Module 2: Solving Recurrence Relations of order 2

Linear 1st order Recurrence Relation

- We say a recurrence relation is linear if f is a linear function or in other words,

  $a_n = f(a_{n-1}, \ldots, a_{n-k}) = s_1 a_{n-1} + \cdots + s_k a_{n-k} + f(n)$,
  where s_i, $f(n)$ are real numbers.
- The recurrence relation is ***homogeneous*** if $f(n)=0$;
- The order of the recurrence relation is determined by k. A recurrence relation is of ***order k*** if $a_n = f(a_{n-1}, \ldots, a_{n-k})$.
- A recurrence relation is of ***First Order*** if a_n depends **only** on one previous term.
- We will discuss how to solve linear recurrence relations of orders 1 and 2.

Characteristic Equation

- Consider a homogeneous, linear recurrence relation with constant coefficients: $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ra_{n-r}$.
- Suppose, $a_k = x^k$ is a solution of the recurrence relation, for any positive integer value of $k \in [1..n]$.
- Then $x^n = c_1x^{n-1} + c_2x^{n-2} + \cdots + c_rx^{n-r}$.
- Ignoring the trivial solution $x=0$, we obtain the polynomial equation $x^r - c_1x^{r-1} - c_2x^{r-2} - \cdots - c_r = 0$.
- *This polynomial degree r equation is called the* ***characteristic equation*** *for the given recurrence relation.*
- It has r roots in general.

Solving Linear Recurrence Relation

- Let's consider, a two-ordered linear recurrence relation $F_n = AF_{n-1} + BF_{n-2}$, where A, B are real number coefficients.
- The characteristic equation for the above recurrence relation is: $x^2 - Ax - B = 0$
- Three cases may occur while finding the roots
  - Case 1: If the equation factors as $(x-x_1)(x-x_2)=0$ and it produces two distinct real roots x_1 and x_2, then $F_n = ax_1^n + bx_2^n$ is the solution $\forall\, n \geq 1$.
    - *Solution for a k-ordered linear recurrence relation is $F_n = a_1x_1^n + a_2x_2^n + \ldots + a_kx_k^n$ where $x_1, x_2, \ldots, x_k$ are the distinct roots of a k-order generative equation.*

Solving Linear Recurrence Relation

- Case 2: If the equation factors as $(x-x_1)^2=0$ and it produces single real root x_1, then $F_n=(a+bn)x_1^n$ is the solution ∀ n ≥ 1.
  - *Solution for all equal roots in a k-ordered relation will be $F_n=(a_1+a_2n+a_3n^2+\ldots+a_kn^{k-1})(-1)^n$.*
- Case 3: If the equation produces two distinct complex roots x_1, x_2: $x_1=r\theta$ and $x_2=r(-\theta)$ then $F_n = r^n(a\cos(n\theta)+b\sin(n\theta))$ is the solution ∀ n ≥ 1.

- In all the cases, a and b are constants.

Solving Linear Recurrence Relation

- **EX 9.** Solve the recurrence relation $F_n = 5F_{n-1} - 6F_{n-2}$, where $F_0=1$ and $F_1=4$.
- The characteristic equation for the above recurrence relation is $x^2 - 5x + 6 = 0$. Therefore, $(x-3)(x-2)=0$
- The solution for the recurrence relation is $F_n = a3^n + b2^n$, where a and b are two constants.
  - $F_0 = a*3^0 + b*2^0 = 1$, and
  - $F_1 = a*3^1 + b*2^1 = 4$
- Solving these two equations we get, a=2, b=-1.
- Therefore, the final solution is $F_n = 2*3^n - 2^n$

Solving Linear Recurrence Relation

- **EX 10.** Solve the recurrence relation $F_n = 10F_{n-1} - 25F_{n-2}$, where $F_0=3$ and $F_1=17$.
- The characteristic equation for the above recurrence relation is $x^2 - 10x + 25 = 0$. Therefore, $(x-5)^2=0$
- The solution for the recurrence relation is $F_n = (a + bn)5^n$, where a and b are two constants.
  - $F_0 = a*5^0 + b*0*5^0 = a = 3$, and
  - $F_1 = a*5^1 + b*1*5^1 = 5a+5b = 17$
- Solving these two equations we get, $a=3$, $b=2/5=0.4$.
- Therefore, the final solution is $F_n = 3*5^n + 0.4*n*5^n$

Solving Linear Recurrence Relation

- **EX 11.** Solve the recurrence relation $F_n = 2F_{n-1} - 2F_{n-2}$, where $F_0=1$ and $F_1=3$.
- The characteristic equation for the above recurrence relation is $x^2-2x+2 = 0$. Therefore, the roots are $x_1 = 1+i$, and $x_1 = 1-i$, where $i = \sqrt{-1}$
- In polar form, $x_1=r\theta$, and $x_2=r(-\theta)$ where, $r=\sqrt{2}$ and $\theta=\pi/4$.
- Hence, the solution to the recurrence relation will be of the format $F_n=(\sqrt{2})^n(a\cos(n*\pi/4)+b\sin(n*\pi/4))$
  - $F_0 = (\sqrt{2})^0(a\cos(0*\pi/4)+b\sin(0*\pi/4)) = a = 1$, and
  - $F_1 = (\sqrt{2})^1(a\cos(1*\pi/4)+b\sin(1*\pi/4)) = a + b = 3$
- Solving these two equations we get, a=1, b=2.
- $\therefore$ the final solution is $F_n = (\sqrt{2})^n(\cos(n*\pi/4)+2\sin(n*\pi/4))$

Solving Linear Recurrence Relation

- **Ex. 12:** Solve the recurrence relation $F_n=-F_{n-1}+4F_{n-2}+4F_{n-3}$ with the initial conditions $F_0 = 8$, $F_1 = 6$, and $F_2 = 26$.
- Solution: The characteristic equation is $r^3 + r^2 - 4r - 4 = 0$. Therefore, $(r+1)(r+2)(r-2) = 0$. The roots for the equation are $x_1=-1$, $x_2=-2$, and $x_3=2$.
- Therefore, the solution is of the format $F_n = a(-1)^n + b(-2)^n + c2^n$
  - $F_0 = a(-1)^0 + b(-2)^0 + c2^0 = a+b+c=8$
  - $F_1 = a(-1)^1 + b(-2)^1 + c2^1 = -a-2b+2c = 6$
  - $F_2 = a(-1)^2 + b(-2)^2 + c2^2 = a+4b+4c = 26$
- So, a = 2, b = 1, and c = 5.
- The solution is therefore $F_n = 2(-1)^n + (-2)^n + 5*2^n$

Solving Linear Recurrence Relation

- **EX 13.** Solve the recurrence relation $F_n = 8F_{n-2} - 16F_{n-4}$, where $F_0=1$ and $F_1=4$, $F_2=28$, $F_3=32$.
- Solution: The characteristic equation is $r^4 - 8r^2 + 16 = 0$. Therefore, $(r^2 - 4)^2 = (r-2)^2 * (r+2)^2 = 0$. There are two distinct roots $r_1 = 2$ and $r_2 = -2$ with multiplicities 2.
- Therefore, the solution is of the format $F_n = (a+bn)(2)^n + (c+dn)(-2)^n$
    - $F_0 = (a+b*0)(2)^0 + (c+d*0)(-2)^0 = a+c = 1$
    - $F_1 = (a+b)(2)^1 + (c+d)(-2)^1 = 2(a+b) -2(c+d) = 4$
    - $F_2 = (a+2b)(2)^2 + (c+2d)(-2)^2 = 4(a+c)+8(b+d) = 28$
    - $F_3 = (a+3b)(2)^3 + (c+3d)(-2)^3 = 8(a+c)+24(b+d) = 32$
- So, a= 1, b = 2, c = 0 and d = 1 and the final solution is *$F_n = (1 + 2n) 2^n + n (-2)^n$*

Exhausted! ☺

Image Courtesy: Depositphotos

Practise Problem

Q01: Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ $(n \geq 3)$ with initial conditions $a_1 = 0$, $a_2 = 6$.

Q02: Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$ $(n \geq 3)$ with initial conditions $a_1 = 1$, $a_2 = 3$.

Q03: Solve the Fibonacci recurrence relation $a_n = a_{n-1} + a_{n-2}$ with the consecutive initial conditions $a_0 = 1$ and $a_1 = 1$.

Q04: Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with the initial conditions $a_0 = 2$ and $a_1 = 7$.

Q05: Solve the recurrence relation $a_n = -3*a_{n-1} - 3a_{n-2} - a_{n-3}$ with the initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

Exercise to Solve

ES01: Find the generating function for the solutions to $h_n = 4h_{n-1} - 3h_{n-2}$, $h_0=2$, $h_1=5$, and use it to find a formula for h_n.

ES02: Find the generating function for the solutions to $h_n = 3h_{n-1} + 4h_{n-2}$, $h_0=h_1=1$, and use it to find a formula for h_n.

ES03: Find the generating function for the solutions to $h_n = 2h_{n-1} + 3^n$, $h_0=0$, and use it to find a formula for h_n.

ES04: Find the generating function for the solutions to $h_n = h_{n-1} + h_{n-2}$, $h_0=1$, $h_1=3$, and use it to find a formula for h_n.

ES05: Find the generating function for the solutions to $h_n = 3h_{n-1} + 4h_{n-2}$, $h_0=0$, $h_1=1$, and use it to find a formula for h_n.

Questions?