

A CLEAN TITLE

LUCAS KERBS



A Fun Subtitle

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Ohana means family.
Family means nobody gets left behind, or forgotten.
— Lilo & Stitch

Dedicated to the loving memory of Rudolf Miede.
1939 – 2005

ABSTRACT

Short summary of the contents in English...a great guide by Kent Beck how to write good abstracts can be found here:

<https://plg.uwaterloo.ca/~migod/research/beck00PSLA.html>

*We have seen that computer programming is an art,
because it applies accumulated knowledge to the world,
because it requires skill and ingenuity, and especially
because it produces objects of beauty.*

— Donald E. Knuth [3]

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Put your acknowledgments here.

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PART I: *NAME*

INTRODUCTION

1.1 FUNCTIONAL CALCULUS

Functional Calculus refers to the process of extending the domain of a function on \mathbb{R} to include matrices (or in some cases operators). The most basic formulation uses the fact that the space $n \times n$ matrices forms a ring and so there is a natural way to evaluate polynomials $f \in \mathbb{C}[x]$. If we require that $A \in M_n(\mathbb{C})$ is self-adjoint—and hence diagonalizable as $A = U\Lambda U^*$ —then it is a standard result that:

$$\begin{aligned} f(A) &= a_n A^n + \cdots + a_1 A + a_0 \\ &= a_n (U\Lambda U^*)^n + \cdots + a_1 U\Lambda U^* + a_0 \\ &= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 \\ &= U(a_n \Lambda^n + \cdots + a_1 \Lambda + a_0) U^* \\ &= U(f(\Lambda)) U^* \end{aligned}$$

Further, since Λ is diagonal and f is a polynomial,

$$f\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right) = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}$$

Therefore, given a self-adjoint matrix A and a polynomial $f \in \mathbb{C}[x]$

$$f(A) = Uf(\Lambda)U^* = U \operatorname{diag}\{f(\lambda_1), \dots, f(\lambda_n)\} U^*$$

Since self-adjoint matrices play such a vital role in free analysis, we will let $\mathbb{H}_n \subset M_n(\mathbb{C})$ denote the set of $n \times n$ -matrices over \mathbb{C} . With the polynomial case in mind, we can extend a function $g : [a, b] \rightarrow \mathbb{C}$ to a function on self adjoint (normal?) matrices with their spectrum in $[a, b]$. Let A be such a matrix (diagonalized by the unitary matrix U), and define

$$g(A) = U \begin{bmatrix} g(\lambda_1) & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^*$$

Thus, for each $n \in \mathbb{N}$, g induces a function on the self-adjoint $n \times n$ matrices with spectrum in $[a, b]$. The natural ordering **explain why natural?** on self-adjoint matrices is called the **Loewner Order**:

Definition i.1 (Loewner Ordering). *For like size self-adjoint matrices, we say that $A \preceq B$ if $B - A$ is positive semidefinite and $A \prec B$ if $B - A$ is positive definite.*

With this ordering in place, we can extend many of the familiar function theoretic properties (monotonicity, convexity) to these matrix-values functions. In fact, these properties are defined identically to their classical counterpart: We say that a function is *matrix-monotone* if $A \preceq B$ implies that $f(A) \preceq f(B)$ and *matrix-convex* (or *nc-convex*) if

$$f\left(\frac{X+Y}{2}\right) \preceq \frac{f(X)+f(Y)}{2}$$

for every pair of like-size matrices for which f is defined. These conditions are rather restrictive (since they must hold for matrices of *all* sizes) so many functions which are convex/monotone (in the traditional sense) fail to be matrix-convex/monotone. For example, $f(x) = x^4$ fails to be nc-convex. **Below is an example from FCAC (Helton). What is the best way to refer to this?**

Indeed, if

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$\frac{X^4 + Y^4}{2} - \left(\frac{1}{2}X + \frac{1}{2}Y\right)^4 = \begin{bmatrix} 164 & 120 \\ 120 & 84 \end{bmatrix}$$

Which is not positive definite! Thus x^4 fails to be convex on even 2×2 matrices.

nc-positive if positive for all matrices

Further, a number of the standard constructions lift identically in this functional calculus.

Definition i.2 (Directional Derivative). *The derivative of f in the direction H is*

$$Df(X)[H] = \lim_{t \rightarrow 0} \frac{f(X + tH) - f(X)}{t}$$

where H and X are like-size self-adjoint matrices.

Often, the best way to compute these directional derivatives is via an equivalent formulation:

$$Df(X)[H] = \left. \frac{df(X + tH)}{dt} \right|_{t=0}$$

This version allows us to more easily define higher order derivatives

$$D^{(k)}f(X)[H] = \left. \frac{d^{(k)}f(X + tH)}{d^{(k)}t} \right|_{t=0}$$

Despite nc-convexity being so restrictive, Lemma 12 in [2] shows that the standard characterization of convexity via the second derivative: a function f is convex if and only if $D^2f(X)[H]$ is nc-positive. Unlike the classical case, however, the only convex polynomials are of degree ≤ 1 .

¹ See [2] for details.

1.2 EXTENDING MULTI-VARIABLE FUNCTIONS

We can extend this same functional calculus to functions of several variables, although the details are a bit more subtle. We could simply “plug in” at tuple of matrices to a standard multivariable polynomial ring over \mathbb{R} or \mathbb{C} , but this ignores the noncommutativity of $M_n(\mathbb{C})$. In light of this, let $x = (x_1, \dots, x_g)$ be a g -tuples of noncommuting formal variables. The formal variables x_1, \dots, x_n are *free* in the sense that there are no nontrivial relations between them.² A **word** in x is a product of these variables (e.g. $x_1 x_3 x_1 x_4^2$ or $x_1^2 x_5^3$). An **nc-polynomial** in x is a formal finite linear combination of words in x with coefficients in your favorite field. We use $\mathbb{R}\langle x \rangle$ and $\mathbb{C}\langle x \rangle$ to denote the set of nc-polynomials in x over \mathbb{R} or \mathbb{C} respectively.

With $\mathbb{C}\langle x \rangle$ constructed, we can define the functional calculus. Given a word $w(x) = x_{i_1}^{p_1} \cdots x_{i_d}^{p_d}$ and a g -tuple of self-adjoint matrices, X , we can evaluate w on X via $w(X) = X_{i_1}^{p_1} \cdots X_{i_d}^{p_d}$. Since our nc-polynomials are linear combinations of these words, we can extend this evaluation to evaluation of entire polynomials. Algebraically, we have a natural evaluation map: Given some $f \in \mathbb{C}\langle x \rangle$ and $X = (X_1, \dots, X_n)$ a g -tuple of self-adjoint matrices, define

$$\begin{aligned} \varepsilon_f : \mathbb{H}_n^g &\longrightarrow M_n(\mathbb{C}) \\ X &\longmapsto f(X). \end{aligned}$$

In the context of these multivariate functions, our definition of the Directional Derivative still makes sense (although our direction H now becomes a tuple of directions). We also inherit (from multi-variable calculus) a notion of the **gradient** of a function—but this will require a bit more work.

1.3 THE NATURAL INVOLUTION ON NC-POLYNOMIALS

[Short section on the involution. Could also put this in the algebra errata](#)

1.4 MATRIX UNIVERSES

[Be sure to include the “dot product” here](#)

1.5 THE TOPOLOGY OF MATRIX UNIVERSES

[How much detail here? Just what we are doing or the basics of the other topologies? This could also talk about the nc varieties/singular sets](#)

² This becomes important in the eventual functional calculus—matrices *do* have non-trivial relations. See section [ALGEBRAIC CONSTRUCTION] for the details.

1.6 TRACIAL FUNCTIONS AND UNIQUENESS OF THE GRADIENT

Now that we have \mathcal{M}^d , we can work with general functions on our matrix universe. As a whole, free analysis is concerned with so-called *free functions*, which respect the direct sums and unitary conjugation. **Do they need to be graded?**

Definition i.3 (Free Function). *A function $f : D \rightarrow \mathcal{M}^{\text{something}}$ is called free if*

1. $f(X \oplus Y) = f(X) \oplus f(Y)$
2. $f(UXU^*) = f(U)f(X)f(U^*)$ where X and U are like-size and U is unitary.

The two other classes of functions we are concerned with are those that act like the trace and the determinant:

Definition i.4 (Determinantal Free Function). *A function $f : D \rightarrow \mathbb{C}$ is a determinantal free function if*

1. $f(X \oplus Y) = f(X)f(Y)$
2. $f(UXU^*) = f(X)$ where X and U are like-size and U is unitary.

Definition i.5 (Tracial Free Function). *A function $f : D \rightarrow \mathbb{C}$ is a tracial free function if*

1. $f(X \oplus Y) = f(X) + f(Y)$
2. $f(UXU^*) = f(X)$ where X and U are like-size and U is unitary.

It is only these tracial functions which inherit the gradient mentioned above. Similarly to traditional multivariable calculus we define the gradient via its relationship to the directional derivative:

Definition i.6 (Free Gradient). *Given a tracial free function f , the free gradient, ∇f , is the unique free function satisfying*

$$\text{tr}(H \cdot \nabla f(X)) = \text{tr} Df(X)[H]$$

It is not-at-all obvious that such a ∇f should be unique—after all any linear combination of commutator is has trace zero. **should I explain this?** In the case that f is a single-variable function we can replace ∇f with the traditional derivative, f' , as seen in [4, Thm 3.3].

Theorem i.7. *Let $f : (a, b) \rightarrow \mathbb{R}$ be a C^1 function. Then*

$$\text{tr} Df(X)[H] = \text{tr}(Hf'(X))$$

The proof in [4] simply asserts the uniqueness of a function $g(X)$ and then shows that $g(x) = f'(x)$ for $x \in (a, b)$. Instead, we can construct such a g and recover the theorem along the way:

Proof. We start with a construction from Bhatia's Matrix Analysis: Let $f \in C^1(I)$ and define $f^{[1]}$ on $I \times I$ by

$$f^{[1]}(\lambda, \mu) = \begin{cases} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} & \lambda \neq \mu \\ f'(\lambda) & \lambda = \mu. \end{cases}$$

We call $f^{[1]}(\lambda, \mu)$ the *first divided difference* of f at (λ, μ) . If Λ is a diagonal matrix with entries $\{\lambda_i\}$, We may extend f to accept Λ by defining the (i, j) -entry of $f^{[1]}(\Lambda)$ to be $f^{[1]}(\lambda_i, \lambda_j)$. If A is a self adjoint matrix with $A = U\Lambda U^*$, then we define $f^{[1]}(A) = Uf^{[1]}(\Lambda)U^*$. Now we borrow a theorem from Bhatia [1]:

Theorem i.8 (Bhatia V.3.3). *Theorem numbering?* Let $f \in C^1(I)$ and let A be a self adjoint matrix with all eigenvalues in I . Then

$$Df(A)[H] = f^{[1]}(A) \circ H,$$

where \circ denotes the Schur-product³ in a basis where A is diagonal.

That is, if $A = U\Lambda U^*$, then

$$Df(A)[H] = U \left(f^{[1]}(\Lambda) \circ (U^* H U) \right) U^*.$$

To prove our claim, we need to take the trace of both sides. Since trace is invariant under a change of basis, it is clear that

$$\text{tr} Df(A)[H] = \text{tr} \left(f^{[1]}(\Lambda) \circ (U^* H U) \right).$$

If $U = u_{ij}$, $U^* = \bar{u}_{ij}$ and $H = h_{ij}$, then the (i, j) -entry of $U^* H U$ is

$$(U^* H U)_{ij} = \bar{u}_{ik} h_{k\ell} u_{\ell j}$$

Where we sum over the duplicate indices k and ℓ . While the structure of $f^{[1]}(\Lambda)$ is a bit unruly, our diagonal entries are $f'(\lambda)$. This means that when we take the trace of the Schur product, we have

$$\sum_k \sum_\ell \sum_i f'(\lambda_i) \bar{u}_{ik} h_{k\ell} u_{\ell i}$$

Now consider the matrix product $U \text{diag}\{f'(\lambda_1), \dots, f'(\lambda_n)\} U^* H$. Since one of our terms is diagonal, the trace of this multiplication is simple:

$$\text{tr} U \text{diag}\{f'(\lambda_1), \dots, f'(\lambda_n)\} U^* H = \sum_k \sum_\ell \sum_i u_{ik} f'(\lambda_k) \bar{u}_{k\ell} h_{\ell i}$$

Since $u_{ik}, \bar{u}_{k\ell}, h_{\ell i} \in \mathbb{C}$ they commute. We can then relabel our indices $i \mapsto \ell$ $\ell \mapsto k$ $k \mapsto i$ to get

$$\text{tr} U \text{diag}\{f'(\lambda_1), \dots, f'(\lambda_n)\} U^* H = \sum_k \sum_\ell \sum_i f'(\lambda_i) \bar{u}_{ik} h_{k\ell} u_{\ell i},$$

³ Entrywise

So, for every direction H , we have that $\text{tr}(U \text{diag}\{f'(\lambda_1), \dots, f'(\lambda_n)\} U^* H) = \text{tr}\left(f^{[1]}(\Lambda) \circ (U^* H U)\right)$. **overfull hbox :eyeroll:** By picking the “correct” H^4 , we conclude that our unique quantity $g(X)$ is $U \text{diag}\{f'(\lambda_1), \dots, f'(\lambda_n)\} U^*$. But, recall that $X = U \Lambda U$ so, in the functional calculus, $g(X) = f'(X)$. This recovers theorem 3.3 of [4] as we have constructed a g such that

$$\text{tr } Df(X)[H] = \text{tr } H g(X)$$

■

Connect this to the gradient and this section is done

⁴ See example EXAMPLE NUMBER for details

Part II

THE SHOWCASE

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Part III

APPENDIX

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