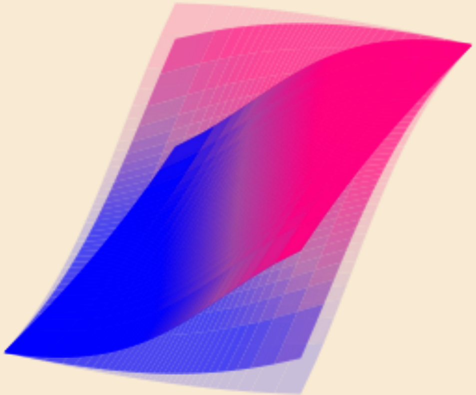
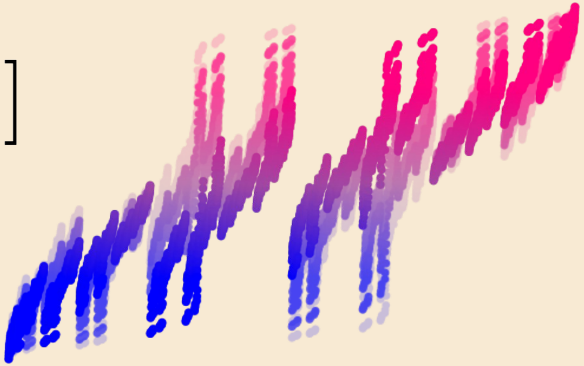
Equivalence of neural field dynamics with different embedding dimensionality

Nicole Vadot
(Supervisor)
Dr. Valentin Schmutz
(Supervisor, LCN director)
Prof. Wulfram Gerstner
(External expert)
Prof. Romain Veltz



$$S : \mathbb{R}^p \mapsto [0, 1]$$

$$\Longleftrightarrow$$



Neural field in $\mathbb{R}^p$

Neural field in $[0, 1]$

Online: https://ninivert.github.io/lcnthesis/

recap: neural fields & embedding spaces

activation (firing rate)

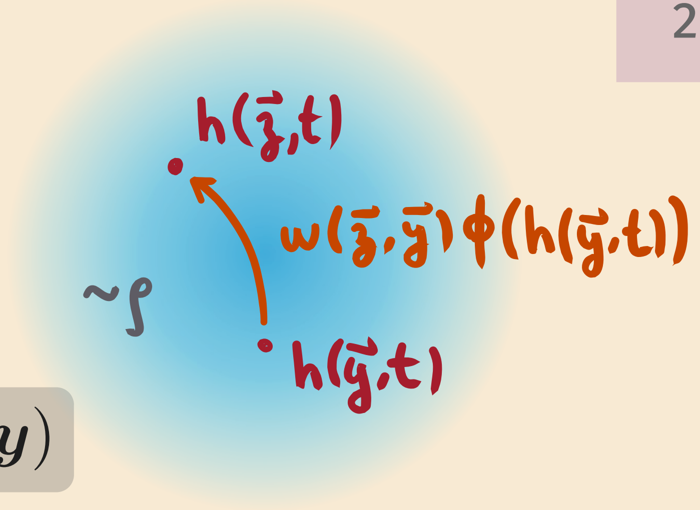
$$\partial_t h(\boldsymbol{z},t) = -h(\boldsymbol{z},t) + \int_{\mathbb{R}^p} w(\boldsymbol{z},\boldsymbol{y})\phi(h(\boldsymbol{y},t))\rho(\mathrm{d}\boldsymbol{y})$$

neural field

embedding space

connectivity kernel

population density



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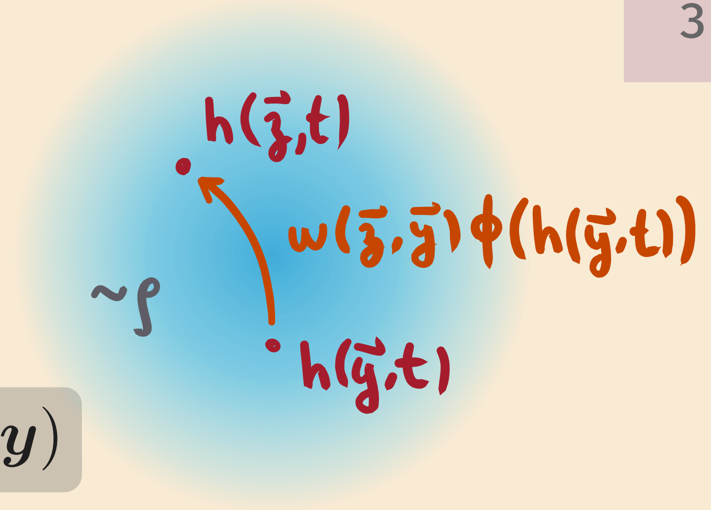

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population density

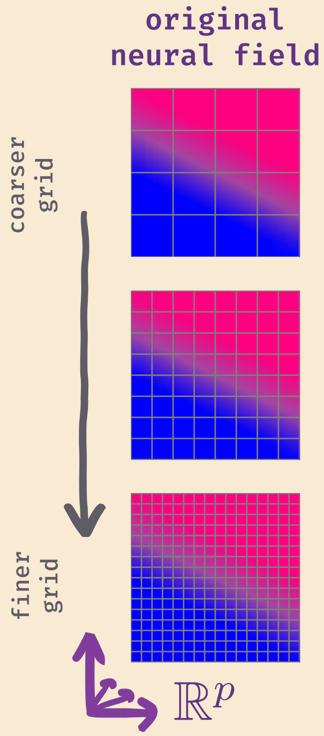
by redefining, we can "hide" the density in the kernel:
⇒ we get a **neural field in [0,1]^p**

$$\partial_t h_U(\boldsymbol{v},t) = -h_U(\boldsymbol{v},t) + \int_{[0,1]^p} w_U(\boldsymbol{v},\boldsymbol{u})\phi(h_U(\boldsymbol{u},t))\mathrm{d}\boldsymbol{u}$$

problem statement

neural field in p dimensions

$$\partial_t h_U(t,\boldsymbol{v}) = -h_U(t,\boldsymbol{v}) + \int_{[0,1]^p} w_U(\boldsymbol{v},\boldsymbol{u})\phi(h_U(t,\boldsymbol{u}))\mathrm{d}\boldsymbol{u}$$



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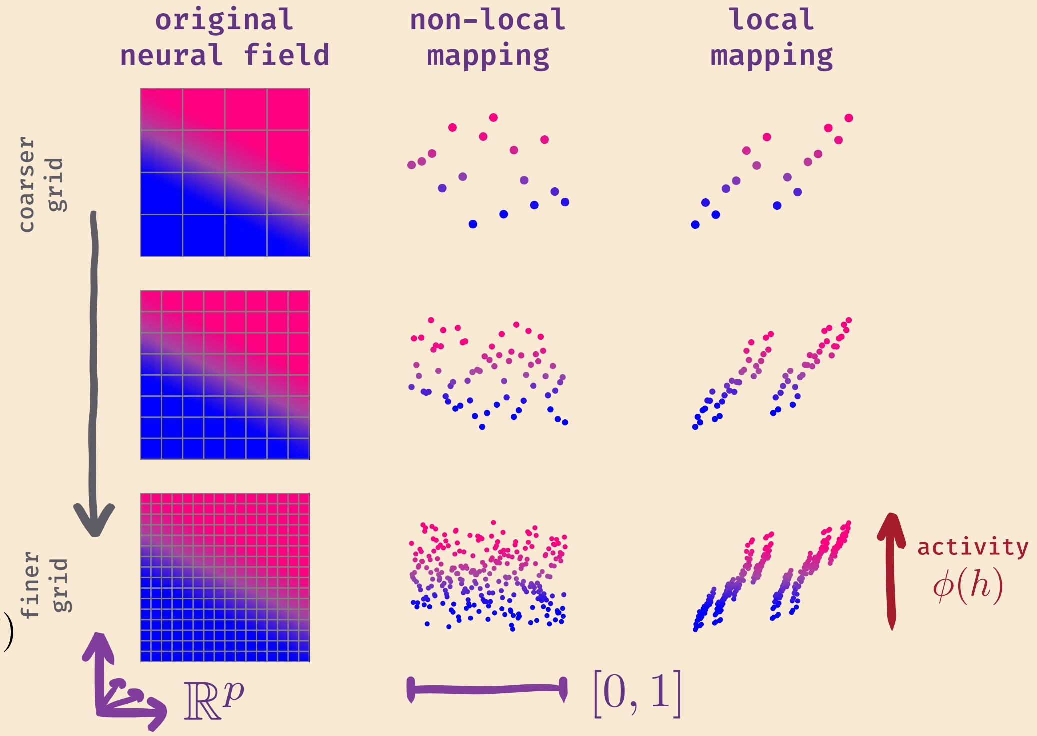
map to neural field in one dimension?

$$\partial_t \tilde{h}(\alpha,t) = -\tilde{h}(\alpha,t) + \int_{[0,1]} \tilde{w}(\alpha,\beta)\phi(\tilde{h}(\beta,t))\left[\lambda \circ S^{-1}\right](\mathrm{d}\beta)$$



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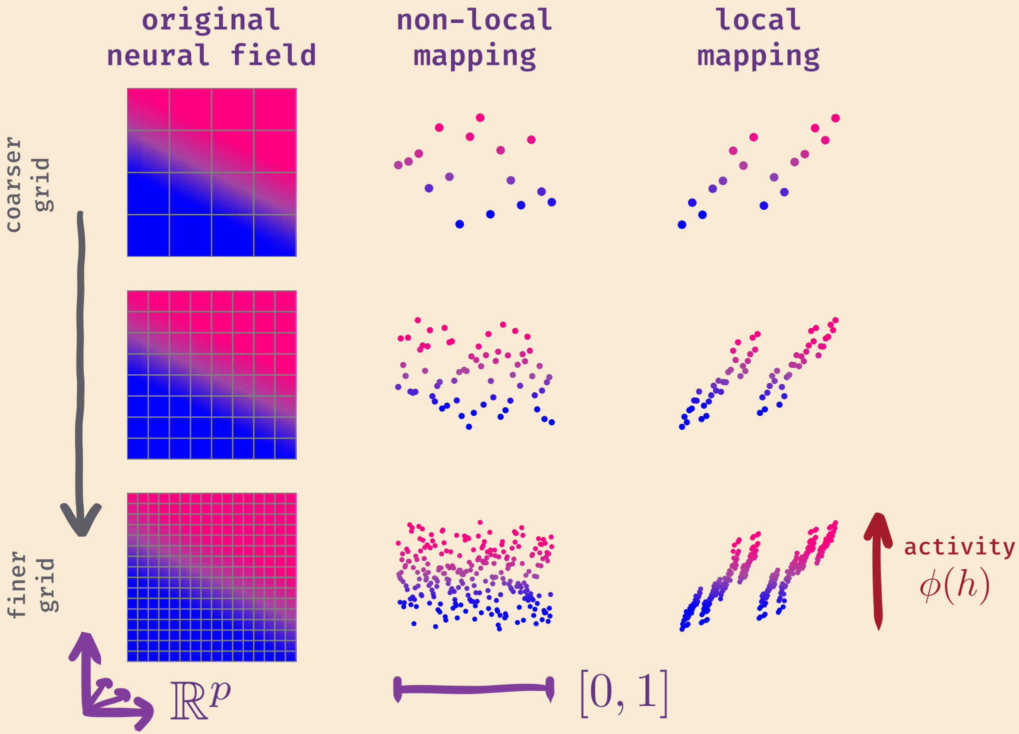

- → "regularity" of the activity is what allows writing the neural field
- → can an equivalent neural field in [0,1] also be regular?
- → what conditions on mapping S to preserve regularity? (→ locality)

smoothness of the kernel: what properties on S^{-1} can we hope for?

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kernel of the [0,1] embedding
defined as: $\tilde{w}(\alpha,\beta) = w_U(S^{-1}(\alpha), S^{-1}(\beta))$

→ w_U is highly regular, but
$\tilde{w}$ might not be, it depends on S^{-1}

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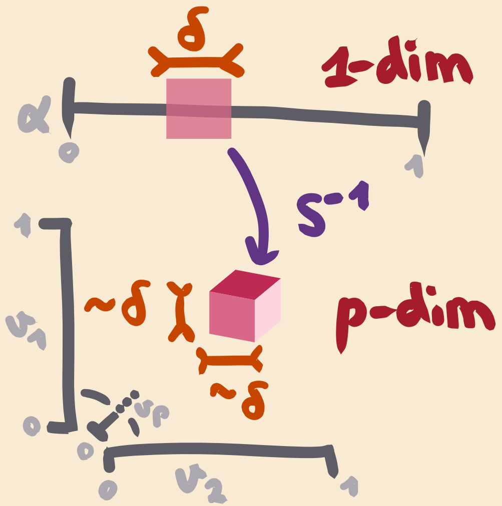
→ this is tricky



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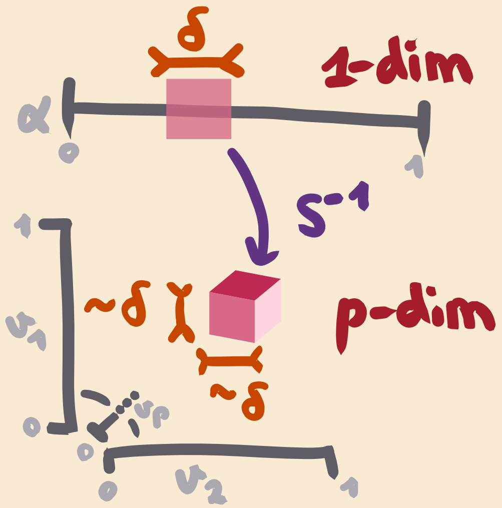
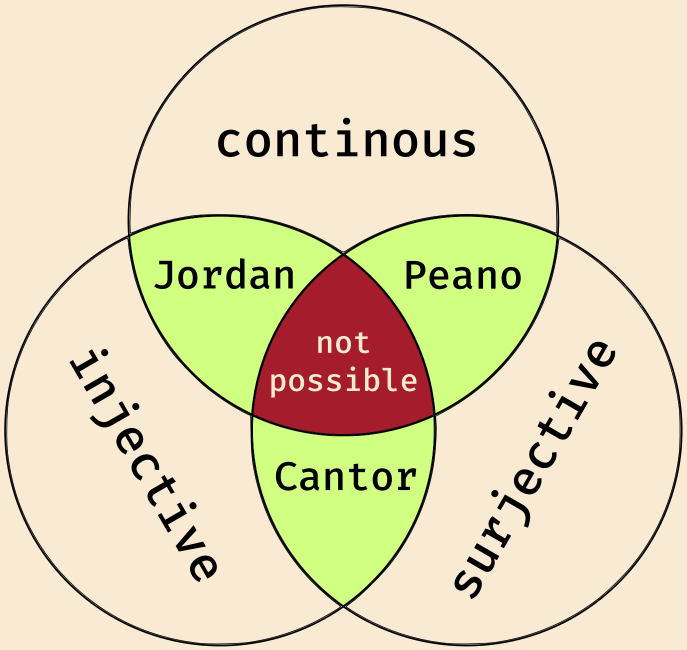




↑ Netto's theorem states there are no continuous bijections $S^{-1}: [0,1] \mapsto [0,1]^2$

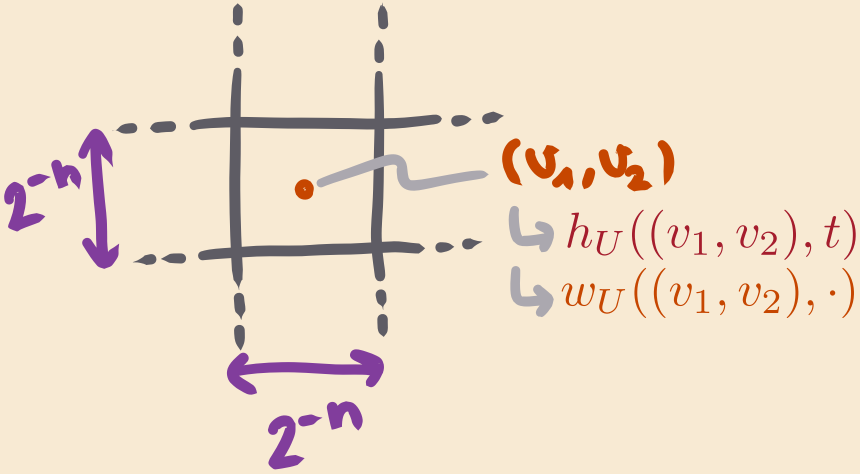
S as the limit of a sequence of mappings S^n – motivation

why? → **grid discretization of the neural field in p dimensions**

→ 2^n bins along each dimension

→ case p=2, positions of bins given by: $\boldsymbol{v_i} = (v_{1,i}, v_{2,i}),\ i \in \{1, \cdots 4^n\}$

↓ each bin "samples" the neural field and kernel



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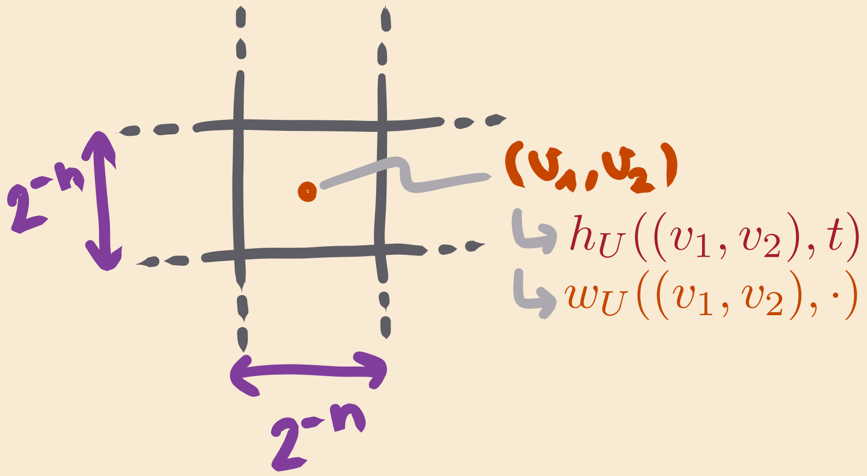

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→ **we simulate the discretized neural field**

$$\frac{\mathrm{d}}{\mathrm{d}t} h_i(t) = -h_i(t) + \sum_{j=1}^{4^n} J_{ij}\phi(h_j(t))$$

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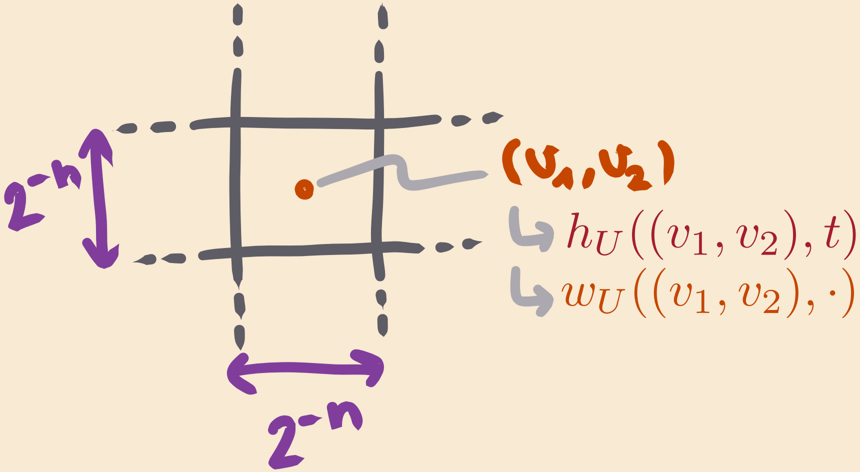
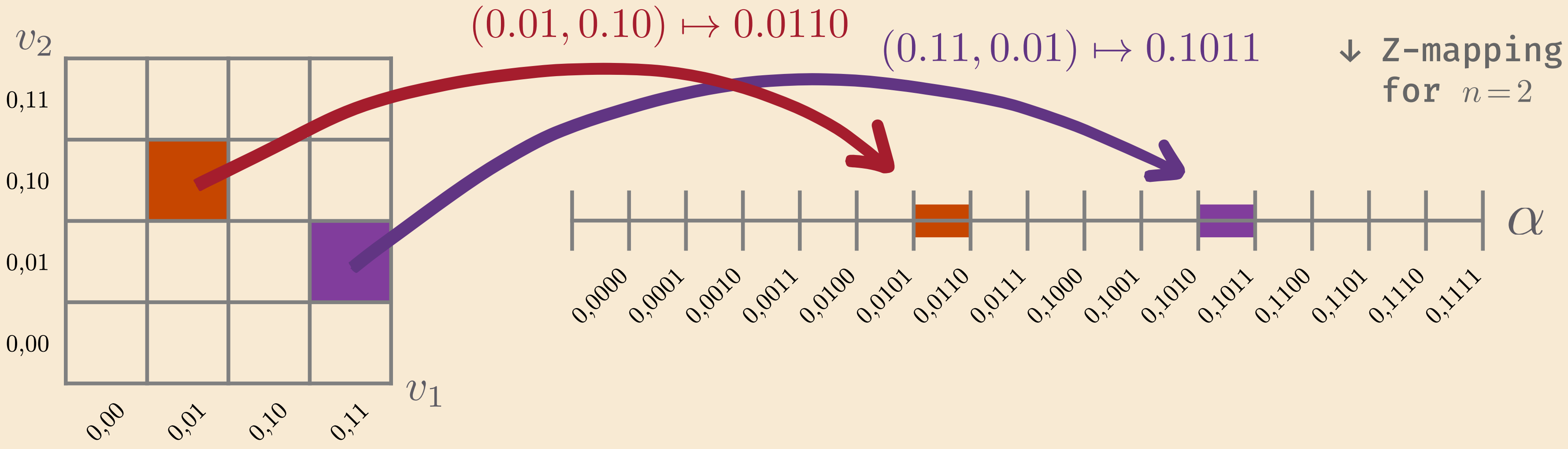


$\mathcal{S}$ as the limit of a sequence of mappings $\mathcal{S}^n$ – definition



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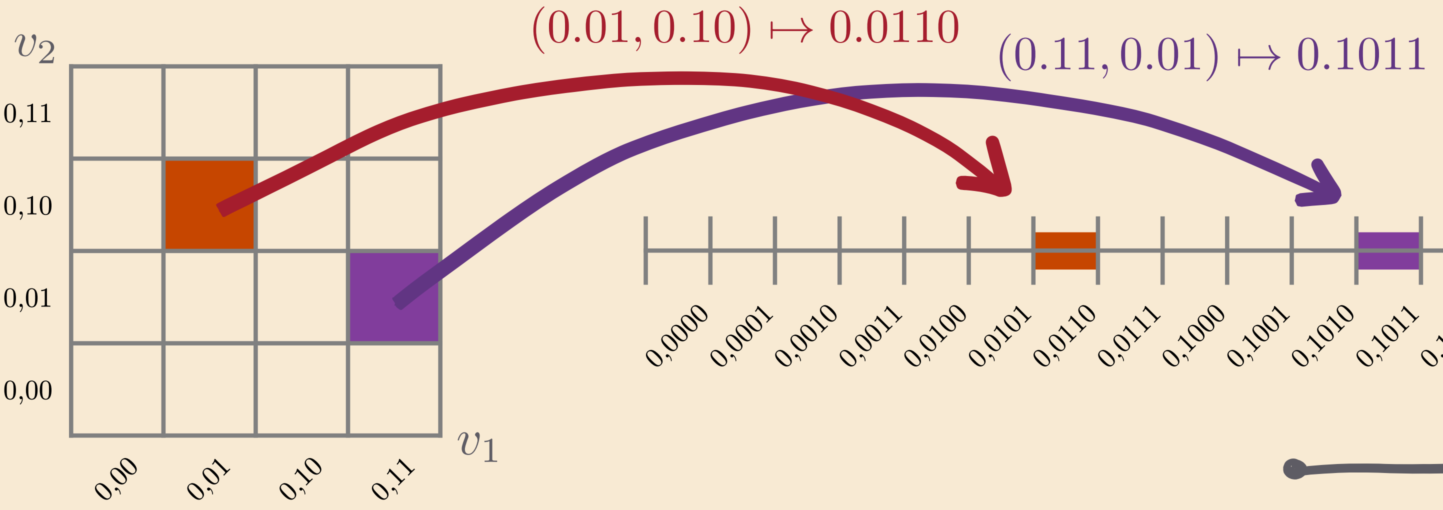
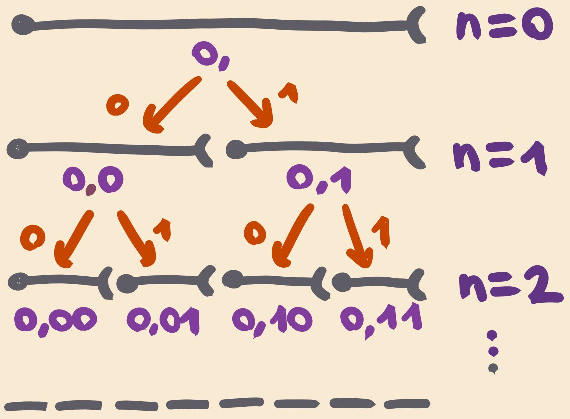


↓ Z-mapping for $n=2$

→ S^n maps 4ⁿ squares of size $2^{-n} \times 2^{-n}$ to 4ⁿ segments of size 4^{-n}

→ bins are located using binary expansions →

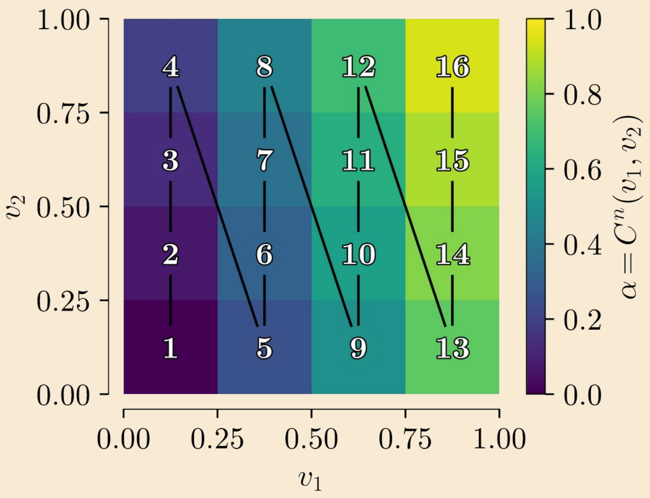


$$S^n : (v_1^{(n)}, v_2^{(n)}) = (0.b_1^1 b_2^1 \cdots b_n^1, 0.b_1^2 b_2^2 \cdots b_n^2)$$
$$\mapsto \alpha^{(n)} = 0.b_1 b_2 \cdots b_{2n}$$

a naive mapping: Column mapping

→ enumerate the populations columnwise

Column mapping for $n=2$ →



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$$\alpha^{(n)} = C(v_1^{(n)}, v_2^{(n)}) = 0.b_1^1 b_2^1 \cdots b_n^1 b_1^2 b_2^2 \cdots b_n^2$$
$$= \sum_{k=1}^{n} b_k^1 2^{-k} + b_k^2 2^{-(n+k)}$$

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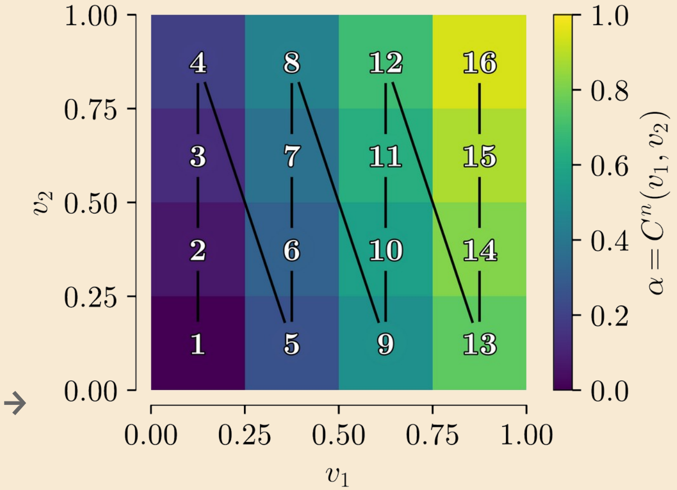

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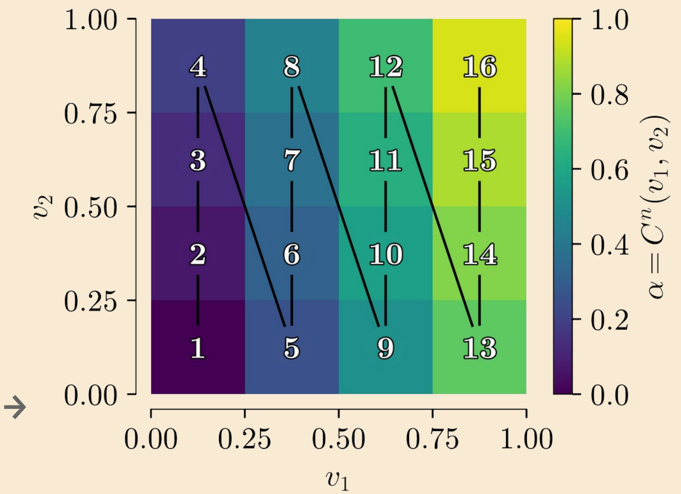


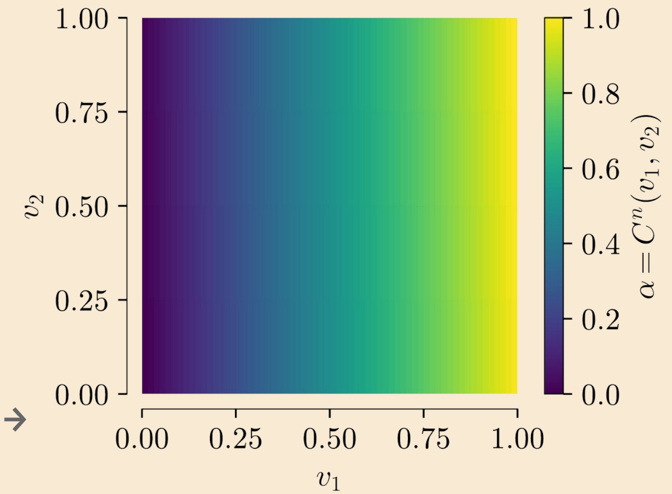
→ at finite-n, inverse is

$$v_1^{(n)} = C_1^{-1}(\alpha^{(n)}) = 0.b_1 b_2 \cdots b_n = \sum_{k=1}^{n} b_k 2^{-k}$$

$$v_2^{(n)} = C_2^{-1}(\alpha^{(n)}) = 0.b_{n+1} b_{n+2} \cdots b_{2n} = \sum_{k=1}^{n} b_{n+k} 2^{-k}$$

→ converges to a projection on the first axis

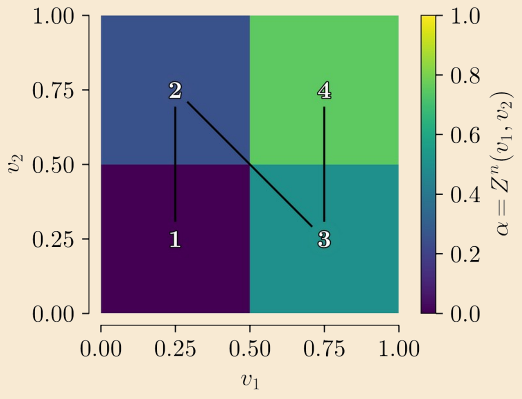
Column mapping for $n \to \infty$ →



a less naive mapping: Z-mapping

→ (recursively) draw Z shapes

↓ $n = 1$



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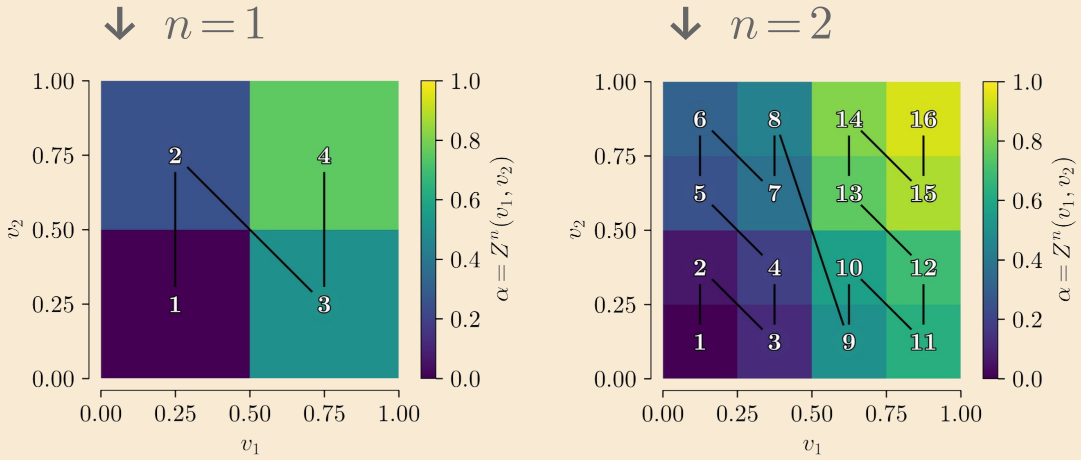

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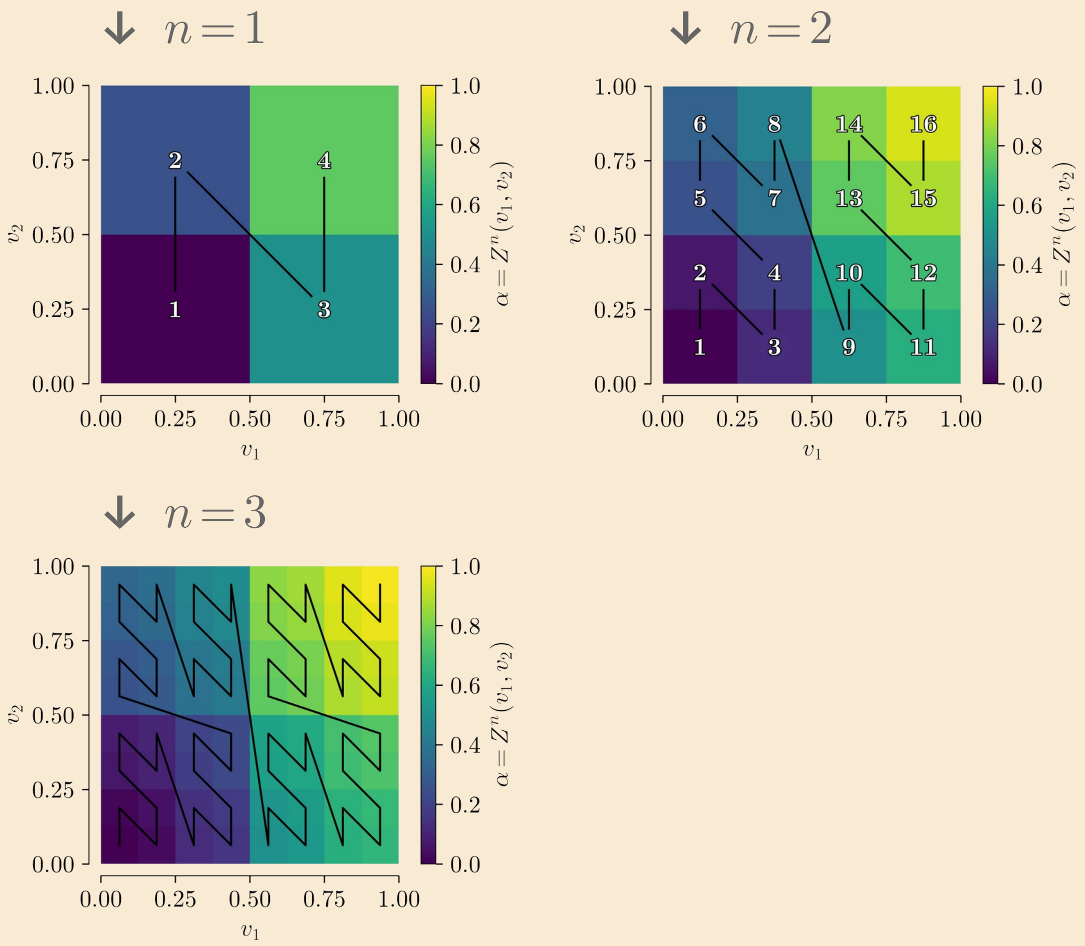

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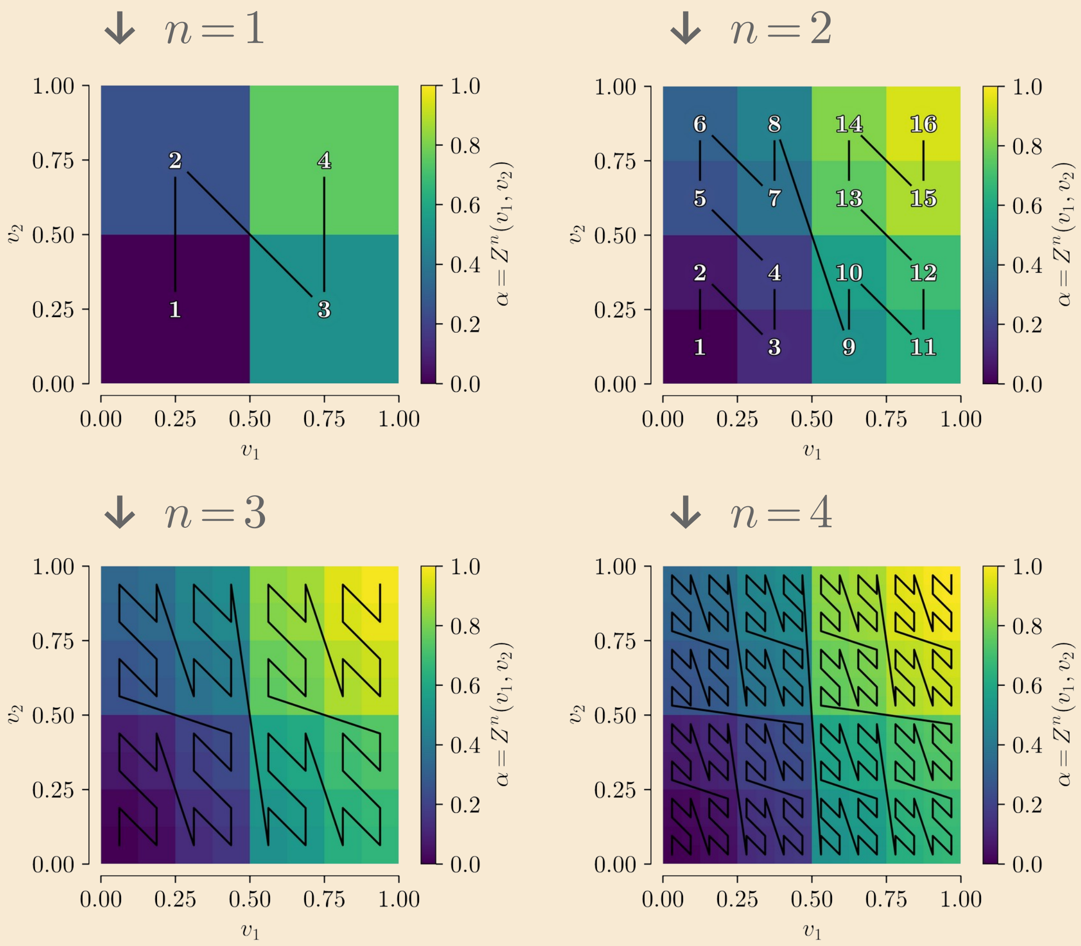

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$$\alpha^{(n)} = Z(v_1^{(n)}, v_2^{(n)})$$
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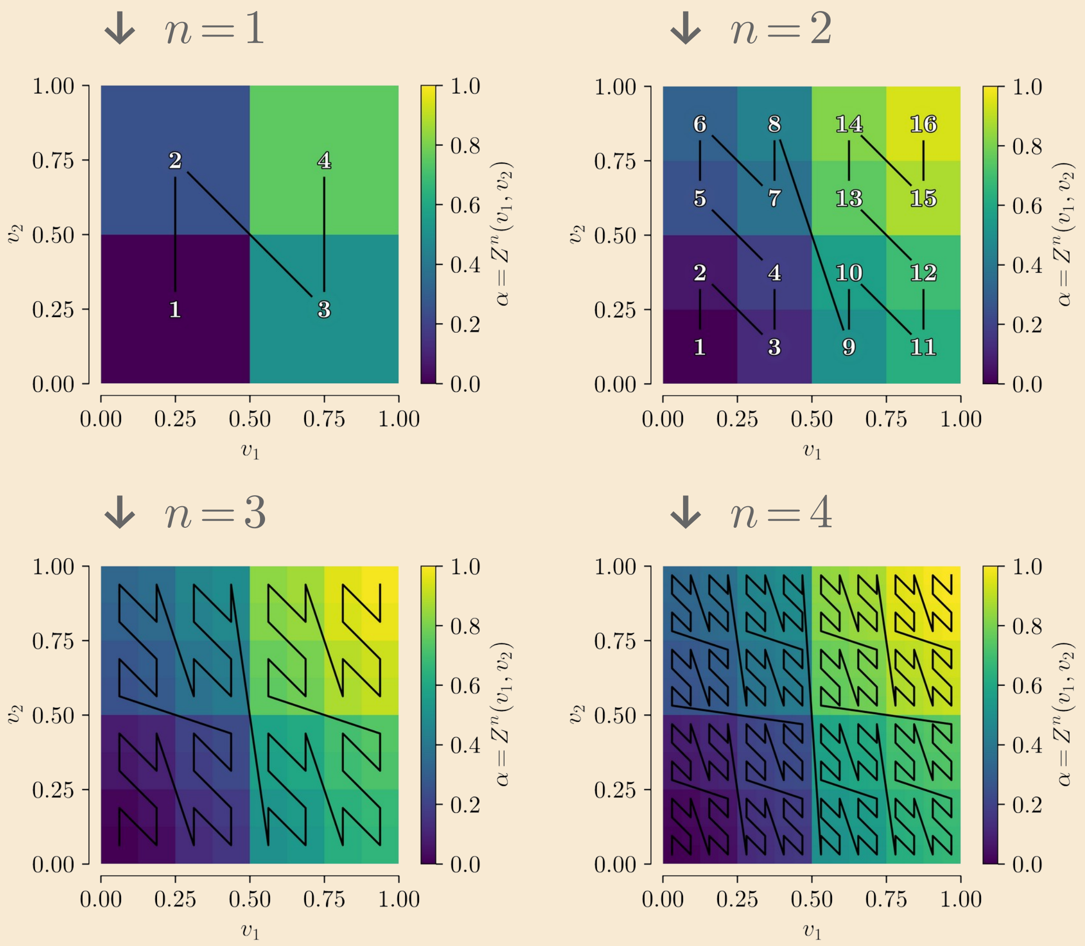

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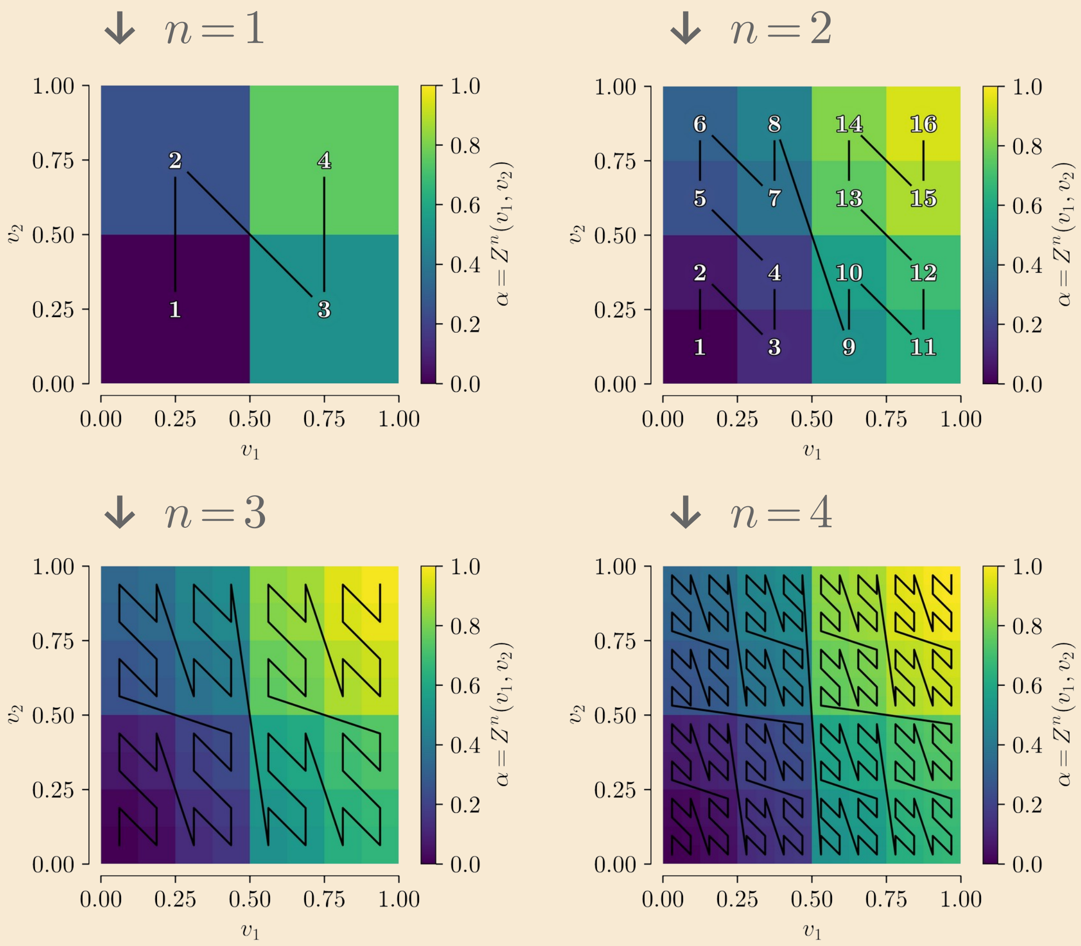
$$v_2^{(n)} = Z_2^{-1}(\alpha^{(n)}) = 0.b_2 b_4 \cdots b_{2n}$$

- converges to a bijection between the segment and the square



coarse-graining: how to simulate the 1-dimensional neural field

→ naive numerical kernel for [0,1]

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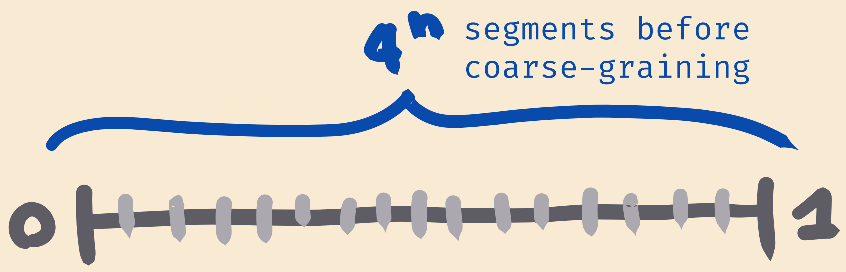

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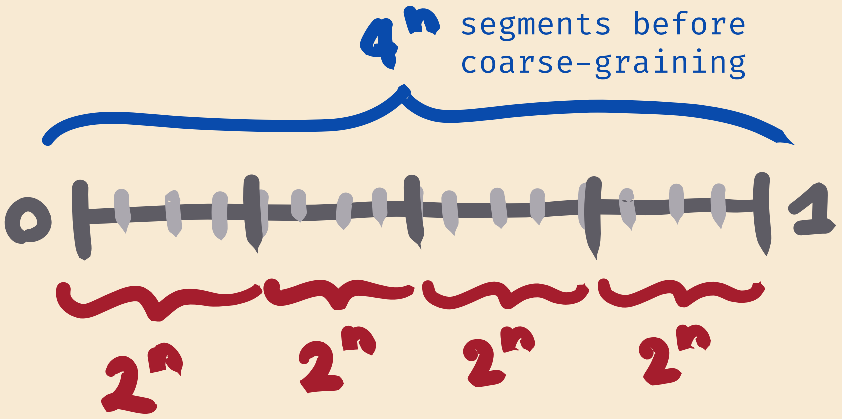

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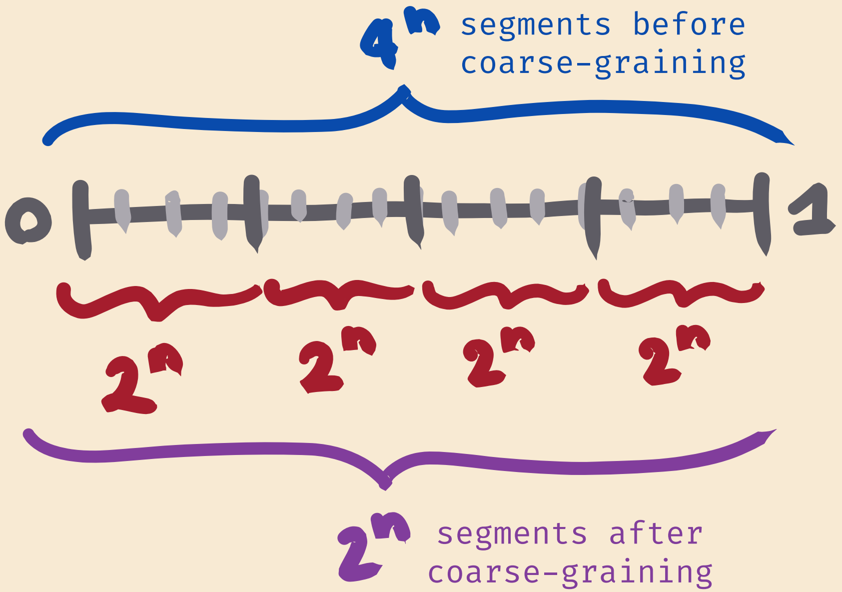

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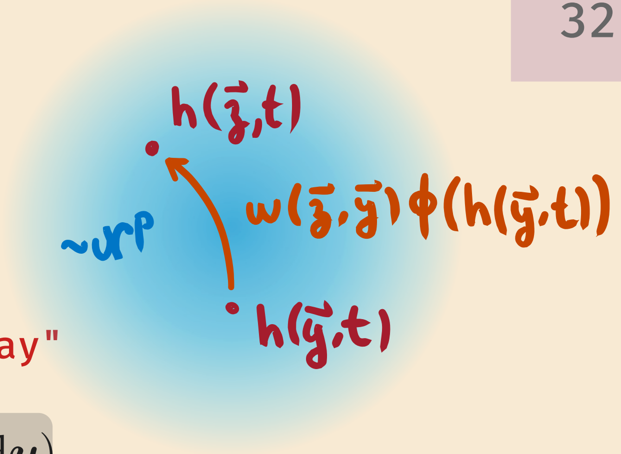
→ low-rank cycling gaussian neural field



"delay"

$$\partial_t h(\boldsymbol{z},t) = -h(\boldsymbol{z},t) + \sum_{\mu=1}^{p} \int_{\mathbb{R}^p} z_{\mu+1}\tilde{\phi}(y_\mu)\phi(h(\boldsymbol{y},t-\delta))\mathcal{N}^p(\mathrm{d}\boldsymbol{y})$$

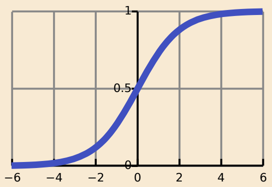
kernel has rank p

"rolling" induces recurrent drive to the next pattern

population density follows a p-dimensional Gaussian (zero mean, no covariance)

$$w(\boldsymbol{z},\boldsymbol{y}) = \sum_{\mu=1}^{p} z_\mu \tilde{\phi}(y_\mu)$$

$$\phi(h) = \frac{1}{1+\mathrm{e}^{-h}}$$



→ cycling behavior with "duration" δ

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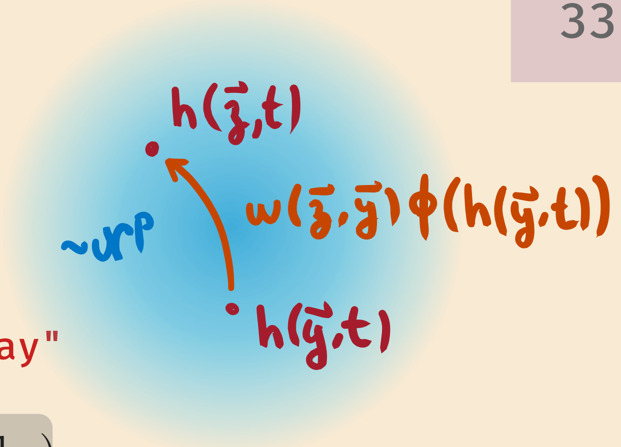

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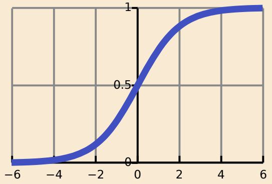
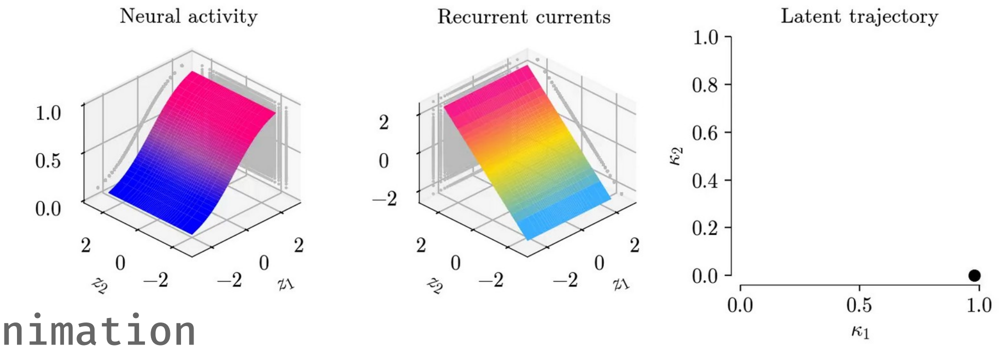


→ cycling behavior with "duration" δ

→ we get latent trajectories for free:

projection of the field onto the patterns

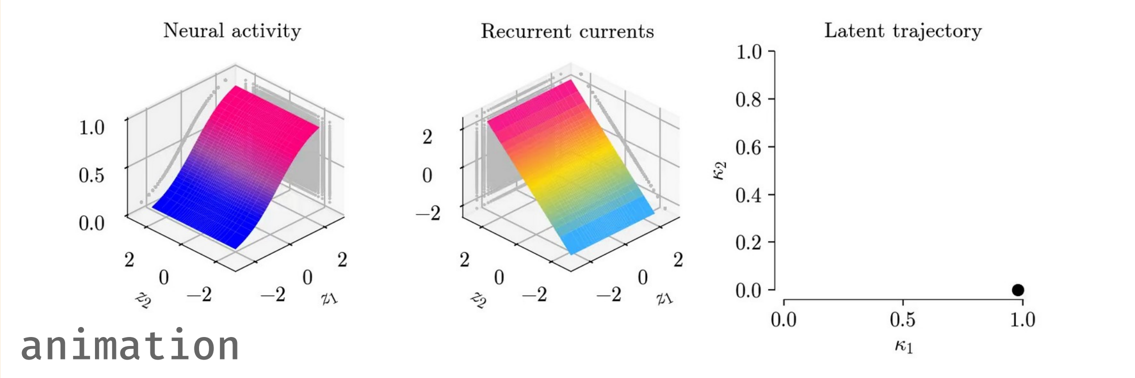
$$\kappa_\mu(t) = \int_{\mathbb{R}^p} y_\mu h(\boldsymbol{y},t)\mathcal{N}^p(\mathrm{d}\boldsymbol{y})$$



animation

← original neural field for $p=2, \delta=6, h(z_1, z_2, 0) = z_1$

⇒ cycling behavior



← original neural field for $p=2, \delta=6, h(z_1, z_2, 0) = z_1$

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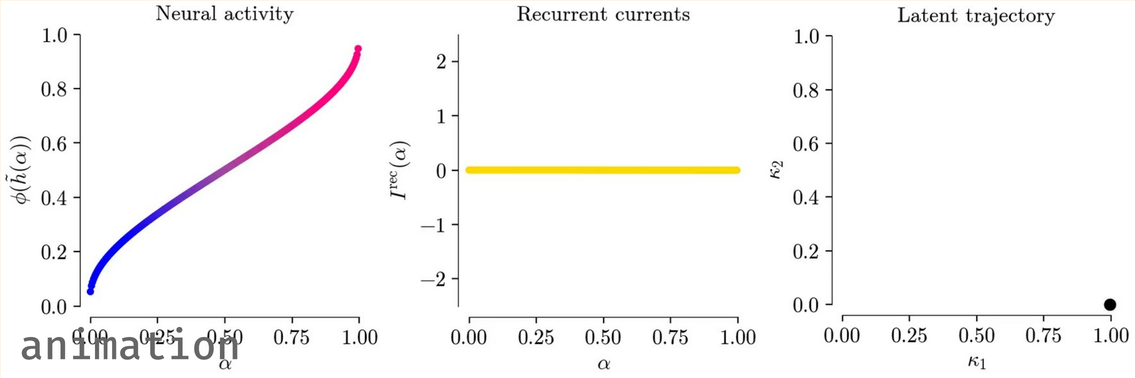
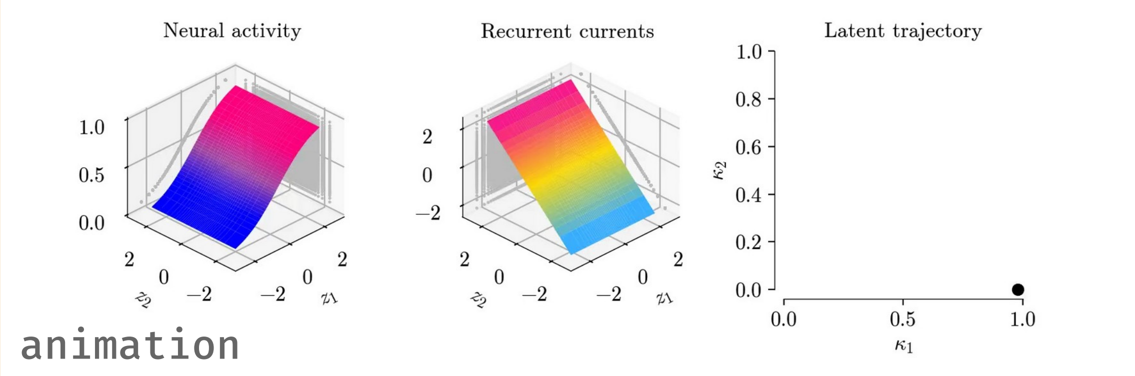


← mapped with Column mapping, then coarse-grained

⇒ quick decay



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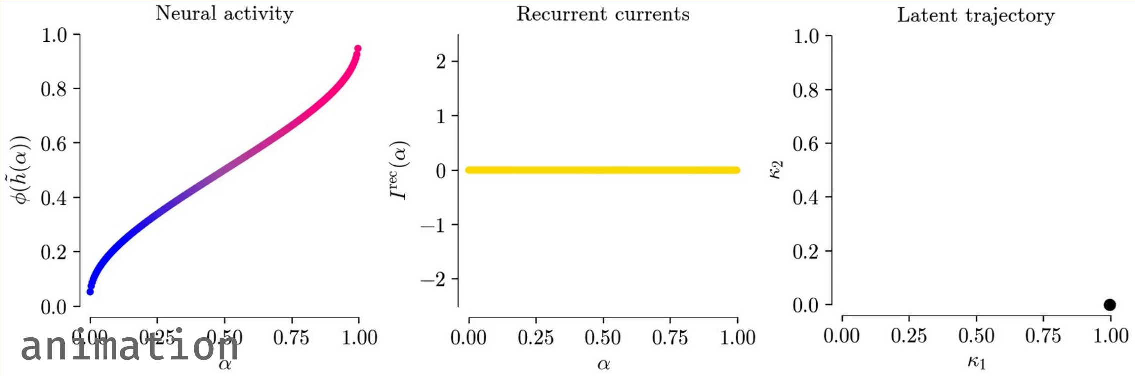

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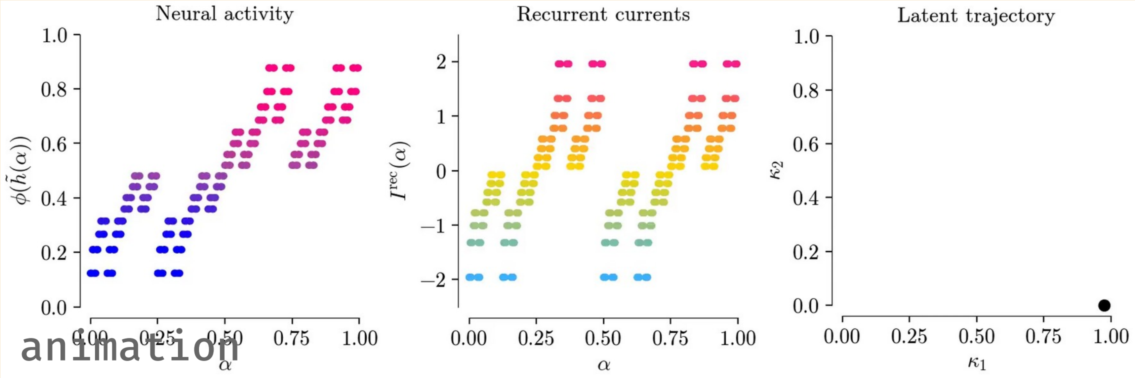
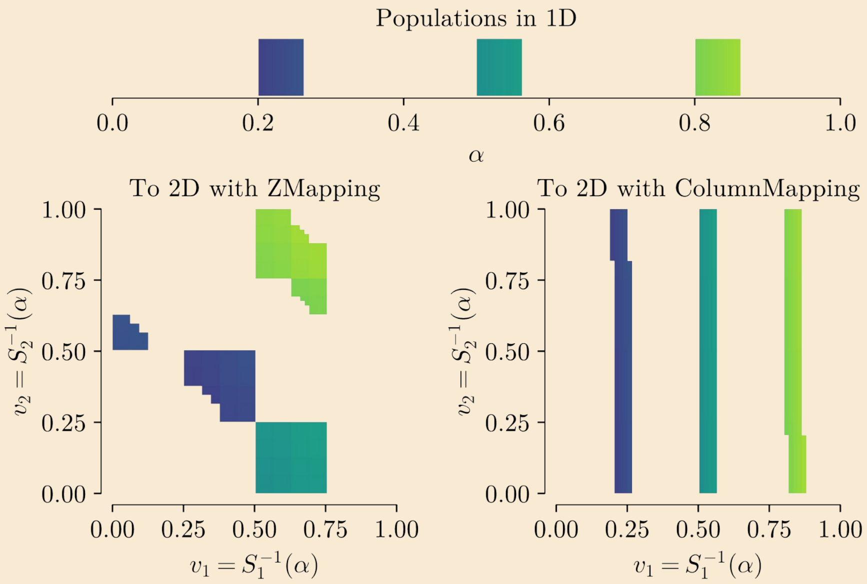


← mapped with Z-mapping, then coarse-grained

⇒ identical cycling behavior

the notion of locality – motivation



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recall: $\tilde{w}(\alpha, \beta) = w_U(S^{-1}(\alpha), S^{-1}(\beta))$

← 2D populations corresponding to three small segments (each of length 1/16) of 1D populations

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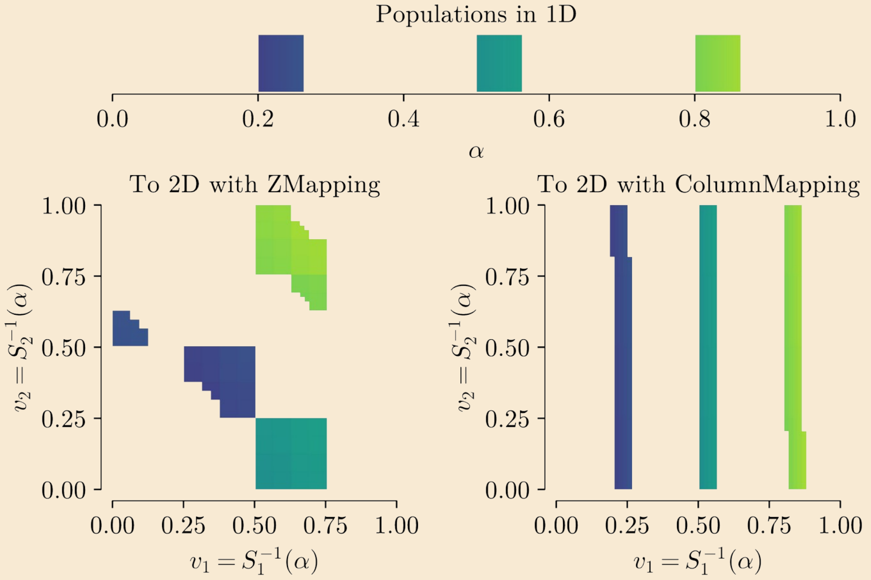

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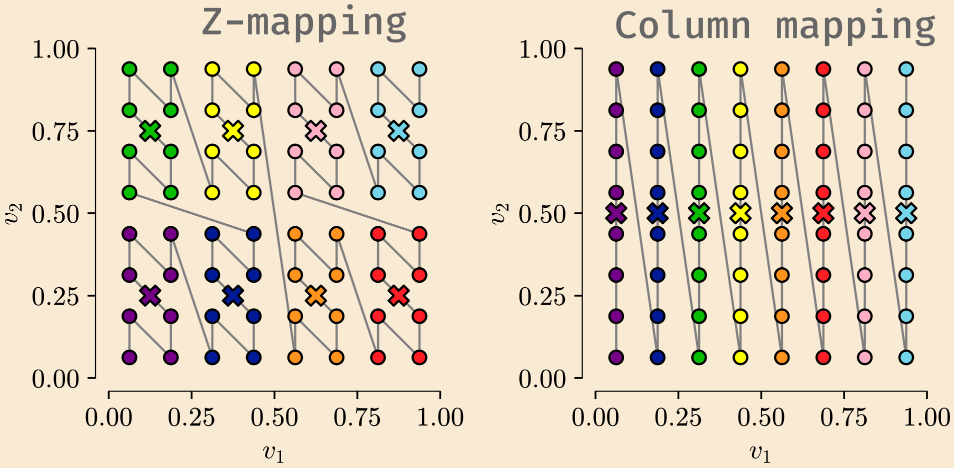
→ what positions in 2D get "averaged together" ?

colour-coded square populations corresponding to binned segment populations before coarse-graining →



the notion of locality - formalism

→ average variation inside each coarse-graining bin

$$V_n(S^{-1}) = \frac{1}{2^n}\sum_{i=1}^{2^n} \sup_{\alpha,\alpha' \in \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]^2} \|S^{-1}(\alpha') - S^{-1}(\alpha)\|_1$$

average over all bins

every pair of positions inside each bin

distance in the p-dimensional embedding

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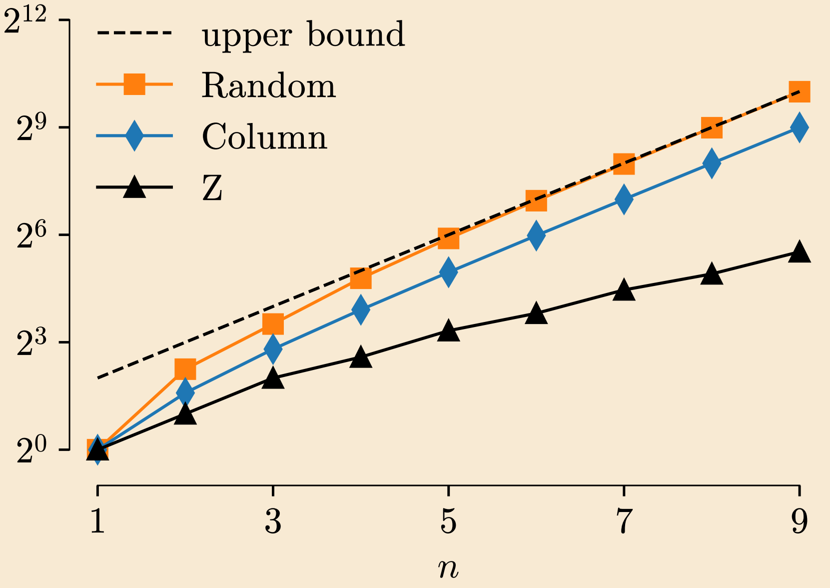
→ scaling behavior of numerator

- Random mapping saturates bound
- Column mapping ~ 2^n ⇒ $V_n(C^{-1}) > 0$
- Z-mapping ~ $2^{n/2}$ ⇒ $V_n(Z^{-1}) \to 0$

↓ (numerator of the) average binned variation as a function of n



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3) bound integrand by supremum and use locality

$$|\mathrm{NI}_n - \mathrm{AI}| \leq L(\alpha)V_n(S^{-1}) \xrightarrow{n\to\infty} 0$$

(bonus) iterating mapping and coarse-graining from p-dim to 1-dim

- $\tilde{w}$ is regular enough that we can again apply a mapping and do coarse-graining
- **"local mappings conserve regularity"**

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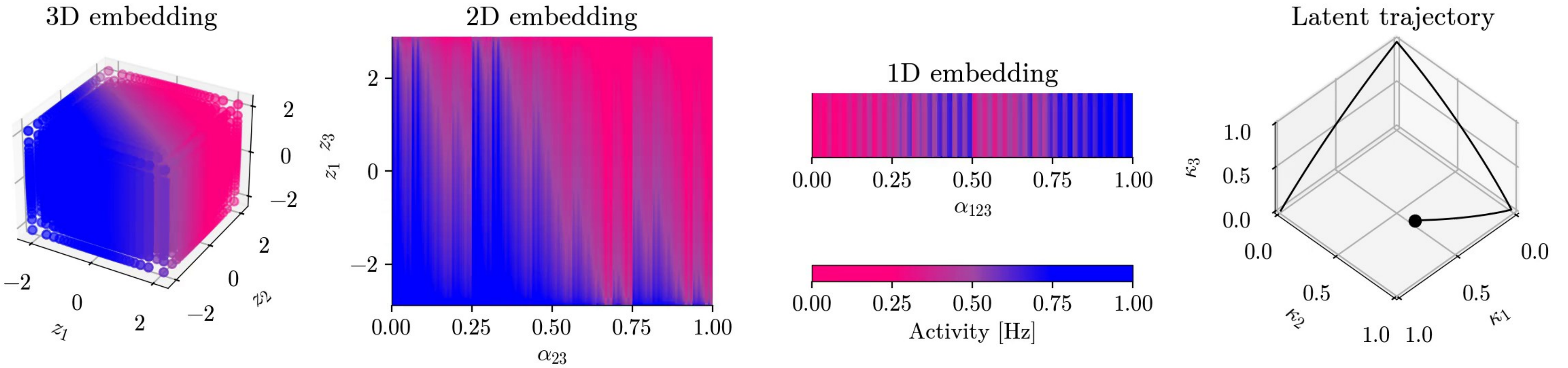
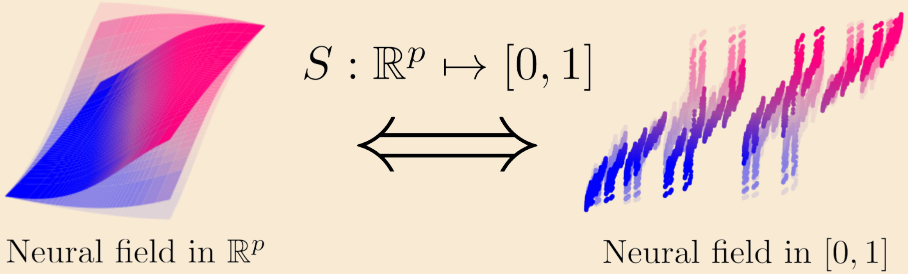
- numerical demo with p=3: from $[0,1]^3$ to $[0,1]^2$ to $[0,1]$



summary & conclusion

- example of neural fields in p dimensions that can be mapped to **equivalent neural fields** in [0,1]
- locality quantifies the **conservation of "regularity"**
- **coarse-graining** numerically enforces the notion of locality: non local mappings are destroyed
- the Z-mapping can be **approximated numerically**, and the numerical integral on [0,1] approaches the analytical integral on [0,1]



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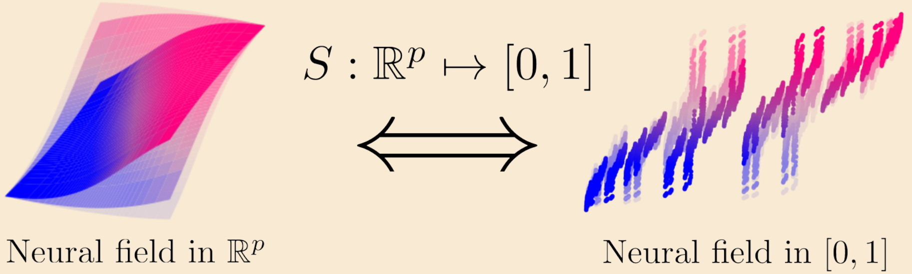

Thank you!