Yiming Hua
Q15

1.

a) $\boxed{\text{Same}}$, because they differ only by a global phase of -1.

b) $\boxed{\text{Same}}$, differ by a global phase of i

c). $\langle\phi| \equiv \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)^{\dagger} = \frac{1}{\sqrt{2}}(1,1)$

$|\psi\rangle \equiv \frac{1}{\sqrt{2}}(-|0\rangle+i|1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ i\end{pmatrix}$

$P_\psi = |\langle\phi|\psi\rangle|^2 = \frac{1}{4}|1+i|^2$

$$\boxed{P_\psi = \frac{1}{2}}$$

d). $P_\psi = |\langle\phi|\psi\rangle|^2$

let $|\phi\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$, $|\psi\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$

$P_\psi = \left|\frac{1}{\sqrt{2}}(1,1)\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ 1\end{pmatrix}\right|^2$

$= \frac{1}{4} * 0$

$\boxed{P_\psi = 0}$

e) Same, global phase -1.

f) Let $|\phi\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle+i|1\rangle)$ $|\psi\rangle \equiv \frac{1}{\sqrt{2}}(i|1\rangle-|0\rangle)$

$P_\psi = |\langle\phi|\psi\rangle|^2$

$= \left|\frac{1}{\sqrt{2}}(-i,1)\frac{1}{\sqrt{2}}\begin{pmatrix}i\\ -1\end{pmatrix}\right|^2$

$\boxed{P_\psi = 0}$

g) Let $|\phi\rangle \equiv \frac{1}{\sqrt{2}}(|i\rangle + |-i\rangle)$, $|\psi\rangle \equiv \frac{1}{\sqrt{2}}(|-\rangle + |+\rangle)$

$\therefore |\phi\rangle = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\left[(|0\rangle + i|1\rangle) + (|0\rangle - i|1\rangle)\right]$

$= \frac{1}{2} \cdot 2|0\rangle$

$= |0\rangle$

$|\psi\rangle = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\left[(|0\rangle + |1\rangle) + (|0\rangle - |1\rangle)\right]$

$= \frac{1}{2} \cdot 2|0\rangle$

$= |0\rangle$

$\therefore$ $\boxed{\text{Same states.}}$

h) Let $|\phi\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$, $|\psi\rangle \equiv \frac{1}{\sqrt{2}}(e^{i\pi/4}|0\rangle + |1\rangle)$

$e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ (Euler's Identity)

$\therefore \langle\phi| = \frac{1}{\sqrt{2}}(1, 1-i)$, $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i \\ 1 \end{pmatrix}$

$P_\psi = |\langle\phi|\psi\rangle|^2 = \left|\frac{1}{2}\left((1+i)+(1-i)\right)\right|^2$

$\boxed{P_\psi = 1}$

2. a). $|1\rangle$ by definition $= c\,|1\rangle$ for some modulus one complex number c.

$$\frac{1}{\sqrt{2}}(|+\rangle + e^{i\theta}|-\rangle)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + e^{i\theta}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

$$= \frac{1}{2}(|0\rangle + |1\rangle) + \frac{e^{i\theta}}{2}(|0\rangle - |1\rangle)$$

$$= \frac{1+e^{i\theta}}{2}|0\rangle + \frac{1-e^{i\theta}}{2}|1\rangle, \quad = c\,|1\rangle$$

$$\therefore \quad \frac{1+e^{i\theta}}{2} = 0$$

$$e^{i\theta} = -1$$

$$\cos\theta + i\sin\theta = -1$$

$$\boxed{\theta = (2n+1)\pi, \quad n \in \mathbb{Z}}$$

b) $\begin{cases} e^{i\theta}\left|-i\right\rangle = c\left|-i\right\rangle \\ \left|i\right\rangle = ce^{-i\theta}\left|-i\right\rangle \end{cases}$ for any modulus complex c.

$$\therefore \begin{cases} e^{i\theta} = c \\ 1 = ce^{-i\theta} \end{cases}$$

$$c = e^{i\theta}$$

So state 2, which is $\frac{1}{\sqrt{2}}(\left|-i\right\rangle + e^{-i\theta}\left|i\right\rangle)$ differs from state 1, which is $\frac{1}{\sqrt{2}}(\left|i\right\rangle + e^{i\theta}\left|-i\right\rangle)$ by a global phase $e^{i\theta}$. $\therefore$ $\boxed{\text{Any value of } \theta}$ will work.

c) $\boxed{\text{Any value of } \theta}$ can make the pair of states equivalent since they share a global phase of $e^{i\theta}$.

Assume normalized.
So $|a|^2+|b|^2=1$.

2. For a given state $|\psi\rangle$, if $|\psi\rangle = a|\phi_1\rangle + b|\phi_2\rangle$, measuring it w/ measurement basis $\{|\phi_1\rangle, |\phi_2\rangle\}$ will generate output states of either $|\phi_1\rangle$ or $|\phi_2\rangle$ w/ corresponding probabilities of $|a|^2$ and $|b|^2$.

* I will write the answers in the form of:

$$\boxed{P_{(\text{outcome state})} = \text{probability}}$$

b/c $P_{(\phi_1)} = |\langle\phi_1|\psi\rangle|^2$

$= |\langle\phi_1|(a|\phi_1\rangle + b|\phi_2\rangle)|^2$

$= |\underbrace{\langle\phi_1|a|\phi_1\rangle}_{= a\langle\phi_1|\phi_2\rangle = a} + \underbrace{\langle\phi_1|b|\phi_2\rangle}_{\text{out of phase} \ \therefore = 0}|^2$

$= |a|^2$.

a) $$\boxed{\begin{cases} P_{(|0\rangle)} = \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4} \\ P_{(|1\rangle)} = \left|\frac{1}{2}\right|^2 = \frac{1}{4} \end{cases}}$$

b) $$\boxed{\begin{cases} P_{(|1\rangle)} = \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4} \\ P_{(|0\rangle)} = \left|-\frac{1}{2}\right|^2 = \frac{1}{4} \end{cases}}$$

c). $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

$\therefore$ $$\boxed{\begin{cases} P_{(|0\rangle)} = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \\ P_{(|1\rangle)} = \left|\frac{-i}{2}\right|^2 = \frac{1}{2} \end{cases}}$$

d) $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$$\boxed{\begin{cases} P(|+\rangle) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \\ P(|-\rangle) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \end{cases}}$$

e). $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1+i}{2}|i\rangle + \frac{1-i}{2}|-i\rangle$

$$\therefore \boxed{\begin{array}{l} P(|i\rangle) = \left|\frac{1+i}{2}\right|^2 = \frac{1}{2} \\ P(|-i\rangle) = \left|\frac{1-i}{2}\right|^2 = \frac{1}{2} \end{array}}$$

f). $|1\rangle = \frac{-i}{\sqrt{2}}|i\rangle + \frac{i}{\sqrt{2}}|-i\rangle$

$$\boxed{\begin{cases} P(|i\rangle) = \left|\frac{-i}{\sqrt{2}}\right|^2 = \frac{1}{2} \\ P(|-i\rangle) = \left|\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2} \end{cases}}$$

g) $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

let a, b s.t. $a \cdot \left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right) + b\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right)$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\begin{cases} \frac{a}{2} + \frac{\sqrt{3}}{2}b = \frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}}{a} - \frac{b}{2} = \frac{1}{\sqrt{2}} \end{cases} \Rightarrow \begin{cases} a = \frac{\sqrt{3}+1}{2\sqrt{2}} \\ b = \frac{\sqrt{3}-1}{2\sqrt{2}} \end{cases}$$

$$\therefore |+\rangle = \frac{\sqrt{3}+1}{2\sqrt{2}}\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right) + \frac{\sqrt{3}-1}{3\sqrt{2}}\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right)$$

$$\boxed{\therefore \begin{cases} P\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right) = \left|\frac{\sqrt{3}+1}{2\sqrt{2}}\right|^2 = \frac{4+2\sqrt{3}}{8} \\ P\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right) = \left|\frac{\sqrt{3}-1}{3\sqrt{2}}\right|^2 = \frac{4-2\sqrt{3}}{8} \end{cases}}$$

4. $|W\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $|V\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\langle W | V \rangle = \sum_{i=1}^{n} w_i v_i$$

$$= 1 - 1$$

$$= 0.$$

$\therefore$ $|W\rangle$ and $|V\rangle$ are orthogonal

$$\frac{|W\rangle}{\|W\|} = \frac{|W\rangle}{\sqrt{\langle W|W\rangle}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{|V\rangle}{\|V\rangle} = \frac{|V\rangle}{\sqrt{\langle V|V\rangle}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

5. a) Let $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Given $\begin{cases} U|0\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \\ U|1\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle \end{cases}$, $\begin{cases} |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$

We have $\begin{cases} a = \frac{1}{2} \\ c = \frac{\sqrt{3}}{2} \end{cases}$ $\begin{cases} b = \frac{\sqrt{3}}{2} \\ d = -\frac{1}{2} \end{cases}$

$$\therefore \boxed{U = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}}$$

$$U^{\dagger}U = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I.$$

$\therefore$ U is a unitary operator.

b) $\begin{cases} U|+\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \\ U|-\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle \end{cases}$ $\quad |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

Let U be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{cases} \frac{1}{\sqrt{2}}(a+b) = \frac{1}{2} \\ \frac{1}{\sqrt{2}}(c+d) = \frac{\sqrt{3}}{2} \end{cases} \qquad \begin{cases} \frac{4}{\sqrt{2}}(a-b) = \frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}}(c-d) = -\frac{1}{2} \end{cases}$$

$\hookrightarrow$ using Mathematica

$$\boxed{U = \frac{1}{4}\begin{bmatrix} \sqrt{2}+\sqrt{6}, & \sqrt{2}-\sqrt{6} \\ \sqrt{6}-\sqrt{2}, & \sqrt{2}+\sqrt{6} \end{bmatrix}}$$

$$U^{\dagger}U = \frac{1}{4}\begin{bmatrix} \sqrt{2}+\sqrt{6}, & \sqrt{6}-\sqrt{2} \\ \sqrt{2}-\sqrt{6}, & \sqrt{2}+\sqrt{6} \end{bmatrix}\begin{bmatrix} \sqrt{2}+\sqrt{6}, & \sqrt{2}-\sqrt{6} \\ \sqrt{6}-\sqrt{2}, & \sqrt{2}+\sqrt{6} \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I.$$

So U is a unitary operator.