Searching for Holes in the Matrix Universe

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Spring 2022

Searching for Holes in the Matrix Universe



1. Eventual goal: lift the tools of algebraic topology to spaces of matrices

2. If we have a fixed size, use classical results but this doesn't work

- for multiple sizes.

 3. For multiple (which hopefully I can convince you is interesting)
- 3. For multiple (which hopefully I can convince you is interesting) we need to develop some hefty tools
- $4.\,$ To do so, we need to go back to our mathematical roots

Searching for Holes in the Matrix Universe

Part I: Objects and Maps

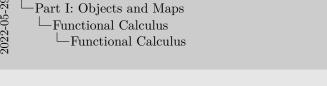
Objects and Maps

Searching for Holes in the Matrix Universe \sqsubseteq Part I: Objects and Maps

Objects and Maps

- 1. That's right—objects and maps.
- 2. Our Naive attempt involves that looking at lifting functions on \mathbb{R} or \mathbb{C} to accept matrices as their input.

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.



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• The naive attempt is modeled after the behavior of polynomials. We could plug in an arbitrary matrix into a polynomial without too much trouble, but requiring SA gives us some special

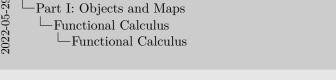
Let $f \in \mathbb{R}[x]$ and $A \in M_t(\mathbb{C})$ be self adjoint.

• Since we are SA, we can diagonalize

behavior.

• Watch what happens when we plug this into our polynomial

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint. A is diagonalizable as $A = U\Lambda U^*$



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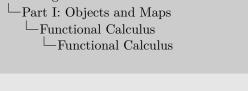
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Searching for Holes in the Matrix Universe

• Watch what happens when we plug this into our polynomial

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint. A is diagonalizable as $A = U\Lambda U^*$

$$f(A) = a_n A^n + \dots + a_1 A + a_0 I_k$$



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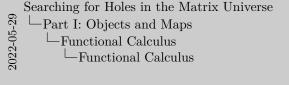


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Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Functional Calculus
Functional Calculus

Let $f \in \mathbb{R}[x]$ and $A \in M_0(\mathbb{C})$ be self adjoint. A is diagonalizable as $A - UAU^*$ $f(A) = a_a A^* + \cdots + a_1 A + a_b I_k$ $= a_a (UAU^*)^* + \cdots + a_k I_k UA^* + a_b I_k$ $= a_a UA^* U^* + \cdots + a_k UAU^* + a_b I_k$

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Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Functional Calculus
Functional Calculus

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint. A is diagonalizable as $A = U\Lambda U^*$ $f(A) = a_k A^* + \cdots + a_1 A + a_k I_k$ $= a_k (U\Lambda U^*)^* + \cdots + a_1 U\Lambda U^* + a_k I_k$ $= a_k U\Lambda^* U^* + \cdots + a_1 U\Lambda U^* + a_k I_k$ $= U (a_k \Lambda^* + \cdots + a_1 \Lambda + a_k I_k) U^*$

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Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Functional Calculus
Functional Calculus



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$$f\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right)$$

Searching for Holes in the Matrix Universe -Part I: Objects and Maps -Functional Calculus -Functional Calculus



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$$f\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right) = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}$$

Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Functional Calculus
Functional Calculus

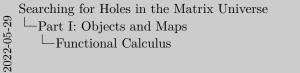


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Searching for Holes in the Matrix Universe Part I: Objects and Maps Functional Calculus

Let \mathbb{H}_n be the set of $n \times n$ self adjoint matrices, and define

$$\mathbb{H} = \bigcup_{n \in \mathbb{N}} \mathbb{H}_n, \qquad \mathcal{M} = \bigcup_{n \in \mathbb{N}} M_n(\mathbb{C})$$





- With the polynomial case in mind, we can extend a general function. First, a piece of notation
- Lets grab a function on the real line and the self adjoint matrices with their spectrum in that domain
- \bullet Then we can lift g by emulating the behavior of polynomials.
- The natural question now is "what can we do with these functions"

Searching for Holes in the Matrix Universe
- Part I: Objects and Maps

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Definition:

Functional Calculus

Let $g:[a,b]\to\mathbb{C}$ and $D\subset\mathbb{H}$ denote the set of self adjoint matrices with their spectrum in [a,b].

Searching for Holes in the Matrix Universe

—Part I: Objects and Maps

—Functional Calculus

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Searching for Holes in the Matrix Universe
- Part I: Objects and Maps

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Definition:

Let $g:[a,b]\to\mathbb{C}$ and $D\subset\mathbb{H}$ denote the set of self adjoint matrices with their spectrum in [a,b]. Then

$$g: D \longrightarrow \mathcal{M}$$

$$X = U\Lambda U^* \longmapsto U \begin{bmatrix} g(\lambda_1) & & & \\ & \ddots & & \\ & & g(\lambda_n) \end{bmatrix} U^*.$$

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Part I: Objects and Maps
Functional Calculus



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Directional Derivative

Definition: Directional Derivative

Fix some $X \in \mathbb{H}_n$. The derivative of f at X in the direction $H \in M_n(\mathbb{C})$ is

$$Df(X)[H] = \lim_{t \to 0} \frac{f(X + tH) - f(X)}{t}$$

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Part I: Objects and Maps
Functional Calculus
Directional Derivative

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- We can define a directional derivative—as long as we are careful to have the direction in the same "level-wise" slice.
- Notice that, with some special attention to what operation we are carrying our, this is the exact same definition as classic multivariable calculus.
- There is another formulation that is (generally) more useful for computation

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Alternatively,

$$Df(X)[H] = \frac{df(X+tH)}{dt}\bigg|_{t=0}$$

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—Part I: Objects and Maps

—Functional Calculus

—Directional Derivative

Directional Derivative

Definition: Directional Derivative $H \in M_{\alpha}(\Sigma)$ is \mathbb{Z}_{n} . The derivative of f at X in the direction $H \in M_{\alpha}(\Sigma)$ is \mathbb{Z}_{n} . The G is H in the direction of H is H in H

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$$g(X + tH) =$$

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Part I: Objects and Maps

Functional Calculus

Example: $g(x) = x^3$

Searching for Holes in the Matrix Universe

g(X+tH)=

- Now we consider an example. Since Df(X)[H] is linear, we can just work with a single monomial
- First we expand $(x + th)^3$ —but we can't use the binomial theorem since x and h don't commute
- Once we expand, we take standard derivatives w.r.t t—treating X and H as formal symbols.

$$g(X + tH) = X^{3} + tX^{2}H + tXHX + t^{2}XH^{2} + tHX^{2} + t^{2}HXH + t^{2}H^{2}X + t^{3}H^{3}.$$

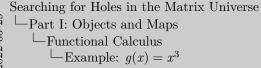
Searching for Holes in the Matrix Universe -Part I: Objects and Maps -Functional Calculus -Example: $g(x) = x^3$



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$$g(X + tH) = X^{3} + tX^{2}H + tXHX + t^{2}XH^{2} + tHX^{2} + t^{2}HXH + t^{2}H^{2}X + t^{3}H^{3}.$$

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Searching for Holes in the Matrix Universe
- Part I: Objects and Maps

Functional Calculus

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From here, we can calculate:

$$\frac{d}{dt}g(X + tH) = X^{2}H + XHX + 2tXH^{2} + HX^{2} + 2tHXH + 2tH^{2}X + 3t^{2}H^{3}$$

Searching for Holes in the Matrix Universe Part I: Objects and Maps

Functional Calculus

Example: $g(x) = x^3$

 $g(X + iB) = X^3 + iX^2B + iXBX + i^2XB^2$ $+ iBX^2 + i^2BXB + i^2B^2X + i^2B^3$ in here, we can calculate: $\frac{d}{dt}g(X + iB) = X^2B + XBX + 2iXB^2 + BX^2$ $+ 2iBXB + 2iB^2X + 3i^2B^3$

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Searching for Holes in the Matrix Universe

Part I: Objects and Maps
Functional Calculus

Example: $g(x) = x^3$

$$g(X + tH) = X^{3} + tX^{2}H + tXHX + t^{2}XH^{2} + tHX^{2} + t^{2}HXH + t^{2}H^{2}X + t^{3}H^{3}.$$

From here, we can calculate:

$$\frac{d}{dt}g(X+tH) = X^{2}H + XHX + 2tXH^{2} + HX^{2} + 2tHXH + 2tH^{2}X + 3t^{2}H^{3}$$
$$+ 2tHXH + 2tH^{2}X + 3t^{2}H^{3}$$
$$\frac{d^{2}}{dt^{2}}g(X+tH) = 2XH^{2} + 2HXH + 2H^{2}X + 6tH^{3}$$

Searching for Holes in the Matrix Universe Part I: Objects and Maps

Functional Calculus

Example: $g(x) = x^3$

 $g(X + IH) = X^3 + IX^2H + IXHX + i^2XH^2$ $+IHX^3 + i^2HXH + i^2H^2X + i^2H^3$, here, we can calculate: $\frac{d}{dx}g(X + IH) = X^2H + XHX + 2IXH^2 + HX^2$ $+2IXXH + 2IIX^2 + 3I^2H^3$ $\frac{d^2}{dx}g(X + IH) = XXH^2 + 2HXH + 2H^2X + 4iH^3$

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From here, we can calculate:

$$\frac{d}{dt}g(X+tH) = X^{2}H + XHX + 2tXH^{2} + HX^{2} + 2tHXH + 2tH^{2}X + 3t^{2}H^{3}$$
$$+ 2tHXH + 2tH^{2}X + 3t^{2}H^{3}$$
$$\frac{d^{2}}{dt^{2}}g(X+tH) = 2XH^{2} + 2HXH + 2H^{2}X + 6tH^{3}$$
$$\frac{d^{3}}{dt^{3}}g(X+tH) = 6H^{3}.$$

Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Functional Calculus
Example: $q(x) = x^3$

$$\begin{split} g(X+BI) &= X^3 + tX^2H + tXHX + t^2XH^2 \\ &+ tHX^2 + t^2HX^2 + t^2HX + t^2H^2, \end{split}$$
 no here, we can exhaust $\frac{d}{dt}g(X+HI) = X^2H + XHX + 2XH^2 + HX^2 \\ &+ \frac{d}{dt}g(X+HI) = X^2H + XHX + 2XH^2 + HX^2 \\ &+ 2XHXH + 2HX^2 + 32H^2 + 32H^2 \\ &+ 2XHXH + 2HX^2 + 32H^2 + 32H^2 \\ &+ 2HX^2 + 2HX^2 + 2HX^2 + 4H^2 X + 6H^3 \\ &+ \frac{d}{dt}g(X+HI) = 2H^2 + 2HXH + 2H^2 X + 6H^3 \\ &+ \frac{d}{dt}g(X+HI) = 6H^3. \end{split}$

- Now we consider an example. Since Df(X)[H] is linear, we can just work with a single monomial
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Part I: Objects and Maps

Searching for Holes in the Matrix Universe

And so the first 3 directional derivatives are:

-Part I: Objects and Maps -Functional Calculus \sqsubseteq Example: $g(x) = x^3$

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Searching for Holes in the Matrix Universe



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$$Df(X)[H] = X^2H + XHX + HX^2$$

Part I: Objects and Maps

Functional Calculus

Example: $g(x) = x^3$

Searching for Holes in the Matrix Universe

And so the first 3 directional derivatives are: $Df(X)[H] = X^2H + XHX + HX^2 \label{eq:def}$

And so the first 3 directional derivatives are:

$$Df(X)[H] = X^2H + XHX + HX^2$$

$$D^{(2)}f(X)[H] = 2XH^2 + 2HXH + 2H^2X$$

Searching for Holes in the Matrix Universe Part I: Objects and Maps Functional Calculus Example: $g(x) = x^3$

And so the first 3 directional derivatives are: $Df(X)[H]=X^2H+XHX+HX^2$ $D^{(2)}f(X)[H]=2XH^2+2HXH+2H^2X$

And so the first 3 directional derivatives are:

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Searching for Holes in the Matrix Universe Part I: Objects and Maps

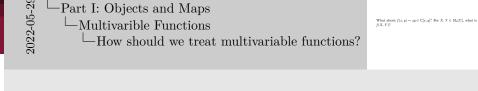
—Functional Calculus

—Example: $g(x) = x^3$

. 3(-) -

And so the finst 3 directional derivatives are: $Df(X)[H]=X^2H+XBX+HX^2$ $D^{(2)}f(X)[H]=2XH^2+2HXH+2H^2X$ $D^{(3)}f(X)[H]=6H^2$

What about $f(x, y) = xy \in \mathbb{C}[x, y]$? For $X, Y \in M_n(\mathbb{C})$, what is f(X, Y)?



Searching for Holes in the Matrix Universe

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Part I: Objects and Maps

Multivarible Functions

How should we treat multivariable functions?

Searching for Holes in the Matrix Universe

What about $f(x, y) = xy \in \mathbb{C}[x, y]$? For $X, Y \in M_n(\mathbb{C})$, what is f(X, Y)? $XY \qquad YX$

Searching for Holes in the Matrix Universe

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-Part I: Objects and Maps —Multivarible Functions What about $f(x, y) = xy \in \mathbb{C}[x, y]$? For $X, Y \in M_n(\mathbb{C})$, what is How should we treat multivariable functions?

 $XY = YX = \frac{1}{2}(XY + YX)$

Searching for Holes in the Matrix Universe

What about $f(x, y) = xy \in \mathbb{C}[x, y]$? For $X, Y \in M_n(\mathbb{C})$, what is f(X, Y)?

$$XY \qquad YX \qquad \frac{1}{2}(XY + YX)$$

Clearly $\mathbb{C}[x, y]$ is the wrong space!

Part I: Objects and Maps

Multivarible Functions

How should we treat multivariable functions?

Searching for Holes in the Matrix Universe

What about $f(x,y)=xy\in \mathbb{C}[x,y]^T$ for $X,Y\in M_n(\mathbb{C}),$ what is $f(X,Y)^T$ $XY = YX = \frac{1}{2}(XY+YX)$ Clearly $\mathbb{C}[x,y]$ is the wrong space?

We must construct a newspace!

Part I: Objects and Maps

Multivarible Functions

nc Polynomials

nc Polynomials

We must construct a newspace

Searching for Holes in the Matrix Universe

• Let $x = (x_1, \ldots, x_d)$ be a tuple of formal variables.

Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Multivarible Functions
Inc Polynomials

We must construct a new space! • Let $x=(x_1,\dots,x_d)$ be a tuple of formal variables

nc Polynomials

We must construct a newspace!

- Let $x = (x_1, \ldots, x_d)$ be a tuple of formal variables.
- A word in x is a product of these variables.

Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Multivarible Functions
nc Polynomials

We must construct a new space! • Let $x=(x_1,\dots,x_d)$ be a tuple of formal variables. • A word in x is a product of these variables.

nc Polynomials

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 - e.g. $x_1 x_3 x_1 x_4^2$ $x_2^4 x_5^3$

Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Multivarible Functions

nc Polynomials

nc Polynomials

We must construct a newspace! • Let $x = (x_1, \dots, x_d)$ be a tuple of formal variables. • A word in x_i is a product of these variables. • $x_i = x_1 x_2 x_1 x_d^2 = x_d^2 x_d^2$

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- An **nc polynomial** is a linear combination of words in x over your favorite field.

Searching for Holes in the Matrix Universe
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nc Polynomials

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$$x_1 x_3 x_1 x_4^2$$
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Let $\mathbb{R}\langle x\rangle$ and $\mathbb{C}\langle x\rangle$ denote the set of nc polynomials over \mathbb{R} and

Searching for Holes in the Matrix Universe -Part I: Objects and Maps -Multivarible Functions -nc Polynomials

nc Polynomials

- We must construct a newspace Let x = (x₁,...,x_d) be a tuple of formal variables A word in x is a product of these variables
- e.g. $x_1x_2x_1x_2^2 = x_2^4x_1^3$ An nc polynomial is a linear combination of words in z over your favorite field.
- Let $\mathbb{R}(x)$ and $\mathbb{C}(x)$ denote the set of nc polynomials over \mathbb{R} and

Searching for Holes in the Matrix Universe

Example: $f(x, y) = x^2 - xyx - 1 \in \mathbb{R}\langle x, y \rangle$

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

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Searching for Holes in the Matrix Universe

-Part I: Objects and Maps

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$$f(X, Y) = X^2 - XYX + I_2$$

Part I: Objects and Maps

Multivarible Functions

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$$= \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}^{2} - \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Multivarible Functions

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$$= \begin{bmatrix} -11 & -4 \\ -4 & 1 \end{bmatrix}.$$

Searching for Holes in the Matrix Universe Part I: Objects and Maps Multivarible Functions Example: $f(x,y) = x^2 - xyx - 1 \in \mathbb{R}\langle x,y\rangle$

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Multivarible Functions
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• Mention that directional derivatives still work

Searching for Holes in the Matrix Universe

-Part I: Objects and Maps

• nc polynomials also do the unitary thing — that is worth mentioning

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$
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$$f(X \oplus X, Y \oplus Y) =$$

- Part I: Objects and Maps

 Multivarible Functions

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└Multivarible Functions

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Searching for Holes in the Matrix Universe Part I: Objects and Maps Multivarible Functions Example: $f(x,y) = x^2 - xyx - 1 \in \mathbb{R}\langle x,y\rangle$

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$$f(X \oplus X, Y \oplus Y) = \begin{bmatrix} -11 & -4 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & -11 & -4 \\ 0 & 0 & -4 & 1 \end{bmatrix}$$
$$= f(X, Y) \oplus f(X, Y).$$

Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Multivarible Functions

Example: $f(x,y) = x^2 - xyx - 1 \in \mathbb{R}\langle x,y\rangle$

- $$\begin{split} X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} & \text{and} & Y = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \\ f(X \oplus X, Y \oplus Y) = \begin{bmatrix} -11 & -4 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -f(X, Y) \otimes f(X, Y). \end{split}$$
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Searching for Holes in the Matrix Universe

Part I: Objects and Maps

Matrix Universe

Objects and Maps A Second Attempt

Searching for Holes in the Matrix Universe
Part I: Objects and Maps
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Objects and Maps $_{A \text{ Second Attempt}}$

1. Second attempt takes polynomials as an example object instead of as a thing to lift

Some Definitions

Definition: Matrix Universe
The g-dimensional Matrix Universe is

$$\mathcal{M}^g = \bigcup_{n \in \mathbb{N}} (M_n(\mathbb{C}))^g$$

Searching for Holes in the Matrix Universe

—Part I: Objects and Maps

—Matrix Universe

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Searching for Holes in the Matrix Universe

Part I: Objects and Maps

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Definition: Free Set

We say $D \subset \mathcal{M}^g$ is a **free set** if it is closed with respect to direct sums and unitary conjugation. That is,

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 $1 X, Y \in D \text{ means } X \oplus Y \in D.$

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- ② For X, U like-size matrices with U unitary and $X \in D$, then $UXU^* = (UX_1U^*, \dots, UX_qU^*) \in D$.

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 \emptyset $X, Y \in D$ means $X \oplus Y \in D$. \emptyset For X, U like-size matrices with U unitary and $X \in D$, then $UXU^* = (UX_1U^*, ..., UX_nU^*) \in D$.