Homework 1

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Sorry for the inconvenience of the positions of figures, this is my first time using Latex so I still have not figured out how to put them into good positions.

Problem 1.
(a) Please see Figure 1 -12 for the log-log scale of the truncation error ϵ vs. Δx^{-1}.
By comparing different sets of schemes, we can find that ϵ_{TR} will decay faster with higher order of polynomial interpolant$(N-1)$.
And Staggered-Centered scheme has the fastest decay rate.
Note that Figure 9 should be the same as Figure 12.

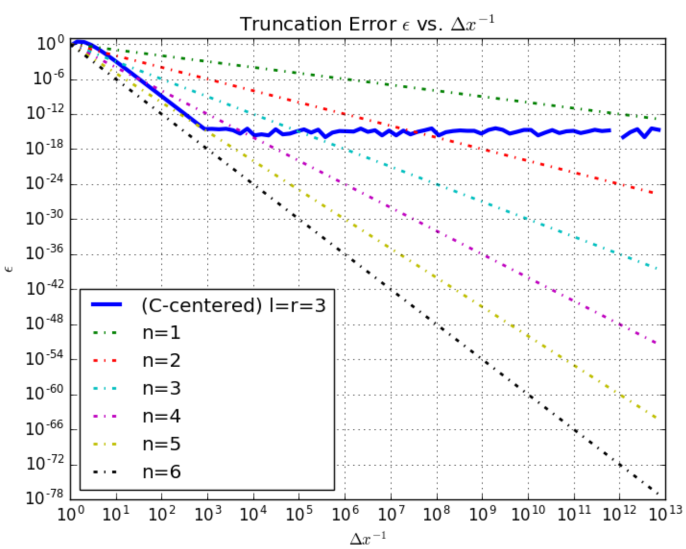

Figure 1: Collocated-Centered $l = r = 3$

(b) As you can see from the plots, the order of accuracy $\boxed{\text{does not}}$ always correspond to the order of the polynomial interpolant.
For a given polynomial order p, the minimum order of accuracy is p and the maximum order

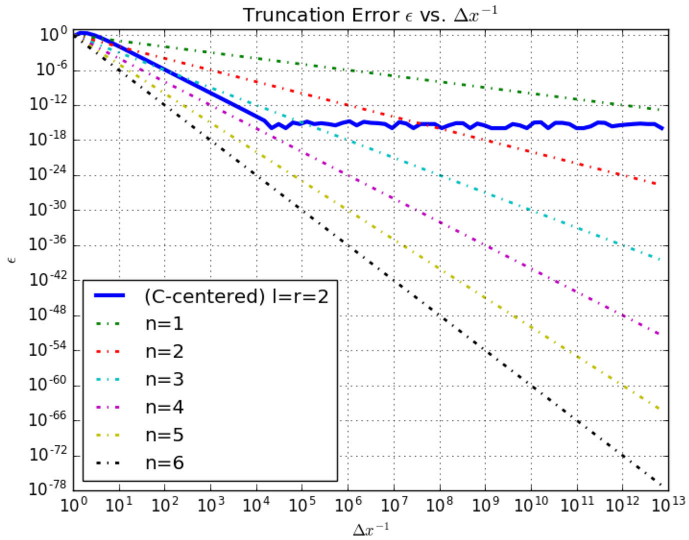

Figure 2: Collocated-Centered $l = r = 2$

of accuracy is $p + 1$. But if N is large enough, these two orders should be the same.

Problem 2.
(a) Please see Figure 13 - 16 for spy plots for the four schemes respectively.

(b) Please see Figure 17 - 18 for the log-log scale of the RMS vs. Δx^{-1} of 3rd and 5th order polynomial reconstructions respectively.
Please noted that if I increase my iteration times of the 5th-order to 8 (I am using 7 right now), the system will produce warning: RankWarning: Polyfit may be poorly conditioned, warnings.warn(msg, RankWarning)

Problem 3.
(a) By using Pade' scheme, we need to use 5th-order Pade' scheme to derive it.
As illustrated in the lecture notes, we expand the derivation to 5 points:

$$\alpha_1\{u_{i-2} = u_i - u_i^{(1)} \cdot 2\Delta x + u_i^{(2)} \cdot \frac{(2\Delta x)^2}{2!} - u_i^{(3)} \cdot \frac{(2\Delta x)^3}{3!} + u_i^{(4)} \cdot \frac{(2\Delta x)^4}{4!} - u_i^{(5)} \cdot \frac{(2\Delta x)^5}{5!} + u_i^{(6)} \cdot \frac{(2\Delta x)^6}{6!}\}$$

$$\alpha_2\{u_{i-1} = u_i - u_i^{(1)} \cdot \Delta x + u_i^{(2)} \cdot \frac{(\Delta x)^2}{2!} - u_i^{(3)} \cdot \frac{(\Delta x)^3}{3!} + u_i^{(4)} \cdot \frac{(\Delta x)^4}{4!} - u_i^{(5)} \cdot \frac{(2\Delta x)^5}{5!} + u_i^{(6)} \cdot \frac{(\Delta x)^6}{6!}\}$$

$$\alpha_3\{u_i = u_i\}$$

$$\alpha_4\{u_{i+1} = u_i + u_i^{(1)} \cdot \Delta x + u_i^{(2)} \cdot \frac{(\Delta x)^2}{2!} + u_i^{(3)} \cdot \frac{(\Delta x)^3}{3!} + u_i^{(4)} \cdot \frac{(\Delta x)^4}{4!} + u_i^{(5)} \cdot \frac{(2\Delta x)^5}{5!} + u_i^{(6)} \cdot \frac{(\Delta x)^6}{6!}\}$$

$$\alpha_5\{u_{i-2} = u_i + u_i^{(1)} \cdot 2\Delta x + u_i^{(2)} \cdot \frac{(2\Delta x)^2}{2!} + u_i^{(3)} \cdot \frac{(2\Delta x)^3}{3!} + u_i^{(4)} \cdot \frac{(2\Delta x)^4}{4!} + u_i^{(5)} \cdot \frac{(2\Delta x)^5}{5!} + u_i^{(6)} \cdot \frac{(2\Delta x)^6}{6!}\}$$

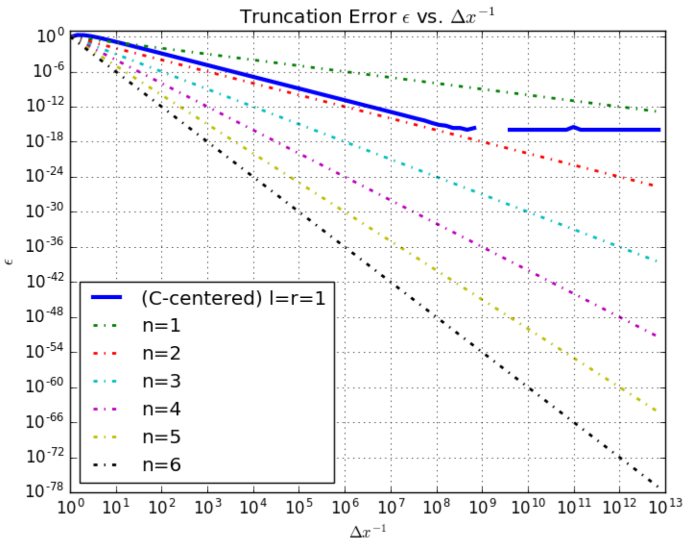

Figure 3: Collocated-Centered $l = r = 1$

$$\beta_1\{\Delta x^3 \cdot u_{i+1}^{(3)} = 0 + \Delta x^3[u_i^{(3)} + u_i^{(4)} \cdot \Delta x^4 + u_i^{(5)} \cdot \frac{(\Delta x)^2}{2!} + u_i^{(6)} \cdot \frac{(\Delta x)^3}{3!}]\}$$

$$\beta_3\{\Delta x^3 \cdot u_{i-1}^{(3)} = 0 + \Delta x^3[u_i^{(3)} - u_i^{(4)} \cdot \Delta x^4 + u_i^{(5)} \cdot \frac{(\Delta x)^2}{2!} - u_i^{(6)} \cdot \frac{(\Delta x)^3}{3!}]\}$$

We multiple α_1, α_2, α_3, α_4, α_5, β_1 and β_3 on both sides and add them together and, combine like terms with u_i, $u_i^{(1)}$, $u_i^{(2)}$, $u_i^{(3)}$, $u_i^{(4)}$, $u_i^{(5)}$ and $u_i^{(6)}$, we can get:

$$\begin{aligned}
&\Rightarrow (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \cdot u_i \\
&+ (-2\alpha_1 - \alpha_2 + \alpha_4 + 2\alpha_5) \cdot u_i^{(1)}\Delta x \\
&+ (2\alpha_1 + \tfrac{1}{2}\alpha_2 + \tfrac{1}{2}\alpha_4 + 2\alpha_5) \cdot u_i^{(2)}\Delta x^2 \\
&+ (-\tfrac{4}{3}\alpha_1 - \tfrac{1}{6}\alpha_2 + \tfrac{1}{6}\alpha_4 + \tfrac{3}{4}\alpha_5 + \beta_1 + \beta_3) \cdot u_i^{(3)}\Delta x^3 \\
&+ (\tfrac{2}{3}\alpha_1 + \tfrac{1}{24}\alpha_2 + \tfrac{1}{24}\alpha_4 + \tfrac{2}{3}\alpha_5 + \beta_1 + \beta_3) \cdot u_i^{(4)}\Delta x^4 \\
&+ (-\tfrac{4}{15}\alpha_1 + \tfrac{1}{120}\alpha_2 - \tfrac{1}{120}\alpha_4 + \tfrac{4}{15}\alpha_5 + \tfrac{1}{2}\beta_1 - \tfrac{1}{2}\beta_3) \cdot u_i^{(5)}\Delta x^5 \\
&+ (\tfrac{4}{45}\alpha_1 + \tfrac{1}{720}\alpha_2 + \tfrac{1}{720}\alpha_4 + \tfrac{4}{45}\alpha_5 + \tfrac{1}{6}\beta_1 - \tfrac{1}{6}\beta_3) \cdot u_i^{(6)}\Delta x^6
\end{aligned}$$

Because we want to solve the 3rd-order derivative, the coefficient in front of all $u_i^{(m)}$ should be zero except the coefficient in front of $u_i^{(3)}$ is one.
So we have the above seven equations and seven unknowns, solve the linear system, we can got:

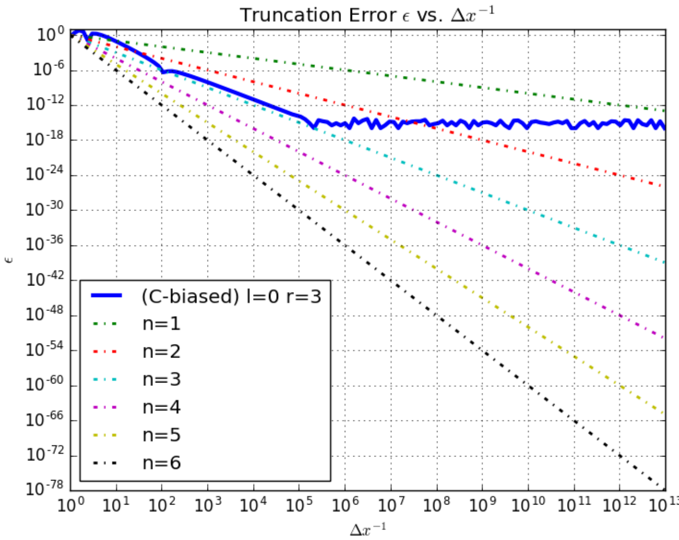

Figure 4: Collocated-Biased $l = 0$ $r = 3$

$$\begin{cases} \alpha_1 = -1 \\ \alpha_2 = 2 \\ \alpha_3 = 0 \\ \alpha_4 = -2 \\ \alpha_5 = 1 \\ \beta_1 = -\frac{1}{2} \\ \beta_3 = \frac{1}{2} \end{cases}$$

Insert back the αs and βs in the original equation, we can get:

$\Rightarrow -u_{i-2} + 2u_{i-1} - 2u_{i+1} + u_{i+2} - \frac{1}{2}\Delta x^3 \cdot u_{i+1}^{(3)} + \frac{1}{2}\Delta x^3 \cdot u_{i-1}^{(3)} = \sigma(\Delta x^3)$
$\Rightarrow \frac{-u_{i-2}+2u_{i-1}-2u_{i+1}+u_{i+2}}{\Delta x^3} = \frac{1}{2}(u_{i+1}^{(3)} - u_{i-1}^{(3)})$

So the order of accuracy is 3.

I do not have time but I think I know the basic concept for the rest of problem 3.
$\Rightarrow$
Then we can write the above equation in a form of:
$\Rightarrow \underline{L}\ \underline{\hat{u^{(3)}}} = \underline{R}\ \underline{U}$
$\Rightarrow \underline{\hat{u^{(3)}}} = \underline{L^{-1}}\ \underline{R}\ \underline{U}$
$\Rightarrow \underline{\hat{u^{(3)}}} = \underline{D}\ \underline{U}$
Where $\underline{D}$ should be a full rank matrix.

Also, we have to insert the three boundary conditions in the first, second last and last row of matrix $\underline{D}$ like Problem 2(b) and use the linear solver to get $\underline{\hat{u^{(3)}}}$ at the end.

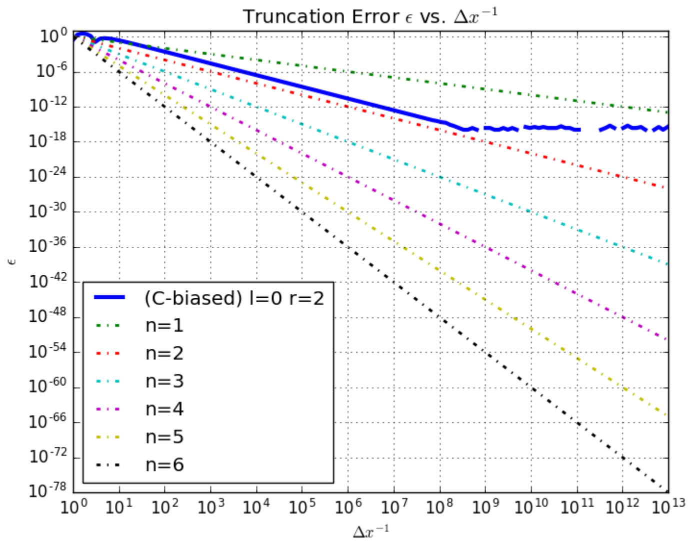

Figure 5: Collocated-Biased $l = 0$ $r = 2$

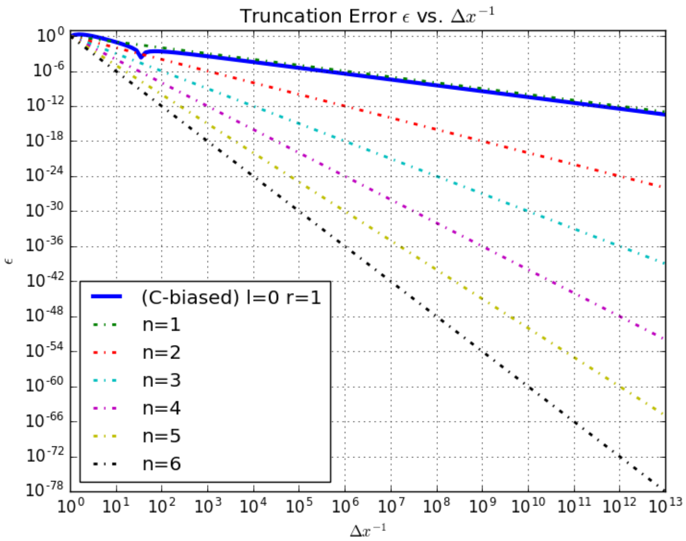

Figure 6: Collocated-Biased $l = 0$ $r = 1$

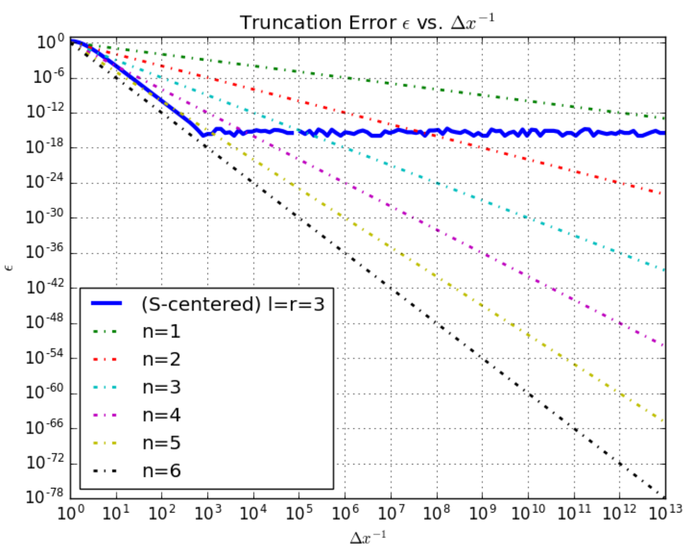

Figure 7: Staggered-Centered $l = r = 3$

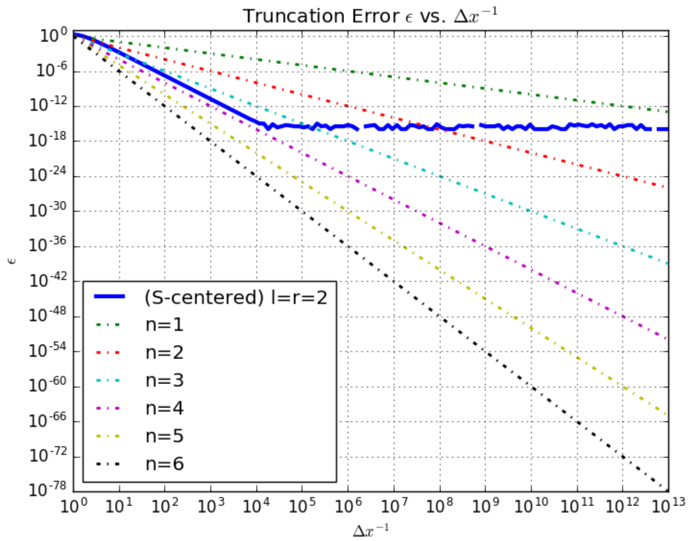

Figure 8: Staggered-Centered $l = r = 2$

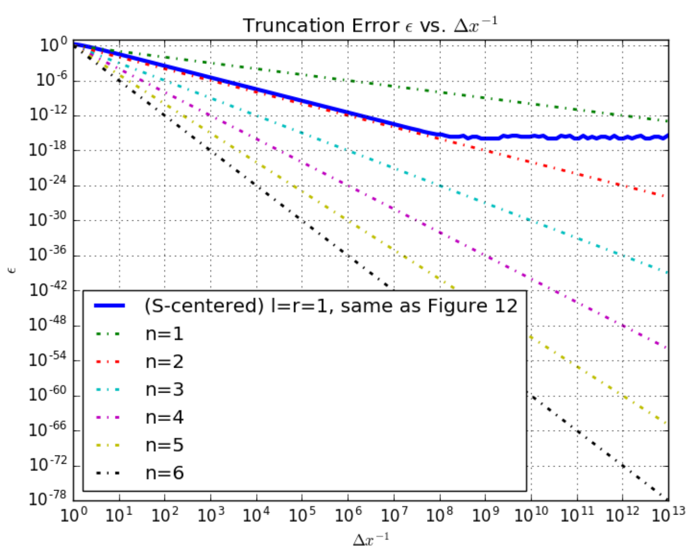

Figure 9: Staggered-Centered $l = r = 1$

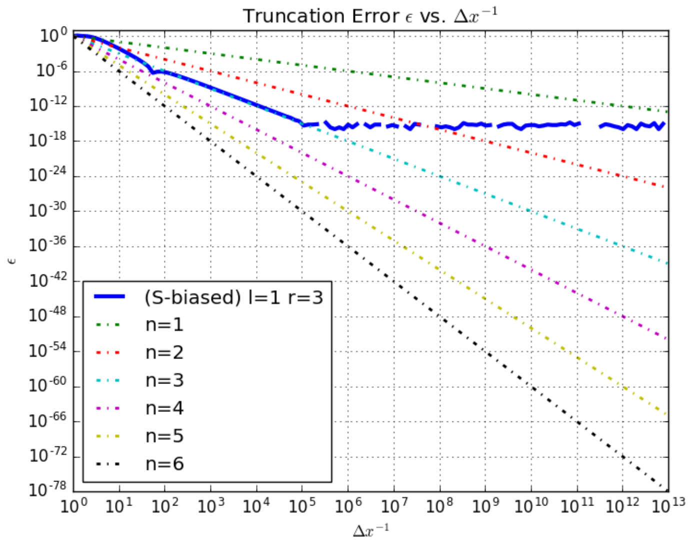

Figure 10: Staggered-Biased $l = 0$ $r = 3$

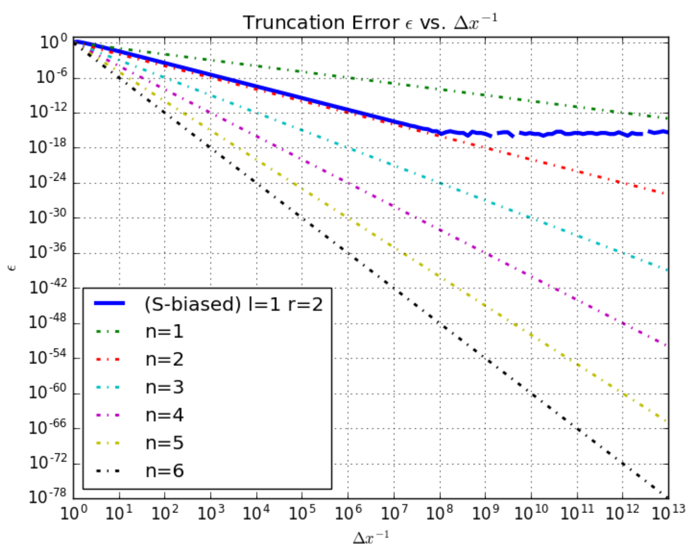

Figure 11: Staggered-Biased $l = 0$ $r = 2$

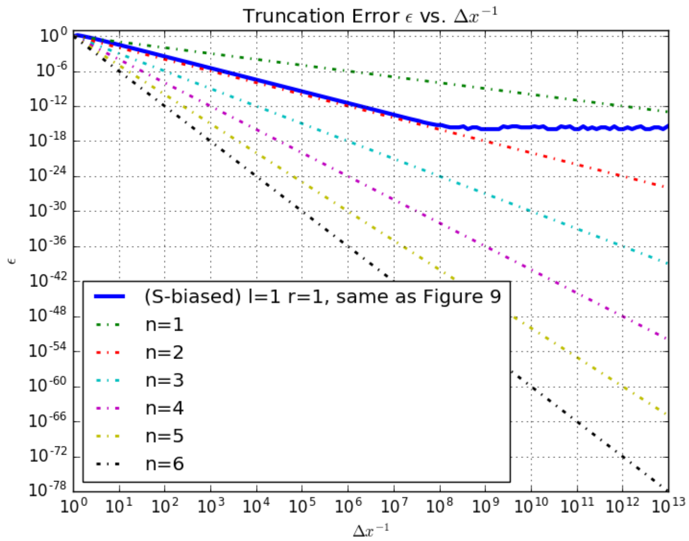

Figure 12: Staggered-Biased $l = 0$ $r = 1$

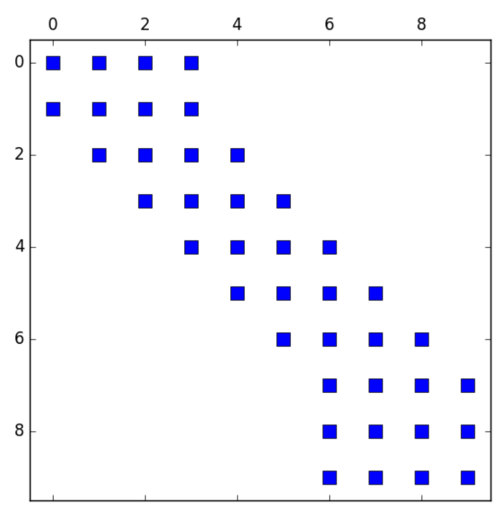

Figure 13: Spy Plot of 1st-order Derivatives with 3rd-order Polynomial Reconstructions

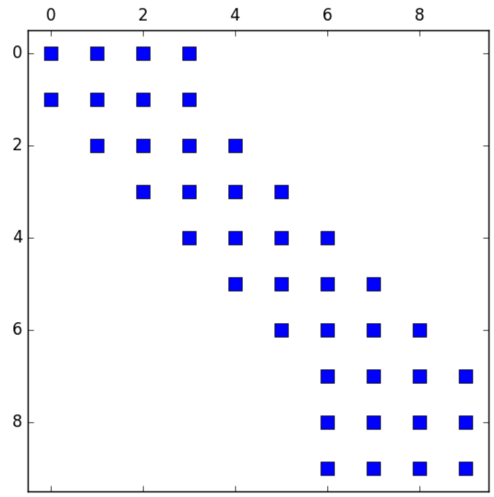

Figure 14: Spy Plot of 3rd-order Derivatives with 3rd-order Polynomial Reconstructions

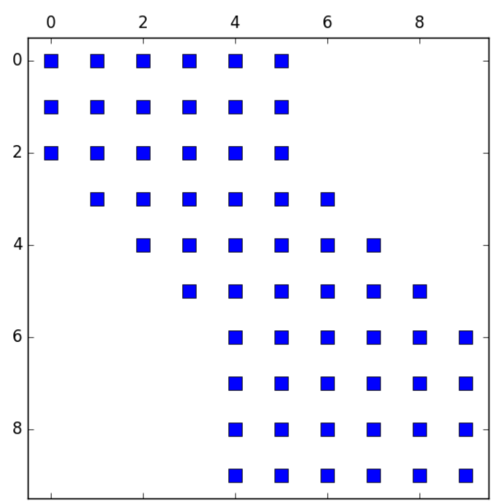

Figure 15: Spy Plot of 1st-order Derivatives with 5th-order Polynomial Reconstructions

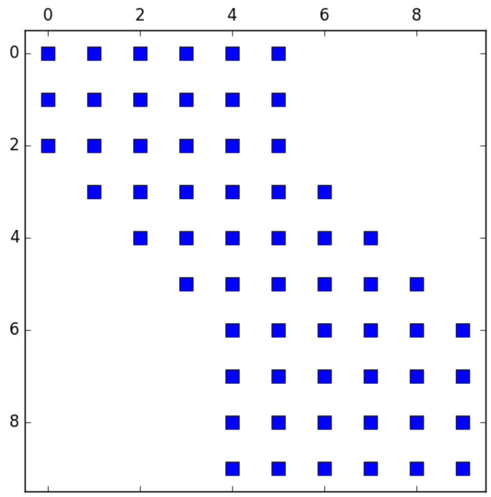

Figure 16: Spy Plot of 3rd-order Derivatives with 5th-order Polynomial Reconstructions

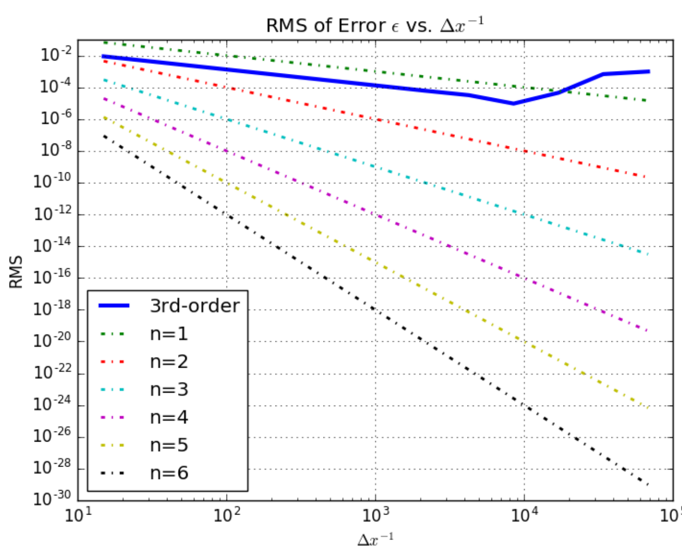

Figure 17: 3rd-order Polynomial Reconstructions Scheme

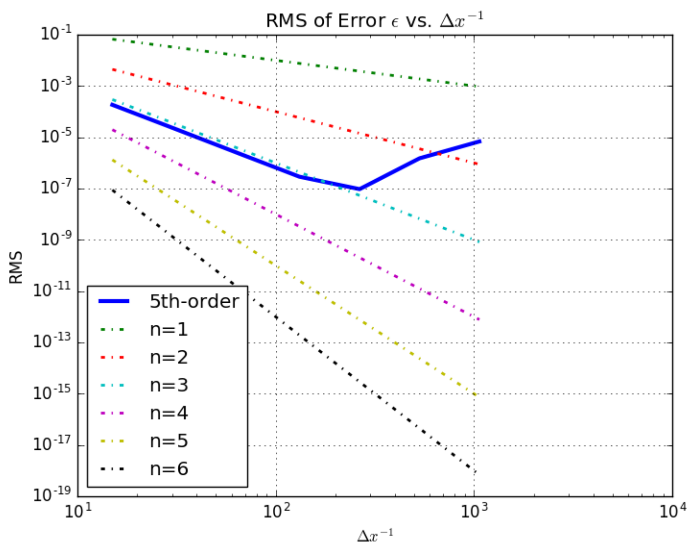

Figure 18: 5th-order Polynomial Reconstructions Scheme