近独立子组成的系统：把系统分成不同的子系（粒子，粒子的某个自由度，元激发）

单原子气体 $\hat{H}=\sum_i \frac{\hat{p}_i^2}{2m}$ 　多原子分子气体：$\hat{H}=\sum_i\left(\frac{\hat{p}_i^2}{2M}+\frac{\hat{J}_i^2}{2I}+\hat{h}_{iv}\right)$ 　声子：$\hat{H}=E_0+\sum_i \hbar\omega_i |n_i\rangle\langle n_i|$

注：从单粒子态到系统态时需考虑粒子全同性

Boltzmann 统计：

$$W_{Bol}(\{a_l\})=\frac{N!}{\prod_l a_l!}\prod_l w_l^{a_l}$$ 　最可几分布 $\delta\ln\Omega-\beta\delta E-\alpha\delta N=0$

$$a_l=w_l \boxed{e^{-\alpha}} e^{-\beta\varepsilon_l}=\boxed{\frac{N}{Z_1}} w_l e^{-\beta\varepsilon_l}=N\frac{w_l e^{-\beta\varepsilon_l}}{\sum_l w_l e^{-\beta\varepsilon_l}}$$ 　$N=\sum_l a_l=\frac{N}{Z_1}\sum_l w_l e^{-\beta\varepsilon_l}$

⇒引入 Z，求解系统宏观参量（α 难求但 N 易知）.

$Z=Z_1^N$ 无相互作用的定域粒子 ⇔ $Z=Z_1^N/N!$ 无相互作用的非定域粒子（无相互作用的经典粒子）

$$U=-\left(\frac{\partial\ln Z}{\partial\beta}\right)_V=\sum_l a_l\varepsilon_l=\sum_l w_l e^{-\alpha-\beta\varepsilon_l}\varepsilon_l=\frac{N}{Z_1}\sum_l w_l\varepsilon_l e^{-\beta\varepsilon_l}$$

$$S=k_B\ln Z+\frac{U}{T}-k_B\ln N!$$

$$F=-\frac{1}{\beta}(\ln Z-\ln N!)$$ 非定域性 　$p=\frac{1}{\beta}\left(\frac{\partial\ln Z}{\partial V}\right)_\beta$ 　$\mu=\left(\frac{\partial F}{\partial N}\right)_{\beta,V}$

Fermion 统计：

$$W_F(\{a_l\})=\prod_l\frac{w_l!}{a_l!(w_l-a_l)!}$$

$$a_l=w_l/(e^{\alpha+\beta\varepsilon_l}+1)\qquad \alpha=-\beta\mu$$

⇒引入 Ξ，求解系统宏观参量（α,β 需要知道）

$$\Xi=\prod_l\left(1+e^{-\alpha-\beta\varepsilon_l}\right)^{w_l}$$

$$U=-\left(\frac{\partial\ln\Xi}{\partial\beta}\right)_{\alpha,V}=k_BT^2\left(\frac{\partial\ln\Xi}{\partial T}\right)_{\mu,V}+N\mu$$

$$S=k_B(\ln\Xi+\alpha N+\beta U)$$

$$N=-\left(\frac{\partial\ln\Xi}{\partial\alpha}\right)_{\beta,V}\qquad p=\frac{1}{\beta}\left(\frac{\partial\ln\Xi}{\partial V}\right)_{\alpha,\beta}$$

Boson 统计：

$$W_B(\{a_l\})=\prod_l\frac{(a_l+w_l-1)!}{a_l!(w_l-1)!}$$

$$a_l=w_l/(e^{\alpha+\beta\varepsilon_l}-1)$$

$$\Xi=\prod_l\left(1-e^{-\alpha-\beta\varepsilon_l}\right)^{-w_l}$$

双原子气体 $\hat{h}=\frac{\hat{p}_c^2}{2m}\underbrace{-\frac{\hbar^2}{2\mu}\partial_x^2+\frac{1}{2}\mu\omega^2x^2+V_0}_{\text{振动}}\underbrace{-\frac{\hbar^2}{2I}\left[\left(\frac{1}{\sin\theta}\partial\theta\right)^2+\frac{1}{\sin^2\theta}\partial_\phi^2\right]}_{\text{转动：与全同性有关}}$

$$Z_t=V\left(\frac{2\pi m}{\beta h^2}\right)^{\frac{3}{2}}\qquad Z_v=\sum_{n=0}^\infty e^{-\beta\hbar\omega(n+\frac{1}{2})}$$

$$Z_r=\begin{cases}\overset{\text{电子自旋}}{w_s}\sum_{l=0}(2l+1)e^{-\beta\frac{l(l+1)\hbar^2}{2I}} & \text{（非全同，无近似）}\\ w_s\left(1+3e^{-\frac{2\Theta_r}{T}}\right) & \text{（非全同，低温取前两项）}\\ w_s\int_0^\infty e^{-\frac{\Theta_r l(l+1)}{T}}d[l(l+1)] & \text{（非全同，高温变积分）}\\ \sum_{l=1,3,\dots}3(2l+1)e^{-\frac{\Theta_r l(l+1)}{T}} & \text{（全同，正氢）}\\ \sum_{l=0,2,\dots}1(2l+1)e^{-\frac{\Theta_r l(l+1)}{T}} & \text{（全同，仲氢）}\end{cases}$$

简并划分：$\lambda\ll1$ ⇒弱简并，全同性不重要

$e^{-\alpha}=e^{\beta\mu}$

$\lambda\lesssim1$ ⇒玻色强简并

$\lambda>1$ ⇒费米强简并

弱简并理想气体：$\ln\Xi=Z_1(T,V)F_{5/2}^{(\mp)}(\lambda)$ $\begin{cases}\text{上：Fermion}\\ \text{下：Boson}\end{cases}$

w_s：对光子：$w_s=2$，对电子 $w_s=2s+1=2$

$$N=Z_1(T,V)\boxed{F_{3/2}^{(\mp)}(\lambda)}\triangleq y\left(y=\frac{N}{Z_1(T,V)}\text{已知}\right)$$

$\boxed{\text{BEC}}$: $\ln\Xi = -w_g\ln(1-e^{-\alpha}) - \int_0^\infty g(\varepsilon)\ln[1-e^{-\alpha-\beta\varepsilon}]\,d\varepsilon$

(强简并玻色气体) $= -w_g\ln(1-\lambda) + Z_1(T,V)F^{(+)}_{5/2}(\lambda)$

$$N = \frac{w_g\lambda}{1-\lambda} + Z_1(T,V)F^{(+)}_{3/2}(\lambda) \qquad \lambda_{max}=1 \Leftrightarrow \mu=0$$

$$= N_0 + N\left(\frac{T}{T_c}\right) \qquad T=T_c\text{时}\quad N = Z_1(T_c,V)F^{(+)}_{3/2}(1)$$

$\boxed{\text{光子气体}}$: $\mu=0 \quad \varepsilon = cp \quad g(\varepsilon) = 2\int\delta(\varepsilon-cp)\boxed{\frac{d\vec{r}\cdot d\vec{p}}{h^3}}$ 3D. $\boxed{\frac{dx\,dp}{h}}$ 1D.

$\ln\Xi = -\frac{V}{\pi^2c^3}\int_0^\infty d\omega\,\omega^2\ln(1-e^{-\hbar\omega\beta})$ 保留此形式,不积分,求 $U(T) = -\left(\frac{\partial\ln\Xi}{\partial\beta}\right)_V$

$\boxed{\text{强简并费米气体}}$: $g(\varepsilon) = \int\delta\left(\varepsilon-\frac{p^2}{2m}\right)\boxed{\frac{d\vec{r}\cdot d\vec{p}}{h^3}}$ or other form

$$\ln\Xi = 4\pi V\left(\frac{2m}{h^2}\right)^{\frac{3}{2}}\int_0^\infty\sqrt{\varepsilon}\ln(1+e^{-\alpha-\beta\varepsilon})\,d\varepsilon$$

$$= 4\pi V\left(\frac{2m}{h^2}\right)^{\frac{3}{2}}\frac{2\beta}{3}\int_0^\infty\varepsilon^{\frac{3}{2}}f(\varepsilon)\,d\varepsilon$$

Sommerfeld展开: $I = \int_0^\infty N(\varepsilon)f(\varepsilon)\,d\varepsilon = \int_0^\mu N(\varepsilon)\,d\varepsilon + \frac{\pi^2}{6}(kT)^2N'(\mu)$

$\boxed{\text{Ising平均场理论}}$: $\hat{H} = -\frac{1}{2}\sum_{i\neq j}J_{ij}\hat{\vec{S}}_i\cdot\hat{\vec{S}}_j - \sum_i g\mu_B\vec{H}\cdot\hat{\vec{S}}_i$

$$= -\frac{J}{2}\sum_{\langle ij\rangle}\hat{\sigma}_i\hat{\sigma}_j - g\mu_BH\sum_i\hat{\sigma}_i$$

令 $\sigma_i = \bar{\sigma} + \sigma_i - \bar{\sigma} \quad (\bar{\sigma}+\delta\sigma_i)$

$\hat{H} = -Jz\bar{\sigma}\sum_i\sigma_i - g\mu_BH\sum_i\sigma_i + NJz\bar{\sigma}^2/2$ $\quad z$为最邻近格点数

$\bar{\sigma} = \tanh\{\beta\mu_Bg[H+Jz\bar{\sigma}/g\mu_B]\}$

$\Xi = \sum_N e^{\beta\mu N}Z(\beta,N,V) = \sum_N e^{\beta\mu N}\sum_{E_i}\Omega(E_i,N,V)e^{-\beta E_i} = \sum_{N=0}^{\infty}\frac{1}{N!}e^{\beta\mu N}\int e^{-\beta\hat{H}}\frac{d\vec{r}\,d\vec{p}}{h^{Nd}}$ (经典极限 非定域)

微正则系综的经典极限 $\Omega(E,N,V) = \begin{cases}\int\delta(E-\hat{H})\frac{d\vec{r}\,d\vec{p}}{h^{Nd}} & \text{定域}\\ \frac{1}{N!}\int\delta(E-\hat{H})\frac{d\vec{r}\,d\vec{p}}{h^{Nd}} & \text{非定域}\end{cases}$

$dS = \frac{dE}{T} + \frac{p}{T}dV - \frac{\mu}{T}dN$ 故 T, p, μ 由 S 的偏微分得到

正则系综 $Z(T,N,V) = \sum_s e^{-\beta E_s} = \sum_{E_i}\sum_{\{s|E_s=E_i\}}e^{-\beta E_i} = \sum_{E_i}\Omega(E_i,N,V)e^{-\beta E_i}$

经典极限 $Z(T,N,V) = \begin{cases}\int e^{-\beta\hat{H}}\frac{d\vec{r}\cdot d\vec{p}}{h^{Nd}} & \text{定域}\\ \frac{1}{N!}\int e^{-\beta\hat{H}}\frac{d\vec{r}\cdot d\vec{p}}{h^{Nd}} & \text{非定域}\end{cases}$

$$\Delta N^2 = \left(\frac{\partial^2\ln\Xi}{\partial\alpha^2}\right)_{\beta,V}$$

$$\Delta E^2 = \left(\frac{\partial^2\ln Z}{\partial\beta^2}\right)_V \ (\text{算}\ \overline{E^2})$$

$$\bar{a}_i = -\frac{\partial\ln Z}{\partial(\beta\varepsilon_i)}$$

$$\bar{a}_i = -\frac{\partial\ln\Xi}{\partial(\beta\varepsilon_i)}$$

$$\Delta a_i^2 = \frac{\partial^2\ln\Xi}{\partial(\beta\varepsilon_i)^2} = \frac{\partial^2\ln Z}{\partial(\beta\varepsilon_i)^2}$$

Tips: 题目要求涨落 ⇒ 用系综

N,T,V = 正则 $\begin{cases}\text{经典粒子: 几率法}\\ \left.\begin{array}{l}\text{玻色子}\\ \text{费米子}\end{array}\right\}\text{系综}\end{cases}$