4.20

$u_1 = (1, 2, 5)$ $u_2 = (1, 3, 1)$ $u_3 = (2, 5, 7)$ $u_4 = (3, 1, 4)$

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 5 & 1 \\ 5 & 1 & 7 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad R_2 - 2R_1$$

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & -5 \\ 5 & 1 & 7 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad R_3 - 5R_1 \quad \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & -5 \\ 0 & -4 & -3 & -11 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

linear dependent

$$\begin{matrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 1 & 9 \end{matrix}$$

$c_1 + c_2 + 2c_3 + 3c_4 = 0$

$c_2 + c_3 - 5c_4 = 0$

$c_3 + 9c_4 = 0$ $\qquad$ $c_3 = -9c_4$

$c_3 = -9t$ $\qquad$ $c_4 = t$

$c_2 - 9t - 5t = 0$

$c_2 = 14t$ $\qquad$ $14 - 18 + 3 = 1$

$c_1 + 14t - 2 \cdot 9t + 3t = 0$

$c_1 - t = 0$ $\qquad$ $c_1 = t$

$t = 1$ $\qquad$ $c_1 = 1$

$c_2 = 14$

$c_3 = -9$

$c_4 = 1.$

$1 \cdot u_1 + 14u_2 - 9 \cdot u_3 + 1 \cdot u_4 = 0 \Rightarrow$

$(u_1 \dots u_n)$ lin. depen.

$u=(1,-3)$ $\quad$ $v=(-2,6)$

$c_1 = 2c_2$

$\boxed{c_1 = 2,\ c_2 = 1}$

$c_1 \neq 0$

$c_2 \neq 0$

$c_1 u + c_2 v = 0$

4.17 Linear dependence

Determine wether or not u and v are linearly dependent

b) $u=(1,-3)$ $v=(-2,6)$ $\quad u \cdot c_1 + u_2 \cdot v_2$

$$\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad R_2 + 3R_1 \quad \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$18-6$

$$2 \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix} + 1 \cdot \begin{pmatrix} -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix} + \begin{pmatrix} -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$c_1 - 2c_2 = 0$

$c_1 = 2c_2$!

$10u + 5v = 0$

$c_1 - 2c_2 = 0 \Rightarrow c_1 = 2c_2$

$c_2 = 5$

$c_1 = 10$

c) $u=(1,2,-3)$ $v=(4,5,-6)$

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \cancel{c_3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad R_2 - 2R_1$$

$$\begin{pmatrix} 1 & 4 \\ 0 & -3 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \cancel{c_3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad R_3 + 3R_1 \quad \begin{pmatrix} 1 & 4 \\ 0 & -3 \\ 0 & -15 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \quad R_2 + 5R_3$$

$$\begin{pmatrix} 1 & 4 \\ 0 & 0 \\ 0 & -15 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \quad \frac{R_3}{-15}$$

$c_1 + 4c_2 = 0$ $\quad$ $c_1 + 4 \cdot 0 = 0$

$c_2 = 0$ $\quad$ $c_1 = 0$

$-15c_3 = 0$

$c_3 = 15$ or $c_3 = 1$

are linearly dependent, where

(c) $u = (1, 2, -3)$, $v = (4, 5, -6)$

[illegible] c_1, c_2 [illegible]

$$c_1 u + c_2 \cdot v = 0$$

$$\underset{u}{\begin{pmatrix}1\\2\\-3\end{pmatrix}} c_1 + \underset{v}{\begin{pmatrix}4\\5\\-6\end{pmatrix}} \cdot c_2 = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 4\\2 & 5\\-3 & -6\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{aligned} c_1 + 4c_2 &= 0\\ 2c_1 + 5c_2 &= 0\\ -3c_1 - 6c_2 &= 0\end{aligned}$$

$R_2 - 2R_1$

$R_3 + 3R_1$

$$\begin{pmatrix}1 & 4\\0 & -3\\0 & 6\end{pmatrix} \quad R_2 \cdot 2 + R_3 \quad \begin{pmatrix}1 & 4\\0 & -3\\0 & 0\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{aligned} c_1 + 4c_2 &= 0\\ 0c_1 - 3c_2 &= 0\\ 0c_1 + 0c_2 &= 0\end{aligned} \qquad \begin{aligned} c_1 + 4 \cdot 0 &= 0\\ \boxed{c_2 = 0}&\\ \boxed{c_1 = 0}&\end{aligned}$$

$c_1 = 0 \quad c_2 = 0$

u, v Lin. Indep.

$$\begin{pmatrix} 1 & 2 & 1 \\ -3 & -4 & -5 \\ 2 & -1 & 7 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} \quad R_3 - 2R_1$$

$$\begin{pmatrix} 1 & 2 & 1 \\ -3 & -4 & -5 \\ 0 & -5 & 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix} \quad R_2 + 3R_1 \quad \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & -5 & 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

[illegible]

$2R_3 + 5R_2$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{matrix} 2 \\ 1 \\ 3 \end{matrix}$$

$C_1 + 2c_2 + C_3 = 2$

$2c_2 - 3c_3 = 1$ $\qquad c_3 = 3$

$C_3 = 3$

$2C_2 - 3 \cdot 3 = 1$

$2c_2 - 9 = 1$

$2c_2 = 10$

$C_2 = \frac{10}{2} = 5$

$C_1 + 2 \cdot 5 + 3 = 2$

$C_1 = 2 - 13$

$C_1 = -9$

d) $u = (2, 4, -8)$ $v = (3, 6, -12)$

$u_1 \cdot v_1 + u_2 v_2 + u_3 v_3 = 0$

$$\begin{pmatrix} 2 & 3 \\ 4 & 6 \\ -8 & -12 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad R_3 + 2R_2$$

$$\begin{pmatrix} 2 & 3 \\ 4 & 6 \\ -8 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \cancel{R_3 + 2R_1} \\ R_3 + 4R_1 \end{matrix} \quad \begin{pmatrix} 2 & 3 \\ 4 & 6 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 - 2R_1$

$$\begin{pmatrix} 2 & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad 2c_1 + 3c_2 = 0. \quad c_1 = -2$$

$c_2 = 0$

$c_3 = 0$

4.5

$V = (2, -5, 3)$ in $\mathbb{R}^3$ as a linear combination of the vectors.

$u_1 = (1, -3, 2)$ $u_2 = (2, -4, -1)$ $u_3 = (1, -5, 7)$

$c_1 u_1 + c_2 u_2 + c_3 u_3 = V$

$$c_1 \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$$

$\longrightarrow$

Problem 1:
Write down the truth tables for the following expressions:

i) $\neg(A \wedge B)$,

ii) $(\neg A) \vee (\neg B)$,

iii) $(\, \neg(A \wedge B) \,) \iff (\, (\neg A) \vee (\neg B) \,)$,

iv) $A \implies (B \vee C)$.

Problem 2:
How many elements do the following sets contain?

i) $\{1, 3\}$, ii) $\{2, 2\}$, iii) $\{\{1\}\}$, iv) $\{1, \{1\}\}$

v) $\{\,\}$, vi) $\{\{\,\}\}$, vii) $\{\{1, 1\}, \{1\}, 1\}$.

$\neg$ negation.

" უარყოფა

A	B	$A \wedge B$	$\neg(A \wedge B)$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

ii) $(\neg A) \vee (\neg B)$,

A	B	$\neg A$	$\neg B$	$(\neg A) \vee (\neg B)$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

A	B	$(\neg A) \vee (\neg B)$
0	0	1
0	1	1
1	0	1
1	1	0

A	B	A⇔B
T	T	T
T	F	F
F	T	F
F	F	T

Fill out the following truth tables:

1.

A	B	$\neg A \iff \neg B$
T	T	
T	F	
F	T	
F	F	

A	B	$\neg A$	$\neg B$	$(\neg A) \Leftrightarrow (\neg B)$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

R^3 — dim $= 3$

Basis and Dimension

4.24. Determine whether or not each of the following form a basis of $\mathbf{R}^3$:

(a) $(1,1,1)$, $(1,0,1)$;

(b) $(1,2,3)$, $(1,3,5)$, $(1,0,1)$, $(2,3,0)$;

(c) $(1,1,1)$, $(1,2,3)$, $(2,-1,1)$;

(d) $(1,1,2)$, $(1,2,5)$, $(5,3,4)$.

(a and b) No, because a basis of $\mathbf{R}^3$ must contain exactly three elements because $\dim \mathbf{R}^3 = 3$.

(c) The three vectors form a basis if and only if they are linearly independent. Thus, form the matrix whose rows are the given vectors, and row reduce the matrix to echelon form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

The echelon matrix has no zero rows; hence, the three vectors are linearly independent, and so they do form a basis of $\mathbf{R}^3$.

a)

(c) $(1,1,1)$, $(1,2,3)$, $(2,-1,1)$;
u_1, u_2, u_3,

$$x \cdot u_1 + y \cdot u_2 + z \cdot u_3 = 0$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow{\substack{R_2 - R_1 \\ R_3 - R_1}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 2 & -1 \end{bmatrix} \; R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

$-1 - (-6) = -1 + 6 = 5$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) 3-ha $\mathbb{R}^3$, (ii) 3 ... 3 ... 3

(iii) $\underbrace{u_1, u_2, u_3}_{3}$ — lin indep

B basis $= \{u_1, u_2, u_3\}$

Extend $\{u_1 = (1,1,1,1), u_2 = (2,2,3,4)\}$ to a basis of $\mathbf{R}^4$.

First form the matrix with rows u_1 and u_2, and reduce to echelon form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Then $w_1 = (1,1,1,1)$ and $w_2 = (0,0,1,2)$ span the same set of vectors as spanned by u_1 and u_2. Let $u_3 = (0,1,0,0)$ and $u_4 = (0,0,0,1)$. Then w_1, u_3, w_2, u_4 form a matrix in echelon form. Thus, they are linearly independent, and they form a basis of $\mathbf{R}^4$. Hence, u_1, u_2, u_3, u_4 also form a basis of $\mathbf{R}^4$.

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \quad \begin{matrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \\ 0 & 2 \end{matrix} \quad \begin{matrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{matrix} \quad \begin{matrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} \quad \begin{matrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix} \qquad \underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}$$

$\mathbb{R}^4$