CHAPTER TWELVE

Sturm–Liouville Boundary-Value Problems and Fourier Series

In Chapter 1 we encountered boundary-value problems consisting of a second-order linear differential equation and two supplementary conditions which the solution of the equation must satisfy. In this chapter we shall consider a special kind of boundary-value problem known as a *Sturm–Liouville problem.* Our study of this type of problem will introduce us to several important concepts including *characteristic function, orthogonality,* and *Fourier series.* These concepts are frequently employed in the applications of differential equations to physics and engineering. In Chapter 14 we shall use them to obtain solutions of boundary-value problems which involve partial differential equations.

12.1 STURM–LIOUVILLE PROBLEMS

A. Definition and Examples

Our first concern in this chapter is a study of the special type of two-point boundary-value problem given in the following definition:

DEFINITION

We consider a boundary-value problem which consists of

1. a second-order homogeneous linear differential equation of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0, \tag{12.1}$$

where p, q, and r are real functions such that p has a continuous derivative, q and r are *continuous, and $p(x) > 0$ and $r(x) > 0$ for all x on a real interval $a \le x \le b$; and λ is a parameter independent of x; and*

2. *two supplementary conditions*

$$\begin{aligned} A_1 y(a) + A_2 y'(a) &= 0, \\ B_1 y(b) + B_2 y'(b) &= 0, \end{aligned} \tag{12.2}$$

where A_1, A_2, B_1, and B_2 are real constants such that A_1 and A_2 are not both zero and B_1 and B_2 are not both zero.

This type of boundary-value problem is called a Sturm–Liouville problem (*or* Sturm–Liouville system).

Two important special cases are those in which the supplementary conditions (12.2) are either of the form

$$y(a) = 0, \qquad y(b) = 0 \tag{12.3}$$

or of the form

$$y'(a) = 0, \qquad y'(b) = 0. \tag{12.4}$$

▶ Example 12.1

The boundary-value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \tag{12.5}$$

$$y(0) = 0, \qquad y(\pi) = 0 \tag{12.6}$$

is a Sturm–Liouville problem. The differential equation (12.5) may be written

$$\frac{d}{dx}\left[1 \cdot \frac{dy}{dx}\right] + [0 + \lambda \cdot 1]y = 0$$

and hence is of the form (12.1), where $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$. The supplementary conditions (12.6) are of the special form (12.3) of (12.2).

▶ Example 12.2

The boundary-value problem

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + [2x^2 + \lambda x^3]y = 0, \tag{12.7}$$

$$\begin{aligned} 3y(1) + 4y'(1) &= 0, \\ 5y(2) - 3y'(2) &= 0, \end{aligned} \tag{12.8}$$

is a Sturm–Liouville problem. The differential equation (12.7) is of the form (12.1), where $p(x) = x$, $q(x) = 2x^2$, and $r(x) = x^3$. The conditions (12.8) are of the form (12.2), where $a = 1$, $b = 2$, $A_1 = 3$, $A_2 = 4$, $B_1 = 5$, and $B_2 = -3$.

Let us now see what is involved in solving a Sturm–Liouville problem. We must find a function f which satisfies both the differential equation (12.1) and the two supplementary conditions (12.2). Clearly one solution of *any* problem of this type is the *trivial* solution ϕ such that $\phi(x) = 0$ for all values of x. Equally clear is the fact that this trivial solution is not very useful. We shall therefore focus our attention on the search for *nontrivial* solutions of the problem. That is, we shall attempt to find functions, *not identically zero*, which satisfy both the differential equation (12.1) and the two conditions (12.2). We shall see that the existence of such nontrivial solutions depends upon the value of the parameter λ in the differential equation (12.1). To illustrate this, let us return to the Sturm–Liouville problem of Example 12.1 and attempt to find nontrivial solutions.

▶ Example 12.3

Find nontrivial solutions of the Sturm–Liouville problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \tag{12.5}$$

$$y(0) = 0, \qquad y(\pi) = 0. \tag{12.6}$$

Solution. We shall consider separately the three cases $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$. In each case we shall first find the general solution of the differential equation (12.5). We shall then attempt to determine the two arbitrary constants in this solution so that the supplementary conditions (12.6) will also be satisfied.

Case 1: $\lambda = 0$. In this case the differential equation (12.5) reduces at once to

$$\frac{d^2y}{dx^2} = 0$$

and so the general solution is

$$y = c_1 + c_2 x. \tag{12.9}$$

We now apply the conditions (12.6) to the solution (12.9). Applying the first condition $y(0) = 0$, we obtain $c_1 = 0$. Applying the second condition $y(\pi) = 0$, we find that $c_1 + c_2\pi = 0$. Hence, since $c_1 = 0$, we must also have $c_2 = 0$. Thus in order for the solution (12.9) to satisfy the conditions (12.6), we must have $c_1 = c_2 = 0$. But then the solution (12.9) becomes the solution y such that $y(x) = 0$ for all values of x. Thus if the parameter $\lambda = 0$, the only solution of the given problem is the trivial solution.

Case 2: $\lambda < 0$. The auxiliary equation of the differential equation (12.5) is $m^2 + \lambda = 0$ and has the roots $\pm\sqrt{-\lambda}$. Since in this case $\lambda - 0$, these roots are real and unequal. Denoting $\sqrt{-\lambda}$ by α, we see that for $\lambda < 0$ the general solution of (12.5) is of the form

$$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}. \tag{12.10}$$

We now apply the conditions (12.6) to the solution (12.10). Applying the first condition $y(0) = 0$, we obtain

$$c_1 + c_2 = 0. \tag{12.11}$$

Applying the second condition $y(\pi) = 0$, we find that

$$c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} = 0. \tag{12.12}$$

We must thus determine c_1 and c_2 such that the system consisting of (12.11) and (12.12) is satisfied. Thus in order for the solution (12.10) to satisfy the conditions (12.6), the constants c_1 and c_2 must satisfy the system of Equations (12.11) and (12.12). Obviously $c_1 = c_2 = 0$ is a solution of this system; but these values of c_1 and c_2 would only give the trivial solution of the given problem. We must therefore seek nonzero values of c_1 and c_2 which satisfy (12.11) and (12.12). By Chapter 7, Section 7.5C, Theorem A, this system has nonzero solutions only if the determinant of coefficients is zero. Therefore we must have

$$\begin{vmatrix} 1 & 1 \\ e^{\alpha\pi} & e^{-\alpha\pi} \end{vmatrix} = 0.$$

But this implies that $e^{\alpha\pi} = e^{-\alpha\pi}$ and hence that $\alpha = 0$. Thus in order for a nontrivial function of the form (12.10) to satisfy the conditions (12.6) we must have $\alpha = 0$. Since $\alpha = \sqrt{-\lambda}$, we must then have $\lambda = 0$. But $\lambda < 0$ in this case. Thus there are no nontrivial solutions of the given problem in the case $\lambda < 0$.

Case 3: $\lambda > 0$. Since $\lambda > 0$ here, the roots $\pm\sqrt{-\lambda}$ of the auxiliary equation of (12.5) are the conjugate-complex numbers $\pm\sqrt{\lambda}i$. Thus in this case the general solution of (12.5) is of the form

$$y = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x. \tag{12.13}$$

We now apply the conditions (12.6) to this general solution. Applying the first condition $y(0) = 0$, we obtain

$$c_1 \sin 0 + c_2 \cos 0 = 0$$

and hence $c_2 = 0$. Applying the second condition $y(\pi) = 0$, we find that

$$c_1 \sin\sqrt{\lambda}\pi + c_2 \cos\sqrt{\lambda}\pi = 0.$$

Since $c_2 = 0$, this reduces at once to

$$c_1 \sin\sqrt{\lambda}\pi = 0 \tag{12.14}$$

We must therefore satisfy (12.14). At first glance it appears that we can do this in either of two ways: we can set $c_1 = 0$ or we can set $\sin\sqrt{\lambda}\pi = 0$. However, if we set $c_1 = 0$, then (since $c_2 = 0$ also) the solution (12.13) reduces immediately to the unwanted trivial solution. Thus to obtain a *nontrivial* solution we can *not* set $c_1 = 0$ but rather we *must* set

$$\sin\sqrt{\lambda}\pi = 0. \tag{12.15}$$

If $k > 0$, then $\sin k\pi = 0$ only if k is a positive integer $n = 1, 2, 3, \ldots$. Thus in order to satisfy (12.15), we must have $\sqrt{\lambda} = n$, where $n = 1, 2, 3, \ldots$. Therefore, in order that the differential equation (12.5) have a nontrivial solution of the form (12.13) satisfying the conditions (12.6), we must have

$$\lambda = n^2, \qquad \text{where } n = 1, 2, 3, \ldots. \tag{12.16}$$

In other words, the parameter λ in (12.5) must be a member of the infinite sequence

$$1, 4, 9, 16, \ldots, n^2, \ldots.$$

Summary. If $\lambda \leq 0$, the Sturm–Liouville problem consisting of (12.5) and (12.6) does *not* have a nontrivial solution; if $\lambda > 0$, a nontrivial solution can exist only if λ is one of the values given by (12.16). We now note that if λ *is* one of the values (12.16), then the problem *does have* nontrivial solutions. Indeed, from (12.13) we see that nontrivial solutions corresponding to $\lambda = n^2$ ($n = 1, 2, 3, \ldots$) are given by

$$y = c_n \sin nx \qquad (n = 1, 2, 3, \ldots), \tag{12.17}$$

where c_n ($n = 1, 2, 3, \ldots$) is an arbitrary nonzero constant. That is, the functions defined by $c_1 \sin x, c_2 \sin 2x, c_3 \sin 3x, \ldots$, where $c_1, c_2, c_3, \ldots$ are arbitrary nonzero constants, are nontrivial solutions of the given problem.

B. Characteristic Values and Characteristic Functions

Example 12.3 shows that the existence of nontrivial solutions of a Sturm–Liouville problem does indeed depend upon the value of the parameter λ in the differential equation of the problem. Those values of the parameter for which nontrivial solutions do exist, as well as the corresponding nontrivial solutions themselves, are singled out by the following definition:

DEFINITION

Consider the Sturm–Liouville problem consisting of the differential equation (12.1) and the supplementary conditions (12.2). The values of the parameter λ in (12.1) for which there exist nontrivial solutions of the problem are called the characteristic values *of the problem. The corresponding nontrivial solutions themselves are called the* characteristic functions *of the problem.**

► Example 12.4

Consider again the Sturm–Liouville problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \tag{12.5}$$

$$y(0) = 0, \qquad y(\pi) = 0. \tag{12.6}$$

In Example 12.3 we found that the values of λ in (12.5) for which there exist nontrivial solutions of this problem are the values

$$\lambda = n^2, \qquad \text{where } n = 1, 2, 3, \ldots. \tag{12.16}$$

These then are the characteristic values of the problem under consideration. The characteristic functions of the problem are the corresponding nontrivial solutions

$$y = c_n \sin nx \qquad (n = 1, 2, 3, \ldots), \tag{12.17}$$

where c_n ($n = 1, 2, 3, \ldots$) is an arbitrary nonzero constant.

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* The characteristic values are also called *eigenvalues;* and the characteristic functions are also called *eigenfunctions.*

▶ Example 12.5

Find the characteristic values and characteristic functions of the Sturm–Liouville problem

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\frac{\lambda}{x}y=0. \tag{12.18}$$

$$y'(1)=0, \qquad y'(e^{2\pi})=0, \tag{12.19}$$

where we assume that the parameter λ in (12.18) is nonnegative.

Solution. We consider separately the cases $\lambda = 0$ and $\lambda > 0$. If $\lambda = 0$, the differential equation (12.18) reduces to

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right]=0.$$

The general solution of this differential equation is

$$y = C\ln|x| + C_0,$$

where C and C_0 are arbitrary constants. If we apply the conditions (12.19) to this general solution, we find that both of them require that $C = 0$ but neither of them imposes any restriction upon C_0. Thus for $\lambda = 0$ we obtain the solutions $y = C_0$, where C_0 is an arbitrary constant. These are nontrivial solutions for all choices of $C_0 \neq 0$. Thus $\lambda = 0$ is a characteristic value and the corresponding characteristic functions are given by $y = C_0$, where C_0 is an arbitrary nonzero constant.

If $\lambda > 0$, we see that for $x \neq 0$ this equation is equivalent to the Cauchy-Euler equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\lambda y=0. \tag{12.20}$$

Letting $x = e^t$, Equation (12.20) transforms into

$$\frac{d^2y}{dt^2}+\lambda y=0. \tag{12.21}$$

Since $\lambda > 0$, the general solution of (12.21) is of the form

$$y = c_1\sin\sqrt{\lambda}t + c_2\cos\sqrt{\lambda}t.$$

Thus for $\lambda > 0$ and $x > 0$ the general solution of (12.18) may be written

$$y = c_1\sin(\sqrt{\lambda}\ln x) + c_2\cos(\sqrt{\lambda}\ln x). \tag{12.22}$$

We now apply the supplementary conditions (12.19). From (12.22) we find that

$$\frac{dy}{dx}=\frac{c_2\sqrt{\lambda}}{x}\cos(\sqrt{\lambda}\ln x)-\frac{c_2\sqrt{\lambda}}{x}\sin(\sqrt{\lambda}\ln x) \tag{12.23}$$

for $x > 0$. Applying the first condition $y'(1) = 0$ of (12.19) to (12.23), we have

$$c_1\sqrt{\lambda}\cos(\sqrt{\lambda}\ln 1) - c_2\sqrt{\lambda}\sin(\sqrt{\lambda}\ln 1) = 0$$

or simply $c_1\sqrt{\lambda} = 0$. Thus we must have

$$c_1 = 0. \tag{12.24}$$

Applying the second condition $y'(e^{2\pi}) = 0$ of (12.19) to (12.23), we obtain

$$c_1\sqrt{\lambda}e^{-2\pi}\cos(\sqrt{\lambda}\ln e^{2\pi}) - c_2\sqrt{\lambda}e^{-2\pi}\sin(\sqrt{\lambda}\ln e^{2\pi}) = 0.$$

Since $c_1 = 0$ by (12.24) and $\ln e^{2\pi} = 2\pi$, this reduces at once to

$$c_2\sqrt{\lambda}e^{-2\pi}\sin(2\pi\sqrt{\lambda}) = 0.$$

Since $c_1 = 0$, the choice $c_2 = 0$ would lead to the trivial solution.

We must have $\sin(2\pi\sqrt{\lambda}) = 0$ and hence $2\pi\sqrt{\lambda} = n\pi$, where $n = 1, 2, 3, \ldots$. Thus in order to satisfy the second condition (12.19) nontrivially we must have

$$\lambda = \frac{n^2}{4} \qquad (n = 1, 2, 3, \ldots). \tag{12.25}$$

Corresponding to these values of λ we obtain for $x > 0$ the nontrivial solutions

$$y = C_n \cos\left(\frac{n \ln x}{2}\right) \qquad (n = 1, 2, 3, \ldots), \tag{12.26}$$

where the $C_n (n = 1, 2, 3, \ldots)$ are arbitrary nonzero constants.

Thus the values

$$\lambda = 0, \frac{1}{4}, 1, \frac{9}{4}, 4, \frac{25}{4}, \ldots, \frac{n^2}{4}, \ldots,$$

given by (12.25) for $n \geq 0$, are the characteristic values of the given problem. The functions

$$C_0, C_1 \cos\left(\frac{\ln x}{2}\right), C_2 \cos(\ln x), C_3 \cos\left(\frac{3 \ln x}{2}\right), \ldots,$$

given by (12.26) for $n \geq 0$, where $C_0, C_1, C_2, C_3, \ldots$ are arbitrary nonzero constants, are the corresponding characteristic functions.

For each of the Sturm–Liouville problems of Examples 12.3 and 12.5 we found an infinite number of characteristic values. We observe that in each of these problems the infinite set of characteristic values thus found can be arranged in a monotonic increasing sequence*

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

such that $\lambda_n \to +\infty$ as $n \to +\infty$. For example, the characteristic values of the problem of Example 12.3 can be arranged in the monotonic increasing sequence

$$1 < 4 < 9 < 16 < \cdots \tag{12.27}$$

such that $\lambda_n = n^2 \to +\infty$ as $n \to +\infty$. We also note that in each problem there is a one-parameter family of characteristic functions corresponding to each characteristic value, and any two characteristic functions corresponding to the same characteristic value are merely nonzero constant multiples of each other. For example, in the problem of Example 12.3 the one-parameter family of characteristic functions corresponding to the characteristic value n^2 is $c_n \sin nx$, where $c_n \neq 0$ is the parameter.

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* An infinite sequence $\{x_n\}$ is said to be monotonic increasing if $x_{n+1} \geq x_n$ for every n.

We might now inquire whether or not all Sturm–Liouville problems of the type under consideration possess characteristic values and characteristic functions having the properties noted in the preceding paragraph. We can answer in the affirmative by stating the following important theorem.

THEOREM 12.1

Hypothesis. *Consider the Sturm–Liouville problem consisting of*

1. the differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0, \tag{12.1}$$

where p, q, and r are real functions such that p has a continuous derivative, q and r are continuous, and $p(x) > 0$ and $r(x) > 0$ for all x on a real interval $a \le x \le b$; and λ is a parameter independent of x; and

2. the conditions

$$\begin{aligned} A_1 y(a) + A_2 y'(a) &= 0, \\ B_1 y(b) + B_2 y'(b) &= 0, \end{aligned} \tag{12.2}$$

where A_1, A_2, B_1, and B_2 are real constants such that A_1 and A_2 are not both zero and B_1 and B_2 are not both zero.

Conclusions

1. There exists an infinite number of characteristic values λ_n of the given problem. These characteristic values λ_n can be arranged in a monotonic increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

such that $\lambda_n \to +\infty$ as $n \to +\infty$.

2. Corresponding to each characteristic value λ_n there exists a one-parameter family of characteristic functions ϕ_n. Each of these characteristic functions is defined on $a \le x \le b$, and any two characteristic functions corresponding to the same characteristic value are nonzero constant multiples of each other.

3. Each characteristic function ϕ_n corresponding to the characteristic value λ_n ($n = 1, 2, 3, \ldots$) has exactly $(n - 1)$ zeros in the open *interval $a < x < b$.*

We regard the proof of this theorem as beyond the scope of this book; therefore we omit it.

▶ Example 12.6

Consider again the Sturm–Liouville problem of Examples 12.3 and 12.4,

$$\frac{d^2y}{dx^2} + \lambda y = 0, \tag{12.5}$$

$$y(0) = 0, \qquad y(\pi) = 0. \tag{12.6}$$

We have already noted the validity of Conclusions 1 and 2 of Theorem 12.1 for this problem. The infinite number of characteristic values $\lambda_n = n^2$ $(n = 1, 2, 3, \dots)$ can be arranged in the unbounded monotonic increasing sequence indicated by (12.27); and the characteristic functions $c_n \sin nx (c_n \neq 0)$, corresponding to $\lambda_n = n^2$, possess the properties stated.

We now illustrate Conclusion 3 by showing that each characteristic function $c_n \sin nx$ corresponding to $\lambda_n = n^2$ has exactly $(n - 1)$ zeros in the open interval $0 < x < \pi$. We know that $\sin nx = 0$ if and only if $nx = k\pi$, where k is an integer. Thus the zero of $c_n \sin nx$ are given by

$$x = \frac{k\pi}{n} \qquad (k = 0, \pm 1, \pm 2, \dots). \tag{12.28}$$

The zeros (12.28) which lie in the open interval $0 < x < \pi$ are precisely those for which $k = 1, 2, 3, \dots, n - 1$. Thus, just as Conclusion 3 asserts, each characteristic function $c_n \sin nx$ has precisely $(n - 1)$ zeros in the open interval $0 < x < \pi$.

Exercises

Find the characteristic values and characteristic functions of each of the following Sturm–Liouville problems.

1. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \qquad y(0) = 0, \qquad y\left(\dfrac{\pi}{2}\right) = 0.$

2. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \qquad y(0) = 0, \qquad y'(\pi) = 0.$

3. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \qquad y(0) = 0, \qquad y(L) = 0, \quad \text{where } L > 0.$

4. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \qquad y'(0) = 0, \qquad y'(L) = 0, \quad \text{where } L > 0.$

5. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \qquad y(0) = 0. \qquad y(\pi) - y'(\pi) = 0.$

6. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \qquad y(0) - y'(0) = 0, \qquad y(\pi) - y'(\pi) = 0.$

7. $\dfrac{d}{dx}\left[x\dfrac{dy}{dx}\right] + \dfrac{\lambda}{x} y = 0, \qquad y(1) = 0, \qquad y(e^\pi) = 0.$

8. $\dfrac{d}{dx}\left[x\dfrac{dy}{dx}\right] + \dfrac{\lambda}{x} y = 0, \qquad y(1) = 0, \qquad y'(e^\pi) = 0.$

9. $\dfrac{d}{dx}\left[(x^2 + 1)\dfrac{dy}{dx}\right] + \dfrac{\lambda}{x^2 + 1} y = 0, \qquad y(0) = 0, \qquad y(1) = 0.$

   [*Hint*: Let $x = \tan t$.]

10. $\dfrac{d}{dx}\left[\dfrac{1}{3x^2 + 1}\dfrac{dy}{dx}\right] + \lambda(3x^2 + 1)y = 0, \qquad y(0) = 0, \qquad y(\pi) = 0.$

    [*Hint*: Let $t = x^3 + x$.]