Searching for Holes in the Matrix Universe

Lucas Kerbs

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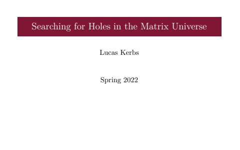

- Eventual goal: lift the tools of algebraic topology to spaces of matrices
- If we only consider 2×2 matrices we can use classical theory
- The moment we want more than one size, things the classical theory breaks down
- Today we will develop some *fairly heftly* tools to do just that
- Along the way, hopefully I can convince you that this is an interesting question.
- To do so, we need to go back to our mathematical roots

Part I: Objects and Maps

A Naive Attempt

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- That's right—objects and maps.
- Our Naive attempt involves that looking at lifting functions on $\mathbb{R}$ or $\mathbb{C}$ to accept matrices as their input.
- An operator theorist would call this a "functional calculus"

Functional Calculus

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- You might say that SA is unnecessary bc we can already evaluate a polynomial on a matrix

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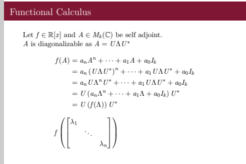

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Let $\mathbb{H}_n$ be the set of $n \times n$ self adjoint matrices, and define

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- With the polynomial case in mind, we can extend a general function. First, a piece of notation
- Lets grab a function on the real line and the self adjoint matrices with their spectrum in that domain
- Then we can lift g by emulating the behavior of polynomials.
- unwrap a self adjoint matrix, apply g to the diagonal, then wrap it back up
- Something to notice about this functional calculus—it treats direct sums *very* well
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Important: In this functional calculus,

$$g(X \oplus Y) = g(X) \oplus g(Y)$$

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- Notice that, with some special attention to what operation we are carrying our, this is the exact same definition as classic multivariable calculus.
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Example: $g(x) = x^3$

$$g(X + tH) =$$

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- Now we consider an example. Since $Df(X)[H]$ is linear, we can just work with a single monomial
- First we expand $(x + th)^3$—but we can't use the binomial theorem since x and h don't commute
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\frac{d^2}{dt^2} g(X+tH) &= 2XH^2 + 2HXH + 2H^2X + 6tH^3
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$g(X+tH) = X^3 + tX^2H + tXHX + t^2XH^2 + tHX^2 + t^2HXH + t^2H^2X + t^3H^3.$
From here, we can calculate:
$\frac{d}{dt} g(X+tH) = X^2H + XHX + 2tXH^2 + HX^2 + 2tHXH + 2tH^2X + 3t^2H^3$
$\frac{d^2}{dt^2} g(X+tH) = 2XH^2 + 2HXH + 2H^2X + 6tH^3$

- Now we consider an example. Since $Df(X)[H]$ is linear, we can just work with a single monomial
- First we expand $(x+th)^3$—but we can't use the binomial theorem since x and h don't commute
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Searching for Holes in the Matrix Universe
└Part I: Objects and Maps
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Part I.5: Objects and Maps

A Second Attempt

Part I.5: Objects and Maps
A Second Attempt

- In seeking a more general theory we need to leave the world of this “SA functional calculus” behind.
- Rather than lifting functions to be matrix valued, we will define *new* objects that behave like those we just looked at.

Some Definitions

Definition:
The g-dimensional **Matrix Universe** is

$$\mathcal{M}^g = \bigcup_{n \in \mathbb{N}} (M_n(\mathbb{C}))^g$$

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Searching for Holes in the Matrix Universe
└Part I: Objects and Maps
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- In math we often think about substructures that capture the implicit structure our space (subgroup, subspace, etc)
- In the nc setting, this is a *free set*, also called nc set
- direct sums and unitary conjugation are component wise
- If you see a D, you can assume that it is a free set.
- A subscript denotes a level-wise slice
- Note that this requires a lot of structure on free sets—we want to put a name to these structure.

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Searching for Holes in the Matrix Universe
└Part I: Objects and Maps
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- The fiber is all the points “above” a given point. Conceptually, we imagine identification along the fiber—this will become important when we start doing topology
- The envelope (which will be less important to us) is the unitary smearing of the fiber at each level.
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Searching for Holes in the Matrix Universe
└Part I: Objects and Maps
└Matrix Universe
└A bit of topology

A bit of topology

- If we are going to look for holes and build up the algebraic topology, we need a point set topology first—so there is a natural question.
- Bad news: there isn't a natural choice
- There are a handful of candidates (fine, fat, free, nc Zariski). I wish we had time to go into detail.
- For us, free sets are open if their level-wise restriction is open
- Other point-set characterizations are similar.

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2022-06-03
Searching for Holes in the Matrix Universe
└Part I: Objects and Maps
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└What the natural functions on $\mathcal{M}^g$?

What the natural functions on $\mathcal{M}^g$?

- We have our objects, but what are the maps?
- Free functions are defined to be anything that behaves like a polynomial
- tracial functions look like traces
- For both of these maps, the directional derivative is defined identically as before—but tracial functions get something extra.

What the natural functions on $\mathcal{M}^g$?

Definition:
A function $f : D \to \mathcal{M}^{\hat{g}}$ is called **free** if

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Definition:
Given a tracial function f, the free gradient, ∇f, is the unique free function satisfying

$$\operatorname{tr}\left(H \cdot \nabla f(X)\right) = Df(X)[H],$$

where, if $A = (A_1, \ldots, A_g)$ and $B = (B_1, \ldots, B_g)$ are tuples of like-size matrices then $A \cdot B = \sum_{i=1}^{g} A_i B_i$.

Searching for Holes in the Matrix Universe
└Part I: Objects and Maps
└Uniqueness of the Gradient

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- The ∇ of a free function is the unique free function satisfy this equation—where the * is just like the dot product
- Whenever you see tr(*) I want you to think of the inner product—it is slightly distinct but it will make a lot of things make more sense
- Some of you may be hesitant at the fact that I claim ∇ is unique. Why should this be true?

Why should ∇f be unique?

Theorem (Trace Duality)

Let f, g be free functions $\mathcal{M}^g \to \mathcal{M}^{\tilde{g}}$. If $\operatorname{tr}(H \cdot f) = \operatorname{tr}(H \cdot g)$ for all tuples H, then $f = g$.

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- $f = g$ whenever the domains overlap
- In the vector space setting—with an inner product—this is a fairly immediate result. You would show it by picking vectors of all 0's and a single 1.
- You prove this identically—but with coordinate matrices instead of coordinate vectors.
- Before we look at the algebraic topology, we need to take a brief trip to complex variable land

Part II: Analytic Continuation and Monodromy

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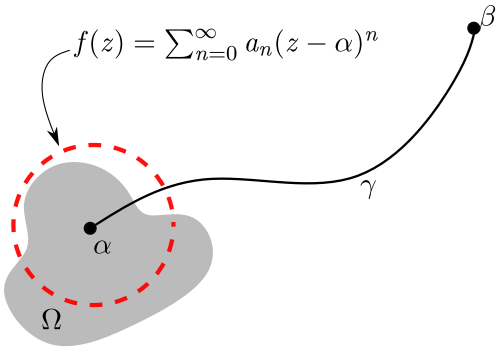
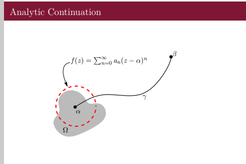
- analytic continuation and monodromy is the link between complex analysis and topology.

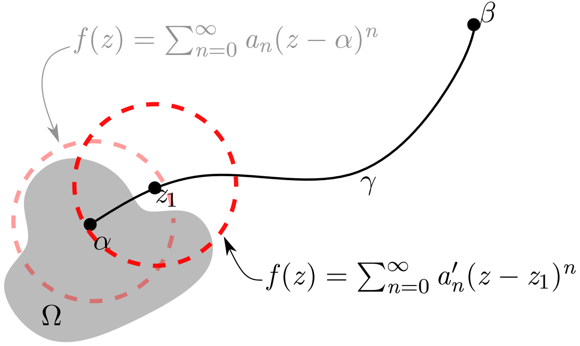
Analytic Continuation

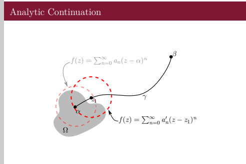




- Lets say we have some analytic function defined on Ω and a curve γ taking α to β.
- We can expand a power series about α with some radius of convergence. But since f is analytic, we can expand about some any point on γ that is still in the red disk

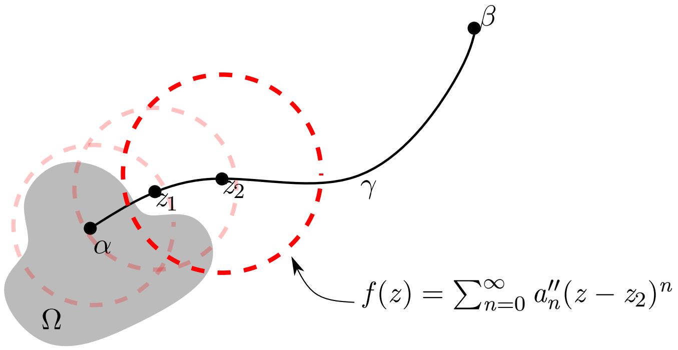
Analytic Continuation

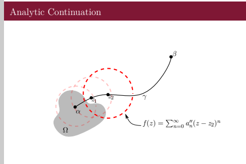




- But now we have a new radius of convergence! Importantly this will agree with the original function on that initial overlap
- We can keep doing this!

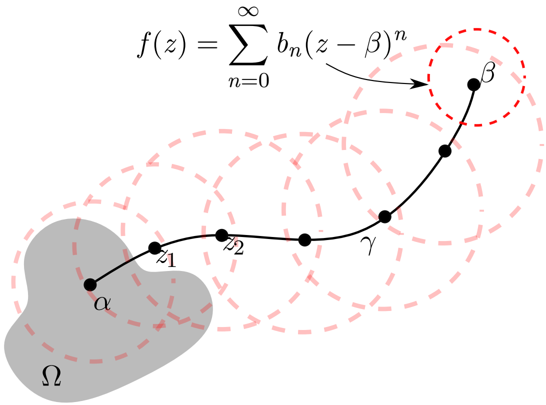
Analytic Continuation

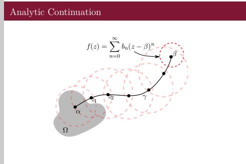




- Now we have a third power series representation for f—once again, it will agree with our last expasions where those two disks overlap.
- As long as γ stays away from any potential poles, we can keep doing this all the way to β

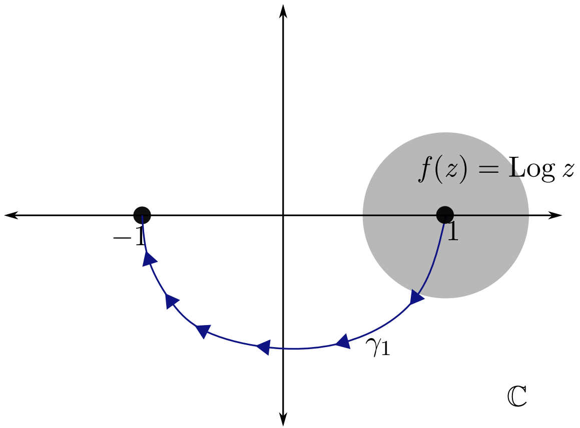
Analytic Continuation

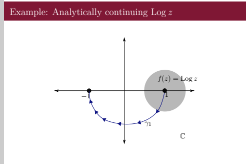




- After repeatedly expanding, we finally have an analytic function at β!
- As we said, we need γ to avoid poles, but what can we say about the uniqueness of the analytic function at β?

Example: Analytically continuing Log z





- Now we have another example: consider the principle branch of the complex logarithm.
- if we analytically continue along γ_1, then we can evaluate $\text{Log}(-1)$.

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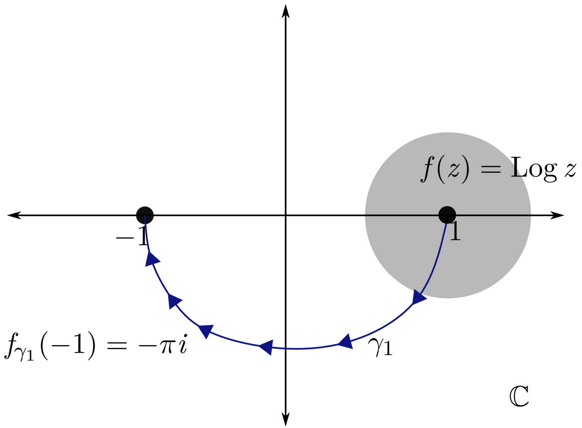

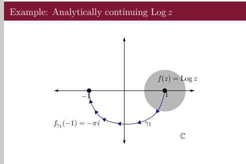

- when we do this, we see that the $\text{Log}(-1) = -\pi i$.
- But what about the other way? What if we continued along a path that went through the UHP?

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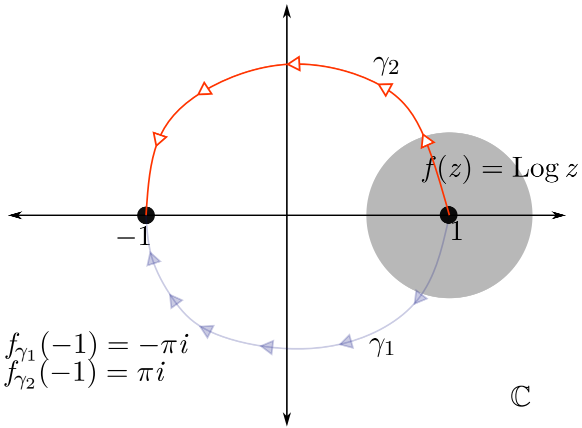
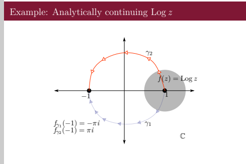


2022-06-03
Searching for Holes in the Matrix Universe
└Part II: Monodromy
└Analytic Continuation
└Example: Analytically continuing Log z



- when we do this, we get $\text{Log}(-1) = \pi i$! They disagree!
- what's even stranger is what happens when we keep going on a circle around the origin.

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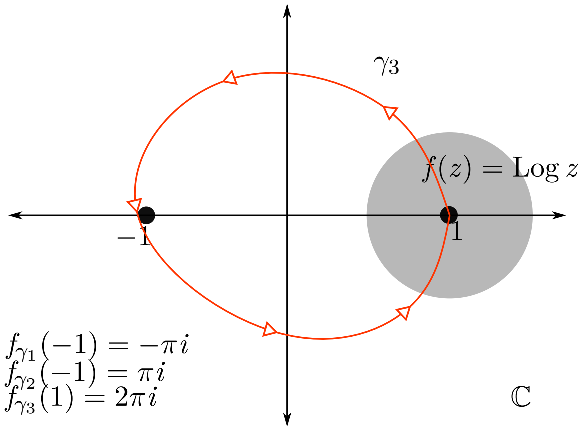

Searching for Holes in the Matrix Universe
└Part II: Monodromy
└Analytic Continuation
└Example: Analytically continuing Log z

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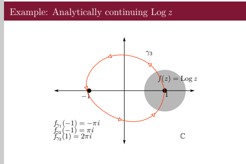

- amazingly, when you continue all the way around and compute $f(1)$, you get $2\pi i$—not 1.
- What is going on here? when are two analytic continuations equal?

When are two analytic continuations equal?

2022-06-03
Searching for Holes in the Matrix Universe
└Part II: Monodromy
└Monodromy
└When are two analytic continuations equal?

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- It is in answering this question that we see the deep link between analytic continuation and topology.
- Monodromy 1: the paths are homotopic and the function analytically continues along all of the intermediate paths, then you are golden!
- a picture for those who like that!
- Note that this tells us that analytic contiuation searches for holes
- when seek to lift this idea to a nc case, it will serve much better to consider an alternate characterization

When are two analytic continuations equal?

Theorem (Monodromy I)

Let γ_1, γ_2 be two paths from α to β and Γ_s be a fixed-endpoint homotopy between them. If f can be continued along Γ_s for all $s \in [0,1]$, then the continuations along γ_1 and γ_2 agree at β.

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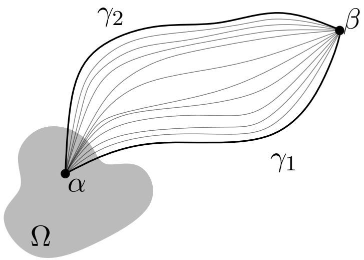

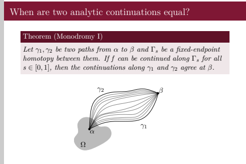

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Theorem (Monodromy II)

Let $U \subset \mathbb{C}$ be a disk in $\mathbb{C}$ centered at z_0 and $f : U \to \mathbb{C}$ an analytic function. If W is an open, simply connected set containing U and f continues along any path $\gamma \subset W$ starting at z_0, then f has a unique extension to all of W.

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- We require simply connected in the big set! Thus, holes get in the way of a unique extension.
- Obviously we don't have the time to go into explicit detail about why, but it turn out that you can realize the fundamental group of some open, connected, subset of $\mathbb{C}$ simply by looking at the analytic continuation of functions!

What about the nc case?

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- Now that we have this (very power) theorem from classical complex analysis, if we want an nc analogue—what is an "analytic" free function.
- As with our other characterizations, a free function is analytic if it is analytic as a function on each D_n.
- Even more surprisingly, we have a *wild* characterization due to Agler and McCarthy
- This is incredibly powerful—I will let you draw your own conclusion as to what is says about the underlying point set topology.
- With our question answered, we can present the free analogue.

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Let $f : D \to \mathcal{M}^{\hat{g}}$ be a free function. If f is locally bounded on each D_n, then f is an analytic free function.

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- all you need to have a unique extension is to continue along every path!
- The larger set doesn't have to be simply connected! This is huge!
- While this theorem is amazing, it is the bearer of bad news

Consequences of Free Monodromy

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- There are two major (and related) consequences to free monodromy
- First, free functions cannot detect holes via analytic continuation
- Therefore, if we want a fundamental group that is governed by analytic continuation, we need to look elsewhere
- before we transition to the fundamental group, any questions.

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Part III: Homotopy

Searching for Holes in the Matrix Universe
└Part III: Homotopy

2022-06-03

Part III: Homotopy

- In part III we look at two types of fundamental group.
- Since free monodromy says that free functions won't give us a π_1 we start by looking at a fundamental group that is divorced from analytic continuation

Definition:
A continuous function $\gamma : [0,1] \to D$ **essentially takes** X to Y if

$$\gamma(0) = X^{\oplus \ell}, \text{ for some } \ell \in \mathbb{N}$$
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- recall that the fiber of a point in $\mathcal{M}^g$ is all direct sum copies of that point—and futher that we consider everything in the fiber somehow “the same”
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- In order for us to create a group out of these paths, we a concatination product.

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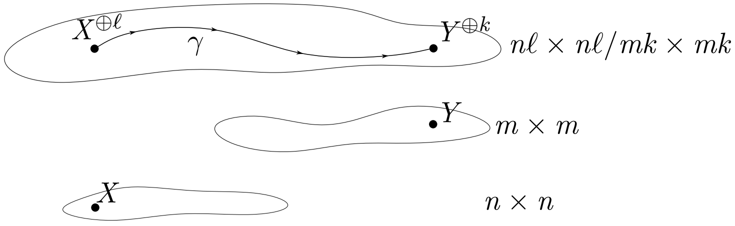

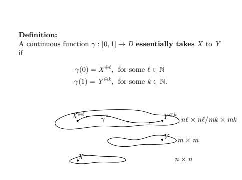

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Given γ essentially taking X to Y and β taking Z to W, define

$$\gamma \oplus \beta(t) = \begin{bmatrix} \gamma(t) & 0 \\ 0 & \beta(t) \end{bmatrix}.$$

Searching for Holes in the Matrix Universe
└Part III: Homotopy
└A First Fundamental Group

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- To concatenate the paths, the definition is almost identical—first you do one path twice as fast, then you do the other
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Let γ and β be paths taking X to Y and Y to Z respectively. We define their product to be the path essentially taking X to Z given by

$$\beta\gamma(t) = \begin{cases} \gamma^{\oplus k}(2t) & t \in [0, 0.5) \\ \beta^{\oplus \ell}(2t-1) & t \in [0.5, 1] \end{cases}$$

where k and ℓ are positive integers chosen to make $\gamma^{\oplus k}$ and $\beta^{\oplus \ell}$ like size matrices for each $t \in [0, 1]$.

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- That was all we needed to create the “full” fundamental group.
- group of paths up to homotopy equivalence and direct sum of paths
- You can show that this is abelian and divisible—but computationally we are totally stuck—we don’t have any tools to compute π *full*
- Instead, we can look at analytic continuation of *tracial* functions and see if that can get us anything.

Let $D \subset \mathcal{M}^g$ be a connected, open, free set. If there exists a nonempty, simply-connected, open, free $B \subset D$, then we say that D is **anchored**.

2022-06-03

Searching for Holes in the Matrix Universe
└Part III: Homotopy
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Trace Equivalence

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Let $B \subset D$ be an anchor and fix $X \in B$. If γ and β both essentially take X to Y, we say they are **trace equivalent** if, for every global germ f and every path δ taking Y to Z,

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2022-06-03

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Part IV: Cohomology

- Cohomology is going to help us a lot when it comes to characterizing—and eventually computing π_1^{tr}.
- Just a note, this is by far the most technical section—so bear with me.
- before we jump into the particular cohomological theory we are going to be using, lets speedrun a review of cohomology.

2022-06-03
Searching for Holes in the Matrix Universe
└Part IV: Cohomology
└A Short Review of Cohomology

- In traditional homology, the boundarmy homomorphisms *decrease the index.*
- For cohomology, the boundary morphism go the other way—the index *increases*
- Generally, we consider the chain groups to be groups of functions into some abelian group, but that isn't always the case
- once you have this co-chain complex, you compute the cohomology groups exactly the same way you did before.
- the **kernel** of one map mod the **image** of the previous one.

Traditional homology considers a complex of the form

$$\cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} \cdots$$

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In general, C^k is a group of functions into some abelian group.

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$$\cdots \xrightarrow{d_{n-2}} C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \xrightarrow{d_{n+1}} \cdots$$

In general, C^k is a group of functions into some abelian group.

The k-th cohomology group is

$$H^k = \frac{\ker d_i}{\operatorname{Im} d_{i-1}}$$

Searching for Holes in the Matrix Universe
└Part IV: Cohomology
└A Short Review of Cohomology

2022-06-03

Traditional homology considers a complex of the form

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- In traditional homology, the boundarmy homomorphisms *decrease the index.*
- For cohomology, the boundary morphism go the other way—the index *increases*
- Generally, we consider the chain groups to be groups of functions into some abelian group, but that isn't always the case
- once you have this co-chain complex, you compute the cohomology groups exactly the same way you did before.
- the **kernel** of one map mod the **image** of the previous one.

Let D be an anchored set. Denote the set of (globally defined) tracial functions on D by $\mathcal{T}(D)$ and the set of free functions by $\mathcal{F}(D)$.

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- Lets build a complex! We start with two sets of functions.
- Since the gradient of a **tracial** function is a free one, we have the first map!
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2022-06-03

Searching for Holes in the Matrix Universe
└Part IV: Cohomology
└Tracial Cohomology

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- Since we are looking at π_1, we need to create H^1—which is the homology of *this* part.
- The image of ∇ is very easy—a free function is exact if it there is some *global* tracial function such that $\nabla f = g$
- The kernel is a bit harder—we say closed if it follows this condition.
- if you view $\mathrm{tr}(*)$ as the inner product and $Dg(X)[H]$ as something like the jacobian evaluated on a direction, then this is just the classical condition of a closed!
- Now we define the first *tracial* fundamental group as the closed functions mod the exact ones

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A free function $g : D \to \mathcal{M}^g$ is **exact** if there exists a tracial function $f : D \to \mathbb{C}$ such that $\nabla f = g$.

2022-06-03

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for all directions H, K.

2022-06-03
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└Part IV: Cohomology
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- Since f continues along every path, so does ∇f.
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2022-06-03

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- Before we continue, I want to give a look at the light at the end of the tunnel. We are going to use the tracial cohomology group to show that π_1^{tr} injects in $\mathbb{C}$ and prove a major structure theorem.
- Like any good theorem, it all depends on some technical lemma—here is ours
- So what is this saying? $f(\alpha\beta) - f(\alpha)$ measure how analytic continuation changes the value of $f(\alpha)$. With this in mind, it isn't hard to believe the lemma.

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Lemma (Kerbs)

Let D be an anchored set. For any $\alpha, \beta \in \pi_1^{\text{tr}}(D)$ and global germ f,

$$f(\alpha\beta) - f(\alpha) = f(\beta) - f(\tau)$$

where τ is the constant path.

Searching for Holes in the Matrix Universe
└Part IV: Cohomology
└Injecting into $\mathbb{C}$

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For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{\mathrm{tr}}(D)$, define

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2022-06-03

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- Lets define $c^f(\gamma)$ to be the measure of how analytic continuation along γ changes f.
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2022-06-03

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└Part IV: Cohomology
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- We map π_1 into a product of $\mathbb{C}$'s with one for every unique image of a global germ under ∇.
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- The first thing we need is that, in π_1 you can direct sum a τ on either side. All you need for this is a fixed end pt homotopy of some rotation matrices
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2022-06-03
Searching for Holes in the Matrix Universe
└Part IV: Cohomology
└Characterizing $\pi_1^{\mathrm{tr}}(D)$
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2022-06-03

Searching for Holes in the Matrix Universe
└Part IV: Cohomology
└Characterizing $\pi_1^{\mathrm{tr}}(D)$

- Under trace equivalence, we can direct sum as many γ's we want, so we grab k of them
- Since diagonal, we can separate this to be a single γ and a whole lot of τ's
- But now we can go through and flip these one by one to put the γ first
- If we collect terms, we get a formula for k-th roots!

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Searching for Holes in the Matrix Universe
└Part IV: Cohomology
└Characterizing $\pi_1^{\mathrm{tr}}(D)$

2022-06-03

- Under trace equivalence, we can direct sum as many γ's we want, so we grab k of them
- Since diagonal, we can separate this to be a single γ and a whole lot of τ's
- But now we can go through and flip these one by one to put the γ first
- If we collect terms, we get a formula for k-th roots!

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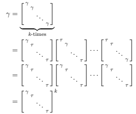

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Theorem (Pascoe 2020)

For D an anchored free set, $\pi_1^{\mathrm{tr}}(D)$ is a torsion free, abelian, divisible group. That is,

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- Putting this all together, we know that π_1^{tr} is abelian, divisible, and torsion free. Thanks to a "fundamental structure theorem" this means that π_1^{tr} is isomorphic to some number of copies of $\mathbb{Q}$!
- "But I promised we would be able to compute! This is just a structure theorem!"

Part V: Computing $\pi_1^{\mathrm{tr}}(D)$

Searching for Holes in the Matrix Universe
└Computing $\pi_1^{\mathrm{tr}}(D)$

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- In classical theory, we have VC and MV—neither of these exist here. But we have something better: universal properties.

Let D be an anchored, path connected set such that each D_n is nonempty and choose an anchor $B \subset D$ such that each B_n is also nonempty.

Searching for Holes in the Matrix Universe
└Computing $\pi_1^{\text{tr}}(D)$
└The Direct Limit Approach

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Let D be an anchored, path connected set such that each D_n is nonempty and choose an anchor $B \subset D$ such that each B_n is also nonempty.

- Here are our niceness condition for our computation to work.
- as before, the $_n$ denotes restricting to the $n \times n$ level.
- We are a quotient bc these are all paths in D_n but there might be some paths that are distinct in $\pi_1(D_n)$ but not in $\pi_1^{\text{tr}}(D)_n$.
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$$\pi_1^{\mathrm{tr}}(D)_1 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_2 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_6 \hookrightarrow \cdots \hookrightarrow \pi_1^{\mathrm{tr}}(D)_{n!} \hookrightarrow \cdots$$

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- starting at the scalar level, lets run through all of the natural inclusions, muliplying by the next positive integer each time.
- If you're familiar with direct limits, its not hard to see that π_1^{tr} is the direct limit of this sequence
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Let $GL = \bigcup_{n\in\mathbb{N}} GL_n(\mathbb{C})$.

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- using the direct limit approach, we want look at the 1x1 slice.
- the determinant a scalar is itself, so our set is $\mathbb{C} \setminus \{0\}$, which has fundamental group $\mathbb{Z}$. The only way to be at torsion free subgroup/quotient of $\mathbb{Z}$ is just to be $\mathbb{Z}$.
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Thus, $\pi_1^{\mathrm{tr}}(GL)_2 \simeq \mathbb{Z}\left[\frac{1}{2}\right]$.

2022-06-03

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- That's it! The algebraic topology of free sets is in its infancy but results look promising!
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