Searching for Holes in the Matrix Universe

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Objects and Maps

A Naive Attempt

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$$f\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right) = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}$$

Introduction and Overview

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- A homogeneous ideal I in R is said to be of type $(n; d_1, \dots, d_r)$ if I is generated by generic forms f_i of degree d_i for $i = 1, \dots, r$.
- In other words we can write $I = (f_1, \dots, f_r)$ where each f_i is in some sense "random."

Zariski Open Sets

- An ideal generated by a sequence of f_i's of degrees d_i are chosen "at random." Meaning that we can view $\prod_{i=1}^{r} R_{d_i}$ as an affine space for which the coordinates are the coefficients of the polynomials in the sequence.

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- The set of coefficients where our d_i-forms are generic is open in the Zariski topology. A concern is that the empty set is open in the topology. But if we were to find at least one such ideal, then there are infinitely many.
- We will not become preoccupied with the Zariski topology happening in the background, but will move forward thinking of our choices as "random".

Absolute Value of a Generating Function

Given degrees d_i for $i = 1, \ldots, r$ we can produce a generating function for the forms. Let $\left|\sum_{i=0}^{\infty} a_i t^i\right|$ be the series $\sum_{i=0}^{\infty} b_i t^i$ where

$$b_i = \begin{cases} a_i, & \text{if } a_i > 0 \text{ for all } 0 \leq j \leq i \\ 0, & \text{otherwise} \end{cases}$$

So in the absolute value of a series, one a term becomes nonpositive, it and every term after it is set equal to 0.

Fröberg's Conjecture

In 1985, Fröberg conjectured that ideals generated by generic forms exhibit minimal Hilbert behavior. Recall that the Hilbert Function is another invariant that measures“size” of an ideal. Fröberg's conjecture states that

Conjecture 1.1.(Fröberg's Conjecture)
If k is an infinite field and I is generated by a generic sequence of polynomials of degrees $d_1, \ldots, d_r$, then

$$H_{R/I}(t) = \left| \frac{\prod_{i=1}^{r} \left(1 - t^{d_i}\right)}{(1-t)^n} \right|$$

where H is the Hilbert function.

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- The desired Hilbert function will place constraints on our choices. In particular, there is a homogeneous system of linear equations in our choices for coefficients whose solution set must be avoided.

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- In our later examples, we will see how our underlying linear algebra affects the corresponding free resolutions and Betti tables. These connections are the main theme explored in this thesis.

Equivalent Conjecture

Fröberg's conjecture is equivalent to the following conjecture:

Conjecture 1.2.
If k is an infinite field and $R = k[x_1, \ldots, x_n]$, and $d_1, \ldots, d_r$ are non-negative integers, then a generic sequence of polynomials of polynomials of degrees $d_1, \ldots, d_r$ is semi-regular.

The reason for this shift in conjecture is that semi-regular polynomials are more intuitive to work with. We are able to learn about the structure of our solution in terms of the generators themselves.

Small Cases

- For a particular small set of $\{d_1, \ldots, d_r\}$, the problem devolves into a simple case; it is enough to show there exists a semi-regular homogeneous ideal for which the Hilbert series agrees because then our Zariski set is non-empty.

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- To solve for such an ideal can be checked by asking a computer to try all monomials with "random" coefficients of d_i.
- Specify any n for the number of variables and a list of forms with degrees d_i. For every specific case tried an ideal can be produced given enough time, but to prove Fröberg's conjecture in general has proven quite difficult.

Known cases

In the list below n is the number of variables in the polynomial ring and r is the number of forms. Fröberg's conjecture is known to be true for

- $r \leq n$
- $n = 2$
- $n = 3$
- $r = n + 1$ with char $k = 0$
- $d_1, \cdots = d_r = 3$ and $n \leq 8$

This conjecture is interesting because it is wide open even though any particular case of small integers is immediately knowable.

Preliminaries

Basic Definitions

Throughout this paper $R = k[x_1, \ldots, x_n]$, with the natural grading by degree, k denotes the base field of R. The number r always denotes the number of forms in a sequence of interest in R.

Monomial Definition

- Let $R = k[x_1, \ldots, x_n]$. We say an element $p \in R$ is a **monomial** of degree d if $p = \prod_{i=1}^{n} x_i^{d_i}$ for $d_i \in \mathbb{N} \cup \{0\}$ where $\sum_{i=1}^{n} d_i = d$. We say 1 is a monomial of degree 0 and the zero polynomial has degree -1

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A **polynomial** in R are sums of monomials with coefficients in k.

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For example, if $R = \mathbb{R}[x, y, z]$ then a homogeneous element p of degree 2 would be

$$p = a_1 x^2 + a_2 xy + a_3 xz + a_4 y^2 + a_5 yz + a_6 z^2$$

where $a_1, \ldots, a_6 \in \mathbb{R}$ and at least one $a_i \neq 0$.

Homeogeneous Ideal Definition

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Since $R = k[x_1, \ldots, x_n]$ is know to be Noetherian, then every ideal of R can be finitely generated. For any homogeneous ideal I there exists $f_1, \ldots, f_r$ such that $F = (f_1, \ldots, f_r)$. For our purposes, if we state $I = (f_1, \ldots, f_r)$ we shall assume that I has already been reduced to a set of minimum generators.

Graded Free Resolutions

Definitions

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In other words, for the polynomial ring $R = k[x_1, \ldots, x_n]$, R_i is the k-vector space of the homogeneous polynomials of degree i. So we can express R by

$$R = \bigoplus_{i=1}^{\infty} R_i$$

The $\dim_k R_i$ is the dimension of the i^{th} graded piece of R as a k-vector space.

Definitions

- If A, B, and C are R-Modules, and $\alpha : A \to B, \beta : B \to C$ are homomorphisms, then a pair of homomorphisms

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

  is **exact** if the image of α is equal to $\ker \beta$. In general, a sequence of maps between modules of the form

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  is exact if each pair of consecutive maps is exact.
- A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

  where α is an injection, β is a surjection, and the image of α is the kernel of β.

Definitions

- A **complex of R-Modules** is a sequence of modules F_i and maps $F_i \to F_{i-1}$ such that the compositions $F_{i+1} \to F_i \to F_{i-1}$ are all zero. The homology of this complex at F_i is the module

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- A **free resolution** of an R-Module M is a complex

$$\mathscr{F} : \ \dots \longrightarrow F_n \xrightarrow{\phi_n} \dots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M \longrightarrow 0$$

of free R modules such that $\mathscr{F}$ is exact.

Definitions

- If $R = R_0 \oplus R_1 \oplus \ldots$ is a graded ring then a **graded module** over R is a module M with decomposition

$$M = \bigoplus_{-\infty}^{\infty} M_i$$

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A graded module allows us to keep track of elements degree-wise.

- A resolution $\mathscr{F}$ is a **graded free resolution** if R is a graded ring, the F_i are graded free modules, and the maps are homogeneous maps of degree 0.

Example of a Graded Free Resolution

Let $R = k[x, y]$ and let $I = (x, y)$. Then if we have the following mapping

$$R^2 \xrightarrow[\begin{bmatrix} x \\ y \end{bmatrix}]{} R \longrightarrow R/I$$

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Then we are mapping degree 0 elements in R to degree 1 elements. Notice this sequence is exact, but it is not homogeneous because a degree 0 element gets mapped to a degree 1 element. We will need to fix this to give us a homogeneous sequence as well.

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- Define $M(d)$ to b the altered graded module M shifted in its grading d steps. Then $M(d) \simeq M$ as a module and having grading defined by $M(d)_e = M_{d+e}$. Note that $M(d)$ is sometimes called the **dth Twist of M**.

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So in order to preserve our degrees, we need to grade our left module by 1. So our free resolution becomes

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This grading takes the degree of our map and brings it down by 1. Thus $1 \mapsto x, 1 \mapsto y$ maps a degree 0 element to a degree 0 element as desired.

Hilbert Series

Defintions

- Let M be a finitely generated graded module over $k[x_1, \ldots, x_r]$ with grading generated in positive degrees. The numerical function

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$$H_{R/I}(t) = \sum_{i=0}^{\infty} \dim_k (R/I)_i t^i$$
- Given a series $\sum_{i=0}^{\infty} a_i t^i$, $a_i \in Z$ for all i, let $\left|\sum_{i=0}^{\infty} a_i t^i\right|$ be the series $\sum_{i=0}^{\infty} b_i t^i$ where
$$b_i = \begin{cases} a_i, & \text{if } a_j > 0 \text{ for all } 0 \leq j \leq i \\ 0, & \text{otherwise} \end{cases}$$

Definitions

- A sequence of elements $f_1, \ldots, f_r$ in a ring R is a **regular sequence** on R if the ideal $(f_1, \ldots, f_r)$ is proper and for each i, the image of f_{i+i} is a non-zero divisor in $R/(f_1, \ldots, f_i)$.

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- Let $R = k[x_1, \ldots, x_n]$ and let I be a homogeneous ideal. A nonzero form $f \in R_d$ is called **semi-regular** on R/I if the multiplication maps $(R/I)_{a-d} \xrightarrow{f} (R/I)_a$ are linear maps of maximal rank for all a.

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- A sequence of forms $f_1, \ldots, f_r$ in R with degrees $d_1, \ldots, d_r$ is called a **semi-regular sequence** if f_i is semi-regular on $R/(f_1, \ldots, f_{i-1})$ for all $i = 1, \ldots, r$.

Semi-Regular

An ideal being semi-regular leads to a nice generating function for its Hilbert series. A main take away of why this property is so attractive, is that if we have a semi-regular sequence for our ideal I then we can systematically compute the Hilbert series for R/I.

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The Hilbert series of R/I where I is generated by a semi-regular sequence of forms of degrees $d_1, \ldots, d_r$ is

$$H_{R/I}(t) = \left| \frac{\prod_{i=1}^{r} (1 - t^{d_i})}{(1 - t)^n} \right|$$

Frame Title

Betti Tables and Their Uses

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- If I is an ideal in R, then R/I has a minimal graded free resolution
$$\cdots \to F_1 \to F_0 \to R/I$$
where the F_i are free R-modules. The $\mathbf{i, j}$**th graded Betti number** of R/I is $\beta_{i,j}(R/I)$ which is equal to the dimension, as a k-vector space, of the jth graded piece of F_i.

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So we have $\beta_{i,j}(R/I)$ equals the number of degree j generates in any minimally generated set of the R-module F_i. Moreover, i represents the place in our free resolution while j represents the grading on each copy of our ring that is present at the ith place in our resolution.

Hilbert's Syzygy Theorem

Theorem (Hilbert Syzygy Theorem)

If $R = k[x_1, \ldots, x_n]$, then every finitely generated graded R-module has a finite graded free resolution of length $\leq n$, by finitely generated free modules.

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It follows from Hilbert's Syzygy theorem, $\beta_{i,j}(R/I) = 0$ for $i > n$. Note that $F_0 = R$ and so $\beta_{0,0}(R/I) = 1$ since R is generated by $1 \in R$ as an R-module. Therefore $\beta_{0,j}(R/I) = 0$ for all $j \neq 0$. Since our free resolution has minimal grading, it follows that $B_{i,j}(R/I) = 0$ whenever $i > j$.

Definitions

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- The **Poincaré series** $P_{R/I}(s,t) = \sum_{i=0}^{n} \sum_{j=0}^{\infty} \beta_{i,j} s^i t^j$ is the generating series of the graded Betti numbers.

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- The **Betti Table** of R/I is a table with $\rho + 1$ rows and $n+1$ columns where the i, jth entry, counting from zero, is $\beta_{i,i+j}(R/I)$.

Betti Table Example

total:	1	3	3	1
0:	1	.	.	.
1:	.	1	.	.
2:	.	1	.	.
3:	.	.	1	.
4:	.	1	.	.
5:	.	.	1	.
6:	.	.	1	.
7:	.	.	.	1

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The third row second column the entry $\beta_{2,3} = \beta_{2,2+1} = 1$ corresponds to a 3rd degree basis element.

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$$R(-10) \longrightarrow \begin{matrix} R(-5) \\ \oplus \\ R(-7) \\ \oplus \\ R(-8) \end{matrix} \longrightarrow \begin{matrix} R(-2) \\ \oplus \\ R(-3) \\ \oplus \\ R(-5) \end{matrix} \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

Poincaré Series

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So

$$P_{R/I}(-1,t) = (1-t)^n \left| \frac{\prod_{i=1}^{r} \left(1-t^{d_i}\right)}{(1-t)^n} \right| = \left| \prod_{i=1}^{r} \left(1-t^{d_i}\right) \right|$$

Koszul Complexes and Special Cases

Defining K_i

- Let $f_1, \ldots, f_r$ be a sequence of homogeneous polynomials of degrees $d_1, \ldots, d_r$.

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For example, let $f_1, \ldots, f_5$ be a sequence of homogeneous polynomials of degrees $d_1 = 2$, $d_2 = 3$, $d_3 = 3$, $d_4 = 3$, $d_5 = 4$ if $i = 3$ and $\sigma = \{1, 4, 5\}$. Then

$$\deg \sigma = \sum_{h \in \sigma} d_h = d_1 + d_4 + d_5 = 2 + 3 + 4 = 9.$$

Definition

- The **Koszul complex** is defined by

$$\cdots \to K_2 \to K_1 \to K_0$$

where if $\sigma = \{\sigma_1 < \sigma_2 < \cdots < \sigma_i\}$ has order $i > 0$ then the image of κ_σ in K_{i-1} is $\sum_{h=1}^{i} (-1)^{i+h} f_{\sigma_h} \kappa_{\sigma - \sigma_h}$.