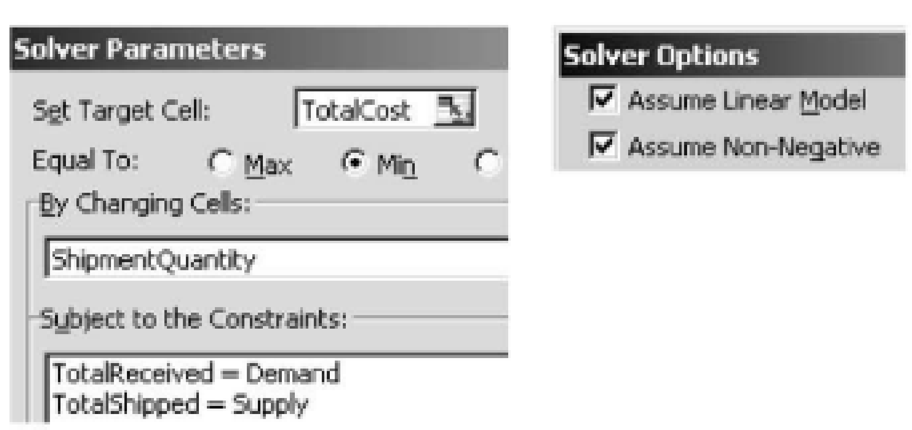
linear programming problem, addressing these questions also is a suitable starting point for formulating this kind of problem on a spreadsheet. The design of the spreadsheet then revolves around laying out this information and the associated data in a logical way.

To illustrate, consider the P & T Co. problem again. The decisions to be made are the number of truckloads of peas to ship from each cannery to each warehouse. The constraints on these decisions are that the total amount shipped from each cannery must equal its output (the supply) and the total amount received at each warehouse must equal its allocation (the demand). The overall measure of performance is the total shipping cost, so the objective is to minimize this quantity.

This information leads to the spreadsheet model shown in Fig. 8.4. All the data provided in Table 8.2 are displayed in the following data cells: UnitCost (D5:G7), Supply (J12:J14), and Demand (D17:G17). The decisions on shipping quantities are given by the changing cells, ShipmentQuantity (D12:G14). The output cells are TotalShipped (H12:H14) and TotalReceived (D15:G15), where the SUM functions entered into these cells are shown near the bottom of Fig. 8.4. The constraints, TotalShipped (H12:H14) = Supply (J12:J14) and TotalReceived (D15:G15) = Demand (D17:G17), have been specified on the spreadsheet and entered into the Solver dialogue box. The target cell is TotalCost (J17), where its SUMPRODUCT function is shown in the lower right-hand corner of Fig. 8.4. The Solver dialogue box specifies that the objective is to minimize this target cell. One of the selected Solver options (Assume Non-Negative) specifies that all shipment quantities must be nonnegative. The other one (Assume Linear Model) indicates that this transportation problem is also a linear programming problem.

To begin the process of solving the problem, any value (such as 0) can be entered in each of the changing cells. After clicking on the Solve button, the Solver will use the simplex method to solve the transportation problem and determine the best value for each of

	A	B	C	D	E	F	G	H	I	J
1	P&T Co. Distribution Problem									
2										
3		Unit Cost		Destination (Warehouse)						
4				Sacramento	Salt Lake City	Rapid City	Albuquerque			
5		Source	Bellingham	$464	$513	$654	$867			
6		(Cannery)	Eugene	$352	$416	$690	$791			
7			Albert Lea	$995	$682	$388	$685			
8										
9										
10		Shipment Quantity		Destination (Warehouse)						
11		(Truckloads)		Sacramento	Salt Lake City	Rapid City	Albuquerque	Total Shipped		Supply
12		Source	Bellingham	0	20	0	55	75	=	75
13		(Cannery)	Eugene	80	45	0	0	125	=	125
14			Albert Lea	0	0	70	30	100	=	100
15			Total Received	80	65	70	85			
16				=	=	=	=			Total Cost
17			Demand	80	65	70	85			$ 152,535



Range Name	Cells
Demand	D17:G17
ShipmentQuantity	D12:G14
Supply	J12:J14
TotalCost	J17
TotalReceived	D15:G15
TotalShipped	H12:H14
UnitCost	D5:G7

	H
11	Total Shipped
12	=SUM(D12:G12)
13	=SUM(D13:G13)
14	=SUM(D14:G14)

	C	D	E	F	G
15	Total Received	=SUM(D12:D14)	=SUM(E12:E14)	=SUM(F12:F14)	=SUM(G12:G14)

	J
16	Total Cost
17	=SUMPRODUCT(UnitCost,ShipmentQuantity)

■ **FIGURE 8.4**
A spreadsheet formulation of the P & T Co. problem as a transportation problem, including the target cell TotalCost (J17) and the other output cells TotalShipped (H12:H14) and TotalReceived (D15:G15), as well as the specifications needed to set up the model. The changing cells ShipmentQuantity (D12:G14) show the optimal shipping plan obtained by the Solver.

the decision variables. This optimal solution is shown in ShipmentQuantity (D12:G14) in Fig. 8.4, along with the resulting value $152,535 in the target cell TotalCost (J17).

Note that the Solver simply uses the general simplex method to solve a transportation problem rather than a streamlined version that is specially designed for solving transportation problems very efficiently, such as the transportation simplex method presented in the next section. Therefore, a software package that includes such a streamlined version should solve a large transportation problem much faster than the Excel Solver.

We mentioned earlier that some problems do not quite fit the model for a transportation problem because they violate the requirements assumption, but that it is possible to reformulate such a problem to fit this model by introducing a dummy destination or a dummy source. When using the Excel Solver, it is not necessary to do this reformulation since the simplex method can solve the original model where the supply constraints are in $\leq$ form or the demand constraints are in $\geq$ form. (The Excel files for the next two examples in your OR Courseware illustrate spreadsheet formulations that retain either the supply constraints or the demand constraints in their original inequality form.) However, the larger the problem, the more worthwhile it becomes to do the reformulation and use the transportation simplex method (or equivalent) instead with another software package.

The next two examples illustrate how to do this kind of reformulation.

An Example with a Dummy Destination

The NORTHERN AIRPLANE COMPANY builds commercial airplanes for various airline companies around the world. The last stage in the production process is to produce the jet engines and then to install them (a very fast operation) in the completed airplane frame. The company has been working under some contracts to deliver a considerable number of airplanes in the near future, and the production of the jet engines for these planes must now be scheduled for the next four months.

To meet the contracted dates for delivery, the company must supply engines for installation in the quantities indicated in the second column of Table 8.7. Thus, the cumulative number of engines produced by the end of months 1, 2, 3, and 4 must be at least 10, 25, 50, and 70, respectively.

The facilities that will be available for producing the engines vary according to other production, maintenance, and renovation work scheduled during this period. The resulting monthly differences in the maximum number that can be produced and the cost (in millions of dollars) of producing each one are given in the third and fourth columns of Table 8.7.

Because of the variations in production costs, it may well be worthwhile to produce some of the engines a month or more before they are scheduled for installation, and this possibility is being considered. The drawback is that such engines must be stored until the scheduled installation (the airplane frames will not be ready early) at a storage cost of $15,000 per month (including interest on expended capital) for each engine,[1] as shown in the rightmost column of Table 8.7.

The production manager wants a schedule developed for the number of engines to be produced in each of the four months so that the total of the production and storage costs will be minimized.

Formulation. One way to formulate a mathematical model for this problem is to let x_j be the number of jet engines to be produced in month j, for $j = 1, 2, 3, 4$. By using only

[1]For modeling purposes, assume that this storage cost is incurred at the *end of the month* for just those engines that are being held over into the next month. Thus, engines that are produced in a given month for installation in the same month are assumed to incur no storage cost.

TABLE 8.7 Production scheduling data for Northern Airplane Co.

Month	Scheduled Installations	Maximum Production	Unit Cost* of Production	Unit Cost* of Storage
1	10	25	1.08	0.015
2	15	35	1.11	0.015
3	25	30	1.10	0.015
4	20	10	1.13	

*Cost is expressed in millions of dollars.

these four decision variables, the problem can be formulated as a linear programming problem that does *not* fit the transportation problem type. (See Prob. 8.2-18.)

On the other hand, by adopting a different viewpoint, we can instead formulate the problem as a transportation problem that requires *much* less effort to solve. This viewpoint will describe the problem in terms of sources and destinations and then identify the corresponding x_{ij}, c_{ij}, s_i, and d_j. (See if you can do this before reading further.)

Because the units being distributed are jet engines, each of which is to be scheduled for production in a particular month and then installed in a particular (perhaps different) month,

$$\begin{aligned}
\text{Source } i &= \text{production of jet engines in month } i \ (i = 1, 2, 3, 4) \\
\text{Destination } j &= \text{installation of jet engines in month } j \ (j = 1, 2, 3, 4) \\
x_{ij} &= \text{number of engines produced in month } i \text{ for installation in month } j \\
c_{ij} &= \text{cost associated with each unit of } x_{ij} \\
&= \begin{cases} \text{cost per unit for production and any storage} & \text{if } i \le j \\ ? & \text{if } i > j \end{cases} \\
s_i &= ? \\
d_j &= \text{number of scheduled installations in month } j.
\end{aligned}$$

The corresponding (incomplete) parameter table is given in Table 8.8. Thus, it remains to identify the missing costs and the supplies.

Since it is impossible to produce engines in one month for installation in an earlier month, x_{ij} must be zero if $i > j$. Therefore, there is no real cost that can be associated with such x_{ij}. Nevertheless, in order to have a well-defined transportation problem to which the solution procedure of Sec. 8.2 can be applied, it is necessary to assign some value for the unidentified costs. Fortunately, we can use the Big M method introduced in Sec. 4.6 to

TABLE 8.8 Incomplete parameter table for Northern Airplane Co.

		Cost per Unit Distributed				
		Destination				
		1	2	3	4	Supply
Source	1	1.080	1.095	1.110	1.125	?
	2	?	1.110	1.125	1.140	?
	3	?	?	1.100	1.115	?
	4	?	?	?	1.130	?
Demand		10	15	25	20	

assign this value. Thus, we assign a *very* large number (denoted by M for convenience) to the unidentified cost entries in Table 8.8 to force the corresponding values of x_{ij} to be zero in the final solution.

The numbers that need to be inserted into the supply column of Table 8.8 are not obvious because the "supplies," the amounts produced in the respective months, are not fixed quantities. In fact, the objective is to solve for the most desirable values of these production quantities. Nevertheless, it is necessary to assign some fixed number to every entry in the table, including those in the supply column, to have a transportation problem. A clue is provided by the fact that although the supply constraints are not present in the usual form, these constraints do exist in the form of upper bounds on the amount that can be supplied, namely,

$$x_{11} + x_{12} + x_{13} + x_{14} \le 25,$$
$$x_{21} + x_{22} + x_{23} + x_{24} \le 35,$$
$$x_{31} + x_{32} + x_{33} + x_{34} \le 30,$$
$$x_{41} + x_{42} + x_{43} + x_{44} \le 10.$$

The only change from the standard model for the transportation problem is that these constraints are in the form of inequalities instead of equalities.

To convert these inequalities to equations in order to fit the transportation problem model, we use the familiar device of *slack variables,* introduced in Sec. 4.2. In this context, the slack variables are allocations to a single **dummy destination** that represent the *unused production capacity* in the respective months. This change permits the supply in the transportation problem formulation to be the total production capacity in the given month. Furthermore, because the demand for the dummy destination is the total unused capacity, this demand is

$$(25 + 35 + 30 + 10) - (10 + 15 + 25 + 20) = 30.$$

With this demand included, the sum of the supplies now equals the sum of the demands, which is the condition given by the *feasible solutions property* for having feasible solutions.

The cost entries associated with the dummy destination should be zero because there is no cost incurred by a fictional allocation. (Cost entries of M would be *inappropriate* for this column because we do not want to force the corresponding values of x_{ij} to be zero. In fact, these values need to sum to 30.)

The resulting final parameter table is given in Table 8.9, with the dummy destination labeled as destination 5(D). By using this formulation, it is quite easy to find the optimal production schedule by the solution procedure described in Sec. 8.2. (See Prob. 8.2-10 and its answer in the back of the book.)

TABLE 8.9 Complete parameter table for Northern Airplane Co.

		Cost per Unit Distributed					
		Destination					
		1	2	3	4	5(D)	Supply
Source	1	1.080	1.095	1.110	1.125	0	25
	2	M	1.110	1.125	1.140	0	35
	3	M	M	1.100	1.115	0	30
	4	M	M	M	1.130	0	10
Demand		10	15	25	20	30	

An Example with a Dummy Source

METRO WATER DISTRICT is an agency that administers water distribution in a large geographic region. The region is fairly arid, so the district must purchase and bring in water from outside the region. The sources of this imported water are the Colombo, Sacron, and Calorie rivers. The district then resells the water to users in the region. Its main customers are the water departments of the cities of Berdoo, Los Devils, San Go, and Hollyglass.

It is possible to supply any of these cities with water brought in from any of the three rivers, with the exception that no provision has been made to supply Hollyglass with Calorie River water. However, because of the geographic layouts of the aqueducts and the cities in the region, the cost to the district of supplying water depends upon both the source of the water and the city being supplied. The variable cost per acre foot of water (in tens of dollars) for each combination of river and city is given in Table 8.10. Despite these variations, the price per acre foot charged by the district is independent of the source of the water and is the same for all cities.

The management of the district is now faced with the problem of how to allocate the available water during the upcoming summer season. In units of 1 million acre feet, the amounts available from the three rivers are given in the rightmost column of Table 8.10. The district is committed to providing a certain minimum amount to meet the essential needs of each city (with the exception of San Go, which has an independent source of water), as shown in the *minimum needed* row of the table. The *requested* row indicates that Los Devils desires no more than the minimum amount, but that Berdoo would like to buy as much as 20 more, San Go would buy up to 30 more, and Hollyglass will take as much as it can get.

Management wishes to allocate *all* the available water from the three rivers to the four cities in such a way as to at least meet the essential needs of each city while minimizing the total cost to the district.

Formulation. Table 8.10 already is close to the proper form for a parameter table, with the rivers being the sources and the cities being the destinations. However, the one basic difficulty is that it is not clear what the demands at the destinations should be. The amount to be received at each destination (except Los Devils) actually is a decision variable, with both a lower bound and an upper bound. This upper bound is the amount requested unless the request exceeds the total supply remaining after the minimum needs of the other cities are met, in which case this *remaining supply* becomes the upper bound. Thus, insatiably thirsty Hollyglass has an upper bound of

$$(50 + 60 + 50) - (30 + 70 + 0) = 60.$$

Unfortunately, just like the other numbers in the parameter table of a transportation problem, the demand quantities must be *constants,* not bounded decision variables. To

TABLE 8.10 Water resources data for Metro Water District

	Cost (Tens of Dollars) per Acre Foot				
	Berdoo	**Los Devils**	**San Go**	**Hollyglass**	**Supply**
Colombo River	16	13	22	17	50
Sacron River	14	13	19	15	60
Calorie River	19	20	23	—	50
Minimum needed	30	70	0	10	(in units of 1
Requested	50	70	30	∞	million acre feet)

begin resolving this difficulty, temporarily suppose that it is not necessary to satisfy the minimum needs, so that the upper bounds are the only constraints on amounts to be allocated to the cities. In this circumstance, can the requested allocations be viewed as the demand quantities for a transportation problem formulation? After one adjustment, yes! (Do you see already what the needed adjustment is?)

The situation is analogous to Northern Airplane Co.'s production scheduling problem, where there was *excess supply capacity.* Now there is *excess demand capacity.* Consequently, rather than introducing a *dummy destination* to "receive" the unused supply capacity, the adjustment needed here is to introduce a **dummy source** to "send" the *unused demand capacity.* The imaginary supply quantity for this dummy source would be the amount by which the sum of the demands exceeds the sum of the real supplies:

$$(50 + 70 + 30 + 60) - (50 + 60 + 50) = 50.$$

This formulation yields the parameter table shown in Table 8.11, which uses units of million acre feet and tens of millions of dollars. The cost entries in the *dummy* row are zero because there is no cost incurred by the fictional allocations from this dummy source. On the other hand, a huge unit cost of M is assigned to the Calorie River–Hollyglass spot. The reason is that Calorie River water cannot be used to supply Hollyglass, and assigning a cost of M will prevent any such allocation.

Now let us see how we can take each city's minimum needs into account in this kind of formulation. Because San Go has no minimum need, it is all set. Similarly, the formulation for Hollyglass does not require any adjustments because its demand (60) exceeds the dummy source's supply (50) by 10, so the amount supplied to Hollyglass from the *real* sources will be *at least 10* in any feasible solution. Consequently, its minimum need of 10 from the rivers is guaranteed. (If this coincidence had not occurred, Hollyglass would need the same adjustments that we shall have to make for Berdoo.)

Los Devils' minimum need equals its requested allocation, so its *entire* demand of 70 must be filled from the real sources rather than the dummy source. This requirement calls for the Big M method! Assigning a huge unit cost of M to the allocation from the dummy source to Los Devils ensures that this allocation will be zero in an optimal solution.

Finally, consider Berdoo. In contrast to Hollyglass, the dummy source has an adequate (fictional) supply to "provide" at least some of Berdoo's minimum need in addition to its extra requested amount. Therefore, since Berdoo's minimum need is 30, adjustments must be made to prevent the dummy source from contributing more than 20 to Berdoo's total demand of 50. This adjustment is accomplished by splitting Berdoo into two destinations, one having a demand of 30 with a unit cost of M for any allocation from the dummy source and the other having a demand of 20 with a unit cost of zero for the dummy source allocation. This formulation gives the final parameter table shown in Table 8.12.

TABLE 8.11 Parameter table without minimum needs for Metro Water District

		Cost (Tens of Millions of Dollars) per Unit Distributed				
		Destination				
		Berdoo	Los Devils	San Go	Hollyglass	Supply
Source	Colombo River	16	13	22	17	50
	Sacron River	14	13	19	15	60
	Calorie River	19	20	23	M	50
	Dummy	0	0	0	0	50
Demand		50	70	30	60	

■ **TABLE 8.12** Parameter table for Metro Water District

			Cost (Tens of Millions of Dollars) per Unit Distributed					
			Destination					
			Berdoo (min.) 1	Berdoo (extra) 2	Los Devils 3	San Go 4	Hollyglass 5	Supply
Source	Colombo River	1	16	16	13	22	17	50
	Sacron River	2	14	14	13	19	15	60
	Calorie River	3	19	19	20	23	M	50
	Dummy	4(D)	M	0	M	0	0	50
Demand			30	20	70	30	60	

This problem will be solved in Sec. 8.2 to illustrate the solution procedure presented there.

Generalizations of the Transportation Problem

Even after the kinds of reformulations illustrated by the two preceding examples, some problems involving the distribution of units from sources to destinations fail to satisfy the model for the transportation problem. One reason may be that the distribution does not go directly from the sources to the destinations but instead passes through transfer points along the way. The Distribution Unlimited Co example in Sec. 3.4 (See Fig. 3.13) illustrates such a problem. In this case, the sources are the two factories and the destinations are the two warehouses. However, a shipment from a particular factory to a particular warehouse may first get transferred at a distribution center, or even at the other factory or the other warehouse, before reaching its destination. The unit shipping costs differ for these different shipping lanes. Furthermore, there are upper limits on how much can be shipped through some of the shipping lanes. Although it is not a transportation problem, this kind of problem still is a special type of linear programming problem, called the *minimum cost flow problem*, that will be discussed in Sec. 9.6. The *network simplex method* described in Sec. 9.7 provides an efficient way of solving minimum cost flow problems. A minimum cost flow problem that does not impose any upper limits on how much can be shipped through the shipping lanes is referred to as a *transshipment problem.* Section 23.1 on the book's website is devoted to discussing transshipment problems.

In other cases, the distribution may go directly from sources to destinations, but other assumptions of the transportation problem may be violated. The *cost assumption* will be violated if the cost of distributing units from any particular source to any particular destination is a nonlinear function of the number of units distributed. The *requirements assumption* will be violated if either the supplies from the sources or the demands at the destinations are not fixed. For example, the final demand at a destination may not become known until after the units have arrived and then a nonlinear cost is incurred if the amount received deviates from the final demand. If the supply at a source is not fixed, the cost of producing the amount supplied may be a nonlinear function of this amount. For example, a fixed cost may be part of the cost associated with a decision to open up a new source. Considerable research has been done to generalize the transportation problem and its solution procedure in these kinds of directions.[2]

[2]For example, see K. Holmberg and H. Tuy: "A Production-Transportation Problem with Stochastic Demand and Concave Production Costs," *Mathematical Programming Series A,* **85:** 157–179, 1999.

8.2 A STREAMLINED SIMPLEX METHOD FOR THE TRANSPORTATION PROBLEM

Because the transportation problem is just a special type of linear programming problem, it can be solved by applying the simplex method as described in Chap. 4. However, you will see in this section that some tremendous computational shortcuts can be taken in this method by exploiting the special structure shown in Table 8.6. We shall refer to this streamlined procedure as the **transportation simplex method.**

As you read on, note particularly how the special structure is exploited to achieve great computational savings. This will illustrate an important OR technique—streamlining an algorithm to exploit the special structure in the problem at hand.

Setting Up the Transportation Simplex Method

To highlight the streamlining achieved by the transportation simplex method, let us first review how the general (unstreamlined) simplex method would set up a transportation problem in tabular form. After constructing the table of constraint coefficients (see Table 8.6), converting the objective function to maximization form, and using the Big M method to introduce artificial variables $z_1, z_2, \ldots, z_{m+n}$ into the $m + n$ respective equality constraints (see Sec. 4.6), typical columns of the simplex tableau would have the form shown in Table 8.13, where all entries *not shown* in these columns are *zeros.* [The one remaining adjustment to be made before the first iteration of the simplex method is to algebraically eliminate the nonzero coefficients of the initial (artificial) basic variables in row 0.]

After any subsequent iteration, row 0 then would have the form shown in Table 8.14. Because of the pattern of 0s and 1s for the coefficients in Table 8.13, by the *fundamental insight* presented in Sec. 5.3, u_i and v_j would have the following interpretation:

u_i = multiple of *original* row i that has been subtracted (directly or indirectly) from *original* row 0 by the simplex method during all iterations leading to the current simplex tableau.

v_j = multiple of *original* row $m + j$ that has been subtracted (directly or indirectly) from *original* row 0 by the simplex method during all iterations leading to the current simplex tableau.

TABLE 8.13 Original simplex tableau before simplex method is applied to transportation problem

Basic Variable	Eq.	Coefficient of: Z	$\cdots$	x_{ij}	$\cdots$	z_i	$\cdots$	z_{m+j}	$\cdots$	Right side
Z	(0)	-1		c_{ij}		M		M		0
	(1)									
	$\vdots$									
z_i	(i)	0		1		1				s_i
	$\vdots$									
z_{m+j}	$(m + j)$	0		1				1		d_j
	$\vdots$									
	$(m + n)$									

TABLE 8.14 Row 0 of simplex tableau when simplex method is applied to transportation problem

Basic Variable	Eq.	Coefficient of: Z	$\cdots$	x_{ij}	$\cdots$	z_i	$\cdots$	z_{m+j}	$\cdots$	Right Side
Z	(0)	-1		$c_{ij} - u_i - v_j$		$M - u_i$		$M - v_j$		$-\sum_{i=1}^{m} s_i u_i - \sum_{j=1}^{n} d_j v_j$

Using the duality theory introduced in Chap. 6, another property of the u_i and v_j is that they are the *dual variables.*[3] If x_{ij} is a nonbasic variable, $c_{ij} - u_i - v_j$ is interpreted as the rate at which Z will change as x_{ij} is increased.

The Needed Information. To lay the groundwork for simplifying this setup, recall what information is needed by the simplex method. In the initialization, an initial BF solution must be obtained, which is done artificially by introducing artificial variables as the initial basic variables and setting them equal to s_i and d_j. The optimality test and step 1 of an iteration (selecting an entering basic variable) require knowing the current row 0, which is obtained by subtracting a certain multiple of another row from the preceding row 0. Step 2 (determining the leaving basic variable) must identify the basic variable that reaches zero first as the entering basic variable is increased, which is done by comparing the current coefficients of the entering basic variable and the corresponding right side. Step 3 must determine the new BF solution, which is found by subtracting certain multiples of one row from the other rows in the current simplex tableau.

Greatly Streamlined Ways of Obtaining This Information. Now, how does the *transportation simplex method* obtain the same information in much simpler ways? This story will unfold fully in the coming pages, but here are some preliminary answers.

First, *no artificial variables* are needed, because a simple and convenient procedure (with several variations) is available for constructing an initial BF solution.

Second, the current row 0 can be obtained *without using any other row* simply by calculating the current values of u_i and v_j directly. Since each basic variable must have a coefficient of zero in row 0, the current u_i and v_j are obtained by solving the set of equations

$$c_{ij} - u_i - v_j = 0 \qquad \text{for each } i \text{ and } j \text{ such that } x_{ij} \text{ is a basic variable.}$$

(We will illustrate this straightforward procedure later when discussing the optimality test for the transportation simplex method.) The special structure in Table 8.13 makes this convenient way of obtaining row 0 possible by yielding $c_{ij} - u_i - v_j$ as the coefficient of x_{ij} in Table 8.14.

Third, the leaving basic variable can be identified in a simple way without (explicitly) using the coefficients of the entering basic variable. The reason is that the special structure of the problem makes it easy to see how the solution must change as the entering basic variable is increased. As a result, the new BF solution also can be identified immediately *without any algebraic manipulations* on the rows of the simplex tableau. (You will see the details when we describe how the transportation simplex method performs an iteration.)

The grand conclusion is that *almost the entire simplex tableau* (and the work of maintaining it) *can be eliminated*! Besides the input data (the c_{ij}, s_i, and d_j values), the only

[3]It would be easier to recognize these variables as dual variables by relabeling all these variables as y_i and then changing all the signs in row 0 of Table 8.14 by converting the objective function back to its original minimization form.

information needed by the transportation simplex method is the current BF solution,[4] the current values of u_i and v_j, and the resulting values of $c_{ij} - u_i - v_j$ for nonbasic variables x_{ij}. When you solve a problem by hand, it is convenient to record this information for each iteration in a **transportation simplex tableau**, such as shown in Table 8.15. (Note carefully that the values of x_{ij} and $c_{ij} - u_i - v_j$ are distinguished in these tableaux by circling the former but not the latter.)

The Resulting Great Improvement in Efficiency. You can gain a fuller appreciation for the great difference in efficiency and convenience between the simplex and the transportation simplex methods by applying both to the same small problem (see Prob. 8.2-17). However, the difference becomes even more pronounced for large problems that must be solved on a computer. This pronounced difference is suggested somewhat by comparing the sizes of the simplex and the transportation simplex tableaux. Thus, for a transportation problem having m sources and n destinations, the simplex tableau would have $m + n + 1$ rows and $(m + 1)(n + 1)$ columns (excluding those to the left of the x_{ij} columns), and the transportation simplex tableau would have m rows and n columns (excluding the two extra informational rows and columns). Now try plugging in various values for m and n (for example, $m = 10$ and $n = 100$ would be a rather typical medium-size transportation problem), and note how the ratio of the number of cells in the simplex tableau to the number in the transportation simplex tableau increases as m and n increase.

Initialization

Recall that the objective of the initialization is to obtain an initial BF solution. Because all the functional constraints in the transportation problem are *equality* constraints, the simplex method would obtain this solution by introducing artificial variables and using them as the initial basic variables, as described in Sec. 4.6. The resulting basic solution

TABLE 8.15 Format of a transportation simplex tableau

		Destination					
		1	**2**	···	***n***	**Supply**	u_i
Source	1	c_{11}	c_{12}	···	c_{1n}	s_1	
	2	c_{21}	c_{22}	···	c_{2n}	s_2	
	⋮	···	···	···	···	⋮	
	m	c_{m1}	c_{m2}	···	c_{mn}	s_m	
Demand		d_1	d_2	···	d_n	$Z =$	
	v_j						

Additional information to be added to each cell:

If x_{ij} is a basic variable	*If x_{ij} is a nonbasic variable*
c_{ij}	c_{ij}
x_{ij}	$c_{ij} - u_i - v_j$

[4]Since nonbasic variables are automatically zero, the current BF solution is fully identified by recording just the values of the basic variables. We shall use this convention from now on.

actually is feasible only for a revised version of the problem, so a number of iterations are needed to drive these artificial variables to zero in order to reach the real BF solutions. The transportation simplex method bypasses all this by instead using a simpler procedure to directly construct a real BF solution on a transportation simplex tableau.

Before outlining this procedure, we need to point out that the number of basic variables in any basic solution of a transportation problem is one fewer than you might expect. Ordinarily, there is one basic variable for each functional constraint in a linear programming problem. For transportation problems with m sources and n destinations, the number of functional constraints is $m + n$. However,

Number of basic variables $= m + n - 1$.

The reason is that the functional constraints are equality constraints, and this set of $m + n$ equations has one *extra* (or *redundant*) equation that can be deleted without changing the feasible region; i.e., any one of the constraints is automatically satisfied whenever the other $m + n - 1$ constraints are satisfied. (This fact can be verified by showing that any supply constraint exactly equals the sum of the demand constraints minus the sum of the *other* supply constraints, and that any demand equation also can be reproduced by summing the supply equations and subtracting the other demand equations. See Prob. 8.2-19.) Therefore, any *BF solution* appears on a transportation simplex tableau with exactly $m + n - 1$ circled *nonnegative* allocations, where the sum of the allocations for each row or column equals its supply or demand.[5]

The procedure for constructing an initial BF solution selects the $m + n - 1$ basic variables one at a time. After each selection, a value that will satisfy one additional constraint (thereby eliminating that constraint's row or column from further consideration for providing allocations) is assigned to that variable. Thus, after $m + n - 1$ selections, an entire basic solution has been constructed in such a way as to satisfy all the constraints. A number of different criteria have been proposed for selecting the basic variables. We present and illustrate three of these criteria here, after outlining the general procedure.

General Procedure[6] for Constructing an Initial BF Solution. To begin, all source rows and destination columns of the transportation simplex tableau are initially under consideration for providing a basic variable (allocation).

1. From the rows and columns still under consideration, select the next basic variable (allocation) according to some criterion.
2. Make that allocation large enough to exactly use up the remaining supply in its row or the remaining demand in its column (whichever is smaller).
3. Eliminate that row or column (whichever had the smaller remaining supply or demand) from further consideration. (If the row and column have the same remaining supply and demand, then arbitrarily select the *row* as the one to be eliminated. The column will be used later to provide a *degenerate* basic variable, i.e., a circled allocation of zero.)
4. If only one row or only one column remains under consideration, then the procedure is completed by selecting every *remaining* variable (i.e., those variables that were neither previously selected to be basic nor eliminated from consideration by eliminating

[5]However, note that any feasible solution with $m + n - 1$ nonzero variables is *not necessarily* a basic solution because it might be the weighted average of two or more degenerate BF solutions (i.e., BF solutions having some basic variables equal to zero). We need not be concerned about mislabeling such solutions as being basic, however, because the transportation simplex method constructs only legitimate BF solutions.

[6]In Sec. 4.1 we pointed out that the simplex method is an example of the algorithms (systematic solution procedures) so prevalent in OR work. Note that this procedure also is an algorithm, where each successive execution of the (four) steps constitutes an iteration.

their row or column) associated with that row or column to be basic with the only feasible allocation. Otherwise, return to step 1.

Alternative Criteria for Step 1

1. *Northwest corner rule:* Begin by selecting x_{11} (that is, start in the northwest corner of the transportation simplex tableau). Thereafter, if x_{ij} was the last basic variable selected, then next select $x_{i,j+1}$ (that is, move one column to the *right*) if source i has any supply remaining. Otherwise, next select $x_{i+1,j}$ (that is, move one row *down*).

Example. To make this description more concrete, we now illustrate the general procedure on the Metro Water District problem (see Table 8.12) with the northwest corner rule being used in step 1. Because $m = 4$ and $n = 5$ in this case, the procedure would find an initial BF solution having $m + n - 1 = 8$ basic variables.

As shown in Table 8.16, the first allocation is $x_{11} = 30$, which exactly uses up the demand in column 1 (and eliminates this column from further consideration). This first iteration leaves a supply of 20 remaining in row 1, so next select $x_{1,1+1} = x_{12}$ to be a basic variable. Because this supply is no larger than the demand of 20 in column 2, all of it is allocated, $x_{12} = 20$, and this row is eliminated from further consideration. (Row 1 is chosen for elimination rather than column 2 because of the parenthetical instruction in step 3.) Therefore, select $x_{1+1,2} = x_{22}$ next. Because the remaining demand of 0 in column 2 is less than the supply of 60 in row 2, allocate $x_{22} = 0$ and eliminate column 2.

Continuing in this manner, we eventually obtain the entire *initial BF solution* shown in Table 8.16, where the circled numbers are the values of the basic variables ($x_{11} = 30, \ldots, x_{45} = 50$) and all the other variables (x_{13}, etc.) are nonbasic variables equal to zero. Arrows have been added to show the order in which the basic variables (allocations) were selected. The value of Z for this solution is

$$Z = 16(30) + 16(20) + \cdots + 0(50) = 2{,}470 + 10M.$$

2. *Vogel's approximation method:* For each row and column remaining under consideration, calculate its **difference,** which is defined as *the arithmetic difference between the smallest and next-to-the-smallest unit cost c_{ij} still remaining in that row or column.* (If two unit costs tie for being the smallest remaining in a row or column, then

TABLE 8.16 Initial BF solution from the Northwest Corner Rule

		Destination						
		1	**2**	**3**	**4**	**5**	**Supply**	u_i
Source	1	16 30	16 20	13	22	17	50	
	2	14	14 0	13 60	19	15	60	
	3	19	19	20 10	23 30	M 10	50	
	4(D)	M	0	M	0	0 50	50	
Demand		30	20	70	30	60	$Z = 2{,}470 + 10M$	
	v_j							

the *difference* is 0.) In that row or column having the *largest difference,* select the variable having the *smallest remaining unit cost.* (Ties for the largest difference, or for the smallest remaining unit cost, may be broken arbitrarily.)

Example. Now let us apply the general procedure to the Metro Water District problem by using the criterion for Vogel's approximation method to select the next basic variable in step 1. With this criterion, it is more convenient to work with parameter tables (rather than with complete transportation simplex tableaux), beginning with the one shown in Table 8.12. At each iteration, after the difference for every row and column remaining under consideration is calculated and displayed, the largest difference is circled and the smallest unit cost in its row or column is enclosed in a box. The resulting selection (and value) of the variable having this unit cost as the next basic variable is indicated in the lower right-hand corner of the current table, along with the row or column thereby being eliminated from further consideration (see steps 2 and 3 of the general procedure). The table for the next iteration is exactly the same except for deleting this row or column and subtracting the last allocation from its supply or demand (whichever remains).

Applying this procedure to the Metro Water District problem yields the sequence of parameter tables shown in Table 8.17, where the resulting initial BF solution consists of the eight basic variables (allocations) given in the lower right-hand corner of the respective parameter tables.

This example illustrates two relatively subtle features of the general procedure that warrant special attention. First, note that the final iteration selects *three* variables (x_{31}, x_{32}, and x_{33}) to become basic instead of the single selection made at the other iterations. The reason is that only *one* row (row 3) remains under consideration at this point. Therefore, step 4 of the general procedure says to select *every* remaining variable associated with row 3 to be basic.

Second, note that the allocation of $x_{23} = 20$ at the next-to-last iteration exhausts *both* the remaining supply in its row *and* the remaining demand in its column. However, rather than eliminate both the row and column from further consideration, step 3 says to eliminate *only the row,* saving the column to provide a *degenerate* basic variable later. Column 3 is, in fact, used for just this purpose at the final iteration when $x_{33} = 0$ is selected as one of the basic variables. For another illustration of this same phenomenon, see Table 8.16 where the allocation of $x_{12} = 20$ results in eliminating only row 1, so that column 2 is saved to provide a degenerate basic variable, $x_{22} = 0$, at the next iteration.

Although a zero allocation might seem irrelevant, it actually plays an important role. You will see soon that the transportation simplex method must know *all* $m + n - 1$ basic variables, including those with value zero, in the current BF solution.

3. *Russell's approximation method:* For each source row i remaining under consideration, determine its $\overline{u}_i$, which is the largest unit cost c_{ij} still remaining in that row. For each destination column j remaining under consideration, determine its $\overline{v}_j$, which is the largest unit cost c_{ij} still remaining in that column. For each variable x_{ij} not previously selected in these rows and columns, calculate $\Delta_{ij} = c_{ij} - \overline{u}_i - \overline{v}_j$. Select the variable having the *largest* (in absolute terms) *negative* value of Δ_{ij}. (Ties may be broken arbitrarily.)

Example. Using the criterion for Russell's approximation method in step 1, we again apply the general procedure to the Metro Water District problem (see Table 8.12). The results, including the sequence of basic variables (allocations), are shown in Table 8.18.

At iteration 1, the largest unit cost in row 1 is $\overline{u}_1 = 22$, the largest in column 1 is $\overline{v}_1 = M$, and so forth. Thus,

$$\Delta_{11} = c_{11} - \overline{u}_1 - \overline{v}_1 = 16 - 22 - M = -6 - M.$$

TABLE 8.17 Initial BF solution from Vogel's approximation method

		Destination						
		1	**2**	**3**	**4**	**5**	**Supply**	**Row Difference**
Source	1	16	16	13	22	17	50	3
	2	14	14	13	19	15	60	1
	3	19	19	20	23	*M*	50	0
	4(*D*)	*M*	0	*M*	$\boxed{0}$	0	50	0
Demand		30	20	70	30	60	Select $x_{44} = 30$	
Column difference		2	14	0	19	15	Eliminate column 4	

		Destination					
		1	**2**	**3**	**5**	**Supply**	**Row Difference**
Source	1	16	16	13	17	50	3
	2	14	14	13	15	60	1
	3	19	19	20	*M*	50	0
	4(*D*)	*M*	0	*M*	$\boxed{0}$	20	0
Demand		30	20	70	60	Select $x_{45} = 20$	
Column difference		2	14	0	15	Eliminate row 4(*D*)	

		Destination					
		1	**2**	**3**	**5**	**Supply**	**Row Difference**
Source	1	16	16	$\boxed{13}$	17	50	3
	2	14	14	13	15	60	1
	3	19	19	20	*M*	50	0
Demand		30	20	70	40	Select $x_{13} = 50$	
Column difference		2	2	0	2	Eliminate row 1	

		Destination					
		1	**2**	**3**	**5**	**Supply**	**Row Difference**
Source	2	14	14	13	$\boxed{15}$	60	1
	3	19	19	20	*M*	50	0
Demand		30	20	20	40	Select $x_{25} = 40$	
Column difference		5	5	7	$M - 15$	Eliminate column 5	

		Destination				
		1	**2**	**3**	**Supply**	**Row Difference**
Source	2	14	14	$\boxed{13}$	20	1
	3	19	19	20	50	0
Demand		30	20	20	Select $x_{23} = 20$	
Column difference		5	5	7	Eliminate row 2	

		Destination				
		1	**2**	**3**	**Supply**	
Source	3	19	19	20	50	
Demand		30	20	0	Select $x_{31} = 30$ $x_{32} = 20$ $x_{33} = 0$	$Z = 2{,}460$

Calculating all the Δ_{ij} values for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$ shows that $\Delta_{45} = 0 - 2M$ has the largest negative value, so $x_{45} = 50$ is selected as the first basic variable (allocation). This allocation exactly uses up the supply in row 4, so this row is eliminated from further consideration.

Note that eliminating this row changes $\bar{v}_1$ and $\bar{v}_3$ for the next iteration. Therefore, the second iteration requires recalculating the Δ_{ij} with $j = 1, 3$ as well as eliminating $i = 4$. The largest negative value now is

$$\Delta_{15} = 17 - 22 - M = -5 - M,$$

so $x_{15} = 10$ becomes the second basic variable (allocation), eliminating column 5 from further consideration.

The subsequent iterations proceed similarly, but you may want to test your understanding by verifying the remaining allocations given in Table 8.18. As with the other procedures in this (and other) section(s), you should find your IOR Tutorial useful for doing the calculations involved and illuminating the approach. (See the interactive procedure for finding an initial BF solution.)

Comparison of Alternative Criteria for Step 1. Now let us compare these three criteria for selecting the next basic variable. The main virtue of the northwest corner rule is that it is quick and easy. However, because it pays no attention to unit costs c_{ij}, usually the solution obtained will be far from optimal. (Note in Table 8.16 that $x_{35} = 10$ even though $c_{35} = M$.) Expending a little more effort to find a good initial BF solution might greatly reduce the number of iterations then required by the transportation simplex method to reach an optimal solution (see Probs. 8.2-7 and 8.2-9). Finding such a solution is the objective of the other two criteria.

Vogel's approximation method has been a popular criterion for many years,[7] partially because it is relatively easy to implement by hand. Because the *difference* represents the minimum extra unit cost incurred by failing to make an allocation to the cell having the smallest unit cost in that row or column, this criterion does take costs into account in an effective way.

Russell's approximation method provides another excellent criterion[8] that is still quick to implement on a computer (but not manually). Although it is unclear as to which is more

TABLE 8.18 Initial BF solution from Russell's approximation method

Iteration	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	$\bar{u}_4$	$\bar{v}_1$	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_4$	$\bar{v}_5$	Largest Negative Δ_{ij}	Allocation
1	22	19	M	M	M	19	M	23	M	$\Delta_{45} = -2M$	$x_{45} = 50$
2	22	19	M		19	19	20	23	M	$\Delta_{15} = -5 - M$	$x_{15} = 10$
3	22	19	23		19	19	20	23		$\Delta_{13} = -29$	$x_{13} = 40$
4		19	23		19	19	20	23		$\Delta_{23} = -26$	$x_{23} = 30$
5		19	23		19	19		23		$\Delta_{21} = -24$*	$x_{21} = 30$
6										Irrelevant	$x_{31} = 0$
											$x_{32} = 20$
											$x_{34} = 30$
											$Z = 2{,}570$

*Tie with $\Delta_{22} = -24$ broken arbitrarily.

[7]N. V. Reinfeld and W. R. Vogel: *Mathematical Programming,* Prentice-Hall, Englewood Cliffs, NJ, 1958.

[8]E. J. Russell: "Extension of Dantzig's Algorithm to Finding an Initial Near-Optimal Basis for the Transportation Problem," *Operations Research,* **17:** 187–191, 1969.

effective *on average,* this criterion *frequently* does obtain a better solution than Vogel's. (For the example, Vogel's approximation method happened to find the optimal solution with Z = 2,460, whereas Russell's misses slightly with Z = 2,570.) For a large problem, it may be worthwhile to apply both criteria and then use the better solution to start the iterations of the transportation simplex method.

One distinct advantage of Russell's approximation method is that it is patterned directly after step 1 for the transportation simplex method (as you will see soon), which somewhat simplifies the overall computer code. In particular, the $\overline{u}_i$ and $\overline{v}_j$ values have been defined in such a way that the relative values of the $c_{ij} - \overline{u}_i - \overline{v}_j$ *estimate* the relative values of $c_{ij} - u_i - v_j$ that will be obtained when the transportation simplex method reaches an optimal solution.

We now shall use the initial BF solution obtained in Table 8.18 by Russell's approximation method to illustrate the remainder of the transportation simplex method. Thus, our *initial transportation simplex tableau* (before we solve for u_i and v_j) is shown in Table 8.19.

The next step is to check whether this initial solution is optimal by applying the *optimality test.*

Optimality Test

Using the notation of Table 8.14, we can reduce the standard optimality test for the simplex method (see Sec. 4.3) to the following for the transportation problem:

> **Optimality test:** A BF solution is optimal if and only if $c_{ij} - u_i - v_j \geq 0$ for every (i, j) such that x_{ij} is nonbasic.[9]

Thus, the only work required by the optimality test is the derivation of the values of u_i and v_j for the current BF solution and then the calculation of these $c_{ij} - u_i - v_j$, as described below.

TABLE 8.19 Initial transportation simplex tableau (before we obtain $c_{ij} - u_i - v_j$) from Russell's approximation method

Iteration 0		**Destination 1**	**2**	**3**	**4**	**5**	**Supply**	u_i
Source	1	16	16	13 (40)	22	17 (10)	50	
	2	14 (30)	14	13 (30)	19	15	60	
	3	19 (0)	19 (20)	20	23 (30)	M	50	
	4(D)	M	0	M	0	0 (50)	50	
Demand		30	20	70	30	60	$Z = 2{,}570$	
	v_j							

[9]The one exception is that two or more equivalent degenerate BF solutions (i.e., identical solutions having different degenerate basic variables equal to zero) can be optimal with only some of these basic solutions satisfying the optimality test. This exception is illustrated later in the example (see the identical solutions in the last two tableaux of Table 8.23, where only the latter solution satisfies the criterion for optimality).

Since $c_{ij} - u_i - v_j$ is required to be zero if x_{ij} is a basic variable, u_i and v_j satisfy the set of equations

$$c_{ij} = u_i + v_j \qquad \text{for each } (i, j) \text{ such that } x_{ij} \text{ is basic.}$$

There are $m + n - 1$ basic variables, and so there are $m + n - 1$ of these equations. Since the number of unknowns (the u_i and v_j) is $m + n$, one of these variables can be assigned a value arbitrarily without violating the equations. The choice of this one variable and its value does not affect the value of any $c_{ij} - u_i - v_j$, even when x_{ij} is nonbasic, so the only (minor) difference it makes is in the ease of solving these equations. A convenient choice for this purpose is to select the u_i that has the *largest number of allocations in its row* (break any tie arbitrarily) and to assign to it the value zero. Because of the simple structure of these equations, it is then very simple to solve for the remaining variables algebraically.

To demonstrate, we give each equation that corresponds to a basic variable in our initial BF solution.

$$\begin{array}{lll}
x_{31}: & 19 = u_3 + v_1. & \text{Set } u_3 = 0, \text{ so } v_1 = 19, \\
x_{32}: & 19 = u_3 + v_2. & \qquad\qquad\quad v_2 = 19, \\
x_{34}: & 23 = u_3 + v_4. & \qquad\qquad\quad v_4 = 23. \\
x_{21}: & 14 = u_2 + v_1. & \text{Know } v_1 = 19, \text{ so } u_2 = -5. \\
x_{23}: & 13 = u_2 + v_3. & \text{Know } u_2 = -5, \text{ so } v_3 = 18. \\
x_{13}: & 13 = u_1 + v_3. & \text{Know } v_3 = 18, \text{ so } u_1 = -5. \\
x_{15}: & 17 = u_1 + v_5. & \text{Know } u_1 = -5, \text{ so } v_5 = 22. \\
x_{45}: & 0 = u_4 + v_5. & \text{Know } v_5 = 22, \text{ so } u_4 = -22.
\end{array}$$

Setting $u_3 = 0$ (since row 3 of Table 8.19 has the largest number of allocations—3) and moving down the equations one at a time immediately give the derivation of values for the unknowns shown to the right of the equations. (Note that this derivation of the u_i and v_j values depends on which x_{ij} variables are *basic variables* in the current BF solution, so this derivation will need to be repeated each time a new BF solution is obtained.)

Once you get the hang of it, you probably will find it even more convenient to solve these equations without writing them down by working directly on the transportation simplex tableau. Thus, in Table 8.19 you begin by writing in the value $u_3 = 0$ and then picking out the circled allocations (x_{31}, x_{32}, x_{34}) in that row. For each one you set $v_j = c_{3j}$ and then look for circled allocations (except in row 3) in these columns (x_{21}). Mentally calculate $u_2 = c_{21} - v_1$, pick out x_{23}, set $v_3 = c_{23} - u_2$, and so on until you have filled in all the values for u_i and v_j. (Try it.) Then calculate and fill in the value of $c_{ij} - u_i - v_j$ for each nonbasic variable x_{ij} (that is, for each cell without a circled allocation), and you will have the completed initial transportation simplex tableau shown in Table 8.20.

We are now in a position to apply the optimality test by checking the values of $c_{ij} - u_i - v_j$ given in Table 8.20. Because two of these values ($c_{25} - u_2 - v_5 = -2$ and $c_{44} - u_4 - v_4 = -1$) are negative, we conclude that the current BF solution is not optimal. Therefore, the transportation simplex method must next go to an iteration to find a better BF solution.

An Iteration

As with the full-fledged simplex method, an iteration for this streamlined version must determine an entering basic variable (step 1), a leaving basic variable (step 2), and then identify the resulting new BF solution (step 3).

TABLE 8.20 Completed initial transportation simplex tableau

Iteration 0		Destination 1	2	3	4	5	Supply	u_i
Source	1	16 +2	16 +2	13 (40)	22 +4	17 (10)	50	−5
	2	14 (30)	14 0	13 (30)	19 +1	15 −2	60	−5
	3	19 (0)	19 (20)	20 +2	23 (30)	M $M-22$	50	0
	4(*D*)	M $M+3$	0 +3	M $M+4$	0 −1	0 (50)	50	−22
Demand		30	20	70	30	60	$Z = 2{,}570$	
	v_j	19	19	18	23	22		

Step 1: Find the Entering Basic Variable. Since $c_{ij} - u_i - v_j$ represents the rate at which the objective function will change as the nonbasic variable x_{ij} is increased, the entering basic variable must have a *negative* $c_{ij} - u_i - v_j$ value to decrease the total cost Z. Thus, the candidates in Table 8.20 are x_{25} and x_{44}. To choose between the candidates, select the one having the larger (in absolute terms) negative value of $c_{ij} - u_i - v_j$ to be the entering basic variable, which is x_{25} in this case.

Step 2: Find the Leaving Basic Variable. Increasing the entering basic variable from zero sets off a *chain reaction* of compensating changes in other basic variables (allocations), in order to continue satisfying the supply and demand constraints. The first basic variable to be decreased to zero then becomes the leaving basic variable.

With x_{25} as the entering basic variable, the chain reaction in Table 8.20 is the relatively simple one summarized in Table 8.21. (We shall always indicate the entering basic variable by placing a boxed plus sign in the center of its cell while leaving the corresponding value of $c_{ij} - u_i - v_j$ in the lower right-hand corner of this cell.) Increasing x_{25} by some amount requires decreasing x_{15} by the same amount to restore the demand of 60 in column 5. This change then requires increasing x_{13} by this same amount to restore the

TABLE 8.21 Part of initial transportation simplex tableau showing the chain reaction caused by increasing the entering basic variable x_{25}

		Destination	3	4	5	Supply
Source	1	…	13 (40) +	22 +4	17 (10) −	50
	2	…	13 (30) −	19 +1	15 $\boxed{+}$ −2	60
		…	…	…	…	
Demand			70	30	60	

supply of 50 in row 1. This change then requires decreasing x_{23} by this amount to restore the demand of 70 in column 3. This decrease in x_{23} successfully completes the chain reaction because it also restores the supply of 60 in row 2. (Equivalently, we could have started the chain reaction by restoring this supply in row 2 with the decrease in x_{23}, and then the chain reaction would continue with the increase in x_{13} and decrease in x_{15}.)

The net result is that cells (2, 5) and (1, 3) become **recipient cells,** each receiving its additional allocation from one of the **donor cells,** (1, 5) and (2, 3). (These cells are indicated in Table 8.21 by the plus and minus signs.) Note that cell (1, 5) had to be the donor cell for column 5 rather than cell (4, 5), because cell (4, 5) would have no recipient cell in row 4 to continue the chain reaction. [Similarly, if the chain reaction had been started in row 2 instead, cell (2, 1) could not be the donor cell for this row because the chain reaction could not then be completed successfully after necessarily choosing cell (3, 1) as the next recipient cell and either cell (3, 2) or (3, 4) as its donor cell.] Also note that, except for the entering basic variable, *all* recipient cells and donor cells in the chain reaction must correspond to *basic* variables in the current BF solution.

Each donor cell decreases its allocation by exactly the same amount as the entering basic variable (and other recipient cells) is increased. Therefore, the donor cell that starts with the smallest allocation—cell (1, 5) in this case (since $10 < 30$ in Table 8.21)—must reach a zero allocation first as the entering basic variable x_{25} is increased. Thus, x_{15} becomes the leaving basic variable.

In general, there always is just *one* chain reaction (in either direction) that can be completed successfully to maintain feasibility when the entering basic variable is increased from zero. This chain reaction can be identified by selecting from the cells having a basic variable: first the donor cell in the *column* having the entering basic variable, then the recipient cell in the row having this donor cell, then the donor cell in the column having this recipient cell, and so on until the chain reaction yields a donor cell in the *row* having the entering basic variable. When a column or row has more than one additional basic variable cell, it may be necessary to trace them all further to see which one must be selected to be the donor or recipient cell. (All but this one eventually will reach a dead end in a row or column having no additional basic variable cell.) After the chain reaction is identified, *the donor cell having the smallest allocation automatically provides the leaving basic variable*. (In the case of a tie for the donor cell having the smallest allocation, any one can be chosen arbitrarily to provide the leaving basic variable.)

Step 3: Find the New BF Solution. The *new BF solution* is identified simply by adding the value of the leaving basic variable (before any change) to the allocation for each recipient cell and subtracting *this same amount* from the allocation for each donor cell. In Table 8.21 the value of the leaving basic variable x_{15} is 10, so the portion of the transportation simplex tableau in this table changes as shown in Table 8.22 for the new solution. (Since x_{15} is nonbasic in the new solution, its new allocation of zero is no longer shown in this new tableau.)

We can now highlight a useful interpretation of the $c_{ij} - u_i - v_j$ quantities derived during the optimality test. Because of the shift of 10 allocation units from the donor cells to the recipient cells (shown in Tables 8.21 and 8.22), the total cost changes by

$$\Delta Z = 10(15 - 17 + 13 - 13) = 10(-2) = 10(c_{25} - u_2 - v_5).$$

Thus, the effect of increasing the entering basic variable x_{25} from zero has been a cost change at the rate of -2 per unit increase in x_{25}. This is precisely what the value of $c_{25} - u_2 - v_5 = -2$ in Table 8.20 indicates would happen. In fact, another (but less efficient) way of deriving $c_{ij} - u_i - v_j$ for each nonbasic variable x_{ij} is to identify the chain reaction caused by increasing this variable from 0 to 1 and then to calculate the resulting

TABLE 8.22 Part of second transportation simplex tableau showing the changes in the BF solution

		Destination				
			3	4	5	Supply
Source	1	…	13 (50)	22	17	50
	2	…	13 (20)	19	15 (10)	60
		…	…	…	…	
Demand			70	30	60	

cost change. This intuitive interpretation sometimes is useful for checking calculations during the optimality test.

Before completing the solution of the Metro Water District problem, we now summarize the rules for the transportation simplex method.

Summary of the Transportation Simplex Method

Initialization: Construct an initial BF solution by the procedure outlined earlier in this section. Go to the optimality test.

Optimality test: Derive u_i and v_j by selecting the row having the largest number of allocations, setting its $u_i = 0$, and then solving the set of equations $c_{ij} = u_i + v_j$ for each (i, j) such that x_{ij} is basic. If $c_{ij} - u_i - v_j \geq 0$ for every (i, j) such that x_{ij} is *nonbasic,* then the current solution is optimal, so stop. Otherwise, go to an iteration.

Iteration:

1. Determine the entering basic variable: Select the nonbasic variable x_{ij} having the *largest* (in absolute terms) *negative* value of $c_{ij} - u_i - v_j$.
2. Determine the leaving basic variable: Identify the chain reaction required to retain feasibility when the entering basic variable is increased. From the donor cells, select the basic variable having the *smallest* value.
3. Determine the new BF solution: Add the value of the leaving basic variable to the allocation for each recipient cell. Subtract this value from the allocation for each donor cell.

Continuing to apply this procedure to the Metro Water District problem yields the complete set of transportation simplex tableaux shown in Table 8.23. Since all the $c_{ij} - u_i - v_j$ values are nonnegative in the fourth tableau, the optimality test identifies the set of allocations in this tableau as being optimal, which concludes the algorithm.

It would be good practice for you to derive the values of u_i and v_j given in the second, third, and fourth tableaux. Try doing this by working directly on the tableaux. Also check out the chain reactions in the second and third tableaux, which are somewhat more complicated than the one you have seen in Table 8.21.

Special Features of This Example

Note three special points that are illustrated by this example. First, the initial BF solution is *degenerate* because the basic variable $x_{31} = 0$. However, this degenerate basic variable causes no complication, because cell (3, 1) becomes a *recipient cell* in the second tableau, which increases x_{31} to a value greater than zero.

TABLE 8.23 Complete set of transportation simplex tableaux for the Metro Water District problem

Iteration 0		Destination 1	2	3	4	5	Supply	u_i
Source	1	16 / +2	16 / +2	13 / 40+	22 / +4	17 / 10−	50	−5
	2	14 / 30	14 / 0	13 / 30−	19 / +1	15 / + −2	60	−5
	3	19 / 0	19 / 20	20 / +2	23 / 30	*M* / *M* − 22	50	0
	4(*D*)	*M* / *M* + 3	0 / +3	*M* / *M* + 4	0 / −1	0 / 50	50	−22
Demand		30	20	70	30	60	$Z = 2{,}570$	
	v_j	19	19	18	23	22		

Iteration 1		Destination 1	2	3	4	5	Supply	u_i
Source	1	16 / +2	16 / +2	13 / 50	22 / +4	17 / +2	50	−5
	2	14 / 30−	14 / 0	13 / 20	19 / +1	15 / 10+	60	−5
	3	19 / 0+	19 / 20	20 / +2	23 / 30−	*M* / *M* − 20	50	0
	4(*D*)	*M* / *M* + 1	0 / +1	*M* / *M* + 2	0 / + −3	0 / 50−	50	−20
Demand		30	20	70	30	60	$Z = 2{,}550$	
	v_j	19	19	18	23	20		

Iteration 2		Destination 1	2	3	4	5	Supply	u_i
Source	1	16 / +5	16 / +5	13 / 50	22 / +7	17 / +2	50	−8
	2	14 / +3	14 / +3	13 / 20−	19 / +4	15 / 40+	60	−8
	3	19 / 30	19 / 20	20 / + −1	23 / 0−	*M* / *M* − 23	50	0
	4(*D*)	*M* / *M* + 4	0 / +4	*M* / *M* + 2	0 / 30+	0 / 20−	50	−23
Demand		30	20	70	30	60	$Z = 2{,}460$	
	v_j	19	19	21	23	23		

TABLE 8.23 (*Continued*)

Iteration 3		Destination 1	2	3	4	5	Supply	u_i
Source	1	16 / +4	16 / +4	13 / (50)	22 / +7	17 / +2	50	−7
	2	14 / +2	14 / +2	13 / (20)	19 / +4	15 / (40)	60	−7
	3	19 / (30)	19 / (20)	20 / (0)	23 / +1	M / $M - 22$	50	0
	4(*D*)	M / $M + 3$	0 / +3	M / $M + 2$	0 / (30)	0 / (20)	50	−22
Demand		30	20	70	30	60	$Z = 2{,}460$	
	v_j	19	19	20	22	22		

Second, another degenerate basic variable (x_{34}) arises in the third tableau because the basic variables for *two* donor cells in the second tableau, cells (2, 1) and (3, 4), *tie* for having the smallest value (30). (This tie is broken arbitrarily by selecting x_{21} as the leaving basic variable; if x_{34} had been selected instead, then x_{21} would have become the degenerate basic variable.) This degenerate basic variable does appear to create a complication subsequently, because cell (3, 4) becomes a *donor cell* in the third tableau but has nothing to donate! Fortunately, such an event actually gives no cause for concern. Since zero is the amount to be added to or subtracted from the allocations for the recipient and donor cells, these allocations do not change. However, the degenerate basic variable does become the leaving basic variable, so it is replaced by the entering basic variable as the circled allocation of zero in the fourth tableau. This change in the set of basic variables changes the values of u_i and v_j. Therefore, if any of the $c_{ij} - u_i - v_j$ had been negative in the fourth tableau, the algorithm would have gone on to make *real* changes in the allocations (whenever all donor cells have nondegenerate basic variables).

Third, because none of the $c_{ij} - u_i - v_j$ turned out to be negative in the fourth tableau, the equivalent set of allocations in the third tableau is optimal also. Thus, the algorithm executed one more iteration than was necessary. This extra iteration is a flaw that occasionally arises in both the transportation simplex method and the simplex method because of degeneracy, but it is not sufficiently serious to warrant any adjustments to these algorithms.

If you would like to see additional (smaller) examples of the application of the transportation simplex method, two are available. One is the demonstration provided for the transportation problem area in your OR Tutor. In addition, the Worked Examples section of the book's website includes **another example** of this type. Also provided in your IOR Tutorial are both an interactive procedure and an automatic procedure for the transportation simplex method.

Now that you have studied the transportation simplex method, you are in a position to check for yourself how the algorithm actually provides a proof of the *integer solutions property* presented in Sec. 8.1. Problem 8.2-20 helps to guide you through the reasoning.

8.3 THE ASSIGNMENT PROBLEM

The **assignment problem** is a special type of linear programming problem where **assignees** are being assigned to perform **tasks.** For example, the assignees might be employees who need to be given work assignments. Assigning people to jobs is a common application of the assignment problem.[10] However, the assignees need not be people. They also could be machines, or vehicles, or plants, or even time slots to be assigned tasks. The first example below involves machines being assigned to locations, so the tasks in this case simply involve holding a machine. A subsequent example involves plants being assigned products to be produced.

To fit the definition of an assignment problem, these kinds of applications need to be formulated in a way that satisfies the following assumptions.

1. The number of assignees and the number of tasks are the same. (This number is denoted by n.)
2. Each assignee is to be assigned to exactly *one* task.
3. Each task is to be performed by exactly *one* assignee.
4. There is a cost c_{ij} associated with assignee i ($i = 1, 2, \ldots, n$) performing task j ($j = 1, 2, \ldots, n$).
5. The objective is to determine how all n assignments should be made to minimize the total cost.

Any problem satisfying all these assumptions can be solved extremely efficiently by algorithms designed specifically for assignment problems.

The first three assumptions are fairly restrictive. Many potential applications do not quite satisfy these assumptions. However, it often is possible to reformulate the problem to make it fit. For example, *dummy assignees* or *dummy tasks* frequently can be used for this purpose. We illustrate these formulation techniques in the examples.

Prototype Example

The JOB SHOP COMPANY has purchased three new machines of different types. There are four available locations in the shop where a machine could be installed. Some of these locations are more desirable than others for particular machines because of their proximity to work centers that will have a heavy work flow to and from these machines. (There will be no work flow *between* the new machines.) Therefore, the objective is to assign the new machines to the available locations to minimize the total cost of materials handling. The estimated cost in dollars per hour of materials handling involving each of the machines is given in Table 8.24 for the respective locations. Location 2 is not considered suitable for machine 2, so no cost is given for this case.

To formulate this problem as an assignment problem, we must introduce a *dummy machine* for the extra location. Also, an extremely large cost M should be attached to the assignment of machine 2 to location 2 to prevent this assignment in the optimal solution. The resulting assignment problem *cost table* is shown in Table 8.25. This cost table contains all the necessary data for solving the problem. The optimal solution is to assign machine 1 to location 4, machine 2 to location 3, and machine 3 to location 1, for a total cost of \$29 per hour. The dummy machine is assigned to location 2, so this location is available for some future real machine.

[10]For example, see L. J. LeBlanc, D. Randels, Jr., and T. K. Swann: "Heery International's Spreadsheet Optimization Model for Assigning Managers to Construction Projects," *Interfaces,* **30**(6): 95–106, Nov.–Dec. 2000. Page 98 of this article also cites seven other applications of the assignment problem.

TABLE 8.24 Materials-handling cost data ($) for Job Shop Co.

		Location			
		1	**2**	**3**	**4**
	1	13	16	12	11
Machine	2	15	—	13	20
	3	5	7	10	6

TABLE 8.25 Cost table for the Job Shop Co. assignment problem

		Task (Location)			
		1	**2**	**3**	**4**
	1	13	16	12	11
Assignee	2	15	M	13	20
(Machine)	3	5	7	10	6
	4(*D*)	0	0	0	0

We shall discuss how this solution is obtained after we formulate the mathematical model for the general assignment problem.

The Assignment Problem Model

The mathematical model for the assignment problem uses the following decision variables:

$$x_{ij} = \begin{cases} 1 & \text{if assignee } i \text{ performs task } j, \\ 0 & \text{if not,} \end{cases}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Thus, each x_{ij} is a *binary variable* (it has value 0 or 1). As discussed at length in the chapter on integer programming (Chap. 11), binary variables are important in OR for representing *yes/no decisions.* In this case, the yes/no decision is: Should assignee i perform task j?

By letting Z denote the total cost, the assignment problem model is

$$\text{Minimize} \qquad Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij},$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \text{for } i = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \text{for } j = 1, 2, \ldots, n,$$

and

$$x_{ij} \geq 0, \qquad \text{for all } i \text{ and } j$$
$$(x_{ij} \text{ binary,} \qquad \text{for all } i \text{ and } j).$$

The first set of functional constraints specifies that each assignee is to perform exactly one task, whereas the second set requires each task to be performed by exactly one assignee. If we delete the parenthetical restriction that the x_{ij} be binary, the model clearly is a special type of linear programming problem and so can be readily solved. Fortunately, for reasons about to unfold, we *can* delete this restriction. (This deletion is the reason that the assignment problem appears in this chapter rather than in the integer programming chapter.)

Now compare this model (without the binary restriction) with the transportation problem model presented in the third subsection of Sec. 8.1 (including Table 8.6). Note how similar their structures are. In fact, the assignment problem is just a special type of transportation problem where the *sources* now are *assignees* and the *destinations* now are *tasks* and where

Number of sources m = number of destinations n,

Every supply $s_i = 1$,

Every demand $d_j = 1$.

Now focus on the **integer solutions property** in the subsection on the transportation problem model. Because s_i and d_j are integers ($= 1$) now, this property implies that *every BF solution* (including an optimal one) is an *integer* solution for an assignment problem. The functional constraints of the assignment problem model prevent any variable from being greater than 1, and the nonnegativity constraints prevent values less than 0. Therefore, by deleting the binary restriction to enable us to solve an assignment problem as a linear programming problem, the resulting BF solutions obtained (including the final optimal solution) *automatically* will satisfy the binary restriction anyway.

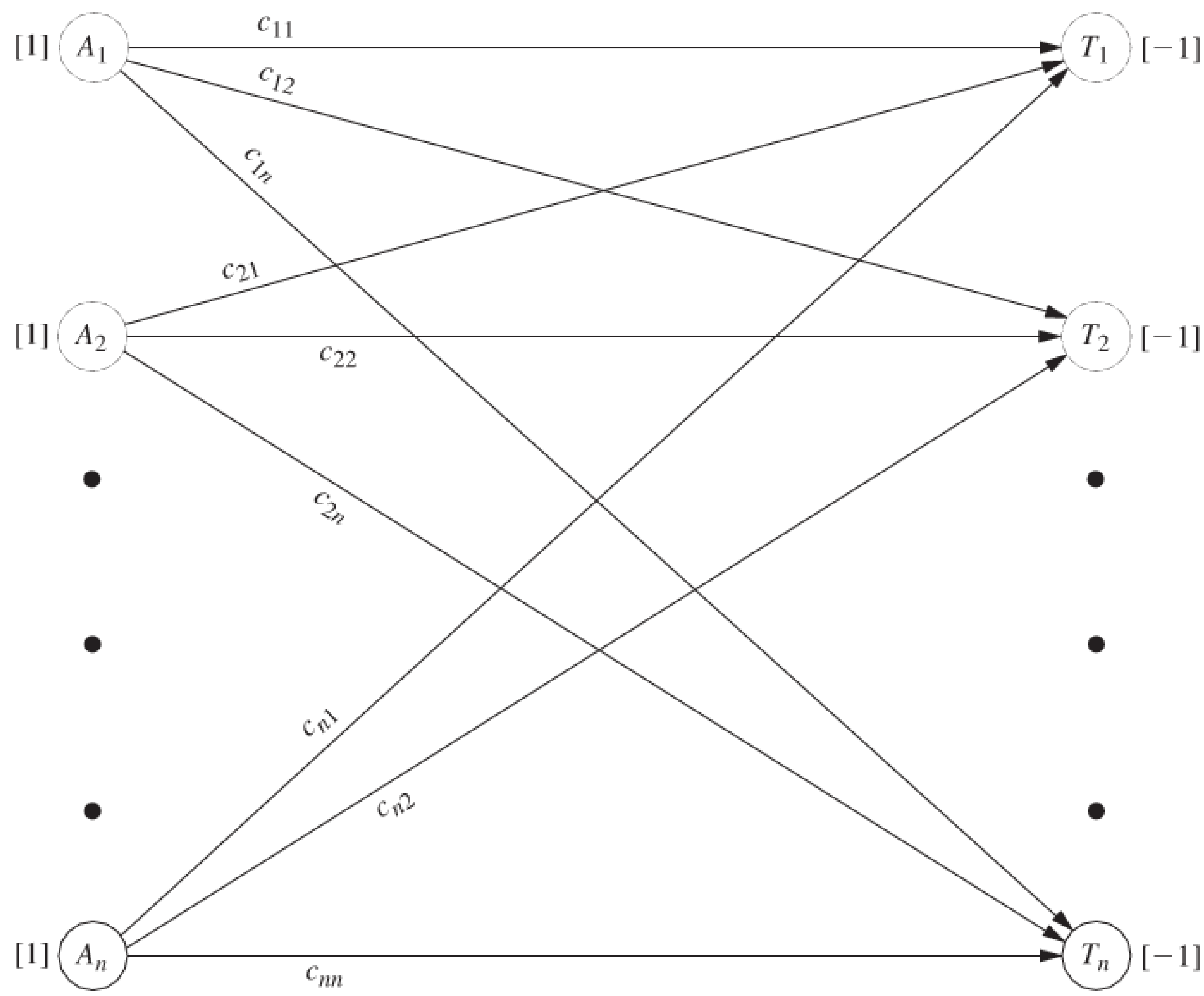
Just as the transportation problem has a network representation (see Fig. 8.3), the assignment problem can be depicted in a very similar way, as shown in Fig. 8.5. The first column now lists the n assignees and the second column the n tasks. Each number in a square bracket indicates the number of assignees being provided at that location in the network, so the values are automatically 1 on the left, whereas the values of -1 on the right indicate that each task is using up one assignee.

For any particular assignment problem, practitioners normally do not bother writing out the full mathematical model. It is simpler to formulate the problem by filling out a cost table (e.g., Table 8.25), including identifying the assignees and tasks, since this table contains all the essential data in a far more compact form.

Problems occasionally arise that do not quite fit the model for an assignment problem because certain assignees will be assigned to more than one task. In this case, the problem can be reformulated to fit the model by splitting each such assignee into separate (but identical) new assignees where each new assignee will be assigned to exactly one task. (Table 8.29 will illustrate this for a subsequent example.) Similarly, if a task is to be performed by multiple assignees, that task can be split into separate (but identical) new tasks where each new task is to be performed by exactly one assignee according to the reformulated model. The Worked Examples section of the book's website provides **another example** that illustrates both cases and the resulting reformulation to fit the model for an assignment problem. An alternative formulation as a transportation problem also is shown.

Solution Procedures for Assignment Problems

Alternative solution procedures are available for solving assignment problems. Problems that aren't much larger than the Job Shop Co. example can be solved very quickly by the



■ **FIGURE 8.5**
Network representation of the assignment problem.

general simplex method, so it may be convenient to simply use a basic software package (such as Excel and its Solver) that only employs this method. If this were done for the Job Shop Co. problem, it would not have been necessary to add the dummy machine to Table 8.25 to make it fit the assignment problem model. The constraints on the number of machines assigned to each location would be expressed instead as

$$\sum_{i=1}^{3} x_{ij} \leq 1 \qquad \text{for } j = 1, 2, 3, 4.$$

As shown in the Excel files for this chapter, a spreadsheet formulation for this example would be very similar to the formulation for a transportation problem displayed in Fig. 8.4 except now all the supplies and demands would be 1 and the demand constraints would be ≤ 1 instead of $= 1$.

However, large assignment problems can be solved much faster by using more specialized solution procedures, so we recommend using such a procedure instead of the general simplex method for big problems.

Because the assignment problem is a special type of transportation problem, one convenient and relatively fast way to solve any particular assignment problem is to apply the transportation simplex method described in Sec. 8.2. This approach requires converting the cost table to a parameter table for the equivalent transportation problem, as shown in Table 8.26*a*.

For example, Table 8.26*b* shows the parameter table for the Job Shop Co. problem that is obtained from the cost table of Table 8.25. When the transportation simplex method is applied to this transportation problem formulation, the resulting optimal solution has

TABLE 8.26 Parameter table for the assignment problem formulated as a transportation problem, illustrated by the Job Shop Co. example

(a) General Case

		Cost per Unit Distributed				
		Destination				
		1	**2**	⋯	***n***	**Supply**
Source	1	c_{11}	c_{12}	⋯	c_{1n}	1
	2	c_{21}	c_{22}	⋯	c_{2n}	1
	⋮	⋯	⋯	⋯	⋯	⋮
	$m = n$	c_{n1}	c_{n2}	⋯	c_{nn}	1
Demand		1	1	⋯	1	

(b) Job Shop Co. Example

		Cost per Unit Distributed				
		Destination (Location)				
		1	**2**	**3**	**4**	**Supply**
Source (Machine)	1	13	16	12	11	1
	2	15	M	13	20	1
	3	5	7	10	6	1
	4(*D*)	0	0	0	0	1
Demand		1	1	1	1	

basic variables $x_{13} = 0$, $x_{14} = 1$, $x_{23} = 1$, $x_{31} = 1$, $x_{41} = 0$, $x_{42} = 1$, $x_{43} = 0$. (You are asked to verify this solution in Prob. 8.3-6.). The degenerate basic variables ($x_{ij} = 0$) and the assignment for the dummy machine ($x_{42} = 1$) do not mean anything for the original problem, so the real assignments are machine 1 to location 4, machine 2 to location 3, and machine 3 to location 1.

It is no coincidence that this optimal solution provided by the transportation simplex method has so many degenerate basic variables. For any assignment problem with n assignments to be made, the transportation problem formulation shown in Table 8.26*a* has $m = n$, that is, both the number of sources (m) and the number of destinations (n) in this formulation equal the number of assignments (n). Transportation problems in general have $m + n - 1$ basic variables (allocations), so every BF solution for this particular kind of transportation problem has $2n - 1$ basic variables, but exactly n of these x_{ij} equal 1 (corresponding to the n assignments being made). Therefore, since all the variables are binary variables, there always are $n - 1$ degenerate basic variables ($x_{ij} = 0$). As discussed at the end of Sec. 8.2, degenerate basic variables do not cause any major complication in the execution of the algorithm. However, they do frequently cause *wasted iterations,* where nothing changes (same allocations) except for the labeling of which allocations of zero correspond to degenerate basic variables rather than nonbasic variables. These wasted iterations are a major drawback to applying the transportation simplex method in this kind of situation, where there *always* are so many degenerate basic variables.

Another drawback of the transportation simplex method here is that it is purely a *general-purpose* algorithm for solving all transportation problems. Therefore, it does nothing to exploit the additional special structure in this special type of transportation problem ($m = n$, every $s_i = 1$, and every $d_j = 1$). Fortunately, specialized algorithms have been developed to fully streamline the procedure for solving just assignment problems. These algorithms operate directly on the cost table and do not bother with degenerate basic variables. When a computer code is available for one of these algorithms, it generally should be used in preference to the transportation simplex method, especially for really big problems.[11]

Section 8.4 describes one of these specialized algorithms (called the *Hungarian algorithm*) for solving only assignment problems very efficiently.

[11]For an article comparing various algorithms for the assignment problem, see J. L. Kennington and Z. Wang: "An Empirical Analysis of the Dense Assignment Problem: Sequential and Parallel Implementations," *ORSA Journal on Computing,* **3:** 299–306, 1991.

Your IOR Tutorial includes both an interactive procedure and an automatic procedure for applying this algorithm.

Example—Assigning Products to Plants

The BETTER PRODUCTS COMPANY has decided to initiate the production of four new products, using three plants that currently have excess production capacity. The products require a comparable production effort per unit, so the available production capacity of the plants is measured by the number of units of any product that can be produced per day, as given in the rightmost column of Table 8.27. The bottom row gives the required production rate per day to meet projected sales. Each plant can produce any of these products, *except* that Plant 2 *cannot* produce product 3. However, the variable costs per unit of each product differ from plant to plant, as shown in the main body of Table 8.27.

Management now needs to make a decision on how to split up the production of the products among plants. Two kinds of options are available.

Option 1: Permit *product splitting,* where the same product is produced in more than one plant.
Option 2: Prohibit *product splitting.*

This second option imposes a constraint that can only increase the cost of an optimal solution based on Table 8.27. On the other hand, the key advantage of Option 2 is that it eliminates some *hidden costs* associated with product splitting that are not reflected in Table 8.27, including extra setup, distribution, and administration costs. Therefore, management wants both options analyzed before a final decision is made. For Option 2, management further specifies that every plant should be assigned at least one of the products.

We will formulate and solve the model for each option in turn, where Option 1 leads to a transportation problem and Option 2 leads to an assignment problem.

Formulation of Option 1. With product splitting permitted, Table 8.27 can be converted directly to a parameter table for a transportation problem. The plants become the sources, and the products become the destinations (or vice versa), so the supplies are the available production capacities and the demands are the required production rates. Only two changes need to be made in Table 8.27. First, because Plant 2 cannot produce product 3, such an allocation is prevented by assigning to it a huge unit cost of M. Second, the total capacity $(75 + 75 + 45 = 195)$ exceeds the total required production $(20 + 30 + 30 + 40 = 120)$, so a dummy destination with a demand of 75 is needed to balance these two quantities. The resulting parameter table is shown in Table 8.28.

The optimal solution for this transportation problem has basic variables (allocations) $x_{12} = 30$, $x_{13} = 30$, $x_{15} = 15$, $x_{24} = 15$, $x_{25} = 60$, $x_{31} = 20$, and $x_{34} = 25$, so

TABLE 8.27 Data for the Better Products Co. problem

		Unit Cost ($) for Product				Capacity Available
		1	2	3	4	
	1	41	27	28	24	75
Plant	2	40	29	—	23	75
	3	37	30	27	21	45
Production rate		20	30	30	40	

TABLE 8.28 Parameter table for the transportation problem formulation of Option 1 for the Better Products Co. problem

		Cost per Unit Distributed					
		Destination (Product)					
		1	2	3	4	5(D)	Supply
Source (Plant)	1	41	27	28	24	0	75
	2	40	29	M	23	0	75
	3	37	30	27	21	0	45
Demand		20	30	30	40	75	

Plant 1 produces all of products 2 and 3.
Plant 2 produces 37.5 percent of product 4.
Plant 3 produces 62.5 percent of product 4 and all of product 1.

The total cost is $Z = \$3{,}260$ per day.

Formulation of Option 2. Without product splitting, each product must be assigned to just one plant. Therefore, producing the products can be interpreted as the tasks for an assignment problem, where the plants are the assignees.

Management has specified that every plant should be assigned at least one of the products. There are more products (four) than plants (three), so one of the plants will need to be assigned two products. Plant 3 has only enough excess capacity to produce one product (see Table 8.27), so *either* Plant 1 or Plant 2 will take the extra product.

To make this assignment of an extra product possible within an assignment problem formulation, Plants 1 and 2 each are split into two assignees, as shown in Table 8.29.

The number of assignees (now five) must equal the number of tasks (now four), so a *dummy task* (product) is introduced into Table 8.29 as 5(D). The role of this dummy task is to provide the fictional second product to either Plant 1 or Plant 2, whichever one receives only one real product. There is no cost for producing a fictional product so, as usual, the cost entries for the dummy task are zero. The one exception is the entry of M in the last row of Table 8.29. The reason for M here is that Plant 3 must be assigned a real product (a choice of product 1, 2, 3, or 4), so the Big M method is needed to prevent the assignment of the fictional product to Plant 3 instead. (As in Table 8.28, M also is used to prevent the infeasible assignment of product 3 to Plant 2.)

The remaining cost entries in Table 8.29 are *not* the unit costs shown in Tables 8.27 or 8.28. Table 8.28 gives a transportation problem formulation (for Option 1), so unit costs

TABLE 8.29 Cost table for the assignment problem formulation of Option 2 for the Better Products Co. problem

		Task (Product)				
		1	2	3	4	5(D)
Assignee (Plant)	1*a*	820	810	840	960	0
	1*b*	820	810	840	960	0
	2*a*	800	870	M	920	0
	2*b*	800	870	M	920	0
	3	740	900	810	840	M

are appropriate there, but now we are formulating an assignment problem (for Option 2). For an assignment problem, the cost c_{ij} is the *total* cost associated with assignee i performing task j. For Table 8.29, the *total cost* (per day) for Plant i to produce product j is the unit cost of production *times* the number of units produced (per day), where these two quantities for the multiplication are given separately in Table 8.27. For example, consider the assignment of Plant 1 to product 1. By using the corresponding unit cost in Table 8.28 (\$41) and the corresponding demand (number of units produced per day) in Table 8.28 (20), we obtain

Cost of Plant 1 producing one unit of product 1	= \$41
Required (daily) production of product 1	= 20 units
Total (daily) cost of assigning plant 1 to product 1	= 20 (\$41)
	= \$820

so 820 is entered into Table 8.29 for the cost of either Assignee 1a or 1b performing Task 1.

The optimal solution for this assignment problem is as follows:

Plant 1 produces products 2 and 3.
Plant 2 produces product 1.
Plant 3 produces product 4.

Here the dummy assignment is given to Plant 2. The total cost is $Z = \$3{,}290$ per day.

As usual, one way to obtain this optimal solution is to convert the cost table of Table 8.29 to a parameter table for the equivalent transportation problem (see Table 8.26) and then apply the transportation simplex method. Because of the identical rows in Table 8.29, this approach can be streamlined by combining the five assignees into three sources with supplies 2, 2, and 1, respectively. (See Prob. 8.3-5.) This streamlining also decreases by two the number of degenerate basic variables in every BF solution. Therefore, even though this streamlined formulation no longer fits the format presented in Table 8.26*a* for an assignment problem, it is a more efficient formulation for applying the transportation simplex method.

Figure 8.6 shows how Excel and its Solver can be used to obtain this optimal solution, which is displayed in the changing cells Assignment (C19:F21) of the spreadsheet. Since the general simplex method is being used, there is no need to fit this formulation into the format for either the assignment problem or transportation problem model. Therefore, the formulation does not bother to split Plants 1 and 2 into two assignees each, or to add a dummy task. Instead, Plants 1 and 2 are given a supply of 2 each, and then $\leq$ signs are entered into cells H19 and H20 as well as into the corresponding constraints in the Solver dialogue box. There also is no need to include the Big M method to prohibit assigning product 3 to Plant 2 in cell E20, since this dialogue box includes the constraint that E20 = 0. The target cell TotalCost (I24) shows the total cost of \$3,290 per day.

Now look back and compare this solution to the one obtained for Option 1, which included the splitting of product 4 between Plants 2 and 3. The allocations are somewhat different for the two solutions, but the total daily costs are virtually the same (\$3,260 for Option 1 versus \$3,290 for Option 2). However, there are hidden costs associated with product splitting (including the cost of extra setup, distribution, and administration) that are not included in the objective function for Option 1. As with any application of OR, the mathematical model used can provide only an approximate representation of the total problem, so management needs to consider factors that cannot be incorporated into the model before it makes a final decision. In this case, after evaluating the disadvantages of product splitting, management decided to adopt the Option 2 solution.

	A	B	C	D	E	F	G	H	I
1	Better Products Co. Production Planning Problem (Option 2)								
2									
3		**Unit Cost**	Product 1	Product 2	Product 3	Product 4			
4		Plant 1	$41	$27	$28	$24			
5		Plant 2	$40	$29	-	$23			
6		Plant 3	$37	$30	$27	$21			
7									
8		Required Production	20	30	30	40			
9									
10									
11		**Cost ($/day)**	Product 1	Product 2	Product 3	Product 4			
12		Plant 1	$820	$810	$840	$960			
13		Plant 2	$800	$870	-	$920			
14		Plant 3	$740	$900	$810	$840			
15									
16									
17							Total		
18		**Assignment**	Product 1	Product 2	Product 3	Product 4	Assignments		Supply
19		Plant 1	0	1	1	0	2	<=	2
20		Plant 2	1	0	0	0	1	<=	2
21		Plant 3	0	0	0	1	1	=	1
22		Total Assigned	1	1	1	1			
23			=	=	=	=			Total Cost
24		Demand	1	1	1	1			$3,290

Solver Parameters

Set Target Cell: TotalCost

Equal To: ○ Max ◉ Min ○

By Changing Cells:

Assignment

Subject to the Constraints:

$E$20 = 0
$G$19:$G$20 <= $I$19:$I$20
$G$21 = $I$21
TotalAssigned = Demand

	B	C	D	E	F
11	**Cost ($/day)**	Product 1	Product 2	Product 3	Product 4
12	Plant 1	=C4*C$8	=D4*D$8	=E4*E$8	=F4*F$8
13	Plant 2	=C5*C$8	=D5*D$8	-	=F5*F$8
14	Plant 3	=C6*C$8	=D6*D$8	=E6*E$8	=F6*F$8

Solver Options

☑ Assume Linear Model
☑ Assume Non-Negative

	G
17	Total
18	Assignments
19	=SUM(C19:F19)
20	=SUM(C20:F20)
21	=SUM(C21:F21)

	B	C	D	E	F
22	Total Assigned	=SUM(C19:C21)	=SUM(D19:D21)	=SUM(E19:E21)	=SUM(F19:F21)

Range Name	Cells
Assignment	C19:F21
Cost	C12:F14
Demand	C24:F24
RequiredProduction	C8:F8
Supply	I19:I21
TotalAssigned	C22:F22
TotalAssignments	G19:G21
TotalCost	I24
UnitCost	C4:F6

	I
23	Total Cost
24	=SUMPRODUCT(Cost,Assignment)

FIGURE 8.6
A spreadsheet formulation of Option 2 for the Better Products Co. problem as a variant of an assignment problem. The target cell is TotalCost (I24) and the other output cells are Cost (C12:F14), TotalAssignments (G19:G21), and TotalAssigned (C22:F22), where the equations entered into these cells are shown below the spreadsheet. The values of 1 in the changing cells Assignment (C19:F21) display the optimal production plan obtained by the Solver.

8.4 A SPECIAL ALGORITHM FOR THE ASSIGNMENT PROBLEM

In Sec. 8.3, we pointed out that the transportation simplex method can be used to solve assignment problems but that a *specialized* algorithm designed for such problems should be more efficient. We now will describe a classic algorithm of this type. It is called the **Hungarian algorithm** (or *Hungarian method*) because it was developed by Hungarian mathematicians. We will focus just on the key ideas without filling in all the details needed for a complete computer implementation.

The Role of Equivalent Cost Tables

The algorithm operates directly on the *cost table* for the problem. More precisely, it converts the original cost table into a series of *equivalent* cost tables until it reaches one where an optimal solution is obvious. This final equivalent cost table is one consisting of only *positive* or *zero* elements where all the assignments can be made to the zero element positions. Since the total cost cannot be negative, this set of assignments with a zero total cost is clearly optimal. The question remaining is how to convert the original cost table into this form.

The key to this conversion is the fact that one can add or subtract any constant from every element of a row or column of the cost table without really changing the problem. That is, an optimal solution for the new cost table must also be optimal for the old one, and conversely.

Therefore, the algorithm begins by subtracting the smallest number in each row from every number in the row. This *row reduction* process will create an equivalent cost table that has a zero element in every row. If this cost table has any columns without a zero element, the next step is to perform a *column reduction* process by subtracting the smallest number in each such column from every number in the column.[12] The new equivalent cost table will have a zero element in every row and every column. If these zero elements provide a complete set of assignments, these assignments constitute an optimal solution and the algorithm if finished.

To illustrate, consider the cost table for the Job Shop Co. problem given in Table 8.25. To convert this cost table into an equivalent cost table, suppose that we begin the row reduction process by subtracting 11 from every element in row 1, which yields:

	1	2	3	4
1	2	5	1	0
2	15	M	13	20
3	5	7	10	6
4(D)	0	0	0	0

Since any feasible solution must have exactly one assignment in row 1, the total cost for the new table must always be exactly 11 less than for the old table. Hence, the solution which minimizes total cost for one table must also minimize total cost for the other.

Notice that, whereas the original cost table had only strictly positive elements in the first three rows, the new table has a zero element in row 1. Since the objective is to obtain enough strategically located zero elements to yield a complete set of assignments, this process should be continued on the other rows and columns. Negative elements are to be avoided, so the constant to be subtracted should be the minimum element in the row or column. Doing this for rows 2 and 3 yields the following equivalent cost table:

	1	2	3	4
1	2	5	1	$\boxed{0}$
2	2	M	$\boxed{0}$	7
3	$\boxed{0}$	2	5	1
4(D)	0	$\boxed{0}$	0	0

This cost table has all the zero elements required for a complete set of assignments, as shown by the four boxes, so these four assignments constitute an *optimal solution* (as

[12]The individual rows and columns actually can be reduced in any order, but starting with all the rows and then doing all the columns provides one systematic way of executing the algorithm.

claimed in Sec. 8.3 for this problem). The total cost for this optimal solution is seen in Table 8.25 to be $Z = 29$, which is just the sum of the numbers that have been subtracted from rows 1, 2, and 3.

Unfortunately, an optimal solution is not always obtained quite so easily, as we now illustrate with the assignment problem formulation of Option 2 for the Better Products Co. problem shown in Table 8.29.

Because this problem's cost table already has zero elements in every row but the last one, suppose we begin the process of converting to equivalent cost tables by subtracting the minimum element in each column from every entry in that column. The result is shown below.

	1	2	3	4	5(D)
1a	80	0	30	120	0
1b	80	0	30	120	0
2a	60	60	M	80	0
2b	60	60	M	80	0
3	0	90	0	0	M

Now *every* row and column has at least one zero element, but a complete set of assignments with zero elements is *not* possible this time. In fact, the maximum number of assignments that can be made in zero element positions in only 3. (Try it.) Therefore, one more idea must be implemented to finish solving this problem that was not needed for the first example.

The Creation of Additional Zero Elements

This idea involves a new way of creating *additional* positions with zero elements without creating any negative elements. Rather than subtracting a constant from a *single* row or column, we now add or subtract a constant from a *combination* of rows and columns.

This procedure begins by drawing a set of lines through some of the rows and columns in such a way as to *cover all the zeros*. This is done with a *minimum* number of lines, as shown in the next cost table.

	1	2	3	4	5(D)
1a	80	0	30	120	0
1b	80	0	30	120	0
2a	60	60	M	80	0
2b	60	60	M	80	0
3	0	90	0	0	M

Notice that the minimum element not crossed out is 30 in the two top positions in column 3. Therefore, subtracting 30 from every element in the entire table, i.e., from every row or from every column, will create a new zero element in these two positions. Then, in order to restore the previous zero elements and eliminate negative elements, we add 30 to each row or column with a line covering it—row 3 and columns 2 and 5(D). This yields the following equivalent cost table.

	1	2	3	4	5(D)
1a	50	0	0	90	0
1b	50	0	0	90	0
2a	30	60	M	50	0
2b	30	60	M	50	0
3	0	120	0	0	M

A shortcut for obtaining this cost table from the preceding one is to subtract 30 from just the elements without a line through them and then add 30 to every element that lies at the intersection of two lines.

Note that columns 1 and 4 in this new cost table have only a single zero element and they both are in the same row (row 3). Consequently, it now is possible to make four assignments to zero element positions, but still not five. (Try it.) In general, the minimum number of lines needed to cover all zeros equals the maximum number of assignments that can be made to zero element positions. Therefore, we repeat the above procedure, where four lines (the same number as the maximum number of assignments) now are the minimum needed to cover all zeros. One way of doing this is shown below.

	1	2	3	4	5(D)
1a	~~50~~	~~0~~	~~0~~	~~90~~	~~0~~
1b	~~50~~	~~0~~	~~0~~	~~90~~	~~0~~
2a	30	60	M	50	~~0~~
2b	30	60	M	50	~~0~~
3	~~0~~	~~120~~	~~0~~	~~0~~	~~M~~

The minimum element not covered by a line is again 30, where this number now appears in the first position in both rows 2a and 2b. Therefore, we subtract 30 from every *uncovered* element and add 30 to every *doubly covered* element (except for ignoring elements of M), which gives the following equivalent cost table.

	1	2	3	4	5(D)
1a	50	$\boxed{0}$	0	90	30
1b	50	0	$\boxed{0}$	90	30
2a	$\boxed{0}$	30	M	20	0
2b	0	30	M	20	$\boxed{0}$
3	0	120	0	$\boxed{0}$	M

This table actually has several ways of making a complete set of assignments to zero element positions (several optimal solutions), including the one shown by the five boxes. The resulting total cost is seen in Table 8.29 to be

$$Z = 810 + 840 + 800 + 0 + 840 = 3{,}290.$$

We now have illustrated the entire algorithm, as summarized below.

Summary of the Hungarian Algorithm

1. Subtract the smallest number in each row from every number in the row. (This is called *row reduction*.) Enter the results in a new table.
2. Subtract the smallest number in each column of the new table from every number in the column. (This is called *column reduction*.) Enter the results in another table.
3. Test whether an optimal set of assignments can be made. You do this by determining the minimum number of lines needed to cover (i.e., cross out) all zeros. Since this minimum number of lines equals the maximum number of assignments that can be made to zero element positions, if the minimum number of lines equals the number of rows, an optimal set of assignments is possible. (If you find that a complete set of assignments to zero element positions is not possible, this means that you did not reduce the number of lines covering all zeros down to the minimum number.) In that case, go to step 6. Otherwise go on to step 4.
4. If the number of lines is less than the number of rows, modify the table in the following way:
   a. Subtract the smallest uncovered number from every uncovered number in the table.
   b. Add the smallest uncovered number to the numbers at intersections of covering lines.
   c. Numbers crossed out but not at the intersections of cross-out lines carry over unchanged to the next table.
5. Repeat steps 3 and 4 until an optimal set of assignments is possible.
6. Make the assignments one at a time in positions that have zero elements. Begin with rows or columns that have only one zero. Since each row and each column needs to receive exactly one assignment, cross out both the row and the column involved after each assignment is made. Then move on to the rows and columns that are not yet crossed out to select the next assignment, with preference again given to any such row or column that has only one zero that is not crossed out. Continue until every row and every column has exactly one assignment and so has been crossed out. The complete set of assignments made in this way is an optimal solution for the problem.

Your IOR Tutorial provides an interactive procedure for applying this algorithm efficiently. An automatic procedure in included as well.

8.5 CONCLUSIONS

The linear programming model encompasses a wide variety of specific types of problems. The general simplex method is a powerful algorithm that can solve surprisingly large versions of any of these problems. However, some of these problem types have such simple formulations that they can be solved much more efficiently by *streamlined* algorithms that exploit their *special structure.* These streamlined algorithms can cut down tremendously on the computer time required for large problems, and they sometimes make it computationally feasible to solve huge problems. This is particularly true for the two types of linear programming problems studied in this chapter, namely, the transportation problem and the assignment problem. Both types have a number of common applications, so it is important to recognize them when they arise and to use the best available algorithms. These special-purpose algorithms are included in some linear programming software packages.

We shall reexamine the special structure of the transportation and assignment problems in Sec. 9.6. There we shall see that these problems are special cases of an important class of linear programming problems known as the *minimum cost flow problem.* This problem has the interpretation of minimizing the cost for the flow of goods through a network. A streamlined version of the simplex method called the *network simplex method* (described in Sec. 9.7) is widely used for solving this type of problem, including its various special cases.

A supplementary chapter (Chap. 23) on the book's website describes various additional special types of linear programming problems. One of these, called the *transshipment problem,* is a generalization of the transportation problem which allows shipments from any source to any destination to first go through intermediate transfer points. Since the transshipment problem also is a special case of the minimum cost flow problem, we will describe it further in Sec. 9.6.

Much research continues to be devoted to developing streamlined algorithms for special types of linear programming problems, including some not discussed here. At the same time, there is widespread interest in applying linear programming to optimize the operation of complicated large-scale systems. The resulting formulations usually have special structures that can be exploited. Being able to recognize and exploit special structures is an important factor in the successful application of linear programming.

SELECTED REFERENCES

1. Dantzig, G. B., and M. N. Thapa: *Linear Programming 1: Introduction,* Springer, New York, 1997, chap. 8.
2. Hall, R. W.: *Handbook of Transportation Science*, 2nd ed., Kluwer Academic Publishers (now Springer), Boston, 2003.
3. Hillier, F. S., and M. S. Hillier: *Introduction to Management Science: A Modeling and Case Studies Approach with Spreadsheets,* 3rd ed., McGraw-Hill/Irwin, Burr Ridge, IL, 2008, chap. 15.

LEARNING AIDS FOR THIS CHAPTER ON OUR WEBSITE (www.mhhe.com/hillier)

Worked Examples:

Examples for Chapter 8

A Demonstration Example in OR Tutor:

The Transportation Problem

Interactive Procedures in IOR Tutorial:

Enter or Revise a Transportation Problem
Find Initial Basic Feasible Solution—for Interactive Method
Solve Interactively by the Transportation Simplex Method
Solve an Assignment Problem Interactively

Automatic Procedures in IOR Tutorial:

Solve Automatically by the Transportation Simplex Method
Solve an Assignment Problem Automatically

An Excel Add-in:

Premium Solver for Education

"Ch. 8—Transsp. & Assignment" Files for Solving the Examples:

Excel Files
LINGO/LINDO File
MPL/CPLEX File

Glossary for Chapter 8

Supplement to this Chapter:

A Case Study with Many Transportation Problems

See Appendix 1 for documentation of the software.

PROBLEMS

The symbols to the left of some of the problems (or their parts) have the following meaning:

D: The demonstration example just listed may be helpful.
I: We suggest that you use the relevant interactive procedure in IOR Tutorial (the printout records your work).
C: Use the computer with any of the software options available to you (or as instructed by your instructor) to solve the problem.

An asterisk on the problem number indicates that at least a partial answer is given in the back of the book.

8.1-1. Read the referenced article that fully describes the OR study summarized in the application vignette in Sec. 8.1. Briefly describe how the model for the transportation problem was applied in this study. Then list the various financial and nonfinancial benefits that resulted from this study.

8.1-2. The Childfair Company has three plants producing child push chairs that are to be shipped to four distribution centers. Plants 1, 2, and 3 produce 12, 17, and 11 shipments per month, respectively. Each distribution center needs to receive 10 shipments per month. The distance from each plant to the respective distributing centers is given below:

		Distance			
		Distribution Center			
		1	2	3	4
Plant	1	800 miles	1,300 miles	400 miles	700 miles
	2	1,100 miles	1,400 miles	600 miles	1,000 miles
	3	600 miles	1,200 miles	800 miles	900 miles

The freight cost for each shipment is $100 plus 50 cents per mile.

How much should be shipped from each plant to each of the distribution centers to minimize the total shipping cost?

(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.
(b) Draw the network representation of this problem.
C **(c)** Obtain an optimal solution.

8.1-3.* Tom would like 3 pints of home brew today and an additional 4 pints of home brew tomorrow. Dick is willing to sell a maximum of 5 pints total at a price of $3.00 per pint today and $2.70 per pint tomorrow. Harry is willing to sell a maximum of 4 pints total at a price of $2.90 per pint today and $2.80 per pint tomorrow.

Tom wishes to know what his purchases should be to minimize his cost while satisfying his thirst requirements.

(a) Formulate a *linear programming* model for this problem, and construct the initial simplex tableau (see Chaps. 3 and 4).
(b) Formulate this problem as a *transportation problem* by constructing the appropriate parameter table.
C **(c)** Obtain an optimal solution.

8.1-4. The Versatech Corporation has decided to produce three new products. Five branch plants now have excess product capacity. The unit manufacturing cost of the first product would be $41, $39, $42, $38, and $39 in Plants 1, 2, 3, 4, and 5, respectively. The unit manufacturing cost of the second product would be $55, $51, $56, $52, and $53 in Plants 1, 2, 3, 4, and 5, respectively. The unit manufacturing cost of the third product would be $48, $45, and $50 in Plants 1, 2, and 3, respectively, whereas Plants 4 and 5 do not have the capability for producing this product. Sales forecasts indicate that 700, 1,000, and 900 units of products 1, 2, and 3, respectively, should be produced per day. Plants 1, 2, 3, 4, and 5 have the capacity to produce 400, 600, 400, 600, and 1,000 units daily, respectively, regardless of the product or combination of products involved. Assume that any plant having the capability and capacity to produce them can produce any combination of the products in any quantity.

Management wishes to know how to allocate the new products to the plants to minimize total manufacturing cost.

(a) Formulate this problem as a *transportation problem* by constructing the appropriate parameter table.

C **(b)** Obtain an optimal solution.

C **8.1-5.** Reconsider the P & T Co. problem presented in Sec. 8.1. You now learn that one or more of the shipping costs per truckload given in Table 8.2 may change slightly before shipments begin.

Use the Excel Solver to generate the Sensitivity Report for this problem. Use this report to determine the allowable range for each of the unit costs. What do these allowable ranges tell P & T management?

8.1-6. The Onenote Co. produces a single product at three plants for four customers. The three plants will produce 60, 80, and 40 units, respectively, during the next time period. The firm has made a commitment to sell 40 units to customer 1, 60 units to customer 2, and at least 20 units to customer 3. Both customers 3 and 4 also want to buy as many of the remaining units as possible. The net profit associated with shipping a unit from plant i for sale to customer j is given by the following table:

		Customer			
		1	2	3	4
Plant	1	$800	$700	$500	$200
	2	$500	$200	$100	$300
	3	$600	$400	$300	$500

Management wishes to know how many units to sell to customers 3 and 4 and how many units to ship from each of the plants to each of the customers to maximize profit.

(a) Formulate this problem as a transportation problem where the objective function is to be maximized by constructing the appropriate parameter table that gives unit profits.

(b) Now formulate this transportation problem with the usual objective of minimizing total cost by converting the parameter table from part (*a*) into one that gives unit costs instead of unit profits.

(c) Display the formulation in part (*a*) on an Excel spreadsheet.

C **(d)** Use this information and the Excel Solver to obtain an optimal solution.

C **(e)** Repeat parts (*c*) and (*d*) for the formulation in part (*b*). Compare the optimal solutions for the two formulations.

8.1-7. The Move-It Company has two plants producing forklift trucks that then are shipped to three distribution centers. The production costs are the same at the two plants, and the cost of shipping for each truck is shown for each combination of plant and distribution center:

		Distribution Center		
		1	2	3
Plant	A	$800	$700	$400
	B	$600	$800	$500

A total of 60 forklift trucks are produced and shipped per week. Each plant can produce and ship any amount up to a maximum of 50 trucks per week, so there is considerable flexibility on how to divide the total production between the two plants so as to reduce shipping costs. However, each distribution center must receive exactly 20 trucks per week.

Management's objective is to determine how many forklift trucks should be produced at each plant, and then what the overall shipping pattern should be to minimize total shipping cost.

(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.

(b) Display the transportation problem on an Excel spreadsheet.

C **(c)** Use the Excel Solver to obtain an optimal solution.

8.1-8. Redo Prob. 8.1-7 when any distribution center may receive any quantity between 10 and 30 forklift trucks per week in order to further reduce total shipping cost, provided only that the total shipped to all three distribution centers must still equal 60 trucks per week.

8.1-9. The MJK Manufacturing Company must produce two products in sufficient quantity to meet contracted sales in each of the next three months. The two products share the same production facilities, and each unit of both products requires the same amount of production capacity. The available production and storage facilities are changing month by month, so the production capacities, unit production costs, and unit storage costs vary by month. Therefore, it may be worthwhile to overproduce one or both products in some months and store them until needed.

For each of the three months, the second column of the following table gives the maximum number of units of the two products combined that can be produced on Regular Time (RT) and on Overtime (O). For each of the two products, the subsequent columns give (1) the number of units needed for the contracted sales, (2) the cost (in thousands of dollars) per unit produced on Regular Time, (3) the cost (in thousands of dollars) per unit produced on Overtime, and (4) the cost (in thousands of dollars) of storing each extra unit that is held over into the next month. In each case, the numbers for the two products are separated by a slash /, with the number for Product 1 on the left and the number for Product 2 on the right.

Month	Maximum Combined Production RT	Maximum Combined Production OT	Product 1/Product 2 Sales	Unit Cost of Production ($1,000's) RT	Unit Cost of Production ($1,000's) OT	Unit Cost of Storage ($1,000's)
1	10	3	5/3	15/16	18/20	1/2
2	8	2	3/5	17/15	20/18	2/1
3	10	3	4/4	19/17	22/22	

The production manager wants a schedule developed for the number of units of each of the two products to be produced on Regular Time and (if Regular Time production capacity is used up) on Overtime in each of the three months. The objective is to minimize the total of the production and storage costs while meeting the contracted sales for each month. There is no initial inventory, and no final inventory is desired after the three months.

(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.

C **(b)** Obtain an optimal solution.

8.2-1. Consider the transportation problem having the following parameter table:

		Destination 1	Destination 2	Destination 3	Supply
Source	1	15	9	13	7
	2	11	*M*	17	5
	3	9	11	9	3
Demand		7	3	5	

(a) Use Vogel's approximation method manually (don't use the interactive procedure in IOR Tutorial) to select the first basic variable for an initial BF solution.

(b) Use Russell's approximation method manually to select the first basic variable for an initial BF solution.

(c) Use the northwest corner rule manually to construct a complete initial BF solution.

D,I **8.2-2.*** Consider the transportation problem having the following parameter table:

		Destination 1	2	3	4	5	Supply
Source	1	2	4	6	5	7	4
	2	7	6	3	*M*	4	6
	3	8	7	5	2	5	6
	4	0	0	0	0	0	4
Demand		4	4	2	5	5	

Use each of the following criteria to obtain an initial BF solution. Compare the values of the objective function for these solutions.

(a) Northwest corner rule.

(b) Vogel's approximation method.

(c) Russell's approximation method.

D,I **8.2-3.** Consider the transportation problem having the following parameter table:

		Destination 1	2	3	4	5	6	Supply
Source	1	13	10	22	29	18	0	5
	2	14	13	16	21	*M*	0	6
	3	3	0	*M*	11	6	0	7
	4	18	9	19	23	11	0	4
	5	30	24	34	36	28	0	3
Demand		3	5	4	5	6	2	

Use each of the following criteria to obtain an initial BF solution. Compare the values of the objective function for these solutions.

(a) Northwest corner rule.

(b) Vogel's approximation method.

(c) Russell's approximation method.

8.2-4. Consider the transportation problem having the following parameter table:

		Destination 1	2	3	4	Supply
Source	1	7	4	1	4	1
	2	4	6	7	2	1
	3	8	5	4	6	1
	4	6	7	6	3	1
Demand		1	1	1	1	

(a) Notice that this problem has three special characteristics: (1) number of sources = number of destinations, (2) each supply = 1, and (3) each demand = 1. Transportation problems with these characteristics are of a special type called the assignment problem (as described in Sec. 8.3). Use the integer solutions property to explain why this type of transportation problem can be interpreted as assigning sources to destinations on a one-to-one basis.

(b) How many basic variables are there in every BF solution? How many of these are degenerate basic variables (= 0)?

D,I **(c)** Use the northwest corner rule to obtain an initial BF solution.

I **(d)** Construct an initial BF solution by applying the general procedure for the initialization step of the transportation simplex method. However, rather than using one of the three criteria for step 1 presented in Sec. 8.2, use the minimum cost criterion given next for selecting the next basic variable. (With the corresponding interactive routine in your OR Courseware, choose the *Northwest Corner Rule,* since this choice actually allows the use of any criterion.)

> **Minimum cost criterion:** From among the rows and columns still under consideration, select the variable x_{ij} having the smallest unit cost c_{ij} to be the next basic variable. (Ties may be broken arbitrarily.)

D,I **(e)** Starting with the initial BF solution from part (*c*), interactively apply the transportation simplex method to obtain an optimal solution.

8.2-5. Consider the prototype example for the transportation problem (the P & T Co. problem) presented at the beginning of Sec. 8.1. Verify that the solution given there actually is optimal by applying just the *optimality test* portion of the transportation simplex method to this solution.

8.2-6. Consider the transportation problem having the following parameter table:

		Destination					
		1	2	3	4	5	Supply
Source	1	8	6	3	7	5	20
	2	5	M	8	4	7	30
	3	6	3	9	6	8	30
	4(D)	0	0	0	0	0	20
Demand		25	25	20	10	20	

After several iterations of the transportation simplex method, a BF solution is obtained that has the following basic variables: $x_{13} = 20$, $x_{21} = 25$, $x_{24} = 5$, $x_{32} = 25$, $x_{34} = 5$, $x_{42} = 0$, $x_{43} = 0$, $x_{45} = 20$. Continue the transportation simplex method for *two more* iterations by hand. After two iterations, state whether the solution is optimal and, if so, why.

D,I **8.2-7.*** Consider the transportation problem having the following parameter table:

		Destination				
		1	2	3	4	Supply
Source	1	3	7	6	4	5
	2	2	4	3	2	2
	3	4	3	8	5	3
Demand		3	3	2	2	

Use each of the following criteria to obtain an initial BF solution. In each case, interactively apply the transportation simplex method, starting with this initial solution, to obtain an optimal solution. Compare the resulting number of iterations for the transportation simplex method.

(a) Northwest corner rule.
(b) Vogel's approximation method.
(c) Russell's approximation method.

D,I **8.2-8.** The Cost-Less Corp. supplies its four retail outlets from its four plants. The shipping cost per shipment from each plant to each retail outlet is given below.

		Unit Shipping Cost Retail Outlet			
		1	2	3	4
Plant	1	\$700	\$800	\$500	\$200
	2	\$200	\$900	\$100	\$400
	3	\$400	\$500	\$300	\$100
	4	\$200	\$100	\$400	\$300

Plants 1, 2, 3, and 4 make 10, 20, 20, and 10 shipments per month, respectively. Retail outlets 1, 2, 3, and 4 need to receive 20, 10, 10, and 20 shipments per month, respectively.

The distribution manager, Randy Smith, now wants to determine the best plan for how many shipments to send from each plant to the respective retail outlets each month. Randy's objective is to minimize the total shipping cost.

(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.
(b) Use the northwest corner rule to construct an initial BF solution.
(c) Starting with the initial basic solution from part (*b*), interactively apply the transportation simplex method to obtain an optimal solution.

8.2-9. The Energetic Company needs to make plans for the energy systems for a new building.

The energy needs in the building fall into three categories: (1) electricity, (2) heating water, and (3) heating space in the building. The daily requirements for these three categories (all measured in the same units) are

Electricity	30 units
Water heating	20 units
Space heating	50 units

The three possible sources of energy to meet these needs are electricity, natural gas, and a solar heating unit that can be installed on the roof. The size of the roof limits the largest possible solar heater to 40 units, but there is no limit to the electricity and natural gas available. Electricity needs can be met only by purchasing electricity (at a cost of \$50 per unit). Both other energy needs can be met by any source or combination of sources. The unit costs are

	Electricity	Natural Gas	Solar Heater
Water heating	\$150	\$110	\$70
Space heating	\$140	\$100	\$90

The objective is to minimize the total cost of meeting the energy needs.

(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.

D,I **(b)** Use the northwest corner rule to obtain an initial BF solution for this problem.

D,I **(c)** Starting with the initial BF solution from part (*b*), interactively apply the transportation simplex method to obtain an optimal solution.

D,I **(d)** Use Vogel's approximation method to obtain an initial BF solution for this problem.

D,I **(e)** Starting with the initial BF solution from part (*d*), interactively apply the transportation simplex method to obtain an optimal solution.

I **(f)** Use Russell's approximation method to obtain an initial BF solution for this problem.

D,I **(g)** Starting with the initial BF solution obtained from part (*f*), interactively apply the transportation simplex method to obtain an optimal solution. Compare the number of iterations required by the transportation simplex method here and in parts (*c*) and (*e*).

D,I **8.2-10.*** Interactively apply the transportation simplex method to solve the Northern Airplane Co. production scheduling problem as it is formulated in Table 8.9.

D,I **8.2-11.*** Reconsider Prob. 8.1-2.

(a) Use the northwest corner rule to obtain an initial BF solution.

(b) Starting with the initial BF solution from part (*a*), interactively apply the transportation simplex method to obtain an optimal solution.

D,I **8.2-12.** Reconsider Prob. 8.1-3*b*. Starting with the northwest corner rule, interactively apply the transportation simplex method to obtain an optimal solution for this problem.

D,I **8.2-13.** Reconsider Prob. 8.1-4. Starting with the northwest corner rule, interactively apply the transportation simplex method to obtain an optimal solution for this problem.

D,I **8.2-14.** Reconsider Prob. 8.1-6. Starting with Russell's approximation method, interactively apply the transportation simplex method to obtain an optimal solution for this problem.

8.2-15. Reconsider the transportation problem formulated in Prob. 8.1-7*a*.

D,I **(a)** Use each of the three criteria presented in Sec. 8.2 to obtain an initial BF solution, and time how long you spend for each one. Compare both these times and the values of the objective function for these solutions.

C **(b)** Obtain an optimal solution for this problem. For each of the three initial BF solutions obtained in part (*a*), calculate the percentage by which its objective function value exceeds the optimal one.

D,I **(c)** For each of the three initial BF solutions obtained in part (*a*), interactively apply the transportation simplex method to obtain (and verify) an optimal solution. Time how long you spend in each of the three cases. Compare both these times and the number of iterations needed to reach an optimal solution.

8.2-16. Follow the instructions of Prob. 8.2-15 for the transportation problem formulated in Prob. 8.1-7*a*.

8.2-17. Consider the transportation problem having the following parameter table:

		Destination		
		1	2	Supply
Source	1	8	5	4
	2	6	4	2
Demand		3	3	

(a) Using your choice of a criterion from Sec. 8.2 for obtaining the initial BF solution, solve this problem manually by the transportation simplex method. (Keep track of your time.)

(b) Reformulate this problem as a general linear programming problem, and then solve it manually by the *simplex method*. Keep track of how long this takes you, and contrast it with the computation time for part (*a*).

8.2-18. Consider the Northern Airplane Co. production scheduling problem presented in Sec. 8.1 (see Table 8.7). Formulate this problem as a general linear programming problem by letting the decision variables be x_j = number of jet engines to be produced in month j ($j = 1, 2, 3, 4$). Construct the initial simplex tableau for this formulation, and then contrast the size (number of rows and columns) of this tableau and the corresponding tableaux used to solve the transportation problem formulation of the problem (see Table 8.9).

8.2-19. Consider the general linear programming formulation of the transportation problem (see Table 8.6). Verify the claim in Sec. 8.2 that the set of $(m + n)$ functional constraint equations (m supply constraints and n demand constraints) has one *redundant* equation; i.e., any one equation can be reproduced from a linear combination of the other $(m + n - 1)$ equations.

8.2-20. When you deal with a transportation problem where the supply and demand quantities have *integer* values, explain why the steps of the transportation simplex method guarantee that all the basic variables (allocations) in the BF solutions obtained must have integer values. Begin with why this occurs with the initialization step when the general procedure for constructing an *initial* BF solution is used (regardless of the criterion for selecting the next basic variable). Then given a *current* BF solution that is integer, next explain why Step 3 of an iteration must obtain a new BF solution that also is integer. Finally, explain how the initialization step can be used to construct *any* initial BF solution, so the transportation simplex method actually gives a proof of the integer solutions property presented in Sec. 8.1.

8.2-21. A contractor, Susan Meyer, has to haul gravel to three building sites. She can purchase as much as 18 tons at a gravel pit in the north of the city and 14 tons at one in the south. She needs 10, 5, and 10 tons at sites 1, 2, and 3, respectively. The purchase price per ton at each gravel pit and the hauling cost per ton are given in the table below.

	Hauling Cost per Ton at Site			
Pit	**1**	**2**	**3**	**Price per Ton**
North	\$100	\$190	\$160	\$300
South	\$180	\$110	\$140	\$420

Susan wishes to determine how much to haul from each pit to each site to minimize the total cost for purchasing and hauling gravel.

(a) Formulate a linear programming model for this problem. Using the Big M method, construct the initial simplex tableau ready to apply the simplex method (but do not actually solve).

(b) Now formulate this problem as a transportation problem by constructing the appropriate parameter table. Compare the size of this table (and the corresponding transportation simplex tableau) used by the transportation simplex method with the size of the simplex tableaux from part (*a*) that would be needed by the simplex method.

D **(c)** Susan Meyer notices that she can supply sites 1 and 2 completely from the north pit and site 3 completely from the south pit. Use the optimality test (but no iterations) of the transportation simplex method to check whether the corresponding BF solution is optimal.

D,I **(d)** Starting with the northwest corner rule, interactively apply the transportation simplex method to solve the problem as formulated in part (*b*).

(e) As usual, let c_{ij} denote the unit cost associated with source i and destination j as given in the parameter table constructed in part (*b*). For the optimal solution obtained in part (*d*), suppose that the value of c_{ij} for each basic variable x_{ij} is fixed at the value given in the parameter table, but that the value of c_{ij} for each nonbasic variable x_{ij} possibly can be altered through bargaining because the site manager wants to pick up the business. Use sensitivity analysis to determine the *allowable range* for each of the latter c_{ij}, and explain how this information is useful to the contractor.

C **8.2-22.** Consider the transportation problem formulation and solution of the Metro Water District problem presented in Secs. 8.1 and 8.2 (see Tables 8.12 and 8.23).

The numbers given in the parameter table are only estimates that may be somewhat inaccurate, so management now wishes to do some what-if analysis. Use the Excel Solver to generate the Sensitivity Report. Then use this report to address the following questions. (In each case, assume that the indicated change is the only change in the model.)

(a) Would the optimal solution in Table 8.23 remain optimal if the cost per acre foot of shipping Calorie River water to San Go were actually \$200 rather than \$230?

(b) Would this solution remain optimal if the cost per acre foot of shipping Sacron River water to Los Devils were actually \$160 rather than \$130?

(c) Must this solution remain optimal if the costs considered in parts (*a*) and (*b*) were simultaneously changed from their original values to \$215 and \$145, respectively?

(d) Suppose that the supply from the Sacron River and the demand at Hollyglass are decreased simultaneously by the same amount. Must the shadow prices for evaluating these changes remain valid if the decrease were 0.5 million acre feet?

8.2-23. Without generating the Sensitivity Report, adapt the sensitivity analysis procedure presented in Secs. 6.6 and 6.7 to conduct the sensitivity analysis specified in the four parts of Prob. 8.2-22.

8.3-1. Consider the assignment problem having the following cost table.

		Task			
		1	**2**	**3**	**4**
Assignee	A	8	6	5	7
	B	6	5	3	4
	C	7	8	4	6
	D	6	7	5	6

(a) Draw the network representation of this assignment problem.

(b) Formulate this problem as a transportation problem by constructing the appropriate parameter table.

(c) Display this formulation on an Excel spreadsheet.

C **(d)** Use the Excel Solver to obtain an optimal solution.

8.3-2. Four cargo ships will be used for shipping goods from one port to four other ports (labeled 1, 2, 3, 4). Any ship can be used for making any one of these four trips. However, because of differences in the ships and cargoes, the total cost of loading, transporting, and unloading the goods for the different ship-port combinations varies considerably, as shown in the following table:

		Port			
		1	**2**	**3**	**4**
Ship	1	\$500	\$400	\$600	\$700
	2	\$600	\$600	\$700	\$500
	3	\$700	\$500	\$700	\$600
	4	\$500	\$400	\$600	\$600

The objective is to assign the four ships to four different ports in such a way as to minimize the total cost for all four shipments.

(a) Describe how this problem fits into the general format for the assignment problem.

C **(b)** Obtain an optimal solution.

(c) Reformulate this problem as an equivalent transportation problem by constructing the appropriate parameter table.

D,I **(d)** Use the northwest corner rule to obtain an initial BF solution for the problem as formulated in part (*c*).

D,I **(e)** Starting with the initial BF solution from part (*d*), interactively apply the transportation simplex method to obtain an optimal set of assignments for the original problem.

D,I **(f)** Are there other optimal solutions in addition to the one obtained in part (*e*)? If so, use the transportation simplex method to identify them.

8.3-3. Reconsider Prob. 8.1-4. Suppose that the sales forecasts have been revised downward to 280, 400, and 350 units per day of products 1, 2, and 3, respectively, and that each plant now has the capacity to produce all that is required of any one product. Therefore, management has decided that each new product should be assigned to only one plant and that no plant should be assigned more than one product (so that three plants are each to be assigned one product, and two plants are to be assigned none). The objective is to make these assignments so as to minimize the *total* cost of producing these amounts of the three products.

(a) Formulate this problem as an assignment problem by constructing the appropriate cost table.

C **(b)** Obtain an optimal solution.

(c) Reformulate this assignment problem as an equivalent transportation problem by constructing the appropriate parameter table.

D,I **(d)** Starting with Vogel's approximation method, interactively apply the transportation simplex method to solve the problem as formulated in part (*c*).

8.3-4.* The coach of an age group swim team needs to assign swimmers to a 200-yard medley relay team to send to the Junior Olympics. Since most of his best swimmers are very fast in more than one stroke, it is not clear which swimmer should be assigned to each of the four strokes. The five fastest swimmers and the best times (in seconds) they have achieved in each of the strokes (for 50 yards) are

Stroke	Carl	Chris	David	Tony	Ken
Backstroke	37.7	32.9	33.8	37.0	35.4
Breaststroke	43.4	33.1	42.2	34.7	41.8
Butterfly	33.3	28.5	38.9	30.4	33.6
Freestyle	29.2	26.4	29.6	28.5	31.1

The coach wishes to determine how to assign four swimmers to the four different strokes to minimize the sum of the corresponding best times.

(a) Formulate this problem as an assignment problem.

C **(b)** Obtain an optimal solution.

8.3-5. Consider the assignment problem formulation of Option 2 for the Better Products Co. problem presented in Table 8.29.

(a) Reformulate this problem as an equivalent transportation problem with three sources and five destinations by constructing the appropriate parameter table.

(b) Convert the optimal solution given in Sec. 8.3 for this assignment problem into a complete BF solution (including degenerate basic variables) for the transportation problem formulated in part (*a*). Specifically, apply the "General Procedure for Constructing an Initial BF Solution" given in Sec. 8.2. For each iteration of the procedure, rather than using any of the three alternative criteria presented for step 1, select the next basic variable to correspond to the next assignment of a plant to a product given in the optimal solution. When only one row or only one column remains under consideration, use step 4 to select the remaining basic variables.

(c) Verify that the optimal solution given in Sec. 8.3 for this assignment problem actually is optimal by applying just the optimality test portion of the transportation simplex method to the complete BF solution obtained in part (*b*).

(d) Now reformulate this assignment problem as an equivalent transportation problem with five sources and five destinations by constructing the appropriate parameter table. Compare this transportation problem with the one formulated in part (*a*).

(e) Repeat part (*b*) for the problem as formulated in part (*d*). Compare the BF solution obtained with the one from part (*b*).

D,I **8.3-6.** Starting with Vogel's approximation method, interactively apply the transportation simplex method to solve the Job Shop Co. assignment problem as formulated in Table 8.26*b*. (As stated in Sec. 8.3, the resulting optimal solution has $x_{14} = 1$, $x_{23} = 1$, $x_{31} = 1$, $x_{42} = 1$, and all other $x_{ij} = 0$.)

8.3-7. Reconsider Prob. 8.1-7. Now assume that distribution centers 1, 2, and 3 must receive exactly 10, 20, and 30 units per week, respectively. For administrative convenience, management has decided that each distribution center will be supplied totally by a single plant, so that one plant will supply one distribution center and the other plant will supply the other two distribution centers. The choice of these assignments of plants to distribution centers is to be made solely on the basis of minimizing total shipping cost.

(a) Formulate this problem as an assignment problem by constructing the appropriate cost table, including identifying the corresponding assignees and tasks.

C **(b)** Obtain an optimal solution.

(c) Reformulate this assignment problem as an equivalent transportation problem (with four sources) by constructing the appropriate parameter table.

C **(d)** Solve the problem as formulated in part (*c*).

(e) Repeat part (*c*) with just two sources.

C **(f)** Solve the problem as formulated in part (*e*).

8.3-8. Consider the assignment problem having the following cost table.

		Job		
		1	2	3
Person	A	5	7	4
	B	3	6	5
	C	2	3	4

The optimal solution is A-3, B-1, C-2, with $Z = 10$.

C **(a)** Use the computer to verify this optimal solution.

(b) Reformulate this problem as an equivalent transportation problem by constructing the appropriate parameter table.

C **(c)** Obtain an optimal solution for the transportation problem formulated in part (*b*).

(d) Why does the optimal BF solution obtained in part (*c*) include some (degenerate) basic variables that are not part of the optimal solution for the assignment problem?

(e) Now consider the *nonbasic* variables in the optimal BF solution obtained in part (*c*). For each nonbasic variable x_{ij} and the corresponding cost c_{ij}, adapt the sensitivity analysis procedure for general linear programming (see Case 2*a* in Sec. 6.7) to determine the *allowable range for* c_{ij}.

8.3-9. Consider the linear programming model for the general assignment problem given in Sec. 8.3. Construct the table of constraint coefficients for this model. Compare this table with the one for the general transportation problem (Table 8.6). In what ways does the general assignment problem have more special structure than the general transportation problem?

I **8.4-1.** Reconsider the assignment problem presented in Prob. 8.3-2. Manually apply the Hungarian algorithm to solve this problem. (You may use the corresponding interactive procedure in your IOR Tutorial.)

I **8.4-2.** Reconsider Prob. 8.3-4. See its formulation as an assignment problem in the answers given in the back of the book. Manually apply the Hungarian algorithm to solve this problem. (You may use the corresponding interactive procedure in your IOR Tutorial.)

I **8.4-3.** Reconsider the assignment problem formulation of Option 2 for the Better Products Co. problem presented in Table 8.29. Suppose that the cost of having Plant 1 produce product 1 is reduced from 820 to 720. Solve this problem by manually applying the Hungarian algorithm. (You may use the corresponding interactive procedure in your IOR Tutorial.)

I **8.4-4.** Manually apply the Hungarian algorithm (perhaps using the corresponding interactive procedure in your IOR Tutorial) to solve the assignment problem having the following cost table:

		Job		
		1	2	3
Person	1	M	8	7
	2	7	6	4
	3(D)	0	0	0

I **8.4-5.** Manually apply the Hungarian algorithm (perhaps using the corresponding interactive procedure in your IOR Tutorial) to solve the assignment problem having the following cost table:

		Task			
		1	2	3	4
Assignee	A	4	1	0	1
	B	1	3	4	0
	C	3	2	1	3
	D	2	2	3	0

I **8.4-6.** Manually apply the Hungarian algorithm (perhaps using the corresponding interactive procedure in your IOR Tutorial) to solve the assignment problem having the following cost table:

		Task			
		1	2	3	4
Assignee	A	5	8	6	7
	B	9	5	7	8
	C	5	9	8	4
	D	6	3	5	9

CASES

CASE 8.1 Shipping Wood to Market

Alabama Atlantic is a lumber company that has three sources of wood and five markets to be supplied. The annual availability of wood at sources 1, 2, and 3 is 15, 20, and 15 million board feet, respectively. The amount that can be sold annually at markets 1, 2, 3, 4, and 5 is 11, 12, 9, 10, and 8 million board feet, respectively.

In the past the company has shipped the wood by train. However, because shipping costs have been increasing, the alternative of using ships to make some of the deliveries is being investigated. This alternative would require the company to invest in some ships. Except for these investment costs, the shipping costs in thousands of dollars per million board feet by rail and by water (when feasible) would be the following for each route:

	Unit Cost by Rail ($1,000's) Market					Unit Cost by Ship ($1,000's) Market				
Source	**1**	**2**	**3**	**4**	**5**	**1**	**2**	**3**	**4**	**5**
1	61	72	45	55	66	31	38	24	—	35
2	69	78	60	49	56	36	43	28	24	31
3	59	66	63	61	47	—	33	36	32	26

The capital investment (in thousands of dollars) in ships required for each million board feet to be transported annually by ship along each route is given as follows:

	Investment for Ships ($1,000's) Market				
Source	**1**	**2**	**3**	**4**	**5**
1	275	303	238	—	285
2	293	318	270	250	265
3	—	283	275	268	240

Considering the expected useful life of the ships and the time value of money, the equivalent uniform annual cost of these investments is one-tenth the amount given in the table. The objective is to determine the overall shipping plan that minimizes the total equivalent uniform annual cost (including shipping costs).

You are the head of the OR team that has been assigned the task of determining this shipping plan for each of the following three options.

Option 1: Continue shipping exclusively by rail.
Option 2: Switch to shipping exclusively by water (except where only rail is feasible).
Option 3: Ship by either rail or water, depending on which is less expensive for the particular route.

Present your results for each option. Compare.

Finally, consider the fact that these results are based on *current* shipping and investment costs, so the decision on the option to adopt now should take into account management's projection of how these costs are likely to change in the future. For each option, describe a scenario of future cost changes that would justify adopting that option now.

(*Note:* Data files for this case are provided on the book's website for your convenience.)

PREVIEWS OF ADDED CASES ON OUR WEBSITE (www.mhhe.com/hillier)

CASE 8.2 Continuation of the Texago Case Study

The supplement to this chapter on the book's website presents a case study of how the Texago Corp. solved many transportation problems to help make its decision regarding where to locate its new oil refinery. Management now needs to address the question of whether the capacity of the new refinery should be made somewhat larger than originally planned. This will require formulating and solving some additional transportation problems. A key part of the analysis then will involve combining two transportation problems into a single linear programming model that simultaneously considers the shipping of crude oil from the oil fields to the refineries and the shipping of final product from the refineries to the distribution centers. A memo to management summarizing your results and recommendations also needs to be written.

CASE 8.3 Project Pickings

This case focuses on a series of applications of the assignment problem for a pharmaceutical manufacturing company. The decision has been made to undertake five research and development projects to attempt to develop new drugs that will treat five specific types of medical ailments. Five senior scientists are available to lead these projects as project directors. The problem now is to decide on how to assign these scientists to the projects on a one-to-one basis. A variety of likely scenarios need to be considered.