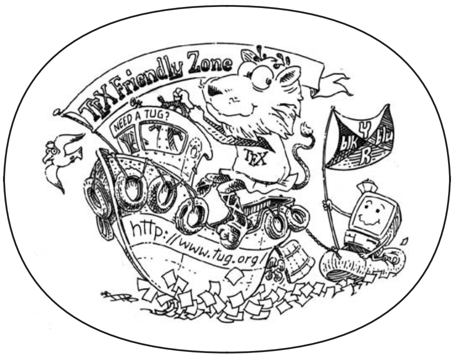
A CLEAN TITLE

LUCAS KERBS



A Fun Subtitle

February 2022 – LucasThesis v1

Ohana means family.
Family means nobody gets left behind, or forgotten.

— Lilo & Stitch

Dedicated to the loving memory of Rudolf Miede.

1939 – 2005

ABSTRACT

Short summary of the contents in English... a great guide by Kent Beck how to write good abstracts can be found here:

https://plg.uwaterloo.ca/~migod/research/beckOOPSLA.html

We have seen that computer programming is an art,
because it applies accumulated knowledge to the world,
because it requires skill and ingenuity, and especially
because it produces objects of beauty.

— Donald E. Knuth [6]

ACKNOWLEDGMENTS

Put your acknowledgments here.

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1 Members of GuIT (Gruppo Italiano Utilizzatori di TeX e LaTeX)

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Part I

OBJECTS AND THE MAPS BETWEEN THEM

"Young man, in mathematics you don't understand things. You just get used to them"

— John von Neumann

1

A FIRST ATTEMPT

1.1 INTRODUCTION

As a note to the reader (and myself): things written in blue denote things I want to add/expand upon and things writteng in red denote things that I need to add/find out/fix

Gotta do this at some point. Maybe here we define things like $\mathcal{U}_n$.

1.2 FUNCTIONAL CALCULUS

Functional Calculus refers to the process of extending the domain of a function on $\mathbb{R}$ to include matrices (or in some cases operators). The most basic formulation uses the fact that the space $n \times n$ matrices forms a ring and so there is a natural way to evaluate polynomials $f \in \mathbb{C}[x]$. If we require that $A \in M_n(\mathbb{C})$ is self-adjoint—and hence diagonalizable as $A = U\Lambda U^*$—then it is a standard result that:

$$\begin{aligned} f(A) &= a_n A^n + \cdots + a_1 A + a_0 I_n \\ &= a_n \left(U\Lambda U^*\right)^n + \cdots + a_1 U\Lambda U^* + a_0 I_n \\ &= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 I_n \\ &= U\left(a_n \Lambda^n + \cdots + a_1 \Lambda + a_0 I_n\right) U^* \\ &= U\left(f(\Lambda)\right) U^* \end{aligned}$$

Further, since Λ is diagonal and f is a polynomial,

$$f\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right) = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}$$

Therefore, given a self-adjoint matrix A and a polynomial $f \in \mathbb{C}[x]$

$$f(A) = Uf(\Lambda)U^* = U \operatorname{diag}\{f(\lambda_1), \ldots, f(\lambda_n)\}\, U^*$$

Notice that can simply substitute A in for x without any trouble as long as we transform the constant term $a_0 \mapsto a_0 I_n$ when evaluation on $n \times n$ matrices.[1] Since self-adjoint matrices play such a vital role in free analysis, we will let $\mathbb{H}_n \subset M_n(\mathbb{C})$ denote the set of $n \times n$-matrices over $\mathbb{C}$. With the polynomial case in mind, we can extend a function $g : [a, b] \to \mathbb{C}$ to a function on self adjoint matrices with their spectrum

1 Technically we have $a_0 \mapsto a_0 \otimes I_n$ but they are identical in this case. It is common in free analysis to tensor by I_n to make matrices the same size.

in $[a, b]$. Let A be such a matrix (diagonalized by the unitary matrix U), and define

$$g(A) = U \begin{bmatrix} g(\lambda_1) & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^*$$

Thus, for each $n \in \mathbb{N}$, g induces a function on the self-adjoint $n \times n$ matrices with spectrum in $[a, b]$. The natural ordering on self-adjoint matrices is called the **Loewner Order**:

Definition i.1 (Loewner Ordering)**.** *For like size self-adjoint matrices, we say that $A \preceq B$ if $B - A$ is positive semidefinite and $A \prec B$ is $B - A$ is positive definite.*

With this ordering in place, we can extend many of the familiar function theoretic properties (monotonicity, convexity) to these matrix-values functions. In fact, these properties are defined identically to their classical counterpart: We say that a function is *matrix-monotone* if $A \preceq B$ implies that $f(A) \preceq f(B)$ and *matrix-convex* (or *nc-convex*) if

$$f\left(\frac{X+Y}{2}\right) \preceq \frac{f(X) + f(Y)}{2}$$

for every pair of like-size matrice for which f is defined. These condition are rather restrictive (since the must hold for matrices of *all* sizes) so many functions which are convex/monotone (in the traditional sense) fail to be matrix-convex/monotone. For a full treatment of nc-convexity, see [5]. To illustrate the restrictiveness of nc-convexity, we will steal an example from Helton. (I don't like this phrasing. I don't mind the word steal, its just awkward)

Example i.2. *In contrast to the real (or even complex) case, $f(x) = x^4$ fails to be nc-convex. Indeed, if*

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \qquad \textit{and} \qquad Y = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$\frac{X^4 + Y^4}{2} - \left(\frac{1}{2}X + \frac{1}{2}Y\right)^4 = \begin{bmatrix} 164 & 120 \\ 120 & 84 \end{bmatrix}$$

Which is not positive definite! Thus x^4 fails to be convex on even 2×2 matrices.

Further, a number of the standard constructions lift identically in this functional calculus.

Definition i.3 (Directional Derivative)**.** *The derivative of f in the direction H is*

$$Df(X)[H] = \lim_{t \to 0} \frac{f(X + tH) - f(X)}{t}$$

where H and X are like-size self-adjoint matices.

Often, the best way to compute these directional derivatives is via an equivalent formulation:

$$Df(X)[H] = \left. \frac{df(X + tH)}{dt} \right|_{t=0}$$

This version allows us to more easily define higher order derivatives

$$D^{(k)} f(X)[H] = \left. \frac{d^{(k)} f(X + tH)}{d^{(k)} t} \right|_{t=0}$$

Example i.4. *Just as in the classical case, the directional derivative is linear, so we will only show a calculation of a monomial. Let $f(x) = x^3$. Since X and H do not commute,*

$$\begin{aligned} f(X + tH) = X^3 + tX^2H + tXHX + t^2XH^2 \\ + tHX^2 + t^2HXH + t^2H^2X + t^3H^3. \end{aligned}$$

From here, we can calculate:

$$\begin{aligned} \frac{d}{dt} f(X + tH) = X^2H + XHX + 2tXH^2 + HX^2 \\ + 2tHXH + 2tH^2X + 3t^2H^3 \end{aligned}$$

$$\frac{d^2}{dt^2} f(X + tH) = 2XH^2 + 2HXH + 2H^2X + 6tH^3$$

$$\frac{d^3}{dt^3} f(X + tH) = 6H^3.$$

And so the first 3 directional derivatives are:

$$Df(X)[H] = X^2H + XHX + HX^2$$

$$D^{(2)} f(X)[H] = 2XH^2 + 2HXH + 2H^2X$$

$$D^{(3)} f(X)[H] = 6H^3$$

In general, the k-th derivative of a polynomial is degree k as a polynomial in H.

Just as in the classical case, the second derivative tells gives us information about the convexity of a function. A function $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is said to be **positive** if $0 \preceq A \implies 0 \preceq f(A)$. In the functional calculus, we say that f is **nc positive** if it is positive as a map on $M_n(\mathbb{C})$ for all n. Despite nc-convexity being so restrictive, Lemma 12 in [5] shows that the standard characterization of convexity via the second derivative: a function f is convex if and only if $D^2 f(X)[H]$ is nc-positive. Unlike the classical case, however, the only convex polynomials are of degree 2.[2]

1.3 EXTENDING MULTI-VARIABLE FUNCTIONS

We can extend this same functional calculus to functions of several variables, although the details are a bit more subtle. We could simply "plug in" at tuple of matrices to a standard multivariable polynomial ring over $\mathbb{R}$ or $\mathbb{C}$, but this ignores the noncommutativity of $M_n(\mathbb{C})$. In light of this, let $x = (x_1, \ldots x_g)$ be a g-tuples of noncommuting formal variables. The formal variables $x_1, \ldots, x_n$ are *free* in the sense that there are no nontrivial relations between them.[3] A **word** in x is a product of these variables (e. g.$x_1 x_3 x_1 x_4^2$ or $x_1^2 x_5^3$). An **nc-polynomial** in x is a formal finite linear combination of words in x with coefficients in your favorite field. We use $\mathbb{R}\langle x \rangle$ and $\mathbb{C}\langle x \rangle$ to denote the set of nc-polynomials in x over $\mathbb{R}$ or $\mathbb{C}$ respectively.

With $\mathbb{C}\langle x \rangle$ constructed, we can define the functional calculus. Given a word $w(x) = x_{i_1}^{p_1} \cdots x_{i_d}^{p_d}$ and a g-tuple of self-adjoint matrices, X, we can evaluate w on X via $w(X) = X_{i_1}^{p_1} \cdots X_{i_d}^{p_d}$. Since our nc-polynomials are linear combinations of these words, we can extend this evaluation to evaluation of entire polynomials. Algebraically, we have a natural evaluation map: Given some $f \in \mathbb{C}\langle x \rangle$ and $X = \left(X_1, \ldots, X_g\right)$ a g-tuple of self-adjoint matrices, define

$$\begin{aligned} \varepsilon_f : \mathbb{H}_\bullet{}^g &\longrightarrow M_\bullet(\mathbb{C}) \\ X &\longmapsto f(X). \end{aligned}$$

Notice that our functions are **graded** in the sense that if X is a tuple of $n \times n$ matrices, then $f(X)$ is also a tuple of $n \times n$ matrices.

Example i.5. *Let $f(x, y) = x^2 - xyx + 1 \in \mathbb{R}\langle x, y \rangle$. If we define*

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \qquad \textit{and} \qquad Y = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

2 See [5] for details.

3 This becomes important in the eventual functional calculus—matrices *do* have nontrivial relations. See section [ALGEBRAIC CONSTRUCTION] for the details.

as before, then

$$\begin{aligned} f(X,Y) &= X^2 - XYX + I_2 \\ &= \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}^2 - \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -11 & -4 \\ -4 & 1 \end{bmatrix}. \end{aligned}$$

Additionally,

$$\begin{aligned} f(X \oplus X, Y \oplus Y) &= \begin{bmatrix} -11 & -4 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & -11 & -4 \\ 0 & 0 & -4 & 1 \end{bmatrix} \\ &= f(X,Y) \oplus f(X,Y). \end{aligned}$$

It is no accident that polynomials handle direct sums of matrices well. As in the classical case, they are the "well behaved" example which we would like general objects to emulate. In the next chapter, we will define free functions—which behave like nc polynomials.

In the context of these multivariate functions, our definition of the Directional Derivative still makes sense (although our direction H now becomes a tuple of directions). We also inherit (from multi-variable calculus) a notion of the **gradient** of a function—but this will require a bit more work.

1.3.1 *The Natural Involution on nc-Polynomials*

Given our ring of nc polynomials, we may define an involution * which we may view as an extension of the conjugate transpose. Let * reverse the order of words (i. e.$(x_1x_3x_2^2)^* = x_2^2x_3x_1$) and extend linearly to all of $\mathbb{R}\langle x \rangle$. We consider the formal variables $x_1, \dots, x_n$ *symmetric* in the sense that $x_i^* = x_i$. We say that a polynomial $p \in \mathbb{R}\langle x \rangle$ is symmetric if $p^* = p$. For example, if

$$p(x) = 5x_1^2x_3x_2 + x_3x_2x_3 \qquad q(x) = 3x_2x_1x_2 + x_3^2 - x_1,$$

then a cursory inspection tells that q is symmetric while p is not.

Notice that the majority of the previous two sections breaks down if we try to extend functions to non-self-adjoint matrices. The act of "plugging in" a tuple of arbitry matrices to some element of $\mathbb{R}\langle x \rangle$ via the same functional calculus described above still works, but $\mathbb{R}\langle x \rangle$ is no longer the natural algebra for these evaluations.

Let $x = (x_1, \dots, x_g)$ be formal variables and let $x^* = (x_1^*, \dots, x_g^*)$ denote their formal adjoints. Once again, we let the ring $\mathbb{R}\langle x, x^* \rangle$ be the finite formal sums of words in $x_1, x_1^*, \dots, x_g, x_g^*$ with coefficients in $\mathbb{R}$. Endow $\mathbb{R}\langle x, x^*\rangle$ with an involution * which sends $x_i \mapsto x_i^*$ and $x_i^* \mapsto x_i$ and reverses the order of words extended linearly. Notice that this involution behaves identically to the adjoint with respect to products and sums of matrices. This new ring inherits a natural functional calculus just like that in section 1.3 except it can accept *any* matrix as an input instead of simply self-adjoint matrices.

Example i.6. *Let* $f(x,y) = x^*y - xy^*x + 2$. *Then*

$$f^*(x,y) = y^*x - x^*yx^* + 2.$$

Evaluating f *on a pair of non self-adjoint matrices is left to the reader.*

1.3.2 *Matrices of nc-Polynomials*

It is occasionally useful in the larger theory of free analysis (e. g. when construction the free topology in section 2.3.1) to consider matrices where the matrices are nc polynomials. Formally, let $\mathbb{R}\langle x\rangle^{k\times k}$ denote the set of $k \times k$ matrices with entries in $\mathbb{R}\langle x\rangle$.[4] We can naturally extend the involution * on $\mathbb{R}\langle x\rangle$ to our matrices by applying * component wise and taking the transpose of the matrix.[5]

Given some $\delta \in \mathbb{R}\langle x\rangle^{k\times k}$ a matrix of nc polynomials, and $X \in \mathbb{H}_n^g$ there is a natural evaluation map.

$$\begin{aligned} \varepsilon_\delta : \mathbb{H}_n{}^g &\longrightarrow M_{nk}(\mathbb{C}) \\ X &\longmapsto \delta(X) \end{aligned}$$

given by evaluating each polynomial in δ at X and then viewing the result at a block $k \times k$ where each block is an $n \times n$ matrix.

Example i.7. *Define* $\delta \in \mathbb{R}\langle x, y\rangle^{2\times 2}$ *as*

$$\delta(x,y) = \begin{bmatrix} x^2 - xyx + 1 & xy - yx \\ x^4 & y^3 - 5xy + 3 \end{bmatrix}$$

Then

$$\delta^*(x,y) = \begin{bmatrix} x^2 - xyx + 1 & yx - xy \\ x^4 & y^3 - 5yx + 3 \end{bmatrix}$$

For an evaluation, we will once again let

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

4 Some sources additionally consider non-square matrices but this is rare.
5 We could likewise define $\mathbb{R}\langle x, x^*\rangle$ and extend the corresponding involution.

We already know what the evaluations of the first column from examples i.2 and i.5, so we need only complute the second column.

$$XY - YX = \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$$

$$Y^3 - 5XY + 3 = \begin{bmatrix} -29 & 0 \\ -20 & 3 \end{bmatrix}$$

And thus

$$\delta(X, Y) = \begin{bmatrix} -11 & -4 & 0 & -2 \\ -4 & 1 & 4 & 0 \\ 164 & 120 & -29 & 0 \\ 120 & 84 & -20 & 3 \end{bmatrix}.$$

2 A SECOND ATTEMPT

2.1 MATRIX UNIVERSES

Beyond the functional calculus, it becomes useful to construct general functions on spaces of matrices—to do so, we must make this idea of "spaces of matrices" concrete. The largest such space is the so-called **Matrix Universe**—consisting of g-tuples of matrices of all sizes:

$$\mathcal{M}^g = \bigcup_{n=1}^{\infty} (M_n(\mathbb{C}))^g$$

By convention, when we consider some $X = (X_1, \ldots, X_g) \in \mathcal{M}^g$, we require that the X_i are all the same size. Since $\mathcal{M}^g$ is such a large set, we often want to deal with subsets that still carry some of the implicit structure of $\mathcal{M}^g$.

Definition i.8 (Free Set)**.** *We say $D \subset \mathcal{M}^g$ is a* ***free set*** *(also called an nc set) if it is closed with respect to direct sums and unitary conjugation. That it*

1. *$X, Y \in D$ means $X \oplus Y \in D$.*
2. *For X, U like-size matrices with U unitary and $X \in D$, then $UXU^* = (UX_1U^*, \ldots, UX_gU^*) \in D$.*

For the remainder of this text, D will denote some free set. Using the terminology of [8], let $D_n = D \cap M_n(\mathbb{C})^g$ be the level-wise slice of all $n \times n$ matrices in D. We say that D is **nc-open**[1] (resp. **connected**, **simply connected**, **bounded**) if each D_n is open (resp. connected, simply connected, bounded). Finally, we say that D is **differentiable** if each D_n is an open C^1 manifold where the complex tangent space of every $X \in D_n$ is all of $M_n(\mathbb{C})^g$.

In the context of sections 1.2 and 1.3 , the domains in the functional calculus were $\mathbb{H}^g = \bigcup_{n=1}^{\infty} \mathbb{H}_n{}^g$. $\mathbb{H}^g$ is a differentiable a free set with two connected components.

On $\mathcal{M}^g$, we define a product that resembles the inner product on $\mathbb{C}^n$. Given $A, B \in \mathcal{M}^g$ which are g-tuples of $n \times n$ matrices:

$$\cdot : \mathcal{M}^g \times \mathcal{M}^g \longrightarrow M_n(\mathbb{C})$$
$$\cdot(A, B) = A \cdot B \longmapsto \sum_{i=1}^{g} A_i B_i$$

1 The topology of $\mathcal{M}^g$ is still in flux and there is not a canonical topology. See section 2.3 for the details

James uses this product, but like what in the world is going on with it???

$tr(A \otimes Id)$ But its more complicated than that bc A is a "row vector" of sorts

2.2 TRACIAL FUNCTIONS AND UNIQUENESS OF THE GRADIENT

Now that we have $\mathcal{M}^d$, we can work with general functions on our matrix universe. As a whole, free analysis is concerned with so-called *free functions*, which respect the direct sums and unitary conjugation. Do they need to be graded?

Definition i.9 (Free Function). *A function $f : D \to \mathcal{M}^{something}$ is called free if*

1. $f(X \oplus Y) = f(X) \oplus f(Y)$
2. $f(UXU^*) = f(U)f(X)f(U^*)$ *where X and U are like-size and U is unitary.*

The two other classes of functions we are concerned with are those that act like the trace and the determinant:

Definition i.10 (Determinantal Free Function). *A function $f : D \to \mathbb{C}$ is a **determinantal free function** if*

1. $f(X \oplus Y) = f(X)f(Y)$
2. $f(UXU^*) = f(X)$ *where X and U are like-size and U is unitary.*

Definition i.11 (Tracial Free Function). *A function $f : D \to \mathbb{C}$ is a **tracial free function** if*

1. $f(X \oplus Y) = f(X) + f(Y)$
2. $f(UXU^*) = f(X)$ *where X and U are like-size and U is unitary.*

Given a free function of any type, we can define the directional derivative (Definition i.3) identically. It is worth noting that, while they share the moniker of *free*, determinantal and tracial functions are *not* free functions. It is only these tracial functions which inherit the gradient mentioned above. Similarly to traditional multivariable calculus we define the gradient via its relationship to the directional derivative:

Definition i.12 (Free Gradient). *Given a tracial free function f, the free gradient, ∇f, is the unique free function satisfying*

$$\operatorname{tr}(H \cdot \nabla f(X)) = \operatorname{tr} Df(X)[H]$$

It is not-at-all obvious that such a ∇f should be unique—after all any linear combination of commutator is has trace zero. should I explain this? In the case that f is a single-variable function we can replace ∇f with the traditional derivative, f', as seen in [10, Thm 3.3].

Theorem i.13. *Let $f : (a, b) \to \mathbb{R}$ be a C^1 function. Then*

$$\operatorname{tr} Df(X)[H] = \operatorname{tr}\left(Hf'(X)\right)$$

The proof in [10] simply asserts the uniqueness of a function $g(X)$ and then shows that $g(x) = f'(x)$ for $x \in (a, b)$. Instead, we can construct such a g and recover the theorem along the way:

Proof. We start with a construction from Bhatia's Matrix Analysis: Let $f \in C^1(I)$ and define $f^{[1]}$ on $I \times I$ by

$$f^{[1]}(\lambda, \mu) = \begin{cases} \frac{f(\lambda)-f(\mu)}{\lambda-\mu} & \lambda \neq \mu \\ f'(\lambda) & \lambda = \mu. \end{cases}$$

We call $f^{[1]}(\lambda, \mu)$ the *first divided difference* of f at (λ, μ). If Λ is a diagonal matrix with entries $\{\lambda_i\}$, We may extend f to accept Λ by defining the (i, j)-entry of $f^{[1]}(\Lambda)$ to be $f^{[1]}(\lambda_i, \lambda_j)$. If A is a self adjoint matrix with $A = U\Lambda U^*$, then we define $f^{[1]}(A) = Uf^{[1]}(\Lambda)U^*$. Now we borrow a theorem from Bhatia [4]:

Theorem i.14 (Bhatia V.3.3)**.** *Theorem numbering? Let $f \in C^1(I)$ and let A be a self adjoint matrix with all eigenvalues in I. Then*

$$Df(A)[H] = f^{[1]}(A) \circ H,$$

where $\circ$ denotes the Schur-product[2] in a basis where A is diagonal.

That is, if $A = U\Lambda U^*$, then

$$Df(A)[H] = U\left(f^{[1]}(\Lambda) \circ (U^*HU)\right)U^*.$$

To prove our claim, we need to take the trace of both sides. Since trace is invariant under a change of basis, it is clear that

$$\operatorname{tr} Df(A)[H] = \operatorname{tr}\left(f^{[1]}(\Lambda) \circ (U^*HU)\right).$$

If $U = u_{ij}$, $U^* = \overline{u}_{ij}$ and $H = h_{ij}$, then the (i, j)-entry of U^*HU is

$$(U^*HU)_{ij} = \overline{u}_{ik}h_{k\ell}u_{\ell j}$$

Where we sum over the duplicate indices k and ℓ. While the structure of $f^{[1]}(\Lambda)$ is a bit unruly, our diagonal entries are $f'(\lambda)$. This means that when we take the trace of the Schur product, we have

$$\sum_k \sum_\ell \sum_i f'(\lambda_i)\overline{u}_{ik}h_{k\ell}u_{\ell i}$$

Now consider the matrix product $U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\}\, U^*H$. Since one of our terms is diagonal, the trace of this multiplication is simple:

$$\operatorname{tr} U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\}\, U^*H = \sum_k \sum_\ell \sum_i u_{ik}f'(\lambda_k)\overline{u}_{k\ell}h_{\ell i}$$

2 Entrywise

Since $u_{ik}, \overline{u}_{k\ell}, h_{\ell i} \in \mathbb{C}$ they commute. We can then relabel our indices $i \mapsto \ell$ $\ell \mapsto k$ $k \mapsto i$ to get

$$\operatorname{tr}\ U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\}\, U^* H = \sum_k \sum_\ell \sum_i f'(\lambda_i) \overline{u}_{ik} h_{k\ell} u_{\ell i},$$

So, for every direction H, we have that $\operatorname{tr}\left(U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\}\, U^* H\right) = \operatorname{tr}\left(f^{[1]}(\Lambda) \circ (U^* H U)\right)$. overfull hbox :eyeroll: By picking the "correct" H[3], we conclude that our unique quantity $g(X)$ is $U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\}\, U^*$. But, recall that $X = U\Lambda U$ so, in the functional calculus, $g(X) = f'(X)$. This recovers theorem 3.3 of [10] as we have constructed a g such that

$$\operatorname{tr}\ Df(X)[H] = \operatorname{tr} Hg(X)$$

■

With our theorem proven, we turn our attention back to the ∇f. The single variable case motivates that ∇f should correspond to the standard gradient from vector calculus. With some work, the above proof lifts the multi-variable case. It will be instructive, however, to consider a different proof.

Theorem i.15 (Trace Duality)**.** *Let f, g be free functions $\mathcal{M}^g \to \mathcal{M}^{\tilde{g}}$. If $\operatorname{tr} H \cdot f = \operatorname{tr} H \cdot g$ for all tuples H, then $f = g$.*

Proof. Since the trace relation holds for all H, we may choose our H carefully to show the equality of f and g. Say that $H, f(X), g(X)$ are g-tuples of matrices—we will first show that $f_1 = g_1$ and we will do so entry by entry. Let E_{ij} be the matrix will all zeroes and a 1 in the (i,j)-entry. Now let $H = (E_{ji}, 0, \ldots, 0)$. So $\operatorname{tr} E_{ji} f_1(X) = \operatorname{tr} E_{ji} g_1(X)$. In our products, the only elements on the diagonal are $(f_1(X))_{ij}$ and $(g_1(X))_{ij}$, so when we take the trace we have $(f_1(X))_{ij} = (g_1(X))_{ij}$. If we do this for every (i,j), we see that $f_1(X) = g_1(X)$. Similarly, we can choose $H = (0, E_{ji}, 0, \ldots, 0)$ for each i, j to show that $f_2(X) = g_2(X)$ and so on. Since $f(X) = g(X)$ for each $X \in \mathcal{M}^g$, it follows that $f = g$. ■

Admittedly, there is a slight complication that is overlooked in the above proof when it comes to the domains of f and g. Where these domains overlap, we can consider them as the same function (and therefore ∇f is unique) but if f is defined on D and g is defined on $\tilde{D}$, then the above proof only holds on $D \cap \tilde{D}$. Examples of such f and g abound when considering rational functions, which are explored in THAT SECTION

3 See example EXAMPLE NUMBER for details

2.3 THE TOPOLOGY OF MATRIX UNIVERSES

At the time of writing, there is no "canonical topology" for $\mathcal{M}^g$. For a long time it seemed like the *free* topology (to be defined below) was the obvious choice, but recent work (c.f. [7]) has shown that the free topology does not put enough structure on $\mathcal{M}^g$. See [2] for a full treatment of the common topologies on $\mathcal{M}^g$.

A naive approach to a topology on $\mathcal{M} = \bigcup_n M_n(\mathbb{C})$ would be the disjoint union topology—which is then extended do a topology on $\mathcal{M}^g$ via the product topology. does disjoint union topology "commute" with the product topology? Notice, however that this ignores a significant amount of the implicit structure of nc-sets as we get a disconnected space with countable many connected components. Topologically, this is means that means that

$$H_\bullet(D) = \bigoplus_{n\in\mathbb{N}} H_\bullet(D_n).$$

At first glance, this seems fine enough, but it ignores the fact that for $X \in D$ we require $X^{\oplus k} \in D$ for all k and $U^*XU \in D$ for all unitary U. Should define the similarity envelope at some point In a sense, we think of the all the direct sums of X and its similarity envelope as "the same." In light of this, if $\sim$ is the equivalence relation that $X \sim Y$ if $Y = X^{\oplus k}$ or $Y = U^*XU$ is this actually an equivalence relation? The second statement is immediate but the first isn't an eq. rel., then any useful topological theory on $D \subset \mathcal{M}^g$ should descend to classic theory on $D/\sim$. One needs only look at $H_0(D)$ to see that the naive approach fails to give useful information. It should be the case that $H_0(\mathcal{M}^g)$ is trivial but in the disjoint union topology it is easy to see

$$H_0(\mathcal{M}^g) = \bigoplus_{n\in\mathbb{N}} \mathbb{Z},$$

which does not behave as we would expect.

a note on convergence somewhere.

2.3.1 *Admissible Topologies*

In light of the above discussion, we will present some of the candidate topologies which show some promise in understanding the topology on $\mathcal{M}^g$ and its subsets. We say that a topology τ is **admissible** if it has a basis of nc bounded open sets, D (recall that this means that D is closed under direct sums and unitary conjugation, and that each D_n is a bounded open set in $M_n(\mathbb{C})^g$). The finest admissible topology is the so-called **fine topology**, the basis of which consists of *all* nc open sets.

A slightly more restrictive topology (that seems to show some promise in the eyes of the author) is the **fat** topology. For $n \in \mathbb{N}$, $r \in \mathbb{R}^+$, and $X \in \mathcal{M}^g_n$, we first define a matricial polydisc

$$D_n(X,r) := \{A \in \mathcal{M}^g \mid \max_{1\le i\le g} \|X_i - A_i\| < r\}.$$

Now we sweep D_n through all direct sum copies of X:

$$D(X,r) := \bigcup_{k=1}^{\infty} D_{kn}(X^{\oplus k}, r)$$

Finally, we take the similarity envelope of $D(X,r)$

$$F(X,r) := \bigcup_{n=1}^{\infty} \bigcup_{U \in \mathcal{U}_n} U^* \left(D(X,r) \cap \mathcal{M}_n^g\right) U$$

Both the fine and the fat topologies admit implicit function theorems.

The final candidate topology is the aforementioned **free** topology. Recall that $\mathbb{R}\langle x \rangle$ is the algebra of nc polynomials over the real number and that $\mathbb{R}\langle x \rangle^{k\times k}$ is the set of $k \times k$ matrices with entries in $\mathbb{R}\langle x \rangle$. Let $\delta \in \mathbb{R}\langle x \rangle^{k\times k}$ and define

$$G_\delta = \{x \in \mathcal{M}^g \mid \|\delta(x)\| < 1\}$$

The set of all G_δ as k ranges over $\mathbb{Z}^+$ form the basis for the free topology. Indeed, any $X \in \mathcal{M}^g$ is trivially in one of the G_δ (take $\delta = X$) and with some work one can show that $G_{\delta_1} \cap G_{\delta_2} = G_{\delta_1 \oplus \delta_2}$ (prove this) so we do, indeed, have a basis.

In [1], Agler and McCarthy proved an free analogue of the Oka-Weil theorem: any holomorphic function on a compact set in the free topology can be uniformly approximated by polynomials. Unfortunately, it was later proven in [7] and [3] that the only compact sets in the $\mathcal{M}^g$ are the envelope of finitely many points, trivializing the result of Agler and McCarthy.

It is the opinion of the author that all of these topologies are definitively broken. As shown above, the free topology lacks a wealth of compact sets. The fine topology (and therefore any admissible topology) fails to be T_1, let alone Hausdorff—notice that any open set containing X must also contain $X \oplus X$. Further, given any free function f on an nc-domain D, if f is locally bounded on each D_n then f is analytic (admits a power series representation.) There are two ways so view this result: First, one can accept that analytic functions are a dime dozen on $\mathcal{M}^g$. Alternatively, one can be skeptical that the topological structures put on $\mathcal{M}^g$ are indeed the natural choice. The work of J.E. Pascoe in [8] covered in part ii seeks to solve some of these issues by extending some of the concepts of traditional algebraic topology.

For the rest of this thesis, we will be using the conventions mentioned in section 2.1: $D \subset \mathcal{M}^g$ open if each D_n is open—these are precisely the basic open sets in the fine topology. this is the fine topology, right? or is it just the basic sets? Is the topology generated by these just themselves?

2.4 NC RATIONAL FUNCTIONS

Short review about defining rational functions via equivalence classes. We need this bc rational functions give the nc picard group and divisors and all that

Part II

THE ALGEBRAIC GEOMETRY AND TOPOLOGY OF MATRIX DOMAINS

"Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine."

— Michael Francis Atiyah

3 ZERO SETS AND PRINCIPLE DIVISORS

3.1 TEMP

This needs a name. The goal of this chapter is to develop the whole thing with zero sets, singular sets, and divisors

4 MONODROMY, GLOBAL GERMS, ALGEBRAIC TOPOLOGY

4.1 TEMP

This section needs a better title.

Monodromy $\rightarrow$ Global Germs (maybe with sheaf theoretic language) $\rightarrow$ alg. Topology.

A good thing to end on is the fact that $\pi_1^{tr}(GL) = \mathbb{Q}$ and $\pi_1(G_\Lambda) = \mathbb{Q}^{|\Lambda|}$.

4.2 CLASSICAL MONODROMY

In the study of functions of a single complex variable, many of the central theorems surround the idea of analytic continuation. Given some analytic function f on a domain $\Omega \subset \mathbb{C}$ and a larger domain $\overline{\Omega} \supset \Omega$, we can (with sufficient "niceness" conditions) extend f to an analytic function g on $\overline{\Omega}$. In particular, given some path γ which start in Ω we wish to continue f *along* γ by recomputing the power series on overlapping disks with their centers on γ.

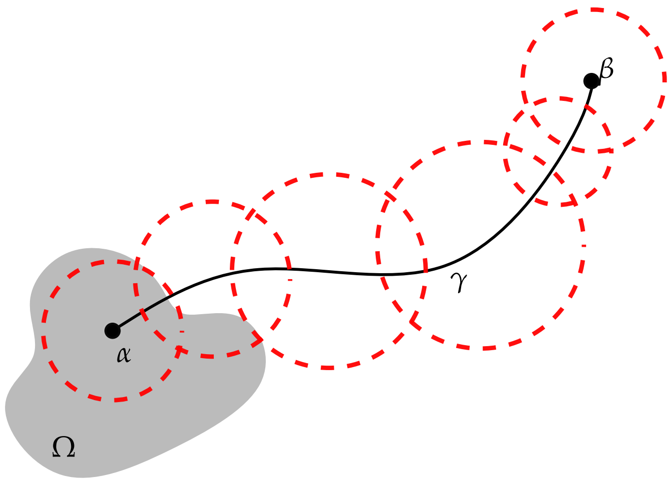

Figure 4.1: Analytic continuation along a curve

Our path γ must avoid any potential poles of f so that we may compute the power series, but the uniqueness of such an extension is not obvious. This is where the aforementioned niceness conditions come into play! For example, consider the follow setup:

Example ii.1. *If we let $f(x) = \operatorname{Log} x$ be the principle branch of the complex logarithm the defined on the right half plane, and continue f along γ_1 and*

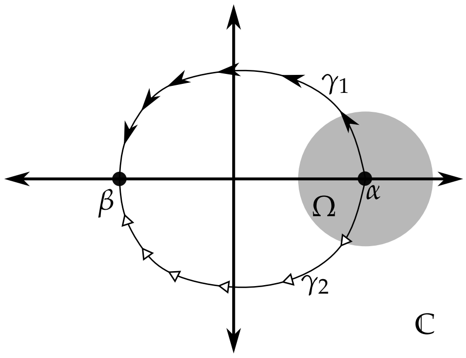

Figure 4.2: Two paths in $\mathbb{C}$

γ_2 we get two functions f_1 and f_2 which are analytic at β, but they don't agree! In this case, $f_1(\beta)$ and $f_2(\beta)$ disagree by exactly $2\pi i$.

The monodromy theorem gives sufficient conditions for the continuation along two curves to agree:

Theorem ii.2 (Monodromy I)**.** *Let γ_1, γ_2 be two paths from α to β and Γ_s be a fixed-endpoint homotopy between them. If f can be continued along Γ_s for all $s \in [0,1]$, then the continuations along γ_1 and γ_2 agree at β.*

In the example above, any homotopy between the two paths must pass through the origin—where $\operatorname{Log} x$ fails to be analytic—and hence the two continuations disagree at β. An equivalent formulation of the monodromy theorem concerns extending a functions to a larger domain:

Theorem ii.3 (Monodromy II)**.** *Let $U \subset \mathbb{C}$ be a disk in $\mathbb{C}$ centered at z_0 and $f : U \to \mathbb{C}$ an analytic function. If W is an open, simply connected set containing U and f continues along any path $\gamma \subset W$ starting at z_0, then f has a unique extension to all of W.*

This second formulation gives another perspective on $\operatorname{Log} x$. In the example, U is a disk around α that stays in the right half plane and W is $\mathbb{C} \setminus \{0\}$. While $\operatorname{Log} x$ continues along any path in $\mathbb{C} \setminus \{0\}$, the larger domain is *not* simply connected, so monodromy fails.

In practice, after the initial exposure in a first course in complex variables, no one computes continuations by hand. This could be a paragraph, but is it necessary?

4.3 FREE MONODROMY

There is an analogous theorem to theorems ii.2 and ii.3 in the free settings initial proven by J.E. Pasocoe in [9]. In the classic case, the larger set W must be simply connected. In the free setting, however, the theorem is much more powerful.

Theorem ii.4 (Free Univeresal Monodromy)**.** *If f is an analytic free function defined on some ball $B \subset D$, for D an open, connected free set. Then f analytically continues along every path in D if and only if f has a unique analytic continuation to all of D.*

Proof (From [9]). The fact that a unique extension to all of D implies that f has a continuation along any γ is immediate.

honestly I don't quite get the proof yet, so this will come later. ∎

In the free case, the "larger" set need not be simply connected. Analytic continuations of free functions, then, cannot be used to detect holes in matrix domains. It will turn out, however, that the tracial and determinental functions introduced in section 2.2 can detect holes and produce an analogue of the fundamental group!

4.4 THE GERM OF FUNCTION

eww the title. I always like Aluffi's chapter called "a bit of algebraic geometry"—could do "a bit of sheaf theory" but I don't want to scare the reader.

As studied in complex analytic and measure theoretic settings, if our space is structured enough functions are defined by their local behavior. This idea can be generalized to arbitrary topological spaces by stealing from sheaf theory fix that wording.

Let X be a topological space. For any open set U we can have $C(U)$, the ring of continuous functions $f : U \to \mathbb{R}$ (where addition and multiplication are defined point-wise) Tracial functions fail to be a ring but they *are* a group—should I just change this to a group?. Given any $V \subset U$, notice that a continuous function f on U, we can restrict f to V and maintain continuity. This gives two maps:

$$\begin{aligned} V &\hookrightarrow U \\ v &\longmapsto v \end{aligned} \qquad\qquad \begin{aligned} C(U) &\hookrightarrow C(V) \\ f &\longmapsto f|_V \end{aligned}$$

Notice that the induced function goes the "other way." This construction is an example of a sheaf of rings[1]—since $C(U)$ has a ring structure. We can similarly define sheaves of abelian groups or sets: to each open set in X we assign a group (or set) such that there are analogous restriction maps. For our purposes, these will always be groups/sets of functions and the restriction maps are the natural ones.

We are interested in the general behavior of continuous functions at some $x \in X$. Define $\mathfrak{C}_x$ to be the set of all functions defined on a neighborhood of x:

$$\mathfrak{C}_x = \{f \in C(U) \mid x \in U \subset X \text{ is open}\}.$$

1 To be completely rigorous, a sheaf needs additional axioms, but the sheaf of continuous functions is one of the prototypical examples so the full defition is not needed in this context.

By convention, we refer to elements of $\mathfrak{C}_x$ as a pair, (f, U) of a continuous function and the open set on which it is defined. In light of the inclusion maps given above, it obvious that $\mathfrak{C}_x$ will have "duplicate" elements. Therefore, we define an equivalence relation on $\mathfrak{C}_x$ by $(f, U) \sim (g, V) \Leftrightarrow$ there exists $W \subset U \cap V$ where $f|_W = g|_W$. In a sheaf-theoretic context, ${}^{\mathfrak{C}_x}\!/\!_{\sim}$ is called the **stalk** at x and elements of the stalk are **germs** at x. If we are dealing with sheaves of groups or sets, this construction remains unchanged! We can still define the stalk at given point. While it will not come into play, it is worth noting that the stalk inherits the algebraic structure of the original sheaf—e. g.for a sheaf of rings (or group), the stalk has a natural ring (group) structure.

Sheafs of rings/groups/sets of functions arise naturally in many areas of mathematics. For example, if X happens to be a smooth manifold, we may replace $C(U)$ with $C^\infty(U)$, the ring of smooth functions into $\mathbb{R}$ and then obtain germs of smooth functions. Similarly, if X is a complex manifold we can construct germs of holomorphic functions.

Example ii.5. *Should I use the same number?*

Consider, again, example ii.1. Our function $f(x) = \operatorname{Log} x$ has a germ in Ω. In particular, both f_1 and f_2 belong to the equivalence class $[(f, \Omega)]$ as all three functions agree on Ω. From this, we see the genesis of the name germ: germs capture the local *behavior of function. Colloquially, this is the "heart" of a function similar to the germ of seed.*[2]

Link to monodromy again?

As usual, lifting this construction to the free context requires some nuance. For $U \subset D$ open, the set of tracial functions on U (denoted $C_{\text{tr}}(U)$) does not for a ring—it is closed under addition but not multiplication. Given two tracial functions, $f, g \in C_{\text{tr}}(U)$, we see that

$$\begin{aligned}
(f+g)(X \oplus Y) &= f(X \oplus Y) + g(X \oplus Y) \\
&= f(X) + f(Y) + g(X) + g(Y) \\
&= (f+g)(X) + (f+g)(Y)
\end{aligned}$$

but,

$$\begin{aligned}
(fg)(X \oplus Y) &= f(X \oplus Y) g(X \oplus Y) \\
&= (f(X) + f(Y))(g(X) + g(Y)) \\
&= (fg)(X) + (fg)(Y) + f(X)g(Y) + f(Y)g(X).
\end{aligned}$$

Thankfully, however, the construction remains unchanged if we substitute a ring of functions for an abelian group of functions (with the identity being $f \equiv 0$ and inverses given by simply negating the output). In the case of determinental and free functions (which play

2 Sheaf theory abounds with agrarian nomenclature. This is my first footnote. Should the thesis have fun math facts like this in the footnotes or should I do away with them?

a lesser role in the theory to be developed) there is not a natural algebraic structure for the corresponding sheaves, so they are simply sheaves of sets.

4.5 THE TRACIAL FUNDAMENTAL GROUP

While Free Monodromy means that free functions cannot detect the topology of free sets, the same is not true for a general tracial function! To do so, however, we will need some definitions. Is this too conversational?

Definition ii.6 (Anchored)**.** *Let $D \subset \mathcal{M}^g$ be a connected, open, free set. If there exists a nonempty, simply-connected, open, free $B \subset D$, then we say that D is* ***anchored****.*

Definition ii.7 (Global Germ)**.** *For D an open set, and $B \subset D$ its anchor, we call a tracial function $f : B \to \mathbb{C}$ a* ***global germ*** *if it analytically continues along every path in D which starts in B.*

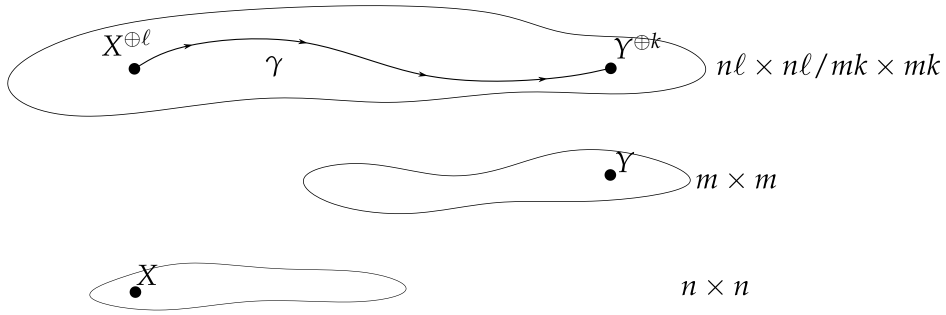

Figure 4.3: A path essentiall taking X to Y

Part III

APPENDIX

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