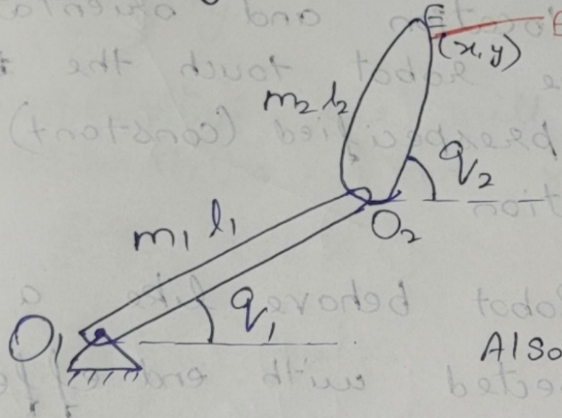
2R Manipulator

↳ Mechanical moving parts
↳ Electrical actuatation
↳ Sense & decide actions.
↳ Usually implemented in code



q_1, q_2 : Joint angles.
Also called Planar elbow manipulator

(x,y): End effector Position
Assume origin at O_1.

↳ Let's assume a motor connected to each link at O_1 & O_2.

↳ Let's assume, we have a way to control either the torques τ_1 and τ_2 applied to the two joints or control angles q_1 & q_2 directly.

↳ We will later study How (Hardware, algorithm, Software), we can control τ_1, τ_2 or q_1, q_2.

Note: angles are sometime q_1, q_2 or P_1 P_2 in various textbooks.

Let's discuss/consider 3 tasks.

Task (T_1): Given arbitrary trajectory of end effector (given (x, y) as a function of time), make the robot follow the trajectory.

Task (T_2): Given a location and orientation on a wall, make the robot touch the ~~ball~~ wall and apply a prespecified (constant) force at that location

Task (T_3): Make the Robot behave like a virtual spring connected with end effector E to a given point (x_0, y_0).

$$x = l_1 \cos q_1 + l_2 \cos q_2$$
$$y = l_1 \sin q_1 + l_2 \sin q_2$$

or using a Simplified notation

~~$x = l_1 \cos q_1 + l_2 \cos q_2$~~

$$\left.\begin{aligned} x &= l_1 c q_1 + l_2 c q_2 \\ y &= l_1 s q_1 + l_2 s q_2 \end{aligned}\right] - (1)$$

Forward kinematics

Differentiating (1),

$$\dot{x} = -l_1 s q_1 \dot{q}_1 - l_2 s q_2 \dot{q}_2$$
$$\dot{y} = l_1 c q_1 \dot{q}_1 + l_2 c q_2 \dot{q}_2$$
$$- (2)$$

In Matrix Form,

$$\overset{A}{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}} = \overset{B}{\begin{bmatrix} -l_1 sq_1 & -l_2 sq_2 \\ l_1 cq_1 & l_2 cq_2 \end{bmatrix}} \overset{C}{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}$$

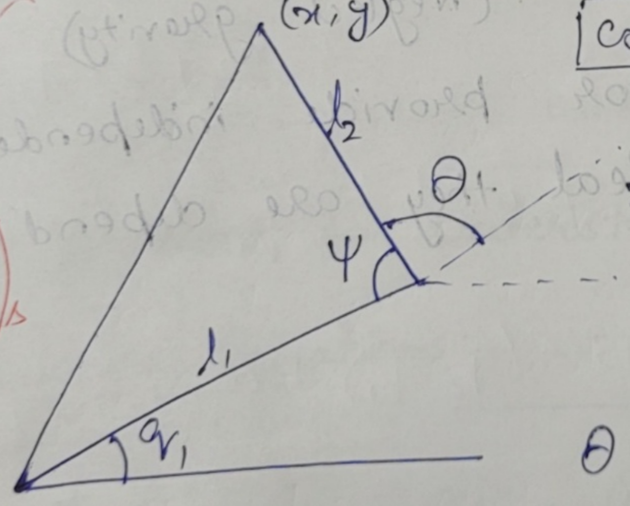
Rotation b/w joint & end effector in terms of velocity.

↳ For task 1, we will need the Reverse Relationships. Given x, y, we need to able to find q_1, q_2.

→ option 1: Solve numerically

→ option 2: Derive a closed form expression (Hard in general, multiple solution)

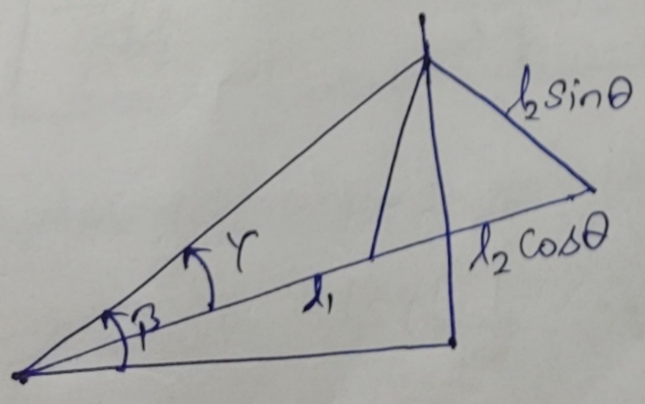
Inverse Kinematics



$\boxed{\text{Cosine Rule}}$ + Switching to ~~actual~~ acute angle

$$x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 \cos\theta$$
$$= l_1^2 + l_2^2 - 2 l_1 l_2 \cos\psi$$

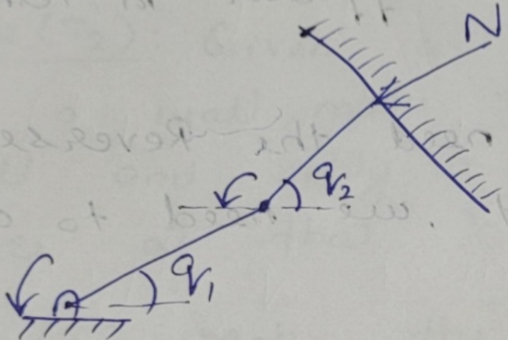
$$\theta = \cos^{-1}\left[\frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right]$$

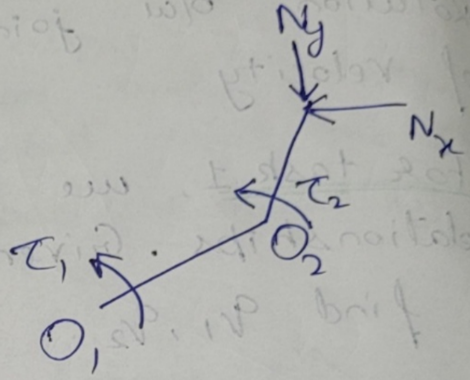


$$q_1 = \beta - \gamma$$
$$= \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2 \cos\theta}\right) \quad \text{— (3)}$$

↳ We will later start using the notations x_d & y_d (q_{1d} & q_{2d}) here for desired values (They are not necessarily actual values).

Task 2 : FBD





Static Equilibrium

Forces applied by Manipulator

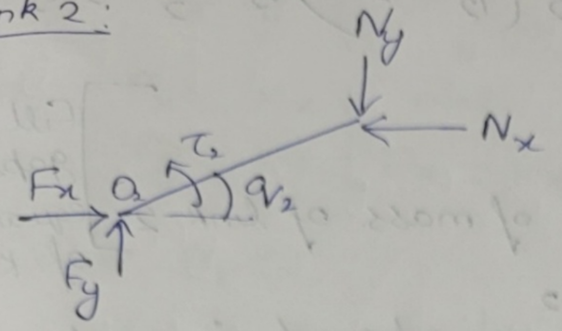
$F_x = -N_x$, $F_y = -N_y$

(Neglect gravity)

(Assume both motor provide independent torque, but in Real they are depend on each other).

FBD of each link separately:

Link 2:

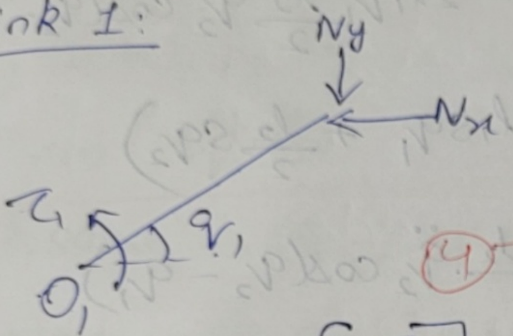


$\Sigma M_{O_2} = 0$

CCW $\leftrightarrow$ +ve

$N_y l_2 cq_2 - N_x l_2 sq_2 = \tau_2$

Link 1:



$\Sigma M_{O_1} = 0$

$$\boxed{\begin{aligned} N_y l_1 cq_1 - N_x l_1 sq_1 &= \tau_1 \\ N_y l_2 cq_1 - N_x l_2 sq_2 &= \tau_2 \end{aligned}} \quad (4)$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 sq_1 & l_1 cq_1 \\ -l_2 sq_2 & l_2 cq_2 \end{bmatrix} \begin{bmatrix} N_x \\ N_y \end{bmatrix}$$

equation (3) along with (4) solves.

For τ_3: Need to understand dynamics.

Lagrange's Equations:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i} \quad (5)$$

Generalized forces derived using Principle of virtual work

Lagrangian, $L = K - V$

K: Kinetic energy, V: Potential Energy

$$K = \underbrace{\frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right)\dot{q}_1^2}_{\text{Pure Rotation of } L_1} + \frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right)\dot{q}_2^2 + \frac{1}{2} m_2 v_{c_2}^2$$

Fill the gaps of equations of K & V

Ne fo wo

$v_{c_2} \rightarrow$ velocity of centre of mass of link 2

$$v_{c_2}^2 = (l_1\dot{q}_1)^2 + \left(\frac{l_2}{2}\dot{q}_2\right)^2 + 2 l_1 \dot{q}_1 \frac{l_2}{2}\dot{q}_2 \cos(q_2 - q_1)$$

$$V = m_1 g \frac{l_1}{2} s q_1 + m_2 g \left(l_1 s q_1 + \frac{l_2}{2} s q_2\right)$$

$$\frac{1}{3} m_1 l_1^2 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + \frac{m_2 l_1 l_2}{2}\ddot{q}_2 \cos(q_2 - q_1)$$
$$- \frac{m_2 l_1 l_2}{2}\dot{q}_2(\dot{q}_2 - \dot{q}_1)\sin(q_2 - q_1) + m_1 g \frac{l_1}{2} c q_1$$
$$+ m_2 g l_1 c q_1 = \tau_1$$

$$\frac{1}{3} m_2 l_2^2 \ddot{q}_2 + \frac{m_2 l_2^2}{4}\ddot{q}_2 + \frac{m_2 l_1 l_2}{2}\ddot{q}_1 \cos(q_2 - q_1)$$
$$- \frac{m_2 l_1 l_2}{2}\dot{q}_1(\dot{q}_2 - \dot{q}_1)\sin(q_2 - q_1) + m_2 g \frac{l_2}{2} s q_2 = \tau_2$$

6

Save this, will need several times later
→ Separate the code to add dynamic & use in animation.

Next, we note that (4) is valid for any forces F_x, F_y. (not just wall forces).

want: $F_x = Kx$ $\quad F_y = Ky$ [more generally, $F_x = K_x(x - x_0)$, $F_y = K_y(y - y_0)$]

from eq (1),

$$F_x = K(l_1 cq_1 + l_2 cq_2)$$
$$F_y = K(l_2 sq_1 + l_2 sq_2)$$

from eq (4),

$$K(l_1 sq_1 + l_2 sq_2) l_2 cq_2 - K(l_1 cq_1 + l_2 cq_2) = \tau_{2s}$$

$$K(l_1 sq_1 + l_2 s q_2) l_1 cq_1 - K(l_1 cq_1 + l_2 cq_2) l_1 sq_1 = \tau_{1s}$$

(7)

⇒ Set motor Torque to be $\tau_1 + \tau_{1s}$ & $\tau_2 + \tau_{2s}$.
Answer to T_3.

other way to solve T_1.... Solve for $q_1 d$ & $q_2 d$ from (3)

$$\downarrow$$
$$\dot{q}_{1d} \quad \ddot{q}_{1d} \quad \dot{q}_{2d}, \ \ddot{q}_{2d}$$
$$\downarrow$$
τ_1 & τ_2 from (6)