HW#5

1. a) $|\Psi^-\rangle = \frac{|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B}{\sqrt{2}}$

$\rho_{AB} = \frac{1}{2}(|00\rangle - |11\rangle)(\langle 00| - \langle 11|)$

$= \frac{1}{2}(|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|)$

$= \frac{1}{2}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\begin{bmatrix}1&0&0&0\end{bmatrix} - \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\begin{bmatrix}0&0&0&1\end{bmatrix} - \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\begin{bmatrix}1&0&0&0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\begin{bmatrix}0&0&0&1\end{bmatrix}\right)$

$= \frac{1}{2}\begin{bmatrix}1&0&0&-1\\0&0&0&0\\0&0&0&0\\-1&0&0&1\end{bmatrix}$

$\rho_A = tr_B(\rho_{AB})$

$= tr_B\left(\frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| - |0\rangle\langle 1| \otimes |0\rangle\langle 1| - |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)\right)$

$= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$

$= \frac{1}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$\rho_B = tr_A(\rho_{AB}) = tr_B(\rho_{AB}) = \frac{1}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix}$

b) $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)$

$\rho_{AB} = \frac{1}{2}(|01\rangle + |10\rangle)(\langle 01| + \langle 10|)$

$= \frac{1}{2}(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|)$

$= \frac{1}{2}\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\begin{bmatrix}0&1&0&0\end{bmatrix} + \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\begin{bmatrix}0&0&1&0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\begin{bmatrix}0&1&0&0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\begin{bmatrix}0&0&1&0\end{bmatrix}\right)$

$= \frac{1}{2}\begin{bmatrix}0&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&0\end{bmatrix}$

$\rho_A = tr_B(\rho_{AB}) = \frac{1}{2}(|0\rangle\langle 0|\langle 1|1\rangle + |0\rangle\langle 1|\langle 0|1\rangle + |1\rangle\langle 0|\langle 1|0\rangle +$
$|1\rangle\langle 1|\langle 0|0\rangle$

$$\mathrm{tr}_B \rho_{AB} = \sum_{ijkl} C_{ijkl} |a_i\rangle\langle a_j| \langle b_l | b_k \rangle$$

$$= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho_B = \rho_A = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. $\langle X \rangle = \langle \psi | X | \psi \rangle = \lambda \langle \psi | \psi \rangle = \lambda$

$$\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \sqrt{\langle \psi | X^2 | \psi \rangle - \lambda^2}$$

$$= \sqrt{\lambda^2 - \lambda^2}$$

$$= 0$$

4. a) let p be $\{p(1), p(2), p(3), \cdots p(d)\}$ w/ $p(1)$ for event d_1 in $[d]$, p_2 for event $d(2)$ in $[d]$ and so on.
Similarly let q be $\{q(1), q(2), \cdots q(e)\}$ w/ $q(1)$ for event e_1 in $[e]$ and so on
Since drawing i from p and drawing j from q are independent, the probability distribution on $[d] \times [e]$, which is $\{(1,1), (1,2), (1, \cdots), (1,e), \cdots, (d,e)\}$ is

$$\begin{bmatrix} p(1,1) \\ p(1,2) \\ p(1,3) \\ \vdots \\ p(1,e) \\ \vdots \\ p(2,e) \\ \vdots \\ p(d,e) \end{bmatrix} = \begin{bmatrix} p(1)p(1) \\ p(1)p(2) \\ p(1)p(3) \\ \vdots \\ p(1)p(e) \\ \vdots \\ p(2)p(e) \\ \vdots \\ p(d)p(e) \end{bmatrix} = \begin{bmatrix} p(1) \cdot q \\ p(2) \cdot q \\ \vdots \\ p(d) \cdot q \end{bmatrix} = p \otimes q \quad \square$$

b) $\rho = \sum_{i=1}^{m} p_i |\psi_i\rangle\langle\psi_i|$ $\qquad \rho' = \sum_{i=1}^{n} q_i |\phi_i\rangle\langle\phi_i|$

$$= \begin{bmatrix} p_1 & & & \\ & p_2 & & \\ & & p_3 & \\ & & & p_m \end{bmatrix} \qquad = \begin{bmatrix} q_1 & & \\ & q_2 & \\ & & \ddots & \\ & & & q_m \end{bmatrix}$$

Use $\otimes$ to join 2 pure states

mixed state $|M\rangle =$

$$\begin{bmatrix} (p_1q_1, |\psi_1\phi_1\rangle), & (p_1q_2, |\psi_1\phi_2\rangle), & \cdots, & (p_1q_n, |\psi_1\phi_n\rangle) \\ (p_2q_1, |\psi_2\phi_1\rangle), & (p_2q_2, |\psi_2\phi_2\rangle), & \cdots, & (p_2q_n |\psi_2\phi_n\rangle) \\ & \vdots & & \\ (p_nq_1, |\psi_n\phi_1\rangle), & (p_nq_2 |\psi_n\phi_2\rangle), & \cdots & (p_nq_n |\psi_n\phi_n\rangle) \end{bmatrix}$$

$$\rho_m = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j |\psi_i \phi_j\rangle$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (p_i |\psi_i\rangle) \otimes (q_j |\phi_j\rangle))$$

$$= P \otimes P'$$

5. a) $\mathbb{E}_p[I_{d\times d}] = tr(\rho I_{d\times d})$
$= tr(\rho)$
$= 1$ by definition

b) $\mathbb{E}_p[\alpha X + \beta Y] = \alpha \mathbb{E}_p[X] + \beta \mathbb{E}_p[Y]$
$= tr(\rho\alpha X + \rho\beta Y)$
$= \alpha\, tr(\rho X) + \beta\, tr(\rho Y)$
$= \alpha \mathbb{E}_p(X) + \beta \mathbb{E}_p(Y)$

If $\alpha, \beta \in \mathbb{R}$, then $(\alpha X)^\dagger = \alpha X^\dagger = \alpha X$ $\Rightarrow \alpha X + \beta Y$ is hermitian
$(\beta Y)^\dagger = \beta Y^\dagger = \beta Y$

If not, then $(\alpha X)^\dagger$ might be $\alpha^\dagger X$
and $(\beta Y)^\dagger$ might be $\beta^\dagger Y$.
So we are not sure if $\alpha X + \beta Y$ is Hermitian or not.

c) For any arbitrary element on the diagnol of A, let the element be $\alpha + \beta i$
Thus for an random element on the diagnol of $A^\dagger A$, it can be written as $(\alpha+\beta i)^\dagger(\alpha+\beta i)$
$= |\alpha|^2 + |\beta|^2 \geq 0$
$tr(A^\dagger A)$ thus sums all such elements and so ≥ 0
$\therefore \mathbb{E}_p(A^\dagger A) = tr(A^\dagger A) \geq 0$.

d) $\mathbb{E}_p[X]\,\mathbb{E}_{p'}[Y] = tr(\rho X)\, tr(\rho' Y)$
$= \left(\sum_{i,j}^{d} \rho_{ij} X_{ij}\right) \cdot \left(\sum_{i,j}^{d} \rho'_{ij} Y_{ij}\right)$
$= \sum_{i,j}^{d} (\rho_{ij} X_{ij} \rho'_{ij} Y_{ij})$
$= tr[(\rho X) \otimes (\rho' Y)]$
$= \mathbb{E}_{\rho\otimes\rho'}[X \otimes Y]$