Searching for Holes in the Matrix Universe

Lucas Kerbs

Spring 2022

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1. Eventual goal: lift the tools of algebraic topology to spaces of matrices
2. If we have a fixed size, use classical results but this doesn't work for multiple sizes.
3. For multiple (which hopefully I can convince you is interesting) we need to develop some hefty tools
4. To do so, we need to go back to our mathematical roots

Part I: Objects and Maps

A Naive Attempt

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1. That's right—objects and maps.
2. Our Naive attempt involves that looking at lifting functions on $\mathbb{R}$ or $\mathbb{C}$ to accept matrices as their input.

Functional Calculus

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.

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└Part I: Objects and Maps
└Functional Calculus
└Functional Calculus

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- The naive attempt is modeled after the behavior of polynomials. We could plug in an arbitrary matrix into a polynomial without too much trouble, but requiring SA gives us some special behavior.
- Since we are SA, we can diagonalize
- Watch what happens when we plug this into our polynomial

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Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
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Let $g : [a, b] \to \mathbb{C}$ and $D \subset \mathbb{H}$ denote the set of self adjoint matrices with their spectrum in $[a, b]$.

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$$\begin{aligned} g : D &\longrightarrow \mathcal{M} \\ X = U \Lambda U^* &\longmapsto U \begin{bmatrix} g(\lambda_1) & & \\ & \ddots & \\ & & g(\lambda_n) \end{bmatrix} U^*. \end{aligned}$$

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Directional Derivative

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Fix some $X \in \mathbb{H}_n$. The derivative of f at X in the direction $H \in M_n(\mathbb{C})$ is

$$Df(X)[H] = \lim_{t \to 0} \frac{f(X + tH) - f(X)}{t}$$

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- Notice that, with some special attention to what operation we are carrying our, this is the exact same definition as classic multivariable calculus.
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- First we expand $(x + th)^3$—but we can't use the binomial theorem since x and h don't commute
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Part I.5: Objects and Maps

A Second Attempt

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1. Second attempt takes polynomials as an example object instead of as a thing to lift

Some Definitions

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The g-dimensional **Matrix Universe** is

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- should I write the defn for fiber and envelope or just say it verbally

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What the natural functions on $\mathcal{M}^g$?

Definition:
A function $f : D \to \mathcal{M}^{\hat{g}}$ is called **free** if

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Definition: *Free Gradient*
Given a tracial free function f, the free gradient, ∇f, is the unique free function satisfying

$$\operatorname{tr}\left(H \cdot \nabla f(X)\right) = Df(X)[H],$$

where, if $A = (A_1, \ldots, A_g)$ and $B = (B_1, \ldots, B_g)$ are tuples of like-size matrices then $A \cdot B = \sum_{i=1}^{g} A_i B_i$.

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└Uniqueness of the Gradient

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Why should ∇f be unique?

Theorem (Trace Duality)

Let f, g be free functions $\mathcal{M}^{g} \to \mathcal{M}^{\tilde{g}}$. If $\operatorname{tr}(H \cdot f) = \operatorname{tr}(H \cdot g)$ for all tuples H, then $f = g$.

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- probably just give the idea of this proof—use the fact that tr(*) is like an inner product and choose cooridinate matrices

Part II: Analytic Continuation and Monodromy

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Analytic Continuation

The pictures!

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The pictures!

When are two analytic continuations equal?

Theorem (Monodromy I)

Let γ_1, γ_2 be two paths from α to β and Γ_s be a fixed-endpoint homotopy between them. If f can be continued along Γ_s for all $s \in [0, 1]$, then the continuations along γ_1 and γ_2 agree at β.

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Theorem (Monodromy II)

Let $U \subset \mathbb{C}$ be a disk in $\mathbb{C}$ centered at z_0 and $f : U \to \mathbb{C}$ an analytic function. If W is an open, simply connected set containing U and f continues along any path $\gamma \subset W$ starting at z_0, then f has a unique extension to all of W.

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Go through the needed constructions – direct sums of paths, analytic continuation of free functions, etc

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└Free Monodromy

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Theorem (Free Universal Monodromy, Pascoe 2020)

Let f be an analytic free function defined on some ball $B \subset D$, for D an open, connected free set. Then f analytically continues along every path in D if and only if f has a unique analytic continuation to all of D.

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Part III: Homotopy

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Definition:
A continuous function $\gamma : [0,1] \to D$ **essentially takes** X to Y if

$$\gamma(0) = X^{\oplus \ell}, \text{ for some } \ell \in \mathbb{N}$$
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└Part III: Homotopy
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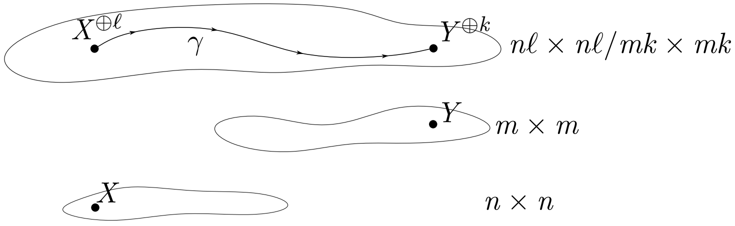

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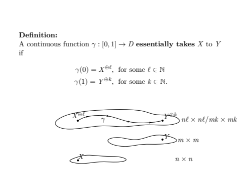

Given γ essentially taking X to Y and β taking Z to W, define

$$\gamma \oplus \beta(t) := \begin{bmatrix} \gamma(t) & 0 \\ 0 & \beta(t) \end{bmatrix}.$$

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Definition:
Let γ and β be paths taking X to Y and Y to Z respectively. We define their product to be the path essentially taking X to Z given by

$$\beta\gamma(t) := \begin{cases} \gamma^{\oplus k}(2t) & t \in [0, 0.5) \\ \beta^{\oplus \ell}(2t-1) & t \in [0.5, 1] \end{cases}$$

where k and ℓ are positive integers chosen to make $\gamma^{\oplus k}$ and $\beta^{\oplus \ell}$ like size matrices for each $t \in [0, 1]$.

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The Full Fundamental Group

For $D \subset \mathcal{M}^g$ a connected free set, the **full fundamenal group**, $\pi_1(D)$, is the group of paths essentially taking X to X up to homotopy equivalence and the relation $\gamma = \gamma^{\oplus k}$.

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- remark about how this misses the link to analytic continuation of functions, but free functions won't do.

Let $D \subset \mathcal{M}^g$ be a connected, open, free set. If there exists a nonempty, simply-connected, open, free $B \subset D$, then we say that D is **anchored**.

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For our purposes, we view γ as coupled with its endpoint. Thus, if γ essentially takes X to Y, then

$$f(\gamma) = \frac{1}{k} f(Y^{\oplus k}).$$

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Trace Equivalence

Definition:
Let $B \subset D$ be an anchor and fix $X \in B$. If γ and β both essentially take X to Y, we say they are **trace equivalent** if, for every global germ f and every path δ taking Y to Z,

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That is, trace equivalent paths are those which cannot be told apart via analytic continuation of global germ.

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Part IV: Cohomology

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└─Part IV: Cohomology
└─A Short Review of Cohomology

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Traditional homology considers a complex of the form

$$\cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} \cdots$$

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The k-th cohomology group is

$$H^k = \frac{\ker d_i}{\operatorname{Im} d_{i-1}}$$

Searching for Holes in the Matrix Universe
└Part IV: Cohomology
└A Short Review of Cohomology

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Traditional homology considers a complex of the form

$$\cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} \cdots$$

While cohomology considers a dual complex

$$\cdots \xrightarrow{d_{n-2}} C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \xrightarrow{d_{n+1}} \cdots$$

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Let D be an anchored set. Denote the set of (globally defined) tracial functions on D by $\mathcal{T}(D)$ and the set of free functions by $\mathcal{F}(D)$.

Searching for Holes in the Matrix Universe
└Part IV: Cohomology
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$$0 \to \mathcal{T}(D) \xrightarrow{\nabla} \mathcal{F}(D) \to \cdots .$$

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└Part IV: Cohomology
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Searching for Holes in the Matrix Universe
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A free function $g : D \to \mathcal{M}^g$ is **exact** if there exists a tracial function $f : D \to \mathbb{C}$ such that $\nabla f = g$.

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The **first tracial cohomology group** is the vector space of closed free functions moduluo the exact free function. We write $H^1_{\operatorname{tr}}(D)$.

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Goal: Show that $\pi_1^{\mathrm{tr}}(D)$ injects into $\mathbb{C}$.

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Searching for Holes in the Matrix Universe
└Part IV: Cohomology
└Injecting into $\mathbb{C}$

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Lemma

Let D be an anchored nc domain. For any $\alpha, \beta \in \pi_1^{\mathrm{tr}}(D)$ and global germ f,

$$f(\alpha\beta) - f(\alpha) = f(\beta) - f(\tau)$$

where τ is the constant path.

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For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{\mathrm{tr}}(D)$, define

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Lemma

The map

$$\Phi : \prod_{\substack{\nabla f \in H^1_{\mathrm{tr}}(D) \\ f \text{ a global germ}}} \pi_1^{\mathrm{tr}}(D) \longrightarrow \prod_{\substack{\nabla f \in H^1_{\mathrm{tr}}(D) \\ f \text{ a global germ}}} \mathbb{C}$$

$$\prod \gamma \longmapsto \prod c^f(\gamma)$$

is an injective homomophism.

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So $\pi_1^{\mathrm{tr}}(D)$ is commutative and torsion free!

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└Part IV: Cohomology
└Characterizing $\pi_1^{\mathrm{tr}}(D)$
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Why?

$$H(t,\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} (\gamma \oplus \gamma_X) \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^*$$

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Searching for Holes in the Matrix Universe
└Part IV: Cohomology
└Characterizing $\pi_1^{\mathrm{tr}}(D)$

Theorem

For D an anchored free set, $\pi_1^{\mathrm{tr}}(D)$ is a torsion free, abelian, divisible group. That is,

$$\pi_1^{\mathrm{tr}}(D) \simeq \bigoplus_{i \in I} \mathbb{Q} = \mathbb{Q}^I$$

for some set I.

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Part V: Computing $\pi_1^{\mathrm{tr}}(D)$

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- hard bc no VC or MV

Let D be an anchored, free, path connected set such that each D_n is nonempty and choose an anchor $B \subset D$ such that each B_n is also nonempty.

Searching for Holes in the Matrix Universe
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Now consider the chain of inclusions:

$$\pi_1^{\mathrm{tr}}(D)_1 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_2 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_6 \hookrightarrow \cdots \hookrightarrow \pi_1^{\mathrm{tr}}(D)_{n!} \hookrightarrow \cdots$$

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The limit of this sequence isomorphic to $\pi_1^{\mathrm{tr}}(D)$!

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Example: $\pi_1^{\mathrm{tr}}(GL)$

Let $GL = \bigcup_{n\in\mathbb{N}} GL_n(\mathbb{C})$.

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Searching for Holes in the Matrix Universe
└ Computing $\pi_1^{\mathrm{tr}}(D)$
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$GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}$, so $\pi_1^{\mathrm{tr}}(D)_1 \simeq \mathbb{Z}$

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Thus, $\pi_1^{\mathrm{tr}}(GL)_2 \simeq \mathbb{Z}\left[\frac{1}{2}\right]$.

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Let $GL = \bigcup_{n\in\mathbb{N}} GL_n(\mathbb{C})$.

$GL_1(\mathbb{C}) = \mathbb{C}\setminus\{0\}$, so $\pi_1^{\mathrm{tr}}(D)_1 \simeq \mathbb{Z}$

Inclusion into $\pi_1^{\mathrm{tr}}(D)_2$ picks up square roots. If $\gamma \in \pi_1^{\mathrm{tr}}(GL)_1$, then

$$\begin{bmatrix}\gamma & \\ & \tau\end{bmatrix} \in \pi_1^{\mathrm{tr}}(GL)_2.$$

Thus, $\pi_1^{\mathrm{tr}}(GL)_2 \simeq \mathbb{Z}\left[\frac{1}{2}\right]$.

Similarly, inclusion into $\pi_1^{\mathrm{tr}}(D)_3$ picks up cube roots:

$$\pi_1^{\mathrm{tr}}(GL)_6 \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$$

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In the n-th inclusion, we pick up n-th roots and so we adjoin $\frac{1}{n}$ to the preceding group. Therefore,

$$\pi_1^{\mathrm{tr}}(GL) \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right]$$

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