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Surrogate models for uncertainty quantification and design optimization

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Surrogate models for uncertainty quantification and design optimization

Bruno Sudret

Contributions from: C.V. Mai, S. Marelli, R. Schöbi, E. Torre, P.-R. Wagner

Chair of Risk, Safety and Uncertainty Quantification
ETH Zurich

$$\Psi\left(\frac{x - \mu}{\sigma}\right)$$

14ème Colloque National en Calcul des Structures
Giens (France) – May 13, 2019

How to cite?

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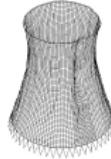
Computational models in engineering

Complex engineering systems are designed and assessed using computational models, a.k.a simulators

A computational model combines:

- A mathematical description of the physical phenomena (governing equations), e.g. mechanics, electromagnetism, fluid dynamics, etc.
- Discretization techniques which transform continuous equations into linear algebra problems
- Algorithms to solve the discretized equations

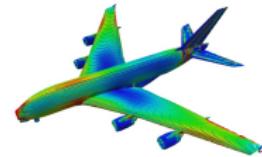
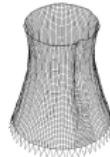
$$\begin{aligned}\operatorname{div} \boldsymbol{\sigma} + \mathbf{f} &= \mathbf{0} \\ \boldsymbol{\sigma} &= \mathbf{D} \cdot \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} &= \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u})^T\end{aligned}$$



Computational models in engineering

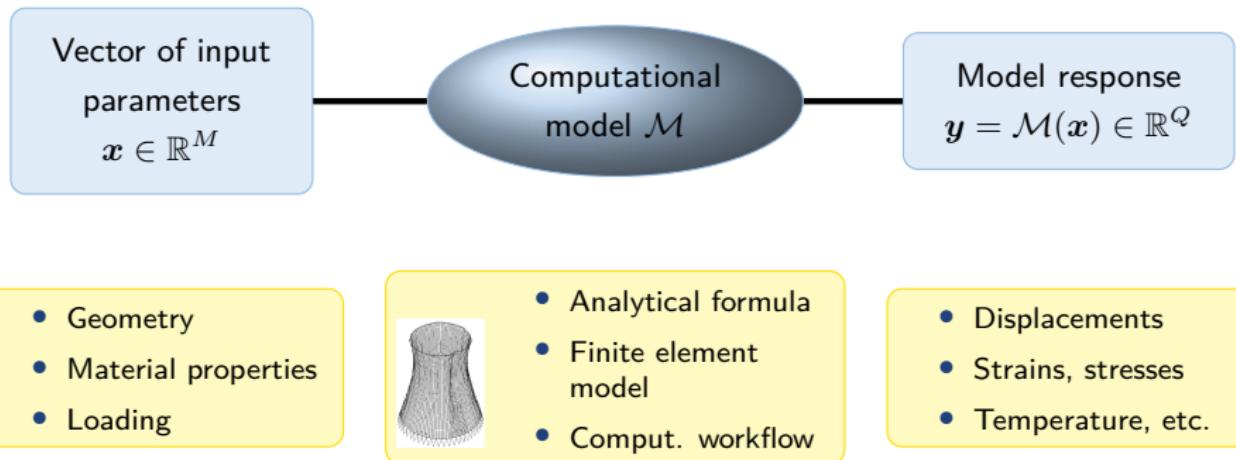
Computational models are used:

- To explore the design space (“**virtual prototypes**”)
- To **optimize** the system (e.g. minimize the mass) under performance constraints
- To assess its **robustness** w.r.t uncertainty and its **reliability**
- Together with experimental data for **calibration** purposes



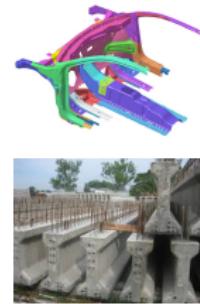
Computational models: the abstract viewpoint

A computational model may be seen as a **black box** program that computes **quantities of interest** (QoI) (a.k.a. **model responses**) as a function of input parameters



Real world is uncertain

- Differences between the **designed** and the **real** system:
 - Dimensions (tolerances in manufacturing)
 - Material properties (e.g. variability of the stiffness or resistance)
- **Unforecast exposures:** exceptional service loads, natural hazards (earthquakes, floods, landslides), climate loads (hurricanes, snow storms, etc.), accidental human actions (explosions, fire, etc.)



Outline

① Introduction

② Uncertainty quantification: why surrogate models?

③ Polynomial chaos expansions

PCE basis

Computing the coefficients

Sparse PCE

Post-processing

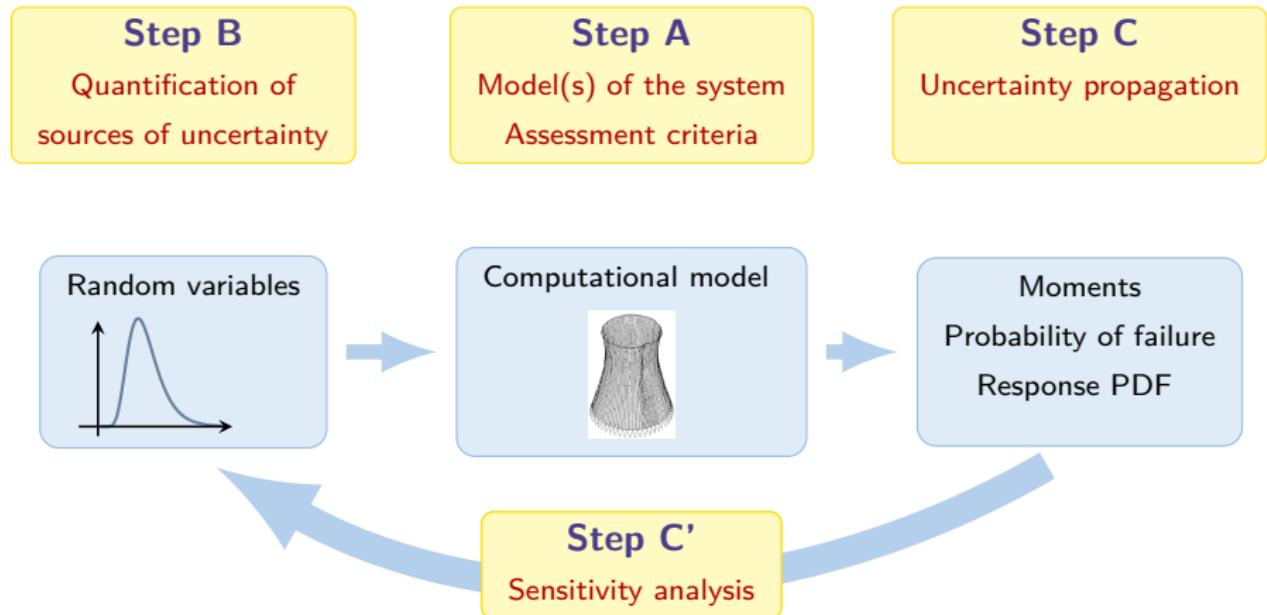
Extensions

④ Kriging (a.k.a Gaussian process modelling)

Kriging equations

Use in structural reliability

Global framework for uncertainty quantification



B. Sudret, *Uncertainty propagation and sensitivity analysis in mechanical models – contributions to structural reliability and stochastic spectral methods (2007)*

Step B: Quantification of the sources of uncertainty

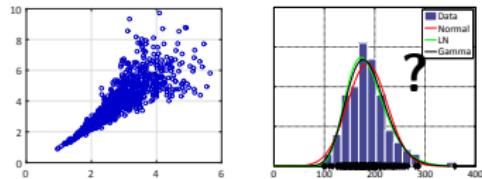
Goal: represent the uncertain parameters based on the available data and information

Probabilistic model f_X

Experimental data is available

- What is the distribution of each parameter ?
- What is the dependence structure ?

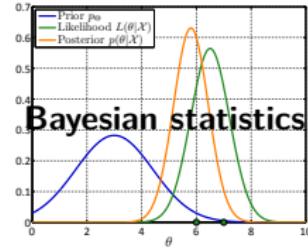
Copula theory



No data is available: expert judgment

- Engineering knowledge (e.g. reasonable bounds and uniform distributions)
- Statistical arguments and literature (e.g. extreme value distributions for climatic events)

Scarce data + expert information



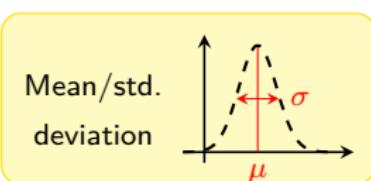
Step C: uncertainty propagation

Goal: estimate the uncertainty / variability of the quantities of interest (QoI) $Y = \mathcal{M}(\mathbf{X})$ due to the input uncertainty $f_{\mathbf{X}}$

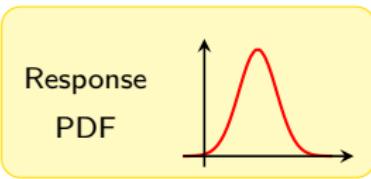
- Output statistics, *i.e.* mean, standard deviation, etc.

$$\mu_Y = \mathbb{E}_{\mathbf{X}} [\mathcal{M}(\mathbf{X})]$$

$$\sigma_Y^2 = \mathbb{E}_{\mathbf{X}} [(\mathcal{M}(\mathbf{X}) - \mu_Y)^2]$$

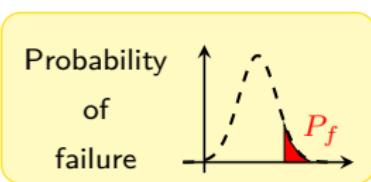


- Distribution of the QoI



- Probability of exceeding an admissible threshold y_{adm}

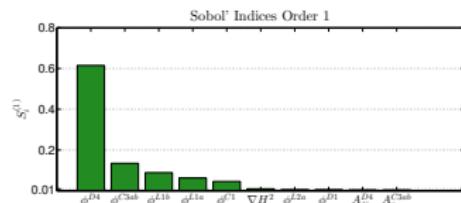
$$P_f = \mathbb{P}(Y \geq y_{adm})$$



Step C': sensitivity analysis

Goal: determine what are the input parameters (or combinations thereof) whose uncertainty explains the variability of the quantities of interest

- **Screening:** detect input parameters whose uncertainty has no impact on the output variability
- **Feature setting:** detect input parameters which allow one to best decrease the output variability when set to a deterministic value
- **Exploration:** detect interactions between parameters, *i.e.* joint effects not detected when varying parameters one-at-a-time



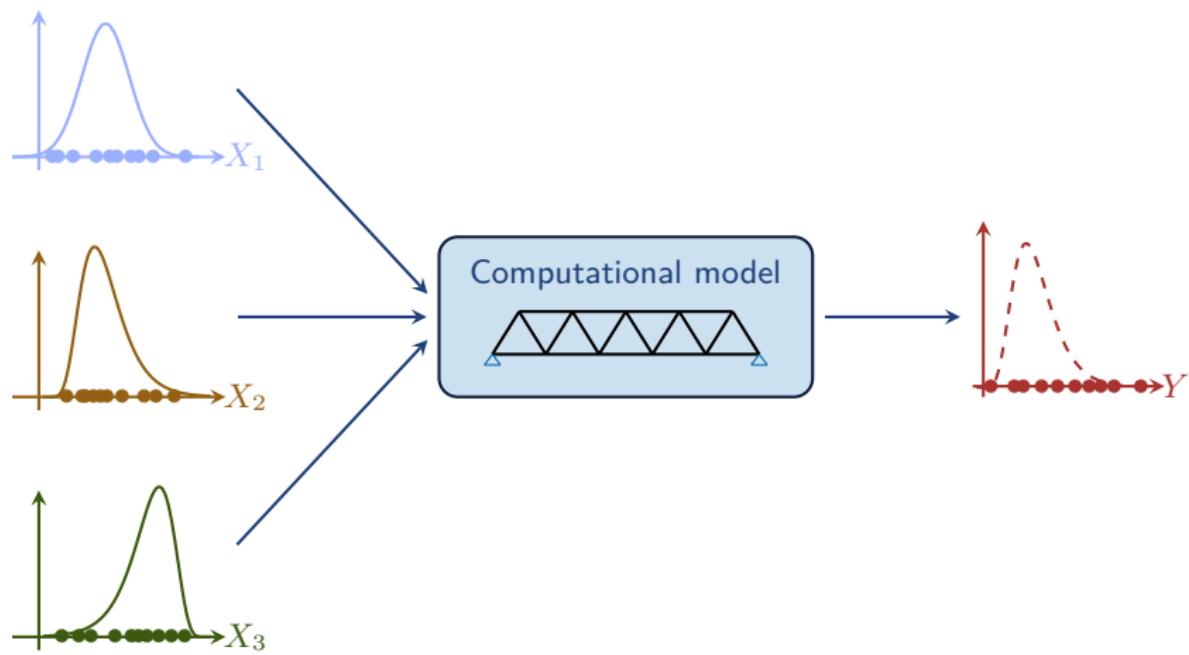
Variance decomposition (Sobol' indices)

Uncertainty propagation using Monte Carlo simulation

Principle: Generate **virtual prototypes** of the system using **random numbers**

- A sample set $\mathcal{X} = \{x_1, \dots, x_n\}$ is drawn according to the input distribution f_x
- For each sample the quantity of interest (resp. performance criterion) is evaluated, say $\mathcal{Y} = \{\mathcal{M}(x_1), \dots, \mathcal{M}(x_n)\}$
- The set of model outputs is used for moments-, distribution- or reliability analysis

Uncertainty propagation using Monte Carlo simulation



Advantages/Drawbacks of Monte Carlo simulation

Advantages

- Universal method: only rely upon sampling random numbers and running repeatedly the computational model
- Sound statistical foundations: convergence when $n \rightarrow \infty$
- Suited to High Performance Computing: “embarrassingly parallel”

Drawbacks

- Statistical uncertainty: results are not exactly reproducible when a new analysis is carried out (handled by computing confidence intervals)
- Low efficiency: convergence rate $\propto n^{-1/2}$

Monte Carlo for reliability analysis

To compute $P_f = 10^{-k}$ with an accuracy of $\pm 10\%$ (coef. of variation of 5%), $4 \cdot 10^{k+2}$ runs are required

Surrogate models for uncertainty quantification

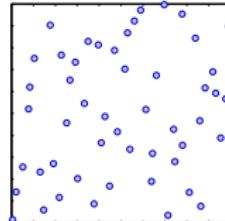
A **surrogate model** $\tilde{\mathcal{M}}$ is an **approximation** of the original computational model \mathcal{M} with the following features:

- It is built from a **limited** set of runs of the original model \mathcal{M} called the **experimental design** $\mathcal{X} = \{\boldsymbol{x}^{(i)}, i = 1, \dots, N\}$
- It assumes some regularity of the model \mathcal{M} and some general functional shape

Name	Shape	Parameters
Polynomial chaos expansions	$\tilde{\mathcal{M}}(\boldsymbol{x}) = \sum_{\alpha \in \mathcal{A}} a_{\alpha} \Psi_{\alpha}(\boldsymbol{x})$	a_{α}
Low-rank tensor approximations	$\tilde{\mathcal{M}}(\boldsymbol{x}) = \sum_{l=1}^R b_l \left(\prod_{i=1}^M v_l^{(i)}(x_i) \right)$	$b_l, z_{k,l}^{(i)}$
Kriging (a.k.a Gaussian processes)	$\tilde{\mathcal{M}}(\boldsymbol{x}) = \beta^T \cdot \mathbf{f}(\boldsymbol{x}) + Z(\boldsymbol{x}, \omega)$	$\beta, \sigma_Z^2, \theta$
Support vector machines	$\tilde{\mathcal{M}}(\boldsymbol{x}) = \sum_{i=1}^m a_i K(\boldsymbol{x}_i, \boldsymbol{x}) + b$	a, b

Ingredients for building a surrogate model

- Select an **experimental design** \mathcal{X} that covers at best the domain of input parameters: **Latin hypercube sampling (LHS)**, **low-discrepancy sequences**
- Run the computational model \mathcal{M} onto \mathcal{X} exactly as in **Monte Carlo simulation**
- Smartly post-process the data $\{\mathcal{X}, \mathcal{M}(\mathcal{X})\}$ through a **learning algorithm**



Name	Learning method
Polynomial chaos expansions	sparse grid integration, least-squares, compressive sensing
Low-rank tensor approximations	alternate least squares
Kriging	maximum likelihood, Bayesian inference
Support vector machines	quadratic programming

Advantages of surrogate models

Usage

$$\mathcal{M}(\boldsymbol{x}) \approx \tilde{\mathcal{M}}(\boldsymbol{x})$$

hours per run seconds for 10^6 runs

Advantages

- Non-intrusive methods: based on runs of the computational model, exactly as in Monte Carlo simulation
- Suited to high performance computing: “embarrassingly parallel”

Challenges

- Need for rigorous validation
- Communication: advanced mathematical background

Efficiency: 2-3 orders of magnitude less runs compared to Monte Carlo

Outline

- ① Introduction
- ② Uncertainty quantification: why surrogate models?
- ③ Polynomial chaos expansions
 - PCE basis
 - Computing the coefficients
 - Sparse PCE
 - Post-processing
 - Extensions
- ④ Kriging (a.k.a Gaussian process modelling)

Polynomial chaos expansions in a nutshell

Ghanem & Spanos (1991; 2003); Xiu & Karniadakis (2002); Soize & Ghanem (2004)

- Consider the input random vector \mathbf{X} ($\dim \mathbf{X} = M$) with given probability density function (PDF) $f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^M f_{X_i}(x_i)$
- Assuming that the random output $Y = \mathcal{M}(\mathbf{X})$ has finite variance, it can be cast as the following polynomial chaos expansion:

$$Y = \sum_{\alpha \in \mathbb{N}^M} y_{\alpha} \Psi_{\alpha}(\mathbf{X})$$

where :

- $\Psi_{\alpha}(\mathbf{X})$: basis functions
- y_{α} : coefficients to be computed (coordinates)
- The PCE basis $\{\Psi_{\alpha}(\mathbf{X}), \alpha \in \mathbb{N}^M\}$ is made of multivariate orthonormal polynomials

Multivariate polynomial basis

Univariate polynomials

- For each input variable X_i , univariate orthogonal polynomials $\{P_k^{(i)}, k \in \mathbb{N}\}$ are built:

$$\left\langle P_j^{(i)}, P_k^{(i)} \right\rangle = \int P_j^{(i)}(u) P_k^{(i)}(u) \mathbf{f}_{X_i}(u) du = \gamma_j^{(i)} \delta_{jk}$$

e.g., Legendre polynomials if $X_i \sim \mathcal{U}(-1, 1)$, Hermite polynomials if $X_i \sim \mathcal{N}(0, 1)$

- Normalization: $\Psi_j^{(i)} = P_j^{(i)} / \sqrt{\gamma_j^{(i)}} \quad i = 1, \dots, M, \quad j \in \mathbb{N}$

Tensor product construction

$$\Psi_{\alpha}(x) \stackrel{\text{def}}{=} \prod_{i=1}^M \Psi_{\alpha_i}^{(i)}(x_i) \quad \mathbb{E} [\Psi_{\alpha}(X) \Psi_{\beta}(X)] = \delta_{\alpha\beta}$$

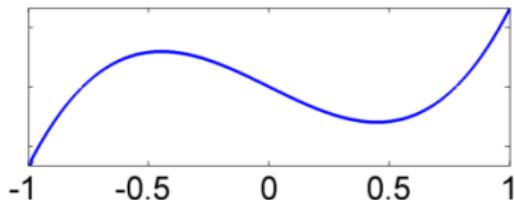
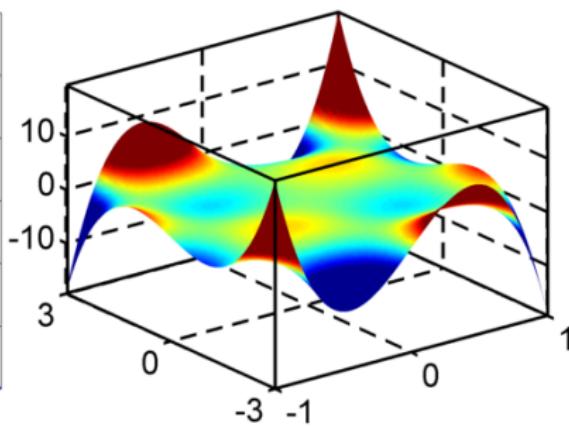
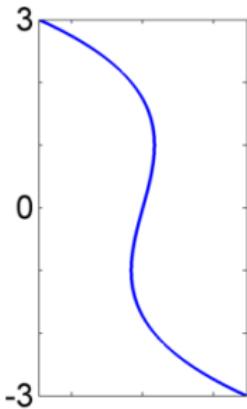
where $\alpha = (\alpha_1, \dots, \alpha_M)$ are multi-indices (partial degree in each dimension)

Example: $M = 2$

Xiu & Karniadakis (2002)

$$\alpha = [3, 3]$$

$$\Psi_{(3,3)}(\boldsymbol{x}) = \tilde{P}_3(x_1) \cdot \tilde{H}e_3(x_2)$$



- $X_1 \sim \mathcal{U}(-1, 1)$: Legendre polynomials
- $X_2 \sim \mathcal{N}(0, 1)$: Hermite polynomials

Dealing with complex input distributions

Independent variables

Input parameters with given marginal CDFs $X_i \sim F_{X_i}$, $i = 1, \dots, M$

- **Arbitrary PCE**: orthogonal polynomial computed numerically on-the-fly
- **Isoprobabilistic transform** through a one-to-one mapping to reduced variables,
e.g. :

$$X_i = F_{X_i}^{-1} \left(\frac{\xi_i + 1}{2} \right) \quad \text{if } \xi_i \sim \mathcal{U}(-1, 1)$$

$$X_i = F_{X_i}^{-1} (\Phi(\xi_i)) \quad \text{if } \xi_i \sim \mathcal{N}(0, 1)$$

General case: addressing dependence

Sklar's theorem (1959)

- The joint CDF is defined through its marginals and **copula**

$$F_{\mathbf{X}}(\mathbf{x}) = \mathcal{C}(F_{X_1}(x_1), \dots, F_{X_M}(x_M))$$

- Rosenblatt or Nataf isoprobabilistic transform is used

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Computing the coefficients by least-square minimization

Isukapalli (1999); Berveiller, Sudret & Lemaire (2006)

Principle

The exact (infinite) series expansion is considered as the sum of a truncated series and a residual:

$$Y = \mathcal{M}(\mathbf{X}) = \sum_{\alpha \in \mathcal{A}} y_\alpha \Psi_\alpha(\mathbf{X}) + \varepsilon_P \equiv \mathbf{Y}^\top \boldsymbol{\Psi}(\mathbf{X}) + \varepsilon_P(\mathbf{X})$$

where : $\mathbf{Y} = \{y_\alpha, \alpha \in \mathcal{A}\} \equiv \{y_0, \dots, y_{P-1}\}$ (P unknown coefficients)

$$\boldsymbol{\Psi}(\mathbf{x}) = \{\Psi_0(\mathbf{x}), \dots, \Psi_{P-1}(\mathbf{x})\}$$

Least-square minimization

The unknown coefficients are estimated by minimizing the mean square residual error:

$$\hat{\mathbf{Y}} = \arg \min \mathbb{E} \left[(\mathbf{Y}^\top \boldsymbol{\Psi}(\mathbf{X}) - \mathcal{M}(\mathbf{X}))^2 \right]$$

Discrete (ordinary) least-square minimization

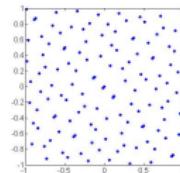
An estimate of the mean square error (sample average) is minimized:

$$\hat{\mathbf{Y}} = \arg \min_{\mathbf{Y} \in \mathbb{R}^P} \frac{1}{n} \sum_{i=1}^n (\mathbf{Y}^\top \Psi(\mathbf{x}^{(i)}) - \mathcal{M}(\mathbf{x}^{(i)}))^2$$

Procedure

- Select a truncation scheme, e.g. $\mathcal{A}^{M,p} = \{\boldsymbol{\alpha} \in \mathbb{N}^M : |\boldsymbol{\alpha}|_1 \leq p\}$
- Select an **experimental design** and evaluate the model response

$$\mathbf{M} = \{\mathcal{M}(\mathbf{x}^{(1)}), \dots, \mathcal{M}(\mathbf{x}^{(n)})\}^\top$$



- Compute the experimental matrix

$$\mathbf{A}_{ij} = \Psi_j(\mathbf{x}^{(i)}) \quad i = 1, \dots, n ; j = 0, \dots, P-1$$

- Solve the resulting **linear system**

$$\hat{\mathbf{Y}} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{M}$$

Simple is beautiful !

Error estimators

- In least-squares analysis, the **generalization error** is defined as:

$$E_{gen} = \mathbb{E} \left[(\mathcal{M}(\mathbf{X}) - \mathcal{M}^{PC}(\mathbf{X}))^2 \right] \quad \mathcal{M}^{PC}(\mathbf{X}) = \sum_{\alpha \in \mathcal{A}} y_{\alpha} \Psi_{\alpha}(\mathbf{X})$$

- The **empirical error** based on the experimental design \mathcal{X} is a poor estimator in case of **overfitting**

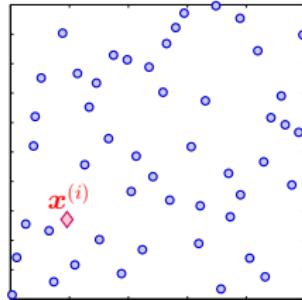
$$E_{emp} = \frac{1}{n} \sum_{i=1}^n (\mathcal{M}(\mathbf{x}^{(i)}) - \mathcal{M}^{PC}(\mathbf{x}^{(i)}))^2$$

- The **coefficient of determination** R^2 is often used as an error estimator:

$$R^2 = 1 - \frac{E_{emp}}{\text{Var}[\mathcal{Y}]} \quad \text{Var}[\mathcal{Y}] = \frac{1}{n} (\mathcal{M}(\mathbf{x}^{(i)}) - \bar{\mathcal{Y}})^2$$

R^2 is a poor estimator of the accuracy of the PCE when there is overfitting

Leave-one-out cross validation



- An experimental design $\mathcal{X} = \{\mathbf{x}^{(j)}, j = 1, \dots, n\}$ is selected
- Polynomial chaos expansions are built using all points but one, i.e. based on $\mathcal{X} \setminus \mathbf{x}^{(i)} = \{\mathbf{x}^{(j)}, j = 1, \dots, n, j \neq i\}$

- Leave-one-out error (PRESS)

$$E_{LOO} \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n \left(\mathcal{M}(\mathbf{x}^{(i)}) - \mathcal{M}^{PC \setminus i}(\mathbf{x}^{(i)}) \right)^2$$

- Analytical derivation from a single PC analysis

$$E_{LOO} = \frac{1}{n} \sum_{i=1}^n \left(\frac{\mathcal{M}(\mathbf{x}^{(i)}) - \mathcal{M}^{PC}(\mathbf{x}^{(i)})}{1 - h_i} \right)^2$$

where h_i is the i -th diagonal term of matrix $\mathbf{A}(\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top$

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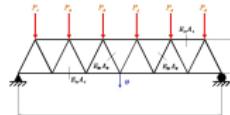
④ Kriging (a.k.a Gaussian process modelling)

Curse of dimensionality

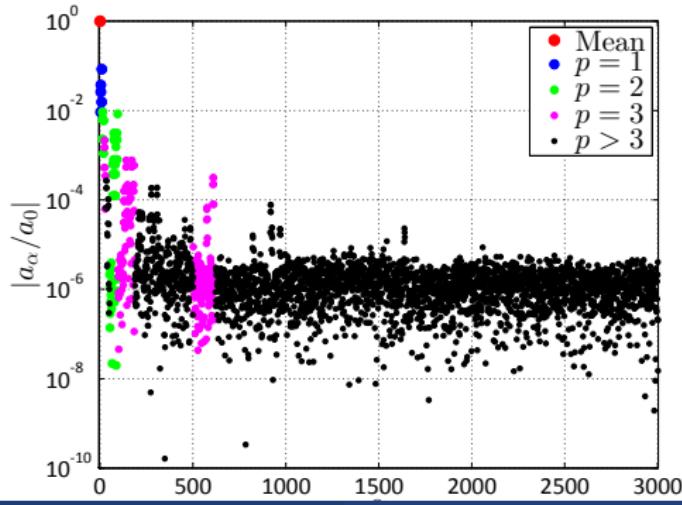
- The cardinality of the truncation scheme $\mathcal{A}^{M,p}$ is $P = \frac{(M+p)!}{M! p!}$
- Typical computational requirements: $n = OSR \cdot P$ where the oversampling rate is $OSR = 2 - 3$

However ... most coefficients are close to zero !

Example



- Elastic truss structure with $M = 10$ independent input variables
- PCE of degree $p = 5$ ($P = 3,003$ coefficients)



Hyperbolic truncation sets

Sparsity-of-effects principle

Blatman & Sudret, Prob. Eng. Mech (2010); J. Comp. Phys (2011)

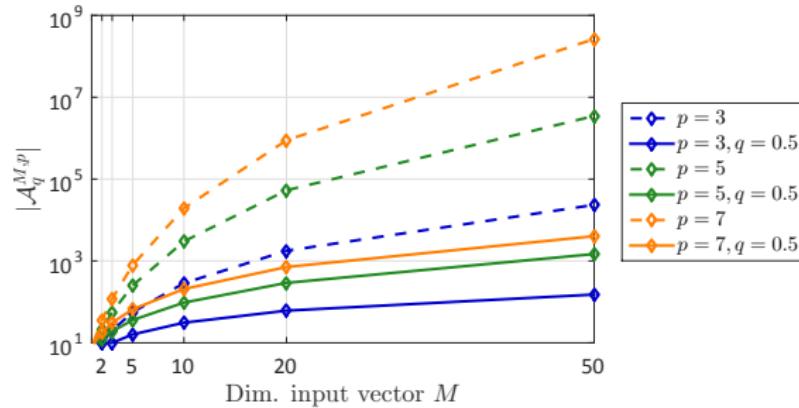
In most engineering problems, only **low-order interactions** between the input variables are relevant

- q -norm of a multi-index α :

$$\|\alpha\|_q \equiv \left(\sum_{i=1}^M \alpha_i^q \right)^{1/q}, \quad 0 < q \leq 1$$

- Hyperbolic truncation sets:

$$\mathcal{A}_q^{M,p} = \{\alpha \in \mathbb{N}^M : \|\alpha\|_q \leq p\}$$



Compressive sensing approaches

Blatman & Sudret (2011); Doostan & Owhadi (2011); Sargsyan *et al.* (2014); Jakeman *et al.* (2015)

- Sparsity in the solution can be induced by ℓ_1 -regularization:

$$\mathbf{y}_\alpha = \arg \min \frac{1}{n} \sum_{i=1}^n (\mathbf{Y}^\top \boldsymbol{\Psi}(\mathbf{x}^{(i)}) - \mathcal{M}(\mathbf{x}^{(i)}))^2 + \lambda \|\mathbf{y}_\alpha\|_1$$

- Different algorithms: LASSO, orthogonal matching pursuit, Bayesian compressive sensing

Least Angle Regression

Efron *et al.* (2004)
Blatman & Sudret (2011)

- Least Angle Regression (LAR) solves the LASSO problem for different values of the penalty constant in a single run without matrix inversion
- Leave-one-out cross validation error allows one to select the best model

Sparse PCE: wrap up

Algorithm 1: LAR-based Polynomial chaos expansion

```

1: Input: Computational budget  $n$ 
2: Initialization
3: Sample experimental design  $\mathcal{X} = \{\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n)}\}$ 
4: Evaluate model response  $\mathcal{Y} = \{\mathcal{M}(\boldsymbol{x}^{(1)}), \dots, \mathcal{M}(\boldsymbol{x}^{(n)})\}$ 
5: PCE construction
6:   for  $p = p_{\min} : p_{\max}$  do
7:     for  $q \in \mathcal{Q}$  do
8:       Select candidate basis  $\mathcal{A}_q^{M,p}$ 
9:       Run LAR for extracting the optimal sparse basis  $\mathcal{A}^*(p, q)$ 
10:      Compute coefficients  $\{y_\alpha, \alpha \in \mathcal{A}^*(p, q)\}$  by OLS
11:      Compute  $e_{\text{LOO}}(p, q)$ 
12:    end
13:  end
14:   $(p^*, q^*) = \arg \min e_{\text{LOO}}(p, q)$ 
15: Return Optimal sparse basis  $\mathcal{A}^*(p, q)$ , PCE coefficients,  $e_{\text{LOO}}(p^*, q^*)$ 

```

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Post-processing sparse PC expansions

Statistical moments

- Due to the orthogonality of the basis functions ($\mathbb{E} [\Psi_\alpha(\mathbf{X})\Psi_\beta(\mathbf{X})] = \delta_{\alpha\beta}$) and using $\mathbb{E} [\Psi_{\alpha \neq 0}] = 0$ the **statistical moments** read:

$$\text{Mean: } \hat{\mu}_Y = y_0$$

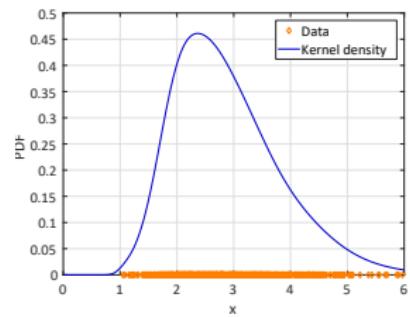
$$\text{Variance: } \hat{\sigma}_Y^2 = \sum_{\alpha \in \mathcal{A} \setminus \mathbf{0}} y_\alpha^2$$

Distribution of the QoI

- The PCE can be used as a **response surface** for sampling:

$$\eta_j = \sum_{\alpha \in \mathcal{A}} y_\alpha \Psi_\alpha(\mathbf{x}_j) \quad j = 1, \dots, n_{big}$$

- The **PDF of the response** is estimated by **histograms** or **kernel smoothing**



Sensitivity analysis

Goal

Sobol' (1993); Saltelli *et al.* (2008)

Global sensitivity analysis aims at quantifying which input parameter(s) (or combinations thereof) influence the most the response variability (variance decomposition)

Hoeffding-Sobol' decomposition $(\mathbf{X} \sim \mathcal{U}([0, 1]^M))$

$$\begin{aligned}\mathcal{M}(\mathbf{x}) &= \mathcal{M}_0 + \sum_{i=1}^M \mathcal{M}_i(x_i) + \sum_{1 \leq i < j \leq M} \mathcal{M}_{ij}(x_i, x_j) + \cdots + \mathcal{M}_{12\dots M}(\mathbf{x}) \\ &= \mathcal{M}_0 + \sum_{\mathbf{u} \subset \{1, \dots, M\}} \mathcal{M}_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}) \quad (\mathbf{x}_{\mathbf{u}} \stackrel{\text{def}}{=} \{x_{i_1}, \dots, x_{i_s}\})\end{aligned}$$

- The summands satisfy the orthogonality condition:

$$\int_{[0,1]^M} \mathcal{M}_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}) \mathcal{M}_{\mathbf{v}}(\mathbf{x}_{\mathbf{v}}) d\mathbf{x} = 0 \quad \forall \mathbf{u} \neq \mathbf{v}$$

Sobol' indices

Total variance:

$$D \equiv \text{Var} [\mathcal{M}(\mathbf{X})] = \sum_{\mathbf{u} \subset \{1, \dots, M\}} \text{Var} [\mathcal{M}_{\mathbf{u}}(\mathbf{X}_{\mathbf{u}})]$$

- Sobol' indices:

$$S_{\mathbf{u}} \stackrel{\text{def}}{=} \frac{\text{Var} [\mathcal{M}_{\mathbf{u}}(\mathbf{X}_{\mathbf{u}})]}{D}$$

- First-order Sobol' indices:

$$S_i = \frac{D_i}{D} = \frac{\text{Var} [\mathcal{M}_i(X_i)]}{D}$$

Quantify the additive effect of each input parameter separately

- Total Sobol' indices:

$$S_i^T \stackrel{\text{def}}{=} \sum_{\mathbf{u} \supset i} S_{\mathbf{u}}$$

Quantify the total effect of X_i , including interactions with the other variables.

Link with PC expansions

Sobol decomposition of a PC expansion

Sudret, CSM (2006); RESS (2008)

Obtained by reordering the terms of the (truncated) PC expansion

$$\mathcal{M}^{\text{PC}}(\boldsymbol{X}) \stackrel{\text{def}}{=} \sum_{\alpha \in \mathcal{A}} y_\alpha \Psi_\alpha(\boldsymbol{X})$$

Interaction sets

For a given $\mathbf{u} \stackrel{\text{def}}{=} \{i_1, \dots, i_s\}$: $\mathcal{A}_{\mathbf{u}} = \{\alpha \in \mathcal{A} : k \in \mathbf{u} \Leftrightarrow \alpha_k \neq 0\}$

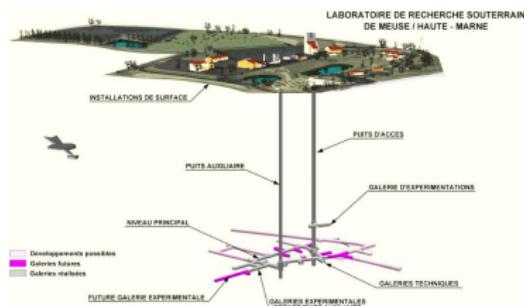
$$\mathcal{M}^{\text{PC}}(\boldsymbol{x}) = \mathcal{M}_0 + \sum_{\mathbf{u} \subset \{1, \dots, M\}} \mathcal{M}_{\mathbf{u}}(\boldsymbol{x}_{\mathbf{u}}) \quad \text{where} \quad \mathcal{M}_{\mathbf{u}}(\boldsymbol{x}_{\mathbf{u}}) \stackrel{\text{def}}{=} \sum_{\alpha \in \mathcal{A}_{\mathbf{u}}} y_\alpha \Psi_\alpha(\boldsymbol{x})$$

PC-based Sobol' indices

$$S_{\mathbf{u}} = D_{\mathbf{u}}/D = \sum_{\alpha \in \mathcal{A}_{\mathbf{u}}} y_\alpha^2 / \sum_{\alpha \in \mathcal{A} \setminus \mathbf{0}} y_\alpha^2$$

The Sobol' indices are obtained analytically, at any order from the coefficients of the PC expansion

Example: sensitivity analysis in hydrogeology



Source: <http://www.futura-sciences.com/>



Source: <http://lexpansion.lexpress.fr/>

- When assessing a **nuclear waste repository**, the Mean Lifetime Expectancy $MLE(x)$ is the time required for a molecule of water at point x to get out of the boundaries of the system

- Computational models have numerous input parameters (in each geological layer) that are **difficult to measure**, and that show **scattering**

Geological model

Joint work with University of Neuchâtel

Deman, Konakli, Sudret, Kerrou, Perrochet & Benabderahmane, Reliab. Eng. Sys. Safety (2016)

- Two-dimensional idealized model of the Paris Basin (25 km long / 1,040 m depth) with 5×5 m mesh (10^6 elements)
- Steady-state flow simulation with Dirichlet boundary conditions:

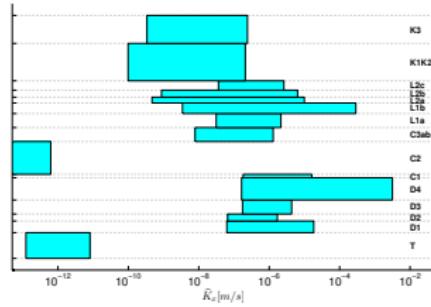
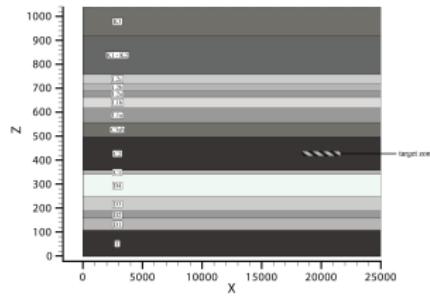
$$\nabla \cdot (\mathbf{K} \cdot \nabla H) = 0$$

- 15 homogeneous layers with uncertainties in:

- Porosity (resp. hydraulic conductivity)
- Anisotropy of the layer properties (inc. dispersivity)
- Boundary conditions (hydraulic gradients)

78 input parameters

Sensitivity analysis

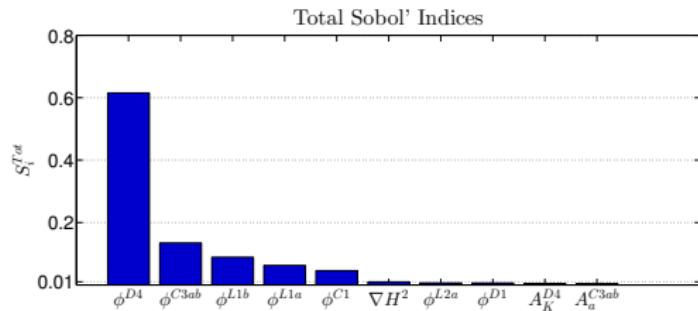


Question

What are the parameters (out of 78) whose uncertainty drives the uncertainty of the prediction of the mean life-time expectancy?

Sensitivity analysis: results

Technique: Sobol' indices computed from polynomial chaos expansions



Parameter	$\sum_j S_j$
ϕ (resp. K_x)	0.8664
A_K	0.0088
θ	0.0029
α_L	0.0076
A_α	0.0000
∇H	0.0057

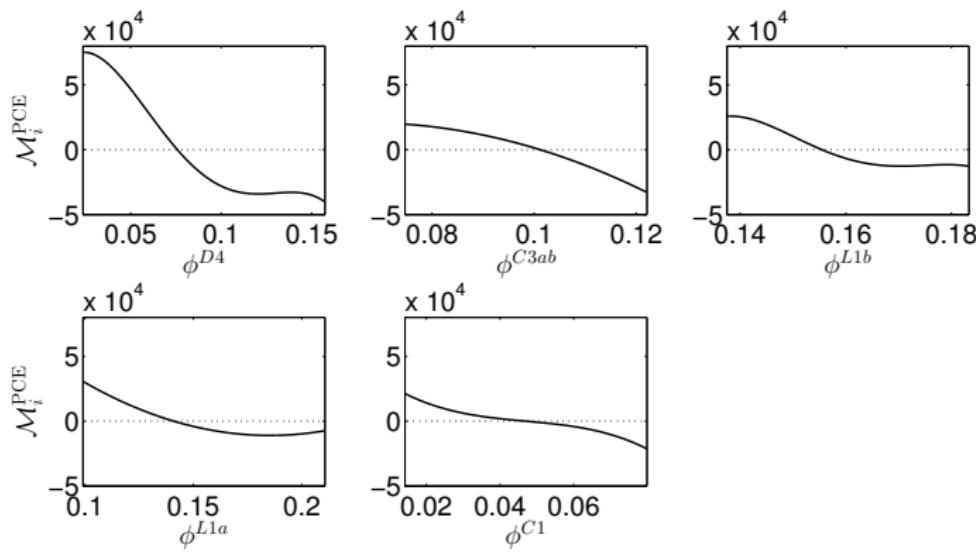
Conclusions

- Only 200 model runs allow one to detect the 10 important parameters out of 78
- Uncertainty in the porosity/conductivity of 5 layers explain 86% of the variability
- Small interactions between parameters detected

Bonus: univariate effects

The **univariate effects** of each variable are obtained as a straightforward post-processing of the PCE

$$\mathcal{M}_i(x_i) \stackrel{\text{def}}{=} \mathbb{E} [\mathcal{M}(\boldsymbol{X}) | X_i = x_i], i = 1, \dots, M$$



Outline

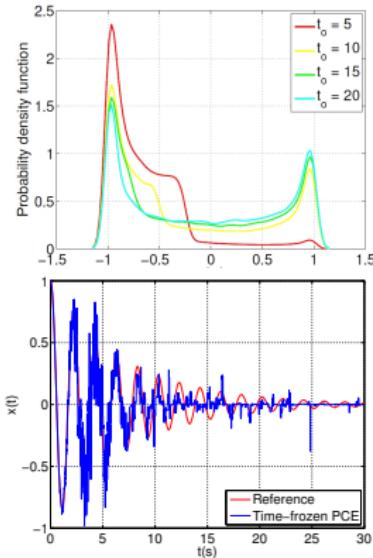
- ① Introduction
- ② Uncertainty quantification: why surrogate models?
- ③ Polynomial chaos expansions
 - PCE basis
 - Computing the coefficients
 - Sparse PCE
 - Post-processing
 - Extensions
- ④ Kriging (a.k.a Gaussian process modelling)

Polynomial chaos expansions in structural dynamics

Spiridonakos et al. (2015); Mai, Spiridonakos, Chatzi & Sudret, IJUQ (2016); Mai & Sudret, SIAM JUQ (2017)

Premise

- For dynamical systems, the complexity of the map $\xi \mapsto \mathcal{M}(\xi, t)$ increases with time.
- Time-frozen PCE does not work beyond first time instants



PC-NARX

- Use of non linear autoregressive with exogenous input models (NARX) to capture the dynamics:

$$y(t) = \mathcal{F}(x(t), \dots, x(t-n_x), y(t-1), \dots, y(t-n_y)) + \epsilon_t \equiv \mathcal{F}(z(t)) + \epsilon_t$$

Earthquake engineering – Bouc-Wen oscillator

Governing equations

Kafali & Grigoriu (2007), Spiridonakos & Chatzi (2015)

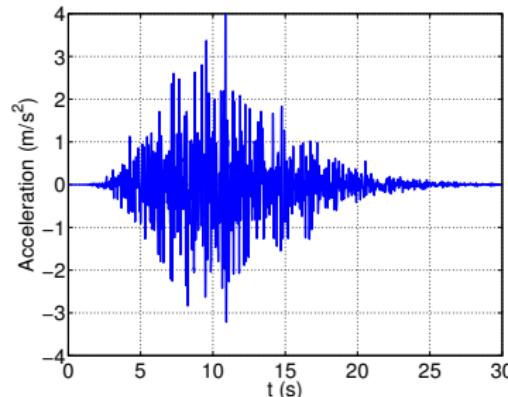
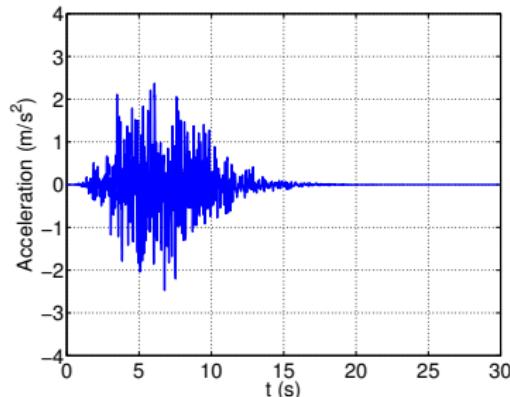
$$\begin{aligned}\ddot{y}(t) + 2\zeta\omega\dot{y}(t) + \omega^2(\rho y(t) + (1 - \rho)z(t)) &= -x(t), \\ \dot{z}(t) &= \gamma\dot{y}(t) - \alpha |\dot{y}(t)| |z(t)|^{n-1} z(t) - \beta \dot{y}(t) |z(t)|^n,\end{aligned}$$

Excitation

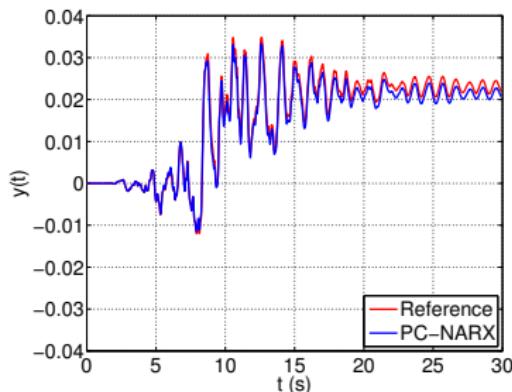
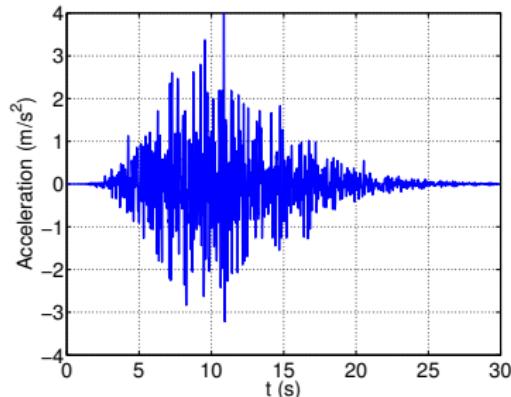
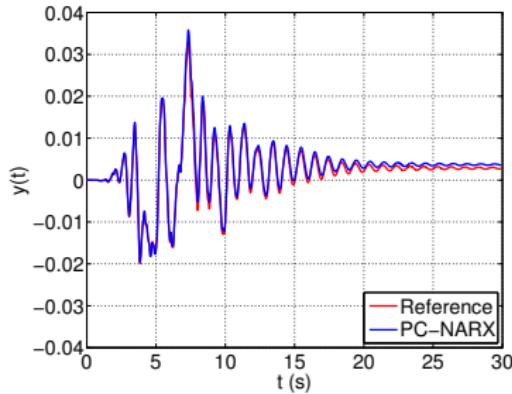
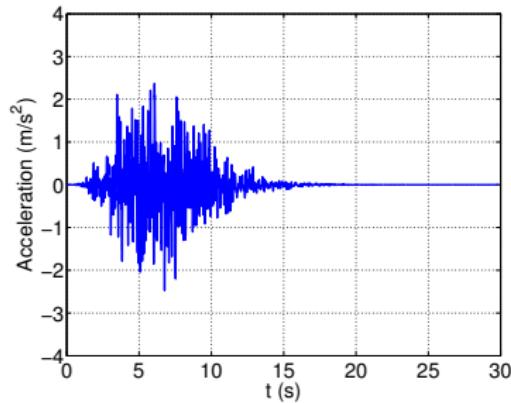
$x(t)$ is generated by a probabilistic ground motion model

Rezaeian & Der Kiureghian (2010)

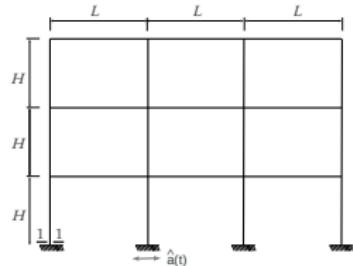
$$x(t) = q(t, \boldsymbol{\alpha}) \sum_{i=1} s_i(t, \boldsymbol{\lambda}(t_i)) U_i$$



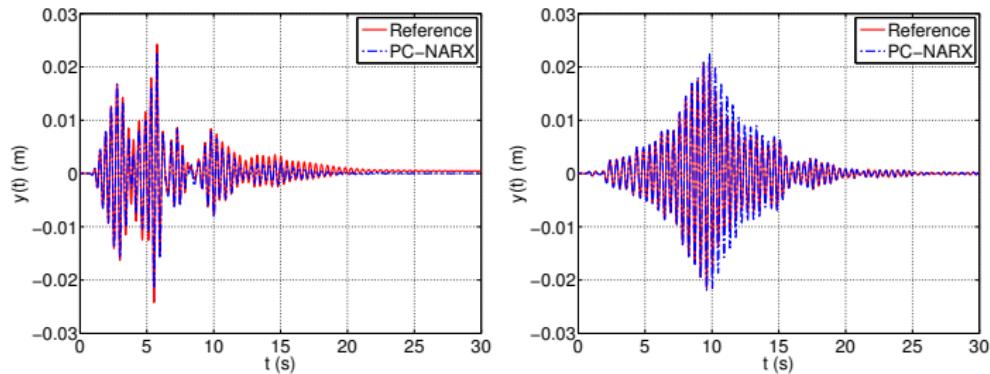
Bouc-Wen model: prediction



Earthquake engineering – frame structure



- 2D steel frame with uncertain properties submitted to synthetic ground motions
- Experimental design of size 300

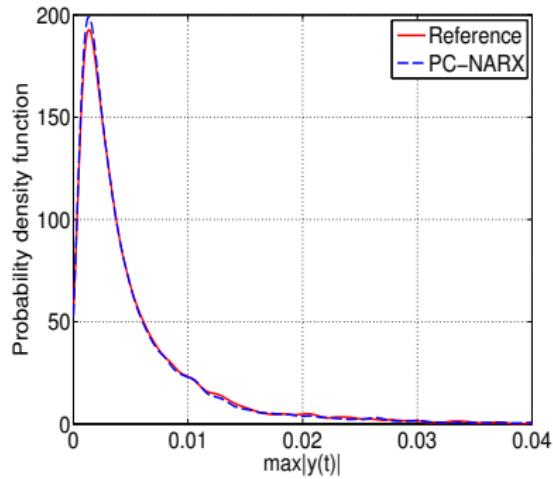


Surrogate model of single trajectories

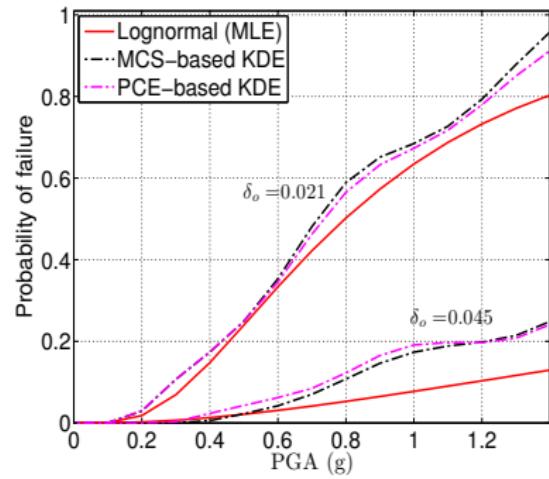
Frame structure – fragility curves

First-storey drift

- PC-NARX calibrated based on 300 simulations
- Reference results obtained from 10,000 Monte Carlo simulations



PDF of max. drift



Fragility curves for two drift thresholds

Outline

- ① Introduction
- ② Uncertainty quantification: why surrogate models?
- ③ Polynomial chaos expansions
- ④ Kriging (a.k.a Gaussian process modelling)
 - Kriging equations
 - Use in structural reliability

Gaussian process modelling (a.k.a Kriging)

Santner, Williams & Notz (2003)

Kriging assumes that $\mathcal{M}(\boldsymbol{x})$ is a trajectory of an underlying Gaussian process

$$\mathcal{M}(\boldsymbol{x}) \approx \mathcal{M}^{(K)}(\boldsymbol{x}) = \boldsymbol{\beta}^T \mathbf{f}(\boldsymbol{x}) + \sigma^2 Z(\boldsymbol{x}, \omega)$$

where:

- $\boldsymbol{\beta}^T \mathbf{f}(\boldsymbol{x})$: trend
- $Z(\boldsymbol{x}, \omega)$: zero mean, unit variance Gaussian process with autocorrelation function, e.g. :

$$R(\boldsymbol{x}, \boldsymbol{x}') = \exp \left(\sum_{k=1}^M - \left(\frac{x_k - x'_k}{\theta_k} \right)^2 \right)$$

- σ^2 : variance



The Gaussian measure artificially introduced is different from the aleatory uncertainty on the model parameters \boldsymbol{X}

Kriging prediction

Unknown parameters

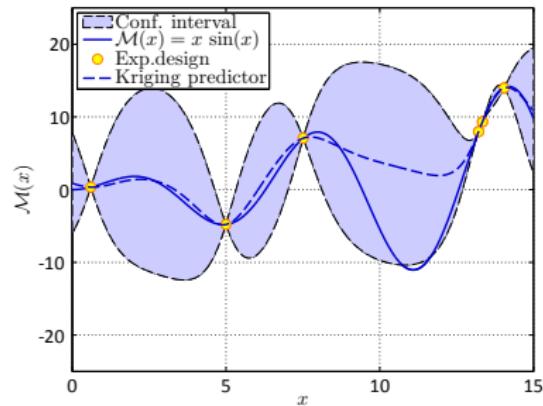
- Parameters $\{\theta, \beta, \sigma^2\}$ are estimated from the experimental design $\mathcal{Y} = \{y_i = \mathcal{M}(\chi_i), i = 1, \dots, n\}$ by maximum likelihood estimation, cross validation or Bayesian calibration

Mean predictor

$$\mu_{\hat{\beta}}(x) = \mathbf{f}(x)^T \hat{\beta} + \mathbf{r}(x)^T \mathbf{R}^{-1} (\mathcal{Y} - \mathbf{F}\hat{\beta})$$

where:

$$\begin{aligned} r_i(x) &= R(x - x^{(i)}, \theta) \\ R_{ij} &= R(x^{(i)} - x^{(j)}, \theta) \\ F_{ij} &= f_j(x^{(i)}) \end{aligned}$$



Kriging variance

$$\sigma_Y^2(x) = \sigma_Y^2 \left(1 - \left\langle \begin{array}{cc} \mathbf{f}(x)^T & \mathbf{r}(x)^T \end{array} \right\rangle \left[\begin{array}{cc} \mathbf{0} & \mathbf{F}^T \\ \mathbf{F} & \mathbf{R} \end{array} \right]^{-1} \left[\begin{array}{c} \mathbf{f}(x) \\ \mathbf{r}(x) \end{array} \right] \right)$$

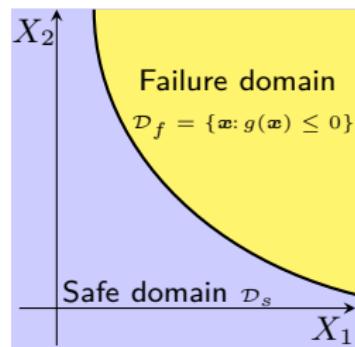
Structural reliability

Problem statement

- A **limit state function** $g : \boldsymbol{x} \in \mathcal{D}_{\boldsymbol{X}} \subset \mathbb{R}^M \mapsto \mathbb{R}$ that describes the **performance** of the system:

- + $g(\boldsymbol{x}) > 0$ means “safe configuration”
- + $g(\boldsymbol{x}) \leq 0$ means “failure”

e.g. Elastic behaviour is defined by: $g(\boldsymbol{x}) = \sigma_Y - \max_{\text{Structure}} \sigma_{VM}(\boldsymbol{x})$



- A probabilistic model (PDF) of the uncertainties: $\boldsymbol{X} \sim f_{\boldsymbol{X}}$

The **probability of failure** is defined by:

$$P_f = \mathbb{P}(g(\boldsymbol{X}) \leq 0) = \int_{\{\boldsymbol{x} \in \mathcal{D}_{\boldsymbol{X}} : g(\boldsymbol{x}) \leq 0\}} f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}$$

Use of Kriging for structural reliability analysis

- From a given experimental design $\mathcal{X} = \{\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n)}\}$, Kriging yields a mean predictor $\mu_{\hat{g}}(\boldsymbol{x})$ and the Kriging variance $\sigma_{\hat{g}}(\boldsymbol{x})$
- The mean predictor is substituted for the “true” limit state function, defining the surrogate failure domain

$$\mathcal{D}_f^0 = \{\boldsymbol{x} \in \mathcal{D}_{\mathbf{X}} : \mu_{\hat{g}}(\boldsymbol{x}) \leq 0\}$$

- The probability of failure is approximated by:

Kaymaz, Struc. Safety (2005)

$$P_f^0 = \mathbb{P} [\mu_{\hat{g}}(\boldsymbol{X}) \leq 0] = \int_{\mathcal{D}_f^0} f_{\mathbf{X}}(\boldsymbol{x}) d\boldsymbol{x} = \mathbb{E} [\mathbf{1}_{\mathcal{D}_f^0}(\boldsymbol{X})]$$

- Monte Carlo simulation can be used on the surrogate model:

$$\widehat{P}_f^0 = \frac{1}{N} \sum_{k=1}^N \mathbf{1}_{\mathcal{D}_f^0}(\boldsymbol{x}_k)$$

Confidence bounds on the probability of failure

Shifted failure domains

Dubourg *et al.*, Struct. Mult. Opt. (2011)

- Let us define a confidence level $(1 - \alpha)$ and $k_{1-\alpha} = \Phi^{-1}(1 - \alpha/2)$, i.e. 1.96 if $1 - \alpha = 95\%$, and:

$$\mathcal{D}_f^- = \{\boldsymbol{x} \in \mathcal{D}_{\boldsymbol{X}} : \mu_{\hat{g}}(\boldsymbol{x}) + k_{1-\alpha} \sigma_{\hat{g}}(\boldsymbol{x}) \leq 0\}$$

$$\mathcal{D}_f^+ = \{\boldsymbol{x} \in \mathcal{D}_{\boldsymbol{X}} : \mu_{\hat{g}}(\boldsymbol{x}) - k_{1-\alpha} \sigma_{\hat{g}}(\boldsymbol{x}) \leq 0\}$$

- Interpretation ($1 - \alpha = 95\%$):
 - If $\boldsymbol{x} \in \mathcal{D}_f^0$ it belongs to the true failure domain with a 50% chance
 - If $\boldsymbol{x} \in \mathcal{D}_f^+$ it belongs to the true failure domain with 95% chance:
conservative estimation

Bounds on the probability of failure

$$\mathcal{D}_f^- \subset \mathcal{D}_f^0 \subset \mathcal{D}_f^+ \quad \Leftrightarrow \quad P_f^- \leq P_f^0 \leq P_f^+$$

Adaptive designs for reliability analysis

Premise

- When using high-fidelity computational models for assessing structural reliability, the goal is to **minimize** the number of runs
- **Adaptive experimental designs** allow one to start from a small ED and **enrich** it with new points in suitable regions (*i.e.* close to the limit state surface)

Enrichment (infill) criterion

Bichon *et al.*, (2008, 2011); Echard *et al.* (2011); Bect *et al.* (2012), Balesdent *et al.* (2013), Morio & Balesdent (2015-16)

The following **learning function** is used:

$$LF(\boldsymbol{x}) = \frac{|\mu_{\hat{g}}(\boldsymbol{x})|}{\sigma_{\hat{g}}(\boldsymbol{x})}$$

- Small if $\mu_{\hat{g}}(\boldsymbol{x}) \approx 0$ (\boldsymbol{x} close to the limit state surface) and/or $\sigma_{\hat{g}}(\boldsymbol{x}) >> 0$ (poor local accuracy)
- The **probability of misclassification** is $\Phi(-LF(\boldsymbol{x}))$
- At each iteration, the new point is: $\boldsymbol{\chi}^* = \arg \min LF(\boldsymbol{x})$

Adaptive Kriging for reliability analysis

Algorithm 2: Adaptive Kriging for reliability analysis

```
1: Initialization
2:   Initial experimental design  $\mathcal{ED} = \{\boldsymbol{\chi}^{(1)}, \dots, \boldsymbol{\chi}^{(n)}\}$ 
3:   Monte Carlo sample  $\mathcal{X} = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\}$ 
4: while NotConverged do
5:   Train a Kriging model  $\widehat{\mathcal{M}}$  on the current experimental design
6:   Compute the probability of failure  $\hat{P}_f^0$ , and its bounds  $[\hat{P}_f^-, \hat{P}_f^+]$  using  $\widehat{\mathcal{M}}$ 
7:   if  $(\hat{P}_f^+ - \hat{P}_f^-)/\hat{P}_f^0 \leq TOL$  then
8:     NotConverged = FALSE
9:   else
10:    Evaluate the learning function  $LF$  on  $\mathcal{X}$ 
11:    Compute the next ED point:  $\boldsymbol{\chi}^* = \arg \min_{\boldsymbol{x} \in \mathcal{X}} LF(\boldsymbol{x})$ 
12:    Update the experimental design:  $\mathcal{ED} \leftarrow \mathcal{ED} \cup \{\boldsymbol{\chi}^*\}$ 
13:   end
14: end
15: Return Probability of failure  $\hat{P}_f^0$  and confidence interval  $[\hat{P}_f^-, \hat{P}_f^+]$ 
```

Series system

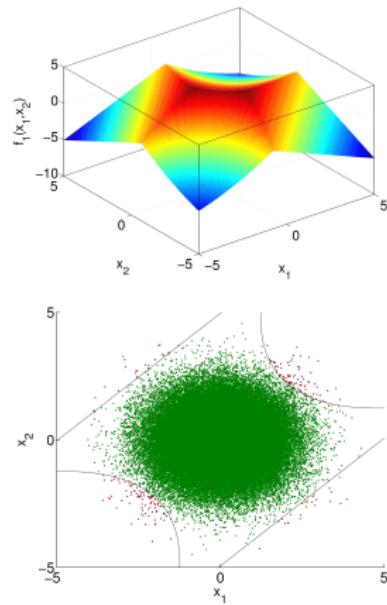
Schöbi, Sudret & Marelli, ASCE J. Risk Unc. (2016)

Consider the system reliability analysis defined by:

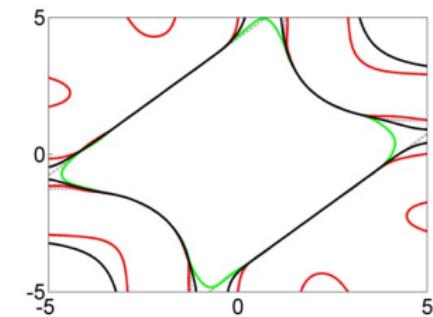
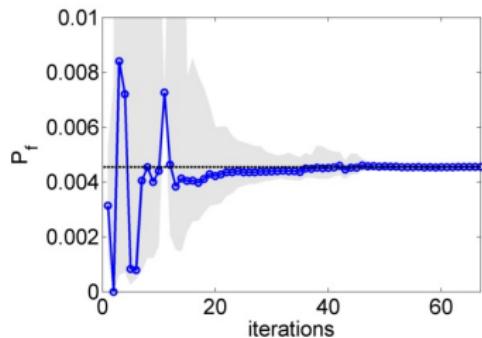
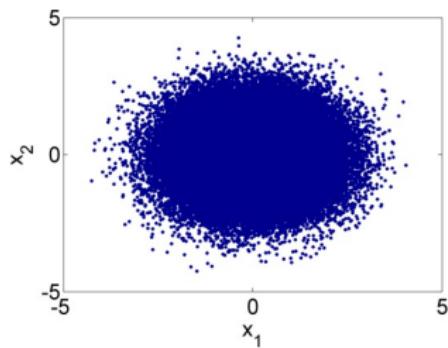
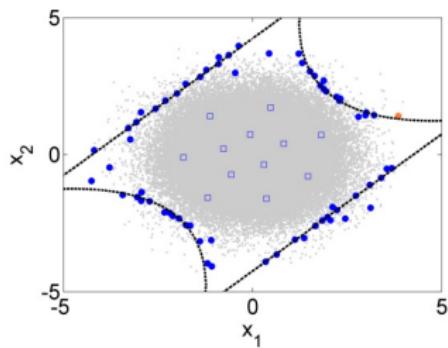
$$g(\mathbf{x}) = \min \begin{pmatrix} 3 + 0.1(x_1 - x_2)^2 - \frac{x_1 + x_2}{\sqrt{2}} \\ 3 + 0.1(x_1 - x_2)^2 + \frac{x_1 + x_2}{\sqrt{2}} \\ (x_1 - x_2) + \frac{6}{\sqrt{2}} \\ (x_2 - x_1) + \frac{6}{\sqrt{2}} \end{pmatrix}$$

where $X_1, X_2 \sim \mathcal{N}(0, 1)$

- Initial design: LHS of size 12 (transformed into the standard normal space)
- In each iteration, one point is added (maximize the probability of missclassification)



Results with PC Kriging

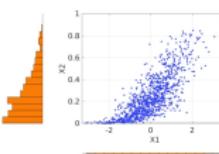
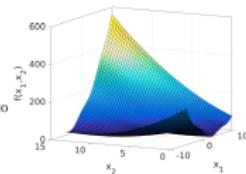
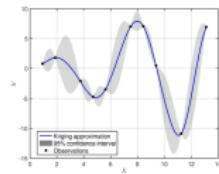
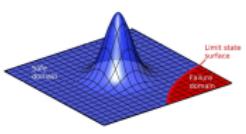
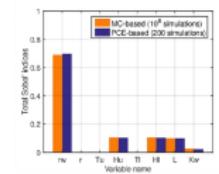
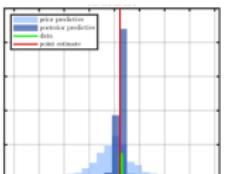


Conclusions

- Surrogate models are unavoidable for solving uncertainty quantification problems involving costly computational models (e.g. finite element models)
- Depending on the analysis, specific surrogates are most suitable: polynomial chaos expansions for distribution- and sensitivity analysis, Kriging (and low-rank tensor approximations) for reliability analysis
- Kriging and PC-Kriging are suitable for adaptive algorithms (enrichment of the experimental design)
- All these techniques are non-intrusive: they rely on experimental designs, the size of which is a user's choice
- They are versatile, general-purpose and field-independent
- All the presented algorithms are available in the general-purpose uncertainty quantification software UQLab

UQLab: The Uncertainty Quantification Software

www.uqlab.com

<h3>PROBABILISTIC INPUT MODELLING</h3> <ul style="list-style-type: none"> Common marginals Support for user-defined marginals Support for bounds on all distributions (including user-defined) Gaussian copula 	<h3>MODELLING FACILITIES</h3> <ul style="list-style-type: none"> Simple text strings MATLAB m-files MATLAB handles UQLINK: easily connect UQLAB to third party modelling software 
<h3>ADVANCED METAMODELLING</h3> <ul style="list-style-type: none"> Sparse degree-adaptive Polynomial Chaos Expansions Gaussian process modelling (Kriging) Polynomial-Chaos Kriging Low-rank tensor approximations Support vector machines 	<h3>RELIABILITY ANALYSIS (RARE EVENT ESTIMATION)</h3> <ul style="list-style-type: none"> FORM/SORM approximation Monte Carlo Simulation (MCS) Importance Sampling Subset Simulation Adaptive Kriging (AK-MCS) 
<h3>SENSITIVITY ANALYSIS</h3> <ul style="list-style-type: none"> Correlation-based indices Standard Regression Coefficients Cotter measure Morris indices Sampling-based Sobol' indices PCE- and LRA-based Sobol' indices Borgonovo δ indices Support for dependent inputs 	<h3>BAYESIAN INVERSION</h3> <ul style="list-style-type: none"> Intuitive problem statement Advanced MCMC algorithms Multi-model support (joint inversion) Support for custom likelihood 

UQLab: The Uncertainty Quantification Software

<http://www.uqlab.com>



- 1,900 downloads
- 950⁺ active users from 77 countries

- Release of V0.9 on July 1st, 2015 (beta version)
- V1.0 on April 28th, 2017
UQLabCore + modules (PCE, Kriging, Sensitivity, Rare events)
- V1.1 on July 1st, 2018
Support vector machines, UQLink
- V1.2 on February 22nd, 2019
Bayesian inversion, UQLib

Country	# Users
United States	343
China	217
France	214
Switzerland	177
Germany	130
United Kingdom	89
Italy	70
India	56
Brazil	50
Canada	50

As of April 23, 2019

UQLab users



Questions ?



**The Uncertainty
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www.rsuq.ethz.ch

Thank you very much for your attention !

References

-  M. Berveiller, B. Sudret, and M. Lemaire.
Stochastic finite elements: a non intrusive approach by regression.
Eur. J. Comput. Mech., 15(1–3):81–92, 2006.
-  G. Blatman and B. Sudret.
An adaptive algorithm to build up sparse polynomial chaos expansions for stochastic finite element analysis.
Prob. Eng. Mech., 25:183–197, 2010.
-  G. Blatman and B. Sudret.
Adaptive sparse polynomial chaos expansion based on Least Angle Regression.
J. Comput. Phys., 230:2345–2367, 2011.
-  A. Doostan and H. Owhadi.
A non-adapted sparse approximation of PDEs with stochastic inputs.
J. Comput. Phys., 230(8):3015–3034, 2011.
-  B. Efron, T. Hastie, I. Johnstone, and R. Tibshirani.
Least angle regression.
Annals of Statistics, 32:407–499, 2004.
-  G. Deman, K. Konakli, B. Sudret, J. Kerrou, P. Perrochet, and H. Benabderrahmane.
Using sparse polynomial chaos expansions for the global sensitivity analysis of groundwater lifetime expectancy in a multi-layered hydrogeological model.
Reliab. Eng. Sys. Safety, 147:156–169, 2016.
-  R. Ghanem and P. Spanos.
Stochastic Finite Elements: A Spectral Approach.
Courier Dover Publications, Mineola, 2nd edition, 2003.



S. S. Isukapalli.

Uncertainty Analysis of Transport-Transformation Models.

PhD thesis, The State University of New Jersey, 1999.



J.D Jakeman, M. S. Eldred, and K. Sargsyan.

Enhancing ℓ_1 -minimization estimates of polynomial chaos expansions using basis selection.

J. Comput. Phys., 289:18–34, 2015.



C. V. Mai, M. D. Spiridonakos, E. N. Chatzi, and B. Sudret.

Surrogate modeling for stochastic dynamical systems by combining nonlinear autoregressive with exogeneous input models and polynomial chaos expansions.

Int. J. Uncer. Quant., 6(4):313–339, 2016.



C. V. Mai and B. Sudret.

Surrogate models for oscillatory systems using sparse polynomial chaos expansions and stochastic time warping.

SIAM/ASA J. Unc. Quant., 5:540–571, 2017.



S. Rezaeian and A. Der Kiureghian.

Simulation of synthetic ground motions for specified earthquake and site characteristics.

Earthq. Eng. Struct. Dyn., 39(10):1155–1180, 2010.



A. Saltelli, M. Ratto, T. Andres, F. Campolongo, J. Cariboni, D. Gatelli, M. Saisana, and S. Tarantola.

Global Sensitivity Analysis – The Primer.

Wiley, 2008.



K. Sargsyan, C. Safta, H. Najm, B. Debusschere, D. Ricciuto, and P. Thornton.

Dimensionality reduction for complex models via Bayesian compressive sensing.

Int. J. Uncertain. Quantificat., 4(1):63–93, 2014.



A. Sklar.

Fonctions de répartition à n dimensions et leurs marges.

Publications de l'Institut de Statistique de L'Université de Paris 8, 8(1):11, 1959.



C. Soize and R. Ghanem.

Physical systems with random uncertainties: chaos representations with arbitrary probability measure.

SIAM J. Sci. Comput., 26(2):395–410, 2004.



I. M. Sobol'.

Sensitivity estimates for nonlinear mathematical models.

Math. Modeling & Comp. Exp., 1:407–414, 1993.



M. D. Spiridonakos and E. N. Chatzi.

Metamodeling of dynamic nonlinear structural systems through polynomial chaos NARX models.

Comput. Struct., 157:99–113, 2015.



B. Sudret.

Uncertainty propagation and sensitivity analysis in mechanical models – Contributions to structural reliability and stochastic spectral methods.

Université Blaise Pascal, Clermont-Ferrand, France, 2007.

Habilitation à diriger des recherches, 173 pages.



B. Sudret.

Global sensitivity analysis using polynomial chaos expansions.

Reliab. Eng. Sys. Safety, 93:964–979, 2008.



D. Xiu and G. E. Karniadakis.

The Wiener-Askey polynomial chaos for stochastic differential equations.

SIAM J. Sci. Comput., 24(2):619–644, 2002.



D. Xiu.

Numerical methods for stochastic computations – A spectral method approach.

Princeton University press, 2010.



Mathieu Balesdent, Jérôme Morio, and Julien Marzat.

Kriging-based adaptive importance sampling algorithms for rare event estimation.

Structural Safety, 44:1–10, 2013.



B. J. Bichon, M. S. Eldred, L. Swiler, S. Mahadevan, and J. McFarland.

Efficient global reliability analysis for nonlinear implicit performance functions.

AIAA Journal, 46(10):2459–2468, 2008.



B. J. Bichon, J. M. McFarland, and S. Mahadevan.

Efficient surrogate models for reliability analysis of systems with multiple failure modes.

Reliab. Eng. Sys. Safety, 96(10):1386–1395, 2011.



V. Dubourg, B. Sudret, and J.-M. Bourinet.

Reliability-based design optimization using Kriging and subset simulation.

Struct. Multidisc. Optim., 44(5):673–690, 2011.



B. Echard, N. Gayton, and M. Lemaire.

AK-MCS: an active learning reliability method combining Kriging and Monte Carlo simulation.

Structural Safety, 33(2):145–154, 2011.



I. Kaymaz.

Application of Kriging method to structural reliability problems.

Structural Safety, 27(2):133–151, 2005.



P. Kersaudy, B. Sudret, N. Varsier, O. Picon, and J. Wiart.

A new surrogate modeling technique combining Kriging and polynomial chaos expansions – Application to uncertainty analysis in computational dosimetry.

J. Comput. Phys., 286:103–117, 2015.



T. J. Santner, B. J. Williams, and W. I. Notz.

The Design and Analysis of Computer Experiments.

Springer, New York, 2003.



R. Schöbi, B. Sudret, and S. Marelli.

Rare event estimation using Polynomial-Chaos-Kriging.

ASCE-ASME J. Risk Uncertainty Eng. Syst., Part A: Civ. Eng., 2016.

D4016002.



R. Schöbi, B. Sudret, and J. Wiart.

Polynomial-chaos-based Kriging.

Int. J. Uncertainty Quantification, 5(2):171–193, 2015.