HW #3

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CMPSCI 690Q: Quantum Information Systems

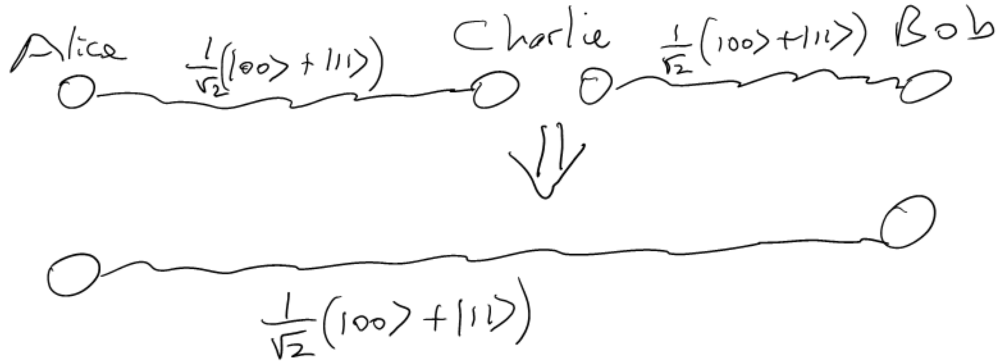
Fall 2019

Problem Set 3

Due date: By end of day Oct. 23 on Moodle

When preparing your written solutions, please show all steps involved in obtaining your answer. This will make it easier to assign partial credit (for partially correct answers) and will also help me (or the grader) in helping you to see where (if anywhere) you made a mistake.

1. Suppose that there are three parties, Alice, Bob, and Charlie. The pairs Alice and Charlie, and Bob and Charlie each share the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ but the pair Alice Bob does not. Charlie's job is use the two qubits that he has belonging to each of these two Bell states to entangle Alice's and Bob's qubits into a shared Bell state (see figure below). Modify the quantum teleportation algorithm to accomplish and show that it works.



2. Exercise 4.37 in NC. Provide a decomposition of the transform

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

into a product of two-level unitaries.

3. Exercise 4.38 in NC. Prove that there exists a $d \times d$ unitary matrix U that cannot be decomposed as a product of fewer than $d-1$ two-level unitary matrices.

4. Exercise 4.39 in NC. Find a quantum circuit using single qubit operations and `CNOT`s that

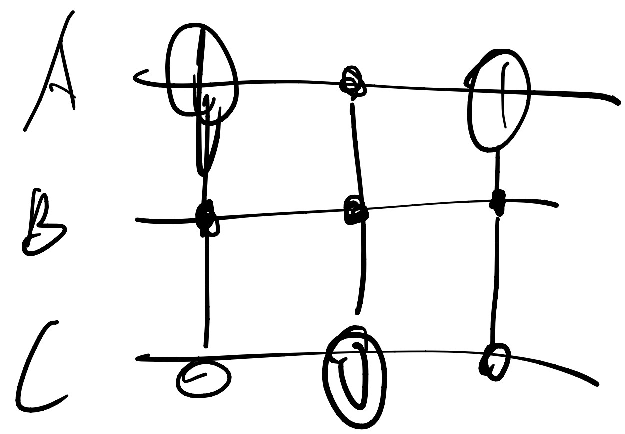
implements the transformation

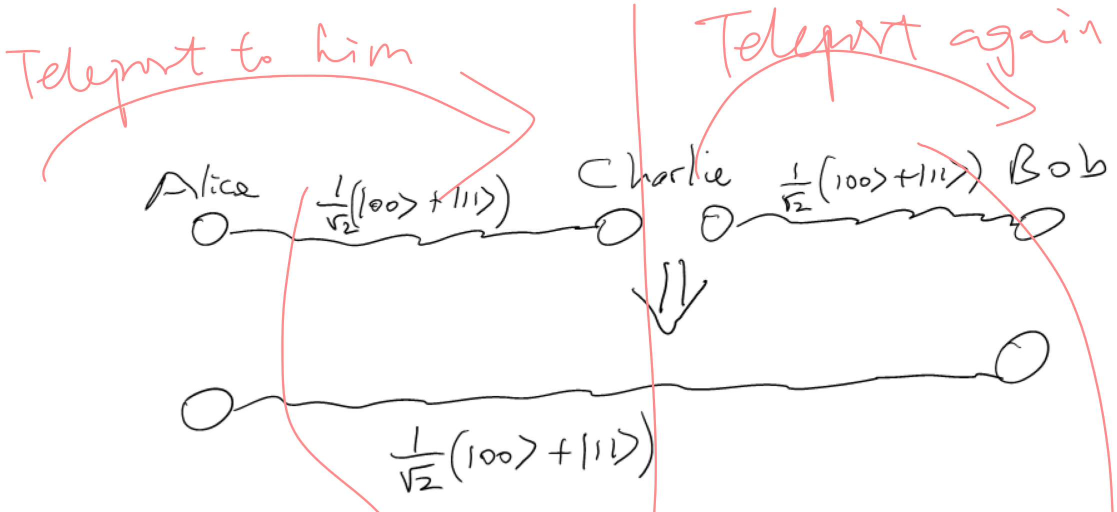
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & d \end{bmatrix}$$

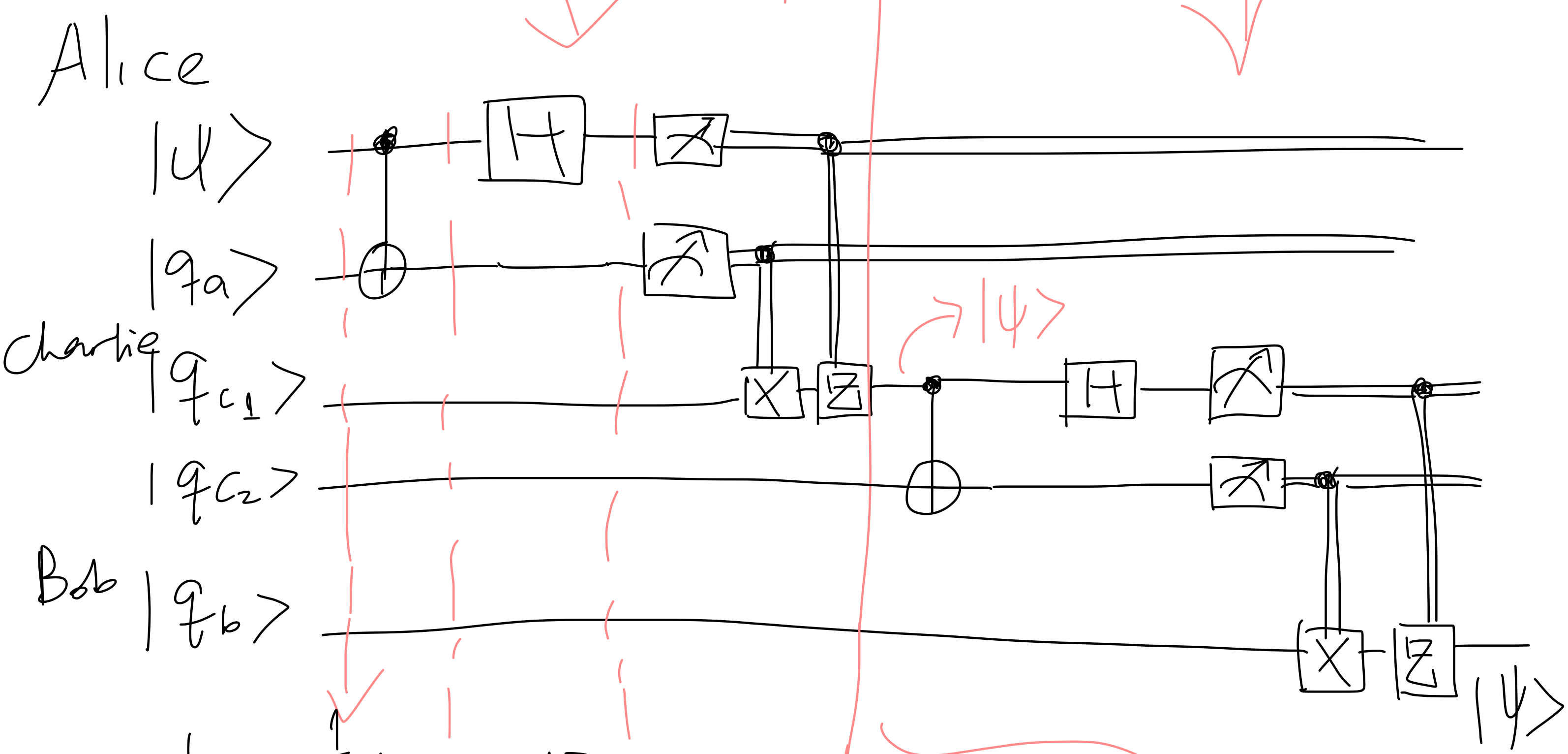
A	B	C
0	1	0
1	1	0
1	1	1

$\tilde{U}$

5. Exercise 5.4 in NC. Recall the "controlled phase gates" (aka controlled R_k gates in NC): if the control qubit is $ket1$ then the unitary R_k is applied to the target qubit where R_k leaves $|0\rangle$ alone and puts a phase of $\bar{\omega}_N^{2^{n-1-j}}$ onto $|1\rangle$. Decompose R_k into single qubit and `CNOT` gates.







$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Stage 1

Stage 2

$$|\psi_1\rangle = (\alpha|0\rangle + \beta|1\rangle)\left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right) \xrightarrow{\text{CNOT}} \frac{\alpha}{\sqrt{2}}|0\rangle(|00\rangle + |11\rangle) + \frac{\beta}{\sqrt{2}}|1\rangle(|10\rangle + |01\rangle)$$

$$|\psi_2\rangle = \frac{\alpha}{2}(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \frac{\beta}{2}(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)$$

↓ Results

$$\left.\begin{array}{ll} 0 & \alpha|00\rangle+\alpha|11\rangle+\beta|10\rangle+\beta|01\rangle \\ 1 & \alpha|00\rangle+\alpha|11\rangle-\beta|10\rangle-\beta|01\rangle \end{array}\right\} A$$

$$\left.\begin{array}{ll} 00 & \alpha|0\rangle+\beta|1\rangle \quad // \; |\psi\rangle \\ 01 & \alpha|1\rangle+\beta|0\rangle \xrightarrow{X} \alpha|0\rangle+\beta|1\rangle \quad // \; |\psi\rangle \\ 10 & \alpha|0\rangle-\beta|1\rangle \xrightarrow{Z} |\psi\rangle \\ 11 & \alpha|1\rangle-\beta|0\rangle \xrightarrow{ZX} |\psi\rangle \end{array}\right\} B$$

For stage 2, just repeat b/c $|q_{c1}\rangle$ is $|\psi\rangle$ after stage 1 $\square$

2).

$$U = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \begin{bmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{bmatrix}$$

$$U_{12} \equiv \begin{array}{cc} 1 & \begin{array}{ccc} 0 & 0 & 0 \end{array} \\ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} & \begin{bmatrix} \frac{a^*}{\sqrt{|a|^2+|b|^2}} & \frac{b^*}{\sqrt{|a|^2+|b|^2}} & 0 \\ \frac{b}{\sqrt{|a|^2+|b|^2}} & \frac{-a}{\sqrt{|a|^2+|b|^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \Rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In[181]:= `MatrixForm[U12.U]`

Out[181]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\frac{1}{2}+\frac{i}{2}}{\sqrt{2}} & 0 & \frac{\frac{1}{2}-\frac{i}{2}}{\sqrt{2}} \\ 0 & \frac{\frac{1}{2}-\frac{i}{2}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{\frac{1}{2}+\frac{i}{2}}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} & \frac{i}{2} \end{pmatrix}$$

a'

c'

→ From this find out U_{13}

$$U_{13} \equiv \begin{array}{cc} 1 & \begin{array}{ccc} 0 & 0 & 0 \end{array} \\ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} & \begin{bmatrix} \frac{a'^*}{\sqrt{|a'|^2+|c'|^2}} & 0 & \frac{c'^*}{\sqrt{|a'|^2+|c'|^2}} \\ 0 & 1 & 0 \\ \frac{c'}{\sqrt{|a'|^2+|c'|^2}} & 0 & \frac{-a'}{\sqrt{|a'|^2+|c'|^2}} \end{bmatrix} \end{array}.$$

In[214]:= $\text{U13} = \{\{\frac{1}{\sqrt{2}} \Big/ \text{Sqrt}[1/2+1/4],\ 0,\ \frac{1}{2} \Big/ \text{Sqrt}[1/2+1/4],\ 0\},\ \{0, 1, 0, 0\},$
$\{\frac{1}{2} \Big/ \text{Sqrt}[1/2+1/4],\ 0,\ \frac{1}{-\sqrt{2}} \Big/ \text{Sqrt}[1/2+1/4],\ 0\},\ \{0, 0, 0, 1\}\};$

```
MatrixForm[
 U13]
```

Out[215]//MatrixForm=

$$\begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ U13

In[184]:= `MatrixForm[U13.U12.U]`

Out[184]//MatrixForm=

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{i}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{i}{2\sqrt{3}} \\ 0 & \frac{\frac{1}{2}-\frac{i}{2}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{\frac{1}{2}+\frac{i}{2}}{\sqrt{2}} \\ 0 & \frac{\frac{3}{2}+\frac{i}{2}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{\frac{3}{2}-\frac{i}{2}}{\sqrt{6}} \\ \frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} & \frac{i}{2} \end{pmatrix}$$

In[216]:= $\text{U14} = \{\{\left(\frac{\sqrt{3}}{2}\right) \Big/ \text{Sqrt}\left[\left(\frac{\sqrt{3}}{2}\right)\text{^}2 + (1/2)\text{^}2\right],\ 0,\ 0,\ (1/2) \Big/ \text{Sqrt}\left[\left(\frac{\sqrt{3}}{2}\right)\text{^}2 + (1/2)\text{^}2\right]\},$
$\{0, 1, 0, 0\},\ \{0, 0, 1, 0\},$
$\{(1/2) \Big/ \text{Sqrt}\left[\left(\frac{\sqrt{3}}{2}\right)\text{^}2 + (1/2)\text{^}2\right],\ 0,\ 0,\ \left(-\frac{\sqrt{3}}{2}\right) \Big/ \text{Sqrt}\left[\left(\frac{\sqrt{3}}{2}\right)\text{^}2 + (1/2)\text{^}2\right]\}\};$

```
MatrixForm[
 U14]
```

Out[217]//MatrixForm=

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

$\Rightarrow$ U14

Similarly,

$$U_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \left(\frac{1}{4}+\frac{i}{4}\right)\sqrt{3} & \frac{3}{4}+\frac{1}{4}i & 0 \\ 0 & \frac{3}{4}-\frac{1}{4}i & -\left(\frac{1}{4}-\frac{i}{4}\right)\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

// Sorry, mathematica has some problem computing $U_{13}U_{12}U_{11}U$ $\Rightarrow$ ~~its~~ result differs from what's manually computed, so I just put my results here.

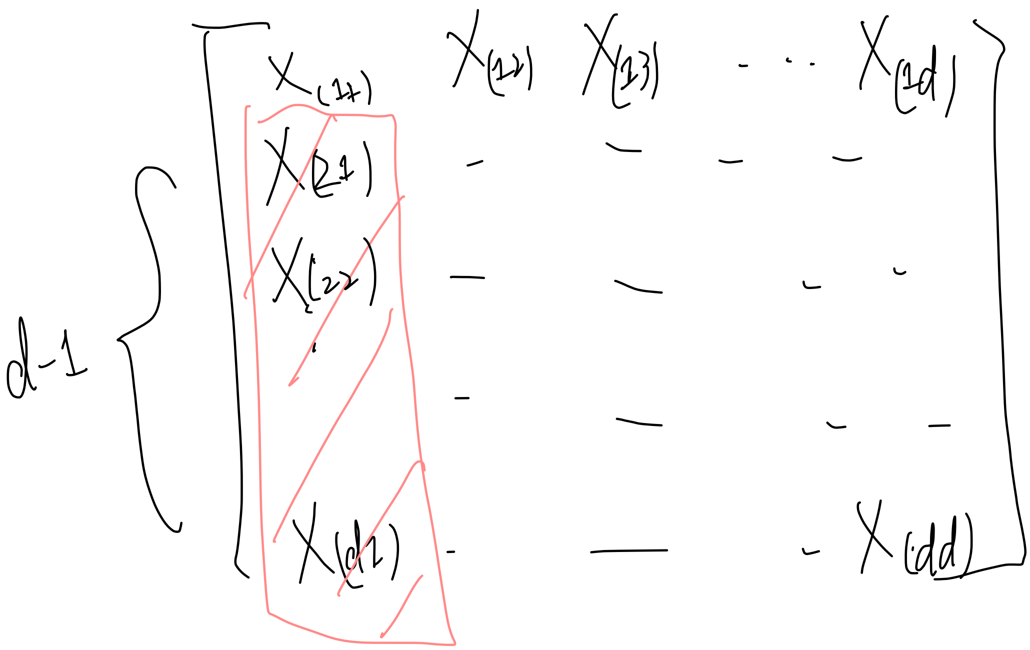
$$U_{24} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}}i \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}}i & 0 & -\sqrt{\frac{2}{3}} \end{bmatrix}$$

$$U_{3,4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

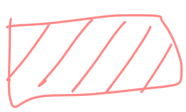
$$U_{34} U_{24} U_{23} U_{14} U_{13} U_{12} U = I$$

3)

3. Exercise 4.38 in NC. Prove that there exists a $d \times d$ unitary matrix U that cannot be decomposed as a product of fewer than $d-1$ two-level unitary matrices.



First we have to make all entries in

 area 0.

If none of the entries in this area is 0, and $T_{jk} T_{jk-1} T_{jk-2} \cdots T_{j2} T_{j-1k} \cdots T_{12} U$

never creates the situation such that

$U_{jk+1} = 0.$

// To elaborate on this, what I meant is such situation mentioned in Neilso Chuang:

// Note that in either case U_1 is a two-level unitary matrix, and when we multiply the matrices out we get

$$U_1 U = \begin{bmatrix} a' & d' & g' \\ 0 & e' & h' \\ c' & f' & j' \end{bmatrix}. \tag{4.46}$$

// The key point to note is that the middle entry in the left hand column is zero. We denote the other entries in the matrix with a generic prime $'$; their actual values do not matter.

Now apply a similar procedure to find a two-level matrix U_2 such that $U_2 U_1 U$ has no entry in the *bottom left* corner. That is, if $c' = 0$ we set

$$U_2 \equiv \begin{bmatrix} a'^* & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.47}$$

$U_1 U$ creates $c' = 0$.

then we have to have at least $d-1$ two level unitary matrices to reduce all entries in the shaded area to be 0.

Apply U_1:

$$\begin{bmatrix} X'_{(11)} & X'_{(12)} & X'_{(13)} & X'_{(14)} & \cdots & X'_{(1d)} \\ 0 & X'_{(22)} & X'_{(23)} & \cdots & & \\ X_{(31)} & & & & & \\ \vdots & & & & \ddots & \\ X_{(d1)} & & & & & X'_{(dd)} \end{bmatrix}$$

Remaining $= d-2$

Apply U_2 :

$$\begin{bmatrix} X''(11) & X''(12) & \cdots & X''(1d) \\ 0 & X''(22) & \cdots & | \\ 0 & 0 & X''(33) & | \\ X''(41) & \ddots & & | \\ \vdots & & \ddots & | \\ X''(d1) & & & X''(dd) \end{bmatrix}$$

Remaining $= d-3$

$\therefore$ Only after applying $d-1$ unitary matrices can we possibly convert this U into I.

↳ when remaining $= 0$.

$\therefore U_n = U_{d-1}$.

$\therefore \exists$ a $d \times d$ unitary matrix that cannot be decomposed w/ fewer than $(d-1)$ 2-level unitary matrices.

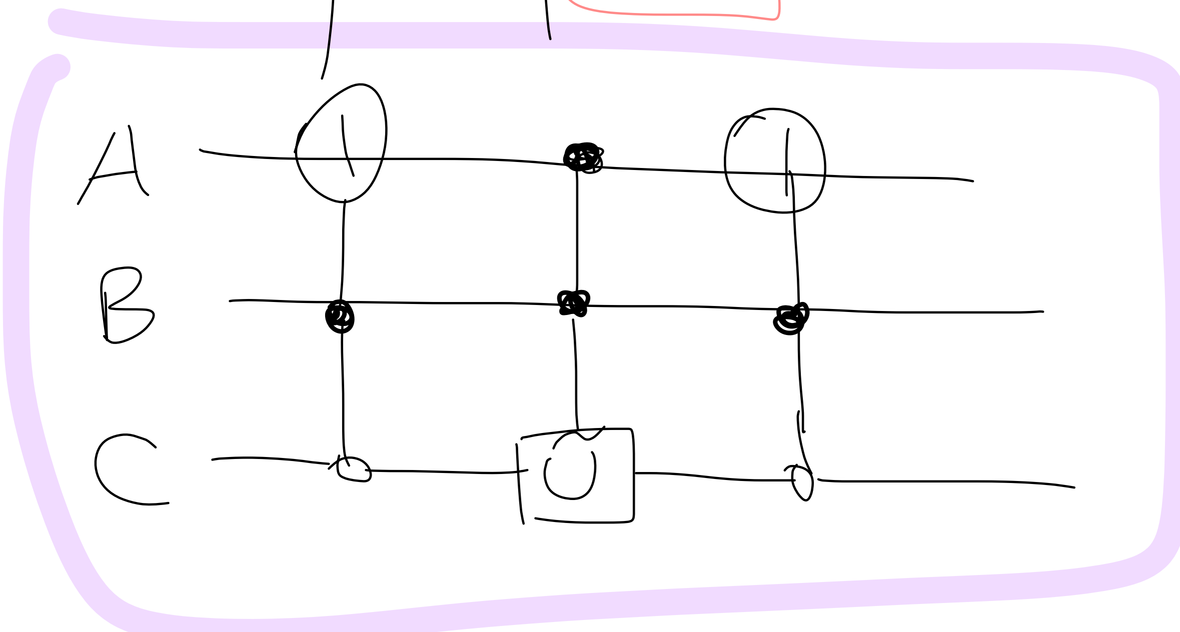
$\square$.

4).

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & d \end{bmatrix},$$

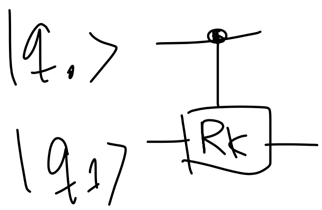
A	B	C
0	1	0
1	1	0
1	1	1

$\tilde{U}$



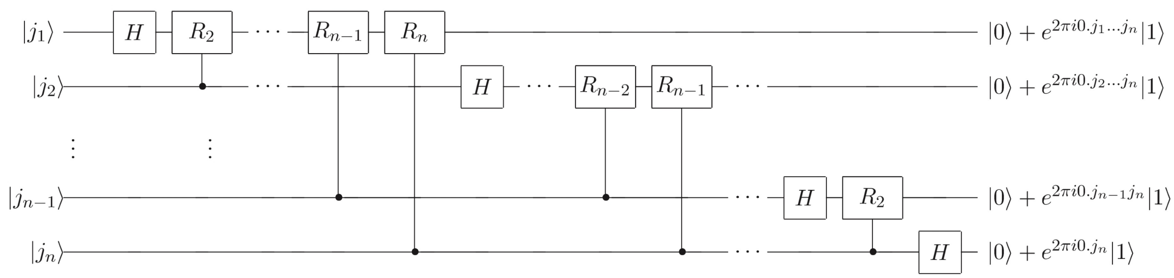
5)

Exercise 5.4: Give a decomposition of the controlled-R_k gate into single qubit and CNOT gates.

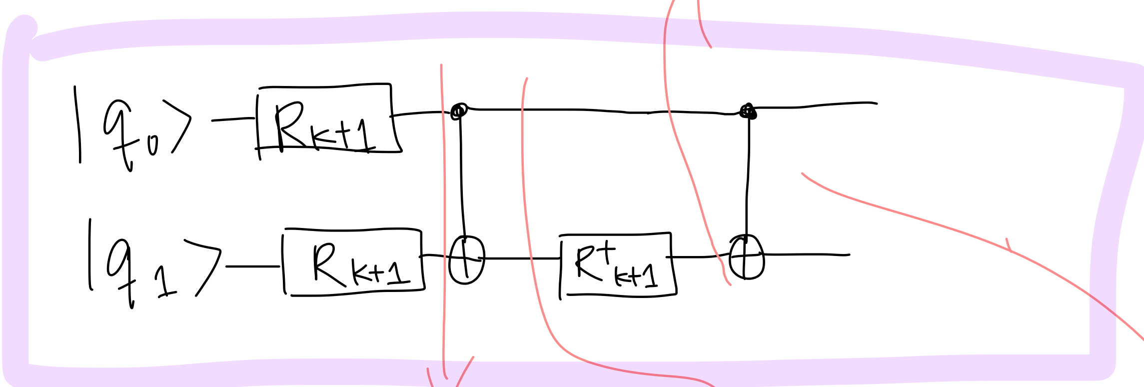


$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{bmatrix}$$

$$\hookrightarrow = \begin{bmatrix} 1 & & & \\ & e^{2\pi i/2^k} & & \\ & & e^{2\pi i/2^k} & \\ & & & e^{2\pi i/2^k} \end{bmatrix}$$



$$\begin{bmatrix} I & \\ & \begin{bmatrix} 1 & \\ & e^{-2\pi i/2^{k+1}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & e^{2\pi i/2^{k+1}} & & \\ & & 1 & \\ & & & e^{2\pi i/2^{k+1}} \end{bmatrix}$$

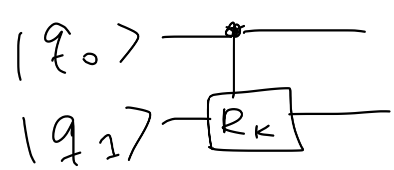


$$\begin{bmatrix} 1 & & & \\ & e^{2\pi i/2^{k+1}} & & \\ & & 1 & \\ & & & e^{2\pi i/2^{k+1}} \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & e^{2\pi i/2^{k+1}} & & \\ & & 1 & \\ & & & e^{2\pi i/2^{k+1}} \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \begin{bmatrix} I & \\ & \begin{bmatrix} 1 & \\ & e^{-2\pi i/2^{k+1}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & e^{2\pi i/2^{k+1}} & & \\ & & 1 & \\ & & & e^{2\pi i/2^{k+1}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & e^{2\pi i/2^k} & & \\ & & e^{2\pi i/2^k} & \\ & & & e^{2\pi i/2^k} \end{bmatrix} =$$



$\square$