CMPSCI 690Q: Quantum Information Systems

Fall 2019

Problem Set 2

Due date: By end of day Oct. 14 on Moodle

When preparing your written solutions, please show all steps involved in obtaining your answer. This will make it easier to assign partial credit (for partially correct answers) and will also help me (orthe grader) in helping you to see where (if anywhere) you made a mistake.

1. (4.4 in NC) Express the Hadamard gate H as a product of R_y and R_z rotations and $e^{i\phi}$ for some ϕ.

2. (NC problems 4.13, 4.14) Prove the following

   (a) $HXH = Z$

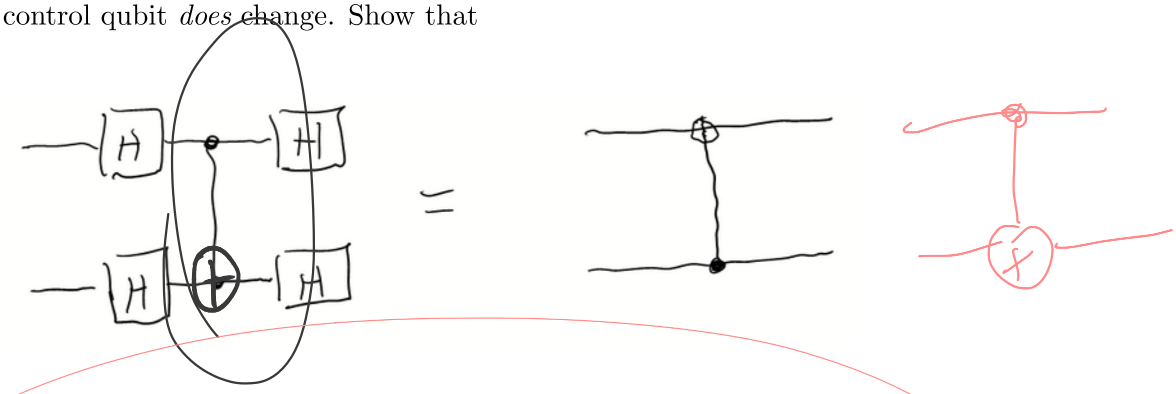
   (b) $HYH = -Y$

   (c) $HZH = X$

   (d) $HTH = R_x(\pi/4)$ up to a global phase.

$R_x(\frac{\pi}{4}) = R(HZH)$

3. Exercise 4.20 in NC. We have described how the CNOT behaves with respect to the computational basis where the state of the control qubit does not change. However, if we work in a different basis the control qubit *does* change. Show that



Introducing basis states $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, use this circuit identity to show the effect of a CNOT with the first qubit as control and second qubit as target to be

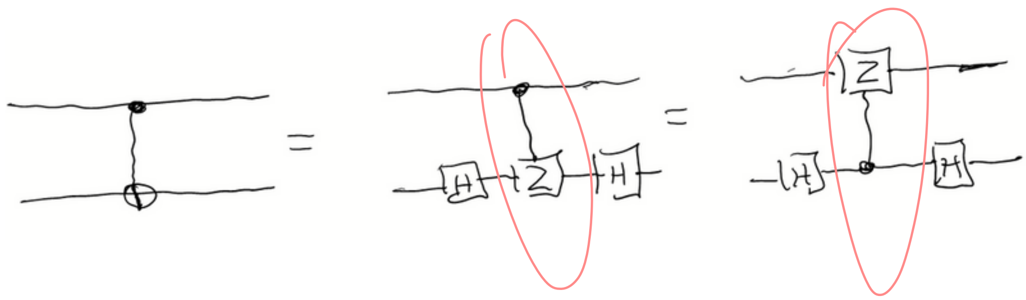
$$|+\rangle|+\rangle \to |+\rangle|+\rangle \tag{1}$$

$$|-\rangle|+\rangle \to |-\rangle|+\rangle \tag{2}$$

$$|+\rangle|-\rangle \to |-\rangle|-\rangle \tag{3}$$

$$|-\rangle|-\rangle \to |+\rangle|-\rangle \tag{4}$$

4. Exercise 4.18 in NC. (Building `CNOT` from controlled-Z gates) Show that the following are equivalent:



5. Exercise 4.25 in NC. (Fredkin gate construction) The Fredkin (controlled swap) gate performs the following operation on 3 qubits

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Give a quantum circuit that uses three Toffoli gates to construct the Fredkin gate (*Hint:* think of the swap gate construction - you can control each gate one at a time).

(b) Show that the first and last Toffoli gates can be repalced by `CNOT` gates.

(c) Now replace the middle Toffoli gate with the circuit given in class for the $C^2(U)$ gate to obtain a Fredkin gate construction using only six two qubit gates.

(d) Can you come up with a simpler construction using only five two-qubit gates?

$$R_x(\theta) = e^{-i\theta\frac{X}{2}} = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_z(\theta) = e^{-i\theta\frac{Z}{2}} = \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix}$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

To get H, start with R_x because R_z is diagonal. Then apply R_z and $e^{i\phi}$ to adjust.

Assume that H can be written in the form of $e^{i\phi} R_{z(\theta_1)} R_{x(\theta_2)} R_{z(\theta_3)}$.

Then we have ↓

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = e^{i\phi}\begin{bmatrix} e^{-i\frac{\theta_1}{2}} & 0 \\ 0 & e^{i\frac{\theta_1}{2}} \end{bmatrix}\begin{bmatrix} \cos\frac{\theta_2}{2} & -i\sin\frac{\theta_2}{2} \\ -i\sin\frac{\theta_2}{2} & \cos\frac{\theta_2}{2} \end{bmatrix}\begin{bmatrix} e^{-i\frac{\theta_3}{2}} & 0 \\ 0 & e^{i\frac{\theta_3}{2}} \end{bmatrix}$$

Using Mathematica,

$$\phi = \frac{\pi}{2}, \quad \theta_1 = \frac{\pi}{2} = \theta_2 = \theta_3$$

$$\boxed{\therefore H = e^{i\frac{\pi}{2}} R_z\left(\frac{\pi}{2}\right) R_x\left(\frac{\pi}{2}\right) R_z\left(\frac{\pi}{2}\right)}$$

Or the system

$$\begin{cases} e^{i\overset{0}{(\alpha - \frac{\beta}{2} - \frac{\delta}{2})}} = 1 \\ -ie^{i\overset{\frac{\pi}{2}}{(\alpha - \frac{\beta}{2} + \frac{\delta}{2})}} = 1 \\ -ie^{i\overset{\frac{\pi}{2}}{(\alpha + \frac{\beta}{2} - \frac{\delta}{2})}} = 1 \\ e^{i\overset{\pi}{(\alpha + \frac{\beta}{2} + \frac{\delta}{2})}} = -1 \end{cases} \rightarrow \begin{cases} \alpha = \frac{\pi}{2} \\ \beta = \frac{\pi}{2} \\ \delta = \frac{\pi}{2}. \end{cases}$$

2.

a)

$$HX = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$ZH = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\therefore HX = ZH$$

$$HXH = ZH \cdot H$$

$$HXH = Z \quad \square.$$

b) $HY = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} i & -i \\ -i & -i \end{bmatrix}$

$YH = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & i \\ i & i \end{bmatrix}$

$\therefore HY = -YH$

$HYH = -YHH$

$HYH = -Y \quad \square$

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c) $HZ = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$XH = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$

$\therefore \quad HZ = XH$

$HZH = XH \cdot H$

$HZH = X \quad \square.$

a). For an arbitrary quantum state $|\psi\rangle$

$$r_1 e^{i\theta}|0\rangle + r_2 e^{i\phi}|1\rangle,$$

$$T|\psi\rangle = r_1 e^{i\theta}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}|0\rangle + r_2 e^{i\phi}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}|1\rangle$$

Since $R_z(\frac{\pi}{4}) = \begin{bmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{bmatrix}$, we rewrite

$T|\psi\rangle$ into $e^{i\frac{\pi}{8}}$.

$$\therefore T|\psi\rangle = r_1 e^{i\theta} e^{i\frac{\pi}{8}}\begin{bmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{bmatrix}|0\rangle$$

$$r_2 e^{i\phi} e^{i\frac{\pi}{8}}\begin{bmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{bmatrix}|1\rangle$$

$$= \underbrace{e^{i\frac{\pi}{8}}}_{\text{global phase}}\left[R_z(\tfrac{\pi}{4})\overbrace{(r_1 e^{i\theta}|0\rangle + r_2 e^{i\phi}|1\rangle)}^{|\psi\rangle}\right]$$

So $T = R_z(\frac{\pi}{4})$. With this we have $HTH = H R_Z(\frac{\pi}{4}) H$

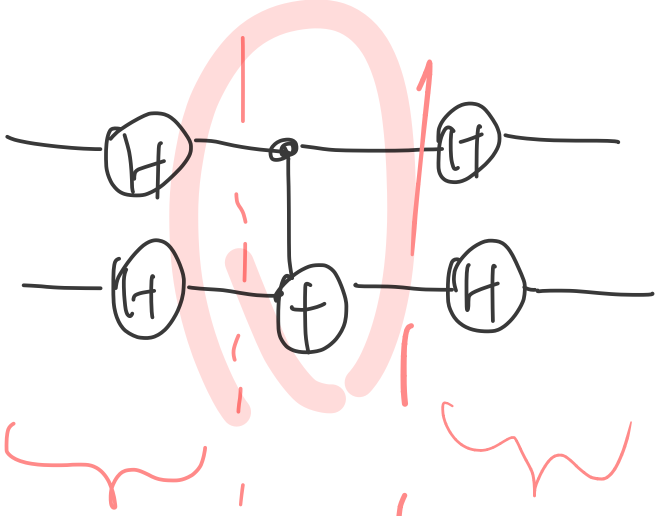
$$= H\left(\cos\frac{\pi}{8} I - i\sin\frac{\pi}{8} Z\right) H$$

$$= \cos\frac{\pi}{8} I H^2 - i\sin\frac{\pi}{8} \cancel{HZH} \times$$

$$= R_Z\left(\frac{\pi}{4}\right)$$

$\therefore HTH = R_Z(\frac{\pi}{4})$ up to a global phase. $\square$.

3.



$= (I \otimes H)(H \otimes I)$

$= \frac{1}{\sqrt{2}} \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}$

$= \frac{1}{\sqrt{2}} \begin{pmatrix} H & H \\ H & -H \end{pmatrix}$

$\begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix}$

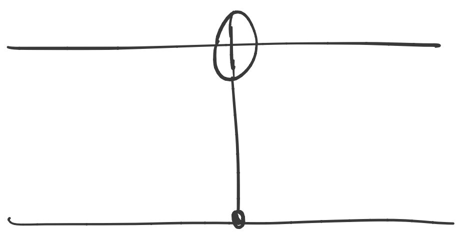
$\frac{1}{\sqrt{2}} \begin{bmatrix} H & H \\ H & -H \end{bmatrix}$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} H & HX \\ H & -HX \end{bmatrix} \begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} I + \cancel{HXH}\, Z & I - \cancel{HXH}\, Z \\ I - \cancel{HXH}\, Z & I + \cancel{HXH}\, Z \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$=$ 

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(1) $|+\rangle|+\rangle \xrightarrow{\text{CNOT}} |+\rangle|+ \oplus +\rangle$

$=$ see next page ↓

$$= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\right)$$

$$|+\rangle|+\oplus+\rangle$$

$$= \frac{1}{2}\left(|0\rangle X^0|0\rangle + |0\rangle X^0|1\rangle + |1\rangle X^1|0\rangle + |1\rangle X^1|1\rangle\right)$$

$$= \frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|1\rangle + |1\rangle|0\rangle\right)$$

$$= |+\rangle|+\rangle$$

$$\therefore \quad |+\rangle|+\rangle \xrightarrow{\text{CNOT}} = |+\rangle|+\rangle \quad \checkmark$$

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(2) $|+\rangle|-\rangle = \frac{1}{2}\left(|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle\right)$

$\swarrow X^0 \quad \swarrow X^0 \quad \swarrow X_1 \quad \swarrow X1$

$$\xrightarrow{\text{CNOT}} \quad = \frac{1}{2}\left(|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|1\rangle\right.$$

$$= \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= |-\rangle|-\rangle \quad \checkmark$$

(3) $|+\rangle|-\rangle = \frac{1}{2}\left(|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle\right)$

$$\xrightarrow{\text{CNOT}} = \frac{1}{2}\left(|0\rangle X^0|0\rangle - |0\rangle X^0|1\rangle + |1\rangle X^1|0\rangle - |1\rangle X^1|1\rangle\right)$$

$$= \frac{1}{2}\left(|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|1\rangle - |1\rangle|0\rangle\right)$$

$$= |-\rangle|-\rangle \quad \checkmark$$

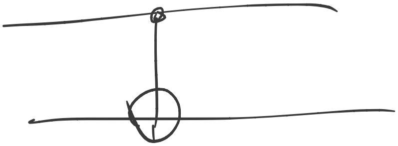
(4) $\overline{|+\rangle|+\rangle} = \frac{1}{2}\left(\underset{\swarrow X^0}{\overline{|0\rangle|0\rangle}} + \underset{\swarrow X^0}{\overline{|0\rangle|1\rangle}} + \underset{\swarrow X^1}{|1\rangle|0\rangle} + \underset{\swarrow X^1}{\overline{|1\rangle|1\rangle}}\right)$

$$\xrightarrow{\text{CNOT}} \frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle - |1\rangle|1\rangle - |1\rangle|0\rangle\right)$$

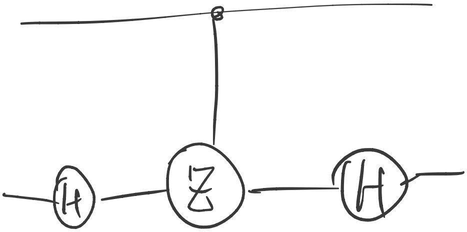
$$= \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

$$= |-\rangle|+\rangle \quad \checkmark$$

4.



$$= \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$



$$= 1 \otimes H \cdot \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} 1 \otimes H$$

$$= \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}$$

$$= \begin{bmatrix} H & 0 \\ 0 & H\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}$$

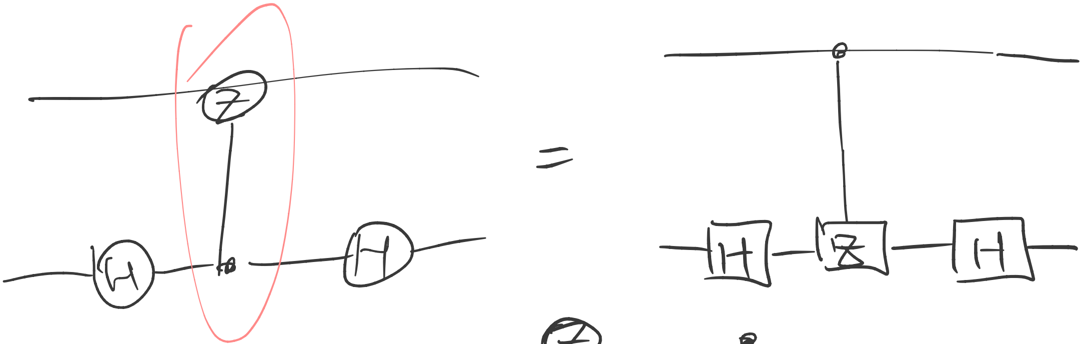
$$= \begin{bmatrix} I & 0 \\ 0 & H\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} H \end{bmatrix}$$

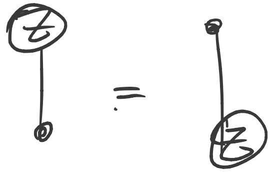
$$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

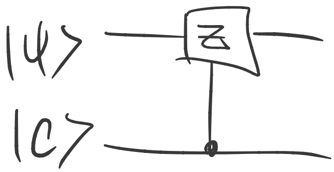
$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



if 

Proof:

1. 

$|\psi\rangle = |0\rangle$ or $|1\rangle$

$|c\rangle = |0\rangle$ or $|1\rangle$

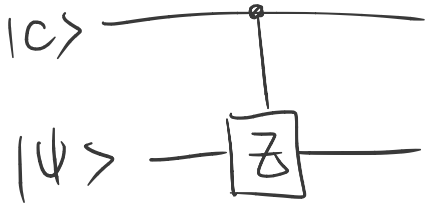
$$|00\rangle \xrightarrow{CZ} Z^0|0\rangle|0\rangle = |00\rangle$$

$$|01\rangle \xrightarrow{CZ} Z^1|0\rangle|1\rangle = |01\rangle$$

$$|10\rangle \xrightarrow{CZ} Z^0|1\rangle|0\rangle = |10\rangle$$

$$|11\rangle \xrightarrow{CZ} Z^1|1\rangle|1\rangle = -|11\rangle$$

2.



$|0\rangle$ or $|1\rangle$

$|0\rangle$ or $|1\rangle$

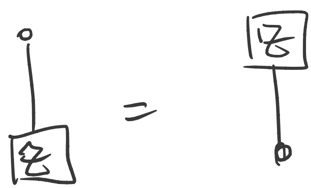
$$|00\rangle \xrightarrow{CZ} |0\rangle Z^0 |0\rangle = |00\rangle$$

$$|01\rangle \xrightarrow{CZ} |0\rangle Z^0 |1\rangle = |01\rangle$$

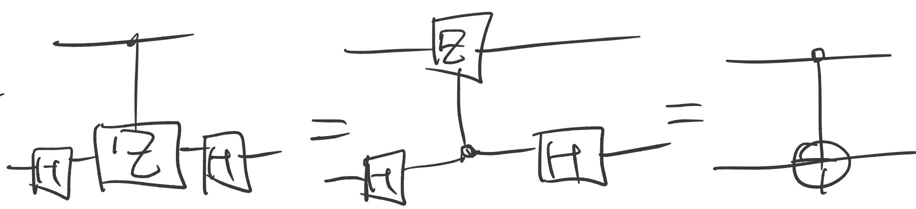
$$|10\rangle \xrightarrow{CZ} |1\rangle Z^1 |0\rangle = |10\rangle$$

$$|11\rangle \xrightarrow{CZ} |1\rangle Z^1 |1\rangle = -|11\rangle$$

Thus



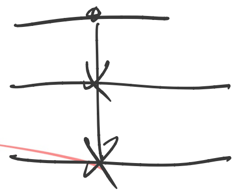
Therefore



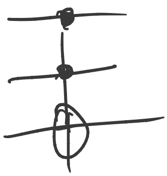
$\square$

5. Fredkin : flips y and z when $c = 1$.

$$\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$



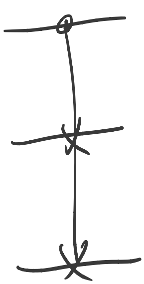
Toffoli :



$$\begin{bmatrix} I & \\ & \begin{matrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{matrix} \end{bmatrix}$$

Acts as substitute of the CNOT flipping role of the Toffoli.

$\therefore$



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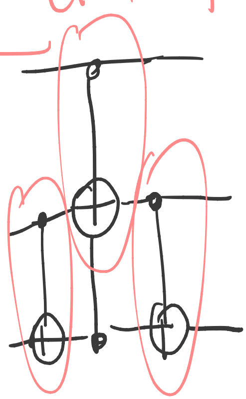
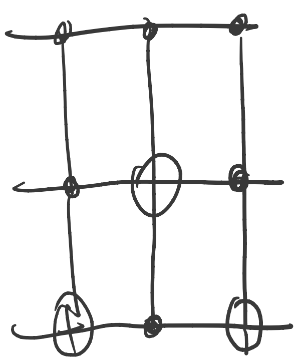


Use Toffoli $\rightarrow$



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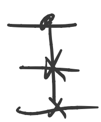
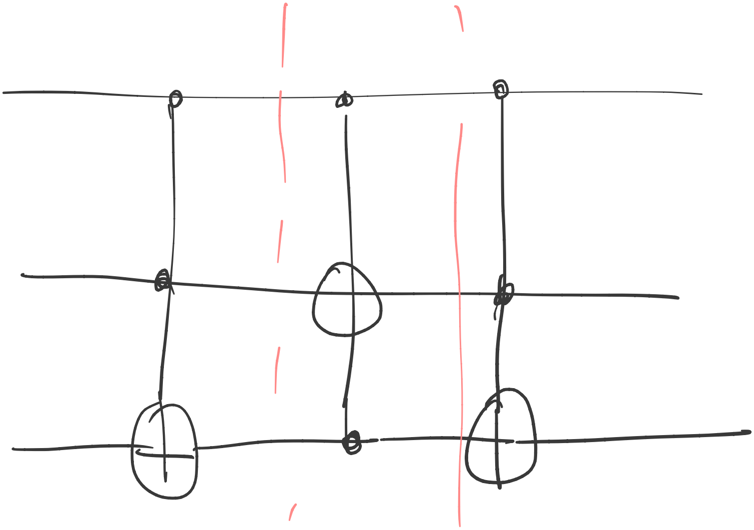


Check



$|\psi_1\rangle = |x, y, z+xy\rangle$

$|\psi_2\rangle = |x, y+xz+x^2y, z+xy\rangle$

$= |x, \bar{x}y+xz, z+xy\rangle$

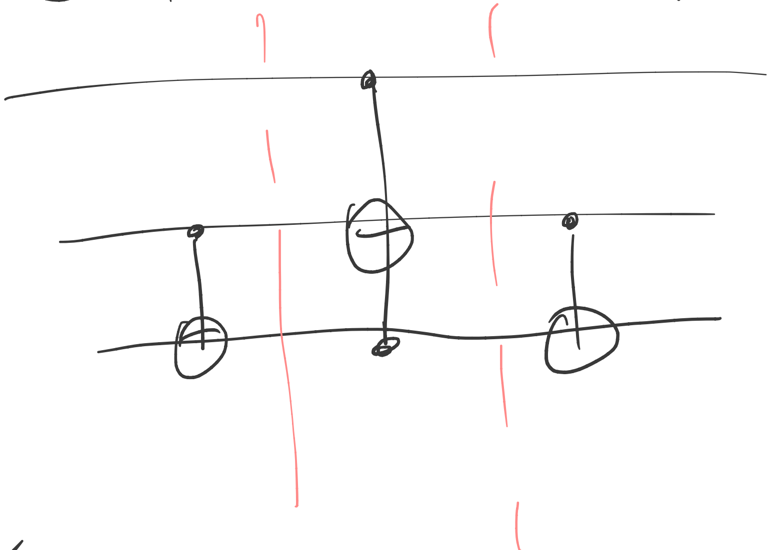
$|\psi_3\rangle = |x, \bar{x}y+xz, z+xy+x\bar{x}y+xz^2\rangle$

$= |x, \bar{x}y+xz, \bar{x}z+xy\rangle$

$\therefore \quad |0\rangle \longrightarrow |0, y, z\rangle$

$|1\rangle \longrightarrow |1, z, y\rangle$ ✓

2. Replace the Toffoli gate on 2 sides with CNOT as shown in previous graph.



Check:

$|\psi_1\rangle = |x, y, z+xy\rangle$

$|\psi_2\rangle = |x, y+xz+x^2y, z+xy\rangle$

$= |x, \bar{x}y + xz, z+xy\rangle$

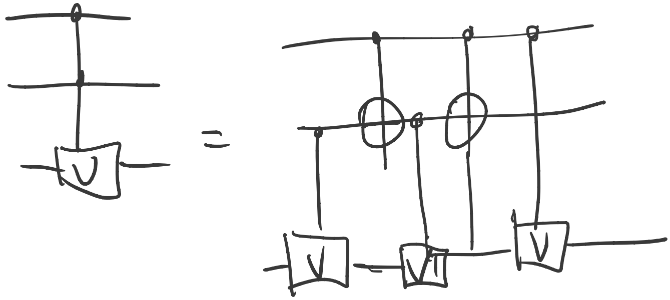
$|\psi_3\rangle = |x, \bar{x}y+xz, z+xy+\bar{x}xy+x^2z\rangle$

$= |x, \bar{x}y+xz, \bar{x}z+xy\rangle$

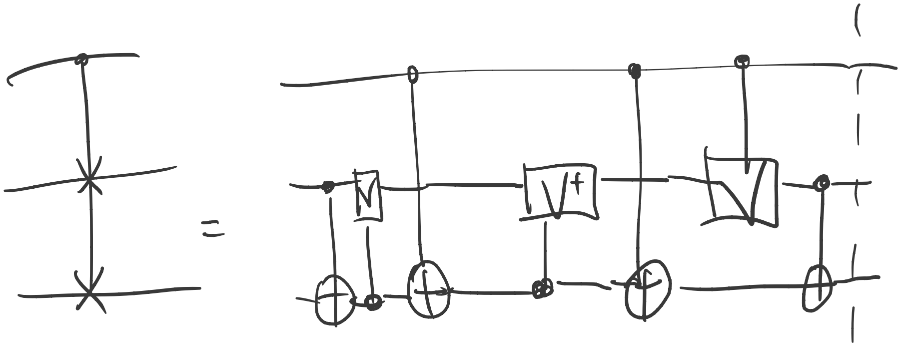
$\Leftarrow$

$\begin{cases} |0\rangle \rightarrow |0, y, z\rangle \\ |1\rangle \rightarrow |1, z, y\rangle \end{cases}$ ✓.

3. $C^2(U)$



$$V = e^{-i\pi/4}(I+iX)/\sqrt{2}. \quad V^2 = X = (V^\dagger)^2$$

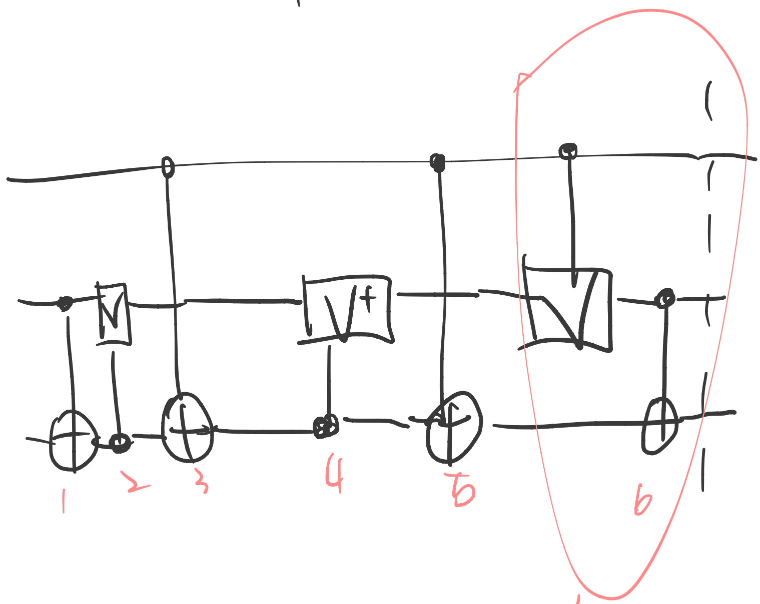


$$\boxed{|\psi_{out}\rangle = |x\rangle \otimes |u\rangle \otimes |u \oplus y \oplus z\rangle,}$$

$$|u\rangle = V^x (V^\dagger)^{x\oplus y\oplus z} V^{y\oplus z} |y\rangle \quad ①$$

$$\begin{cases} x=0, \ |u\rangle = |y\rangle, \ |u\oplus y\oplus z\rangle = |z\rangle \\ x=1, \ |u\rangle = ① = |z\rangle. \end{cases}$$

$\therefore |\psi_{out}\rangle = F|x, y, z\rangle.$



The product of these 2 is still a 2 qubit gate

$\therefore$ Consider this as a 2 qubit gate

$\therefore$ Total : 6. ✓.

d). No. Can't