Decision Theory

BY

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Which Attribute is ”best”?

- We would like to select the attribute that is most useful for classifying examples.
- • ***Information gain*** measures how well a given attribute separates the training examples according to their target classification.
- • ID3 uses this *information gain* measure to select among the candidate attributes at each step while growing the tree.
- • In order to define information gain precisely, we use a measure commonly used in information theory, called ***entropy***
- • ***Entropy*** characterizes the (im)purity of an arbitrary collection of examples.

Information Theory –ID3 (Iterative Dichotomiser 3)

- ID3 algorithm invented by Ross Quinlan and uses information gain as its attribute selection measure
- This measure is based on pioneering work by Claude Shannon on information theory, which studied the value or "information content" of messages
- Let node N represent or hold the tuples of partition D. The attribute with the highest information gain is chosen as the splitting attribute for node N
- This attribute minimizes the information needed to classify the tuples in the resulting partitions and reflects the least randomness or "impurity" in these partitions
- The expected information needed to classify a tuple in D is given by

$$Info(D) = -\sum_{i=1}^{m} p_i \log_2(p_i),$$

- Let D, the data partition, be a training set of class-labeled tuples. Suppose the class label attribute has m distinct values defining m distinct classes, C_i (here I = 1 to m); $p_i = s_i/s$; s= no. of samples; s_i = no. of samples in class label $C_{i\,;}$ *Info*(*D*) is also known as the **entropy** of *D*

ID3--Continued

- suppose we were to partition the tuples in *D* on some attribute *A* having *v* distinct values, $[a_1, a_2, \dots, a_v]$, as observed from the training data. If *A* is discrete-valued, these values correspond directly to the *v* outcomes of a test on *A*. Attribute *A* can be used to split *D* into *v* partitions or subsets, $[D_1, D_2, \dots, D_v]$, where D_j contains those tuples in *D* that have outcome a_j of *A*

$$Info_A(D) = \sum_{j=1}^{v} \frac{|D_j|}{|D|} \times Info(D_j).$$

- Here, $|D_j| / |D|$ = acts as the weight of the *j*thpartition; $Info_A(D)$ is the expected information required to classify a tuple from *D* based on the partitioning by *A*.
- $Info(D_j) = -\sum_{i=1}^{m} p_{ij} \log_2(p_{ij})$; $p_{ij} = s_{ij} / |D_j|$; s_{ij} = no. of samples belongs to class label C_i and having the attribute value a_j

ID3--Continued

- Information gain is defined as the difference between the original information requirement (i.e., based on just the proportion of classes) and the new requirement (i.e., obtained after partitioning on *A*).

$$Gain(A) = Info(D) - Info_A(D).$$

- In other words, *Gain*(*A*) tells us how much would be gained by branching on *A*. It is the expected reduction in the information requirement caused by knowing the value of *A*. The attribute *A* with the highest information gain, *Gain(A*), is chosen as the splitting attribute at node *N.*

Problem statement: Find out Test Attribute

RID	*age*	*income*	*student*	*credit_rating*	*Class: buys_computer*
1	youth	high	no	fair	no
2	youth	high	no	excellent	no
3	middle_aged	high	no	fair	yes
4	senior	medium	no	fair	yes
5	senior	low	yes	fair	yes
6	senior	low	yes	excellent	no
7	middle_aged	low	yes	excellent	yes
8	youth	medium	no	fair	no
9	youth	low	yes	fair	yes
10	senior	medium	yes	fair	yes
11	youth	medium	yes	excellent	yes
12	middle_aged	medium	no	excellent	yes
13	middle_aged	high	yes	fair	yes
14	senior	medium	no	excellent	no

Solution:

- Class P: *buys_computer* = “yes”
- Class N: *buys_computer* = “no”

$$Entropy(D) = -\frac{9}{14}\log_2(\frac{9}{14}) - \frac{5}{14}\log_2(\frac{5}{14}) = 0.940$$

- Compute the expected information requirement for each attribute: start with the attribute *age*

$$Gain(age, D)$$

$$= Entropy(D) - \sum_{v \in \{Youth, Middle-aged, Senior\}} \frac{|S_v|}{|S|} Entropy(S_v)$$

$$= Entropy(D) - \frac{5}{14} Entropy(S_{youth}) - \frac{4}{14} Entropy(S_{middle\_aged}) - \frac{5}{14} Entropy(S_{senior})$$

$$= 0.246$$

$$Gain\ (income, D) = 0.029$$

$$Gain\ (student, D) = 0.151$$

$$Gain\ (credit\_rating, D) = 0.048$$

$Entropy(S_{youth}) = -\sum_{i=1}^{2} p_{i1} \log_2(p_{i1})$

$= - p_{11}\log_2(p_{11}) - p_{21}\log_2(p_{21})$

$= -2/5 \log_2(2/5) – 3/5 \log_2(3/5)$

$= 0.971$

Here, $p_{11} = s_{11}/|D_1| = 2/5$

$p_{21} = s_{21}/|D_1| = 3/5$

log2 X = log10 X / log10 2

$Entropy(S_{middle}) = -\sum_{i=1}^{2} p_{i2} \log_2(p_{i2})$

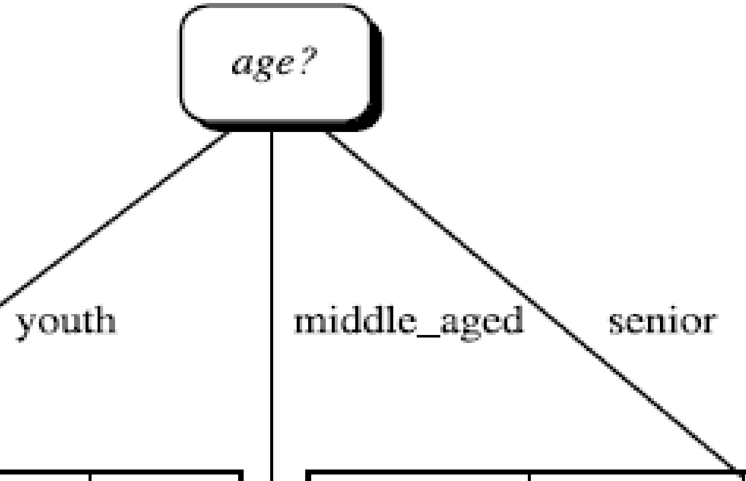
$= - p_{12}\log_2(p_{12}) - p_{22}\log_2(p_{22})$

$= -4/4 \log_2(4/4) – 0/4 \log_2(0/4)$

$= 0$

Here, $p_{12} = s_{12}/|D_2| = 4/4$

$p_{22} = s_{22}/|D_2| = 0/4$

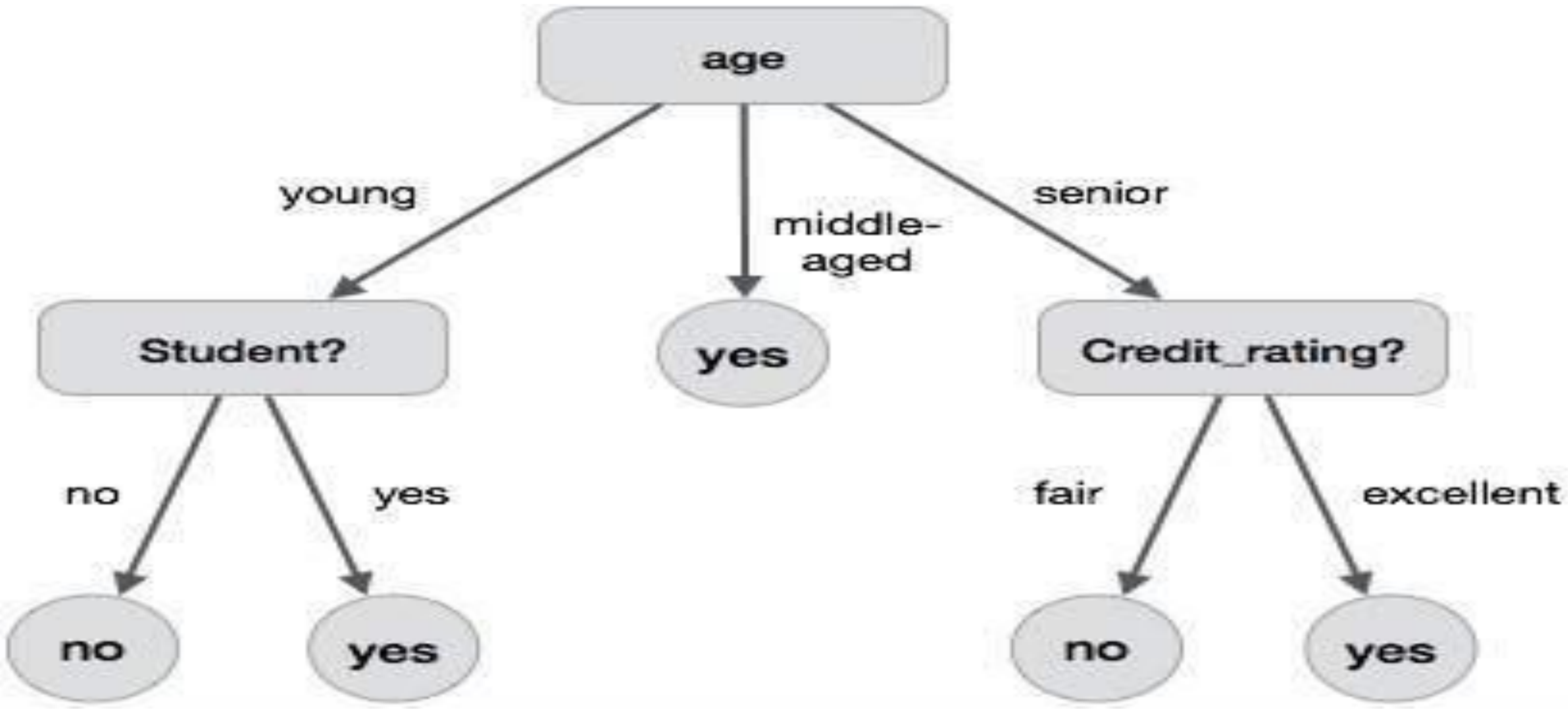


income	*student*	*credit_rating*	*class*
high	no	fair	no
high	no	excellent	no
medium	no	fair	no
low	yes	fair	yes
medium	yes	excellent	yes

income	*student*	*credit_rating*	*class*
high	no	fair	yes
low	yes	excellent	yes
medium	no	excellent	yes
high	yes	fair	yes

income	*student*	*credit_rating*	*class*
medium	no	fair	yes
low	yes	fair	yes
low	yes	excellent	no
medium	yes	fair	yes
medium	no	excellent	no

Decision Tree



X = (age = youth, income = medium, student = yes, credit = fair)   Class label=?

Extracting Rules from Decision Tree

R1: IF *age* = *youth* AND *student* = *no* THEN *buys_computer* = *no*

R2: IF *age* = *youth* AND *student* = *yes* THEN *buys_computer* = *yes*

R3: IF *age* = *middle_aged* THEN *buys_computer* = *yes*

R4: IF *age* = *senior* AND *credit_rating* = *excellent* THEN *buys_computer* = *no*

R5: IF *age* = *senior* AND *credit_rating* = *fair* THEN *buys_computer* = *yes*

Assignment:

Outlook	Temperature	Humidity	Windy	Play
Sunny	Hot	High	False	***No***
Sunny	Hot	High	True	***No***
Overcast	Hot	High	False	***Yes***
Rainy	Mild	High	False	***Yes***
Rainy	Cool	Normal	False	***Yes***
Rainy	Cool	Normal	True	***No***
Overcast	Cool	Normal	True	***Yes***
Sunny	Mild	High	False	***No***
Sunny	Cool	Normal	False	***Yes***
Rainy	Mild	Normal	False	***Yes***
Sunny	Mild	Normal	True	***Yes***
Overcast	Mild	High	True	***Yes***
Overcast	Hot	Normal	False	***Yes***
Rainy	Mild	High	True	***No***

Decision Theory- Naïve Bayes

Supervised Learning

Naïve Bayesian Classifier

According to Bayes' theorem, the probability that we want to compute $P(H|\mathbf{X})$ can be expressed in terms of probabilities $P(H)$, $P(\mathbf{X}|H)$, and $P(\mathbf{X})$ as

$$P(H|\mathbf{X}) = \frac{P(\mathbf{X}|H)\ P(H)}{P(\mathbf{X})},$$

and these probabilities may be estimated from the given data.

Naive Bayesian Classifier

The naive Bayesian classifier works as follows:

Let T be a training set of samples, each with their class labels. There are k classes, $C_1, C_2, \ldots, C_k$. Each sample is represented by an n-dimensional vector, $\mathbf{X} = \{x_1, x_2, \ldots, x_n\}$, depicting n measured values of the n attributes, $A_1, A_2, \ldots, A_n$, respectively.

That is $\mathbf{X}$ is predicted to belong to the class C_i if and only if

$$P(C_i|\mathbf{X}) > P(C_j|\mathbf{X}) \qquad \text{for } 1 \leq j \leq m, \ j \neq i.$$

Thus we find the class that maximizes $P(C_i|\mathbf{X})$. The class C_i for which $P(C_i|\mathbf{X})$ is maximized is called the maximum posteriori hypothesis. By Bayes' theorem

$$P(C_i|\mathbf{X}) = \frac{P(\mathbf{X}|C_i)\ P(C_i)}{P(\mathbf{X})}.$$

As $P(\mathbf{X})$ is the same for all classes, only $P(\mathbf{X}|C_i)P(C_i)$ need be maximized. If the class a priori probabilities, $P(C_i)$, are not known, then it is commonly assumed that the classes are equally likely, that is, $P(C_1) = P(C_2) = \ldots = P(C_k)$, and we would therefore maximize $P(\mathbf{X}|C_i)$. Otherwise we maximize $P(\mathbf{X}|C_i)P(C_i)$. Note that the class a priori probabilities may be estimated by $P(C_i) = \text{freq}(C_i, T)/|T|$.

$$P(\mathbf{X}|C_i) \approx \prod_{k=1}^{n} P(x_k|C_i).$$

The probabilities $P(x_1|C_i), P(x_2|C_i), \ldots, P(x_n|C_i)$ can easily be estimated from the training set. Recall that here x_k refers to the value of attribute A_k for sample $\mathbf{X}$.

(a) If A_k is categorical, then $P(x_k|C_i)$ is the number of samples of class C_i in T having the value x_k for attribute A_k, divided by freq(C_i, T), the number of sample of class C_i in T.

(b) If A_k is continuous-valued, then we typically assume that the values have a Gaussian distribution with a mean μ and standard deviation σ defined by

$$g(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} \exp -\frac{(x-\mu)^2}{2\sigma^2},$$

$$p(x_k|C_i) = g(x_k, \mu_{C_i}, \sigma_{C_i}).$$

We need to compute μ_{C_i} and σ_{C_i}, which are the mean and standard deviation of values of attribute A_k for training samples of class C_i.

In order to predict the class label of $\mathbf{X}$, $P(\mathbf{X}|C_i)P(C_i)$ is evaluated for each class C_i. The classifier predicts that the class label of $\mathbf{X}$ is C_i if and only if it is the class that maximizes $P(\mathbf{X}|C_i)P(C_i)$.

Problem Statement

RID	age	income	student	credit	C_i: buy
1	youth	high	no	fair	C_2: no
2	youth	high	no	excellent	C_2: no
3	middle-aged	high	no	fair	C_1: yes
4	senior	medium	no	fair	C_1: yes
5	senior	low	yes	fair	C_1: yes
6	senior	low	yes	excellent	C_2: no
7	middle-aged	low	yes	excellent	C_1: yes
8	youth	medium	no	fair	C_2: no
9	youth	low	yes	fair	C_1: yes
10	senior	medium	yes	fair	C_1: yes
11	youth	medium	yes	excellent	C_1: yes
12	middle-aged	medium	no	excellent	C_1: yes
13	middle-aged	high	yes	fair	C_1: yes
14	senior	medium	no	excellent	C_2: no

X = (age = youth, income = medium, student = yes, credit = fair)   Class label=?

Solution

We need to maximize $P(\mathbf{X}|C_i)P(C_i)$, for $i = 1, 2$. $P(C_i)$, the a priori probability of each class, can be estimated based on the training samples:

$$P(buy = yes) = \frac{9}{14}$$

$$P(buy = no) = \frac{5}{14}$$

To compute $P(\mathbf{X}|C_i)$, for $i = 1, 2$, we compute the following conditional probabilities:

$$P(age = youth|buy = yes) = \frac{2}{9}$$

$$P(age = youth|buy = no) = \frac{3}{5}$$

$$P(income = medium|buy = yes) = \frac{4}{9}$$

$$P(income = medium | buy = no) = \frac{2}{5}$$

$$P(student = yes | buy = yes) = \frac{6}{9}$$

$$P(student = yes | buy = no) = \frac{1}{5}$$

$$P(credit = fair | buy = yes) = \frac{6}{9}$$

$$P(credit = fair | buy = no) = \frac{2}{5}$$

Using the above probabilities, we obtain

$$\begin{aligned} P(\mathbf{X} | buy = yes) &= P(age = youth | buy = yes) \\ &\quad P(income = medium | buy = yes) \\ &\quad P(student = yes | buy = yes) \\ &\quad P(credit = fair | buy = yes) \\ &= \frac{2}{9}\frac{4}{9}\frac{6}{9}\frac{6}{9} = 0.044. \end{aligned}$$

$$P(\mathbf{X}|buy = no) = \frac{3}{5}\frac{2}{5}\frac{1}{5}\frac{2}{5} = 0.019$$

To find the class that maximizes $P(\mathbf{X}|C_i)P(C_i)$, we compute

$$P(\mathbf{X}|buy = yes)P(buy = yes) = 0.028$$

$$P(\mathbf{X}|buy = no)P(buy = no) = 0.007$$

Thus the naive Bayesian classifier predicts $buy = yes$ for sample $\mathbf{X}$.

CART

Gini Index

- Many alternative measures to Information Gain
- Most popular altermative: Gini index
  - used in e.g., in CART (Classification And Regression Trees)
  - impurity measure (instead of entropy)

$$Gini(S) = 1 - \sum_i p_i^2$$

  - average Gini index (instead of average entropy / information)

$$Gini(S, A) = \sum_i \frac{|S_i|}{|S|} \cdot Gini(S_i)$$

  - Gini Gain
    - could be defined analogously to information gain
    - but typically avg. Gini index is minimized instead of maximizing Gini gain

Dataset

Day	Outlook	Temp.	Humidity	Wind	Decision
1	Sunny	Hot	High	Weak	No
2	Sunny	Hot	High	Strong	No
3	Overcast	Hot	High	Weak	Yes
4	Rain	Mild	High	Weak	Yes
5	Rain	Cool	Normal	Weak	Yes
6	Rain	Cool	Normal	Strong	No
7	Overcast	Cool	Normal	Strong	Yes
8	Sunny	Mild	High	Weak	No
9	Sunny	Cool	Normal	Weak	Yes
10	Rain	Mild	Normal	Weak	Yes
11	Sunny	Mild	Normal	Strong	Yes
12	Overcast	Mild	High	Strong	Yes
13	Overcast	Hot	Normal	Weak	Yes
14	Rain	Mild	High	Strong	No

Gini index

Gini index is a metric for classification tasks in CART. It stores sum of squared probabilities of each class. We can formulate it as illustrated below.

$$\text{Gini} = 1 - \Sigma (P_i)^2 \text{ for } i=1 \text{ to number of classes}$$

Outlook

Outlook is a nominal feature. It can be sunny, overcast or rain. I will summarize the final decisions for outlook feature.

Outlook	Yes	No	Number of instances
Sunny	2	3	5
Overcast	4	0	4
Rain	3	2	5

$\text{Gini}(\text{Outlook=Sunny}) = 1 - (2/5)^2 - (3/5)^2 = 1 - 0.16 - 0.36 = 0.48$

$\text{Gini}(\text{Outlook=Overcast}) = 1 - (4/4)^2 - (0/4)^2 = 0$

$\text{Gini}(\text{Outlook=Rain}) = 1 - (3/5)^2 - (2/5)^2 = 1 - 0.36 - 0.16 = 0.48$

Then, we will calculate weighted sum of gini indexes for outlook feature.

$\text{Gini}(\text{Outlook}) = (5/14) \times 0.48 + (4/14) \times 0 + (5/14) \times 0.48 = 0.171 + 0 + 0.171 = 0.342$

Temperature	Yes	No	Number of instances
Hot	2	2	4
Cool	3	1	4
Mild	4	2	6

Gini(Temp=Hot) = $1 - (2/4)^2 - (2/4)^2 = 0.5$

Gini(Temp=Cool) = $1 - (3/4)^2 - (1/4)^2 = 1 - 0.5625 - 0.0625 = 0.375$

Gini(Temp=Mild) = $1 - (4/6)^2 - (2/6)^2 = 1 - 0.444 - 0.111 = 0.445$

We'll calculate weighted sum of gini index for temperature feature

Gini(Temp) = $(4/14) \times 0.5 + (4/14) \times 0.375 + (6/14) \times 0.445 = 0.142 + 0.107 + 0.190 = 0.439$

Humidity

Humidity is a binary class feature. It can be high or normal.

Humidity	Yes	No	Number of instances
High	3	4	7
Normal	6	1	7

Gini(Humidity=High) = $1 - (3/7)^2 - (4/7)^2 = 1 - 0.183 - 0.326 = 0.489$

Gini(Humidity=Normal) = $1 - (6/7)^2 - (1/7)^2 = 1 - 0.734 - 0.02 = 0.244$

Weighted sum for humidity feature will be calculated next

Gini(Humidity) = $(7/14) \times 0.489 + (7/14) \times 0.244 = 0.367$

Wind

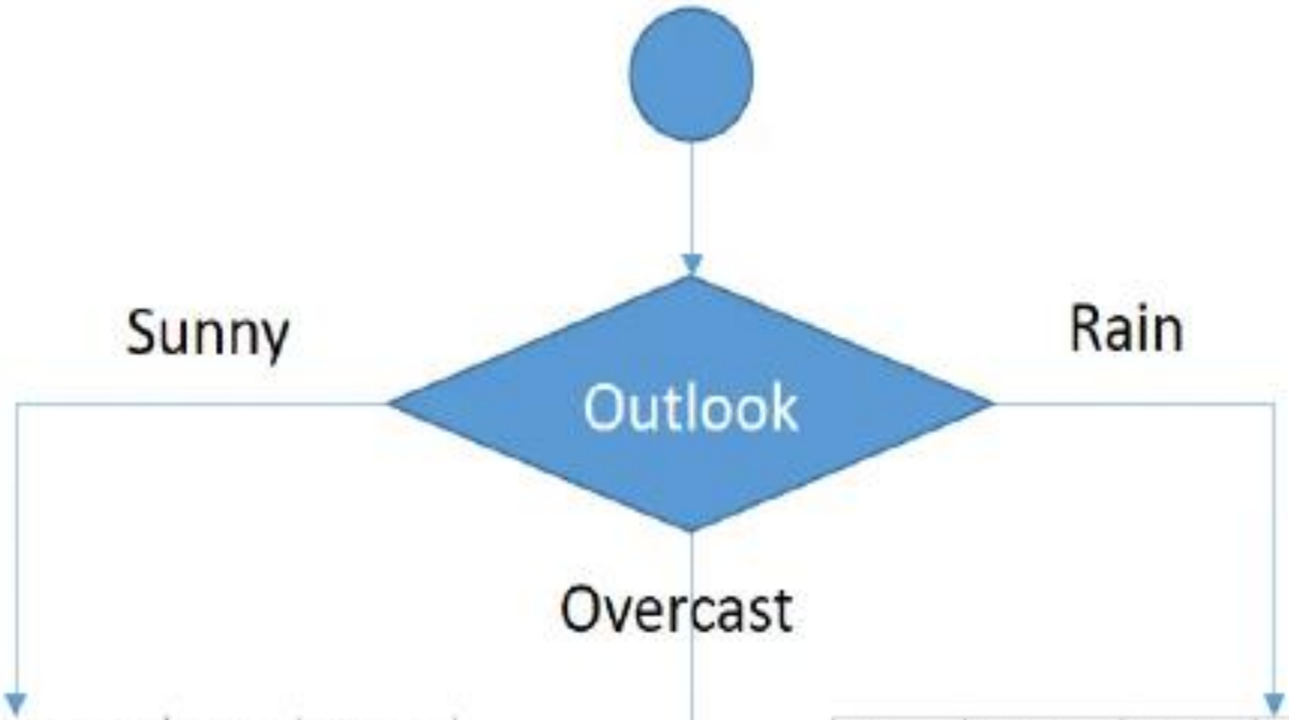
Wind is a binary class similar to humidity. It can be weak and strong.

Wind	Yes	No	Number of instances
Weak	6	2	8
Strong	3	3	6

Gini(Wind=Weak) = $1 - (6/8)^2 - (2/8)^2 = 1 - 0.5625 - 0.062 = 0.375$

Gini(Wind=Strong) = $1 - (3/6)^2 - (3/6)^2 = 1 - 0.25 - 0.25 = 0.5$

Gini(Wind) = $(8/14) \times 0.375 + (6/14) \times 0.5 = 0.428$



Day	Outlook	Temp.	Humidity	Wind	Decision
1	Sunny	Hot	High	Weak	No
2	Sunny	Hot	High	Strong	No
8	Sunny	Mild	High	Weak	No
9	Sunny	Cool	Normal	Weak	Yes
11	Sunny	Mild	Normal	Strong	Yes

Day	Outlook	Temp.	Humidity	Wind	Decision
4	Rain	Mild	High	Weak	Yes
5	Rain	Cool	Normal	Weak	Yes
6	Rain	Cool	Normal	Strong	No
10	Rain	Mild	Normal	Weak	Yes
14	Rain	Mild	High	Strong	No

Day	Outlook	Temp.	Humidity	Wind	Decision
3	Overcast	Hot	High	Weak	Yes
7	Overcast	Cool	Normal	Strong	Yes
12	Overcast	Mild	High	Strong	Yes
13	Overcast	Hot	Normal	Weak	Yes

We will apply same principles to those sub datasets in the following steps.

Focus on the sub dataset for sunny outlook. We need to find the gini index scores for temperature, humidity and wind features respectively.

Day	Outlook	Temp.	Humidity	Wind	Decision
1	Sunny	Hot	High	Weak	No
2	Sunny	Hot	High	Strong	No
8	Sunny	Mild	High	Weak	No
9	Sunny	Cool	Normal	Weak	Yes
11	Sunny	Mild	Normal	Strong	Yes

Gini of temperature for sunny outlook

Temperature	Yes	No	Number of instances
Hot	0	2	2
Cool	1	0	1
Mild	1	1	2

Gini(Outlook=Sunny and Temp.=Hot) = $1 - (0/2)^2 - (2/2)^2 = 0$

Gini(Outlook=Sunny and Temp.=Cool) = $1 - (1/1)^2 - (0/1)^2 = 0$

Gini(Outlook=Sunny and Temp.=Mild) = $1 - (1/2)^2 - (1/2)^2 = 1 - 0.25 - 0.25 = 0.5$

Gini(Outlook=Sunny and Temp.) = $(2/5)\times0 + (1/5)\times0 + (2/5)\times0.5 = 0.2$

Gini of humidity for sunny outlook

Humidity	Yes	No	Number of instances
High	0	3	3
Normal	2	0	2

Gini(Outlook=Sunny and Humidity=High) = $1 - (0/3)^2 - (3/3)^2 = 0$

Gini(Outlook=Sunny and Humidity=Normal) = $1 - (2/2)^2 - (0/2)^2 = 0$

Gini(Outlook=Sunny and Humidity) = $(3/5)\text{x}0 + (2/5)\text{x}0 = 0$

Gini of wind for sunny outlook

Wind	Yes	No	Number of instances
Weak	1	2	3
Strong	1	1	2

Gini(Outlook=Sunny and Wind=Weak) = $1 - (1/3)^2 - (2/3)^2 = 0.266$

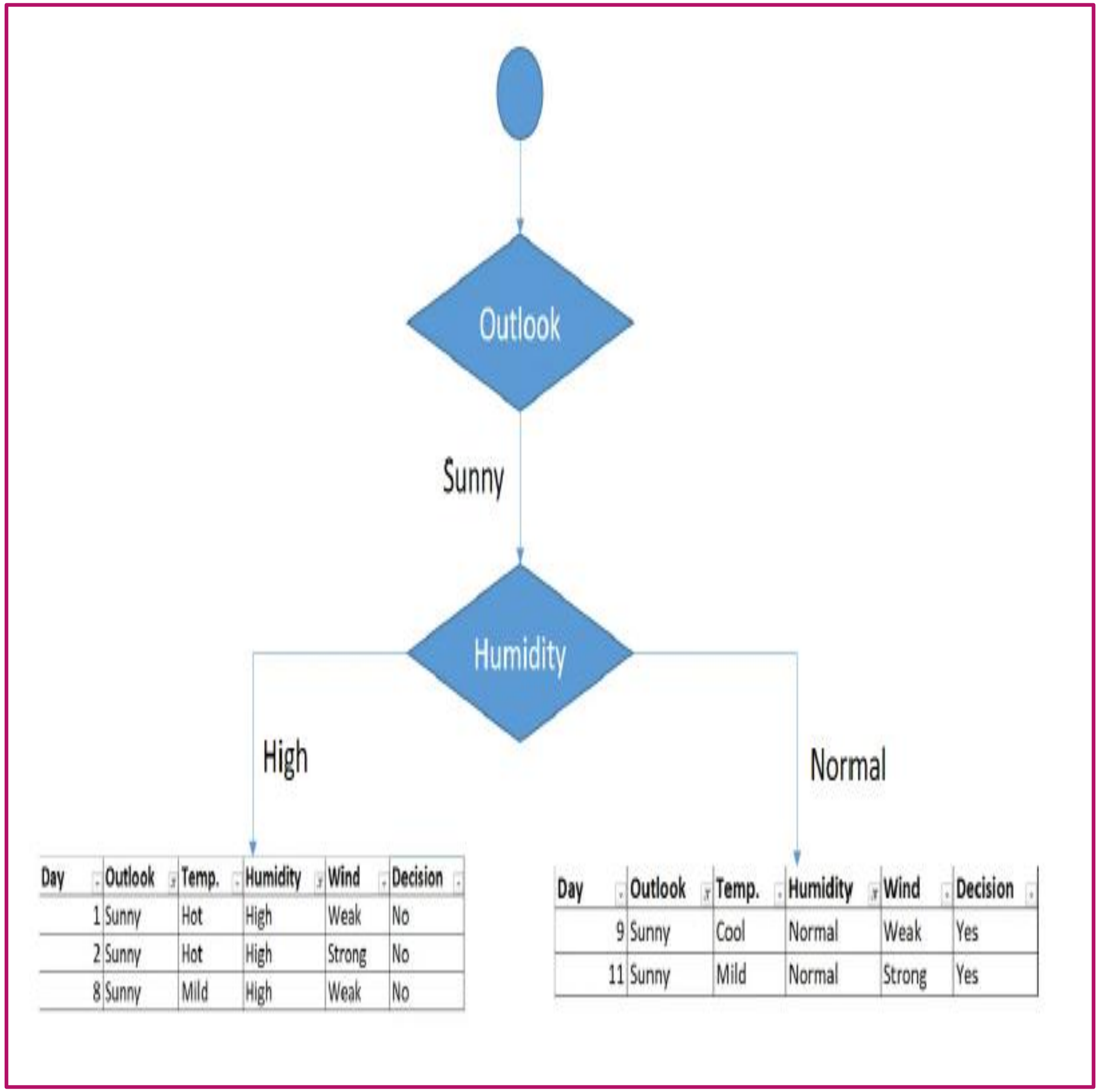
Gini(Outlook=Sunny and Wind=Strong) = $1 - (1/2)^2 - (1/2)^2 = 0.2$

Gini(Outlook=Sunny and Wind) = $(3/5)\text{x}0.266 + (2/5)\text{x}0.2 = 0.466$

Decision for sunny outlook

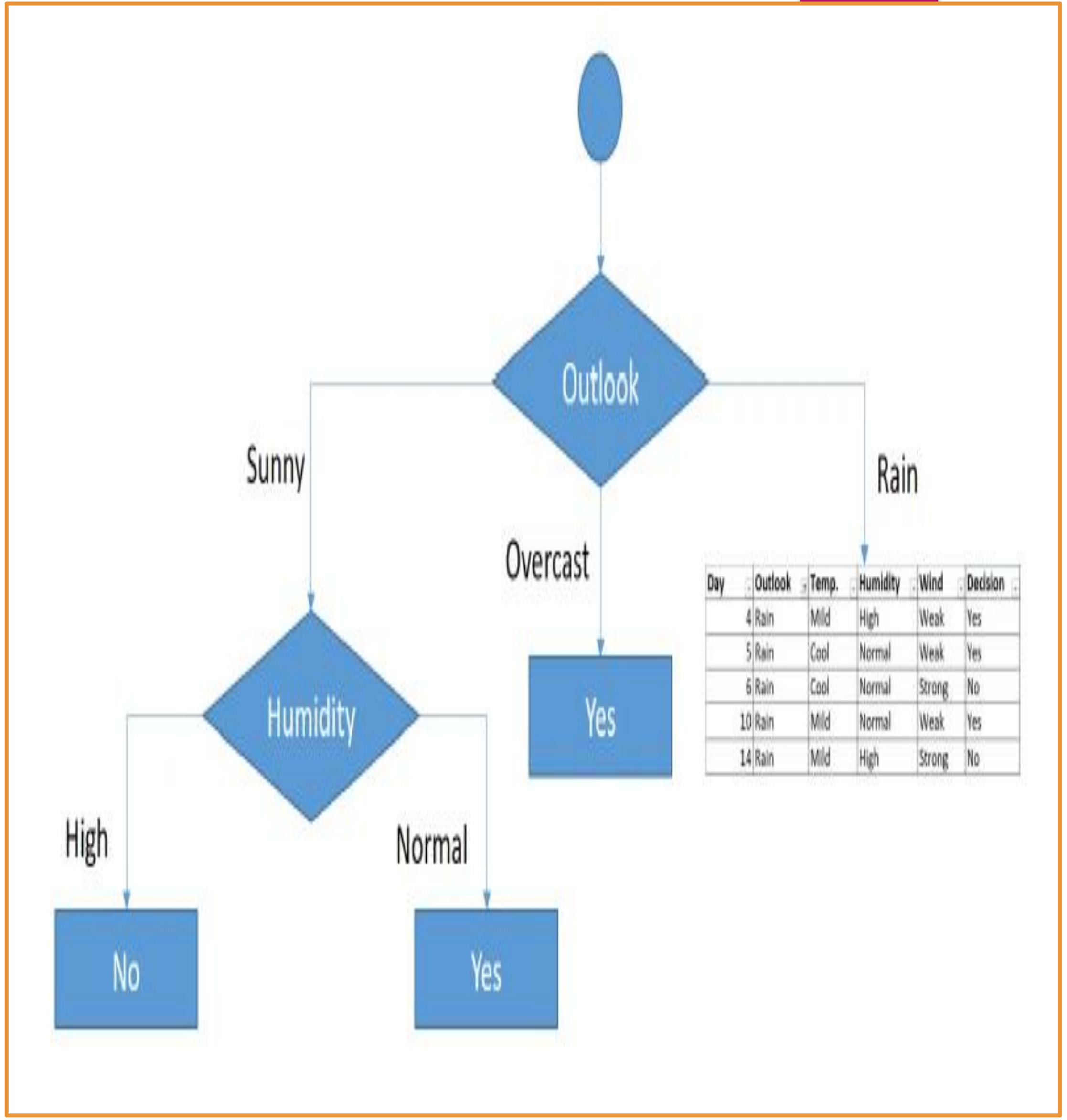
We've calculated gini index scores for feature when outlook is sunny. The winner is humidity because it has the lowest value.

Feature	Gini index
Temperature	0.2
Humidity	0
Wind	0.466



Day	Outlook	Temp.	Humidity	Wind	Decision
1	Sunny	Hot	High	Weak	No
2	Sunny	Hot	High	Strong	No
8	Sunny	Mild	High	Weak	No

Day	Outlook	Temp.	Humidity	Wind	Decision
9	Sunny	Cool	Normal	Weak	Yes
11	Sunny	Mild	Normal	Strong	Yes



Day	Outlook	Temp.	Humidity	Wind	Decision
4	Rain	Mild	High	Weak	Yes
5	Rain	Cool	Normal	Weak	Yes
6	Rain	Cool	Normal	Strong	No
10	Rain	Mild	Normal	Weak	Yes
14	Rain	Mild	High	Strong	No

Rain outlook

Day	Outlook	Temp.	Humidity	Wind	Decision
4	Rain	Mild	High	Weak	Yes
5	Rain	Cool	Normal	Weak	Yes
6	Rain	Cool	Normal	Strong	No
10	Rain	Mild	Normal	Weak	Yes
14	Rain	Mild	High	Strong	No

Gini of temprature for rain outlook

Temperature	Yes	No	Number of instances
Cool	1	1	2
Mild	2	1	3

Gini(Outlook=Rain and Temp.=Cool) = $1 - (1/2)^2 - (1/2)^2 = 0.5$

Gini(Outlook=Rain and Temp.=Mild) = $1 - (2/3)^2 - (1/3)^2 = 0.444$

Gini(Outlook=Rain and Temp.) = $(2/5)\text{x}0.5 + (3/5)\text{x}0.444 = 0.466$

Gini of humidity for rain outlook

Humidity	Yes	No	Number of instances
High	1	1	2
Normal	2	1	3

Gini(Outlook=Rain and Humidity=High) = $1 - (1/2)^2 - (1/2)^2 = 0.5$

Gini(Outlook=Rain and Humidity=Normal) = $1 - (2/3)^2 - (1/3)^2 = 0.444$

Gini(Outlook=Rain and Humidity) = $(2/5) \times 0.5 + (3/5) \times 0.444 = 0.466$

Gini of wind for rain outlook

Wind	Yes	No	Number of instances
Weak	3	0	3
Strong	0	2	2

Gini(Outlook=Rain and Wind=Weak) = $1 - (3/3)^2 - (0/3)^2 = 0$

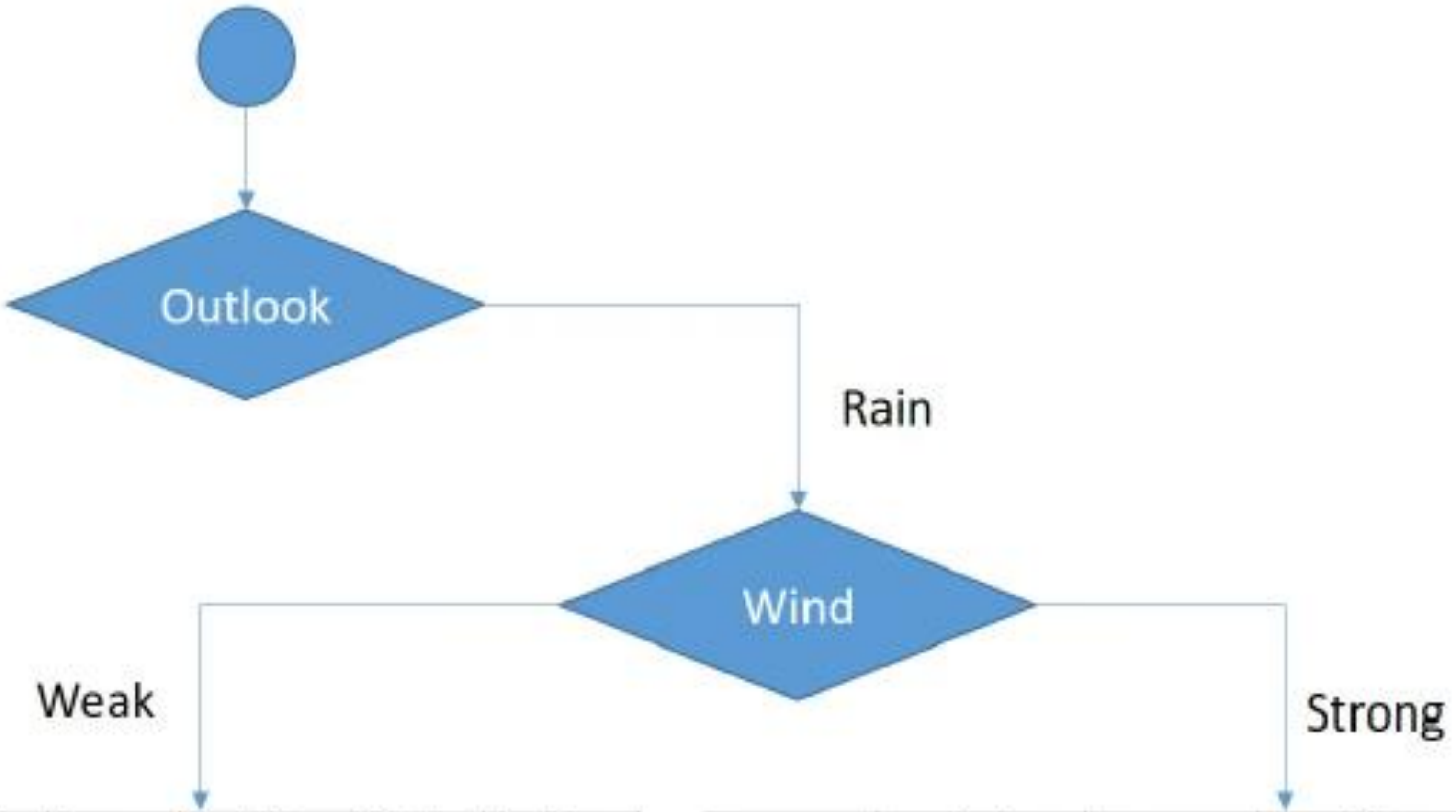
Gini(Outlook=Rain and Wind=Strong) = $1 - (0/2)^2 - (2/2)^2 = 0$

Gini(Outlook=Rain and Wind) = $(3/5) \times 0 + (2/5) \times 0 = 0$

Decision for rain outlook

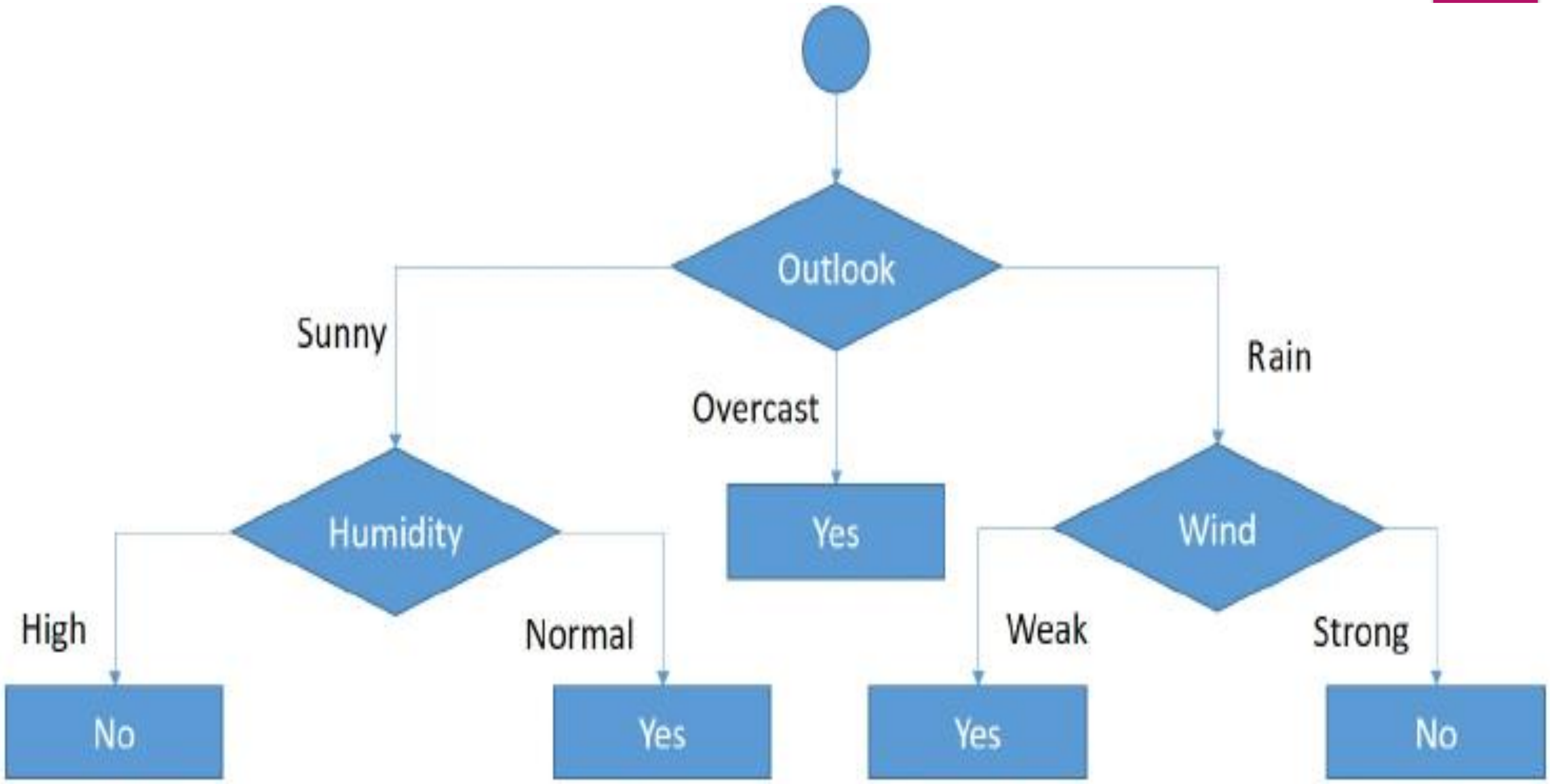
The winner is wind feature for rain outlook because it has the minimum gini index score in features.

Feature	Gini index
Temperature	0.466
Humidity	0.466
Wind	0



Day	Outlook	Temp.	Humidity	Wind	Decision
4	Rain	Mild	High	Weak	Yes
5	Rain	Cool	Normal	Weak	Yes
10	Rain	Mild	Normal	Weak	Yes

Day	Outlook	Temp.	Humidity	Wind	Decision
6	Rain	Cool	Normal	Strong	No
14	Rain	Mild	High	Strong	No


Outlook
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Weak
Strong
No
Yes
Yes
No