Searching for Holes in the Matrix Universe

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Searching for Holes in the Matrix Universe

2022-06-02



- Eventual goal: lift the tools of algebraic topology to spaces of matrices
- If we only consider 2 × 2 matrices we can use classical theory
 The moment we want more than one size, things the classical
- The moment we want more than one size, things the classical theory breaks down
- Today we will develop some fairly heftly tools to do just that
- $\bullet\,$ Along the way, hopefully I can convince you that this is an interesting question.
- To do so, we need to go back to our mathematical roots

Part I: Objects and Maps

Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Part I: Objects and Maps

- That's right—objects and maps.
- Our Naive attempt involves that looking at lifting functions on $\mathbb R$ or $\mathbb C$ to accept matrices as their input.
- An operator theorist would call this a "functional calculus"

Searching for Holes in the Matrix Universe

Part I: Objects and Maps
Functional Calculus
Functional Calculus

Searching for Holes in the Matrix Universe

• Polynomials are the most well behaved functions we have, so lets start with a polynomial and a self adjoint $(A = A^*)$ matrix.

Let $f \in \mathbb{R}[x]$ and $A \in M_t(\mathbb{C})$ be self adjoint.

• You might say that SA is unnecessary be we can already evaluate a polynomial on a matrix

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint. A is diagonalizable as $A = U\Lambda U^*$

2022-06-02 Part I: Objects and Maps -Functional Calculus -Functional Calculus

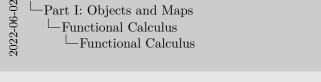
Searching for Holes in the Matrix Universe

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- Since we are SA, we can diagonalize with unitary matrices.

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint. A is diagonalizable as $A = U\Lambda U^*$

$$f(A) = a_n A^n + \dots + a_1 A + a_0 I_k$$



Searching for Holes in the Matrix Universe

Let $f\in\mathbb{R}[x]$ and $A\in M_0(\mathbb{C})$ be self adjoint. A is diagonalizable as $A=UAU^*$ $f(A)=a_aA^a+\cdots+a_1A+a_0I_4$

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- Watch what happens when we plug this into our polynomial need to be careful with the constant term.

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Searching for Holes in the Matrix Universe 2022-06-02 -Part I: Objects and Maps -Functional Calculus -Functional Calculus

Let $f \in \mathbb{R}[x]$ and $A \in M_t(\mathbb{C})$ be self adjoint. A is diagonalizable as $A = U\Lambda U^*$ $= a_-(U\Lambda U^*)^n + \cdots + a_+U\Lambda U^* + a_*I_-$

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Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Functional Calculus
Functional Calculus

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$$\begin{split} f(A) &= a_a A^a + \dots + a_1 A + a_b I_b \\ &= a_a (U\Lambda U^*)^a + \dots + a_1 U\Lambda U^* + a_0 I_0 \\ &= a_a U\Lambda^a U^* - \dots + a_1 U\Lambda U^* + a_0 I_0 U^* \\ &= U \left(a_a \Lambda^a + \dots + a_1 \Lambda + a_0 I_0\right) U^* \end{split}$$

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Part I: Objects and Maps
Functional Calculus
Functional Calculus



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$$f\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right)$$

Searching for Holes in the Matrix Universe -Part I: Objects and Maps -Functional Calculus -Functional Calculus



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Part I: Objects and Maps

Functional Calculus

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- This application along the diagonal is precisely the behavior we want to emulate in the functional calculus.

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Searching for Holes in the Matrix Universe Part I: Objects and Maps

Functional Calculus

Let \mathbb{H}_n be the set of $n \times n$ self adjoint matrices, and define

$$\mathbb{H} = \bigcup_{n \in \mathbb{N}} \mathbb{H}_n, \qquad \mathcal{M} = \bigcup_{n \in \mathbb{N}} M_n(\mathbb{C})$$

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—Part I: Objects and Maps

—Functional Calculus

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- With the polynomial case in mind, we can extend a general function. First, a piece of notation
- Lets grab a function on the real line and the self adjoint matrices with their spectrum in that domain
- Then we can lift g by emulating the behavior of polynomials.
- unwrap a self adjoint matrix, apply g to the diagonal, then wrap it back up
- ullet Something to notice about this functional calculus—it treats direct sums very well
- This is all well and good, but can we do anything with these new functions?

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Part I: Objects and Maps
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Definition:

Let $g:[a,b]\to\mathbb{C}$ and $D\subset\mathbb{H}$ denote the set of self adjoint matrices with their spectrum in [a,b].

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—Part I: Objects and Maps

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Part I: Objects and Maps
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Definition:

Let $g:[a,b]\to\mathbb{C}$ and $D\subset\mathbb{H}$ denote the set of self adjoint matrices with their spectrum in [a,b]. Then

$$g: D \longrightarrow \mathcal{M}$$

$$X = U\Lambda U^* \longmapsto U \begin{bmatrix} g(\lambda_1) & & & \\ & \ddots & & \\ & & g(\lambda_n) \end{bmatrix} U^*.$$

Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Functional Calculus

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- Part I: Objects and Maps

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Important: In this functional calculus,

$$q(X \oplus Y) = q(X) \oplus q(Y)$$

Searching for Holes in the Matrix Universe

Part I: Objects and Maps
Functional Calculus

Let H_n be the set of $n \times n$ self adjoint matrices, and define $H = \bigcup_{n \in \mathbb{N}} H_n(U)$. Definition:
Let g : [n, k] = C and $D \subseteq \mathbb{R}$ denotes the set of self adjoint matrices with respect to $g : D \to M$. $g : D \to M$ $X = U A U' \to U$ Important: In this functional calculus, $g : D \to M$ $X = U A U' \to U$ Important: In this functional calculus,

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Part I: Objects and Maps

Functional Calculus

Directional Derivative

Definition: Directional Derivative

Fix some $X \in \mathbb{H}_n$. The derivative of f at X in the direction $H \in M_n(\mathbb{C})$ is

$$Df(X)[H] = \lim_{t \to 0} \frac{f(X + tH) - f(X)}{t}$$

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Part I: Objects and Maps
Functional Calculus
Directional Derivative

Definition: Directional Derivative Fit some $X \in \mathbb{H}_n$. The derivative of f at X in the direction $H \in M_n(\mathbb{C})$ is $Df(X)[H] = \lim_{t \to \infty} \underbrace{f(X + HH) - f(X)}_{t}$

- We can define a directional derivative—as long as we are careful to have the direction in the same "level-wise" slice.
- Notice that, with some special attention to what operation we are carrying our, this is the exact same definition as classic multivariable calculus.
- There is another formulation that is (generally) more useful for computation

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Alternatively,

$$Df(X)[H] = \frac{df(X+tH)}{dt}\bigg|_{t=0}$$

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Part I: Objects and Maps
Functional Calculus
Directional Derivative

Definition: Directional Derivative Fits some $X \in \mathbb{R}_{+}$. The derivative of f at X in the direction $H \in M_{+}(\mathbb{C})$ is $Df(X)[H] = \lim_{t \to \infty} \frac{f(X + HI) - f(X)}{t}$. Alternatively, $Df(X)[H] = \frac{df(X + HI)}{t}$

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$$g(X + tH) =$$

Part I: Objects and Maps

Functional Calculus

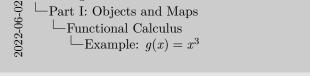
Example: $g(x) = x^3$

Searching for Holes in the Matrix Universe



- Now we consider an example. Since Df(X)[H] is linear, we can just work with a single monomial
- First we expand $(x + th)^3$ —but we can't use the binomial theorem since x and h don't commute
- Once we expand, we take standard derivatives w.r.t t—treating X and H as formal symbols.

$$g(X + tH) = X^{3} + tX^{2}H + tXHX + t^{2}XH^{2} + tHX^{2} + t^{2}HXH + t^{2}H^{2}X + t^{3}H^{3}.$$



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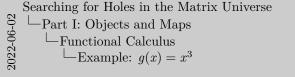
tH) = $X^3 + tX^2H + tXHX + t^2XH^2$ $+ tHX^2 + t^2HXH + t^2H^2X + t^2H^3$.

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From here, we can calculate:





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Searching for Holes in the Matrix Universe
- Part I: Objects and Maps

Functional Calculus

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From here, we can calculate:

$$\frac{d}{dt}g(X + tH) = X^{2}H + XHX + 2tXH^{2} + HX^{2} + 2tHXH + 2tH^{2}X + 3t^{2}H^{3}$$

Searching for Holes in the Matrix Universe Part I: Objects and Maps

Functional Calculus

Example: $g(x) = x^3$

$$\begin{split} g(X+iH) &= X^3 + tX^2H + tXHX + t^2XH^2 \\ &\quad + HX^2 + t^2HXH + t^2H^2X + t^3H^3. \end{split}$$
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$$+ 2tHXH + 2tH^{2}X + 3t^{2}H^{3}$$
$$\frac{d^{2}}{dt^{2}}g(X+tH) = 2XH^{2} + 2HXH + 2H^{2}X + 6tH^{3}$$

Searching for Holes in the Matrix Universe Part I: Objects and Maps

Functional Calculus

Example: $g(x) = x^3$

 $p(X + IH) = X^3 + IX^2H + IXHX + I^2XH^2$ $+ IHX^2 + I^2HXH + I^2H^2X + I^2H^3$, here, we can calculate: $\frac{d}{dt}p(X + IH) = X^2H + XHX + 2IXH^2 + HX^2$ $+ 2IXIXH + 2IH^2X + 3I^2H^2$ $\frac{d}{dt^2}p(X + IH) = 2XH^2 + 2IXIH + 2H^2X + 64H^3$

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From here, we can calculate:

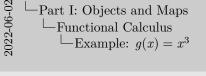
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$$+ 2tHXH + 2tH^{2}X + 3t^{2}H^{3}$$
$$\frac{d^{2}}{dt^{2}}g(X+tH) = 2XH^{2} + 2HXH + 2H^{2}X + 6tH^{3}$$
$$\frac{d^{3}}{dt^{3}}g(X+tH) = 6H^{3}.$$

Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Functional Calculus
Example: $g(x) = x^3$

$$\begin{split} g(X+iH) &= X^3 + LX^2 H + tXHX + t^2 XH^2 \\ &\quad + HX^2 + t^2 HXH + t^2 H^2 X + t^2 H^2, \end{split}$$
 an here, we can calculation $\frac{d}{dt}(X+iH) = X^2 H + XHX + 2tXH^2 + HX^2 \\ &\quad + 2tXH + 2tY^2 X + 2tXH + 2tY^2 X + 3t^2 H^2 \\ &\quad + 2t^2 g(X+iH) = 2XH^2 + 2tIXH + 2H^2 X + 6tH^3 \\ &\quad + 2t^2 g(X+iH) = 6H^3. \end{split}$

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- Now all we have to do is set t=0 in the previous expressions and we get the first three directional derivatives
- Note that they are all in the same direction—mixes derivatives are possible but we won't need them.

And so the first 3 directional derivatives are:

$$Df(X)[H] = X^2H + XHX + HX^2$$

Part I: Objects and Maps

Functional Calculus

Example: $g(x) = x^3$

Searching for Holes in the Matrix Universe

And so the first 3 directional derivatives are: $Df(X)[H] = X^2H + XHX + HX^2 \label{eq:direction}$

- Now all we have to do is set t = 0 in the previous expressions and we get the first three directional derivatives
- Note that they are all in the same direction—mixes derivatives are possible but we won't need them.

Example: $g(x) = x^3$

And so the first 3 directional derivatives are:

$$Df(X)[H] = X^2H + XHX + HX^2$$

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Part I: Objects and Maps

Functional Calculus

Example: $g(x) = x^3$

Searching for Holes in the Matrix Universe

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Searching for Holes in the Matrix Universe Part I: Objects and Maps Functional Calculus Example: $g(x) = x^3$

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Part I.5: Objects and Maps A Second Attempt

Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Matrix Universe

Part I.5: Objects and Maps

- In seeking a more general theory we need to leave the world of this "SA functional calculus" behind.
- Rather than lifting functions to be matrix valued, we will define *new* objects that behave like those we just looked at.

Some Definitions

Definition: Matrix Universe
The g-dimensional Matrix Universe is

$$\mathcal{M}^g = \bigcup_{n \in \mathbb{N}} (M_n(\mathbb{C}))^g$$

Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Matrix Universe

Some Definitions



- Definition of the matrix universe g tuples of matrices of all sizes
- By convetion, tuples are likes size

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Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Matrix Universe

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Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Matrix Universe

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- In math we often think about substructures that capture the implicit structure our space (subgroup, subspace, etc)
- In the nc setting, this is a *free set*, also called nc set
- direct sums and unitary conjugation are component wise
- If you see a D, you can assume that it is a free set.
- A subscript denotes a level-wise slice
- Note that this requires a lot of structure on free sets—we want to put a name to these structure.

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Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Matrix Universe

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For D a free set, define $D_n = D \cap (M_n(\mathbb{C}))^g$.

Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Matrix Universe

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Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Matrix Universe

Definition: Fiber

Given $X \in \mathcal{M}^g$, a tuple of $n \times n$ matrices, the **fiber** of X is the set

$$\{X^{\oplus k} \mid k \in \mathbb{N}\}.$$

Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Matrix Universe

Definition: Filter Given $X\in \mathcal{M}^q$, a tuple of $n\times n$ matrices, the fiber of X is the set $\{X^{\otimes k}\mid k\in\mathbb{N}\}.$

- The fiber is all the points "above" a given point. Conceptually, we imagine identification along the fiber—this will become important when we start doing topology
- The envelope (which will be less important to us) is the unitary smearing of the fiber at each level.

Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Matrix Universe

Definition: Fiber

Given $X \in \mathcal{M}^g$, a tuple of $n \times n$ matrices, the **fiber** of X is the set

$$\{X^{\oplus k} \mid k \in \mathbb{N}\}.$$

Definition: Envelope

Given $X \in \mathcal{M}^g$, a tuple of $n \times n$ matrices, the **envelope** of X is the set

$$\{U^*X^{\oplus k}U \mid k \in \mathbb{N}, U \text{ Unitary}\}.$$

Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Matrix Universe

Definition: Fiber $\{X^{\otimes k} \mid k \in \mathbb{N}\}.$ Obtainition: Let $X \in \mathbb{N}$ be the set $\{X^{\otimes k} \mid k \in \mathbb{N}\}.$ Definition: Enough $x \in \mathbb{N}$ tuple of $x \in \mathbb{N}$ matrices, the envelope of X is the set

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A bit of topology

- topology, we need a point set topology first—so there is a natural question.
 - Bad news: there isn't a natural choice

Searching for Holes in the Matrix Universe

- There are a handful of candidates (fine, fat, free, nc Zariski). I wish we had time to go into detail.
- For us, free sets are open if their level-wise restriction is open
- Other point-set characterizations are similar.

Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Matrix Universe

A bit of topology

What does it mean for $D \subset \mathcal{M}^g$ to be open?

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Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Matrix Universe
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Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Matrix Universe

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Searching for Holes in the Matrix Universe
Part I: Objects and Maps
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Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Matrix Universe

A bit of topology

What does it mean for $D \subset \mathcal{M}^g$ to be open?

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Searching for Holes in the Matrix Universe
Part I: Objects and Maps
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topology

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Searching for Holes in the Matrix Universe

What the natural functions on \mathcal{M}^g ?

Part I: Objects and Maps

Matrix Universe

What the natural functions on \mathcal{M}^g ?

- We have our objects, but what are the maps?
- Free functions are defined to be anything that behaves like a polynomial
- tracial functions look like traces

Searching for Holes in the Matrix Universe

• For both of these maps, the directional derivative is defined identically as before—but tracial functions get something extra.

Definition:

A function $f: D \to \mathcal{M}^{\hat{g}}$ is called **free** if

- 2 $f(UXU^*) = f(U)f(X)f(U^*)$ where X and U are like-size and U is unitary.

Searching for Holes in the Matrix Universe -Part I: Objects and Maps -Matrix Universe What the natural functions on \mathcal{M}^g ?

A function $f: D \rightarrow M^{\frac{1}{2}}$ is called **free** if

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Definition:

A function $f: D \to \mathbb{C}$ is a **tracial function** if

- **1** $f(X \oplus Y) = f(X) + f(Y)$
- 2 $f(UXU^*) = f(X)$ where X and U are like-size and U is unitary.

Searching for Holes in the Matrix Universe hat the natural functions on M^g -Part I: Objects and Maps -Matrix Universe What the natural functions on \mathcal{M}^g ?

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Uniqueness of the Gradient

Definition: Free Gradient

Given a tracial function f, the free gradient, $\nabla f,$ is the unique free function satisfying

$$\operatorname{tr}(H \cdot \nabla f(X)) = Df(X)[H],$$

where, if $A = (A_1, ..., A_g)$ and $B = (B_1, ..., B_g)$ are tuples of like-size matrices then $A \cdot B = \sum_{i=1}^g A_i B_i$.

Searching for Holes in the Matrix Universe

—Part I: Objects and Maps

—Uniqueness of the Gradient

Definition: Free Gradient Given a tracial function f, the free gradient, ∇f , is the unique free function satisfying $\operatorname{tr}(H \cdot \nabla f(X)) = Df(X)[H],$ where, if $A = (A_1, \dots, A_g)$ and $B = (B_1, \dots, B_g)$ are tuples of

- The ∇ of a free function is the unique free function satisfy this equation—where the * is just like the dot product
- Whenever you see tr(*) I want you to think of the inner product—it is slightly distinct but it will make a lot of things make more sense
- Some of you may be hesitant at the fact that I claim ∇ is unique. Why should this be true?

Let f, g be free functions $\mathcal{M}^g \to \mathcal{M}^{\tilde{g}}$. If $\operatorname{tr}(H \cdot f) = \operatorname{tr}(H \cdot g)$ for all tuples H, then f = q.

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Searching for Holes in the Matrix Universe Part I: Objects and Maps -Uniqueness of the Gradient Why should ∇f be unique?



- f = q whenever the domains overlap
- In the vector space setting—with an inner product—this is a fairly immediate result. You would show it by picking vectors of all 0's and a single 1.
 - You prove this identically—but with coordinate matrices instead of coordinate vectors.

Part II: Analytic Continuation and Monodromy

Analytic Continuation

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Figures of generic analytic continuation

—Part II: Monodromy

└─Analytic Continuation

└─Analytic Continuation

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Figures of generic analytic continuation

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Part II: Monodromy

Analytic Continuation

figures of the example of continuing $\log(x)$ through UHP and LHP

Searching for Holes in the Matrix Universe
Part II: Monodromy
Analytic Continuation

figures of the example of continuing $\log(\mathbf{x})$ through UHP and LHP

Theorem (Monodromy I)

Let γ_1, γ_2 be two paths from α to β and Γ_s be a fixed-endpoint homotopy between them. If f can be continued along Γ_s for all $s \in [0, 1]$, then the continuations along γ_1 and γ_2 agree at β .

Searching for Holes in the Matrix Universe

Part II: Monodromy

Monodromy

When are two analytic continuations equal?

Here are two analytic continuations equal? Theorem (Monodromy I) $Lid_{1,1,2}$ be two paths from α to β and Γ , be a finite-adoptical monotopy between the, If I can be continuately Γ , for all $\alpha \in [0,1]$, then the continuations along Γ , δ and γ_{2} agree at β .

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figure of the homotopy

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Part II: Monodromy
Monodromy
When are two analytic continuations equal?

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Theorem (Monodromy II)

Let $U \subset \mathbb{C}$ be a disk in \mathbb{C} centered at z_0 and $f: U \to \mathbb{C}$ an analytic function. If W is an open, simply connected set containing U and f continues along any path $\gamma \subset W$ starting at z_0 , then f has a unique extension to all of W.

Searching for Holes in the Matrix Universe

Part II: Monodromy

Monodromy

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figure of the simply connected larger set

Searching for Holes in the Matrix Universe

Part II: Monodromy

Monodromy

When are two analytic continuations equal?

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Before we look at a free analogue of the monodromy theorem, we need to ask an important question: What does it mean for a free function to be analytic?

2022-06-02 Part II: Monodromy -Free Monodromy What about the nc case?

Searching for Holes in the Matrix Universe

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• be sure to comment on the theorem—it is powerful!

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A: $f: D \to \mathcal{M}^{\hat{g}}$ is analytic if it is analytic as a function of each D_n .

Part II: Monodromy
Free Monodromy
What about the nc case?

Searching for Holes in the Matrix Universe

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Theorem (Agler, McCarthy (2016))

Let $f: D \to \mathcal{M}^{\hat{g}}$ be a free function. If f is locally bounded on each D_n , then f is an analytic free function.

Searching for Holes in the Matrix Universe

• be sure to comment on the theorem—it is powerful!

Searching for Holes in the Matrix Universe

Part II: Monodromy

Free Monodromy

Q: What about the nc case? Can we say anything similar?

Searching for Holes in the Matrix Universe

Part II: Monodromy
Free Monodromy

Searching for Holes in the Matrix Universe └Part II: Monodromy ⊢Free Monodromy

Q: What about the nc case? Can we say anything similar?

Theorem (Free Universal Monodromy, Pascoe 2020)

Let f be an analytic free function defined on some ball $B \subset D$, for D an open, connected free set. Then f analytically continues along every path in D if and only if f has a unique analytic continuation to all of D.

Searching for Holes in the Matrix Universe 2022-06-02 -Part II: Monodromy -Free Monodromy

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2022-06-02 -Part II: Monodromy -Free Monodromy Consequences of Free Monodromy

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onsequences of Free Monodromy

Consequences of Free Monodromy

• Free functions can't detect holes!

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Part II: Monodromy
Free Monodromy
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Searching for Holes in the Matrix Universe

Part II: Monodromy
Free Monodromy
Consequences of Free Monodromy

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sequences of Free Monodromy

If we want a fundamental group governed by analytic continuation, we need to look elsewhere.

Part III: Homotopy

Searching for Holes in the Matrix Universe

—Part III: Homotopy

Part III: Homotopy

Searching for Holes in the Matrix Universe
-Part III: Homotopy
-A First Fundamental Group

Definition:

A continuous function $\gamma:[0,1]\to D$ essentially takes X to Y if

$$\gamma(0) = X^{\oplus \ell}$$
, for some $\ell \in \mathbb{N}$
 $\gamma(1) = Y^{\oplus k}$, for some $k \in \mathbb{N}$.

Searching for Holes in the Matrix Universe

Part III: Homotopy

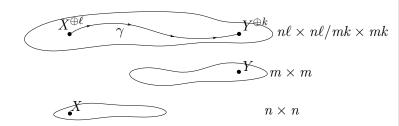
A First Fundamental Group

$$\begin{split} \textbf{Definition:} & \text{A continuous function } \gamma : [0,1] \to D \text{ essentially takes } X \text{ to } Y \\ & \text{if} \\ & \gamma(0) = X^{\otimes \ell}, \text{ for some } \ell \in \mathbb{N} \\ & \gamma(1) = Y^{\otimes k}, \text{ for some } k \in \mathbb{N}. \end{split}$$

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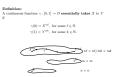


Searching for Holes in the Matrix Universe

Part III: Homotopy

A First Fundamental Group

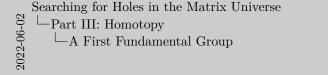
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Searching for Holes in the Matrix Universe └─Part III: Homotopy └─A First Fundamental Group

Given γ essentially taking X to Y and β taking Z to W, define

$$\gamma \oplus \beta(t) := \begin{bmatrix} \gamma(t) & 0 \\ 0 & \beta(t) \end{bmatrix}.$$





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Definition:

Let γ and β be paths taking X to Y and Y to Z respectively. We define their product to be the path essentially taking X to Z given by

$$\beta \gamma(t) := \begin{cases} \gamma^{\oplus k}(2t) & t \in [0, 0.5) \\ \beta^{\oplus \ell}(2t - 1) & t \in [0.5, 1] \end{cases}$$

where k and ℓ are positive integers chosen to make $\gamma^{\oplus k}$ and $\beta^{\oplus \ell}$ like size matrices for each $t \in [0,1]$.

Searching for Holes in the Matrix Universe
Part III: Homotopy
A First Fundamental Group

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The Full Fundamental Group

For $D \subset \mathcal{M}^g$ a connected free set, the **full fundamenal group**, $\pi_1(D)$, is the group of paths essentially taking X to X up to homotopy equivalence and the relation $\gamma = \gamma^{\oplus k}$.

Searching for notes in the Matrix Universe	
Part III: Homotopy	
└A First Fundamental Group	
└─The Full Fundamental Group	

$\subset M^{g}$ a connected free set, the full fundamen $\pi_{1}(D)$, is the group of paths essentially taking	
comotopy equivalence and the relation $\gamma = \gamma^{\oplus k}$.	

The Full Fundamental Group

• remark about how this misses the link to analytic continuation of functions, but free functions won't do.

Searching for Holes in the Matrix Universe Part III: Homotopy A Second Fundamental Group

Let $D \subset \mathcal{M}^g$ be a connected, open, free set. If there exists a nonempty, simply-connected, open, free $B \subset D$, then we say that D is **anchored**.

Searching for Holes in the Matrix Universe

Part III: Homotopy

A Second Fundamental Group

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Searching for Holes in the Matrix Universe Part III: Homotopy A Second Fundamental Group

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For D an anchored set, and $B \subset D$ its anchor, we call a tracial function $f: B \to \mathbb{C}$ a **global germ** if it analytically continues along every path in D which starts in B.

Searching for Holes in the Matrix Universe

Part III: Homotopy

A Second Fundamental Group

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For our purposes, we view γ as coupled with its endpoint. Thus, if γ essentially takes X to Y, then

$$f(\gamma) = \frac{1}{k} f(Y^{\oplus k}).$$

Searching for Holes in the Matrix Universe

Part III: Homotopy

A Second Fundamental Group

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Trace Equivalence

Definition:

Let $B \subset D$ be an anchor and fix $X \in B$. If γ and β both essentially take X to Y, we say they are **trace equivalent** if, for every global germ f and every path δ taking Y to Z,

$$f(\delta \gamma) = f(\delta \beta).$$

Searching for Holes in the Matrix Universe
Part III: Homotopy
A Second Fundamental Group
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Searching for Holes in the Matrix Universe
Part III: Homotopy
A Second Fundamental Group
Trace Equivalence

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Searching for Holes in the Matrix Universe

Searching for Holes in the Matrix Universe

• remark that the choice of base point is irrelevant. and that it is a

The Tracial Fundamental Group

quotient of the full fundamental group

The Tracial Fundamental Group

Let $D \subset \mathcal{M}^g$ be an anchored space with B is anchor. For $X \in B$ define $\pi_1^{\mathrm{tr}}(D)$ to be the group of trace equivalent paths essentially taking X to X.

Searching for Holes in the Matrix Universe
Part III: Homotopy
LA Second Fundamental Group
The Tracial Fundamental Group

Searching for Holog in the Matrix Universe

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The Tracial Fundamental Group

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Searching for Holes in the Matrix Universe
Part III: Homotopy
A Second Fundamental Group
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Searching for Holes in the Matrix Universe

-Part IV: Cohomology

Part IV: Cohomology

Part IV: Cohomology

Searching for Holes in the Matrix Universe

Part IV: Cohomology

A Short Review of Cohomology

Searching for Holes in the Matrix Universe
Part IV: Cohomology
A Short Review of Cohomology

Searching for Holes in the Matrix Universe Part IV: Cohomology A Short Review of Cohomology

Traditional homology considers a complex of the form

$$\cdots \stackrel{\partial_{n-1}}{\longleftarrow} C_{n-1} \stackrel{\partial_n}{\longleftarrow} C_n \stackrel{\partial_{n+1}}{\longleftarrow} C_{n+1} \stackrel{\partial_{n+2}}{\longleftarrow} \cdots$$

Searching for Holes in the Matrix Universe
Part IV: Cohomology
A Short Review of Cohomology

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Searching for Holes in the Matrix Universe Part IV: Cohomology A Short Review of Cohomology

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While cohomology considers a dual complex

$$\cdots \xrightarrow{d_{n-2}} C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \xrightarrow{d_{n+1}} \cdots$$

Searching for Holes in the Matrix Universe
Part IV: Cohomology
A Short Review of Cohomology

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Searching for Holes in the Matrix Universe Part IV: Cohomology

LA Short Review of Cohomology

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In general, C^k is a group of functions into some abelian group.

Searching for Holes in the Matrix Universe
Part IV: Cohomology
A Short Review of Cohomology

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Searching for Holes in the Matrix Universe Part IV: Cohomology

A Short Review of Cohomology

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The k-th cohomology group is

$$H^k = \frac{\ker d_i}{\operatorname{Im} d_{i-1}}$$

Searching for Holes in the Matrix Universe

Part IV: Cohomology

A Short Review of Cohomology

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 \cdots $\stackrel{\partial n-1}{\partial n}$ $C_{n-1} \stackrel{\partial n}{\partial n}$ $C_n \stackrel{\partial n+1}{\partial n}$ $C_{n+1} \stackrel{\partial n+1}{\partial n}$ While cohomology considers a dual complex $\stackrel{\partial n-2}{\partial n}$ $C_{n-1} \stackrel{\partial n-1}{\partial n}$ $C_n \stackrel{\partial n}{\partial n}$ $\stackrel{\partial n}{\partial n}$ $\stackrel{\partial n-1}{\partial n}$ $\stackrel{\partial n}{\partial n}$

In general, \mathbb{C}^k is a group of functions into some abelian group. The k-th cohomology group is

 $H^{k} = \frac{\ker d_{i}}{\operatorname{Im} d_{i-1}}$

Let D be an anchored set. Denote the set of (globally defined) tracial functions on D by $\mathcal{T}(D)$ and the set of free functions by $\mathcal{F}(D)$.

Searching for Holes in the Matrix Universe

Part IV: Cohomology

Tracial Cohomology

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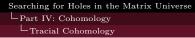
Let D be an anchored set. Denote the set of (globally defined) tracial functions on D by $\mathcal{T}(D)$ and the set of free functions by $\mathcal{F}(D)$. For $f \in \mathcal{T}(D)$, $\nabla f \in \mathcal{F}(D)$ —so we have the beginnings of a chain complex!

$$0 \to \mathcal{T}(D) \xrightarrow{\nabla} \mathcal{F}(D) \to \cdots$$

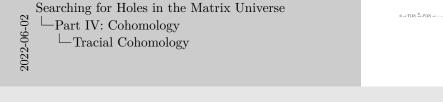
Searching for Holes in the Matrix Universe
Part IV: Cohomology
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test text

Searching for Holes in the Matrix Universe
- Part IV: Cohomology
- Tracial Cohomology

$$0 \to \mathcal{T}(D) \xrightarrow{\nabla} \mathcal{F}(D) \to \cdots$$

A free function $g: D \to \mathcal{M}^g$ is **exact** if there exists a tracial function $f: D \to \mathbb{C}$ such that $\nabla f = g$.



• test text

Searching for Holes in the Matrix Universe

Part IV: Cohomology

Tracial Cohomology

$$0 \to \mathcal{T}(D) \xrightarrow{\nabla} \mathcal{F}(D) \to \cdots$$

A free function $g: D \to \mathcal{M}^g$ is **exact** if there exists a tracial function $f: D \to \mathbb{C}$ such that $\nabla f = g$.

A free function $q: D \to \mathcal{M}^g$ is **closed** if

$$\operatorname{tr}(K \cdot Dq(X)[H]) = \operatorname{tr}(H \cdot Dq(X)[K])$$

for all directions H, K.



• test text

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D 6 111

Definition: The **first tracial cohomology group** is the vector space of closed free functions moduluo the exact free function. We write $H^1_{tr}(D)$.

Searching for Holes in the Matrix Universe

Part IV: Cohomology

Tracial Cohomology

Tracial Cohomology

Between $(D \to A)(D) \to A(D)$ Tracial Cohomology

Tracial Cohomology

Tracial Cohomology

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Part IV: Cohomology
Tracial Cohomology
What out global germs?

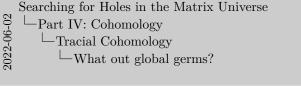
Searching for Holes in the Matrix Universe

 • For $f\colon B\to \mathbb{C}$ a global germ, since f analytically continues along every path, so does $\nabla f.$

What out global germs'

What out global germs?

- For $f: B \to \mathbb{C}$ a global germ, since f analytically continues along every path, so does ∇f .
- Free Monodromy means that ∇f has a global extension.



For f: B → C a global germ, since f analytically continue along every path, so does ∇f.
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- Important: If f is a global germ, then ∇f is not necessarily exact since $\mathcal{T}(D)$ is the set of tracial functions defined on all of D.

Searching for Holes in the Matrix Universe
Part IV: Cohomology
Tracial Cohomology
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Goal: Show that $\pi_1^{\mathrm{tr}}(D)$ injects into \mathbb{C} .

Searching for Holes in the Matrix Universe $\begin{tabular}{l} \line \l$

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Lemma

Let D be an anchored nc domain. For any $\alpha, \beta \in \pi_1^{tr}(D)$ and global germ f,

$$f(\alpha\beta) - f(\alpha) = f(\beta) - f(\tau)$$

where τ is the constant path.

Searching for Holes in the Matrix Universe
Part IV: Cohomology
Injecting into C

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D be an enchand ne domain. For any $\alpha, \beta \in \pi_1^{\mathrm{tr}}(D)$ and if gerns f, $f(\alpha\beta) - f(\alpha) = f(\beta) - f(\tau)$ $r\tau$ is the constant path.

Searching for Holes in the Matrix Universe

Part IV: Cohomology

Injecting into C

For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{tr}(D)$, define

$$c^f(\gamma) = f(\gamma) - f(\tau)$$

Part IV: Cohomology
Injecting into C

Searching for Holes in the Matrix Universe

For D an anchored set, $X\in B_1$ the base point, f a global germ and $\gamma\in\pi_1^{p'}(D),$ define $c^f(\gamma)=f(\gamma)-f(\tau)$

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$$c^f(\gamma) = f(\gamma) - f(\tau)$$

 c^f gives us a homomorphism into $\mathbb{C}!$

Searching for Holes in the Matrix Universe Part IV: Cohomology Injecting into \mathbb{C}

For D an anchored set, $X\in B_1$ the base point, f a global germ, and $\gamma\in u_1^{\alpha}(D)$, define $c'(\gamma)=f(\gamma)-f(\tau)$ of gives us a homomorphism into $\mathbb C!$

Searching for Holes in the Matrix Universe Lart IV: Cohomology

└Injecting into C

For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{tr}(D)$, define

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 c^f gives us a homomorphism into $\mathbb{C}!$

$$c^f(\gamma_1\gamma_2) = f(\gamma_1\gamma_2) - f(\tau)$$

Searching for Holes in the Matrix Universe Leading For Holes in the Matrix Universe Leading L

For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^*(D)$, define $e^t(\gamma) = f(\gamma) - f(r)$ e^t gives us a homomorphism into $\mathbb{C}!$ $e^t(\gamma\gamma) = f(\gamma\gamma\gamma) - f(r)$

Searching for Holes in the Matrix Universe - Part IV: Cohomology

Linjecting into C

For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{tr}(D)$, define

$$c^f(\gamma) = f(\gamma) - f(\tau)$$

 c^f gives us a homomorphism into $\mathbb{C}!$

$$c^{f}(\gamma_1 \gamma_2) = f(\gamma_1 \gamma_2) - f(\tau)$$
$$= f(\gamma_1 \gamma_2) - f(\gamma_1) + f(\gamma_1) - f(\tau)$$

Searching for Holes in the Matrix Universe Part IV: Cohomology Injecting into \mathbb{C}

For D an archaeot set, X, B, the base point, f a global germ, and $\gamma \in S_1^{pr}(D)$, define $c^{\ell}(\gamma) = f(\gamma) - f(r)$ of gives us a homomorphism into C1 $c^{\ell}(\gamma \gamma \gamma) = f(\gamma \gamma \gamma) - f(r)$ $-f(\gamma \gamma \gamma) - f(\gamma \gamma) - f(\gamma \gamma) - f(r)$

Searching for Holes in the Matrix Universe └Part IV: Cohomology

└Injecting into C

For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{\mathrm{tr}}(D)$, define

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Searching for Holes in the Matrix Universe 2022-06-02 -Part IV: Cohomology -Injecting into \mathbb{C}

For D an anchored set, $X \in B_1$ the base point, f a global germ $c^f(\gamma) = f(\gamma) - f(\tau)$ cf gives us a homomorphism into C!

 $c^f(\gamma_1\gamma_2) = f(\gamma_1\gamma_2) - f(\tau)$

 $= f(\gamma_1\gamma_2) - f(\gamma_1) + f(\gamma_1) - f(\tau)$ $= f(\gamma_2) - f(\tau) + f(\gamma_1) - f(\tau)$

Searching for Holes in the Matrix Universe Part IV: Cohomology

 \sqcup Injecting into $\mathbb C$

For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{tr}(D)$, define

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$$= f(\gamma_{2}) - f(\tau) + f(\gamma_{1}) - f(\tau)$$

$$= c^{f}(\gamma_{2}) + c^{f}(\gamma_{1}).$$

Searching for Holes in the Matrix Universe Part IV: Cohomology Injecting into \mathbb{C}

For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{p_\ell}(D)$, define $\dot{\sigma}^\ell(\gamma) = f(\gamma) - f(\tau)$ $\dot{\sigma}^\ell$ gives us a homomorphism into C!

 $c^{f}(\gamma_{1}\gamma_{2}) = f(\gamma_{1}\gamma_{2}) - f(\tau)$ $= f(\gamma_{1}\gamma_{2}) - f(\gamma_{1}) + f(\gamma_{1}) - f(\tau)$ $= f(\gamma_{2}) - f(\tau) + f(\gamma_{1}) - f(\tau)$ $= c^{f}(\gamma_{2}) + c^{f}(\gamma_{1}).$ Searching for Holes in the Matrix Universe

Part IV: Cohomology

Injecting into C

is an injective homomophism.

Lemma

The map

$$\Phi: \prod_{\substack{\nabla f \in H^1_{\mathrm{tr}}(D) \\ f \text{ a global germ}}} \pi_1^{\mathrm{tr}}(D) \longrightarrow \prod_{\substack{\nabla f \in H^1_{\mathrm{tr}}(D) \\ f \text{ a global germ}}} \mathbb{C}$$

Searching for Holes in the Matrix Universe Part IV: Cohomology Linjecting into \mathbb{C}



Lemma

└Injecting into C

The map

$$\Phi: \prod_{\substack{\nabla f \in H^1_{\mathrm{tr}}(D) \\ f \text{ a global germ}}} \pi_1^{\mathrm{tr}}(D) \longrightarrow \prod_{\substack{\nabla f \in H^1_{\mathrm{tr}}(D) \\ f \text{ a global germ}}} \mathbb{C}$$

$$\prod \gamma \longmapsto \prod c^f(\gamma)$$

is an injective homomophism.

So $\pi_1^{\rm tr}(D)$ is commutative and torsion free!

Searching for Holes in the Matrix Universe Part IV: Cohomology Injecting into \mathbb{C}



Searching for Holes in the Matrix Universe

 $\pi_1^{tr}(D)$ is divisible

Searching for Holes in the Matrix Universe

$\pi_1^{\mathrm{tr}}(D)$ is divisible

For any
$$\gamma \in \pi_1^{tr}(D)$$
, $\gamma \oplus \tau = \tau \oplus \gamma$.

Part IV: Cohomology
Characterizing $\pi_1^{\text{tr}}(D)$ $\pi_1^{\text{tr}}(D)$ is divisible

Searching for Holes in the Matrix Universe



$\pi_1^{\mathrm{tr}}(D)$ is divisible

For any
$$\gamma \in \pi_1^{\mathrm{tr}}(D)$$
,

Why?

$$H(t,\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} (\gamma \oplus \gamma_X) \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^*$$

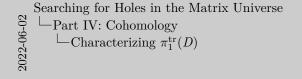
 $\gamma \oplus \tau = \tau \oplus \gamma$.

is a homotopy between the paths.

Searching for Holes in the Matrix Universe

Part IV: Cohomology

Characterizing $\pi_1^{\mathrm{tr}}(D)$ $\pi_1^{\mathrm{tr}}(D)$ is divisible $\pi_1^{\mathrm{tr}}(D)$ $\pi_1^{\mathrm{tr}}(D)$ is divisible



Searching for Holes in the Matrix Universe Part IV: Cohomology

 \sqsubseteq Characterizing $\pi_1^{\mathrm{tr}}(D)$

$$\gamma = \underbrace{\begin{bmatrix} \gamma \\ \ddots \\ \gamma \end{bmatrix}}_{k+1\text{-times}} \\
= \begin{bmatrix} \gamma \\ \tau \\ \ddots \\ \tau \end{bmatrix} \begin{bmatrix} \tau \\ \gamma \\ \ddots \\ \tau \end{bmatrix} \cdots \begin{bmatrix} \tau \\ \tau \\ \ddots \\ \gamma \end{bmatrix}$$

Searching for Holes in the Matrix Universe \square Part IV: Cohomology \square Characterizing $\pi_1^{\rm tr}(D)$



Searching for Holes in the Matrix Universe

Searching for Holes in the M \sqsubseteq Part IV: Cohomology \sqsubseteq Characterizing $\pi_1^{\mathrm{tr}}(D)$

$$\gamma = \underbrace{\begin{bmatrix} \gamma & \gamma & \cdots & \gamma \\ \vdots & \ddots & \gamma \end{bmatrix}}_{k+1\text{-times}}$$

$$= \begin{bmatrix} \gamma & \tau & \cdots & \gamma \\ \vdots & \ddots & \gamma \end{bmatrix} \begin{bmatrix} \tau & \gamma & \cdots & \tau \\ \vdots & \ddots & \gamma \end{bmatrix} \cdots \begin{bmatrix} \tau & \tau & \cdots & \gamma \\ \vdots & \ddots & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & \tau & \cdots & \gamma & \gamma \\ \vdots & \ddots & \gamma & \gamma \end{bmatrix} \begin{bmatrix} \gamma & \tau & \cdots & \gamma \\ \vdots & \ddots & \gamma & \gamma \end{bmatrix}$$

Searching for Holes in the Matrix Universe Part IV: Cohomology — Characterizing $\pi_1^{\mathrm{tr}}(D)$



Searching for Holes in the Matrix Universe $\dot{}$

$$\vdash$$
 Part IV: Cohomology \vdash Characterizing $\pi_1^{\operatorname{tr}}(D)$

$$\gamma = \underbrace{\begin{bmatrix} \gamma & \gamma & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \end{bmatrix}}_{k+1\text{-times}}$$

$$= \begin{bmatrix} \gamma & & & \\ & \ddots & \\ & & \ddots \\ & & \end{bmatrix} \begin{bmatrix} \gamma & & & \\ & \ddots & \\ & & \ddots \\ & & \end{bmatrix} \cdots \begin{bmatrix} \gamma & & & \\ & \ddots & \\ & & \ddots \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & & & \\ & \ddots & \\ & & \ddots \\ & & \end{bmatrix}_{k}^{k}$$

$$= \begin{bmatrix} \gamma & & & \\ & \ddots & \\ & & \ddots \\ & & \end{bmatrix}_{x}^{k}$$

Searching for Holes in the Matrix Universe Part IV: Cohomology Characterizing $\pi_1^{\rm tr}(D)$



Theorem

For D an anchored free set, $\pi_1^{tr}(D)$ is a torsion free, abelian, divisible group. That is,

$$\pi_1^{\mathrm{tr}}(D) \simeq \bigoplus_{i \in I} \mathbb{Q} = \mathbb{Q}^I$$

for some set I.

Searching for Holes in the Matrix Universe Part IV: Cohomology Characterizing $\pi_1^{\rm tr}(D)$

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-Computing $\pi_1^{\rm tr}(D)$

Part V: Computing $\pi_1^{\text{tr}}(D)$

• hard bc no VC or MV

Searching for Holes in the Matrix Universe $\ \ \Box$ Computing $\pi_1^{\rm tr}(D)$ $\ \Box$ The Direct Limit Approach

Let D be an anchored, free, path connected set such that each D_n is nonempty and choose an anchor $B \subset D$ such that each B_n is also nonempty.

Searching for Holes in the Matrix Universe Computing $\pi_1^{tr}(D)$ The Direct Limit Approach

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Let $\pi_1^{\text{tr}}(D)_n$ is the subgroup of paths in D_n . Note that $\pi_1^{\text{tr}}(D)_1$ is a quotient of $\pi_1(D_n)$.

Searching for Holes in the Matrix Universe Computing $\pi_1^{\text{tr}}(D)$ The Direct Limit Approach

Let D be an anchored, free, path connected set such that each D_a is nonempty and choose an anchor $B \subset D$ such that each B_a is also nonempty. Let $\pi_1^{n}(D)_a$ is the subgroup of paths in D_a . Note that $\pi_2^{n}(D)_1$ is a consticut of $\pi_1(D)$. Computing $\pi_1^{o_1}(D)$ The Direct Limit Approach

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Let $\pi_1^{\text{tr}}(D)_n$ is the subgroup of paths in D_n . Note that $\pi_1^{\text{tr}}(D)_1$ is a quotient of $\pi_1(D_n)$.

There is a natural inclusion

$$\pi_1^{\operatorname{tr}}(D)_n \longrightarrow \pi_1^{\operatorname{tr}}(D)_{kn}$$

$$\gamma \longmapsto \gamma^{\oplus k}$$

for all k.

Searching for Holes in the Matrix Universe Computing $\pi_1^{\rm tr}(D)$ The Direct Limit Approach

Let D be an anchored, free, path connected set such that each D_c is nonempty and choose an anchor $B \subset D$ such that each B_c is also nonempty. Let $\pi^{p}(D_c)_{b}$ is the subgroup of paths in D_c . Note that $\pi^{p}_{c}(D_{b})_{c}$ is a quotient of $\pi_{c}(D_{c})_{c}$. There is a natural inclusion $\pi^{p}_{c}(D_{b})_{c} \longrightarrow \pi^{p}_{c}(D_{b})_{b}$

.

for all k

Searching for Holes in the Matrix Universe \sqsubseteq Computing $\pi_1^{\rm tr}(D)$ \sqsubseteq The Direct Limit Approach

Now consider the chain of inclusions:

$$\pi_1^{\mathrm{tr}}(D)_1 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_2 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_6 \hookrightarrow \cdots \hookrightarrow \pi_1^{\mathrm{tr}}(D)_{n!} \hookrightarrow \cdots$$

Searching for Holes in the Matrix Universe Γ_0^{Tr} —Computing $\pi_1^{\text{tr}}(D)$ —The Direct Limit Approach

Now consider the chain of inclusions: $\pi_1^{II}(D)_1 \hookrightarrow \pi_1^{II}(D)_2 \hookrightarrow \pi_1^{II}(D)_6 \hookrightarrow \cdots \hookrightarrow \pi_1^{II}(D)_{dl} \hookrightarrow \cdots$

Now consider the chain of inclusions:

$$\pi_1^{\mathrm{tr}}(D)_1 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_2 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_6 \hookrightarrow \cdots \hookrightarrow \pi_1^{\mathrm{tr}}(D)_{n!} \hookrightarrow \cdots$$

The limit of this sequence isomorphic to $\pi_1^{\text{tr}}(D)$!

Searching for Holes in the Matrix Universe Computing $\pi_1^{tr}(D)$ The Direct Limit Approach

Now consider the chain of inclusions: $\pi_1^{tr}(D)_1 \hookrightarrow \pi_1^{tr}(D)_2 \hookrightarrow \pi_1^{tr}(D)_6 \hookrightarrow \cdots \hookrightarrow \pi_1^{tr}(D)_{el} \hookrightarrow \cdots$

The limit of this sequence isomorphic to $\pi_1^{tr}(D)!$

Example: $\pi_1^{\mathrm{tr}}(GL)$

Let
$$GL = \bigcup_{n \in \mathbb{N}} GL_n(\mathbb{C})$$
.

Searching for Holes in the Matrix Universe

as well.

Example: $\pi_1^{tr}(GL)$

Searching for Holes in the Matrix Universe

Example: $\pi_1^{\mathrm{tr}}(\mathit{GL})$

Let
$$GL = \bigcup_{n \in \mathbb{N}} GL_n(\mathbb{C})$$
.

$$GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}$$
. Since $\pi_1(GL_1) \simeq \mathbb{Z}$, $\pi_1^{\mathrm{tr}}(GL)_1 \simeq \mathbb{Z}$ as well.

Computing $\pi_1^{\text{tr}}(D)$ An Example

Example: $\pi_1^{\text{tr}}(GL)$

Searching for Holes in the Matrix Universe

Let $GL = \bigcup_{n \in \mathbb{N}} GL_n(\mathbb{C})$. $GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. Since $\pi_1(GL_1) \simeq \mathbb{Z}$, $\pi_1^{\mathrm{pt}}(GL)_1 \simeq \mathbb{Z}$ as well.

Example: $\pi_1^{\mathrm{tr}}(\mathit{GL})$

Let
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. Since $\pi_1(GL_1) \simeq \mathbb{Z}$, $\pi_1^{\mathrm{tr}}(GL)_1 \simeq \mathbb{Z}$ as well.

Inclusion into $\pi_1^{\text{tr}}(GL)_2$ picks up square roots. If $\gamma \in \pi_1^{\text{tr}}(GL)_1$, then

$$\begin{bmatrix} \gamma & \\ & \tau \end{bmatrix} \in \pi_1^{\mathrm{tr}}(\mathit{GL})_2.$$

Searching for Holes in the Matrix Universe Computing $\pi_1^{\mathrm{tr}}(D)$ An Example

Example: $\pi_1^{\mathrm{tr}}(GL)$

se: π_I (GL)

Let $GL = \bigcup_{a \in \mathbb{N}} GL_a(\mathbb{C}).$ $GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}.$ Since $\pi_1(GL_1) \simeq \mathbb{Z}$, $\pi_1^{\mathrm{tr}}(GL)_1 \simeq \mathbb{Z}$ as well. Inclusion into $\pi_1^{\mathrm{tr}}(GL)_2$ pids up square roots. If $\gamma \in \pi_1^{\mathrm{tr}}(GL)$ then $\begin{bmatrix} \gamma & \\ & \gamma \end{bmatrix} \in \pi_1^{\mathrm{tr}}(GL)_2.$

Example: $\pi_1^{\mathrm{tr}}(GL)$

Let
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Inclusion into $\pi_1^{\mathrm{tr}}(GL)_2$ picks up square roots. If $\gamma \in \pi_1^{\mathrm{tr}}(GL)_1$, then

$$\begin{bmatrix} \gamma & \\ & \tau \end{bmatrix} \in \pi_1^{\mathrm{tr}}(\mathit{GL})_2.$$

Thus,
$$\pi_1^{\operatorname{tr}}(GL)_2 \simeq \mathbb{Z}\left[\frac{1}{2}\right]$$
.

Searching for Holes in the Matrix Universe Computing $\pi_1^{\rm tr}(D)$ An Example Example: $\pi_1^{\rm tr}(GL)$

Let $GL = \bigcup_{n \in \mathbb{N}} GL_n(\mathbb{C})$. $GL_n(\mathbb{C}) = \mathbb{C} \setminus \{0\}. \text{ Since } \pi_1(GL_n) \simeq \mathbb{Z} \text{ and } GL_n \simeq \mathbb{Z} \text{ so well}.$

Let $GL = \bigcup_{u \in \mathbb{N}} GL_u(\mathbb{C})$, $GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. Since $\pi_1(GL_1) \simeq \mathbb{Z}$, $\pi_1^{p_1}(GL)_1 \simeq \mathbb{Z}$ as Inclusion into $\pi_1^{p_1}(GL)_2$ picks up square roots. If $\gamma \in \pi_1^{p_1}$ then $\begin{bmatrix} \gamma & \\ & \gamma \end{bmatrix} \in \pi_1^{p_1}(GL)_2$ Thus, $\pi_1^{p_1}(GL)_2 \simeq \mathbb{Z} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Similarly, inclusion into $\pi_1^{\text{tr}}(GL)_{3!}$ picks up cube roots:

$$\pi_1^{\mathrm{tr}}(GL)_6 \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$$

Searching for Holes in the Matrix Universe Γ_0^{Tr} —Computing $\pi_1^{\text{tr}}(D)$ —An Example

Similarly, inclusion into $\pi_1^{tr}(GL)_3$; picks up cube roots: $\pi_1^{tr}(GL)_6\simeq \mathbb{Z}\left[\frac{1}{2},\frac{1}{3}\right]$

LAn Example

$$\pi_1^{\mathrm{tr}}(GL)_6 \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$$

In the *n*-th inclusion, we pick up *n*-th roots and so we adjoin $\frac{1}{n}$ to the preceding group. Therefore,

$$\pi_1^{\mathrm{tr}}(\mathit{GL}) \simeq \mathbb{Z}\left[rac{1}{2},rac{1}{3},rac{1}{4},\dots
ight]$$

Searching for Holes in the Matrix Universe Computing $\pi_1^{tr}(D)$ An Example

Similarly, inclusion into $\sigma_1^{tr}(GL)_B$ picks up cube roots: $\sigma_1^{tr}(GL)_B = \mathbb{Z}\left[\frac{1}{2},\frac{1}{3}\right]$ In the a-th inclusion, we pick up a-th roots and so we adjoin $\frac{1}{a}$ to the preceding group. Therefore, $T(GL) \gg \mathbb{Z}\left[\frac{1}{a},\frac{1}{a},\dots\right]$ LAn Example

$$\pi_1^{\mathrm{tr}}(\mathit{GL})_6 \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$$

In the *n*-th inclusion, we pick up *n*-th roots and so we adjoin $\frac{1}{n}$ to the preceding group. Therefore,

$$\pi_1^{\mathrm{tr}}(\mathit{GL}) \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right] \simeq \mathbb{Q}.$$

Searching for Holes in the Matrix Universe Computing $\pi_1^{\mathrm{tr}}(D)$ An Example

Similarly, inclusion into $\pi_1^{p_1}(GL)_0$ picks up only roots: $\pi_1^{p_1}(GL)_0 \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$ In the n-th inclusion, we pick up n-th roots and so we adjoin $\frac{1}{n}$ to the preceding group. Therefore, $\pi_1^{p_1}(GL) \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \dots\right] \simeq \mathbb{Q}$. Thank You!

Searching for Holes in the Matrix Universe Computing $\pi_1^{tr}(D)$ An Example

Thank You!