COMP6212 (2016/17): Computational Finance Assignment 2 Report

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This report was done to summarize the work for COMP6212 assignment 2.

Task 1: Derive Black-Scholes Equation
Let's define the notation:

K:	strike price	T:	maturity time
S:	value of the underlying asset	t:	current time
r:	risk-free interest rate	$\mathcal{N}(x)$:	cumulative normal distribution

Step 1:
Clearly, $\mathcal{N}'(x)$ is probability density function of normal distribution i.e.

$$\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \quad \mathbf{(1)}$$

Step 2:
Prove $S\mathcal{N}'(d_1) = K\exp\left(-r(T-t)\right)\mathcal{N}'(d_1)$ where

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}} \ \mathbf{(2)}\ and\ \ d_2 = d_1 - \sigma\sqrt{(T-t)} \ \ \mathbf{(3)}$$

proof:
Equation **(1)** gives

$\mathcal{N}'(d_1) = \ e^{\frac{-{d_1}^2}{2}}$

$\mathcal{N}'(d_2) = \ e^{\frac{-{d_2}^2}{2}} = e^{\frac{-(d_1 - \sigma\sqrt{(T-t)})^2}{2}} = \ e^{\frac{-d1^2}{2} + d_1\sigma\sqrt{(T-t)} - \frac{\sigma^2(T-t)}{2}}$

we take logs for both $S\mathcal{N}'(d_1)$ and $K\ \exp\left(-r(T-t)\right)\mathcal{N}'(d_2)$

$\log\left(S\mathcal{N}'(d_1)\right) = \log(s) - \frac{{d_1}^2}{2}$

$\log\Big(K\ exp\big(-r(T-t)\big)\ \mathcal{N}'(d_2)\Big)$

$= \log(K) - r(T-t) - \frac{{d_1}^2}{2} + \ d_1\sigma\sqrt{(T-t)} - \frac{\sigma^2(T-t)}{2}$

$= \log(K) - \frac{{d_1}^2}{2} + \ d_1\sigma\sqrt{(T-t)} - (r + \frac{\sigma^2}{2})(T-t)$

$\log\left(S\mathcal{N}'(d_1)\right) - \log\Big(K\ exp\big(-r(T-t)\big)\ \mathcal{N}'(d_1)\Big)$

$= \log(s) - \frac{{d_1}^2}{2} - (\log(K) - \frac{{d_1}^2}{2} + \ d_1\sigma\sqrt{(T-t)} - (r + \frac{\sigma^2}{2})(T-t))$

$= \log\left(\frac{S}{K}\right) - d_1\sigma\sqrt{(T-t)} + (r + \frac{\sigma^2}{2})(T-t)$

Now substitute **(2)** into above function and it gives

$= \log\left(\frac{S}{K}\right) - \frac{\log\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}\sigma\sqrt{(T-t)} + (r + \frac{\sigma^2}{2})(T-t)$

$= \log\left(\frac{S}{K}\right) - \log\left(\frac{S}{K}\right) - \left(r + \frac{\sigma^2}{2}\right)(T-t) + \left(r + \frac{\sigma^2}{2}\right)(T-t) = 0$

So $\log\left(S\mathcal{N}'(d_1)\right) = \log\Big(K\ exp\big(-r(T-t)\big)\ \mathcal{N}'(d_2)\Big)$ i.e.

$$S\mathcal{N}'(d_1) = K\ \exp\big(-r(T-t)\big)\ \mathcal{N}'(d_2) \ \ \mathbf{(4)}$$

Step 3:

$\frac{\partial d_1}{\partial S} = \frac{1/S}{\sigma\sqrt{(T-t)}} = \frac{1}{\sigma S\sqrt{(T-t)}}$ (**5**)

$\frac{\partial d_2}{\partial S} = \frac{\partial(d_1 - \sigma\sqrt{(T-t)})}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{1}{\sigma S\sqrt{(T-t)}}$ (**6**)

Step 4:

The option price given by

$$c = S\mathcal{N}(d_1) - K\exp\big(-r(T-t)\big)\mathcal{N}(d_2) \quad (\mathbf{7})$$

$\frac{\partial c}{\partial t} = S\mathcal{N}'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}\mathcal{N}(d_2) - Ke^{-r(T-t)}\mathcal{N}'(d_2)\frac{\partial d_2}{\partial t}$

Equation **(4)** gives

$= S\mathcal{N}'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}\mathcal{N}(d_2) - S\mathcal{N}'(d_1)\frac{\partial d_2}{\partial t}$

$= -rKe^{-r(T-t)}\mathcal{N}(d_2) + S\mathcal{N}'(d_1)(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t})$

$= -rKe^{-r(T-t)}\mathcal{N}(d_2) + S\mathcal{N}'(d_1)(\frac{\partial d_1}{\partial t} - (\frac{\partial d_1}{\partial t} - \frac{\sigma}{2\sqrt{(T-t)}}))$

$= -rKe^{-r(T-t)}\mathcal{N}(d_2) - S\mathcal{N}'(d_1)\frac{\sigma}{2\sqrt{(T-t)}}$ (**8**)

$\frac{\partial c}{\partial S} = \mathcal{N}(d_1) + S\mathcal{N}'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}\mathcal{N}'(d_2)\frac{\partial d_2}{\partial S}$

Equation **(4)** gives

$= \mathcal{N}(d_1) + S\mathcal{N}'(d_1)\frac{\partial d_1}{\partial S} - S\mathcal{N}'(d_1)\frac{\partial d_2}{\partial S}$

Equation **(5)** and **(6)** gives

$= \mathcal{N}(d_1) + S\mathcal{N}'(d_1)(\frac{\partial d_1}{\partial S} - \frac{\partial d_2}{\partial S}) = \mathcal{N}(d_1)$ (**9**)

Step 5:

$\frac{\partial^2 c}{\partial^2 S^2} = \mathcal{N}'(d_1)\frac{\partial d_1}{\partial S} = \frac{\mathcal{N}'(d_1)}{\sigma S\sqrt{(T-t)}}$ (**10**)

Black-Scholes models gives

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial^2 S^2} - rc = 0 \quad (\mathbf{11})$$

Now we substitute equation **(8)**, **(9)** and **(10)** into left-hand side (LHS) of (**11)** to see if we can obtain zero at its right-hand side (RHS).

$-rKe^{-r(T-t)}\mathcal{N}(d_2) - S\mathcal{N}'(d_1)\frac{\sigma}{2\sqrt{(T-t)}} + rS\mathcal{N}(d_1) + \frac{1}{2}\sigma^2 S^2\frac{\mathcal{N}'(d_1)}{\sigma S\sqrt{(T-t)}} - rc$

$= -rKe^{-r(T-t)}\mathcal{N}(d_2) + rS\mathcal{N}(d_1) - rc - S\mathcal{N}'(d_1)\frac{\sigma}{2\sqrt{(T-t)}} + \frac{1}{2}\sigma S\frac{\mathcal{N}'(d_1)}{\sqrt{(T-t)}}$

$= r(-Ke^{-r(T-t)}\mathcal{N}(d_2) + rS\mathcal{N}(d_1) - c)$

Equation **(7)** gives

$= r(c - c) = 0$

so LHS = RHS proves call option is the solution to the Black-Scholes differential equation.

Task 2: Evaluation of Option Prices by Black-Scholes Model

Normally the uncertainty of return is measured by volatility σ resulted from stocks. P. Brandimarte had presented a straightforward illustration of volatility estimation using historical stock prices in [2] section 15.4. Let Define:

$n+1$: number of available data

S_i: closed stock price at end of i^{th} time interval, $i = 0,1,2,\dots,n$

τ: length of time interval in year

$u_i =$ $\log\left(\frac{S_i}{S_{i-1}}\right), i = 1,2,\dots,n$

The standard deviation s of u_i is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} u_i^2 + \frac{1}{n(n-1)}(\sum_{i=1}^{n} u_i)^2}$$

The standard deviation s is an estimate of $\sigma\sqrt{\tau}$, hence, estimated volatility $\hat{\sigma} = \frac{s}{\sqrt{\tau}}$ [1].

Suppose $\boldsymbol{S}_w$ is $|w|\times 1$ sub-vector of stock price vector **S** with maturity T where $|w|$ is the window size. In this task, the estimated volatility $\hat{\boldsymbol{\sigma}}$ is computed given from historical data by applying sliding window algorithms. **Algorithm 1** illustrates pseudo code of implementing volatility estimation from $\frac{T}{4}+1$ to T.

Algorithm 1 Sliding Window Algorithm

Input: $\boldsymbol{S}_w$,T

Output: $\hat{\boldsymbol{\sigma}}$

Initialize: $k = \frac{T}{4}+1, \hat{\sigma}_{w_{k-1}} = \emptyset, n+1 = |w| \; and \; \tau = \frac{1}{T}$

While $k \le T$ **do**

For all S_i in $\boldsymbol{S}_{w_k}$ compute $u_i = \log\left(\frac{S_i}{S_{i-1}}\right)$

Compute $s_{w_k} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} u_i^2 + \frac{1}{n(n-1)}(\sum_{i=1}^{n} u_i)^2}$ and $\hat{\sigma}^* = \frac{s_{w_k}}{\sqrt{\tau}}$

Set $\hat{\sigma}_{w_k} = \hat{\sigma}_{w_{k-1}} \cup \hat{\sigma}^*$ and $k = k+1$

End While

Set $\hat{\boldsymbol{\sigma}} = \hat{\sigma}_{w_T}$

Return $\hat{\boldsymbol{\sigma}}$

For estimating call options, the prices are calculated by equation **(2)**, **(3)** and **(7)** whereas the put option prices are using $p = K\exp\big(-r(T-t)\big)\mathcal{N}(-d_2) - S\mathcal{N}(-d_1)$. Fig. 1 shows the comparison between estimated option price using Black-Scholes formula and true price. The difference (i.e. true price – estimated price) in call option indicates that on average, the true call option prices are lower than estimated ones. Conversely, the true put option price are higher than estimated ones. This may result from the payoff formula determined by which option is purchased. Payoff for call option is $\max(S-K,0)$ while that for put option is $\max(K-S,0)$. It is notable that some absolute difference of corresponding call and put pairs that has same strike price has similar interval. For example, call option with K= 2925 has almost identical absolute difference interval as that of put option with K = 2925.

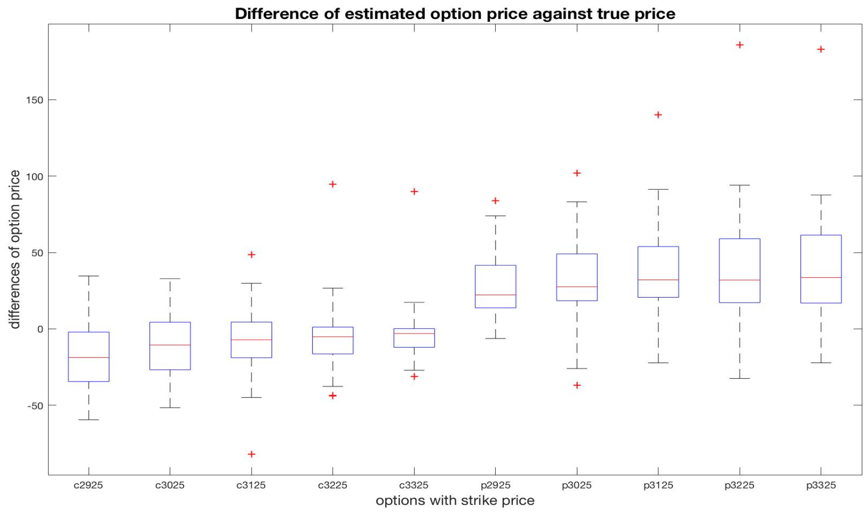

Figure 1: **Comparison of estimated option price using Black-Scholes formula with true price.**

Another observation in Fig. 1 is that the difference in call options is decreasing when the strike price increasing. So the estimation for call option with strike price 3325 is quite closed to its true price. While, the difference in put options has an opposite tendency that estimation for put with strike 2925 has very closed approximation for true price. Therefore, Black-Scholes formula shows superior approximation only in certain strike price. In general, the estimation done by Black-Scholes can give overall trend of payoffs towards call/put options.

Task 3: Implied Volatilities

Task 3 aims to compare implied volatilities with respective historical volatilities and examine the systematic variation of implied volatilities with different strike prices. The implied volatilities are defined as the volatilities implied through observations of option price in the current market [1] and can be obtained by `blsimpv` function in Matlab. J. C. Hull described implied volatility as forward looking because it can track the current market's opinion. Conversely, historical volatilities are backwards looking served as the baseline [1].

Fig. 2 illustrates the scatter plot of implied volatilities against historical ones in 30 consecutive trading days. The historical volatilities tend to be more stable ranging from 10% to 15% while implied volatilities have considerable fluctuation from 5% up to 35%. Apparently, the historical volatilities are significantly lower than the average implied levels. In this case, the option premium is considered to be overvalued as the relative value of option premium is measured via fluctuations of implied volatilities derived from the baseline [4]. Therefore, investors should closely oversee the variation in both volatilities to exploit overestimated or underestimated option premiums.

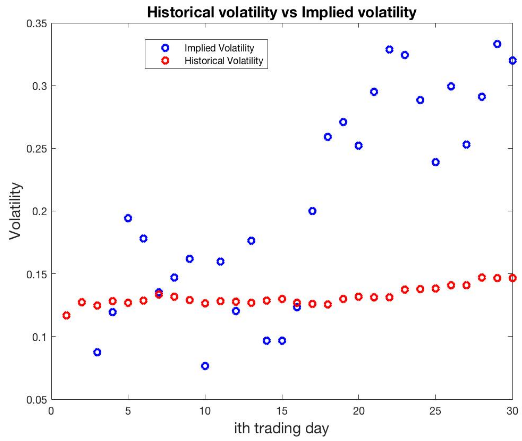

Figure 2: **Comparison between historical volatility and implied volatility. Implied volatilities appear to be higher than historical ones.**

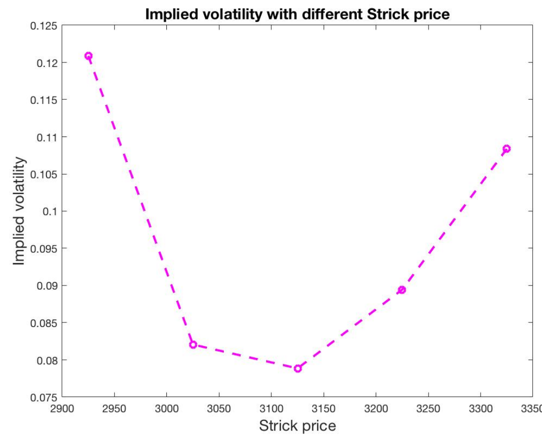

Figure 3: **Volatility smile for European call option with five different strike price.**

Fig.3 shows volatility smile plot only using European call options with five different strike prices and expired on the same date as J.C. Hull had proved it is same for calls and puts in [1] section 2.1. The implied volatilities are increasing when there are extremely low or high strike prices (i.e. option are either in-the-money or out-of-money). Hence, the minimum in the curve indicates the option is at-the-money. In this case, volatilities smile provides a well-reflection on pricing options.

Task 4: Comparing Black-Scholes with Binomial lattice

Suppose the current stock price $S = 3480$ and risk-free interest $r = 0.06$. Given the call option `c3225` parameters (i.e. strike price $K = 3225$ and T= 222) and associated historical volatility $\hat{\sigma}$, Black-Scholes model and Binomial lattice methods are compared in terms of pricing a call option with decreasing δt. As we can see in Fig. 4, the difference between Black-Scholes and Binomial lattice methods when pricing a European call option has dramatically declined with decreasing δt.

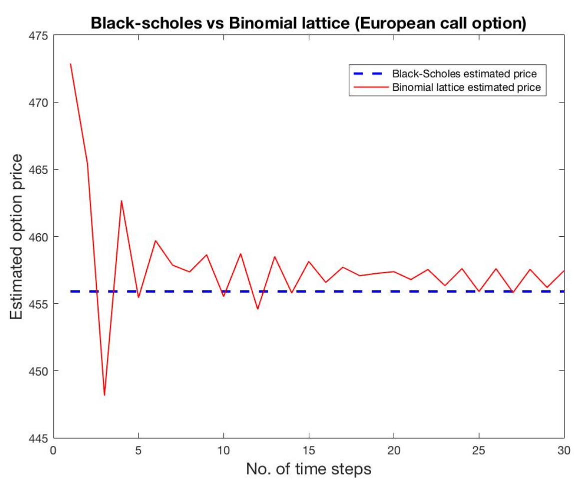

Figure 4: **Black-Scholes call option pricing is approximated by Binomial lattice. The difference between two methods of pricing call options is decreasing when time steps**

Clearly, the stock prices are thought of geometric Brownian motion in both methods described as a stochastic process [3]. Consequently, Binomial lattice model gradually converges to Black-Scholes if the number of time steps increases. In this case, the continuous process within Black-Scholes models might be approximated via discrete Binomial lattice. As opposed to European option, Binomial lattice displays superior accuracy of pricing American-style exercise than Black-Scholes approach as it is possible for traders to exercise American option early than expiration T. To conclude, Binomial lattice can yield a discrete estimation for Black-Scholes model as to European-style exercise.

Task 5: Discussion of American Option using Binomial lattice

Early exercise is an essential attribute in American options in contrast to European ones. Broadly speaking, traders must decide either to terminate the contract to get the immediate payoff (i.e. *intrinsic* value of the option) or to continue the game at every time point. J. C. Hull (2009) had introduced the dynamic programming of Binomial lattice to price an American put option in [1] section 7.2. The partial programming code is shown in **Matlab Code 1**.

Matlab Code 1: Pricing American Put Option by Binomial lattice

```
[…]
for tau = 1: N                                          1.1
      for i = (tau+1): 2: (2*N+1-tau)                   1.2
            hold = p_u *PVals(i+1) + p_d*PVals(i-1);    1.3
            PVals(i) = max (hold, K-SVals(i));          1.4
      end
end
[…]
```

Hence, this is an optimal stopping problem. The terminal payoff values `PVals` should be determined in the initial stage. Next, we should work backwards to compare intrinsic values `K-SVals(i)` against *continuation* values `hold` of the option. The continuation values are computed in **code 1.3** where `p_u` and `p_d` are corresponding risk-neutral probabilities for going-up or going-down cases. Decisions at each time step during backtracking is made in **code 1.4**. So, traders should carry on the option if `hold` exceeds `hold`.

Matlab Code 2: Pricing American Call Option by Binomial lattice

```
[…]
for tau = 1: N                                          2.1
      for i = (tau+1): 2: (2*N+1-tau)                   2.2
            hold = p_u *PVals(i+1) + p_d*PVals(i-1);    2.3
            PVals(i) = max (hold, SVals(i)-K);          2.4
      end
end
[…]
```

On the contrary to the put option, American call option has intrinsic values `SVals(i)-K` in this case. The partial programming code for pricing American call options is shown in **Matlab Code 2**.

Reference:

[1] J. C. Hull, *Options, Futures and Other Derivatives*. Prentice Hall, 2009.
[2] P. Brandimarte, *Numerical Methods in Finance and Economics*. Wiley, 2006.
[3] M. Niranjan, "Computational Finance: Part II: Derivatives Pricing", Lecture notes, School of Electronics and Computer Science, University of Southampton, 2017.
[4] J. Krohnfeldt, "Implied vs. Historical Volatility: The Main Differences", *Investopendia*, 2016. Available at: http://www.investopedia.com/articles/investing-strategy/071616/implied-vs-historical-volatility-main-differences.asp