B. Example: The Vibrating String Problem

We now illustrate the method of separation of variables by applying it to obtain a formal solution of the so-called *vibrating string problem.*

The Physical Problem. Consider a tightly stretched elastic string the ends of which are fixed on the x axis at $x = 0$ and $x = L$. Suppose that for each x in the interval $0 < x < L$ the string is displaced into the xy plane and that for each such x the displacement from the x axis is given by $f(x)$, where f is a known function of x (see Figure 14.1).

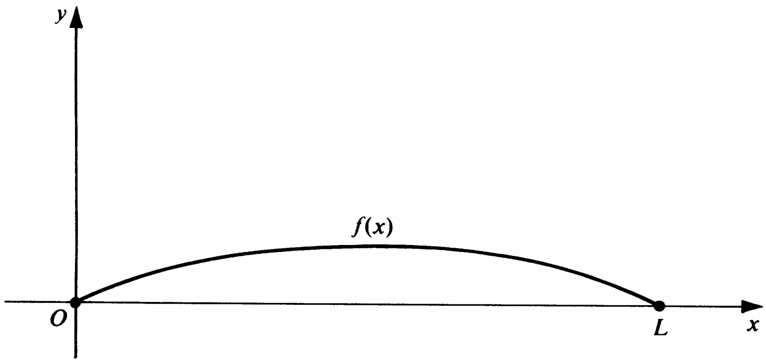

Figure 14.1

Suppose that at $t = 0$ the string is released from the initial position defined by $f(x)$, with an initial velocity given at each point of the interval $0 \leq x \leq L$ by $g(x)$, where g is a known function of x. Obviously the string will vibrate, and its displacement in the y direction at any point x at any time t will be a function of both x and t. We seek to find this displacement as a function of x and t; we denote it by y or $y(x, t)$.

We now make certain assumptions concerning the string, its vibrations, and its surroundings. To begin with, we assume that the string is perfectly flexible, is of constant linear density ρ, and is of constant tension T at all times. Concerning the vibrations, we assume that the motion is confined to the xy plane and that each point on the string moves on a straight line perpendicular to the x axis as the string vibrates. Further, we assume that the displacement y at each point of the string is small compared to the length L and that the angle between the string and the x axis at each point is also sufficiently small. Finally, we assume that no external forces (such as damping forces, for example) act upon the string.

Although these assumptions are not actually valid in any physical problem, nevertheless they are approximately satisfied in many cases. They are made in order to make the resulting mathematical problem more tractable. With these assumptions, then, the problem is to find the displacement y as a function of x and t.

The Mathematical Problem. Under the assumptions stated it can be shown that the displacement y satisfies the *partial differential equation,*

$$\alpha^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}, \tag{14.20}$$

where $\alpha^2 = T/\rho$. This is the one-dimensional wave equation, a special case of which we have already studied in Example 14.3. Since our primary concern here is to illustrate the method of separation of variables, we omit the derivation of the equation.

Since the ends of the string are fixed at $x = 0$ and $x = L$ for all time t, the displacement y must satisfy the *boundary conditions*

$$\begin{aligned} y(0, t) &= 0, \qquad 0 \le t < \infty; \\ y(L, t) &= 0, \qquad 0 \le t < \infty. \end{aligned} \tag{14.21}$$

At $t = 0$ the string is released from the initial position defined by $f(x)$, $0 \le x \le L$, with initial velocity given by $g(x)$, $0 \le x \le L$. Thus the displacement y must also satisfy the *initial conditions*

$$\begin{aligned} y(x, 0) &= f(x), \qquad 0 \le x \le L; \\ \frac{\partial y(x, 0)}{\partial t} &= g(x), \qquad 0 \le x \le L. \end{aligned} \tag{14.22}$$

This, then, is our problem. We must find a function y of x and t which satisfies the partial differential equation (14.20), the boundary conditions (14.21), and the initial conditions (14.22).

Solution. We apply the method of separation of variables. We first make the basic assumption that the differential equation (14.20) has product solutions of the form XT, where X is a function of x only and T is a function of t only. To emphasize this, we write

$$y(x, t) = X(x)T(t). \tag{14.23}$$

We now differentiate (14.23) and substitute into the differential equation (14.20). Differentiating, we find

$$\frac{\partial^2 y}{\partial x^2} = T\frac{d^2 X}{dx^2} \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2} = X\frac{d^2 T}{dt^2};$$

substituting, we obtain

$$\alpha^2 T\frac{d^2 X}{dx^2} = X\frac{d^2 T}{dt^2}.$$

From this we obtain at once

$$\alpha^2\frac{\dfrac{d^2 X}{dx^2}}{X} = \frac{\dfrac{d^2 T}{dt^2}}{T}. \tag{14.24}$$

Since X is a function of x only, the left member of (14.24) is also a function of x only and hence is independent of t. Further, since T is a function of t only, the right member of (14.24) is also a function of t only and hence is independent of x. Since one of the two equal expressions in (14.24) is independent of t and the other one is independent of x, both of them must be equal to a constant k. That is, we have

$$\alpha^2\frac{\dfrac{d^2 X}{dx^2}}{X} = k \quad \text{and} \quad \frac{\dfrac{d^2 T}{dt^2}}{T} = k.$$

From this we obtain the two ordinary differential equations

$$\frac{d^2X}{dx^2} - \frac{k}{\alpha^2} X = 0 \tag{14.25}$$

and

$$\frac{d^2T}{dt^2} - kT = 0. \tag{14.26}$$

Let us now consider the boundary conditions (14.21). Since $y(x, t) = X(x)T(t)$, we see that $y(0, t) = X(0)T(t)$ and $y(L, t) = X(L)T(t)$. Thus the boundary conditions (14.21) take the forms

$$X(0)T(t) = 0, \qquad 0 \le t < \infty;$$

$$X(L)T(t) = 0, \qquad 0 \le t < \infty.$$

Since $T(t) = 0$, $0 \le t < \infty$, would reduce the assumed solution (14.23) to the trivial solution of (14.20), we must have

$$X(0) = 0 \quad \text{and} \quad X(L) = 0. \tag{14.27}$$

Thus the function X in the assumed solution (14.23) must satisfy both the ordinary differential equation (14.25) and the boundary conditions (14.27). That is, the function X must be a nontrivial solution of the Sturm–Liouville problem

$$\frac{d^2X}{dx^2} - \frac{k}{\alpha^2} X = 0, \tag{14.25}$$

$$X(0) = 0, \qquad X(L) = 0. \tag{14.27}$$

We have already solved a special case of this problem in Example 12.3 of Chapter 12. Our procedure here will parallel the treatment in that example. We must first find the general solution of the differential equation (14.25). The form of this general solution depends upon whether $k = 0$, $k > 0$, or $k < 0$.

If $k = 0$, the general solution of (14.25) is of the form

$$X = c_1 + c_2 x. \tag{14.28}$$

We apply the boundary conditions (14.27) to the solution (14.28). The condition $X(0) = 0$ requires that $c_1 = 0$. The condition $X(L) = 0$ becomes $c_1 + c_2 L = 0$. Since $c_1 = 0$, this requires that $c_2 = 0$ also. Thus the solution (14.28) reduces to the trivial solution.

If $k > 0$, the general solution of (14.25) is of the form

$$X = c_1 e^{\sqrt{k}x/\alpha} + c_2 e^{-\sqrt{k}x/\alpha}. \tag{14.29}$$

Applying the boundary conditions (14.27) to the solution (14.29), we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 0, \\ c_1 e^{\sqrt{k}L/\alpha} + c_2 e^{-\sqrt{k}L/\alpha} &= 0. \end{aligned} \tag{14.30}$$

To obtain nontrivial solutions of this system, we must have

$$\begin{vmatrix} 1 & 1 \\ e^{\sqrt{k}L/\alpha} & e^{-\sqrt{k}L/\alpha} \end{vmatrix} = 0.$$

But this implies that $e^{\sqrt{k}L/\alpha} = e^{-\sqrt{k}L/\alpha}$ and hence that $k = 0$, contrary to our assumption in this case. Thus the system (14.30) has no nontrivial solutions, and so the solution (14.29) also reduces to the trivial solution.

Finally, if $k < 0$, the general solution of (14.25) is of the form

$$X = c_1 \sin \frac{\sqrt{-k}x}{\alpha} + c_2 \cos \frac{\sqrt{-k}x}{\alpha}. \tag{14.31}$$

Applying the boundary conditions (14.27) to the solution (14.31), we obtain

$$c_2 = 0$$

and

$$c_1 \sin \frac{\sqrt{-k}L}{\alpha} + c_2 \cos \frac{\sqrt{-k}L}{\alpha} = 0.$$

Since $c_2 = 0$, the latter condition reduces to

$$c_1 \sin \frac{\sqrt{-k}L}{\alpha} = 0.$$

Thus to obtain nontrivial solutions of the form (14.31), we must have

$$\frac{\sqrt{-k}L}{\alpha} = n\pi \qquad (n = 1, 2, 3, \ldots),$$

and so

$$k = -\frac{n^2\pi^2\alpha^2}{L^2} \qquad (n = 1, 2, 3, \ldots). \tag{14.32}$$

We thus find that the constant k must be a negative number of the form (14.32). We recognize these values of k as the characteristic values of the Sturm–Liouville problem under consideration. The corresponding nontrivial solutions (the characteristic functions) of the problem are then

$$X_n = c_n \sin \frac{n\pi x}{L} \qquad (n = 1, 2, 3, \ldots), \tag{14.33}$$

where the c_n $(n = 1, 2, 3, \ldots)$, are arbitrary constants. We thus find that the function X in the assumed solution (14.23) must be of the form (14.33). That is, corresponding to each positive integral value of n, we obtain functions X_n of the form (14.33) which will serve as the function X in the product solution (14.23).

Let us now return to the differential equation (14.26) which the function T in (14.23) must satisfy. Since k must be of the form (14.32), the differential equation (14.26) becomes

$$\frac{d^2T}{dt^2} + \frac{n^2\pi^2\alpha^2}{L^2} T = 0,$$

where $n = 1, 2, 3, \ldots$. For each such value of n, this differential equation has solutions of the form

$$T_n = c_{n,1} \sin \frac{n\pi\alpha t}{L} + c_{n,2} \cos \frac{n\pi\alpha t}{L} \qquad (n = 1, 2, 3, \ldots), \tag{14.34}$$

where the $c_{n,1}$ and $c_{n,2}$ $(n = 1, 2, 3, \ldots)$, are arbitrary constants. Thus the function T in the assumed solution (14.23) must be of the form (14.34). That is, corresponding to each positive integral value of n, we obtain functions T_n of the form (14.34) which will serve as the function T in the product solution (14.23).

Therefore, corresponding to each positive integral value of n $(n = 1, 2, 3, \ldots)$, we obtain solutions

$$X_n T_n = \left[c_n \sin\frac{n\pi x}{L}\right]\left[c_{n,1} \sin\frac{n\pi\alpha t}{L} + c_{n,2} \cos\frac{n\pi\alpha t}{L}\right]$$

which have the product form (14.23).

We set $a_n = c_n c_{n,1}$ and $b_n = c_n c_{n,2}$ $(n = 1, 2, 3, \ldots)$, and write these solutions as

$$y_n(x, t) = \left[\sin\frac{n\pi x}{L}\right]\left[a_n \sin\frac{n\pi\alpha t}{L} + b_n \cos\frac{n\pi\alpha t}{L}\right] \qquad (n = 1, 2, 3, \ldots). \quad (14.35)$$

We point out that each of these solutions (14.35) satisfies both the partial differential equation (14.20) and the two boundary conditions (14.21) for all values of the constants a_n and b_n.

We must now try to satisfy the two initial conditions (14.22). In general no single one of the solutions (14.35) will satisfy these conditions. For example, if we apply the first initial condition (14.22) to a solution of the form (14.35) we must have

$$b_n \sin\frac{n\pi x}{L} = f(x), \qquad 0 \leq x \leq L,$$

where n is some positive integer; and this is clearly impossible unless f happens to be a sine function of the form $A \sin(n\pi x/L)$ for some positive integer n.

What can we do now? By Theorem 14.1 every finite linear combination of solutions of (14.20) is also a solution of (14.20); and by Theorem 14.2, assuming appropriate convergence, an infinite series of solutions of (14.20) is also a solution of (14.20). This suggests that we should form either a finite linear combination or an infinite series of the solutions (14.35) and attempt to apply the initial conditions (14.22) to the "more general" solutions thus obtained. In general no finite linear combination will satisfy these conditions, and we must resort to an infinite series.

We therefore form an infinite series

$$\sum_{n=1}^{\infty} y_n(x, t) = \sum_{n=1}^{\infty} \left[\sin\frac{n\pi x}{L}\right]\left[a_n \sin\frac{n\pi\alpha t}{L} + b_n \cos\frac{n\pi\alpha t}{L}\right]$$

of the solutions (14.35). Assuming appropriate convergence, Theorem 14.2 applies and assures us that the sum of this series is also a solution of the differential equation (14.20). Denoting this sum by $y(x, t)$, we write

$$y(x, t) = \sum_{n=1}^{\infty} \left[\sin\frac{n\pi x}{L}\right]\left[a_n \sin\frac{n\pi\alpha t}{L} + b_n \cos\frac{n\pi\alpha t}{L}\right]. \qquad (14.36)$$

We note that $y(0, t) = 0$ and $y(L, t) = 0$. Thus, assuming appropriate convergence, the function y given by (14.36) satisfies both the differential equation (14.20) and the two boundary conditions (14.21).

Let us now apply the initial conditions (14.22) to the series solution (14.36). The first condition $y(x, 0) = f(x)$, $0 \le x \le L$, reduces (14.36) to

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = f(x), \qquad 0 \le x \le L. \tag{14.37}$$

Thus to satisfy the first initial condition (14.22), we must determine the coefficients b_n so that (14.37) is satisfied. We recognize this as a problem in Fourier sine series (see Section 12.4C). Using (12.54) we find that the coefficients b_n are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, 3, \ldots). \tag{14.38}$$

Thus in order for the series solution (14.36) to satisfy the initial condition $y(x, 0) = f(x)$, $0 \le x \le L$, the coefficients b_n in the series must be given by (14.38).

The only condition which remains to be satisfied is the second initial condition (14.22), which is

$$\frac{\partial y(x, 0)}{\partial t} = g(x), \qquad 0 \le x \le L.$$

From (14.36), we find that

$$\frac{\partial y(x, t)}{\partial t} = \sum_{n=1}^{\infty}\left[\frac{n\pi\alpha}{L}\right]\left[\sin\frac{n\pi x}{L}\right]\left[a_n\cos\frac{n\pi\alpha t}{L} - b_n\sin\frac{n\pi\alpha t}{L}\right].$$

The second initial condition reduces this to

$$\sum_{n=1}^{\infty}\frac{a_n n\pi\alpha}{L}\sin\frac{n\pi x}{L} = g(x), \qquad 0 \le x \le L.$$

Letting $A_n = a_n n\pi\alpha/L$ $(n = 1, 2, 3, \ldots)$, this takes the form

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = g(x), \qquad 0 \le x \le L. \tag{14.39}$$

Thus to satisfy the second initial condition (14.22), we must determine the coefficients A_n so that (14.39) is satisfied. This is another problem in Fourier sine series. Using (12.54) again, we find that the coefficients A_n are given by

$$A_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, 3, \ldots).$$

Since $A_n = a_n n\pi\alpha/L$ $(n = 1, 2, 3, \ldots)$, we find that

$$a_n = \frac{L}{n\pi\alpha}A_n = \frac{2}{n\pi\alpha}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, 3, \ldots). \tag{14.40}$$

Thus in order for the series solution (14.36) to satisfy the second initial condition (14.22), the coefficients a_n in the series must be given by (14.40).

Therefore, the formal solution of the problem consisting of the partial differential equation (14.20), the two boundary conditions (14.21), and the two initial conditions (14.22) is

$$y(x, t) = \sum_{n=1}^{\infty}\left[\sin\frac{n\pi x}{L}\right]\left[a_n\sin\frac{n\pi\alpha t}{L} + b_n\cos\frac{n\pi\alpha t}{L}\right], \tag{14.36}$$

where

$$a_n = \frac{2}{n\pi\alpha}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, 3, \ldots). \tag{14.40}$$

and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \qquad (n = 1, 2, 3, \ldots). \tag{14.38}$$

Summary. We briefly summarize the principal steps in the solution of this problem. The initial step was to assume the product solution XT given by (14.23). This led to the ordinary differential equation (14.25) for the function X and the ordinary differential equation (14.26) for the function T. We then considered the boundary conditions (14.21) and found that they reduced to the boundary conditions (14.27) on the function X. Thus the function X had to be a nontrivial solution of the Sturm–Liouville problem consisting of (14.25) and (14.27). The next step was to solve this Sturm–Liouville problem. We did this and obtained for solutions the functions X_n given by (14.33). We then returned to the differential equation (14.26) for the function T and obtained the solutions T_n given by (14.34). Thus, for each positive integral value of n, we found the product solutions X_nT_n denoted by y_n and given by (14.35). Each of these solutions y_n satisfied both the partial differential equation (14.20) and the boundary conditions (14.21), but no one of them satisfied the initial conditions (14.22). In order to satisfy these initial conditions, we formed an infinite series of the solutions y_n. We thus obtained the formal solution y given by (14.36), in which the coefficients a_n and b_n were arbitrary. We applied the initial conditions to this series solution and thereby determined the coefficients a_n and b_n. We thus obtained the formal solution y given by (14.36), in which the coefficients a_n and b_n are given by (14.40) and (14.38), respectively. We emphasize that this solution is a formal one, for in the process of obtaining it we made assumptions of convergence which we did not justify.

A Special Case. As a particular case of the vibrating string problem, we consider the problem of the so-called *plucked string.* Let us suppose that the string is such that the constant $\alpha^2 = 2500$ and that the ends of the string are fixed on the x axis at $x = 0$ and $x = 1$. Suppose the midpoint of the string is displaced into the xy plane a distance 0.01 in the direction of the positive y axis (see Figure 14.2).

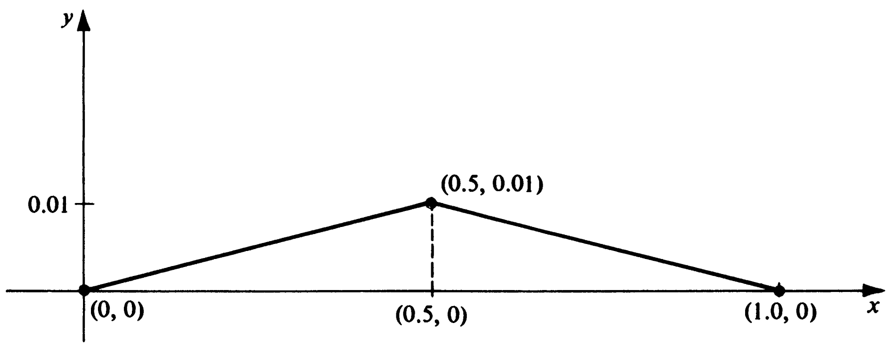

Figure 14.2

Then the displacement from the x axis on the interval $0 \le x \le 1$ is given by $f(x)$, where

$$f(x) = \begin{cases} \dfrac{x}{50}, & 0 \le x \le \dfrac{1}{2}; \\ -\dfrac{x}{50} + \dfrac{1}{50}, & \dfrac{1}{2} \le x \le 1. \end{cases} \tag{14.41}$$

Suppose that at $t = 0$ the string is released from rest from the initial position defined by $f(x)$, $0 \le x \le 1$. Let us find the formal expression (14.36) for the displacement $y(x, t)$ in the y direction, in this special case.

The coefficients a_n and b_n in the expression (14.36) are given by (14.40) and (14.38), respectively. In the special case under consideration we have $\alpha = 50$, $L = 1$, and $f(x)$ given by (14.41). Further, since the string is released from rest, the initial velocity is given by $g(x) = 0$, $0 \le x \le 1$. Therefore from (14.40) we find that

$$a_n = 0 \qquad (n = 1, 2, 3, \ldots).$$

Using (14.38), we find that

$$\begin{aligned} b_n &= 2 \int_0^1 f(x) \sin n\pi x \, dx \\ &= 2 \int_0^{1/2} \frac{x}{50} \sin n\pi x \, dx + 2 \int_{1/2}^1 \left(-\frac{x}{50} + \frac{1}{50} \right) \sin n\pi x \, dx \\ &= \frac{2}{25 n^2 \pi^2} \sin \frac{n\pi}{2} \qquad (n = 1, 2, 3, \ldots). \end{aligned}$$

Hence the even coefficients $b_{2n} = 0$ (n $= 1, 2, 3, \ldots$), and the odd coefficients are given by

$$b_{2n-1} = \frac{(-1)^{n-1} 2}{25\pi^2 (2n-1)^2} \qquad (n = 1, 2, 3, \ldots).$$

Therefore in the special case under consideration the expression (14.36) for the displacement is

$$y(x, t) = \frac{2}{25\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \sin[(2n-1)\pi x] \cos[50(2n-1)\pi t].$$

C. An Example Involving the Laplace Equation

We now consider a second example of the application of the method of separation of variables to a partial differential equations problem. The problem which we shall consider originated from a problem of physics, but we shall not enter into a discussion of this related physical problem. Our sole purpose in presenting this second example is to help the reader to gain greater familiarity with the various details of the method under consideration.

Problem. Apply the method of separation of variables to obtain a formal solution $u(x, y)$ of the problem which consists of the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{14.42}$$

and the four boundary conditions

$$u(0, y) = 0, \qquad 0 \le y \le b; \tag{14.43}$$

$$u(a, y) = 0, \qquad 0 \le y \le b; \tag{14.44}$$

$$u(x, 0) = 0, \qquad 0 \le x \le a; \tag{14.45}$$

$$u(x, b) = f(x), \qquad 0 \le x \le a. \tag{14.46}$$

We point out that the numbers a and b are definite positive constants and the function f is a specified function of x, $0 \le x \le a$.

Formal Solution. We first make the basic assumption that the differential equation (14.42) has product solutions of the form XY, where X is a function of x only and Y is a function of y only. That is, we assume solutions

$$u(x, y) = X(x)Y(y). \tag{14.47}$$

We now differentiate (14.47) and substitute into the differential equation (14.42). Differentiating, we find

$$\frac{\partial^2 u}{\partial x^2} = Y\frac{d^2 X}{dx^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = X\frac{d^2 Y}{dy^2};$$

substituting, we obtain

$$Y\frac{d^2 X}{dx^2} + X\frac{d^2 Y}{dy^2} = 0.$$

From this we obtain at once

$$\frac{\dfrac{d^2 X}{dx^2}}{X} = -\frac{\dfrac{d^2 Y}{dy^2}}{Y}. \tag{14.48}$$

The left member of (14.48) is a function of x only and so is independent of y. The right member of (14.48) is a function of y only and so is independent of x. Therefore the two equal expressions in (14.48) must both be equal to a constant k. Setting each member of (14.48) equal to this constant k, we obtain the two ordinary differential equations

$$\frac{d^2 X}{dx^2} - kX = 0$$

and

$$\frac{d^2 Y}{dy^2} + kY = 0.$$

Let us now consider the four boundary conditions (14.43) through (14.46). The first three of these are homogeneous, but the fourth one is not. Let us attempt to satisfy the three homogeneous conditions first. Since $u(x, y) = X(x)Y(y)$, we see that the three homogeneous conditions (14.43), (14.44), and (14.45) reduce to

$$X(0)Y(y) = 0, \qquad 0 \le y \le b;$$

$$X(a)Y(y) = 0, \qquad 0 \le y \le b;$$

and

$$X(x)Y(0) = 0, \qquad 0 \le x \le a; \text{ respectively.}$$

Since either $X(x) = 0$, $0 \le x \le a$, or $Y(y) = 0$, $0 \le y \le b$, would reduce the assumed solution (14.47) to the trivial solution of (14.42), we must have

$$X(0) = 0, \qquad X(a) = 0, \quad \text{and} \quad Y(0) = 0.$$

Thus the function X in the assumed solution (14.47) must be a nontrivial solution of the Sturm–Liouville problem

$$\frac{d^2X}{dx^2} - kX = 0, \tag{14.49}$$

$$X(0) = 0, \qquad X(a) = 0. \tag{14.50}$$

Further, the function Y in (14.47) must be a nontrivial solution of the problem

$$\frac{d^2Y}{dy^2} + kY = 0, \tag{14.51}$$

$$Y(0) = 0. \tag{14.52}$$

The Sturm–Liouville problem (14.49) and (14.50) is essentially the same as the problem (14.25) and (14.27) which we encountered and solved in connection with the vibrating string problem in Part B of this section. Indeed, the present problem (14.49) and (14.50) is merely the special case of the problem (14.25) and (14.27) in which $\alpha^2 = 1$ and $L = a$. Thus if we set $\alpha^2 = 1$ and $L = a$ in the results (14.32) and (14.33) of the problem (14.25) and (14.27), we shall obtain the desired results for the present problem (14.49) and (14.50). Doing this, we first find from (14.32) that the constant k in (14.49) must be given by

$$k = -\frac{n^2\pi^2}{a^2} \qquad (n = 1, 2, 3, \ldots). \tag{14.53}$$

Then from (14.33) we find that the corresponding nontrivial solutions of the problem (14.49) and (14.50) are

$$X_n = c_n \sin \frac{n\pi x}{a} \qquad (n = 1, 2, 3, \ldots), \tag{14.54}$$

where the c_n $(n = 1, 2, 3, \ldots)$ are arbitrary constants. That is, corresponding to each positive integral value of n, we obtain functions X_n of the form (14.54) which will serve as the function X in the product solution (14.47).

We now return to the problem (14.51) and (14.52) involving the function Y. Since k must be of the form (14.53), the differential equation (14.51) becomes

$$\frac{d^2Y}{dy^2} - \frac{n^2\pi^2}{a^2} Y = 0,$$

where $n = 1, 2, 3, \ldots$. For each such value of n, this differential equation has the general solution

$$Y_n = c_{n,1}e^{n\pi y/a} + c_{n,2}e^{-n\pi y/a} \qquad (n = 1, 2, 3, \ldots),$$

where $c_{n,1}$ and $c_{n,2}$ $(n = 1, 2, 3, \ldots)$, are arbitrary constants. In order to satisfy the condition (14.52), we must have

$$c_{n,1} + c_{n,2} = 0 \qquad (n = 1, 2, 3, \ldots).$$

Thus nontrivial solutions of the problem (14.51) and (14.52) are

$$Y_n = c_{n,1}(e^{n\pi y/a} - e^{-n\pi y/a}) \qquad (n = 1, 2, 3, \ldots),$$

where the $c_{n,1}$ $(n = 1,\ 2,\ 3, \ldots)$, are arbitrary constants. Using the identity $e^{\theta} - e^{-\theta} = 2 \sinh \theta$, we may put these solutions in the form

$$Y_n = c'_{n,1} \sinh \frac{n\pi y}{a} \qquad (n = 1, 2, 3, \ldots), \tag{14.55}$$

where the $c'_{n,1}$ $(n = 1, 2, 3, \ldots)$, are arbitrary constants. Thus, corresponding to each positive integral value of n, we obtain functions Y_n of the form (14.55) which will serve as the function Y in the product solution (14.47).

Hence, corresponding to each positive integral value of $n(n = 1, 2, 3, \ldots)$, we obtain solutions

$$X_nY_n = \left[c_n \sin \frac{n\pi x}{a}\right]\left[c'_{n,1} \sinh \frac{n\pi y}{a}\right]$$

which have the product form (14.47). We set $C_n = c_nc'_{n,1}$ $(n = 1, 2, 3, \ldots)$, and write these solutions as

$$u_n(x, y) = C_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \qquad (n = 1, 2, 3, \ldots). \tag{14.56}$$

Each one of these solutions (14.56) satisfies both the partial differential equation (14.42) and the three homogeneous boundary conditions (14.43), (14.44), and (14.45) for all values of the constant C_n.

We must now apply the nonhomogeneous boundary condition (14.46). In order to do this, we form an infinite series

$$\sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

of the solutions (14.56). Assuming appropriate convergence, Theorem 14.2 applies and assures us that the sum of this series is also a solution of the differential equation (14.42). Denoting this sum by $u(x, y)$, we write

$$u(x, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}. \tag{14.57}$$

We observe that $u(0, y) = 0$, $u(a, y) = 0$, and $u(x, 0) = 0$. Thus, assuming appropriate convergence, the function u given by (14.57) satisfies both the differential equation (14.42) and the three homogeneous boundary conditions (14.43), (14.44), and (14.45).

We now apply the nonhomogeneous boundary condition (14.46) to the series solution (14.57). Doing so, we obtain at once

$$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a} = f(x), \qquad 0 \le x \le a.$$

Letting $A_n = C_n \sinh(n\pi b/a)$ $(n = 1, 2, 3, \ldots)$, this takes the form

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{a} = f(x), \qquad 0 \leq x \leq a. \tag{14.58}$$

Thus in order to satisfy the condition (14.46), we must determine the coefficients A_n so that (14.58) is satisfied. This is a problem in Fourier sine series. Using (12.56), we find that the coefficients A_n are given by

$$A_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx \qquad (n = 1, 2, 3, \ldots).$$

Since $A_n = C_n \sinh(n\pi b/a)$ $(n = 1, 2, 3, \ldots)$, we find that

$$C_n = \frac{A_n}{\sinh \dfrac{n\pi b}{a}} = \frac{2}{a \sinh \dfrac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx \qquad (n = 1, 2, 3, \ldots). \tag{14.59}$$

Thus in order for the series solution (14.57) to satisfy the nonhomogeneous boundary condition (14.46), the coefficients C_n in the series must be given by (14.59).

Therefore the formal solution of the problem consisting of the partial differential equation (14.42) and the four boundary conditions (14.43) through (14.46) is

$$u(x, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}, \tag{14.57}$$

where

$$C_n = \frac{2}{a \sinh \dfrac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx \qquad (n = 1, 2, 3, \ldots). \tag{14.59}$$

D. An Example Involving Bessel Functions

In this example we shall apply the method of separation of variables to a problem in which the partial differential equation has a variable coefficient. As a result of this variable coefficient we shall encounter certain difficulties which were not present in the two previous examples. Further, we shall need to know a few results which we have not yet proved. Whenever such a result is needed, we shall state it without proof.

▶ **Problem** Apply the method of separation of variables to obtain a formal solution $u(x, t)$ of the problem which consists of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} \tag{14.60}$$

and the three conditions

$$1.\quad u(L, t) = 0, \qquad t > 0; \tag{14.61}$$

$$2.\quad u(x, 0) = f(x), \qquad 0 < x < L, \tag{14.62}$$

where f is a prescribed function of x, $0 < x < L$; and

$$3.\quad \lim_{t \to \infty} u(x, t) = 0 \tag{14.63}$$

for each x, $0 \leq x \leq L$.

Formal Solution. We begin by making the basic assumption that the differential equation (14.60) has product solutions of the form

$$u(x, t) = X(x)T(t), \tag{14.64}$$

where X is a function of x only and T is a function of t only. Upon differentiating (14.64) and substituting into the differential equation (14.60), we obtain

$$T\frac{d^2X}{dx^2} + \frac{1}{x}T\frac{dX}{dx} = X\frac{dT}{dt}.$$

From this we obtain at once

$$\frac{1}{X}\left(\frac{d^2X}{dx^2} + \frac{1}{x}\frac{dX}{dx}\right) = \frac{1}{T}\frac{dT}{dt}. \tag{14.65}$$

The left member of (14.65) is a function of x only and so is independent of t. The right member of (14.65) is a function of t only and so is independent of x. Therefore the two equal expressions in (14.65) must both be equal to a constant k. Setting each member of (14.65) equal to this constant k, we obtain the two ordinary differential equations

$$\frac{d^2X}{dx^2} + \frac{1}{x}\frac{dX}{dx} - kX = 0 \tag{14.66}$$

and

$$\frac{dT}{dt} - kT = 0. \tag{14.67}$$

The effect of the variable coefficient $(1/x)$ in the partial differential equation (14.60) appears here, for the ordinary differential equation (14.66) also has this same variable coefficient.

We shall need the general solutions of both of the ordinary differential equations (14.66) and (14.67). Equation (14.67) is the more promising of the two; let us work with it first. We find at once that the general solution of (14.67) is of the form

$$T = Ce^{kt}, \tag{14.68}$$

where C is an arbitrary constant.

Let us now examine the three conditions (14.61), (14.62), and (14.63) to see if any of them will lead to further information about the solution (14.68). The first two of these conditions lead to conditions on the function X. Let us therefore examine the third condition (14.63). Since $u(x, t) = X(x)T(t)$, this condition reduces to

$$X(x)[\lim_{t\to\infty} T(t)] = 0$$

for each x, $0 \le x \le L$. Hence we require that

$$\lim_{t\to\infty} T(t) = 0.$$

From this we see that the constant k in (14.68) must be a negative number. Therefore we set $k = -\lambda^2$, where λ is real and positive. The general solution (14.68) of the differential equation (14.67) now takes the form

$$T = Ce^{-\lambda^2 t}, \tag{14.69}$$

where C is an arbitrary constant.

Let us now return to the differential equation (14.66) for X. Since $k = -\lambda^2$ it now takes the form

$$\frac{d^2X}{dx^2} + \frac{1}{x}\frac{dX}{dx} + \lambda^2 X = 0$$

or equivalently,

$$x^2\frac{d^2X}{dx^2} + x\frac{dX}{dx} + \lambda^2 x^2 X = 0. \tag{14.70}$$

The transformation $\theta = \lambda x$ reduces (14.70) to the equation

$$\theta^2\frac{d^2X}{d\theta^2} + \theta\frac{dX}{d\theta} + \theta^2 X = 0.$$

We readily recognize this equation as the Bessel equation of order zero. Its general solution may be written

$$X = c_1 J_0(\theta) + c_2 Y_0(\theta),$$

where J_0 and Y_0 are the Bessel functions of order zero of the first and second kind, respectively, and c_1 and c_2 are arbitrary constants (see Section 6.3). Thus the general solution of (14.70) may be written

$$X = c_1 J_0(\lambda x) + c_2 Y_0(\lambda x), \tag{14.71}$$

where c_1 and c_2 are arbitrary constants.

Let us now return to the condition (14.63). This condition requires that

$$\lim_{t\to\infty} u(x, t) = 0$$

for *each* x, $0 \le x \le L$. For $x = 0$ this becomes

$$\lim_{t\to\infty} u(0, t) = 0$$

or

$$X(0)[\lim_{t\to\infty} T(t)] = 0.$$

In order to satisfy this condition we must require that $X(0)$ be finite. We recall that $J_0(0) = 1$. However, it can be shown that

$$\lim_{x\to\infty} Y_0(\lambda x) = -\infty.$$

Thus in order for $X(0)$ to be finite we must set $c_2 = 0$. Thus the solution (14.71) reduces to

$$X = c_1 J_0(\lambda x). \tag{14.72}$$

Let us now consider the condition (14.61). We have already noticed that this condition leads to a condition on X. Indeed, it reduces to

$$X(L)T(t) = 0, \qquad t > 0;$$

thus we must have $X(L) = 0$. Applying this to the solution (14.72), we see that λ must satisfy the equation

$$J_0(\lambda L) = 0. \tag{14.73}$$

In Section 6.3B we pointed out that the function J_0 has a damped oscillatory behavior as $x \to +\infty$. Thus the equation $J_0(x) = 0$ has an infinite sequence of positive roots x_n $(n = 1, 2, 3, \ldots)$. Let us arrange these positive roots such that $x_n < x_{n+1}$ $(n = 1, 2, 3, \ldots)$. Then there exists a monotonic increasing sequence of positive numbers

$$\lambda_n = \frac{x_n}{L} \qquad (n = 1, 2, 3, \ldots),$$

each of which satisfies Equation (14.73). Thus corresponding to each positive integer n, the differential equation (14.70) has solutions which satisfy the condition (14.61). These solutions are of the form

$$X_n = c_{1,n} J_0(\lambda_n x) \qquad (n = 1, 2, 3, \ldots), \tag{14.74}$$

where the $c_{1,n}$ $(n = 1, 2, 3, \ldots)$ are arbitrary constants and the λ_n $(n = 1, 2, 3, \ldots)$ are the positive roots of Equation (14.73). That is, corresponding to each positive integer n, we obtain functions X_n of the form (14.74) which will serve as the function X in the product solution (14.64).

Let us now return to the solution (14.69) of the differential equation (14.67). We see that, corresponding to each positive integer n, the differential equation (14.67) has solutions of the form

$$T_n = c_{2,n} e^{-\lambda_n^2 t} \qquad (n = 1, 2, 3, \ldots), \tag{14.75}$$

where the $c_{2,n}$ $(n = 1, 2, 3, \ldots)$ are arbitrary constants and the λ_n $(n = 1, 2, 3, \ldots)$ are the positive roots of Equation (14.73). That is, corresponding to each positive integer n, we obtain functions T_n of the form (14.75) which will serve as the function T in the product solution (14.64).

Hence, corresponding to each positive integral value of n $(n = 1, 2, 3, \ldots)$, we obtain product solutions of the form

$$u_n(x, t) = A_n J_0(\lambda_n x) e^{-\lambda_n^2 t} \qquad (n = 1, 2, 3, \ldots), \tag{14.76}$$

where the $A_n = c_{1,n} c_{2,n}$ $(n = 1, 2, 3, \ldots)$ are arbitrary constants.. Each one of these solutions (14.76) satisfies the partial differential equation (14.60) and the conditions (14.61) and (14.63) for all values of the constant A_n.

We must now apply the initial condition (14.62). In order to do this, we form an infinite series of the solutions (14.76). Assuming appropriate convergence, the sum of this series is also a solution of the partial differential equation (14.60). We denote this sum by $u(x, t)$ and thus write

$$u(x, t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n x) e^{-\lambda_n^2 t}. \tag{14.77}$$

Applying the initial condition (14.62) to the series solution (14.77), we obtain

$$\sum_{n=1}^{\infty} A_n J_0(\lambda_n x) = f(x), \qquad 0 < x < L. \tag{14.78}$$

Thus in order to satisfy the initial condition (14.62), the coefficients A_n must be determined so that (14.78) is satisfied. In other words, we must expand the function f in a series of Bessel functions of the first kind of order zero, valid on the interval $0 < x < L$.

Here we have encountered a new difficulty, and this difficulty leads to matters which are outside the scope of this book. Nevertheless, we shall indicate briefly what can be done. This is one place where we need to know certain of the results which we referred to at the beginning of this section. Let us state them and get on with the problem!

In the first place, it can be shown that if the numbers λ_n $(n = 1, 2, 3, \ldots)$ are the positive roots of the equation $J_0(\lambda L) = 0$, then the set of functions defined by $\{J_0(\lambda_n x)\}$ $(n = 1, 2, 3, \ldots)$ is an orthogonal system with respect to the weight function r such that $r(x) = x$ on the interval $0 \le x \le L$. Therefore,

$$\int_0^L x J_0(\lambda_m x) J_0(\lambda_n x)\, dx = 0 \qquad (m = 1, 2, 3, \ldots; n = 1, 2, 3, \ldots; m \neq n).$$

Further, if $m = n$, we have

$$\int_0^L x[J_0(\lambda_n x)]^2\, dx = \Gamma_n > 0 \qquad (n = 1, 2, 3, \ldots). \tag{14.79}$$

In Section 12.3A we learned how to form a set of *orthonormal* functions from a set of orthogonal characteristic functions of a Sturm–Liouville problem. Applying this procedure to the orthogonal set defined by $\{J_0(\lambda_n x)\}$, we obtain the corresponding orthonormal system $\{\phi_n\}$, where

$$\phi_n(x) = \frac{J_0(\lambda_n x)}{\sqrt{\Gamma_n}} \qquad (n = 1, 2, 3, \ldots), \tag{14.80}$$

and Γ_n $(n = 1, 2, 3, \ldots)$ is given by (14.79). Let us now recall the results of Section 12.3B concerning the formal expansion of a function f in a series

$$\sum_{n=1}^{\infty} c_n \phi_n$$

of orthonormal functions $\{\phi_n\}$. According to (12.37) the coefficients c_n in the expansion of f in the series of orthonormal functions ϕ_n defined by (14.80) are given by

$$c_n = \frac{1}{\sqrt{\Gamma_n}} \int_0^L x f(x) J_0(\lambda_n x)\, dx \qquad (n = 1, 2, 3, \ldots).$$

Thus this expansion takes the form

$$\sum_{n=1}^{\infty} \left[\frac{1}{\sqrt{\Gamma_n}} \int_0^L x f(x) J_0(\lambda_n x)\, dx\right] \frac{J_0(\lambda_n x)}{\sqrt{\Gamma_n}},$$

and we write formally

$$f(x) = \sum_{n=1}^{\infty} \left[\frac{1}{\Gamma_n} \int_0^L x f(x) J_0(\lambda_n x)\, dx\right] J_0(\lambda_n x), \qquad 0 < x < L. \tag{14.81}$$

Comparing (14.78) and (14.81), we see that if the coefficients A_n in (14.78) are given by

$$A_n = \frac{1}{\Gamma_n} \int_0^L x f(x) J_0(\lambda_n x)\, dx \qquad (n = 1, 2, 3, \ldots), \tag{14.82}$$

then the requirement (14.78) will be formally satisfied. We note that the constants Γ_n in (14.82) are given by (14.79). That is,

$$\Gamma_n = \int_0^L x[J_0(\lambda_n x)]^2\, dx \qquad (n = 1, 2, 3, \ldots).$$

This integral can be evaluated in terms of values of the Bessel function of the first kind of order one, J_1. Indeed, it can be shown that

$$\int_0^L x[J_0(\lambda_n x)]^2\,dx = \frac{L^2}{2}[J_1(\lambda_n L)]^2 \qquad (n = 1, 2, 3, \ldots),$$

and thus

$$\Gamma_n = \frac{L^2}{2}[J_1(\lambda_n L)]^2 \qquad (n = 1, 2, 3, \ldots).$$

Thus the coefficients A_n in (14.78) are given by

$$A_n = \frac{2}{L^2[J_1(\lambda_n L)]^2}\int_0^L xf(x)J_0(\lambda_n x)\,dx \qquad (n = 1, 2, 3, \ldots). \tag{14.83}$$

Finally, then, we obtain the formal solution of the problem consisting of the partial differential equation (14.60) and the conditions (14.61), (14.62), and (14.63). The formal solution is

$$u(x, t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n x)e^{-\lambda_n^2 t},$$

where

$$A_n = \frac{2}{L^2[J_1(\lambda_n L)]^2}\int_0^L xf(x)J_0(\lambda_n x)\,dx \qquad (n = 1, 2, 3, \ldots),$$

the λ_n $(n = 1, 2, 3, \ldots)$, are the positive roots of the equation $J_0(\lambda L) = 0$, and J_0 and J_1 denote the Bessel functions of the first kind of orders zero and one, respectively.

Remarks and Observations. At the risk of being unduly repetitious, but with the good intentions of promoting a cautious attitude, we emphasize that the results which we have obtained are strictly formal results. We *assumed* "appropriate convergence" of the series in (14.77), and we have said nothing concerning the convergence of the Bessel function expansion which we obtained for the function f. In order to relieve our consciences concerning the latter point, we state that there do exist classes of functions f such that the expansions (14.78), in which the coefficients A_n are given by (14.83), is valid on the interval $0 < x < L$. The study of these classes is definitely beyond the scope of this book, and we refer the reader to more advanced works for a discussion of this and other pertinent problems of convergence.

Finally, we point out that the problem which we have considered here gives some indication of the types of difficulties which may be encountered if the method of separation of variables is applied to a problem in which the partial differential equation has variable coefficients. The variable coefficient $(1/x)$ in the partial differential equation (14.60) led to the variable coefficient $(1/x)$ in the ordinary differential equation (14.66) which resulted from the separation of the variables. A similar situation occurs in other problems in which the partial differential equation has variable coefficients. In such problems, one or more of the ordinary differential equations which result from the separation process will also contain variable coefficients. Obtaining the general solutions of these ordinary differential equations can then be a formidable task in its own right. But even if these general solutions can be obtained, they may involve functions which will lead to further difficulties when one attempts to apply certain of the supplementary conditions. This sort of difficulty occurred in the problem of this section when we tried to apply the initial condition (14.62). We were forced to consider the problem of expanding the function f in a series of Bessel functions, valid on the interval $0 < x < L$. A similar situation often occurs in other problems which involve a partial differential equation with variable coefficients. In such problems one is faced with the task of expanding a prescribed function f in a series of nonelementary orthonormal functions $\{\phi_n\}$, valid on a certain interval. The set of orthonormal functions $\{\phi_n\}$ might happen to be one of the many such sets which have been carefully studied, or it might turn out to be a set about which little is known. In any case, additional difficulties may occur which will necessitate further study and possibly some research.

Exercises

Use the method of separation of variables to find a formal solution $y(x, t)$ of each of the problems stated in Exercises 1–4

1. $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2}$,

$$y(0, t) = 0, \quad 0 \le t < \infty,$$
$$y(\pi, t) = 0, \quad 0 \le t < \infty,$$
$$y(x, 0) = \sin 2x, \quad 0 \le x \le \pi,$$
$$\frac{\partial y(x, 0)}{\partial t} = 0, \quad 0 \le x \le \pi.$$

2. $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2}$,

$$y(0, t) = 0, \quad 0 \le t < \infty,$$
$$y(3\pi, t) = 0, \quad 0 \le t < \infty,$$
$$y(x, 0) = 2 \sin^3 x, \quad 0 \le x \le 3\pi,$$
$$\frac{\partial y(x, 0)}{\partial t} = 0, \quad 0 \le x \le 3\pi.$$

3. $4\dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2}$,

$$y(0, t) = 0, \quad 0 \le t < \infty,$$
$$y(3, t) = 0, \quad 0 \le t < \infty,$$
$$y(x, 0) = \begin{cases} x, & 0 \le x \le 1, \\ 1, & 1 \le x \le 2, \\ 3 - x, & 2 \le x \le 3, \end{cases}$$
$$\frac{\partial y(x, 0)}{\partial t} = 0, \quad 0 \le x \le 3.$$

4. $4\dfrac{\partial^2 y}{\partial x^2} = 9\dfrac{\partial^2 y}{\partial t^2}$,

$y(0, t) = 0, \quad 0 \le t < \infty,$

$y(\pi, t) = 0, \quad 0 \le t < \infty,$

$y(x, 0) = \sin^2 x, \quad 0 \le x \le \pi,$

$\dfrac{\partial y(x, 0)}{\partial t} = \sin x, \quad 0 \le x \le \pi.$

5. Apply the method of separation of variables to obtain a formal solution $u(x, t)$ of the problem which consists of the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

the boundary conditions

$$u(0, t) = 0, \quad t > 0,$$
$$u(L, t) = 0, \quad t > 0,$$

and the initial condition

$$u(x, 0) = f(x), \qquad 0 \le x \le L,$$

where $L > 0$, and f is a specified function of x, $0 \le x \le L$.

6. Use the method of separation of variables to find a formal solution $u(x, y)$ of the problem which consists of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = -Ae^{-\alpha x},$$

where $A \ge 0$ and $\alpha > 0$, and the conditions

$$u(0, y) = 0, \qquad y > 0,$$
$$u(L, y) = 0, \qquad y > 0,$$
$$u(x, 0) = f(x), \qquad 0 < x < L.$$

[*Hint*: Let $u(x, y) = v(x, y) + \psi(x)$, where ψ is such that v satisfies the "homogeneous" equation

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial y} = 0$$

and the homogeneous boundary conditions

$$v(0, y) = 0, v(L, y) = 0.]$$

7. Use the method of separation of variables to find a formal solution $u(x, y)$ of the problem which consists of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and the boundary conditions

$$u(0, y) = 0, \qquad 0 \le y \le \pi,$$
$$u(\pi, y) = 0, \qquad 0 \le y \le \pi,$$
$$u(x, \pi) = 0, \qquad 0 \le x \le \pi,$$
$$u(x, 0) = f(x), \qquad 0 \le x \le \pi,$$

where f is a specified function of x, $0 \le x \le \pi$.

8. Use the method of separation of variables to obtain a formal solution $u(r, \theta)$ of the problem which consists of the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

the periodicity condition

$$u(r, \theta) = u(r, \theta + 2\pi), \qquad 0 \le r \le L, \quad \text{for all values of } \theta,$$

and the boundary conditions

$u(0, \theta) = \alpha$, where α is finite,

$u(L, \theta) = f(\theta)$, where f is a prescribed function of θ, for all values of θ.

14.3 CANONICAL FORMS OF SECOND-ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

A. Canonical Forms

In this section we restrict our attention to second-order linear partial differential equations of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\, \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0, \tag{14.84}$$

in which the coefficients A, B, C, D, E, and F are real *constants*. This equation is a special case of the more general equation (14.6) in which these coefficients are functions of x and y. In Section 14.1 we classified such equations according to the sign of $B^2 - 4AC$. Using this classification, Equation (14.84) is said to be

1. *hyperbolic* if $B^2 - 4AC > 0$;
2. *parabolic* if $B^2 - 4AC = 0$;
3. *elliptic* if $B^2 - 4AC < 0$.

We shall now show that in each of these three cases Equation (14.84) can be reduced to a more simple form by a suitable change of the independent variables. The simpler forms which result in this way are called *canonical forms* of Equation (14.84). We therefore introduce new independent variables ξ, η by means of the transformation

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y). \tag{14.85}$$

We now compute derivatives of u, regarding ξ and η as intermediate variables, so that $u = u(\xi, \eta)$, where ξ and η are given by (14.85). We first find

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \tag{14.86}$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}. \tag{14.87}$$

Using (14.86), we next determine $\dfrac{\partial^2 u}{\partial x^2}$.

We find

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\\ &= \frac{\partial u}{\partial \xi}\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\right) + \frac{\partial \xi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right) + \frac{\partial u}{\partial \eta}\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial x}\right) + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right).\end{aligned} \tag{14.88}$$

Since $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$,

$$\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\right) \quad \text{is simply} \quad \frac{\partial^2 \xi}{\partial x^2}$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial x}\right) \quad \text{is simply} \quad \frac{\partial^2 \eta}{\partial x^2}.$$

However, in computing $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial \xi}\right)$ and $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial \eta}\right)$, the situation is somewhat more complicated and we must be careful to remember what is involved. We are regarding u as a function of ξ and η, where ξ and η are themselves functions of x and y. That is, $u = u(\xi, \eta)$, where $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$. Thus $\dfrac{\partial u}{\partial \xi}$ and $\dfrac{\partial u}{\partial \eta}$ are regarded as functions of ξ and η, where ξ and η are themselves functions of x and y. That is, $\dfrac{\partial u}{\partial \xi} = u_1(\xi, \eta)$ and $\dfrac{\partial u}{\partial \eta} = u_2(\xi, \eta)$, where in each case $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$. With this in mind, we compute $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial \xi}\right)$. We have

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right) = \frac{\partial}{\partial x}[u_1(\xi, \eta)] = \frac{\partial u_1}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_1}{\partial \eta}\frac{\partial \eta}{\partial x},$$

and since $u_1 = \dfrac{\partial u}{\partial \xi}$, we thus find

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right) = \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta\,\partial \xi}\frac{\partial \eta}{\partial x}.$$

In like manner, we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right)=\frac{\partial^2 u}{\partial \xi\,\partial \eta}\frac{\partial \xi}{\partial x}+\frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}.$$

Substituting these results into (14.88), we thus obtain

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial \xi}{\partial x}\left(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}+\frac{\partial^2 u}{\partial \eta\,\partial \xi}\frac{\partial \eta}{\partial x}\right)\\&\quad+\frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}+\frac{\partial \eta}{\partial x}\left(\frac{\partial^2 u}{\partial \xi\,\partial \eta}\frac{\partial \xi}{\partial x}+\frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\right).\end{aligned}$$

Assuming $u(\xi, \eta)$ has continuous second derivatives with respect to ξ and η, the so-called cross derivatives are equal, and we have

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2+2\frac{\partial^2 u}{\partial \xi\,\partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2\\&\quad+\frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}.\end{aligned}\tag{14.89}$$

In like manner, we find

$$\begin{aligned}\frac{\partial^2 u}{\partial x\,\partial y}&=\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}+\frac{\partial^2 u}{\partial \xi\,\partial \eta}\left[\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}+\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right]\\&\quad+\frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}+\frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x\,\partial y}+\frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x\,\partial y}\end{aligned}\tag{14.90}$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2}&=\frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2+2\frac{\partial^2 u}{\partial \xi\,\partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}+\frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2\\&\quad+\frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2}+\frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}.\end{aligned}\tag{14.91}$$

We now substitute (14.86), (14.87), (14.89), (14.90), and (14.91) into the partial differential equation (14.84), to obtain

$$\begin{aligned}&A\left[\frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2+2\frac{\partial^2 u}{\partial \xi\,\partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2+\frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}\right]\\&\quad+B\left[\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}+\frac{\partial^2 u}{\partial \xi\,\partial \eta}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}+\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)+\frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}+\frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x\,\partial y}+\frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x\,\partial y}\right]\\&\quad+C\left[\frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2+2\frac{\partial^2 u}{\partial \xi\,\partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}+\frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2+\frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2}+\frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}\right]\\&\qquad+D\left[\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}\right]+E\left[\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}\right]+Fu=0.\end{aligned}$$

Rearranging terms, this becomes

$$\left[A\left(\frac{\partial \xi}{\partial x}\right)^2+B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}+C\left(\frac{\partial \xi}{\partial y}\right)^2\right]\frac{\partial^2 u}{\partial \xi^2}+\left[2A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}+B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}+\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)\right.$$

$$+ 2C \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y}\Bigg] \frac{\partial^2 u}{\partial \xi\, \partial \eta} + \left[A\left(\frac{\partial \eta}{\partial x}\right)^2 + B \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2 \right] \frac{\partial^2 u}{\partial \eta^2}$$

$$+ \left[A \frac{\partial^2 \xi}{\partial x^2} + B \frac{\partial^2 \xi}{\partial x\, \partial y} + C \frac{\partial^2 \xi}{\partial y^2} + D \frac{\partial \xi}{\partial x} + E \frac{\partial \xi}{\partial y} \right] \frac{\partial u}{\partial \xi}$$

$$+ \left[A \frac{\partial^2 \eta}{\partial x^2} + B \frac{\partial^2 \eta}{\partial x\, \partial y} + C \frac{\partial^2 \eta}{\partial y^2} + D \frac{\partial \eta}{\partial x} + E \frac{\partial \eta}{\partial y} \right] \frac{\partial u}{\partial \eta} + Fu = 0.$$

Thus using the transformation (14.85), the equation (14.84) is reduced to the form

$$A_1 \frac{\partial^2 u}{\partial \xi^2} + B_1 \frac{\partial^2 u}{\partial \xi\, \partial \eta} + C_1 \frac{\partial^2 u}{\partial \eta^2} + D_1 \frac{\partial u}{\partial \xi} + E_1 \frac{\partial u}{\partial \eta} + F_1 u = 0, \tag{14.92}$$

where

$$\begin{aligned}
A_1 &= A\left(\frac{\partial \xi}{\partial x}\right)^2 + B \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2, \\
B_1 &= 2A \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x}\right) + 2C \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y}, \\
C_1 &= A\left(\frac{\partial \eta}{\partial x}\right)^2 + B \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2, \\
D_1 &= A \frac{\partial^2 \xi}{\partial x^2} + B \frac{\partial^2 \xi}{\partial x\, \partial y} + C \frac{\partial^2 \xi}{\partial y^2} + D \frac{\partial \xi}{\partial x} + E \frac{\partial \xi}{\partial y}, \\
E_1 &= A \frac{\partial^2 \eta}{\partial x^2} + B \frac{\partial^2 \eta}{\partial x\, \partial y} + C \frac{\partial^2 \eta}{\partial y^2} + D \frac{\partial \eta}{\partial x} + E \frac{\partial \eta}{\partial y},
\end{aligned} \tag{14.93}$$

and

$$F_1 = F.$$

We now show that the new equation (14.92) can be simplified by a suitable choice of $\xi(x, y)$ and $\eta(x, y)$ in the transformation (14.85). The choice of these functions ξ and η and the form of the resulting simplified equation (the canonical form) depend upon whether the original partial differential equation (14.84) is hyperbolic, parabolic, or elliptic.

B. The Hyperbolic Equation

Concerning the canonical form in the hyperbolic case, we state and prove the following theorem.

THEOREM 14.3

Hypothesis. *Consider the second-order linear partial differential equation*

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x\, \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = 0, \tag{14.84}$$

where the coefficients A, B, C, D, E, and F are real constants and $B^2 - 4AC > 0$ *so that the equation is hyperbolic.*

Conclusion. *There exists a transformation*

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y) \tag{14.85}$$

of the independent variables in (14.84) so that the transformed equation in the independent variables (ξ, η) *may be written in the canonical form*

$$\frac{\partial^2 u}{\partial \xi\, \partial \eta} = d\frac{\partial u}{\partial \xi} + e\frac{\partial u}{\partial \eta} + fu, \tag{14.94}$$

where d, e, and f are real constants.

If $A \neq 0$, *such a transformation is given by*

$$\begin{aligned} \xi &= \lambda_1 x + y, \\ \eta &= \lambda_2 x + y, \end{aligned} \tag{14.95}$$

where λ_1 *and* λ_2 *are the roots of the quadratic equation*

$$A\lambda^2 + B\lambda + C = 0. \tag{14.96}$$

If $A = 0$, $B \neq 0$, $C \neq 0$, *such a transformation is given by*

$$\begin{aligned} \xi &= x, \\ \eta &= x - \frac{B}{C}y. \end{aligned} \tag{14.97}$$

If $A = 0$, $B \neq 0$, $C = 0$, *such a transformation is merely the identity transformation*

$$\begin{aligned} \xi &= x, \\ \eta &= y. \end{aligned} \tag{14.98}$$

Proof. We shall first show that the transformations given by (14.95), (14.97), and (14.98) actually do reduce Equation (14.84) so that it may be written in the form (14.94) in the three respective cases described in the conclusion. We shall then observe that these three cases cover all possibilities for the hyperbolic equation, thereby completing the proof.

We have seen that a transformation of the form (14.85) reduces Equation (14.84) to the form (14.92), where the coefficients are given by (14.93). In the case $A \neq 0$, we apply the special case of (14.85) given by

$$\begin{aligned} \xi &= \lambda_1 x + y, \\ \eta &= \lambda_2 x + y, \end{aligned} \tag{14.95}$$

where λ_1 and λ_2 are the roots of the quadratic equation

$$A\lambda^2 + B\lambda + C = 0. \tag{14.96}$$

Then the coefficients in the transformed equation (14.92) are given by (14.93), where $\xi(x, y)$ and $\eta(x, y)$ are given by (14.95). Evaluating these coefficients in this case we find that

$$
\begin{aligned}
A_1 &= A\lambda_1^2 + B\lambda_1 + C,\\
B_1 &= 2A\lambda_1\lambda_2 + B(\lambda_1 + \lambda_2) + 2C\\
&= 2A\left(\frac{C}{A}\right) + B\left(-\frac{B}{A}\right) + 2C = \frac{B^2 - 4AC}{-A},\\
C_1 &= A\lambda_2^2 + B\lambda_2 + C,\\
D_1 &= D\lambda_1 + E,\\
E_1 &= D\lambda_2 + E,\\
F_1 &= F.
\end{aligned}
$$

Since λ_1 and λ_2 satisfy the quadratic equation (14.96), we see that $A_1 = 0$ and $C_1 = 0$. Therefore in this case the transformed equation (14.92) is

$$
\left(\frac{B^2 - 4AC}{-A}\right)\frac{\partial^2 u}{\partial \xi\, \partial \eta} + (D\lambda_1 + E)\frac{\partial u}{\partial \xi} + (D\lambda_2 + E)\frac{\partial u}{\partial \eta} + Fu = 0. \tag{14.99}
$$

Since $B^2 - 4AC > 0$ [Equation (14.84) is hyperbolic], the roots λ_1 and λ_2 of (14.96) are real and distinct. Therefore the coefficients in (14.99) are all real. Furthermore, the leading coefficient $(B^2 - 4AC)/(-A)$ is unequal to zero. Therefore we may write the transformed equation (14.99) in the form

$$
\frac{\partial^2 u}{\partial \xi\, \partial \eta} = d\frac{\partial u}{\partial \xi} + e\frac{\partial u}{\partial \eta} + fu,
$$

where the coefficients

$$
d = \frac{A(D\lambda_1 + E)}{B^2 - 4AC}, \qquad e = \frac{A(D\lambda_2 + E)}{B^2 - 4AC}, \qquad f = \frac{AF}{B^2 - 4AC}
$$

are all real. This is the canonical form (14.94).

Now consider the case in which $A = 0$, $B \neq 0$, $C \neq 0$, and apply the special case of (14.85) given by

$$
\begin{aligned}
\xi &= x,\\
\eta &= x - \frac{B}{C}y.
\end{aligned} \tag{14.97}
$$

In this case the coefficients in the transformed equation (14.92) are given by (14.93), where $\xi(x, y)$ and $\eta(x, y)$ are given by (14.97). Evaluating these coefficients (recall $A = 0$ here), we find that

$$
\begin{aligned}
A_1 &= 0,\\
B_1 &= -\frac{B^2}{C} \neq 0,\\
C_1 &= B\left(-\frac{B}{C}\right) + C\left(-\frac{B}{C}\right)^2 = 0,\\
D_1 &= D, \qquad E_1 = D - \frac{EB}{C}, \qquad F_1 = F.
\end{aligned}
$$

Therefore in this case the transformed equation is

$$\left(-\frac{B^2}{C}\right)\frac{\partial^2 u}{\partial\xi\,\partial\eta} + D\frac{\partial u}{\partial\xi} + \left(\frac{DC - EB}{C}\right)\frac{\partial u}{\partial\eta} + Fu = 0.$$

Since $(-B^2)/C \neq 0$, we may write this in the form

$$\frac{\partial^2 u}{\partial\xi\,\partial\eta} = d\frac{\partial u}{\partial\xi} + e\frac{\partial u}{\partial\eta} + fu,$$

where the coefficients

$$d = \frac{CD}{B^2}, \qquad e = \frac{DC - EB}{B^2}, \qquad f = \frac{CF}{B^2}$$

are all real. This is again the canonical form (14.94).

Finally, consider the case in which $A = 0$, $B \neq 0$, $C = 0$. In this case Equation (14.84) is simply

$$B\frac{\partial^2 u}{\partial x\,\partial y} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0,$$

and the identity transformation (14.98) reduces it to

$$B\frac{\partial^2 u}{\partial\xi\,\partial\eta} + D\frac{\partial u}{\partial\xi} + E\frac{\partial u}{\partial\eta} + Fu = 0.$$

Since $B \neq 0$, we may write this in the form

$$\frac{\partial^2 u}{\partial\xi\,\partial\eta} = d\frac{\partial u}{\partial\xi} + e\frac{\partial u}{\partial\eta} + fu,$$

where

$$d = -\frac{D}{B}, \qquad e = -\frac{E}{B}, \qquad f = -\frac{F}{B}.$$

This is again the canonical form (14.94). In effect, in this special case ($A = 0$, $B \neq 0$, $C = 0$) Equation (14.84) may be put in the canonical form (14.94) simply by transposing the appropriate terms and dividing by B.

We now observe that the three special cases considered exhaust all possibilities for the hyperbolic equation (14.84). We first note that either $A = 0$ or $A \neq 0$. All cases in which $A \neq 0$ are covered by the first of the three special cases which we have considered. Turning to the cases in which $A = 0$, it would appear that the following four distinct possibilities deserve consideration: (a) $B \neq 0$, $C \neq 0$; (b) $B \neq 0$, $C = 0$; (c) $B = 0$, $C \neq 0$; and (d) $B = 0$, $C = 0$. We note that (a) and (b) are covered by the second and third of the three special cases which we have considered. Concerning (c) and (d), in both cases $B^2 - 4AC = 0$, contrary to hypothesis. In particular, if (c) holds, Equation (14.84) is parabolic (not hyperbolic); and if (d) holds, Equation (14.84) is of the first order.

We thus observe that the three special cases considered cover all possibilities for the hyperbolic equation (14.84). Thus there always exists a transformation (14.85) which transforms the hyperbolic equation (14.84) into one which may be written in the canonical form (14.94). *Q.E.D*

▶ Example 14.6

Consider the equation

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x\, \partial y} - 5\frac{\partial^2 u}{\partial y^2} + 6\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} - 9u = 0. \tag{14.100}$$

We first observe that $B^2 - 4AC = 36 > 0$ and so Equation (14.100) is hyperbolic. Since $A \neq 0$, we consider the transformation

$$\begin{aligned} \xi &= \lambda_1 x + y, \\ \eta &= \lambda_2 x + y, \end{aligned} \tag{14.95}$$

where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2 + 4\lambda - 5 = 0$. We find that $\lambda_1 = 1$ and $\lambda_2 = -5$, and so the transformation (14.95) is

$$\begin{aligned} \xi &= x + y, \\ \eta &= -5x + y. \end{aligned} \tag{14.101}$$

Applying (14.101) to Equation (14.100), we see that this equation transforms into

$$-36\frac{\partial^2 u}{\partial \xi\, \partial \eta} + 9\frac{\partial u}{\partial \xi} - 27\frac{\partial u}{\partial \eta} - 9u = 0.$$

Dividing by -36 and transposing terms, we obtain the canonical form

$$\frac{\partial^2 u}{\partial \xi\, \partial \eta} = \frac{1}{4}\frac{\partial u}{\partial \xi} - \frac{3}{4}\frac{\partial u}{\partial \eta} - \frac{1}{4}u.$$

C. The Parabolic Equation

We now investigate the canonical form in the parabolic case and obtain the following theorem.

THEOREM 14.4

Hypothesis. *Consider the second-order linear partial differential equation*

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\, \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0, \tag{14.84}$$

where the coefficients A, B, C, D, E, and F are real constants and $B^2 - 4AC = 0$ so that the equation is parabolic.

Conclusion. *There exists a transformation*

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y) \tag{14.85}$$

of the independent variables in (14.84) so that the transformed equation in the *independent variables* (ξ, η) *may be written in the canonical form*

$$\frac{\partial^2 u}{\partial \eta^2} = d\frac{\partial u}{\partial \xi} + e\frac{\partial u}{\partial \eta} + fu, \tag{14.102}$$

where d, e, and f are real constants.

If $A \neq 0$ *and* $C \neq 0$, *such a transformation is given by*

$$\begin{aligned} \xi &= \lambda x + y, \\ \eta &= y, \end{aligned} \tag{14.103}$$

where λ *is the repeated real root of the quadratic equation*

$$A\lambda^2 + B\lambda + C = 0. \tag{14.104}$$

If $A \neq 0$ *and* $C = 0$, *such a transformation is given by*

$$\begin{aligned} \xi &= y, \\ \eta &= x. \end{aligned} \tag{14.105}$$

If $A = 0$ *and* $C \neq 0$, *such a transformation is merely the identity transformation*

$$\begin{aligned} \xi &= x, \\ \eta &= y. \end{aligned} \tag{14.106}$$

Proof. We shall proceed in a manner similar to that by which we proved Theorem 14.3.

If $A \neq 0$ and $C \neq 0$, we apply the transformation (14.103) to obtain the transformed equation (14.92) with coefficients (14.93), where in this case $\xi(x, y)$ and $\eta(x, y)$ are given by (14.103). Evaluating these coefficients we find that

$$\begin{aligned} A_1 &= A\lambda^2 + B\lambda + C, \\ B_1 &= B\lambda + 2C, \\ C_1 &= C, \\ D_1 &= D\lambda + E, \qquad E_1 = E, \qquad F_1 = F. \end{aligned}$$

Since λ satisfies the quadratic equation (14.104), we see at once that $A_1 = 0$. Also, since $B^2 - 4AC = 0$, $\lambda = -B/2A$ and so

$$B_1 = -\frac{B^2}{2A} + 2C = \frac{4AC - B^2}{2A} = 0.$$

Thus in the present case the transformed equation (14.92) is

$$C\frac{\partial^2 u}{\partial \eta^2} + (D\lambda + E)\frac{\partial u}{\partial \xi} + E\frac{\partial u}{\partial \eta} + Fu = 0. \tag{14.107}$$

Since λ is real, all coefficients in (14.107) are real; and since $C \neq 0$, we may write equation (14.107) in the form

$$\frac{\partial^2 u}{\partial \eta^2} = d\frac{\partial u}{\partial \xi} + e\frac{\partial u}{\partial \eta} + fu,$$

where the coefficients

$$d = -\frac{D\lambda + E}{C}, \qquad e = -\frac{E}{C}, \qquad f = -\frac{F}{C}$$

are all real. This is the canonical form (14.102).

If $A \neq 0$ and $C = 0$, we apply the transformation (14.105) to obtain the transformed equation (14.92) with coefficients (14.93), where in this case $\xi(x, y)$ and $\eta(x, y)$ are given by (14.105). Evaluating these coefficients, we obtain

$$\begin{aligned}
&A_1 = C = 0, \\
&B_1 = B = 0 \qquad (\text{since } B^2 - 4AC = 0 \quad \text{and} \quad C = 0), \\
&C_1 = A \neq 0, \\
&D_1 = E, \qquad E_1 = D, \qquad F_1 = F.
\end{aligned}$$

Thus in the case under consideration the transformed equation (14.92) is

$$A\frac{\partial^2 u}{\partial \eta^2} + E\frac{\partial u}{\partial \xi} + D\frac{\partial u}{\partial \eta} + Fu = 0.$$

Since $A \neq 0$, we may write this in the form

$$\frac{\partial^2 u}{\partial \eta^2} = d\frac{\partial u}{\partial \xi} + e\frac{\partial u}{\partial \eta} + fu,$$

where the coefficients $d = -E/A$, $e = -D/A$, and $f = -F/A$ are all real. This is again the canonical form (14.102).

Finally, consider the case in which $A = 0$ and $C \neq 0$. Since $B^2 - 4AC = 0$, we must also have $B = 0$. Therefore in this case Equation (14.84) is simply

$$C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0,$$

and the identity transformation (14.106) reduces it to

$$C\frac{\partial^2 u}{\partial \eta^2} + D\frac{\partial u}{\partial \xi} + E\frac{\partial u}{\partial \eta} + Fu = 0.$$

Since $C \neq 0$, we may write this in the form

$$\frac{\partial^2 u}{\partial \eta^2} = d\frac{\partial u}{\partial \xi} + e\frac{\partial u}{\partial \eta} + fu,$$

where $d = -D/C$, $e = -E/C$, $f = -F/C$. This is again the canonical form (14.102). We thus see that in this special case ($A = 0$, $B = 0$, $C \neq 0$) Equation (14.84) may be put in the canonical form (14.102) simply by transposing the appropriate terms and dividing by C.

Finally, we observe that the three special cases considered exhaust all possibilities for the parabolic equation (14.84). For, either $A = 0$ or $A \neq 0$. If $A \neq 0$, either $C \neq 0$ or $C = 0$. These two possibilities are, respectively, the first and second special cases considered, and so all cases in which $A \neq 0$ are thus covered. If $A = 0$, either $C \neq 0$ or $C = 0$. The first of these two possibilities is the third special case considered. Finally, consider the situation in which $A = 0$ and $C = 0$. Since $B^2 - 4AC = 0$, we must also have $B = 0$ and so Equation (14.84) reduces to a first-order equation.

Therefore there always exists a transformation (14.85) which transforms the parabolic equation (14.84) into one which may be written in the canonical form (14.102). *Q.E.D.*

▶ Example 14.7

Consider the equation

$$\frac{\partial^2 u}{\partial x^2} - 6\frac{\partial^2 u}{\partial x\,\partial y} + 9\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} - u = 0. \tag{14.108}$$

We observe that $B^2 - 4AC = 0$, and so Equation (14.108) is parabolic. Since $A \neq 0$ and $C \neq 0$, we consider the transformation

$$\begin{aligned} \xi &= \lambda x + y, \\ \eta &= y, \end{aligned} \tag{14.103}$$

where λ is the repeated real root of the quadratic equation $\lambda^2 - 6\lambda + 9 = 0$. We find that $\lambda = 3$, and so the transformation (14.103) is

$$\begin{aligned} \xi &= 3x + y, \\ \eta &= y. \end{aligned} \tag{14.109}$$

Applying (14.109) to Equation (14.108), we see that this equation transforms into

$$9\frac{\partial^2 u}{\partial \eta^2} + 9\frac{\partial u}{\partial \xi} + 3\frac{\partial u}{\partial \eta} - u = 0.$$

Dividing by 9 and transposing terms, we obtain the canonical form

$$\frac{\partial^2 u}{\partial \eta^2} = -\frac{\partial u}{\partial \xi} - \frac{1}{3}\frac{\partial u}{\partial \eta} + \frac{1}{9}u.$$

D. The Elliptic Equation

Finally, we prove the following theorem concerning the canonical form in the elliptic case.

THEOREM 14.5

Hypothesis. *Consider the second-order linear partial differential equation*

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0, \tag{14.84}$$

where the coefficients A, B, C, D, E, and F are real constants and $B^2 - 4AC < 0$ so that the equation is elliptic.

Conclusion. *There exists a transformation*

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y) \tag{14.85}$$

of the independent variables in (14.84) so that the transformed equation in the independent variables (ξ, η) *may be written in the canonical form*

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = d\frac{\partial u}{\partial \xi} + e\frac{\partial u}{\partial \eta} + fu, \tag{14.110}$$

where d, e, and f are real constants.

Such a transformation is given by

$$\begin{aligned} \xi &= ax + y, \\ \eta &= bx, \end{aligned} \tag{14.111}$$

where $a \pm bi$ *(a and b real,* $b \neq 0$*) are the conjugate complex roots of the quadratic equation*

$$A\lambda^2 + B\lambda + C = 0 \tag{14.112}$$

Proof. Since $B^2 - 4AC < 0$, we cannot have $A = 0$ in the elliptic case. Thus Equation (14.112) is a full-fledged quadratic equation with two roots. The condition $B^2 - 4AC < 0$ further shows that these two roots must indeed be conjugate complex.

We apply the transformation (14.111) to obtain the transformed equation (14.92) with coefficients (14.93), where in this case $\xi(x, y)$ and $\eta(x, y)$ are given by (14.111). Evaluating these coefficients we find that

$$\begin{aligned} A_1 &= Aa^2 + Ba + C, \\ B_1 &= 2Aab + Bb = b(2Aa + B) \\ C_1 &= Ab^2 \neq 0 \qquad (\text{since } A \neq 0, \quad b \neq 0) \\ D_1 &= Da + E, \qquad E_1 = Db, \qquad F_1 = F. \end{aligned}$$

Since $a + bi$ satisfies the quadratic equation (14.112), we have

$$A(a + bi)^2 + B(a + bi) + C = 0$$

or

$$[A(a^2 - b^2) + Ba + C] + [b(2Aa + B)]i = 0.$$

Therefore

$$A(a^2 - b^2) + Ba + C = 0$$

and

$$b(2Aa + B) = 0.$$

Thus

$$A_1 = Aa^2 + Ba + C = Ab^2$$

and

$$B_1 = 0.$$

Hence the transformed equation (14.92) is

$$Ab^2\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) + (Da + E)\frac{\partial u}{\partial \xi} + Db\frac{\partial u}{\partial \eta} + Fu = 0. \tag{14.113}$$

Since a and b are real, all coefficients in (14.113) are real; and since $Ab^2 \neq 0$, we may write Equation (14.113) in the form

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = d\frac{\partial u}{\partial \xi} + e\frac{\partial u}{\partial \eta} + fu,$$

where the coefficients

$$d = -\frac{Da + E}{Ab^2}, \qquad e = -\frac{D}{Ab}, \qquad f = -\frac{F}{Ab^2}$$

are all real. This is the canonical form (14.110). *Q.E.D.*

► Example 14.8

Consider the equation

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\, \partial y} + 5\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} - 3u = 0. \tag{14.114}$$

We observe that $B^2 - 4AC = -16 < 0$ and so Equation (14.114) is elliptic. We consider the transformation

$$\begin{aligned} \xi &= ax + y, \\ \eta &= bx, \end{aligned} \tag{14.111}$$

where $a \pm bi$ are the conjugate complex roots of the quadratic equation $\lambda^2 + 2\lambda + 5 = 0$. We find that these roots are $-1 \pm 2i$, and so the transformation (14.111) is

$$\begin{aligned} \xi &= -x + y, \\ \eta &= 2x. \end{aligned} \tag{14.115}$$

Applying (14.115) to Equation (14.114), we see that this equation transforms into

$$4\frac{\partial^2 u}{\partial \xi^2} + 4\frac{\partial^2 u}{\partial \eta^2} - 3\frac{\partial u}{\partial \xi} + 2\frac{\partial u}{\partial \eta} - 3u = 0.$$

Dividing by 4 and transposing terms, we obtain the canonical form

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = \frac{3}{4}\frac{\partial u}{\partial \xi} - \frac{1}{2}\frac{\partial u}{\partial \eta} + \frac{3}{4}u.$$

E. Summary

Summarizing, we list in Table 14.1 the canonical forms which we have obtained for the second-order linear partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\, \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0, \tag{14.84}$$

where A, B, C, D, E, and F are real constants.

TABLE 14.1

Type of equation (14.84)	*Canonical form (where d, e, and f are real constants)*
hyperbolic: $B^2 - 4AC > 0$	$\dfrac{\partial^2 u}{\partial\xi\partial\eta} = d\dfrac{\partial u}{\partial\xi} + e\dfrac{\partial u}{\partial\eta} + fu$
parabolic: $B^2 - 4AC = 0$	$\dfrac{\partial^2 u}{\partial\eta^2} = d\dfrac{\partial u}{\partial\xi} + e\dfrac{\partial u}{\partial\eta} + fu$
elliptic: $B^2 - 4AC < 0$	$\dfrac{\partial^2 u}{\partial\xi^2} + \dfrac{\partial^2 u}{\partial\eta^2} = d\dfrac{\partial u}{\partial\xi} + e\dfrac{\partial u}{\partial\eta} + fu$

Exercises

Transform each of the partial differential equations in Exercises 1–10 into canonical form.

1. $\dfrac{\partial^2 u}{\partial x^2} - 5\dfrac{\partial^2 u}{\partial x\,\partial y} + 6\dfrac{\partial^2 u}{\partial y^2} = 0.$

2. $\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial x\,\partial y} + 4\dfrac{\partial^2 u}{\partial y^2} = 0.$

3. $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x\,\partial y} - 8\dfrac{\partial^2 u}{\partial y^2} + 9\dfrac{\partial u}{\partial x} = 0.$

4. $2\dfrac{\partial^2 u}{\partial x^2} + 3\dfrac{\partial^2 u}{\partial x\,\partial y} - 9\dfrac{\partial^2 u}{\partial y^2} + 4\dfrac{\partial u}{\partial x} = 0.$

5. $\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial x\,\partial y} + 13\dfrac{\partial^2 u}{\partial y^2} - 9\dfrac{\partial u}{\partial y} = 0.$

6. $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} + 3\dfrac{\partial u}{\partial x} + 9u = 0.$

7. $6\dfrac{\partial^2 u}{\partial x\,\partial y} + 3\dfrac{\partial^2 u}{\partial y^2} + 2\dfrac{\partial u}{\partial x} = 0.$

8. $\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x\,\partial y} + 3\dfrac{\partial u}{\partial x} + 5\dfrac{\partial u}{\partial y} = 0.$

9. $2\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x\,\partial y} + 5\dfrac{\partial^2 u}{\partial y^2} + u = 0.$

10. $\dfrac{\partial^2 u}{\partial x^2} - 5\dfrac{\partial^2 u}{\partial x\,\partial y} + 5\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y} + 3u = 0.$

11. Show that the transformation

$$\xi = y - \frac{x^2}{2},$$

$$\eta = x,$$

reduces the equation

$$\frac{\partial^2 u}{\partial x^2} + 2x \frac{\partial^2 u}{\partial x \, \partial y} + x^2 \frac{\partial^2 u}{\partial y^2} = 0$$

to

$$\frac{\partial^2 u}{\partial \eta^2} = \frac{\partial u}{\partial \xi}.$$

12. Consider the equation

$$\frac{\partial^2 u}{\partial x^2} + (2x + 3) \frac{\partial^2 u}{\partial x \, \partial y} + 6x \frac{\partial^2 u}{\partial y^2} = 0. \tag{A}$$

(a) Show that for $x = \frac{3}{2}$, Equation (A) reduces to a parabolic equation, and reduce this parabolic equation to canonical form.

(b) Show that for $x \neq \frac{3}{2}$, the transformation

$$\xi = y - 3x,$$
$$\eta = y - x^2$$

reduces Equation (A) to

$$\frac{\partial^2 u}{\partial \xi \, \partial \eta} = \frac{2 \dfrac{\partial u}{\partial \eta}}{4(\eta - \xi) - 9}.$$

14.4 AN INITIAL-VALUE PROBLEM; CHARACTERISTICS

A. An Initial-Value Problem

We shall now consider an initial-value problem for the second-order linear partial differential equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \, \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = 0, \tag{14.84}$$

where A, B, C, D, E, and F are real constants. In Chapter 4 we considered an initial-value problem for linear *ordinary* differential equations. Let us begin by recalling this problem for the second-order homogeneous linear ordinary differential equation with constant coefficients,

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0. \tag{14.116}$$

The problem is to find a solution f of (14.116) such that $f(x_0) = c_0$ and $f'(x_0) = c_1$, where x_0 is some definite real number and c_0 and c_1 are arbitrary real constants. Interpreting this geometrically, the problem is to find a solution curve $y = f(x)$ of (14.116) which passes through the point (x_0, c_0) and whose tangent line has the slope c_1 at this point.

Let us now attempt to formulate an analogous initial-value problem for the *partial* differential equation (14.84). In doing this we shall let ourselves be guided by the geometric interpretation of the initial-value problem for the ordinary differential equation (14.116). In this problem we must find a solution of Equation (14.116) which satisfies certain supplementary conditions (the initial conditions). A solution of Equation (14.116) is a function f of the one real variable x, and this defines a curve $y = f(x)$ (a solution curve) in the xy plane. In the analogous problem for the partial differential equation (14.84), we shall also seek a solution which satisfies certain supplementary conditions. But a solution of Equation (14.84) is a function ϕ of the two real variables x and y, and this defines a surface $u = \phi(x, y)$ (a solution surface) in three-dimensional x, y, u space. Geometrically speaking, then, for the partial differential equation (14.84) we must find a particular solution surface $u = \phi(x, y)$ in x, y, u space. Now in the initial-value problem for the ordinary differential equation (14.116), there are two supplementary requirements which the solution curve $y = f(x)$ in the xy plane must satisfy. First, the solution curve $y = f(x)$ must pass through a prescribed point (x_0, c_0) in the xy plane; and, second, this solution curve must be such that its tangent line has a prescribed slope c_1 at this point. What sort of analogous requirements might be imposed in an initial-value problem for the partial differential equation (14.84)? Instead of prescribing a point in the xy plane through which the solution curve must pass, we would prescribe a curve in x, y, u space through which the solution surface must pass. Instead of prescribing a slope for the tangent line to the solution curve at the prescribed point, we would prescribe a normal direction for the tangent plane to the solution surface along the prescribed curve. We thus formulate an initial-value problem for the partial differential equation (14.84) in the following geometric language.

We seek a solution surface $u = \phi(x, y)$ of the partial differential equation (14.84) which (a) passes through a prescribed curve Γ (the initial curve) in x, y, u space, and (b) is such that its tangent plane has a prescribed normal direction at all points of the initial curve Γ.

Let us now proceed to formulate this initial-value problem analytically. To do so, let us assume that the prescribed curve Γ has the parametric representation

$$x = x(t), \qquad y = y(t), \qquad u = u(t)$$

for all t on some real interval I. Now let $t_0 \in I$ and consider the corresponding point on Γ. This point has coordinates (x_0, y_0, u_0), where $x_0 = x(t_0)$, $y_0 = y(t_0)$, $u_0 = u(t_0)$. If the solution surface $u = \phi(x, y)$ is to pass through Γ at (x_0, y_0, u_0), then we must have $u_0 = \phi(x_0, y_0)$ or, equivalently, $u(t_0) = \phi[x(t_0), y(t_0)]$. Therefore the requirement that the solution surface $u = \phi(x, y)$ pass through the entire curve Γ is expressed by the condition

$$u(t) = \phi[x(t), y(t)] \tag{14.117}$$

for all $t \in I$.

We now consider the requirement that the tangent plane to the solution surface $u = \phi(x, y)$ have a prescribed normal direction at all points of Γ. Let us assume that this prescribed normal direction is given by $[p(t), q(t), -1]$ for all $t \in I$. Recalling that the normal direction to the tangent plane to $u = \phi(x, y)$ is given by

$$[\phi_x(x, y), \phi_y(x, y), -1],*$$

---

* Here and throughout the remainder of the chapter it is convenient to employ subscript notation for the various partial derivatives of ϕ. Thus we denote

$$\frac{\partial \phi}{\partial x} \text{ by } \phi_x, \quad \frac{\partial \phi}{\partial y} \text{ by } \phi_y, \quad \frac{\partial^2 \phi}{\partial x^2} \text{ by } \phi_{xx},$$

$$\frac{\partial^2 \phi}{\partial y \partial x} \text{ by } \phi_{xy}, \text{ and } \frac{\partial^2 \phi}{\partial y^2} \text{ by } \phi_{yy}.$$

we see that the requirement under consideration is expressed by the conditions that

$$\begin{aligned} p(t) &= \phi_x[x(t), y(t)], \\ q(t) &= \phi_y[x(t), y(t)], \end{aligned} \tag{14.118}$$

for all $t \in I$.

We have thus far said nothing concerning the nature of the functions x, y, u which define Γ or the functions p, q which prescribe the normal direction. Let us assume that each of these five functions is analytic for all $t \in I$. We must now observe that the functions p and q cannot be chosen arbitrarily. For, from (14.117) we must have

$$\frac{du(t)}{dt} = \phi_x[x(t), y(t)]\frac{dx(t)}{dt} + \phi_y[x(t), y(t)]\frac{dy(t)}{dt}$$

for all $t \in I$; and since (14.118) must hold for all $t \in I$, this reduces to

$$\frac{du(t)}{dt} = p(t)\frac{dx(t)}{dt} + q(t)\frac{dy(t)}{dt} \tag{14.119}$$

for all $t \in I$. Thus the functions p and q are not arbitrary; rather they must satisfy the identity (14.119).

In our attempt to formulate the initial-value problem analytically, we have thus introduced a set of five functions defined by $\{x(t), y(t), u(t), p(t), q(t)\}$, where (1) $x = x(t)$, $y = y(t)$, $u = u(t)$ is the analytic representation of a prescribed curve Γ in x, y, u space, (2) $[p(t), q(t), -1]$ is the analytic representation of a prescribed normal direction along Γ, and (3) these five functions satisfy the identity (14.119). Such a set of five functions is called a *strip*, and the identity (14.119) is called the *strip condition.*

We may now state the initial-value problem in the following way.

Initial-Value Problem. Consider the second-order linear partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0, \tag{14.84}$$

where A, B, C, D, E, and F are real constants. Let $x(t)$, $y(t)$, $u(t)$, $p(t)$, and $q(t)$ denote five real functions, each of which is analytic for all t on a real interval I. Further, let these five functions be such that the condition

$$\frac{du(t)}{dt} = p(t)\frac{dx(t)}{dt} + q(t)\frac{dy(t)}{dt} \tag{14.119}$$

holds for all $t \in I$. Let R be a region of the xy plane such that $[x(t), y(t)] \in R$ for all $t \in I$.

We seek a solution $\phi(x, y)$ for the partial differential equation (14.84), defined for all $(x, y) \in R$, such that

$$\phi[x(t), y(t)] = u(t), \tag{14.117}$$

and

$$\begin{aligned} \phi_x[x(t), y(t)] &= p(t), \\ \phi_y[x(t), y(t)] &= q(t), \end{aligned} \tag{14.118}$$

for all $t \in I$.

► **Example 14.9**

Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x\, \partial y} - 2\frac{\partial^2 u}{\partial y^2} = 0 \tag{14.120}$$

and the five real functions defined by

$$x(t) = t, \quad y(t) = t^2, \quad u(t) = t^4 + 2t^2, \quad p(t) = 4t, \quad q(t) = 2t^2 \tag{14.121}$$

for all real t. We observe that these five functions satisfy the strip condition (14.119); indeed we have

$$4t^3 + 4t = (4t)(1) + (2t^2)(2t)$$

for all real t. We may therefore interpret the set of five functions defined by (14.121) as a strip and state the following initial-value problem.

We seek a solution $\phi(x, y)$ of the partial differential equation (14.120), defined for all (x, y), such that

$$\begin{aligned} \phi(t, t^2) &= t^4 + 2t^2, \\ \phi_x(t, t^2) &= 4t, \\ \phi_y(t, t^2) &= 2t^2, \end{aligned} \tag{14.122}$$

for all real t.

Geometrically, we seek a solution surface $u = \phi(x, y)$ of the partial differential equation (14.120) which (a) passes through the space curve Γ defined by

$$x = t, \qquad y = t^2, \qquad u = t^4 + 2t^2$$

for all real t, and (b) is such that its tangent plane has the normal direction $(4t, 2t^2, -1)$ at all points of Γ.

We now observe that a solution of this problem is the function ϕ defined by

$$\phi(x, y) = 2x^2 + y^2$$

for all (x, y). In the first place, this function ϕ satisfies the partial differential equation (14.120). For we find that

$$\phi_{xx}(x, y) = 4, \qquad \phi_{yx}(x, y) = 0, \quad \text{and} \quad \phi_{yy}(x, y) = 2,$$

so that

$$\phi_{xx}(x, y) + \phi_{yx}(x, y) - 2\phi_{yy}(x, y) = 0$$

for all (x, y). In the second place, the conditions (14.122) are satisfied. Clearly $\phi(t, t^2) = 2t^2 + t^4$ for all real t. From $\phi_x(x, y) = 4x$, we see that $\phi_x(t, t^2) = 4t$ for all real t; and from $\phi_y(x, y) = 2y$, we see that $\phi_y(t, t^2) = 2t^2$ for all real t.

We shall now examine the initial-value problem from a slightly different viewpoint, again employing useful geometric terminology in our discussion. Suppose, as before, that the curve Γ through which the solution surface $u = \phi(x, y)$ must pass is defined by

$$x = x(t), \qquad y = y(t), \qquad u = u(t) \tag{14.123}$$

for all t on some real interval I. Then the projection of Γ on the xy plane is the curve Γ_0

defined by

$$x = x(t), \qquad y = y(t) \tag{14.124}$$

for all $t \in I$. Employing this curve Γ_0, we may express the initial-value problem in the following form.

We seek a solution $\phi(x, y)$ of the partial differential equation (14.84) such that ϕ, ϕ_x, and ϕ_y assume prescribed values given by $u(t)$, $p(t)$, and $q(t)$, respectively, for each $t \in I$, or, in other words, at each point of Γ_0 [where we assume that the five functions given by $x(t)$, $y(t)$, $u(t)$, $p(t)$, and $q(t)$ satisfy the strip condition (14.119)].

We shall find that this interpretation of the initial-value problem will be useful in the discussion which follows.

B. Characteristics

Continuing our discussion of the initial-value problem, let us now assume that the problem has a unique solution $\phi(x, y)$ defined on a region R (of the xy plane) which includes the curve Γ_0. Let us also assume that at each point of Γ_0 the solution $\phi(x, y)$ has a power series expansion which is valid in some circle about this point. Now let $t_0 \in I$ and let $x_0 = x(t_0)$, $y_0 = y(t_0)$, so that (x_0, y_0) is a point of Γ_0. Then for all (x, y) in some circle K about (x_0, y_0) we have the valid power series expansion

$$\begin{aligned}\phi(x, y) &= \phi(x_0, y_0) + [\phi_x(x_0, y_0)(x - x_0) + \phi_y(x_0, y_0)(y - y_0)] \\ &\quad + \frac{1}{2!}[\phi_{xx}(x_0, y_0)(x - x_0)^2 + 2\phi_{xy}(x_0, y_0)(x - x_0)(y - y_0) \\ &\quad + \phi_{yy}(x_0, y_0)(y - y_0)^2] + \cdots.\end{aligned} \tag{14.125}$$

The solution $\phi(x, y)$ will be determined in the circle K if we can determine the coefficients

$$\phi(x_0, y_0), \qquad \phi_x(x_0, y_0), \quad \phi_y(x_0, y_0), \quad \phi_{xx}(x_0, y_0), \ldots$$

in the expansion (14.125).

Now we already know the first three of these coefficients. For the initial-value problem requires that ϕ, ϕ_x, and ϕ_y assume prescribed values given by $u(t)$, $p(t)$, and $q(t)$, respectively, at each point of Γ_0. Therefore at the point (x_0, y_0) at which $t = t_0$, we have

$$\phi(x_0, y_0) = u(t_0),$$

$$\phi_x(x_0, y_0) = p(t_0),$$

and

$$\phi_y(x_0, y_0) = q(t_0).$$

Let us now attempt to calculate the next three coefficients in (14.125). That is, let us see if we can determine

$$\phi_{xx}(x_0, y_0), \quad \phi_{xy}(x_0, y_0), \quad \phi_{yy}(x_0, y_0). \tag{14.126}$$

To do this, we need to know conditions which these numbers must satisfy. One such condition is readily available: it can be obtained at once from the partial differential equation (14.84)! For, since $\phi(x, y)$ is a solution of (14.84) in the region R, we have

$$A\phi_{xx}(x, y) + B\phi_{xy}(x, y) + C\phi_{yy}(x, y) = -D\phi_x(x, y) - E\phi_y(x, y) - F\phi(x, y)$$

for all $(x, y) \in R$. Thus, since $(x_0, y_0) \in R$, we have the condition

$$\begin{aligned} &A\phi_{xx}(x_0, y_0) + B\phi_{xy}(x_0, y_0) + C\phi_{yy}(x_0, y_0) \\ &\qquad = -D\phi_x(x_0, y_0) - E\phi_y(x_0, y_0) - F\phi(x_0, y_0). \end{aligned} \tag{14.127}$$

Now along the curve Γ_0 we know that

$$\phi_x[x(t), y(t)] = p(t)$$

and

$$\phi_y[x(t), y(t)] = q(t).$$

Differentiating these identities with respect to t, we find that along the curve Γ_0 we have

$$\phi_{xx}[x(t), y(t)]\frac{dx}{dt} + \phi_{xy}[x(t), y(t)]\frac{dy}{dt} = \frac{dp}{dt},$$

$$\phi_{xy}[x(t), y(t)]\frac{dx}{dt} + \phi_{yy}[x(t), y(t)]\frac{dy}{dt} = \frac{dq}{dt}.$$

Since (x_0, y_0) is a point of Γ_0, we thus obtain the two additional conditions

$$x'(t_0)\phi_{xx}(x_0, y_0) + y'(t_0)\phi_{xy}(x_0, y_0) = p'(t_0) \tag{14.128}$$

and

$$x'(t_0)\phi_{xy}(x_0, y_0) + y'(t_0)\phi_{yy}(x_0, y_0) = q'(t_0), \tag{14.129}$$

where $x'(t_0)$, $y'(t_0)$, $p'(t_0)$, and $q'(t_0)$ denote the values of $\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dp}{dt}$, and $\dfrac{dq}{dt}$, respectively, at $t = t_0$.

Let us examine more closely the three conditions (14.127), (14.128), and (14.129). They are actually a system of three linear algebraic equations in the three unknown numbers (14.126); all other quantities involved are known numbers! A necessary and sufficient condition that this system have a unique solution is that the determinant of coefficients be unequal to zero (Section 7.5C, Theorem B). Therefore in order to have a unique solution for the three coefficients (14.126), we must have

$$\Delta(t_0) = \begin{vmatrix} A & B & C \\ x'(t_0) & y'(t_0) & 0 \\ 0 & x'(t_0) & y'(t_0) \end{vmatrix} \neq 0.$$

Conversely, if $\Delta(t_0) \neq 0$, we can determine the three coefficients (14.126) from the linear system consisting of Equations (14.127), (14.128), and (14.129).

We have assumed that the solution $\phi(x, y)$ has a power series expansion of the form (14.125) at *each* point of the curve Γ_0. Thus if the determinant

$$\Delta(t) = \begin{vmatrix} A & B & C \\ \dfrac{dx}{dt} & \dfrac{dy}{dt} & 0 \\ 0 & \dfrac{dx}{dt} & \dfrac{dy}{dt} \end{vmatrix} \neq 0$$

at all points of Γ_0, we can determine the values of ϕ_{xx}, ϕ_{xy}, and ϕ_{yy} all along Γ_0. We can then attempt to find the values of the higher derivatives of $\phi(x, y)$ along Γ_0, employing a procedure similar to that by which we found the values of the three second derivatives. One can show that if $\Delta(t) \neq 0$ at all points of Γ_0, then the values of these higher derivatives can be determined all along Γ_0. In this manner we find the coefficients in the series expansion of $\phi(x, y)$ along Γ_0.

It can be shown that if $\Delta(t) \neq 0$ at all points of Γ_0, then there exists a unique solution of the initial-value problem. On the other hand, if $\Delta(t) = 0$ at all points of Γ_0, then there is either no solution or infinitely many solutions.

Let us now consider a curve Γ_0 at all points of which

$$\Delta(t) = \begin{vmatrix} A & B & C \\ \dfrac{dx}{dt} & \dfrac{dy}{dt} & 0 \\ 0 & \dfrac{dx}{dt} & \dfrac{dy}{dt} \end{vmatrix} = 0.$$

Expanding this determinant, we thus see that

$$A\left(\frac{dy}{dt}\right)^2 - B\frac{dx}{dt}\frac{dy}{dt} + C\left(\frac{dx}{dt}\right)^2 = 0$$

at all points of Γ_0. Therefore Γ_0 must be a curve having an equation $\zeta(x, y) = c_0$ which satisfies

$$A\,dy^2 - B\,dx\,dy + C\,dx^2 = 0. \tag{14.130}$$

We thus investigate the solutions of Equation (14.130). We shall divide this discussion into the following three subcases: (a) $A \neq 0$; (b) $A = 0, C \neq 0$; and (c) $A = 0$, $C = 0$.

Subcase (a): $A \neq 0$. In this case if follows from (14.130) that the curve Γ_0 is defined by $y = \phi(x)$, where $\phi(x)$ satisfies the ordinary differential equation

$$A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0.$$

From this equation we see that

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} \quad \text{or} \quad \frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A}.$$

Thus if $A \neq 0$ the curve Γ_0 must be a member of one of the families of straight lines defined by

$$y = \left[\frac{B + \sqrt{B^2 - 4AC}}{2A}\right]x + c_1 \quad \text{or} \quad y = \left[\frac{B - \sqrt{B^2 - 4AC}}{2A}\right]x + c_2. \tag{14.131}$$

where c_1 and c_2 are arbitrary constants.

Subcase (b): $A = 0, C \neq 0$. In this case it follows from (14.130) that Γ_0 is defined by $x = \theta(y)$, where $\theta(y)$ satisfies the ordinary differential equation

$$C\left(\frac{dx}{dy}\right)^2 - B\frac{dx}{dy} = 0.$$

From this equation we see that

$$\frac{dx}{dy} = \frac{B}{C} \quad \text{or} \quad \frac{dx}{dy} = 0.$$

Thus if $A = 0$ and $C \neq 0$, the curve Γ_0 must be a member of one of the families of straight lines defined by

$$x = \frac{B}{C} y + c_1 \quad \text{or} \quad x = c_2, \tag{14.132}$$

where c_1 and c_2 are arbitrary constants.

Subcase (c): $A = 0$, $C = 0$. In this case it follows at once from (14.130) that the curve Γ_0 must be a member of one of the families of straight lines defined by

$$y = c_1 \quad \text{or} \quad x = c_2, \tag{14.133}$$

where c_1 and c_2 are arbitrary constants.

Thus if Γ_0 is a curve at all points of which $\Delta(t) = 0$, then Γ_0 must be one of the straight lines defined by (14.131), (14.132), or (14.133). Any such curve Γ_0 in the xy plane along which $\Delta(t) = 0$ is called a *characteristic* (or *characteristic base curve*) of the partial differential equation (14.84). In other words, a characteristic of (14.84) is any curve Γ_0 in the xy plane having an equation $\zeta(x, y) = c_0$ which satisfies

$$A\,dy^2 - B\,dx\,dy + C\,dx^2 = 0. \tag{14.130}$$

We have shown that the characteristics of the partial differential equation (14.84) are straight lines. Specifically, if $A \neq 0$, they are the straight lines defined by (14.131); if $A = 0$ and $C \neq 0$, they are those defined by (14.132); and if $A = 0$ and $C = 0$, they are given by (14.133).

Let us consider the connection between the characteristics of the partial differential equation (14.84) and the initial-value problem associated with this equation. Recall that in this problem we seek a solution $\phi(x, y)$ of (14.84) such that ϕ, ϕ_x, and ϕ_y assume prescribed values given by $u(t)$, $p(t)$, and $q(t)$, respectively, at each point of a curve Γ_0 (in the xy plane) defined by $x = x(t)$, $y = y(t)$ for all t on some real interval I.

Further recall that if

$$\Delta(t) = \begin{vmatrix} A & B & C \\ \dfrac{dx}{dt} & \dfrac{dy}{dt} & 0 \\ 0 & \dfrac{dx}{dt} & \dfrac{dy}{dt} \end{vmatrix}$$

is unequal to zero at all points of Γ_0, then this initial-value problem has a unique solution; but if $\Delta(t) = 0$ at all points of Γ_0, then there is either no solution or infinitely many solutions. Now we have defined a characteristic of (14.84) as a curve in the xy plane along which $\Delta(t) = 0$. Thus we may say that if the curve Γ_0 is nowhere tangent to a characteristic, then the initial-value problem has a unique solution; but if Γ_0 is a characteristic, then the problem has either no solution or infinitely many solutions.

We now examine the characteristics of Equation (14.84) in the hyperbolic, parabolic and elliptic cases. We shall find it convenient to consider once again the three subcases (a) $A \neq 0$, (b) $A = 0$, $C \neq 0$, and (c) $A = 0$, $C = 0$.