Induction

- Most of the claims we make about a running time or a space-bound involve an integer parameter n (usually denoting an intuitive notion of the "size" of the problem).
- Moreover, most of these claims are equivalent to saying some statement q(n) is true "for all $n \geq 1$."
- Since this is making a claim about an infinite set of numbers, we cannot justify this exhaustively in a direct fashion.
- We can often justify claims such as those above as true, however, by using the technique of **induction**.
- This technique amounts to showing that
  - for any particular $n \geq 1$, there is a finite sequence of implications that starts with something known to be true and ultimately leads to showing that q(n) is true.
  - Specifically, we begin a justification by induction by showing that q(n) is true for n = 1 (and possibly some other values n = 2, 3, . . . , k, for some constant k)
  - Then we justify that the inductive "step" is true for n ¿ k, namely, we show "if q(j) is true for all $j < n$, then q(n) is true.
  - The combination of these two pieces completes the justification by induction.

Proposition 1:

Consider the Fibonacci function F(n), which is defined such that $F(1) = 1, F(2) = 2$, and $F(n) = F(n-2) + F(n-1)$ for $n > 2$. We claim that $F(n) < 2^n$.

Justification:

We will show our claim is correct by induction.

- **Base cases**: $(n \leq 2)$. $F(1) = 1 < 2 = 2^1$ and $F(2) = 2 < 4 = 2^2$.
- **Induction step**: $(n > 2)$. Suppose our claim is true for all $j < n$. Since both $n-2$ and $n-1$ are less than n, we can apply the inductive assumption (sometimes called the "inductive hypothesis") to imply that
$$F(n) = F(n-2) + F(n-1) < 2^{n-2} + 2^{n-1}$$
  Since
$$2^{n-2} + 2^{n-1} < 2^{n-1} + 2^{n-1} = 2 \times 2^{n-1} = 2^n,$$
  we have that $F(n) < 2^n$, thus showing the inductive hypothesis for n.

Proposition 2:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Justification:

- **Base cases**: $(n = 1)$. Trivial, for $1 = n(n+1)/2,$ if $n = 1$.
- **Induction step**: $(n \geq 2)$. Suppose our claim is true for all $j < n$. Therefore, for $j = n - 1$, we have

  $$\sum_{i=1}^{n-1} i = \frac{(n-1)(n-1+1)}{2} = \frac{(n-1)n}{2}$$

  Hence, we obtain

  $$\sum_{i=1}^{n} i = n + \sum_{i=1}^{n-1} i = n + \frac{(n-1)n}{2} = \frac{2n + n^2 - n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$$

  thereby proving the inductive hypothesis for n.