Multiple points outputs along the beam with Bayesian inference

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1 Problem definition

A simply supported beam under a concentrated force F is shown in Figure 1. Loading positon δ and Elastic modulus E are major sources of uncertainties. Remaining factors keep constant.

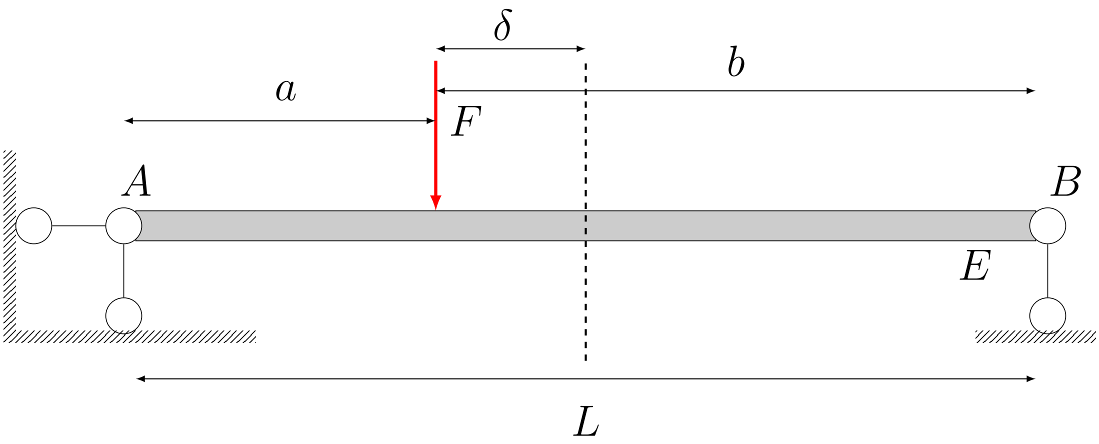

Figure 1: A simply support beam under a concentrated force

a: Distance from point A
b: Distance from point B
L: Beam length
F: Concentrated force on the beam
E: Elastic modulus
I: Cross-section inertia
δ: Loading position away from the center line

Deflections along the beam hold analytical solutions, which can be expressed as Equation 1. In our problem, we hope to reduce the uncertainties of E and δ with 20 points outputs through Bayesian inference.

$$f(x) = \begin{cases} \dfrac{Fbx[(L^2 - b^2) - x^2]}{6LEI} & x \leq a \\ \dfrac{Fb[\frac{L}{b}(x-a)^3 + (L^2 - b^2)]}{6LEI} & x > a \end{cases} \tag{1}$$

2 Parameters and priors

Open source package Uqlab is used. Specific parameters adopted for the beam in this case: $L = 10$m, beam section width $d = 0.15$m, beam section height $h = 0.3$m, concentrated force $F = 43000$N. For the priors' marginal probabilistic distributions, $E \sim \mathcal{N}(13e9,\ 3e9)N/m^2$; $\delta \sim \mathcal{N}(0, 5)m$.

3 Customer defined loglikelihood

We assume the discrepancy model follow Gaussian distribution with unknown residuals σ^2. Likelihood function can be expressed as:

$$\mathcal{L}(\boldsymbol{\theta}, \boldsymbol{\epsilon} \mid \boldsymbol{Y}) = \prod_{i=1}^{N} \frac{1}{(2\pi)^{3/2} \det(\Sigma(\epsilon))^{1/2}} \exp\left(-\frac{1}{2} \left(\boldsymbol{Y_i} - \mathcal{M}(\theta)\right)^{\mathsf{T}} \Sigma(\epsilon)^{-1} \left(\boldsymbol{Y_i} - \mathcal{M}(\theta)\right)\right) \tag{2}$$

When Equation 2 adopts log algorithm, it can be expressed as:

$$\log \mathcal{L}(\boldsymbol{\theta}, \boldsymbol{\epsilon} \mid \boldsymbol{Y}) = \sum_{i}^{N} \left(-\frac{1}{2} \left(\boldsymbol{Y_i} - \mathcal{M}(\theta)\right)^{\mathsf{T}} \Sigma(\epsilon)^{-1} \left(\boldsymbol{Y_i} - \mathcal{M}(\theta)\right)\right) - \frac{3N}{2} \log(2\pi) - \frac{N}{2} \log\left(\det(\Sigma(\epsilon))\right) \tag{3}$$

Because ϵ is unknown, it is considered in the prior θ with predefined distribution $\epsilon \sim \mathcal{N}(0, \sigma^2)$. σ adopts the mean value of the monitored data points along the beam.
Note: All measurement data points along the beams share i.i.d. assumption, i.e., all discrepancy models of the multiple points hold the same prior and σ^2. We assume the beam deflection at different locations has no influence on the monitored data error.

Algorithm 1: Custom defined logLikelihood

Data: *params*: parameters of interests, e.g., Elastic modulus E, loading position δ or discrepancy ϵ; Y: monitored data points

Result: *Loglikeli* $\log \mathcal{L}$: Sum of discrepancy between forward model and monitored data

Initialization;
$E \leftarrow params(:, 1); \delta \leftarrow params(:, 2); \epsilon \leftarrow params(:, 3);$ # Array of variables from $params$;
$Nchain \leftarrow$ Number of rows from *params*, i.e., MCMC chains ;
$Nexp \leftarrow$ Number of rows from Y, i.e., number of experiments ;
for $i \leftarrow 1$ *to* $Nchain$ **do**
  for $j \leftarrow 1$ *to* $Nexp$ **do**
    $\Delta \log \mathcal{L}(i, j) = -\frac{1}{2} \left(\boldsymbol{Y} - \mathcal{M}(\theta)\right)^{\mathsf{T}} \Sigma(\epsilon)^{-1} \left(\boldsymbol{Y} - \mathcal{M}(\theta)\right) - \frac{1}{2} \log\left(\det(\Sigma(\epsilon))\right)$;
    $\log \mathcal{L}(i) \leftarrow \log \mathcal{L}(i) + \Delta \log \mathcal{L}(i, j)$;
    $j = j + 1$;
  end for
  $i = i + 1$
end for

in which, $\Sigma(\epsilon) = \sigma^2 * I$, I denotes the diagonal Unit matrix, σ^2 denotes the variance, det denotes the determination of matrix.

4 Measurement data =Ground truth + error

$E = 15\text{e}9\text{Pa}$, $\delta = 2\text{m}$, measurement data: 20 points measurement data $Y(z, y)$ along the beam, in which z is the position of the beam, y is the deflection of the beam. $D(z, y) =$ Ground truth $Y'(z, y)$ + error. And error $\epsilon = 3\%$ * Ground Truth

$$\epsilon = \mathcal{N}(0, 3\% * Y') \tag{4}$$

$$Y(z, y) = Y'(z, y) + \epsilon \tag{5}$$

There we give 10 sets of individual experiments, and each experiments contains 20 measurements along the beam. Accordingly, Y has dimension of 10×20.

5 Results

Results mainly includes: trace plot of E and δ; comparison between prior and posterior distribution, predictive distribution with 90% and 60 % error band.

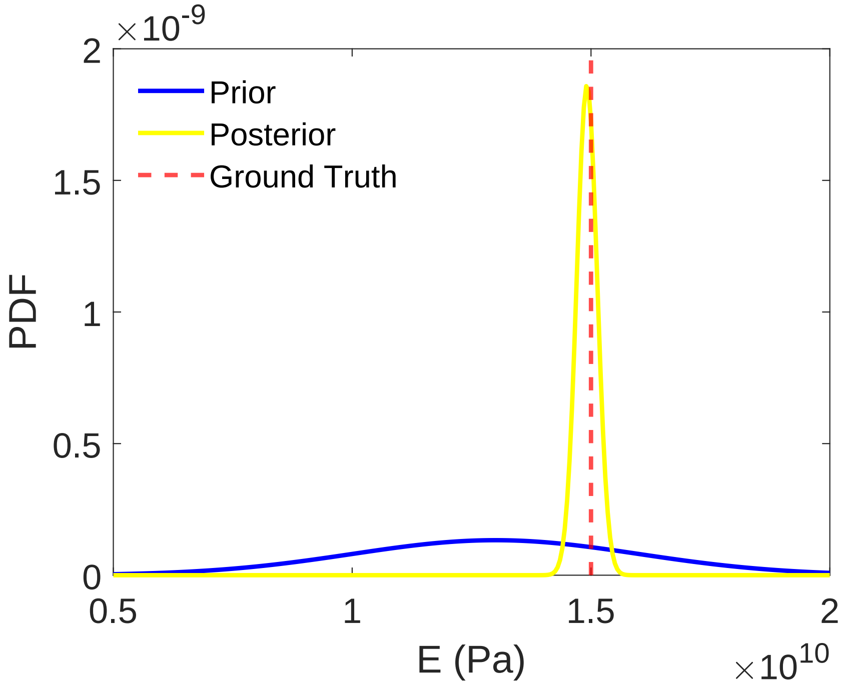

Figure 2: Bayesian inference with E

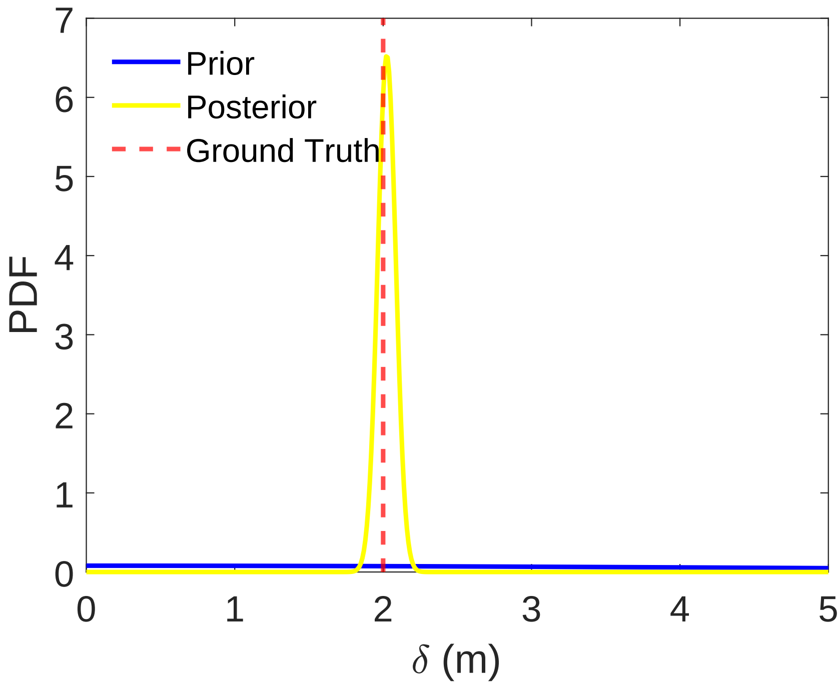

Figure 3: Bayesian inference with δ