Searching for Holes in the Matrix Universe

Lucas Kerbs

Spring 2022

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- Eventual goal: lift the tools of algebraic topology to spaces of matrices
- If we only consider 2×2 matrices we can use classical theory
- The moment we want more than one size, things the classical theory breaks down
- Today we will develop some *fairly heftly* tools to do just that
- Along the way, hopefully I can convince you that this is an interesting question.
- To do so, we need to go back to our mathematical roots

Part I: Objects and Maps

A Naive Attempt

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- That's right—objects and maps.
- Our Naive attempt involves that looking at lifting functions on $\mathbb{R}$ or $\mathbb{C}$ to accept matrices as their input.
- An operator theorist would call this a "functional calculus"

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- You might say that SA is unnecessary bc we can already evaluate a polynomial on a matrix

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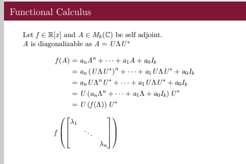

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Let $\mathbb{H}_n$ be the set of $n \times n$ self adjoint matrices, and define

$$\mathbb{H} = \bigcup_{n \in \mathbb{N}} \mathbb{H}_n, \qquad \mathcal{M} = \bigcup_{n \in \mathbb{N}} M_n(\mathbb{C})$$

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- With the polynomial case in mind, we can extend a general function. First, a piece of notation
- Lets grab a function on the real line and the self adjoint matrices with their spectrum in that domain
- Then we can lift g by emulating the behavior of polynomials.
- unwrap a self adjoint matrix, apply g to the diagonal, then wrap it back up
- Something to notice about this functional calculus—it treats direct sums *very* well
- This is all well and good, but can we do anything with these new functions?

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- Notice that, with some special attention to what operation we are carrying our, this is the exact same definition as classic multivariable calculus.
- There is another formulation that is (generally) more useful for computation

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$$g(X + tH) =$$

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- Now we consider an example. Since $Df(X)[H]$ is linear, we can just work with a single monomial
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- Now we consider an example. Since $Df(X)[H]$ is linear, we can just work with a single monomial
- First we expand $(x+th)^3$—but we can't use the binomial theorem since x and h don't commute
- Once we expand, we take standard derivatives w.r.t t—treating X and H as formal symbols.

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g(X+tH) &= X^3 + tX^2H + tXHX + t^2XH^2 \\
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\end{aligned}
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From here, we can calculate:

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└Part I: Objects and Maps
└Functional Calculus
└Example: $g(x) = x^3$

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Part I.5: Objects and Maps

A Second Attempt

Part I.5: Objects and Maps
A Second Attempt

- In seeking a more general theory we need to leave the world of this “SA functional calculus” behind.
- Rather than lifting functions to be matrix valued, we will define *new* objects that behave like those we just looked at.

Some Definitions

Definition: *Matrix Universe*
The g-dimensional **Matrix Universe** is

$$\mathcal{M}^g = \bigcup_{n \in \mathbb{N}} (M_n(\mathbb{C}))^g$$

Searching for Holes in the Matrix Universe
└Part I: Objects and Maps
└Matrix Universe
└Some Definitions

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└Part I: Objects and Maps
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- In math we often think about substructures that capture the implicit structure our space (subgroup, subspace, etc)
- In the nc setting, this is a *free set*, also called nc set
- direct sums and unitary conjugation are component wise
- If you see a D, you can assume that it is a free set.
- A subscript denotes a level-wise slice
- Note that this requires a lot of structure on free sets—we want to put a name to these structure.

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We say $D \subset \mathcal{M}^g$ is a **free set** if it is closed with respect to direct sums and unitary conjugation. That is,

1. $X, Y \in D$ means $X \oplus Y = (X_1 \oplus Y_1, \ldots, X_g \oplus Y_g) \in D$.

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Given $X \in \mathcal{M}^g$, a tuple of $n \times n$ matrices, the **fiber** of X is the set

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Searching for Holes in the Matrix Universe
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- The fiber is all the points “above” a given point. Conceptually, we imagine identification along the fiber—this will become important when we start doing topology
- The envelope (which will be less important to us) is the unitary smearing of the fiber at each level.

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└Part I: Objects and Maps
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A bit of topology

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- If we are going to look for holes and build up the algebraic topology, we need a point set topology first—so there is a natural question.
- Bad news: there isn't a natural choice
- There are a handful of candidates (fine, fat, free, nc Zariski). I wish we had time to go into detail.
- For us, free sets are open if their level-wise restriction is open
- Other point-set characterizations are similar.

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└Part I: Objects and Maps
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- Free functions are defined to be anything that behaves like a polynomial
- tracial functions look like traces
- For both of these maps, the directional derivative is defined identically as before—but tracial functions get something extra.

What the natural functions on $\mathcal{M}^g$?

Definition:
A function $f : D \to \mathcal{M}^{\hat{g}}$ is called **free** if

1. $f(X \oplus Y) = f(X) \oplus f(Y)$
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Given a tracial function f, the free gradient, ∇f, is the unique free function satisfying

$$\operatorname{tr}\left(H \cdot \nabla f(X)\right) = Df(X)[H],$$

where, if $A = (A_1, \dots, A_g)$ and $B = (B_1, \dots, B_g)$ are tuples of like-size matrices then $A \cdot B = \sum_{i=1}^{g} A_i B_i$.

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└Part I: Objects and Maps
└Uniqueness of the Gradient

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- The ∇ of a free function is the unique free function satisfy this equation—where the * is just like the dot product
- Whenever you see tr(*) I want you to think of the inner product—it is slightly distinct but it will make a lot of things make more sense
- Some of you may be hesitant at the fact that I claim ∇ is unique. Why should this be true?

Why should ∇f be unique?

Theorem (Trace Duality)

Let f, g be free functions $\mathcal{M}^g \to \mathcal{M}^{\tilde{g}}$. If $\operatorname{tr}(H \cdot f) = \operatorname{tr}(H \cdot g)$ for all tuples H, then $f = g$.

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- $f = g$ whenever the domains overlap
- In the vector space setting—with an inner product—this is a fairly immediate result. You would show it by picking vectors of all 0's and a single 1.
- You prove this identically—but with coordinate matrices instead of coordinate vectors.

Part II: Analytic Continuation and Monodromy

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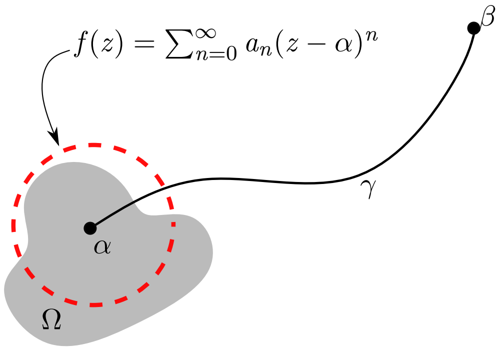
└Part II: Monodromy

Part II: Analytic Continuation and Monodromy

Analytic Continuation



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└Part II: Monodromy
└Analytic Continuation
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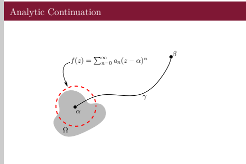

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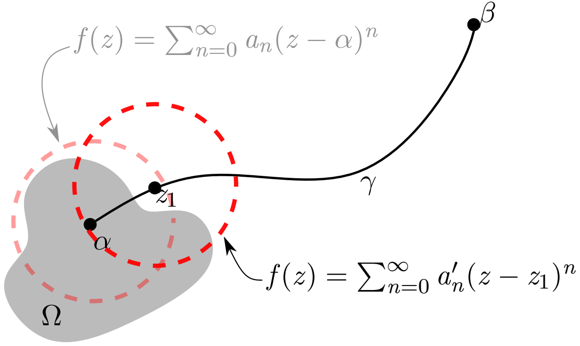

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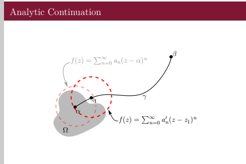

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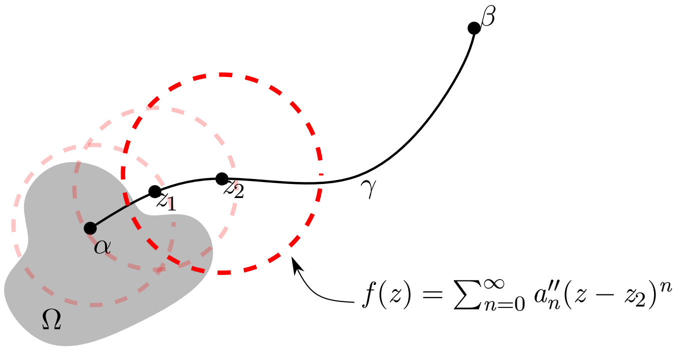

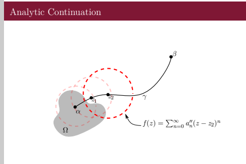

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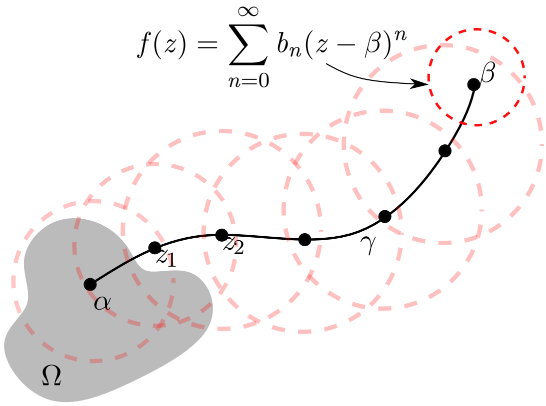

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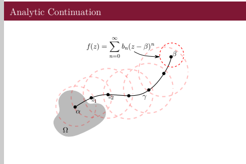
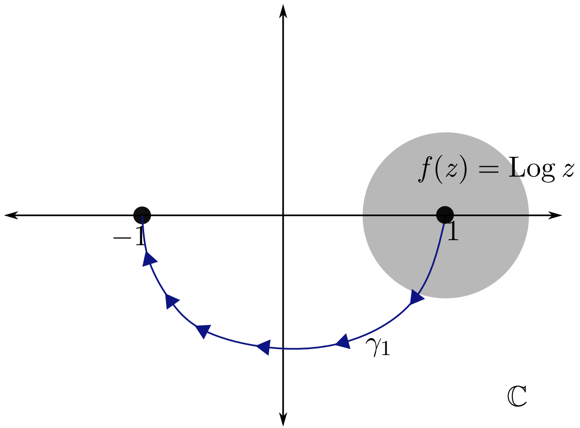
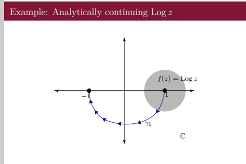


Example: Analytically continuing Log z





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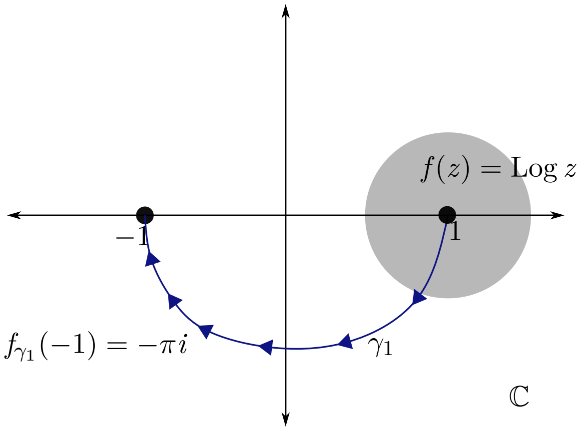

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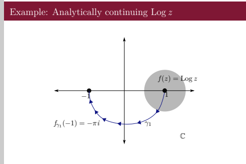

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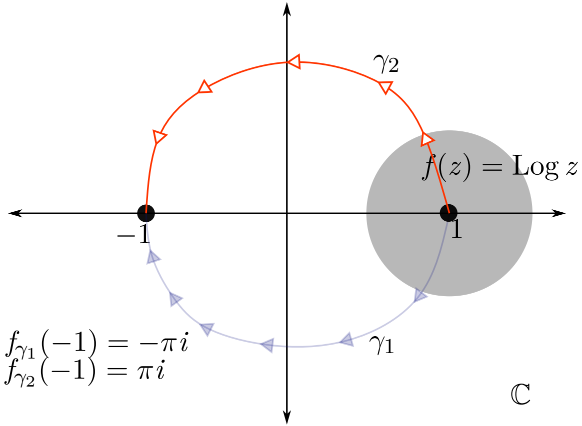

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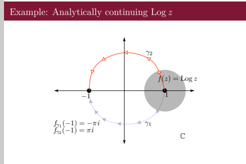
Searching for Holes in the Matrix Universe
└Part II: Monodromy
└Analytic Continuation
└Example: Analytically continuing Log z



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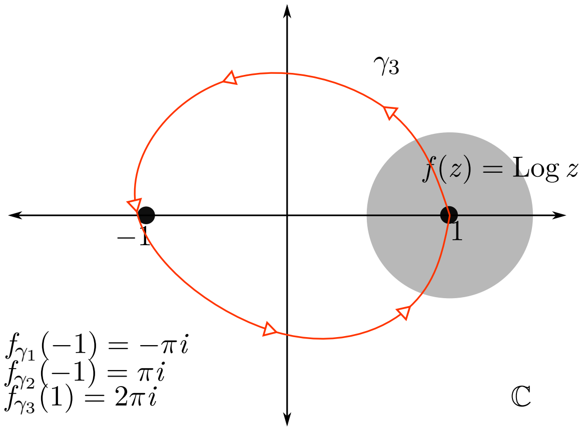

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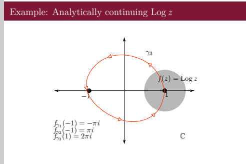

When are two analytic continuations equal?

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└Part II: Monodromy
└Monodromy
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Theorem (Monodromy I)

Let γ_1, γ_2 be two paths from α to β and Γ_s be a fixed-endpoint homotopy between them. If f can be continued along Γ_s for all $s \in [0,1]$, then the continuations along γ_1 and γ_2 agree at β.

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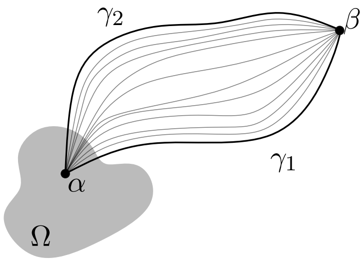

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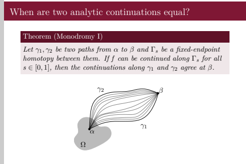

When are two analytic continuations equal?

Theorem (Monodromy II)

Let $U \subset \mathbb{C}$ be a disk in $\mathbb{C}$ centered at z_0 and $f : U \to \mathbb{C}$ an analytic function. If W is an open, simply connected set containing U and f continues along any path $\gamma \subset W$ starting at z_0, then f has a unique extension to all of W.

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Theorem (Agler, McCarthy (2016))

Let $f : D \to \mathcal{M}^{\hat{g}}$ be a free function. If f is locally bounded on each D_n, then f is an analytic free function.

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Theorem (Free Universal Monodromy, Pascoe 2020)

Let f be an analytic free function defined on some ball $B \subset D$, for D an open, connected free set. Then f analytically continues along every path in D if and only if f has a unique analytic continuation to all of D.

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└Part II: Monodromy
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Consequences of Free Monodromy

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Part III: Homotopy

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Searching for Holes in the Matrix Universe
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Definition:
A continuous function $\gamma : [0,1] \to D$ **essentially takes** X to Y if

$$\gamma(0) = X^{\oplus \ell}, \text{ for some } \ell \in \mathbb{N}$$
$$\gamma(1) = Y^{\oplus k}, \text{ for some } k \in \mathbb{N}.$$

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Searching for Holes in the Matrix Universe
└Part III: Homotopy
└A First Fundamental Group

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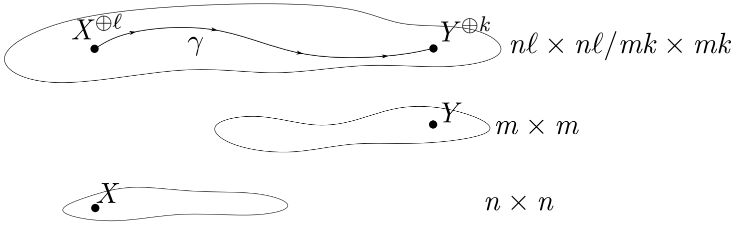

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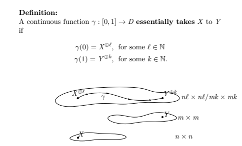

Given γ essentially taking X to Y and β taking Z to W, define

$$\gamma \oplus \beta(t) := \begin{bmatrix} \gamma(t) & 0 \\ 0 & \beta(t) \end{bmatrix}.$$

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$$\beta\gamma(t) := \begin{cases} \gamma^{\oplus k}(2t) & t \in [0, 0.5) \\ \beta^{\oplus \ell}(2t-1) & t \in [0.5, 1] \end{cases}$$

where k and ℓ are positive integers chosen to make $\gamma^{\oplus k}$ and $\beta^{\oplus \ell}$ like size matrices for each $t \in [0, 1]$.

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The Full Fundamental Group

For $D \subset \mathcal{M}^g$ a connected free set, the **full fundamenal group**, $\pi_1(D)$, is the group of paths essentially taking X to X up to homotopy equivalence and the relation $\gamma = \gamma^{\oplus k}$.

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- remark about how this misses the link to analytic continuation of functions, but free functions won't do.

Let $D \subset \mathcal{M}^g$ be a connected, open, free set. If there exists a nonempty, simply-connected, open, free $B \subset D$, then we say that D is **anchored**.

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For D an anchored set, and $B \subset D$ its anchor, we call a tracial function $f : B \to \mathbb{C}$ a **global germ** if it analytically continues along every path in D which starts in B.

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For our purposes, we view γ as coupled with its endpoint. Thus, if γ essentially takes X to Y, then

$$f(\gamma) = \frac{1}{k} f(Y^{\oplus k}).$$

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Trace Equivalence

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Let $B \subset D$ be an anchor and fix $X \in B$. If γ and β both essentially take X to Y, we say they are **trace equivalent** if, for every global germ f and every path δ taking Y to Z,

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Let $D \subset \mathcal{M}^g$ be an anchored space with B is anchor. For $X \in B$ define $\pi_1^{\mathrm{tr}}(D)$ to be the group of trace equivalent paths essentially taking X to X.

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Part IV: Cohomology

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Searching for Holes in the Matrix Universe
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Traditional homology considers a complex of the form

$$\cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} \cdots$$

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$$0 \to \mathcal{T}(D) \xrightarrow{\nabla} \mathcal{F}(D) \to \cdots .$$

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- test text

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A free function $g : D \to \mathcal{M}^g$ is **exact** if there exists a tracial function $f : D \to \mathbb{C}$ such that $\nabla f = g$.

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for all directions H, K.

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- Free Monodromy means that ∇f has a global extension.
- **Important:** If f is a global germ, then ∇f is not necessarily exact since $\mathcal{T}(D)$ is the set of tracial functions defined on *all of D*.

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- For $f : B \to \mathbb{C}$ a global germ, since f analytically continues along every path, so does ∇f.
- Free Monodromy means that ∇f has a global extension.
- **Important:** If f is a global germ, then ∇f is not necessarily exact since $\mathcal{T}(D)$ is the set of tracial functions defined on *all of D*.

Goal: Show that $\pi_1^{\mathrm{tr}}(D)$ injects into $\mathbb{C}$.

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Searching for Holes in the Matrix Universe
└Part IV: Cohomology
└Injecting into $\mathbb{C}$

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Let D be an anchored nc domain. For any $\alpha, \beta \in \pi_1^{\mathrm{tr}}(D)$ and global germ f,

$$f(\alpha\beta) - f(\alpha) = f(\beta) - f(\tau)$$

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$$\Phi : \prod_{\substack{\nabla f \in H^1_{\mathrm{tr}}(D) \\ f \text{ a global germ}}} \pi_1^{\mathrm{tr}}(D) \longrightarrow \prod_{\substack{\nabla f \in H^1_{\mathrm{tr}}(D) \\ f \text{ a global germ}}} \mathbb{C}$$

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$$H(t,\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} (\gamma \oplus \gamma_X) \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^*$$

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Searching for Holes in the Matrix Universe
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Searching for Holes in the Matrix Universe
└Part IV: Cohomology
└Characterizing $\pi_1^{\mathrm{tr}}(D)$

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Theorem

For D an anchored free set, $\pi_1^{\mathrm{tr}}(D)$ is a torsion free, abelian, divisible group. That is,

$$\pi_1^{\mathrm{tr}}(D) \simeq \bigoplus_{i \in I} \mathbb{Q} = \mathbb{Q}^I$$

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Part V: Computing $\pi_1^{\mathrm{tr}}(D)$

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- hard bc no VC or MV

Let D be an anchored, free, path connected set such that each D_n is nonempty and choose an anchor $B \subset D$ such that each B_n is also nonempty.

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Searching for Holes in the Matrix Universe
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└The Direct Limit Approach

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Now consider the chain of inclusions:

$$\pi_1^{\mathrm{tr}}(D)_1 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_2 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_6 \hookrightarrow \cdots \hookrightarrow \pi_1^{\mathrm{tr}}(D)_{n!} \hookrightarrow \cdots$$

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Let $GL = \bigcup_{n\in\mathbb{N}} GL_n(\mathbb{C})$.

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$GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. Since $\pi_1(GL_1) \simeq \mathbb{Z}$, $\pi_1^{\mathrm{tr}}(GL)_1 \simeq \mathbb{Z}$ as well.

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$$\pi_1^{\mathrm{tr}}(GL)_6 \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$$

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