COMP6212 (2016/17): Computational Finance Assignment 1 Report

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This report was done to summarize the work for COMP6212 assignment 1.

Task 1: Efficient Mean-Variance Frontier

Suppose there are three assets A1, A2 and A3 with expected return $\boldsymbol{m}$ and covariance matrix C. 100 random mean-variance portfolios of three securities were generated and displayed in Fig. 1 in terms of E-V space [1]. The scatter in Fig. 1 was indicating an approximate boundary for efficient combinations for the three-assets portfolio.

$$\boldsymbol{m} = \begin{pmatrix} 0.10 \\ 0.20 \\ 1.15 \end{pmatrix} \quad C = \begin{pmatrix} 0.005 & -0.010 & 0.004 \\ -0.010 & 0.040 & -0.002 \\ 0.004 & -0.002 & 0.023 \end{pmatrix}$$

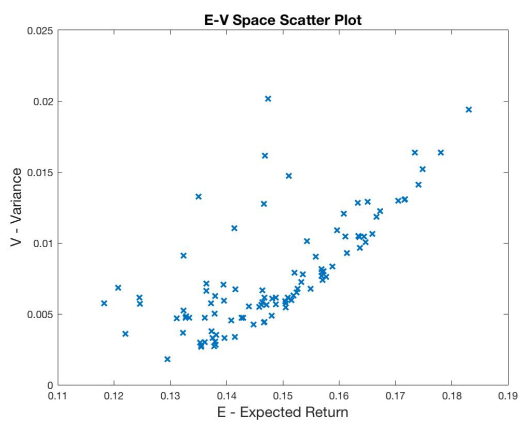

Figure 1: E-V space plot for 100 random portfolios with given $\boldsymbol{m}$ and C.

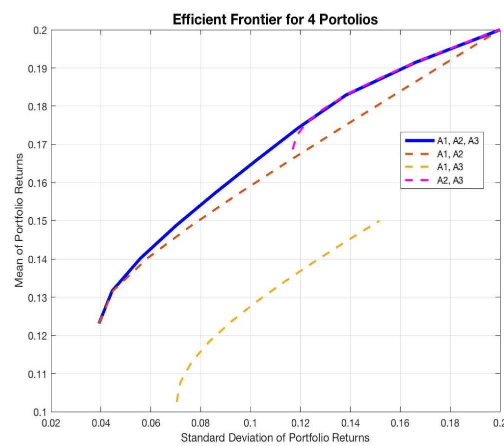

Figure 2: Comparison of efficient frontiers of three-assets model and three two-assets models.

By implementing `frontcon` function, the four distinct efficient frontiers of the three-assets portfolio and three pair-wise models were presented in Fig. 2. Given minimum risk, we can see three-assets portfolio outperformed the other three models. If investor increasing the risk to 0.13 or higher, A2-A3 and A1-A2-A3 models had almost identical return. Likewise, given maximum return, three-assets portfolio had the smallest risk level. In general, diversification in portfolio design is an essential component to achieve financial goals while minimizing investor's risk [2]. However, diversified portfolios cannot guarantee zero risk. So, the portfolio like A1-A2-A3 spreading investment among various securities can somehow provide a 'safe net' to prevent whole portfolio from losing value.

In the `NaiveMV` function in [3], page 384, efficient frontier was composed by a series of efficient portfolios bounded by two extreme efficient portfolios in which one was maximizing return **(1)** while the other was minimizing risk **(2)**. Given N securities, expected return $\boldsymbol{\mu}$ and covariance Σ, we would like to find maximum portfolio return r_{max} without risk constraints.

$$\max_{\boldsymbol{w}} \boldsymbol{w}^T\boldsymbol{\mu} \ subject\ to \sum_{i=1}^{N} w_i = 1\ and\ \boldsymbol{w}_i \geq 0 \qquad (\mathbf{1})$$

Similarly, minimum risk portfolio return r_{MinVar} was found by minimizing risk unconstrained by given return.

$$\min_{\boldsymbol{w}} \boldsymbol{w}^T\boldsymbol{\Sigma}\,\boldsymbol{w} \ subject\ to \sum_{i=1}^{N} w_i = 1\ and\ \boldsymbol{w}_i \geq 0 \qquad (\mathbf{2})$$

Linear Programming `linprog`:

$$\min \boldsymbol{f^T x} \ such\ that \begin{cases} \boldsymbol{Ax} \leq \boldsymbol{b} \\ \boldsymbol{A_{eq}x} = \boldsymbol{b_{eq}} \\ lb \leq \boldsymbol{x} \leq ub \end{cases}$$

`linprog` was used to get the weight $\boldsymbol{w}_{max}$ such that $r_{max} = \boldsymbol{w}_{max}{}^T\boldsymbol{\mu}$. To converse max objective function to min function, we let **f** be $-\boldsymbol{\mu}$ and $\boldsymbol{x}$ be $\boldsymbol{w}$ to get `linprog (`$-\boldsymbol{\mu}$`, [], [], ones(1,N), 1, 0, 0)` based on constraints above. Hence, r_{max} gave an upper bound whereas r_{MinVar} gave a lower bound of returns for the set of efficient portfolios. Hence a set of target returns $\boldsymbol{r}_{tar}$ can be selected by 100 points evenly lying between r_{max} and r_{MinVar}. Subsequently, the set of minimum variances $\boldsymbol{\sigma}_{tar}$ associated with respective $\boldsymbol{r}_{tar}$ were obtained by a `quadprog` problem **(3)**.

$\boldsymbol{\sigma}_{tar}$ can be found by:

$$\min_{\boldsymbol{w}} \boldsymbol{w}^T\boldsymbol{\Sigma}\,\boldsymbol{w} \ subject\ to \sum_{i=1}^{N} w_i = 1,\ \boldsymbol{w}_i \geq 0\ and\ \boldsymbol{w}^T\boldsymbol{\mu} = \boldsymbol{r}_{tar} \qquad (\mathbf{3})$$

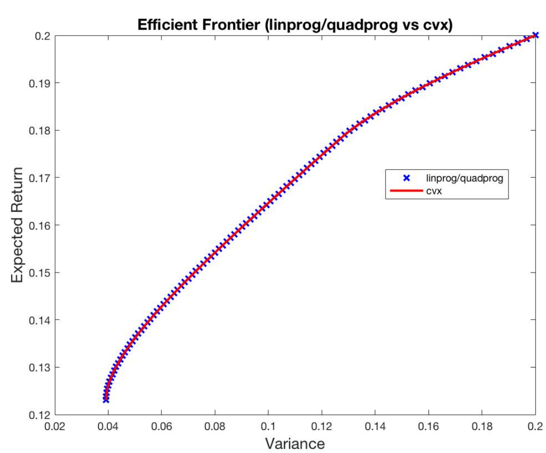

Figure 3: Comparison of efficient frontier plots using `linprog/quadprog` and `cvx` function

With `linprog/quadprog` in `NaiveMV` function replaced by `CVX`, Fig. 3 illustrates comparison between two efficient frontier plots using two different optimization functions. It shows that both methods yielded similar results with almost identical efficient portfolios.

Task 2: Evaluation of Performance

suppose there is a set $\boldsymbol{R}$ of FTSE 100 data for 30 companies from 22 Feb 2014 to 22 Feb 2017 such that $N = \{ n_1, n_2, \ldots., n_{30}\}$ represents 30 companies' stock and $\boldsymbol{R} = \{r_1, r_2, \ldots r_{30}\}$, each vector $\boldsymbol{r}_i$ represents the return for n_i from the entire time series. Average return of each company was to impute missing values if any. As a result, $\boldsymbol{R}$ is a 791×30-dimensional matrix. So I selected stock n_1, n_2, n_3 in my portfolio and computed expected return $\boldsymbol{M}$ and covariance C of submatrix $\boldsymbol{R}_{123} = \{r_1 \quad r_2 \quad r_3\}$ from the first half of the time series. By applying $\boldsymbol{M}$ and C, Markowitz portfolio [1] and naïve 1/N portfolio [4] were employed to construct efficient portfolios with respective weights $\boldsymbol{w}$.

$$\boldsymbol{M} = \begin{pmatrix} 0.629 \\ -0.145 \\ 0.043 \end{pmatrix} \quad C = \begin{pmatrix} 436.471 & 114.970 & 32.612 \\ 114.970 & 197.663 & 17.370 \\ 32.612 & 17.370 & 31.090 \end{pmatrix}$$

- **Markowitz Portfolio:**

$$\min_{w} \boldsymbol{w}^T C\, \boldsymbol{w}\, subject\ to \sum_{i=1}^{3} \boldsymbol{w}_i = 1,\ \boldsymbol{w}_i \geq 0\ and\ \boldsymbol{w}^T \boldsymbol{M} = \rho \quad \textbf{(4)}$$

I chose $\rho = 0.4$ and short selling was not allowed in this case. Markowitz portfolio gave me $\boldsymbol{w}_{mp} = (0.690 \quad 0.000 \quad 0.390)^T$ through optimizing the objective function **(4)**.

- **Naïve 1/N portfolio:**

This strategy simply gave $\boldsymbol{w}_{np} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}^T$

To measure the performance between two strategies, Sharp ratio was calculated for the reminder of the period. Assuming the risk-free rate in UK was 2.5%, Sharp ratio for Markowitz portfolio was $sr_{mp} = -0.056$ whereas $sr_{np} = -0.091$ for naïve strategy. Indeed 1/N portfolio had worse results despite that DeMiguel et al. (2009) pointed out naïve 1/N benchmark had consistent better performance than 14 models discussed in [5]. However, the third contribution in [5] also indicated three conditions that the portfolio strategies outperformed 1/N policy if:
(i) sample mean and variances were estimated from a long estimation window;
(ii) mean-variance efficient portfolio had considerably higher Shape ratio than 1/N strategy;
(iii) we had a relatively small number of assets.

In our example, $\boldsymbol{M}$ and C were estimated from one and half years' data (i.e. condition (i)) and only three stocks were involved in the portfolio (i.e. condition(ii)). Even through Sharpe ratio of

Markowitz portfolio was not proved to be statistically greater than that of 1/N approach (i.e. condition (iii)), I think our simple illustration affirmed the third contribution shown by DeMiguel et al. (2009).

Task 3: Greedy and Sparse Index Tracking

Using above data and notation in Task 2, Task 3 aimed to identify a possible subset $S \subseteq N$ of available stocks to approximate FTSE 100 index. Let vector $\boldsymbol{y} = [y_1 \ \ y_2 \dots \dots \ y_{791}]^T$ such that each y_i represents daily index return in the past three years. The subset selection was discussed in [6] page 21 and attained through minimizing tracking error such that:

$$\min_{\widetilde{\boldsymbol{w}}} \|\boldsymbol{y} - \boldsymbol{R}\widetilde{\boldsymbol{w}}\|_2^2 \ \ subject\ to\ \|\widetilde{\boldsymbol{w}}\|_0 = w_0 \qquad (\mathbf{5})$$

where $\|\widetilde{\boldsymbol{w}}\|_0$ determines $|S|$ i.e. number of non-zero elements in weights $\boldsymbol{w}$. Mean Squared Error (MSE) was used to evaluate the performance of greedy forward selection algorithm and sparse index tracking portfolio as the ultimate goal in this case was to find out which subset can accurately describe index tendency over time span.

- **Greedy Algorithm**

Six companies' stocks were required to depict overall index through six iterations using greedy forward selection algorithm (i.e. $w_0 = 6$ in **(5)**). Let $\boldsymbol{R_s}$ be 791 × $|S|$- dimensional submatrix of $\boldsymbol{R}$ and $\widetilde{\boldsymbol{w}}$ be weights of each stock in *S*. **Algorithm 1** demonstrated the pseudo code of implementing greedy algorithm for our example.

Algorithm 1 Greedy Algorithm

Input: $\boldsymbol{y}, \boldsymbol{R_s}$
Output: S and $\widetilde{\boldsymbol{w}}$
Initialize: $S_0 = \emptyset, G = 6$ and $k = 1$
While $k \le G$ ***do***
  For all $s \in N \setminus S_{k-1}$, *compute*
    $\widetilde{\boldsymbol{w}}_{S_{k-1}} = \min_{\widetilde{\boldsymbol{w}}_{S_{k-1}}} \left\|\boldsymbol{y} - \boldsymbol{R}_{S_{k-1}}\widetilde{\boldsymbol{w}}_{S_{k-1}}\right\|_2^2 \ subject\ to\ \sum_{i=1}^{k} \widetilde{w}_{S_{k-1}i} = 1$
    $\boldsymbol{error}_{S_{k-1}} = \left\|\boldsymbol{y} - \boldsymbol{R}_{S_{k-1}}\widetilde{\boldsymbol{w}}_{S_{k-1}}\right\|_2^2$
  Select $s^* = \min_{s \in N \setminus S_{k-1}} \boldsymbol{error}_{S_{k-1}}$
  Set $S_k = S_{k-1} \cup s^*, \ \ \widetilde{\boldsymbol{w}}_{S_k} = \widetilde{\boldsymbol{w}}_{S_{k-1} \cup s^*}$ and $k = k + 1$
End while
Set $S = S_G$ and $\widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{w}}_{S_G}$
Return S and $\widetilde{\boldsymbol{w}}$

Takeda, A. et al. (2013) also employed greedy algorithm to find S using sparse tracking portfolio model in [6]. The main difference between his work and mine was that if a penalty term was included in the minimizing objective function **(5)**. In our example, $\boldsymbol{R_s}$ was restricted by S and $|S|$ was fixed by 6, so only one stock was determined and added into S at each iteration. Thus, it might be redundant to include penalty to comprise sparsity in this case.

Iteration k	**Subset S_k**	**Iteration k**	**Subset S_k**
1	n_{19}	4	$n_{19}, n_{22}, n_{15}, n_{18}$
2	n_{19}, n_{22}	5	$n_{19}, n_{22}, n_{15}, n_{18}, n_{30}$
3	n_{19}, n_{22}, n_{15}	6	$n_{19}, n_{22}, n_{15}, n_{18}, n_{30}, n_1$

Table 1 : Greedy algorithm selection process at each iteration

Table 1 shows the greedy algorithm process at each iteration by implementing **Algorithm 1**. The selection process terminated at stage $k = 6$ and returned $S = \{n_{19}, n_{22}, n_{15}, n_{18}, n_{30}, n_1\}$ and corresponding weights $\widetilde{\boldsymbol{w}} = (0.876 \;\; 0.602 \; -0.124 \;\; 0.117 \; -0.589)^T$ with MSE = 1.525.

- **Sparse Index Tracking Portfolio**

Brodie, L. et al. (2009) introduced a sparse and stable portfolio by adopting a ℓ_1 regularization to Markowitz objective function **(4)** where a parameter τ was to adjust the relative importance of ℓ_1 penalization. In addition to sparse portfolio construction, they also discussed the equivalent concept into index tracking in [5]. To find S, it is mandatory to acquire $\widetilde{\boldsymbol{w}}_{sp}^{\tau}$ by:

$$\widetilde{\boldsymbol{w}}_{sp}^{\tau} = min_{\widetilde{\boldsymbol{w}}_{sp}} \left\| \boldsymbol{y} - \boldsymbol{R}\widetilde{\boldsymbol{w}}_{sp} \right\|_2^2 + \tau \left\| \widetilde{\boldsymbol{w}}_{sp} \right\|_1 \; subject\ to \sum_{i=1}^{N} \widetilde{w}_{sp_i} = 1 \qquad \textbf{(6)}$$

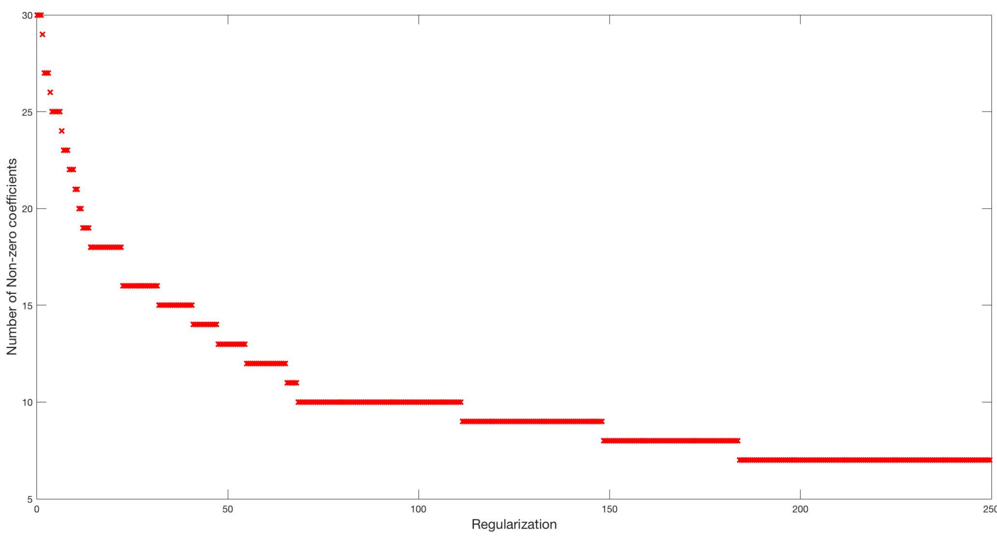

Figure 4: Convergence of number of non-zero coefficients in $\widetilde{w}_{sp}^{\tau}$ when increasing regularization parameter τ

The parameter τ was adjusted to enforce sparsity to portfolio **(6)** so that the number of stocks selected by ℓ_1 regularization was similar to the number obtained by greedy algorithm. Therefore, the components in $\widetilde{\boldsymbol{w}}_{sp}^{\tau}$ would be neglected if its value was less than 0.0001. Fig. 4 shows that there was remarkable convergence of number of non-zero coefficients when τ went to greater value. In this case, $\tau = 185$ was chosen and $\widetilde{\boldsymbol{w}}_{sp}^{185}$ gave 7 non-zero components. Thus, the selected stocks were $S = \{n_4, n_{14}, n_{15}, n_{18}, n_{19}, n_{22}, n_{23}\}$ with associated $\widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{w}}_{sp}^{185} = (0.181 \;\; 0.082 \; -0.007 \;\; 0.124 \;\; 0.315 \;\; 0.437 \;\; 0.018)^T$. The MSE was 1.860.

Both of approaches had a very similar performance to choose S in terms of selecting stocks and MSE. Stocks $n_{15}, n_{18}, n_{19}, n_{22}$ were considered as the best ones to represent index level because they were concurrently selected by above two strategies as well as carrying relatively heavy weights in each $\widetilde{\boldsymbol{w}}$. Furthermore, greedy algorithm may sometimes yield local optimum. So comparison between two methods can further cross validate the collection of stocks that can correctly track the index. Besides, there was only 0.335 difference between two MSEs where greedy algorithm provided a smaller error. In this study, index tracking on problem **(5)** was of priority whereas the profit-and-loss was trivial. Hence, it is more appropriate to use MSE to evaluate two strategies rather than Shape ratio, turnover etc. Clearly, taking target expected return (i.e. $\widetilde{\boldsymbol{w}}^T \boldsymbol{R_s} = \rho$) into consideration would gain more insights into index tracking.

Task 4: Discussion of Lobo et al.

Task 4 presented the portfolio optimization taking transaction cost into consideration and subject to various type of constraints, e.g. shortselling and shortfall constraints, on the feasible portfolios which being illustrated in the work done by Lobo et al. (2007). A specific problem in [8] section 1.6 would be discussed with respect to the objective functions, constraints and implementation.

- **Portfolio Selection with Transaction Costs**

In Lobo, et al.'s paper, transaction cost was thought of a movement in asset weights in the portfolio from current time t_0 to end of financial period t_T. Suppose initially at time t_0, investor had current weights $\boldsymbol{w} = (w_1, \dots, w_n)^T$ for assets and total wealth $\mathbf{1}^T\boldsymbol{w}$. At end of period, the initial holdings $\boldsymbol{w}$ were adjusted by $\boldsymbol{x} = (x_1, \dots, x_n)^T$(i.e. each $asset_i$ had transaction amount x_i at end of period t_T, $x_i > 0$ for buying and $x_i < 0$ for selling), thus, resulting in $(\boldsymbol{w} + \boldsymbol{x})$ for final portfolio. Suppose return at $t_T$was $\boldsymbol{a} = (a_1, \dots, a_n)$where each $asset_i$ had return a_i at t_T. Hence, we can obtain expected return $\bar{a}$ and its covariance matrix Σ where:

$$\bar{\boldsymbol{a}} = \mathbb{E}[\boldsymbol{a}] \ , \ \Sigma = \mathbb{E}[(\boldsymbol{a} - \bar{\boldsymbol{a}})(\boldsymbol{a} - \bar{\boldsymbol{a}})^T]$$

The return of each $riskless\ asset_i$ was denoted by $\bar{a}_i$ and its corresponding covariance in Σ was zero. With resulting weights $(\boldsymbol{w} + \boldsymbol{x})$ at t_T, we can easily compute end of period wealth $\boldsymbol{W} = \boldsymbol{a}^T(\boldsymbol{w} + \boldsymbol{x})$. The expectation and variance of W was given by:

$$\mathbb{E}(\boldsymbol{W}) = \bar{\boldsymbol{a}}^T(\boldsymbol{w} + \boldsymbol{x}), \mathbb{V}ar(\boldsymbol{W}) = (\boldsymbol{w} + \boldsymbol{x})^T\Sigma(\boldsymbol{w} + \boldsymbol{x})$$

Suppose $\phi(\boldsymbol{x})$ be transaction cost function imposed on $\boldsymbol{x}$ and the budget constraint defined by $\mathbf{1}^T\boldsymbol{x} + \phi(\boldsymbol{x}) = 0$. Therefore, investors can select the portfolio by maximizing expected **W** subject to budget constraint:

$$\max_x \bar{\boldsymbol{a}}^T(\boldsymbol{w} + \boldsymbol{x}) \ subject\ to\ \mathbf{1}^T\boldsymbol{x} + \phi(\boldsymbol{x}) \le 0\ and\ \boldsymbol{w} + \boldsymbol{x} \in \mathcal{S} \qquad \textbf{(7)}$$

where $\mathcal{S}$ denoted as set of feasible portfolios. The budget constraint was rewritten as inequality for use of numerical optimization methods [8]. Thus, transaction costs merged into portfolio optimization **(7)** by being attached with objective function and constraint. Besides, Lobe et al. also discussed another portfolio optimization when minimizing total transaction cost

$\phi(\boldsymbol{x})$ appeared to be significant. Suppose r_{min} is the minimum expected return, this problem can be formulated as:

$$min_{\boldsymbol{x}}\ \phi(\boldsymbol{x})\ subject\ to\ \overline{\boldsymbol{a}}^T(\boldsymbol{w}+\boldsymbol{x}) \geq r_{min}\ and\ \boldsymbol{w}+\boldsymbol{x}\ \in \mathcal{S} \qquad (\mathbf{8})$$

- **Portfolio Optimization with Various Constraints**

Suppose total transaction $x_i = (x_i^+ - x_i^-)$ where x_i^+ and x_i^- representing transacted amount in $asset_i$ for buying and selling accordingly, α_i^+ and α_i^- are corresponding cost rate; each $asset_i$ had maximum amount s_i for shortselling. A specific problem **(9)** discussed in [8] section 1.6 was displayed below with its efficient frontier plot in Fig. 5.

$$\begin{aligned}
&max_{\boldsymbol{x}} && \overline{\boldsymbol{a}}^T(\boldsymbol{w}+x^+ - x^-) && \textbf{\textit{(9.1)}} \\
&subject\ to && \mathbf{1}^T(x^+ - x^-) + \textstyle\sum_{i=1}^{n}(\alpha_i^+ x_i^+ + \alpha_i^- x_i^-) \leq 0 && \textbf{\textit{(9.2)}} \\
& && x_i^+, x_i^- \geq 0, i = 1,2,\dots,n && \textbf{\textit{(9.3)}} \quad (9) \\
& && \boldsymbol{w}_i + x_i^+ - x_i^- \geq s_i, i = 1,2,\dots,n && \textbf{\textit{(9.4)}} \\
& && \Phi^{-1}(\eta_j)\left\|\Sigma^{1/2}(\boldsymbol{w}+x^+ - x^-)\right\| \leq \overline{\boldsymbol{a}}^T(\boldsymbol{w}+x^+ - x^-) - W_j^{low}, j = 1,2. && \textbf{\textit{(9.5)}} \\
& && \eta_1 = 80\%, W_1^{low} = 0.9\ (bad\ return); \eta_1 = 97\%, W_1^{low} = 0.7\ (disastrous\ retun) &&
\end{aligned}$$

Problem **(9)** aimed to maximize expected return by imposing objective function **(9.1)** subject to linear transaction cost constraints **(9.2)** & **(9.3)**, individual bound of shortselling s_i on $asset_i$ **(9.4)** and two shortfall risk constraints **(9.5)**. Suppose $\boldsymbol{a} \sim \boldsymbol{\mathcal{N}}(\overline{\boldsymbol{a}}, \boldsymbol{\Sigma})$ and W^{low} was minimum target portfolio return with $\boldsymbol{Prob}(W \geq W^{low}) \geq \eta$ $(\mathbf{10})$. With statistical proof in [8] section 1.5, we can rewrite **(10)** as from of **(9.5)** by imposing standard Gaussian distribution $\Phi(z)$ and assumption $W = \overline{\boldsymbol{a}}^T(\boldsymbol{w}+x^+ - x^-) \sim \boldsymbol{\mathcal{N}}(\mu, \sigma)$.

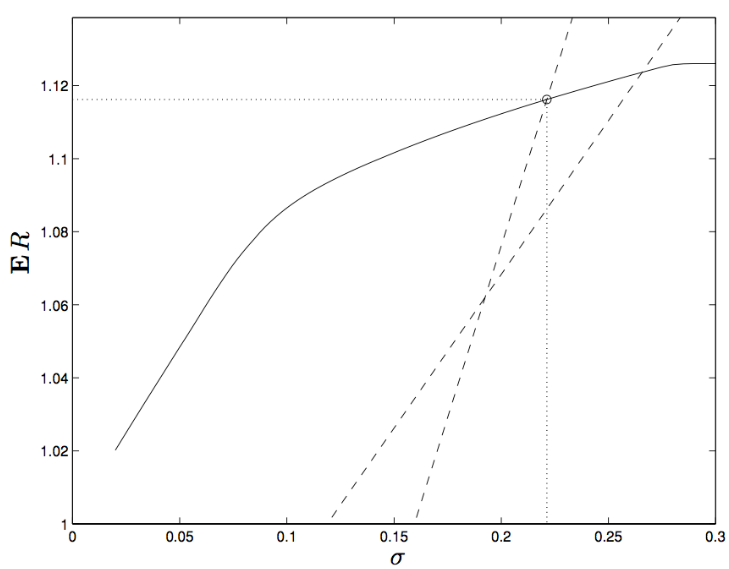

Figure 5: Efficient frontier of problem (9) in [8] with 100 stocks and one riskless asset ignoring constraint (9.5). The dashed lines showed expected return and variance imposed by (9.5). The small circle was the optimal solution with (9.5).

With 100 risky assets and one riskless asset imposed by **(9)**, the efficient frontier of this problem was plotted in Fig. 5 restricting by constraints **(9.2)**, **(9.3)**, **(9.4)**. The steeper dashed line corresponded shortfall probability **(9.5)** of disastrous return. Suppose the maximizing problem **(9)** generated optimal weights $\breve{\boldsymbol{w}} = (\boldsymbol{w} + x^{+} - x^{-})$ and let expected return $E = \overline{\boldsymbol{a}}^{T}\breve{\boldsymbol{w}}$ and standard deviation of return $\breve{\sigma} = \left\|\Sigma^{1/2}\breve{\boldsymbol{w}}\right\|$. So **(9.5)** can be rewritten as $\Phi^{-1}(\eta_j)\breve{\sigma} \le E - W_j^{low}$ where $\Phi^{-1}(\eta_j)$ and W_j^{low} were given constants. Clearly, the shortfall risk constraints were linear in this case. If we replaced the inequality by equality, we can obtain the slope by:

$$slope = \frac{\partial E - W_j^{low}}{\partial \breve{\sigma}} = \Phi^{-1}(\eta_j), j = 1,2$$

Therefore, $\Phi^{-1}(\eta_2 = 97\%)$had a steeper slope presenting consistently with the dashed line in Fig.5. Lobo, et al also pointed out in [8] section 1.5 if $\eta \ge 0.5$, the shortfall constraints can preserve the optimization as convex problem. The intersection of efficient frontier and dashed lines indicated the optimal solution imposed on shortfall risk. Hence, the circle in Fig. 5 implied the maximum return with associated standard deviation restricted by shortfall constraints of disastrous return.

Using above FTSE 100 data, we can formulate a similar portfolio optimization aspects of transaction cost and constraints in **(9)**. Firstly, the current weight $\boldsymbol{w}$ at $t_0 = 22\ Feb\ 2014\ can$ be initialized by naïve 1/N strategy and $\overline{\boldsymbol{a}}^{T}$ can be obtained by taking average of stocks' return at $t_T = 22\ Feb\ 2017$. Then we need to specify transaction cost, shortselling limit and shortfall constraints by choosing appropriate values for α_i^{+}, α_i^{-}, s_i, η_j and W_j^{low}. Hence, the efficient frontier with associated transaction amount $x_i = (x_i^{+} - x_i^{-})$ (ignoring shortfall risk) can be computed via adopting objective **(9.1)** and constraints **(9.2)**, **(9.3)**, **(9.4)** into `NaiveMV` function in [3]. In addition to efficient portfolio, the optimal solution imposed by shortfall risk would be the intersection of the shortfall risk line (i.e. alike the dashed lines in Fig. 5) and efficient frontier.

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