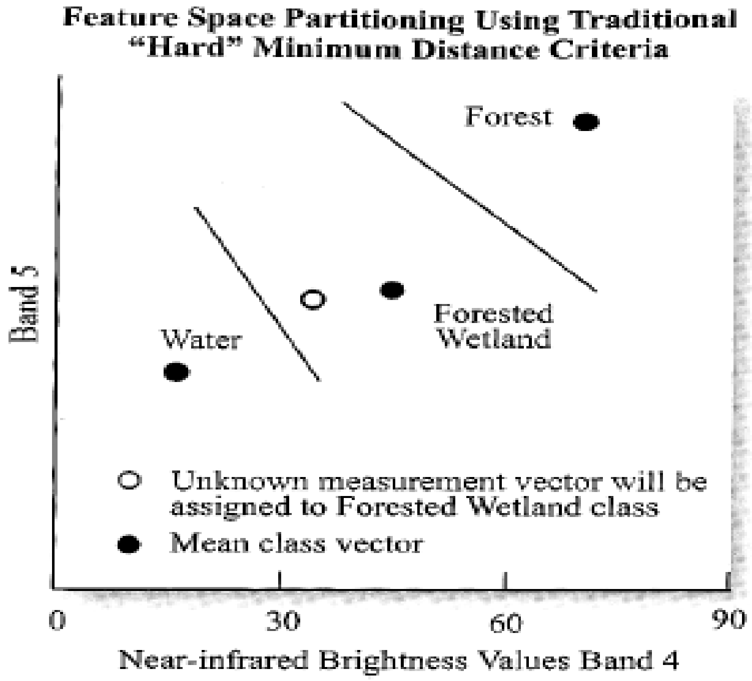
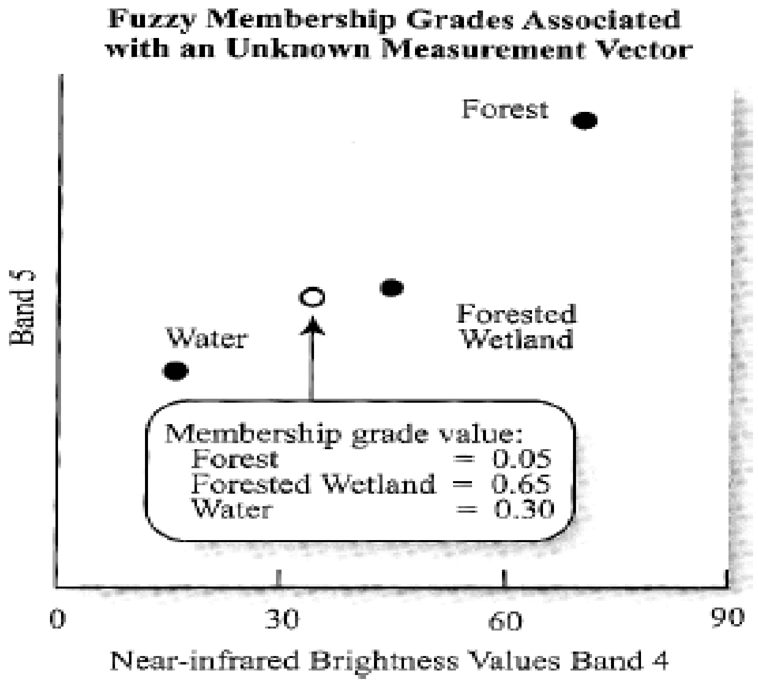
Fuzzy Clustering

Hard vs Soft Classification

Hard- versus soft-classifiers





Why use soft-classifiers?

- Sub-pixel classification
- Uncertainty of classification/scheme
- Incorporating ancillary data (hardeners)

Fuzzy Clustering

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of given data. A fuzzy pseudopartition or fuzzy c-partition of X is a family of fuzzy subsets of X, denoted by $\mathcal{P} = \{A_1, A_2, \ldots, A_c\}$, which satisfies

$$\sum_{i=1}^{c} A_i(x_k) = 1$$

for all $k \in \mathbb{N}_n$ and

$$0 < \sum_{k=1}^{n} A_i(x_k) < n$$

for all $i \in \mathbb{N}_c$, where c is a positive integer.

For instance, given $X = \{x_1, x_2, x_3\}$ and

$$A_1 = .6/x_1 + 1/x_2 + .1/x_3,$$

$$A_2 = .4/x_1 + 0/x_2 + .9/x_3,$$

$$\mathbf{v}_i = \frac{\sum_{k=1}^{n} [A_i(\mathbf{x}_k)]^m \mathbf{x}_k}{\sum_{k=1}^{n} [A_i(\mathbf{x}_k)]^m}$$

$$J_m(\mathcal{P}) = \sum_{k=1}^{n} \sum_{i=1}^{c} [A_i(\mathbf{x}_k)]^m \|\mathbf{x}_k - \mathbf{v}_i\|^2,$$

Fuzzy c-means

The algorithm is based on the assumption that the desired number of clusters c is given and, in addition, a particular distance, a real number $m \in (1, \infty)$, and a small positive number ε, serving as a stopping criterion, are chosen.

Step 1. Let $t = 0$. Select an initial fuzzy pseudopartition $\mathcal{P}^{(0)}$.

Step 2. Calculate the c cluster centers $\mathbf{v}_1^{(t)}, \ldots, \mathbf{v}_c^{(t)}$ by (13.3) for $\mathcal{P}^{(t)}$ and the chosen value of m.

Step 3. Update $\mathcal{P}^{(t+1)}$ by the following procedure: For each $\mathbf{x}_k \in X$, if $\|\mathbf{x}_k - \mathbf{v}_i^{(t)}\|^2 > 0$ for all $i \in \mathbb{N}_c$, then define

$$A_i^{(t+1)}(\mathbf{x}_k) = \left[\sum_{j=1}^{c}\left(\frac{\|\mathbf{x}_k - \mathbf{v}_i^{(t)}\|^2}{\|\mathbf{x}_k - \mathbf{v}_j^{(t)}\|^2}\right)^{\frac{1}{m-1}}\right]^{-1};$$

if $\|\mathbf{x}_k - \mathbf{v}_i^{(t)}\|^2 = 0$ for some $i \in I \subseteq \mathbb{N}_c$, then define $A_i^{(t+1)}(\mathbf{x}_k)$ for $i \in I$ by any nonnegative real numbers satisfying

$$\sum_{i \in I} A_i^{(t+1)}(\mathbf{x}_k) = 1,$$

and define $A_i^{(t+1)}(\mathbf{x}_k) = 0$ for $i \in \mathbb{N}_c - I$.

Step 4. Compare $\mathcal{P}^{(t)}$ and $\mathcal{P}^{(t+1)}$. If $|\mathcal{P}^{(t+1)} - \mathcal{P}^{(t)}| \le \varepsilon$, then stop; otherwise, increase t by one and return to **Step 2**.

In Step 4, $|\mathcal{P}^{(t+1)} - \mathcal{P}^{(t)}|$ denotes a distance between $\mathcal{P}^{(t+1)}$ and $\mathcal{P}^{(t)}$ in the space $\mathbb{R}^{n \times c}$. An example of this distance is

$$|\mathcal{P}^{(t+1)} - \mathcal{P}^{(t)}| = \max_{i \in \mathbb{N}_c, k \in \mathbb{N}_n} |A_i^{(t+1)}(\mathbf{x}_k) - A_i^{(t)}(\mathbf{x}_k)|.$$

P=

	A1	A2	A3	
X1	0.2	0.4	0.4	=1
X2	0.3	0.5	0.2	
X3	0.1	0.9	0.0	
X4	0.8	0.1	0.1	
X5	0.9	0.1	0.0	

A1(X1) =0.2

U_{A1}(X1)=0.2

$$v_i = \frac{\sum_{k=1}^{n}[A_i(x_k)]^m x_k}{\sum_{k=1}^{n}[A_i(x_k)]^m}$$

Sample calculation

- Ai (Xk)   A1(X1)=0.2
- X1=(1,1)=(x11,x12)
- X2=(2,3)
- X3=(-4,-3)
- X4=(0,0)
- X5=(4,-4)

V1=(V11,V12), V2=(V21,V22)

V11= [A1(X1)*A1(X1)*X11 + A1(X2)*A1(X2)*X21 + A1(X3)*A1(X3)*X31 + A1(X4)*A1(X4)*X41 + A1(X5)*A1(X5)*X51]/[A1(X1)*A1(X1) + A1(X2)*A1(X2) + A1(X3)*A1(X3) + A1(X4)*A1(X4) + A1(X5)*A1(X5)]

V11= [0.2*0.2*1 + 0.3*0.3*2 + 0.1*0.1* -4 + 0.8*0.8 *0 + 0.9*0.9 * 4]/ [0.2*0.2+ 0.3*0.3+ 0.1*0.1+ 0.8*0.8+ 0.9*0.9] = 3.42/1.59 = 2.15

V12= -1.86;   V1 = {2.15, -0.02}; V2= ?

P=

	A1	A2	A3	
X1	0.2	0.4	0.4	=1
X2	0.3	0.5	0.2	
X3	0.1	0.9	0.0	
X4	0.8	0.1	0.1	
X5	0.9	0.1	0.0	

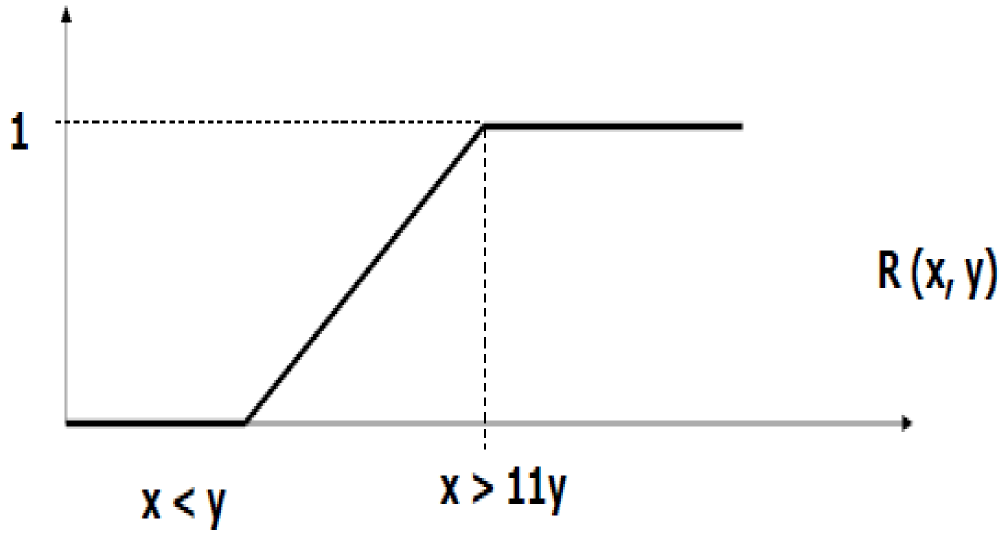
$$v_i = \frac{\sum_{k=1}^{n}[A_i(x_k)]^m x_k}{\sum_{k=1}^{n}[A_i(x_k)]^m}$$

Fuzzy Relation

Definition

- ***Fuzzy relations are mapping elements of one universe, to those of another universe, Y, through the Cartesian product of two universes. X, Universe X = {1, 2, 3}***
- ***R (X, Y)= {[(x, y), μ_R (x, y)] | (x, y) ∈ (X × Y)}***
- ***Where the fuzzy relation R has membership function***
- ***$\mu_R (x, y) = \mu_{A \times B} (x, y) = \min (\mu_A (x), \mu_B (y))$***
- ***It represents the strength of association between elements of the two sets***
- ***Ex: R = "x is considerably larger than y"***
- ***R (X, Y)= Relation between sets X and Y***
- ***R (x, y) = memebership function for the relation R (X, Y)***
- ***R (X, Y) = {R (x, y) / (x, y) | (x, y) ∈ (X × Y)}***

- $$R(x, y) = \begin{cases} 0 & \text{for } x \leq y \\ \{x - y)/(10\text{-}y), & \text{for } y < x \leq 11y \\ 1 & \text{fro } x > 11y \end{cases}$$



Cartesian Product

- *Let $A_1, A_2, \ldots, A_n$ be fuzzy sets in $U_1, U_2, \ldots U_n$, respectively.*

  The Cartesian product of $A_1, A_2, \ldots, A_n$ is a fuzzy set in the space $U_1 \times U_2 \times \ldots \times U_n$ with the membership function as:

  $\mu_{A1 \times A2 \times \ldots \times An}(x_1, x_2, \ldots, x_n) = \min[\mu_{A1}(x_1), \mu_{A2}(x_2), \ldots \mu_{An}(x_n)]$
- *So, the Cartesian product of $A_1, A_2, \ldots, A_n$ are donated by $A_1 \times A_2 \times \ldots \times A_n$*

Crisp Relations

- *The relation between any two sets is the Cartesian product of the elements of $A_1 \times A_2 \times \ldots \times A_n$*
- *For X and Y universes* $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$
- $$\mu_{x \times y}(x, y) = \begin{cases} 1, & (x, y) \in X \times Y \\ 0, & (x, y) \notin X \times Y \end{cases}$$
- *This relation can be represented in a matrix format*

Cartesian Product: Example

- *Let A =* $\{(3, 0.5), (5, 1), (7, 0.6)\}$
- *Let B =* $\{(3, 1), (5, 0.6)\}$
- Find the product
- *The product is all set of pairs from A and B with the minimum associated memberships*
- *A x B =* $\{[(3, 3), \min(0.5, 1)], [(5, 3), \min(1, 1)], [(7, 3), \min(0.6, 1)], [(3, 5), \min(0.5, 0.6)], [(5, 5), \min(1, 0.6)], [(7, 5), \min(0.6, 0.6)]\}$

  $= \{[(3, 3), 0.5], [(5, 3), 1], [(7, 3), 0.6], [(3, 5), 0.5], [(5, 5), 0.6], [(7, 5), 0.6]\}$

Operations on Fuzzy Relations

- *Since the fuzzy relation from X to Y is a fuzzy set in X × Y, then the operations on fuzzy sets can be extended to fuzzy relations. Let R and S be fuzzy relations on the Cartesian space X × Y then:*
- *Union:* $\mu_{R \cup S}(x, y) = \max[\mu_R(x, y), \mu_S(x, y)]$
- *Intersection:* $\mu_{R \cap S}(x, y) = \min[\mu_R(x, y), \mu_S(x, y)]$
- *Complement:* $\mu_{\bar{R}}(x, y) = 1 - \mu_R(x, y)$

- *Assume two Universes:* A = {3, 4, 5} and B = {3, 4, 5, 6, 7}
- $\mu_R(x, y) = \begin{cases} (y-x)/(y+x+2) & \text{if } y > x \\ 0, & \text{if } y \leq x \end{cases}$
- *This can be expressed as follow:*

$$R = \begin{array}{c} 3 \\ 4 \\ 5 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & 0.11 & 0.2 & 0.27 & 0.33 \\ 0 & 0 & 0.09 & 0.17 & 0.23 \\ 0 & 0 & 0 & 0.08 & 0.14 \end{bmatrix} \\ \begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \end{array} \end{array}$$

- *This matrix represents the membership grades between elements in X and Y*
- $\mu_R(x, y)$ = {[*0/(3, 3)*], [*0.11/(3, 4)*], [*0.2/(3, 5)*], ..., [*0.14/(5, 7)*]}

Fuzzy Relations: Example

- Assume two fuzzy sets: $A = \{0.2/x_1 + 0.5/x_2 + 1/x_3\}$

  $B = \{0.3/y_1 + 0.9/y_2\}$

- Find the fuzzy relation (the Cartesian product)

$$A \times B = R = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \\ \end{array} \begin{array}{c} \begin{pmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{pmatrix} \\ \begin{array}{cc} y_1 & y_2 \end{array} \end{array}$$

- Consider the fuzzy relation. Express R using the resolution principle

$$R = \begin{pmatrix} 0.4 & 0.5 & 0 \\ 0.9 & 0.5 & 0 \\ 0 & 0 & 0.3 \\ 0.3 & 0.9 & 0.4 \end{pmatrix}$$

- $R = R_{0.3} + R_{0.4} + R_{0.5} + R_{0.9}$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Composition of Fuzzy Relations

- *Composition of fuzzy relations used to combine fuzzy relations on different product spaces*
- *Having a fuzzy relation; R $(X \times Y)$ and S $(Y \times Z)$, then Composition is used to determine a relation T $(X \times Z)$,*

- *Consider two fuzzy relation; R $(X \times Y)$ and S $(Y \times Z)$, then a relation T $(X \times Z)$, can be expressed as (max-min composition)*

$$T = R \circ S$$

$$\mu_T(x, z) = \text{max-min}\,[\mu_R(x, y), \mu_S(y, z)]$$

$$= \vee\,[\mu_R(x, y) \wedge \mu_S(y, z)]$$

- *If algebraic product is adopted, then max-product composition is adopted:*

$$T = R \circ S$$

$$\mu_T(x, z) = \max\,[\mu_R(x, y) \cdot \mu_S(y, z)]$$

$$= \vee\,[\mu_R(x, y) \cdot \mu_S(y, z)]$$

- *The max-min composition can be interpreted as indicating the strength of the existence of relation between the elements of X and Z*
- *Calculations of (R o S) is almost similar to matrix multiplication*
- *Fuzzy relations composition have the same properties of:*

  Distributivity: R o (S U T) = (R o S) U (R o T)

  Associativity: R o (S o T) = (R o S) o T

- *Assume the following universes:* $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, *and* $Z = \{z_1, z_2, z_3\}$, *with the following fuzzy relations.*

$$R = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0.7 & 0.5 \\ x_2 & 0.8 & 0.4 \end{array} \quad \text{and} \quad S = \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline y_1 & 0.9 & 0.6 & 0.2 \\ y_2 & 0.1 & 0.7 & 0.5 \end{array}$$

- *Find the fuzzy relation between X and Z using the max-min and max-product composition*

- ***By max-min composition***

$$\mu_T(x_1, z_1) = \max[\min(0.7, 0.9), \min(0.5, 0.1)] = 0.7$$

$$T = \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline x_1 & 0.7 & 0.6 & 0.5 \\ x_2 & 0.8 & 0.6 & 0.4 \end{array}$$

- ***By max-product composition***

$$\mu_T(x_2, z_2) = \max[(0.8, 0.6), (0.4, 0.7)] = 0.48$$

$$T = \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline x_1 & 0.63 & 0.42 & 0.25 \\ x_2 & 0.72 & 0.48 & 0.20 \end{array}$$

Let us consider two kinds of troubles a PC may suffer from, viz., the *system hangs while running*, and *the system does not boot*. We symbolize the former by h and the later by b and define the set $A = \{h, b\}$ of PC troubles. Two possible causes of these troubles are *computer virus* (v) and *disc crash*(c) and they form the set $B = \{c, v\}$ of PC trouble makers. And finally, let the sources of the causes mentioned above are *internet* (i) and *obsolescence* (o) and $C = \{i, o\}$ is the set of PC trouble causes. The relation between PC troubles and their causes is expressed by R, a fuzzy relation over $A \times B$. Similarly, S is the fuzzy relation over $B \times C$, i.e., the relation between the causes of troubles and the sources of those causes. The relations R and S in terms of their relation matrices are shown below.

$$R = \begin{array}{c} \\ h \\ b \end{array} \begin{array}{c} \begin{array}{cc} v & c \end{array} \\ \begin{bmatrix} 0.7 & 0.2 \\ 0.5 & 0.8 \end{bmatrix} \end{array}, \qquad S = \begin{array}{c} \\ v \\ c \end{array} \begin{array}{c} \begin{array}{cc} i & o \end{array} \\ \begin{bmatrix} 0.9 & 0.7 \\ 0.1 & 0.2 \end{bmatrix} \end{array}$$

The relation between PC troubles and their ultimate sources, *i.e.*, between A and C, can be computed on the basis of R and S above as the max–min composition $R \circ S$. The first element of $R \circ S$, expressed as $(R \circ S)(h, i)$ is computed as follows.

$$(R \circ S)(h, i) = max\,\{min\,(R\,(h, v), S\,(v, i)), min\,(R\,(h, c), S\,(c, i))\}$$
$$= max\,\{min\,(0.7, 0.9), min\,(0.2, 0.1)\}$$
$$= max\,\{0.7, 0.1\}$$
$$= 0.7$$

The rest of the elements of $R \circ S$ can be found in a similar fashion.

$(R \circ S)(h, o) = 0.7$
$(R \circ S)(b, i) = 0.5$
$(R \circ S)(b, o) = 0.5$

And finally we get,

$$R \bullet S = \begin{array}{c} \\ h \\ b \end{array} \begin{array}{c} \begin{array}{cc} i & o \end{array} \\ \begin{bmatrix} 0.7 & 0.7 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$$

Let A = {Mimi, Bob, Kitty, Jina} be a set of four children, B = {Tintin, Asterix, Phantom, Mickey} be a set of four comic characters, and C = {funny, cute, dreamy} be a set of three attributes. The fuzzy relations $R = x$ Likes y is defined on $A \times B$ and $S = x$ IS y is defined on $B \times C$ as shown in Table 2.20 and Table 2.21. Find $R \circ S$.

Table 2.20. Relation matrix for $R = x$ Likes y

	R ≡ *Likes*			
	Tintin	Asterix	Phantom	Mickey
Mimi	0.8	0.5	0.7	0.8
Bob	0.4	0.9	0.3	0.3
Kitty	0.6	0.7	0.4	0.9
Jina	0.3	0.8	0.2	0.5

Table 2.21. Relation matrix for $S = x$ IS y

	S ≡ *IS*		
	funny	cute	dreamy
Tintin	0.6	0.7	0.3
Asterix	0.8	0.4	0.2
Phantom	0.1	0.2	0.1
Mickey	0.9	0.8	0.3

Table 2.22. Relation matrix for $R \circ S$

	ROS		
	funny	cute	dreamy
Mimi	0.8	0.8	0.3
Bob	0.8	0.4	0.3
Kitty	0.9	0.8	0.3
Jina	0.8	0.5	0.3

Zadeh’s Max-Min rule

Zadeh's Max-Min rule

If x is A then y is B with the implication of Zadeh's max-min rule can be written equivalently as :

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y)$$

Here, Y is the universe of discourse with membership values for all $y \in Y$ is 1, that is , $\mu_Y(y) = 1 \forall y \in Y$.

Suppose $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4\}$

and $A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$

$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$ are two fuzzy sets.

We are to determine $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$

The computation of $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$ is as follows:

$$A \times B = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array} \quad \text{and}$$

$$\bar{A} \times Y = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Therefore,

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.4 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Determine the implication relation :
If x **is A then** y **is B**

This R represents **If** x **is A then** y **is B**

IF x is A THEN y is B ELSE y is C.

The relation R is equivalent to

$$R = (A \times B) \cup (\bar{A} \times C)$$

The membership function of R is given by

$\mu_R(x, y) = max[min\{\mu_A(x), \mu_B(y)\}, min\{\mu_{\bar{A}}(x), \mu_C(y)]$

$X = \{a, b, c, d\}$

$Y = \{1, 2, 3, 4\}$

$A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$

$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$

$C = \{(1, 0), (2, 0.4), (3, 1.0), (4, 0.8)\}$

Determine the implication relation :

If x is A then y is B else y is C

$$\text{Here, } A \times B = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

$$\text{and } \bar{A} \times C = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 1.0 & 0.8 \\ 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 1.0 & 0.8 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Let R : If '*job is risky*' Then '*compensation is high*' be a fuzzy rule. There are four jobs job_1, job_2, job_3 and job_4, constituting the universe $job = \{job_1, job_2, job_3, job_4\}$. Also, there are four categories of compensation c_1, c_2, c_3, and c_4 in ascending order. Hence the universe for compensations is $compensation = \{c_1, c_2, c_3, c_4\}$. The fuzzy sets *risky-job* and *high-compensation* are defined on the universes *job* and *compensation* respectively as given below.

$$risky\text{-}job = \frac{0.3}{job_1} + \frac{0.8}{job_2} + \frac{0.7}{job_3} + \frac{0.9}{job_4}$$

$$high\text{-}compensation = \frac{0.2}{c_1} + \frac{0.4}{c_2} + \frac{0.6}{c_3} + \frac{0.8}{c_4}$$

Using Zadeh's interpretation, the truth value of rule R is expressed by the relation

$R = (\textit{risky-job} \times \textit{high-compensation}) \cup (\overline{\textit{risky} - \textit{job}} \times \textit{compensation})$

Now,

$$\textit{risky-job} \times \textit{high-compensation} = \begin{array}{c} \\ job_1 \\ job_2 \\ job_3 \\ job_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.6 & 0.8 \\ 0.2 & 0.4 & 0.6 & 0.7 \\ 0.2 & 0.4 & 0.6 & 0.8 \end{bmatrix} \end{array},$$

$$\text{and, } \overline{\textit{risky} - \textit{job}} \times \textit{compensation} = \begin{array}{c} \\ job_1 \\ job_2 \\ job_3 \\ job_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 0.7 & 0.7 & 0.7 & 0.7 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.6 & 0.7 \\ 0.2 & 0.4 & 0.6 & 0.8 \end{bmatrix} \end{array},$$

and finally, $R = (\textit{risky-job} \times \textit{high-compensation}) \cup (\overline{\textit{risky} - \textit{job}} \times \textit{compensation})$

$$= \begin{array}{c} job_1 \\ job_2 \\ job_3 \\ job_4 \end{array} \begin{bmatrix} 0.7 & 0.7 & 0.7 & 0.7 \\ 0.2 & 0.4 & 0.6 & 0.8 \\ 0.3 & 0.4 & 0.6 & 0.7 \\ 0.2 & 0.4 & 0.6 & 0.8 \end{bmatrix}$$

Hence the matrix R obtained above embodies the information in the fuzzy implication IF *job is risky* THEN *compensation is high* on the basis of the fuzzy concepts *risky-job* and *high-compensation* as defined above.

Fuzzy number

- Definition: a fuzzy set A on $\mathbf{R}$ must posses at least the following three properties
  - A must be a normal fuzzy set
  - A_α must be a closed interval for every $\alpha \in (o, 1]$ (convex)
  - the support of fuzzy set A, A_{0+}, must be bounded

Operations of Interval

Operation of fuzzy number can be generalized from that of crisp interval. Let's have a look at the operations of interval.

$\forall a_1, a_3, b_1, b_3 \in \Re$

$$A = [a_1, a_3], B = [b_1, b_3]$$

Assuming A and B as numbers expressed as interval, main operations of interval are

(1) Addition

$$[a_1, a_3] (+) [b_1, b_3] = [a_1 + b_1, a_3 + b_3]$$

(2) Subtraction

$$[a_1, a_3] (-) [b_1, b_3] = [a_1 - b_3, a_3 - b_1]$$

(3) Multiplication

$$[a_1, a_3] (\bullet) [b_1, b_3] = [a_1 \bullet b_1 \wedge a_1 \bullet b_3 \wedge a_3 \bullet b_1 \wedge a_3 \bullet b_3, a_1 \bullet b_1 \vee a_1 \bullet b_3 \vee a_3 \bullet b_1 \vee a_3 \bullet b_3]$$

(4) Division

$$[a_1, a_3] (/) [b_1, b_3] = [a_1 / b_1 \wedge a_1 / b_3 \wedge a_3 / b_1 \wedge a_3 / b_3, a_1 / b_1 \vee a_1 / b_3 \vee a_3 / b_1 \vee a_3 / b_3]$$

excluding the case $b_1 = 0$ or $b_3 = 0$

(5) Inverse interval

$$[a_1, a_3]^{-1} = [1 / a_1 \wedge 1 / a_3, 1 / a_1 \vee 1 / a_3]$$

excluding the case $a_1 = 0$ or $a_3 = 0$

Example There are two intervals A and B,

$$A = [3, 5], B = [-2, 7]$$

then following operation might be set.

$$A(+)B = [3-2, \quad 5+7] = [1, \quad 12]$$

$$A(-)B = [3-7, \quad 5-(-2)] = [-4, \quad 7]$$

$$\begin{aligned} A(\bullet)B &= [3\bullet(-2) \wedge 3\bullet 7 \wedge 5\bullet(-2) \wedge 5\bullet 7, \quad 3\bullet(-2) \vee \cdots] \\ &= [-10, \quad 35] \end{aligned}$$

$$\begin{aligned} A(/)B &= [3/(-2) \wedge 3/7 \wedge 5/(-2) \wedge 5/7, \quad 3/(-2) \vee \cdots] \\ &= [-2.5, \quad 5/7] \end{aligned}$$

$$B^{-1} = [-2,7]^{-1} = \left[\frac{1}{(-2)} \wedge \frac{1}{7}, \quad \frac{1}{(-2)} \vee \frac{1}{7}\right] = \left[-\frac{1}{2}, \quad \frac{1}{7}\right]$$

Operations of Fuzzy Number

- α-cut
  - Addition: $A(+)B = \bigcup_{\alpha \in [0,1]} \alpha(A_\alpha(+)B_\alpha)$

$$A = [a_1, a_3] \quad a_1, a_3 \in \Re$$

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}], \forall \alpha \in [0, 1], a_1^{(\alpha)}, a_3^{(\alpha)} \in \Re$$

$$B = [b_1, b_3], \quad b_1, b_3, \in \Re$$

$$B_\alpha = [b_1^{(\alpha)}, b_3^{(\alpha)}], \forall \alpha \in [0, 1], b_1^{(\alpha)}, b_3^{(\alpha)} \in \Re$$

operations between A_α and B_α can be described as follows :

$$[a_1^{(\alpha)}, a_3^{(\alpha)}] (+) [b_1^{(\alpha)}, b_3^{(\alpha)}] = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_3^{(\alpha)} + b_3^{(\alpha)}]$$

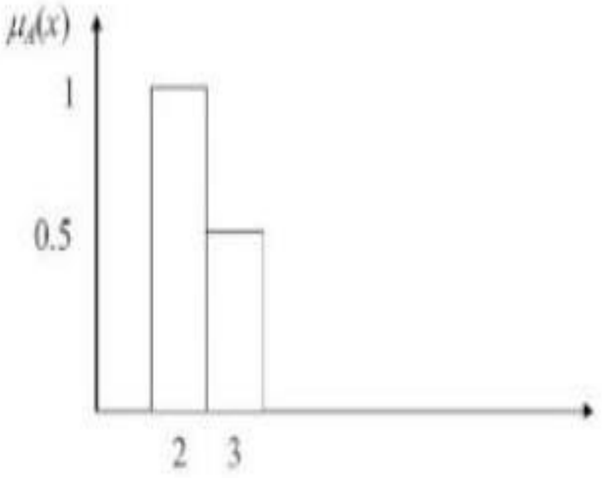
- Subtraction: $A(-)B = \bigcup_{\alpha \in [0,1]} \alpha(A_\alpha(-)B_\alpha)$
- Multiplication: $A(\square)B = \bigcup_{\alpha \in [0,1]} \alpha(A_\alpha(\square)B_\alpha)$
- Division: $A(/)B = \bigcup_{\alpha \in [0,1]} \alpha(A_\alpha(/)B_\alpha)$
- Minimum: $A(\wedge)B = \bigcup_{\alpha \in [0,1]} \alpha(A_\alpha(\wedge)B_\alpha)$
- Maximum: $A(\vee)B = \bigcup_{\alpha \in [0,1]} \alpha(A_\alpha(\vee)B_\alpha)$

Example Addition A(+)B

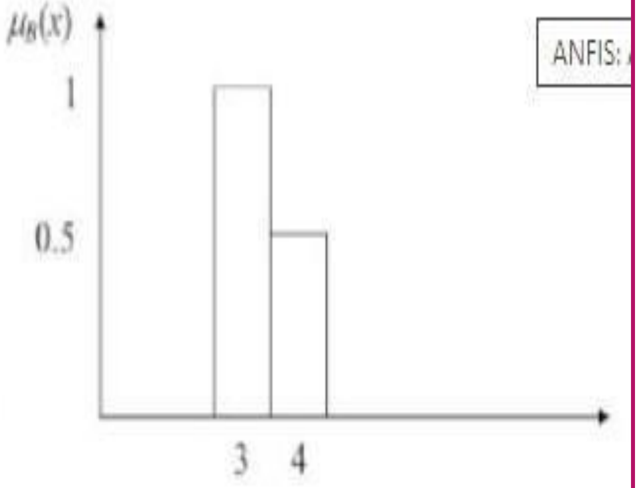
For further understanding of fuzzy number operation, let us consider two fuzzy sets A and B. Note that these fuzzy sets are defined on discrete numbers for simplicity.

$$A = \{(2, 1), (3, 0.5)\},\ B = \{(3, 1), (4, 0.5)\}$$

First of all, our concern is addition between A and B. To induce $A(+)B$, for all $x \in A, y \in B, z \in A(+)B$, we check each case as follows(Fig 5.4) :



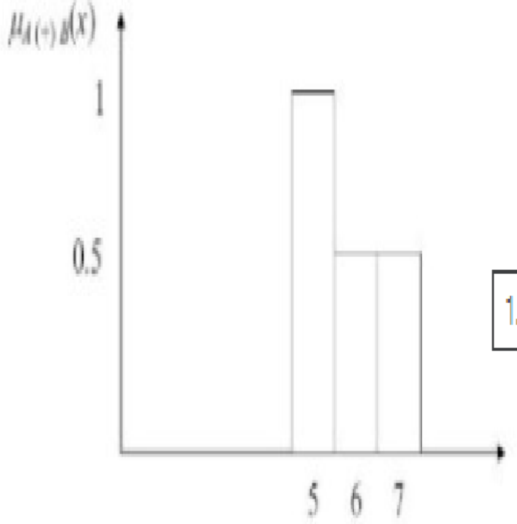
(a) Fuzzy set A



(b) Fuzzy number B

- $\alpha = 0.5$
  - $A_{0.5} = [2,3],\ B_{0.5} = [3,4]$
  - $A_{0.5}(+)B_{0.5} = [5,7]$
- $\alpha = 1.0$
  - $A_{1.0} = 2,\ B_{1.0} = 3$
  - $A_{1.0}(+)B_{1.0} = 5$
- $A(+)B = 0.5(A_{0.5}(+)B_{0.5}) \cup 1.0(A_{1.0}(+)B_{1.0}) = \dfrac{0.5}{[5,7]} \cup \dfrac{1.0}{5}$

$$A(+)B = \frac{1.0}{5} + \frac{0.5}{6} + \frac{0.5}{7}$$



(c) Fuzzy set $A(+)B$

Extension principle

$\forall x, y, z \in \Re$

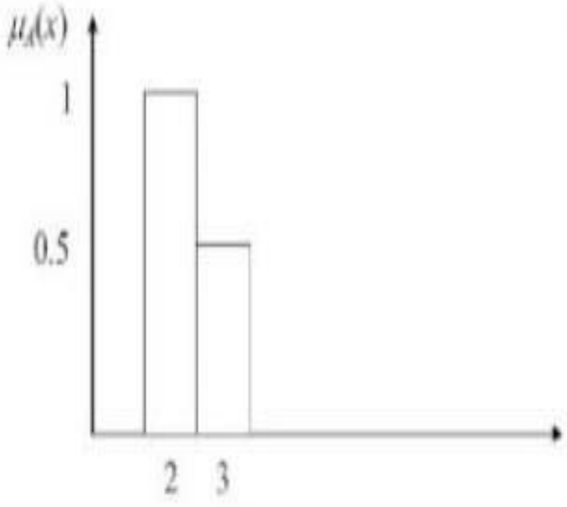
- Addition: $A(+)B$ $\qquad \mu_{A(+)B}(z) = \bigvee_{z=x+y} (\mu_A(x) \wedge \mu_B(y))$
- Subtraction: $A(-)B$ $\qquad \mu_{A(-)B}(z) = \bigvee_{z=x-y} (\mu_A(x) \wedge \mu_B(y))$
- Multiplication: $A(\bullet)B$ $\qquad \mu_{A(\bullet)B}(z) = \bigvee_{z=x\bullet y} (\mu_A(x) \wedge \mu_B(y))$
- Division: $A(/)B$ $\qquad \mu_{A(/)B}(z) = \bigvee_{z=x/y} (\mu_A(x) \wedge \mu_B(y))$
- Minimum: $A(\wedge)B$ $\qquad \mu_{A(\wedge)B}(z) = \bigvee_{z=x\wedge y} (\mu_A(x) \wedge \mu_B(y))$
- Maximum: $A(\vee)B$ $\qquad \mu_{A(\vee)B}(z) = \bigvee_{z=x\vee y} (\mu_A(x) \wedge \mu_B(y))$

Example Addition A(+)B

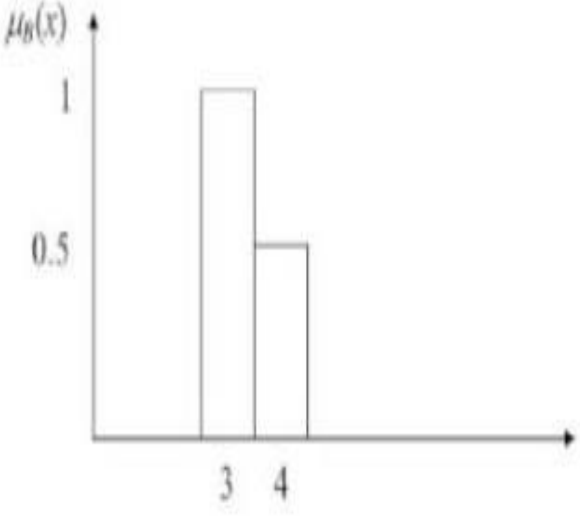
For further understanding of fuzzy number operation, let us consider two fuzzy sets A and B. Note that these fuzzy sets are defined on discrete numbers for simplicity.

$$A = \{(2, 1), (3, 0.5)\},\ B = \{(3, 1), (4, 0.5)\}$$

First of all, our concern is addition between A and B. To induce $A(+)B$, for all $x \in A, y \in B, z \in A(+)B$, we check each case as follows(Fig 5.4) :



(a) Fuzzy set A



(b) Fuzzy number B

i) for $z < 5$,

$$\mu_{A(+)B}(z) = 0$$

ii) $z = 5$

results from $x + y = 2 + 3$

$$\mu_A(2) \wedge \mu_B(3) = 1 \wedge 1 = 1$$

$$\mu_{A(+)B}(5) = \vee_{5=2+3}(1) = 1$$

iii) $z = 6$

results from $x + y = 3 + 3$ or $x + y = 2 + 4$

$$\mu_A(3) \wedge \mu_B(3) = 0.5 \wedge 1 = 0.5$$

$$\mu_A(2) \wedge \mu_B(4) = 1 \wedge 0.5 = 0.5$$

$$\mu_{A(+)B}(6) = \vee_{\substack{6=3+3\\6=2+4}}(0.5, 0.5) = 0.5$$

iv) $z = 7$

results from $x + y = 3 + 4$

$$\mu_A(3) \wedge \mu_B(4) = 0.5 \wedge 0.5 = 0.5$$

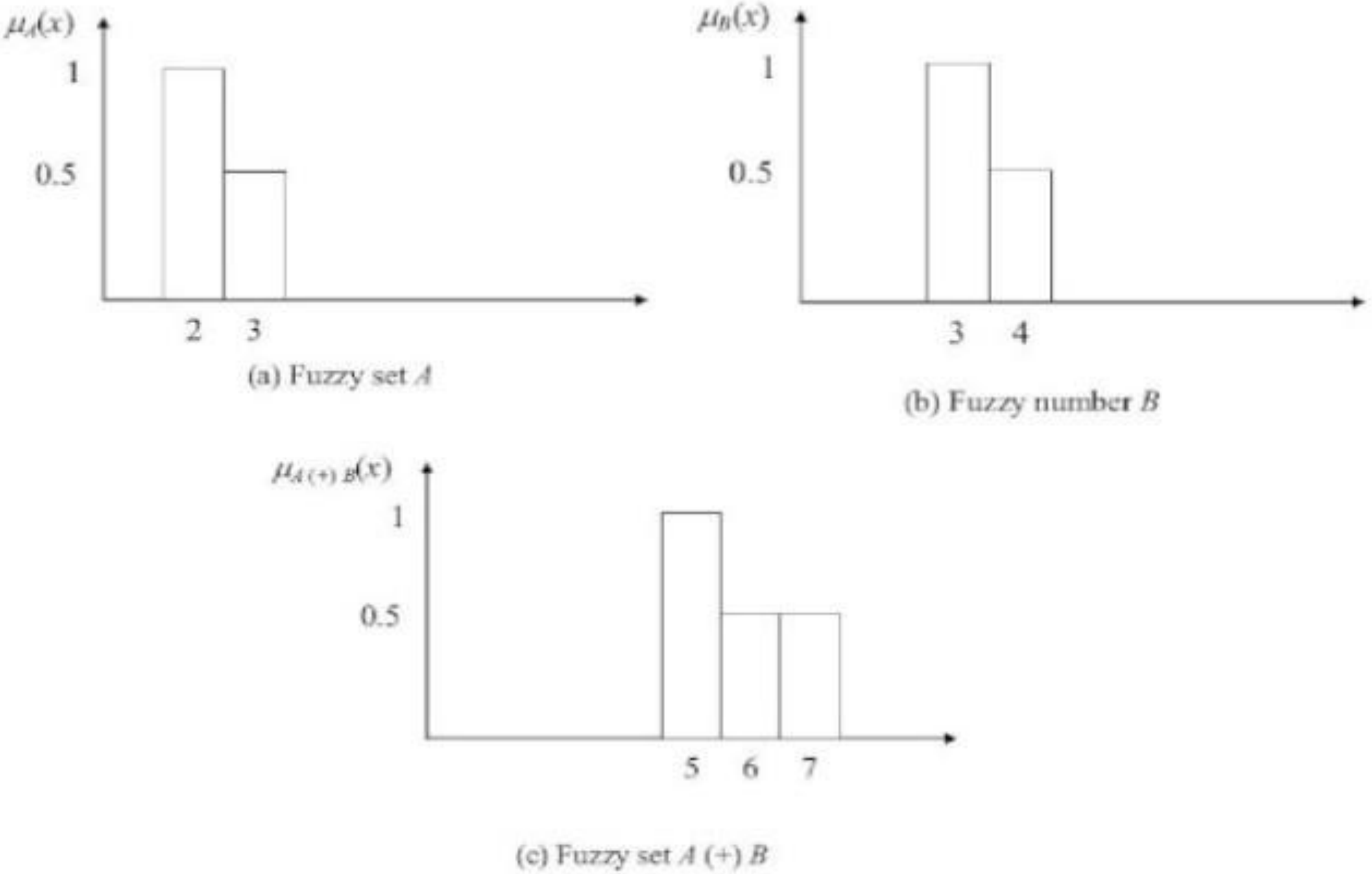
$$\mu_{A(+)B}(7) = \vee_{7=3+4}(0.5) = 0.5$$

v) for $z > 7$

$$\mu_{A(+)B}(z) = 0$$

so $A(+)B$ can be written as

$$A(+)B = \{(5, 1), (6, 0.5), (7, 0.5)\}$$



(a) Fuzzy set A

(b) Fuzzy number B

(c) Fuzzy set A (+) B

Example Subtraction $A(-)B$

Let's manipulate $A(-)B$ between our previously defined fuzzy set A and B. For $x \in A, y \in B, z \in A(-)B$, fuzzy set $A(-)B$ is defined as follows (Fig5. 5).

i) For $z < -2$,

$$\mu_{A(-)B}(z) = 0$$

ii) $z = -2$

results from $x - y = 2 - 4$

$$\mu_A(2) \wedge \mu_B(4) = 1 \wedge 0.5 = 0.5$$

$$\mu_{A(-)B}(-2) = 0.5$$

iii) $z = -1$

results from $x - y = 2 - 3$ or $x - y = 3 - 4$

$$\mu_A(2) \wedge \mu_B(3) = 1 \wedge 1 = 1$$

$$\mu_A(3) \wedge \mu_B(4) = 0.5 \wedge 0.5 = 0.5$$

$$\mu_{A(-)B}(-1) = \bigvee_{\substack{-1=2-3 \\ -1=3-4}} (1, 0.5) = 1$$

iv) $z = 0$

results from $x - y = 3 - 3$

$$\mu_A(3) \wedge \mu_B(3) = 0.5 \wedge 1 = 0.5$$

$$\mu_{A(-)B}(0) = 0.5$$

v) For $z \geq 1$

$$\mu_{A(-)B}(z) = 0$$

so $A(-)B$ is expressed as

$$A(-)B = \{(-2, 0.5), (-1, 1), (0, 0.5)\}$$

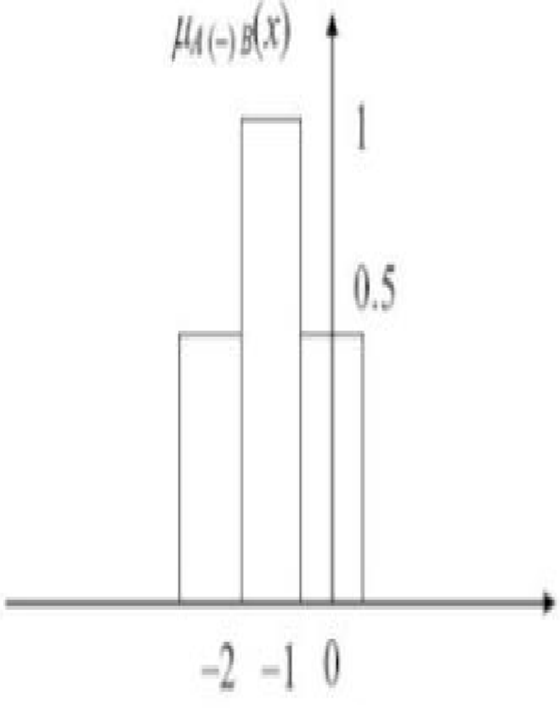


Fig. 5.5. Fuzzy number $A(-)B$

Example Max operation $A(\vee)B$

Let's deal with the operation Max $A(\vee)B$ between A and B

for $x \in A, y \in B, z \in A(\vee)B$, fuzzy set $A(\vee)B$ is defined by $\mu_{A(\vee)B}(z)$.

i) $z \leq 2$

$$\mu_{A(\vee)B}(z) = 0$$

ii) $z = 3$

from $x \vee y = 2 \vee 3$ and $x \vee y = 3 \vee 3$

$$\mu_A(2) \wedge \mu_B(3) = 1 \wedge 1 = 1$$

$$\mu_A(3) \wedge \mu_B(3) = 0.5 \wedge 1 = 0.5$$

$$\mu_{A(\vee)B}(3) = \bigvee_{\substack{3=2\vee 3 \\ 3=3\vee 3}} (1, 0.5) = 1$$

iii) $z = 4$

from $x \vee y = 2 \vee 4$ and $x \vee y = 3 \vee 4$

$$\mu_A(2) \wedge \mu_B(4) = 1 \wedge 0.5 = 0.5$$

$$\mu_A(3) \wedge \mu_B(4) = 0.5 \wedge 0.5 = 0.5$$

$$\mu_{A(\vee)B}(4) = \bigvee_{\substack{4=2\vee 4 \\ 4=3\vee 4}} (0.5, 0.5) = 0.5$$

v) $z > 5$

impossible $\mu_{A(\vee)B}(z) = 0$

so $A(\vee)B$ is defined to be

$$A(\vee)B = \{(3, 1), (4, 0.5)\}$$

so far we have seen the results of operations are fuzzy sets, and thus we come to realize that the extension principle is applied to the operation of fuzzy number.

It will be Continued....