Pco Theoretical Derivations

A Derivation of Binning Bit Cost Bound

We will prove a bound on the number of bits used by binning:

Theorem 1 *Suppose X is a mixture over domain $\{0, \dots, T-1\}$ of s disjoint integer distributions, each of which has a monotonic PMF. Then for any $k > 2s$, there exists a binning of at most k bins such that the expected binned bit cost $\hat{H}$ of a random draw from X satisfies*

$$\hat{H} \leq H + \frac{3s\log_2(T)}{k-2s}\frac{T}{T-1}$$

where H is the base-2 Shannon entropy of X.

And less formally, we will show that this binning consists of bins of roughly equal probability. Also informally, the scaling behavior with imbalanced mixture distributions could be improved in subsection A.4.

To sketch the proof, there will be three main steps. First, we will show that there exists a binning in which every nontrivial bin has probability at most $\frac{3}{2(k-1)}$. Next, we will bound the bit cost of a monotonic distribution using such a binning. Finally, we will extend this to mixtures of monotonic distributions.

A.1 Definitions

Let $P(x)$ be the PMF of a random integer variable X over a finite, contiguous domain of T distinct x values. Additionally, let a *binning* with n bins consist of a list of intervals $([a_1, b_1], \dots, [a_k, b_k])$. Each such bin has various properties:

- *Weight* $w_i = \sum_{x=a_i}^{b_i} P(x)$, the probability mass within the bin.
- *Length* $T_i = b_i - a_i + 1$, the number of x values within the bin. A bin is called *trivial* if its length is 1.
- *Partial entropy* $H_i = \sum_{x=a_i}^{b_i} -P(x)\log_2(P(x))$. It follows that the Shannon entropy H of X is $H = \sum_{i=1}^{k} H_i$.

Finally, define the *bit cost* of a binning to be

$$\hat{H} = \sum_{i=1}^{k} w_i(\log_2(T_i) - \log_2(w_i)). \tag{1}$$

This is a good approximation for the number of bits Pco would use to compress each number in reality. Because $\hat{H}$ describes an expectation, we can use $\hat{H}$ to model both the bit cost of X and also the expected bit cost of an empirical distribution sampled from X. Note that, for the purposes of theoretical analysis, we have made a few simplifications:

- The bit cost of each tANS code is modeled as $-\log_2(w_i)$. This is a good approximation, but the true cost may be a fraction of a bit higher on average due to entropy coding imperfections.

- The bit cost of each offset is modeled as $\log_2(T_i)$, whereas in reality it is $\lceil \log_2(T_i) \rceil$. This difference is not very impactful, especially since our bin optimization step is aware of the true offset cost, and even in antagonistic cases we waste at most one bit per number.

- In practice, we must encode the bins themselves as metadata. One could add a term of kM onto $\hat{H}$ to account for this metadata cost.

A.2 Constructive Bound on Bin Weight

Lemma 1 *If X's PMF is monotonic, then for any k, there exists a binning of X with k or fewer bins such that each bin is either trivial or has weight at most $\frac{3}{2(k-1)}$.*

We will prove this constructively. Suppose without loss of generality that X's PMF is monotonically decreasing. For any x such that $P(x) > \frac{3}{2(k-1)}$, dedicate a trivial bin to that single x value. Suppose there are k_{trivial} of these. Following these, all $k_{\text{nontrivial}}$ remaining bins must be nontrivial by monotonicity.

Now we use a simple algorithm to assign nontrivial bins greedily, until their weight would exceed $\frac{3}{2(k-1)}$:

- Initialize $w_{\text{current}} \leftarrow 0$.

- For each reamining x value, let $w_{\text{proposed}} = w_{\text{current}} + P(x)$, and

  - If $w_{\text{proposed}} \leq \frac{3}{2(k-1)}$, assign $w_{\text{current}} \leftarrow w_{\text{proposed}}$ and add x to the current bin.
  - Otherwise, emit the current bin, start a new one, and set $w_{\text{current}} \leftarrow P(x)$.

Clearly each such bin has weight mass of at most $\frac{3}{2(k-1)}$. And for every bin $[a_i, b_i]$ except the last one, we must have that $w_i + P(b_i + 1) > \frac{3}{2(k-1)}$; otherwise we would have included $b_i + 1$ in the bin. Since the bin is nontrivial, it must contain at least two values, implying that $P(b_i - 1) + P(b_i) + P(b_i + 1) > \frac{3}{2(k-1)}$. By monotonicity, the first two terms must total at least 2/3 of this quantity, giving $w_i > \frac{1}{k-1}$.

So we can bound the sum of all trivial and nontrivial bin weights:

$$\sum w_i > \frac{3k_{\text{trivial}}}{2(k-1)} + \frac{k_{\text{nontrivial}} - 1}{k-1}$$

$$\sum w_i > \frac{k_{\text{trivial}} + k_{\text{nontrivial}} - 1}{k-1}$$

The sum of probabilities must be 1, so

$$k - 1 > k_{\text{trivial}} + k_{\text{nontrivial}} - 1$$

$$k_{\text{trivial}} + k_{\text{nontrivial}} < k$$

In other words, we have used at most k bins, each of which is either trivial or weighed at most $\frac{3}{2(k-1)}$. This completes the proof of Lemma 1.

A.3 Bound for Monotonic PMFs

Lemma 2 *If X's PMF is monotonic, then there exists a binning with k or fewer bins such that*

$$\hat{H} - H \leq \frac{3\log_2(T)}{k-1}\frac{T}{T-1}.$$

We will show that the binning from Lemma 1 satisfies this. Without loss of generality, assume X's PMF is monotonically increasing (this is the opposite from the previous section, but makes the indexing simpler). Consider a bin $[a_i, b_i]$. The partial entropy of this bin is

$$H_i = \sum_{x=a_i}^{b_i} -P(x)\log_2(P(x)). \tag{2}$$

Let $q_i = P(b_i)$, the maximum probability of any x within the bin. Since $P(x) \leq q_i$ within the bin, $-P(x)\log_2(P(x)) \geq -P(x)\log_2(q_i)$. Plugging this into Equation 2,

$$H_i \geq \sum_{x=a_i}^{b_i} -P(x)\log_2(q_i).$$

Which can simply be rewritten as

$$H_i \geq -w_i \log_2(q_i).$$

Supposing that there are k bins, and taking the difference of the total bit cost (1) and these partial entropies,

$$\hat{H} - H \leq \sum_{i=1}^{k} w_i\left(\log_2(T_i) - \log_2(w_i) + \log_2(q_i)\right)$$

By property of the binning we chose and the fact that $P(x)$ is increasing, there is some $k_{\text{nontrivial}}$ such that bins 1 through $k_{\text{nontrivial}}$ are nontrivial, and $k_{\text{nontrivial}} + 1$ through k are trivial. For the trivial bins, $w_i = q_i$ and $T_i = 1$, leaving

$$\hat{H} - H \leq \sum_{i=1}^{k_{\text{nontrivial}}} w_i\left(\log_2(T_i) - \log_2(w_i) + \log_2(q_i)\right)$$

Now we need only consider nontrivial bins. We will further divide these into *shallow* and *deep* bins. Let k_{shallow} be the smallest value such that $\sum_{i=1}^{k_{\text{shallow}}} w_i \geq \frac{3}{2(k-1)(T-1)}$. We can rewrite the inequality as

$$\begin{aligned}\hat{H} - H \leq &\sum_{i=1}^{k_{\text{shallow}}} w_i\left(\log_2(T_i) - \log_2(w_i) + \log_2(q_i)\right) + \\ &\sum_{i=k_{\text{shallow}}+1}^{k_{\text{nontrivial}}} w_i\left(\log_2(T_i) - \log_2(w_i) + \log_2(q_i)\right)\end{aligned} \tag{3}$$

Since $P(x)$ is increasing, each value of x in bin $i \geq 2$ has probability at least q_{i-1}, so

$$T_i \leq \frac{w_i}{q_{i-1}}, \qquad i \geq 2$$

We can use this to bound the deep portion of Equation 3,

$$\hat{H} - H \leq \sum_{i=1}^{k_{\text{shallow}}} \left(\hat{H}_i - H_i\right) + \sum_{i=k_{\text{shallow}}+1}^{k_{\text{nontrivial}}} w_i \left(\log_2\left(\frac{w_i}{q_{i-1}}\right) - \log_2(w_i) + \log_2(q_i)\right)$$

$$\hat{H} - H \leq \sum_{i=1}^{k_{\text{shallow}}} \left(\hat{H}_i - H_i\right) + \sum_{i=k_{\text{shallow}}+1}^{k_{\text{nontrivial}}} w_i \left(\log_2(q_i) - \log_2(q_{i-1})\right)$$

Since each of these bins has $w_i \leq \frac{3}{2(k-1)}$ by Lemma 1,

$$\hat{H} - H \leq \sum_{i=1}^{k_{\text{shallow}}} \left(\hat{H}_i - H_i\right) + \frac{3}{2(k-1)} \sum_{i=k_{\text{shallow}}+1}^{k_{\text{nontrivial}}} \log_2(q_i) - \log_2(q_{i-1})$$

This sum telescopes, leaving

$$\hat{H} - H \leq \sum_{i=1}^{k_{\text{shallow}}} \left(\hat{H}_i - H_i\right) + \frac{3}{2(k-1)} \left(\log_2(q_{k_{\text{nontrivial}}}) - \log_2(q_{k_{\text{shallow}}})\right)$$

Since deep bins are nontrivial, they have $q_i \leq \frac{3}{2(k-1)}$, giving

$$\hat{H} - H \leq \sum_{i=1}^{k_{\text{shallow}}} \left(\hat{H}_i - H_i\right) + \frac{3}{2(k-1)} \left(\log_2\left(\frac{3}{2(k-1)}\right) - \log_2(q_{k_{\text{shallow}}})\right)$$

Now we have a reasonable bound for the deep bins, and we can turn our attention to bounding the shallow bins. All we claim about their partial entropy is that $H_i \leq -w_i \log_2(w_i)$, which follows trivially since $w_i \geq P(x)$ for all x in the bin. This leaves

$$\hat{H} - H \leq \sum_{i=1}^{k_{\text{shallow}}} w_i \log_2(T_i) + \frac{3}{2(k-1)} \left(\log_2\left(\frac{3}{2(k-1)}\right) - \log_2(q_{k_{\text{shallow}}})\right)$$

Let $w_{\text{shallow}} = \sum_{i=1}^{k_{\text{shallow}}}$. Clearly $T_i \leq T$, so

$$\hat{H} - H \leq w_{\text{shallow}} \log_2(T) + \frac{3}{2(k-1)} \left(\log_2\left(\frac{3}{2(k-1)}\right) - \log_2(q_{k_{\text{shallow}}})\right)$$

Also, since mean probability is less than max probability, $q_{k_{\text{shallow}}} \geq w_{\text{shallow}}/(T_1 + T_2 + \ldots, T_{k_{\text{shallow}}}) \geq w_{\text{shallow}}/T$, giving

$$\hat{H} - H \leq w_{\text{shallow}} \log_2(T) + \frac{3}{2(k-1)} \left(\log_2\left(\frac{3}{2(k-1)}\right) - \log_2\left(\frac{w_{\text{shallow}}}{T}\right)\right)$$

$$\hat{H} - H \leq w_{\text{shallow}} \log_2(T) + \frac{3}{2(k-1)} \left(\log_2\left(\frac{3T}{2(k-1)}\right) - \log_2(w_{\text{shallow}})\right)$$

Thinking of the RHS of this inequality as a function of w_{shallow}, it is clear there is a single local minimum. Therefore this bound must be loosest (a maximum on the RHS) when w_{shallow} is at an extremal value. Recall that we selected k_{shallow} to be the smallest value such that $\sum_{i=1}^{k_{\text{shallow}}} w_i \geq \frac{3}{2(k-1)(T-1)}$. By this choice, and by the limited size of our nontrivial bins, we know that

$$w_{\text{shallow}} \in \left[\frac{3}{2(k-1)(T-1)}, \frac{3T}{2(k-1)(T-1)}\right]$$

We need only consider these two bounds, $w_s = \frac{1}{T-1}$ and $w_s = \frac{T}{T-1}$. For convenience, define w_s such that $w_{\text{shallow}} = \frac{3}{2(k-1)} w_s$. Our inequality becomes

$$\hat{H} - H \leq \frac{3}{2(k-1)} \left((w_s + 1) \log_2(T) - \log_2(w_s)\right)$$

Consider the $w_s = w_s = \frac{1}{T-1}$ case. We obtain

$$\hat{H} - H \leq \frac{3}{2(k-1)} \left(\frac{T}{T-1} \log_2(T) + \log_2(T-1)\right)$$

Similarly, when $w_s = \frac{T}{T-1}$, we get

$$\hat{H} - H \leq \frac{3}{2(k-1)} \left(\frac{2T}{T-1} \log_2(T) + \log_2(T-1) - \log_2(T)\right)$$

$$\hat{H} - H \leq \frac{3}{2(k-1)} \left(\frac{T}{T-1} \log_2(T) + \log_2(T-1)\right)$$

The result is identical in either case, so our bound holds universally. We loosen it slightly to make the result more apparent:

$$\hat{H} - H \leq \frac{3\log_2(T)}{k-1} \frac{T}{T-1}$$

This completes the proof of Lemma 2.

A.4 Bound for Mixtures

Suppose X is a mixture of s disjoint, monotonic integer distributions, each with weight u_j. If $k \geq 2s$, we can assign mixture component j a total of $k_j = 2 + \lfloor (k-2s)u_j \rfloor$ bins. Since all u_j sum to 1, the sum of all k_j is at most k. Furthermore, $k_j \geq 1 + (k-2s)u_j$. By Lemma 2, this implies a bit cost of at most

$$\hat{H}_j \leq H_j + \frac{3\log_2(T)}{(k-2s)u_j} \frac{T}{T-1}$$

The total bit cost is $\hat{H} = \sum_{j=1}^{s} u_j(\hat{H}_j - \log_2(u_j))$, and the total entropy is $H = \sum_{j=1}^{s} u_j(H_j - \log_2(u_j))$, so

$$\hat{H} \leq H + \sum_{j=1}^{s} \frac{3\log_2(T)}{(k-2s)} \frac{T}{T-1}$$

$$\hat{H} \leq H + \frac{3s\log_2(T)}{k-2s} \frac{T}{T-1}$$

This completes the proof of Theorem 1.

B Derivation of `IntMult` Bit Cost Approximation

Recall that we defined

- (x_1, x_2, x_3): a triple randomly sampled from the input numbers
- m: $\gcd(x_2 - x_1, x_3 - x_1)$, a proposed value for the parameter to `IntMult` mode
- c, c_m: the total count of triples and count of triples with GCD m, respectively
- $q_i \sim Q_m, r_i \sim R_m$: latents drawn from their respective latent variables, defined as $q_i = \lfloor x_i/m \rfloor, r_i = x_i \mod m$

Note that

$$0 \cong x_2 - x_1 \cong x_3 - x_1 \mod m.$$

Adding x_1 to all sides gives

$$x_1 \cong x_2 \cong x_3 \mod m.$$

Intuitively, this suggests that abnormally frequent values of m are likely to be good parameters. We next make estimates of the bit costs of Q_m and R_m for each m, which we use to choose the lowest-cost value of m. To that end, we count the number of occurrences c_m of each distinct m.

By using some knowledge of binning, we can make an estimate of $\hat{H}[Q_m] - \hat{H}_{\texttt{Classic}}$, the relative bits saved by binning Q_m as opposed to X. Suppose we use the same bins that `Classic` would have used on X, but mapped into the space of Q_m, and that we keep the same bin assignment for each mapped number. Each bin would still have the same count of numbers, and hence identical entropy coding cost. Any `Classic` bin $[a_i, b_i]$ would become $[\lfloor a_i/m \rfloor, \lfloor b_i/m \rfloor]$ in the domain of Q_m. In most cases, this would reduce the number of offset bits by approximately $\log_2(m)$; the only exceptions are the bins that had small domains, with $b_i - a_i < m$. Call these bins and samples with $b_i - a_i < m$ *frequent*. Then conservatively assuming we save 0 bits on these nearly-trivial bins, we obtain a reasonable estimate of

$$\hat{H}[Q_m] - \hat{H}_{\texttt{Classic}} \approx -n'_{\text{infrequent}} \log_2(m)$$

where $n'_{\text{infrequent}}$ is the number of samples that would belong to bins with $b_i - a_i \geq m$ in `Classic` mode.

But histogram computation is computationally expensive, and we can actually avoid choosing bins at by using two more approximations. First, we replace the condition $b_i - a_i \geq m$ with the condition that $\lfloor a_i/m \rfloor = \lfloor b_i/m \rfloor$. This is equivalent up to rounding. This implies that frequent bins contain only a single x value in Q_m's domain. Our second approximation is that bins will contain only a single x value iff their weight is at least $1/256$. We use this constant since Pco's default compression level considers up to 256 bins of roughly equal weight. With these new approximations, we can define $n_{\text{infrequent}}$ to be the count of samples x_i that share their q_i values with less than $n/256$ samples, and claim that $n_{\text{infrequent}} \approx n'_{\text{infrequent}}$. This leaves the computationally practical approximation that

$$\hat{H}[Q_m] - \hat{H}_{\texttt{Classic}} \approx -n_{\text{infrequent}} \log_2(m)$$

For R_m, we can actually make a worst-case estimate of the entropy $H[R_m]$ using only m, c, c_m, and the assumption that Q_m is resembles a uniform distribution over a wide domain. We are constrained by the fact that we found c_m occurrences of m, so we assert that

$$P(\gcd(x_2 - x_1, x_3 - x_1) = m) \approx c_m/c \tag{4}$$

Results from number theory show that in the limit as $n \to \infty$, for a, b drawn uniformly from $1, \ldots, n$,

$$P(0 \cong a \cong b \mod m) = \zeta(2) P(\gcd(a, b) = m)$$

where ζ is the Riemann zeta function [1]. Since we modeled Q_m as a uniform distribution over a wide domain, we can apply this result to Equation 4 to obtain

$$\frac{1}{\zeta(2)} P(0 \cong x_2 - x_1 \cong x_3 - x_1 \mod m) \approx c_m/c$$

$$\frac{1}{\zeta(2)} P(x_1 \cong x_2 \cong x_3 \mod m) \approx c_m/c$$

$$P(x_1 \cong x_2 \cong x_3 \mod m) \approx \min\left(\frac{\zeta(2)c_m}{c}, 1\right)$$

We can evaluate the left-hand side more explicitly by considering all possible values of $r_i = x_i \mod m$:

$$\sum_{r=0}^{m-1} P(R'_m = r)^3 = \min\left(\frac{\zeta(2)c_m}{c}, 1\right)$$

This gives us a very workable constraint. Now we can simply approximate

$$\hat{H}[R_m] \approx \max_{R'_m} H[R'_m], \qquad \sum_{r=0}^{m-1} P(R'_m = r)^3 = \min\left(\frac{\zeta(2)c_m}{c}, 1\right)$$

This maximum occurs when one value of r has high probability p, and the rest are equally likely, giving

$$\hat{H}[R_m] \approx -p\log_2(p) - (m-1)(1-p)\log_2(1-p), \quad \text{s.t.} \quad p^3 + (m-1)(1-p)^3 = \min\left(\frac{\zeta(2)c_m}{c}, 1\right)$$

In practice, we use the method of false position to determine this cubic root p in a quick and numerically stable way. We then plug it in to obtain $\hat{H}[R_m]$.

References

[1] J. Chidambaraswamy and R. Sitaramachandrarao. "On the probability that the values of m polynomials have a given GCD". In: *Journal of Number Theory* 26.3 (1987), pp. 237–245. ISSN: 0022-314X. DOI: https://doi.org/10.1016/0022-314X(87)90081-3. URL: https://www.sciencedirect.com/science/article/pii/0022314X87900813.