Surrogate models for uncertain dynamical systems: applications to earthquake engineering

Bruno Sudret

Chair of Risk, Safety and Uncertainty Quantification

ETH Zurich

Symposium on Uncertainty Quantification in Computational Geosciences

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- Surrogate models (polynomial chaos expansions, Kriging, support vector machines)
- Bayesian model calibration and stochastic inverse problems
- Global sensitivity analysis
- Reliability-based design optimization

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Global framework for uncertainty quantification

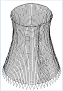
Step B
Quantification of sources of uncertainty

Step A
Model(s) of the system
Assessment criteria

Step C
Uncertainty propagation

Random variables

Computational model



Moments
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Response PDF

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B. Sudret, *Uncertainty propagation and sensitivity analysis in mechanical models – contributions to structural reliability and stochastic spectral methods (2007)*

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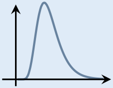

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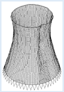

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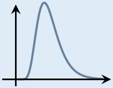

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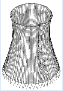

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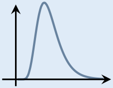

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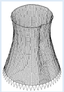

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Surrogate models for uncertainty quantification

A surrogate model $\tilde{\mathcal{M}}$ is an approximation of the original computational model $\mathcal{M}$ with the following features:

- It is built from a limited set of runs of the original model $\mathcal{M}$ called the experimental design $\mathcal{X} = \left\{ \boldsymbol{x}^{(i)},\, i = 1,\, \dots\, , n \right\}$
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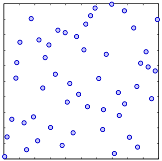
Name	Shape	Parameters
Polynomial chaos expansions	$\tilde{\mathcal{M}}(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} a_{\boldsymbol{\alpha}}\, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{x})$	$\boldsymbol{a}_{\boldsymbol{\alpha}}$
Low-rank tensor approximations	$\tilde{\mathcal{M}}(\boldsymbol{x}) = \sum_{l=1}^{R} b_l \left(\prod_{i=1}^{M} v_l^{(i)}(x_i) \right)$	$b_l,\ z_{k,l}^{(i)}$
Kriging (a.k.a Gaussian processes)	$\tilde{\mathcal{M}}(\boldsymbol{x}) = \boldsymbol{\beta}^{\mathsf{T}} \cdot \boldsymbol{f}(\boldsymbol{x}) + Z(\boldsymbol{x}, \omega)$	$\boldsymbol{\beta}\,,\ \sigma_Z^2\,,\ \boldsymbol{\theta}$
Support vector machines	$\tilde{\mathcal{M}}(\boldsymbol{x}) = \sum_{i=1}^{n} a_i\, K(\boldsymbol{x}_i, \boldsymbol{x}) + b$	$\boldsymbol{a}\,,\ b$

Ingredients for building a surrogate model

- Select an experimental design $\mathcal{X}$ that covers at best the domain of input parameters: Latin hypercube sampling (LHS), low-discrepancy sequences



- Run the computational model $\mathcal{M}$ onto $\mathcal{X}$ exactly as in Monte Carlo simulation

- Smartly post-process the data $\{\mathcal{X}, \mathcal{M}(\mathcal{X})\}$ through a learning algorithm

Name	Learning method
Polynomial chaos expansions	sparse grid integration, least-squares, compressive sensing
Low-rank tensor approximations	alternate least squares
Kriging	maximum likelihood, Bayesian inference
Support vector machines	quadratic programming

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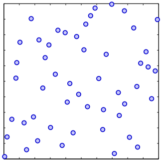

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Advantages of surrogate models

Usage

$$\mathcal{M}(\boldsymbol{x}) \quad \approx \quad \tilde{\mathcal{M}}(\boldsymbol{x})$$

hours per run — seconds for 10^6 runs

Advantages

- Non-intrusive methods: based on runs of the computational model, exactly as in Monte Carlo simulation
- Construction suited to high performance computing: "embarrassingly parallel"

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Challenges

- Need for rigorous validation
- Communication: advanced mathematical background

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Efficiency: 2-3 orders of magnitude less runs compared to Monte Carlo

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2. Polynomial chaos expansions
   - PCE in a nushell
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3. PC-NARX expansions
   - NARX model
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4. Fragility curves
   - Theory
   - Application: steel frame

Polynomial chaos expansions in a nutshell

Ghanem & Spanos (1991); Xiu & Karniadakis (2002); Soize & Ghanem (2004); Lemaître & Knio (2010)

- Consider the input random vector $\boldsymbol{X}$ (dim $\boldsymbol{X} = M$) with given probability density function (PDF) $f_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{i=1}^{M} f_{X_i}(x_i)$
- Assuming that the random output $Y = \mathcal{M}(\boldsymbol{X})$ has finite variance, it can be cast as the following polynomial chaos expansion:

$$Y = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^M} y_{\boldsymbol{\alpha}} \, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{X})$$

  where :

  - $\Psi_{\boldsymbol{\alpha}}(\boldsymbol{X})$: basis functions
  - $y_{\boldsymbol{\alpha}}$: coefficients to be computed (coordinates)

- The PCE basis $\left\{\Psi_{\boldsymbol{\alpha}}(\boldsymbol{X}), \, \boldsymbol{\alpha} \in \mathbb{N}^M\right\}$ is made of multivariate orthonormal polynomials

Computing the coefficients by least-square minimization

Isukapalli (1999); Berveiller, Sudret & Lemaire (2006)

Principle

The exact (infinite) series expansion is considered as the sum of a truncated series and a residual:

$$Y = \mathcal{M}(\boldsymbol{X}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} y_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(\boldsymbol{X}) + \varepsilon_P \equiv \mathbf{Y}^{\mathsf{T}} \boldsymbol{\Psi}(\boldsymbol{X}) + \varepsilon_P(\boldsymbol{X})$$

where : $\mathbf{Y} = \{y_{\boldsymbol{\alpha}}, \, \boldsymbol{\alpha} \in \mathcal{A}\} \equiv \{y_0, \, \dots \, , y_{P-1}\}$ (P unknown coef.)

$$\boldsymbol{\Psi}(\boldsymbol{x}) = \{\Psi_0(\boldsymbol{x}), \, \dots \, , \Psi_{P-1}(\boldsymbol{x})\}$$

Least-square minimization

The unknown coefficients are estimated by minimizing the mean square residual error:

$$\hat{\mathbf{Y}} = \arg\min \mathbb{E}\left[\left(\mathbf{Y}^{\mathsf{T}} \boldsymbol{\Psi}(\boldsymbol{X}) - \mathcal{M}(\boldsymbol{X})\right)^2\right]$$

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Discrete (ordinary) least-square minimization

An estimate of the mean square error (sample average) is minimized:

$$\hat{\mathbf{Y}} = \arg \min_{\mathbf{Y} \in \mathbb{R}^P} \frac{1}{n} \sum_{i=1}^{n} \left(\mathbf{Y}^{\mathsf{T}} \boldsymbol{\Psi}(\boldsymbol{x}^{(i)}) - \mathcal{M}(\boldsymbol{x}^{(i)}) \right)^2$$

Procedure

- Select a truncation scheme, *e.g.* $\mathcal{A}^{M,p} = \left\{ \boldsymbol{\alpha} \in \mathbb{N}^M \; : \; |\boldsymbol{\alpha}|_1 \leq p \right\}$
- Select an experimental design and evaluate the model response

$$\mathbf{M} = \left\{ \mathcal{M}(\boldsymbol{x}^{(1)}), \, \dots \, , \mathcal{M}(\boldsymbol{x}^{(n)}) \right\}^{\mathsf{T}}$$

- Compute the experimental matrix

$$\mathbf{A}_{ij} = \Psi_j \left(\boldsymbol{x}^{(i)} \right) \qquad i = 1, \, \dots \, , n \; ; \; j = 0, \, \dots \, , P - 1$$

- Solve the resulting linear system

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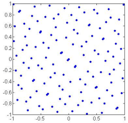

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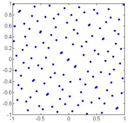

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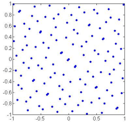

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Simple is beautiful !

Error estimators

- In least-squares analysis, the generalization error is defined as:

$$E_{gen} = \mathbb{E}\left[\left(\mathcal{M}(\boldsymbol{X}) - \mathcal{M}^{\mathsf{PC}}(\boldsymbol{X})\right)^2\right] \qquad \mathcal{M}^{\mathsf{PC}}(\boldsymbol{X}) = \sum_{\boldsymbol{\alpha}\in\mathcal{A}} y_{\boldsymbol{\alpha}}\,\Psi_{\boldsymbol{\alpha}}(\boldsymbol{X})$$

- The empirical error based on the experimental design $\mathcal{X}$ is a poor estimator in case of overfitting

$$E_{emp} = \frac{1}{n}\sum_{i=1}^{n}\left(\mathcal{M}(\boldsymbol{x}^{(i)}) - \mathcal{M}^{\mathsf{PC}}(\boldsymbol{x}^{(i)})\right)^2$$

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Leave-one-out cross validation

- From statistical learning theory, model validation shall be carried out using independent data

$$E_{LOO} = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{\mathcal{M}(\boldsymbol{x}^{(i)}) - \mathcal{M}^{PC}(\boldsymbol{x}^{(i)})}{1 - h_i}\right)^2$$

where h_i is the i-th diagonal term of matrix $\mathbf{A}(\mathbf{A}^{\mathsf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{T}}$

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Why brute-force PCE fails in dynamics?

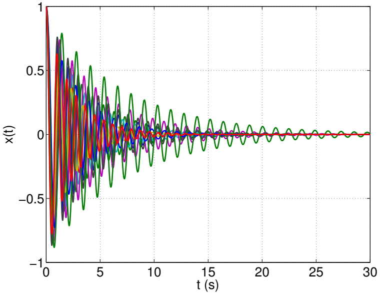
Non-linear SDOF Duffing oscillator

$$\ddot{x}(t) + 2\,\omega\,\zeta\,\dot{x}(t) + \omega^2\,\left(x(t) + \varepsilon\,x^3(t)\right) = 0$$

Initial conditions: $x(0) = 1, \quad \dot{x}(0) = 0$

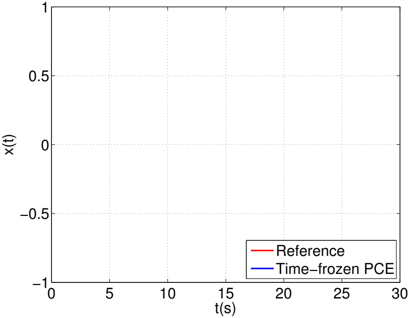
Input: 3 uniform random variables

RV	Distribution	Values
ζ	Uniform	$\mathcal{U}[0.015,\,0.045]$
ω	Uniform	$\mathcal{U}[\pi,\,3\,\pi]$
ε	Uniform	$\mathcal{U}[-0.25,\,-0.75]$

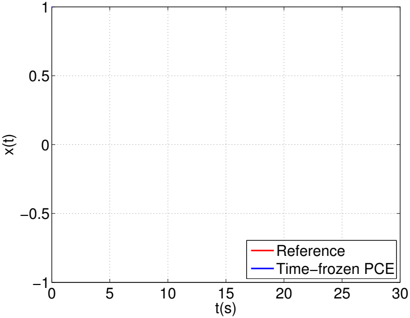


Samples of trajectories

Time-frozen PCE



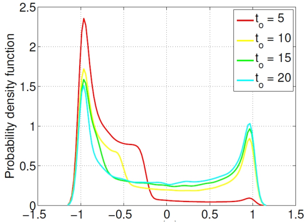
$(\zeta, \omega, \varepsilon) = (0.03, 8.92, -0.34)$

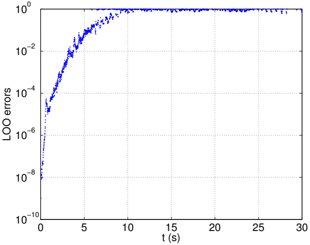


$(\zeta, \omega, \varepsilon) = (0.04, 3.18, -0.33)\mathbf{)}$

Why time-frozen PCE does not work?

- The map $\boldsymbol{\xi} \mapsto \mathcal{M}(\boldsymbol{\xi}, t)$ becomes increasingly non linear with time
- The time-frozen distribution of the output at time t_0 becomes more complex (*e.g.* multimodal)
- Expansions of higher degree would be required to keep sufficient accuracy with time
- For a fixed experimental design, the LOO error blows up





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- NARX model
- Calibration of a PC-NARX model
- Application to Bouc Wen model

4 Fragility curves

Some literature

- Multi-elements PCEs: decomposition of the random space into non-overlapping sub-elements — Wan & Karniadakis, 2005
- Constant phase interpolation: responses interpolated in the phase space — Witteveen & Bijl, 2008
- Asynchronous time integration: intrusive transformed time variable introduced to reduce variability — Le Maître et al., 2010
- Time-dependent PCEs: new random variables added on-the-fly — Gerritsma et al., 2010
- PC flow map composition: long-term response obtained by composing intermediate PCE-based flow maps — Luchtenburg et al., 2014
- PC-NARX: future state determined by current and past states — Spiridonakos & Chatzi, 2015

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Nonlinear AutoRegressive with eXogenous input model

NARX model

Billings, 2013

Based on a time-dependent input excitation $x(t)$ and corresponding system response $y(t)$, the dynamics is captured through:

$$y(t) = \mathcal{F}\left(x(t), \ldots, x(t-n_x), y(t-1), \ldots, y(t-n_y)\right) + \varepsilon_t$$

where:

- $\boldsymbol{z}(t) = \left(x(t), \ldots, x(t-n_x), y(t-1), \ldots, y(t-n_y)\right)^{\mathsf{T}}$ is the vector of current and past values
- n_x and n_y denote the maximum input and output time lags
- $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2(t))$ is the residual error
- $\mathcal{F}(\cdot)$ is a functional of NARX terms, usually linear-in-parameters:

$$y(t) = \sum_{i=1}^{n_g} \vartheta_i \, g_i(\boldsymbol{z}(t)) + \varepsilon_t$$

PC-NARX model

Spiridonakos *et al.* , 2015a,2015b

Computational model with uncertainties

$$y(t, \boldsymbol{\xi}_x, \boldsymbol{\xi}_s) \stackrel{\text{def}}{=} \mathcal{M}(x(t, \boldsymbol{\xi}_x), \boldsymbol{\xi}_s)$$

- $\boldsymbol{\xi}_x$: uncertainty in the input excitation
- $\boldsymbol{\xi}_s$: uncertainty in the system

PC-NARX expansion

$$y(t, \boldsymbol{\xi}) = \sum_{i=1}^{n_g} \vartheta_i(\boldsymbol{\xi})\, g_i(\boldsymbol{z}(t)) + \varepsilon_g(t, \boldsymbol{\xi}) \qquad \boldsymbol{\xi} = (\boldsymbol{\xi}_x, \boldsymbol{\xi}_s)$$

The NARX stochastic coefficients $\vartheta_i(\boldsymbol{\xi})$ are represented by PCEs:

$$\vartheta_i(\boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}_i} \vartheta_{i,\boldsymbol{\alpha}}\, \psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$$

PC-NARX model

Spiridonakos *et al.* , 2015a,2015b

Computational model with uncertainties

$$y(t, \boldsymbol{\xi}_x, \boldsymbol{\xi}_s) \overset{\text{def}}{=} \mathcal{M}(x(t, \boldsymbol{\xi}_x), \boldsymbol{\xi}_s)$$

- $\boldsymbol{\xi}_x$: uncertainty in the input excitation
- $\boldsymbol{\xi}_s$: uncertainty in the system

PC-NARX expansion

$$y(t, \boldsymbol{\xi}) = \sum_{i=1}^{n_g} \vartheta_i(\boldsymbol{\xi})\, g_i(\boldsymbol{z}(t)) + \varepsilon_g(t, \boldsymbol{\xi}) \qquad \boldsymbol{\xi} = (\boldsymbol{\xi}_x, \boldsymbol{\xi}_s)$$

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PC-NARX model

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Interpretation

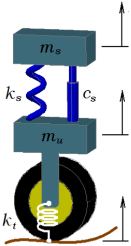
- PC-NARX is a NARX model in which each (random) coefficient is expanded as a PCE
- Compared to time-frozen PCE, a specific dynamics of the random coefficients is imposed
- Similar to flow map composition since the response at current instant is used to predict the response at future instants

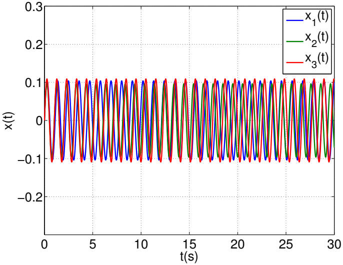
Experimental design

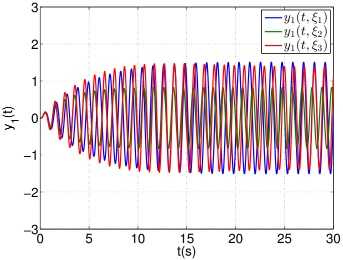
Data

- N realizations of the input excitation, cast as $(x_k[1], \ldots, x_k[T])^\mathsf{T}$, $k = 1, \ldots, N$ (T time instants)
- The corresponding system response computed by a simulator, cast as $(y_k[1], \ldots, y_k[T])^\mathsf{T}$

Example: quarter car model







Deterministic NARX calibration

For a particular realization $\boldsymbol{\xi}_k$

- Select NARX model (candidate terms):

$$\boldsymbol{z}(t) = (x(t), \ldots, x(t - n_x), y(t-1), \ldots, y(t - n_y))^{\mathsf{T}}$$
$$\boldsymbol{\phi}(t) = \{g_i(\boldsymbol{z}(t)),\ i = 1, \ldots, n_g\}^{\mathsf{T}}$$

- Use least angle regression (LARS) to select the best explanatory subset of terms Efron *et al.* , 2004

- Compute the coefficients $\boldsymbol{\vartheta}_k$ by ordinary least-squares

Prediction error (of model $\#k$ on trajectory l)

$$\varepsilon_l^{\#k} = \frac{\sum_{t=1}^{T} (y(t, \boldsymbol{\xi}_l) - \hat{y}^{\#k}(t, \boldsymbol{\xi}_l))^2}{\sum_{t=1}^{T} (y(t, \boldsymbol{\xi}_l) - \bar{y}(t, \boldsymbol{\xi}_l))^2}$$

Common NARX basis

Premise

To expand the NARX coefficients onto a PC basis, it is necessary to have a common NARX model for all trajectories

Procedure

- Select $K \leq N$ trajectories ("NARX learning set"), *e.g.* with the strongest non linear behaviour (peak displacement, velocities, etc.)
- Determine the sparse deterministic NARX models for realizations $k = 1, \ldots, K$, which leads to $P \leq K$ different possible models called $\#1, \ldots, \#P$
- Compute the NARX coefficients of the N trajectories, for each model $\#p$, and evaluate an average error:

$$\varepsilon_p = \frac{1}{N} \sum_{k=1}^{N} \varepsilon_k^{\#p}$$

- Select the final best NARX model that minimizes ε_p

PCE of the NARX coefficients

PCE calibration

- Once a common NARX basis has been found, N realizations of the NARX coefficients are available:

$$\mathcal{ED} = \{\vartheta_{i,k}, i = 1, \ldots, n_g;\ k = 1, \ldots, N\}$$

- n_g different sparse PC expansions are built from this experimental design, using least-angle regression (LAR) Blatman & Sudret, 2011

$$\vartheta_i(\boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}_i} \vartheta_{i,\boldsymbol{\alpha}}\, \psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$$

PC-NARX prediction

- For a new realization of the input parameters $\boldsymbol{\xi}_0$, the NARX coefficients are first evaluated from PCEs
- Then they are plugged into the NARX model

Bouc-Wen model

Governing equations

Kafali & Grigoriu (2007), Spiridonakos & Chatzi (2015)

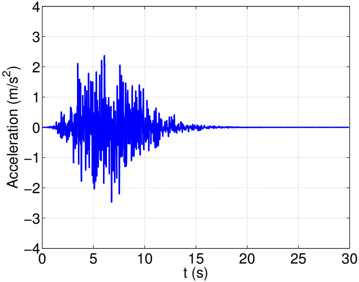
$$\ddot{y}(t) + 2\,\zeta\,\omega\,\dot{y}(t) + \omega^2(\rho\,y(t) + (1-\rho)\,z(t)) = -x(t),$$
$$\dot{z}(t) = \gamma\dot{y}(t) - \alpha\,|\dot{y}(t)|\,|z(t)|^{n-1}\,z(t) - \beta\,\dot{y}(t)\,|z(t)|^{n},$$

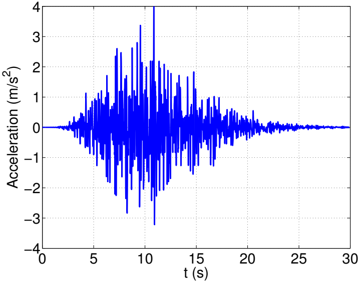
Excitation

$x(t)$ is generated by a probabilistic ground motion model

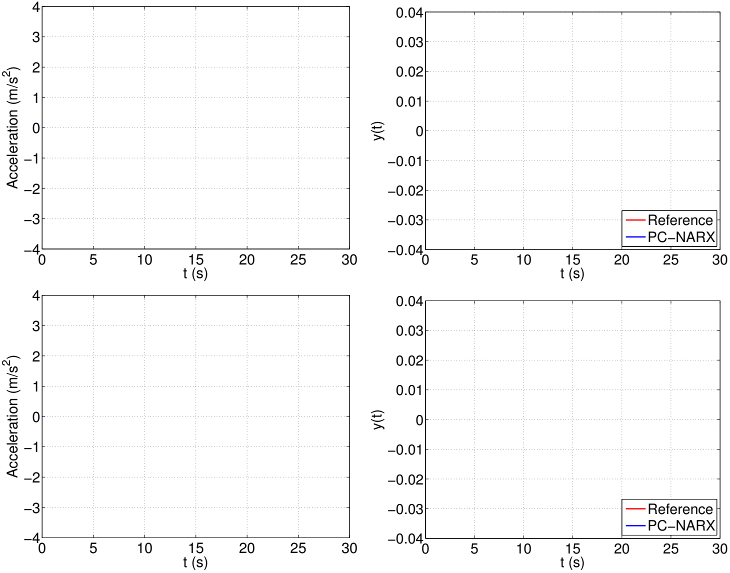
Rezaeian & Der Kiureghian (2010)

$$x(t) = q(t, \boldsymbol{\alpha}) \sum_{i=1}^{n} s_i\,(t, \boldsymbol{\lambda}(t_i))\;U_i$$





Bouc-Wen model: prediction



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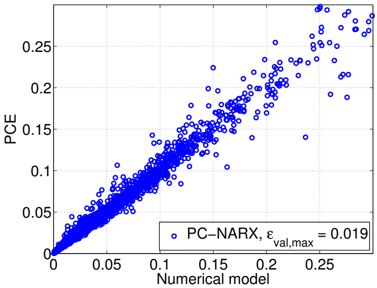
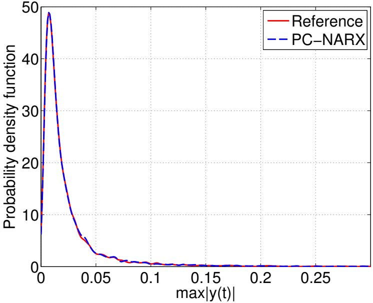


Maximal displacements



PDF of maximal displacements

Outline

1. Introduction
2. Polynomial chaos expansions
3. PC-NARX expansions
4. Fragility curves
   - Theory
   - Application: steel frame

Introduction to fragility curves

- Earthquake engineering aims at assessing the performance of structures and infrastructures w.r.t recorded or potential quakes
- Due to uncertainties in the localization, magnitude, structural behaviour and resistance, etc. probabilistic approaches are commonly used

Fragility curves

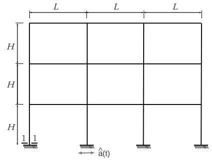
For a given performance criterion $g \leq g_{adm}$, the fragility curve represents the conditional probability of failure given an intensity measure IM:

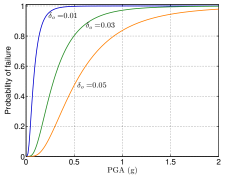
$$\text{Frag}(IM;\, g_{adm}) = \mathbb{P}\left(g \geq g_{adm} \,|\, IM\right)$$

Example

- $g = \max_k \max_{t_i \in [0,T]} |\delta^k_{t_i}|$ (k-th interstorey drift)
- IM: peak ground acceleration (PGA), pseudo-spectral acceleration (PSa), cumulative absolute velocity (CAV), etc.

Fragility curves





Classical approach

- Select a set of ground motions (recorded / synthetic)
- Compute the transient structural response (finite element analysis)
- Assume a parametric shape for the fragility curve, *e.g.* a lognormal shape:

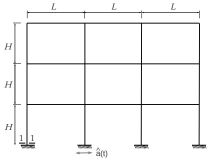
$$\text{Frag}(IM;\,\delta_o) = \mathbb{P}\left(\Delta \geq \delta_o \,|\, IM\right) = \Phi\left(\frac{\log IM - \alpha}{\beta}\right)$$

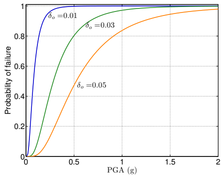
- Fit the parameters (α, β) form data

Limitations

- Predefined shape of the curve
- Subject to epistemic uncertainties when the number of ground motions is small

Fragility curves





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New proposal

- Use non parametric statistics for the fragility curves
- Use surrogate models of the transient analysis based on polynomial chaos expansions

Parametric methods

Linear regression (LR)

Ellingwood (2001)

- Probabilistic demand model:

$$\log \Delta = A \log IM + B + \zeta Z \qquad Z \sim \mathcal{N}(0,1)$$

- A and B determined by ordinary least squares estimation in a log-log scale
- Results in a lognormal-like fragility curve:

$$\begin{aligned}\widehat{\mathsf{Frag}}(IM;\,\delta_o) &= \mathbb{P}\left[\log \Delta \geq \log \delta_o\right] = 1 - \mathbb{P}\left[\log \Delta \leq \log \delta_o\right] \\ &= \Phi\left(\frac{\log IM - \left(\log \delta_o - B\right)/A}{\zeta/A}\right).\end{aligned}$$

Maximum likelihood estimation (ML)

Shinozuka *et al.* (2000)

- Lognormal shape:

$$\widehat{\mathsf{Frag}}(IM;\,\delta_o) = \Phi\left(\frac{\log IM - \log \alpha}{\beta}\right)$$

- Estimation of α and β by maximum likelihood for each δ_o:

$$\mathcal{L}\left(\alpha,\,\beta,\,\{IM_i\}_{i=1}^{N}\right) = \prod_{IM_i:\,\Delta_i \geq \delta_o} \left[\widehat{\mathsf{Frag}}(IM_i;\,\delta_o)\right] \prod_{IM_i:\,\Delta_i < \delta_o} \left[1 - \widehat{\mathsf{Frag}}(IM_i;\,\delta_o)\right]$$

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Non parametric methods

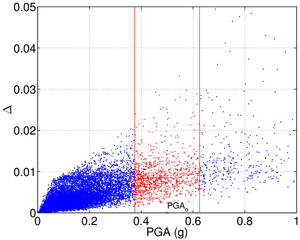
Binned Monte Carlo estimate

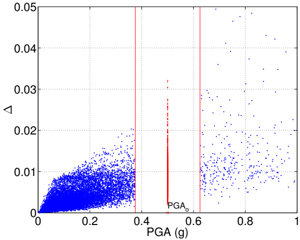
Mai, Konakli & Sudret, Frontiers Struct. Civ. Eng., (2017)

- Suppose N_s analyses are available for $IM = IM_o$, with N_f such that $\Delta \geq \delta_o$. The fragility curve in this point could be estimated by Monte Carlo simulation:

$$\widehat{\text{Frag}}(IM_o) = \frac{N_f\,(IM_o)}{N_s\,(IM_o)}$$

- From the data cloud, a bin centered on IM_o is considered, and points within the beam are "projected" onto the vertical line $IM = IM_o$ by linearization $\widetilde{\Delta_j}(IM_o) = \Delta_j \dfrac{IM_o}{IM_j}$.





Kernel density estimation

Fragility curves as a conditional CCDF

Mai *et al.* , Frontiers Struct. Civ. Eng., (2017)

$$\text{Frag}(a;\, \delta_o) = \mathbb{P}\left(\Delta \geq \delta_o | IM = a\right) = \int_{\delta_o}^{+\infty} f_\Delta(\delta | IM = a)\, \mathrm{d}\delta$$

where:

$$f_\Delta(\delta | IM = a) = \frac{f_{\Delta, IM}(\delta, a)}{f_{IM}(a)}$$

Kernel density estimation

- The joint- and the marginal PDFs are estimated by:

$$\hat{f}_X\left(x\right) = \frac{1}{Nh} \sum_{i=1}^{N} K\left(\frac{x - x_i}{h}\right) \qquad \hat{f}_{\boldsymbol{X}}\left(\boldsymbol{x}\right) = \frac{1}{N\left|\boldsymbol{H}\right|^{1/2}} \sum_{i=1}^{N} K\left(\boldsymbol{H}^{-1/2}(\boldsymbol{x} - \boldsymbol{x}_i)\right)$$

NB: Use of a constant bandwidth in the logarithmic scale

Mai, C., Polynomial chaos expansions for uncertain dynamical systems – Applications in earthquake engineering, PhD Thesis, ETH Zurich, 2016

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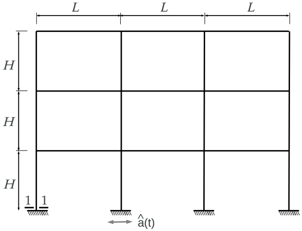
Application : steel frame



- 2D steel frame submitted to synthetic ground motions
- Synthetic earthquakes generated in time domain

Parameter	Distribution	Mean	Standard deviation	C.o.V
f_y (MPa)	Lognormal	264.2878	18.5	0.07
E_0 (MPa)	Lognormal	210000	630	0.03

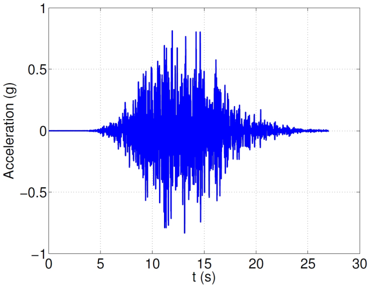
Stochastic ground motion

Stochastic excitation

- Obtained by a modulated filtered white noise process — Rezaeian & Der Kiureghian (2010)

$$x(t) = q(t, \boldsymbol{\alpha}) \sum_{i=1}^{n} s_i \left(t, \boldsymbol{\lambda}(t_i)\right) \cdot \xi_i \qquad \xi_i \sim \mathcal{N}(0,1)$$

- Parameters of the filter $\boldsymbol{\lambda} = \left(\omega_{mid}, \omega', \zeta_f\right)^{\mathsf{T}}$ are calibrated on recorded signals
- Global parameters (Arias intensity I_a, duration D_{5-95}, strong phase peak t_{mid}) are transformed into the parameters $\boldsymbol{\alpha}$ of the modulation function $q(t, \boldsymbol{\alpha})$ (e.g. gamma distribution)



Stochastic ground motion

Parameters of the excitation

Parameter	Distribution	Support	Mean	Standard deviation
I_a (s.g)	Lognormal	$(0, +\infty)$	0.0468	0.164
D_{5-95} (s)	Beta	[5, 45]	17.3	9.31
t_{mid} (s)	Beta	[0.5, 40]	12.4	7.44
$\omega_{mid}/2\pi$ (Hz)	Gamma	$(0, +\infty)$	5.87	3.11
$\omega'/2\pi$ (Hz)	Two-sided exponential	[-2, 0.5]	-0.089	0.185
ζ_f (.)	Beta	[0.02, 1]	0.213	0.143