1. $H(X) = H(p_1, \dots p_n) = -\sum_x p_x \log p_x$ for X with probability distribution $p_X(x)$.

$H(\rho) = -\sum_x \lambda_x \log \lambda_x$ for density matrix $\rho \in \mathbb{C}^{d \times d}$.

Since $\rho = \sum_x p_x |x\rangle\langle x|$,

p_x is the eigenvalue for operator A.

(Defined in p69. N/C)

Therefore $H(\rho) = -\sum_x p_x \log p_x = H(X)$ □

2. a. For a system AB in the entangled state of $(|00\rangle + |11\rangle)/\sqrt{2}$,

$H(\rho AB) = 0$ b/c AB is a pure state

while $H(\rho A) = -tr(\rho_A \log \rho A)$

$= -tr(\frac{I}{2} \log \frac{I}{2})$

$= -tr(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix})$

$= -tr(\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix})$

$= 1$

$\therefore H(\rho A | \rho B) = H(\rho_{AB}) - H(\rho A) = 0 - 1 = -1$

b. $H(X, Y) = H(X) + H(Y|X)$

$H(X) \geq 0$ since by definition entropy of a state is always ≥ 0

$H(Y|X) = -\sum_{xy} p(x,y) \log p(y|x)$ (11.23)

$\because -\log p(y|x) \geq 0$

$\therefore H(Y|X) \geq 0$ as proven by N/C in P507

$\therefore H(X) + H(Y+X) \geq 0$

$H(X, Y) \geq 0$ □

3. $I(A:B) = H(\rho_A) + H(\rho_B) - H(\rho_{AB})$

$H(\rho_{AB}) \geq |H(\rho_A) - H(\rho_B)|$ (11.73)

$\therefore I(A:B) \leq H(\rho_A) + H(\rho_B) - |H(\rho_A) - H(\rho_B)|$

$\rightarrow$ If $H(\rho_A) \geq H(\rho_B)$

Then $I(A:B) \leq 2H(\rho_B)$

$\leq 2\min\{\log d_1, \log d_2\}$

$\rightarrow$ If $H(\rho_A) \leq H(\rho_B)$

Then $I(A:B) \leq 2H(\rho_A)$

$\leq 2\min\{\log d_1, \log d_2\}$

$\therefore I(A:B) \leq 2\min\{\log d_1, \log d_2\}$ $\square$

Stabilizer for Nine-Qubit Code

4.

	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9
M_1	Z	Z							
M_2		Z	Z						
M_3				Z	Z				
M_4					Z	Z			
M_5							Z	Z	
M_6								Z	Z
M_7	X	X	X	X	X	X			
M_8				X	X	X	X	X	X

Pauli operators

Since we would like to see (for the phase errors) the phases in the first block of 3 & the second block of 3 are the same, and the 2nd block of 3 and the 3rd block of 3 are the same, we need 6 X_s for each block

First block of 3 : $|000\rangle \pm |111\rangle$

If we want an eigenvector, then we need to (with eigenvalue ± 1 or $\pm i$, this is why we use the Pauli operators),

we need to flip all 3 at once

$\therefore$ We need 6 X_s for 6 qubits.

If XXX is operated on $|000\rangle \pm |111\rangle$, then the outcome would be $(\pm |000\rangle \pm |111\rangle)$

So if for a state $|\psi\rangle$, $X_1X_2X_3X_4X_5X_6|\psi\rangle = |\psi\rangle$,

any phase flip operation acting on $|\psi\rangle$ will give a -1 global phase when measuring $X_1X_2X_3X_4X_5X_6$

$\Rightarrow X_1X_2X_3X_4X_5X_6 \, Z|\psi\rangle = -Z|\psi\rangle$

Because Z anticommutes with $\begin{cases} X_1X_2X_3X_4X_5X_6 \\ X_4X_5X_6X_7X_8X_9 \end{cases}$

$\therefore$ The syndrome measurement for detecting phase flip errors in the Shor's code corresponds to measuring $\begin{cases} X_1X_2X_3X_4X_5X_6 \\ X_4X_5X_6X_7X_8X_9 \end{cases}$

$\square$

6. $Z_1 Z_2 Z_3 (|000\rangle - |111\rangle)$

$$= Z_1 Z_2 Z_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$= Z_1 Z_2 Z_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \\ \\ \\ \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= |000\rangle + |111\rangle \qquad \square$$