

# A CLEAN TITLE

LUCAS KERBS



A Fun Subtitle

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*Ohana* means family.  
Family means nobody gets left behind, or forgotten.  
— Lilo & Stitch

Dedicated to the loving memory of Rudolf Miede.  
1939 – 2005



## ABSTRACT

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Short summary of the contents in English...a great guide by Kent Beck how to write good abstracts can be found here:

<https://plg.uwaterloo.ca/~migod/research/beck00PSLA.html>



*We have seen that computer programming is an art,  
because it applies accumulated knowledge to the world,  
because it requires skill and ingenuity, and especially  
because it produces objects of beauty.*

— Donald E. Knuth [3]

## ACKNOWLEDGMENTS

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Put your acknowledgments here.

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<sup>1</sup> Members of GuIT (Gruppo Italiano Utilizzatori di T<sub>E</sub>X e L<sup>A</sup>T<sub>E</sub>X)





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## Part I

### OBJECTS AND THE MAPS BETWEEN THEM



## A FIRST ATTEMPT

## 1.1 FUNCTIONAL CALCULUS

Functional Calculus refers to the process of extending the domain of a function on  $\mathbb{R}$  to include matrices (or in some cases operators). The most basic formulation uses the fact that the space  $n \times n$  matrices forms a ring and so there is a natural way to evaluate polynomials  $f \in \mathbb{C}[x]$ . If we require that  $A \in M_n(\mathbb{C})$  is self-adjoint—and hence diagonalizable as  $A = U\Lambda U^*$ —then it is a standard result that:

$$\begin{aligned} f(A) &= a_n A^n + \cdots + a_1 A + a_0 \\ &= a_n (U\Lambda U^*)^n + \cdots + a_1 U\Lambda U^* + a_0 \\ &= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 \\ &= U(a_n \Lambda^n + \cdots + a_1 \Lambda + a_0) U^* \\ &= U(f(\Lambda)) U^* \end{aligned}$$

Further, since  $\Lambda$  is diagonal and  $f$  is a polynomial,

$$f\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right) = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}$$

Therefore, given a self-adjoint matrix  $A$  and a polynomial  $f \in \mathbb{C}[x]$

$$f(A) = Uf(\Lambda)U^* = U \operatorname{diag}\{f(\lambda_1), \dots, f(\lambda_n)\} U^*$$

Since self-adjoint matrices play such a vital role in free analysis, we will let  $\mathbb{H}_n \subset M_n(\mathbb{C})$  denote the set of  $n \times n$ -matrices over  $\mathbb{C}$ . With the polynomial case in mind, we can extend a function  $g : [a, b] \rightarrow \mathbb{C}$  to a function on self adjoint (normal?) matrices with their spectrum in  $[a, b]$ . Let  $A$  be such a matrix (diagonalized by the unitary matrix  $U$ ), and define

$$g(A) = U \begin{bmatrix} g(\lambda_1) & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^*$$

Thus, for each  $n \in \mathbb{N}$ ,  $g$  induces a function on the self-adjoint  $n \times n$  matrices with spectrum in  $[a, b]$ . The natural ordering **explain why natural?** on self-adjoint matrices is called the **Loewner Order**:

**Definition i.1** (Loewner Ordering). *For like size self-adjoint matrices, we say that  $A \preceq B$  if  $B - A$  is positive semidefinite and  $A \prec B$  if  $B - A$  is positive definite.*

With this ordering in place, we can extend many of the familiar function theoretic properties (monotonicity, convexity) to these matrix-values functions. In fact, these properties are defined identically to their classical counterpart: We say that a function is *matrix-monotone* if  $A \preceq B$  implies that  $f(A) \preceq f(B)$  and *matrix-convex* (or *nc-convex*) if

$$f\left(\frac{X+Y}{2}\right) \preceq \frac{f(X)+f(Y)}{2}$$

for every pair of like-size matrices for which  $f$  is defined. These conditions are rather restrictive (since they must hold for matrices of *all* sizes) so many functions which are convex/monotone (in the traditional sense) fail to be matrix-convex/monotone. For example,  $f(x) = x^4$  fails to be nc-convex. **Below is an example from FCAC (Helton). What is the best way to refer to this?**

Indeed, if

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$\frac{X^4 + Y^4}{2} - \left(\frac{1}{2}X + \frac{1}{2}Y\right)^4 = \begin{bmatrix} 164 & 120 \\ 120 & 84 \end{bmatrix}$$

Which is not positive definite! Thus  $x^4$  fails to be convex on even  $2 \times 2$  matrices.

**nc-positive if positive for all matrices**

Further, a number of the standard constructions lift identically in this functional calculus.

**Definition i.2** (Directional Derivative). *The derivative of  $f$  in the direction  $H$  is*

$$Df(X)[H] = \lim_{t \rightarrow 0} \frac{f(X + tH) - f(X)}{t}$$

where  $H$  and  $X$  are like-size self-adjoint matrices.

Often, the best way to compute these directional derivatives is via an equivalent formulation:

$$Df(X)[H] = \left. \frac{df(X + tH)}{dt} \right|_{t=0}$$

This version allows us to more easily define higher order derivatives

$$D^{(k)}f(X)[H] = \left. \frac{d^{(k)}f(X + tH)}{d^{(k)}t} \right|_{t=0}$$

Despite nc-convexity being so restrictive, Lemma 12 in [2] shows that the standard characterization of convexity via the second derivative: a function  $f$  is convex if and only if  $D^2f(X)[H]$  is nc-positive. Unlike the classical case, however, the only convex polynomials are of degree  $\leq 1$ .

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<sup>1</sup> See [2] for details.

## 1.2 EXTENDING MULTI-VARIABLE FUNCTIONS

We can extend this same functional calculus to functions of several variables, although the details are a bit more subtle. We could simply “plug in” at tuple of matrices to a standard multivariable polynomial ring over  $\mathbb{R}$  or  $\mathbb{C}$ , but this ignores the noncommutativity of  $M_n(\mathbb{C})$ . In light of this, let  $x = (x_1, \dots, x_g)$  be a  $g$ -tuples of noncommuting formal variables. The formal variables  $x_1, \dots, x_n$  are *free* in the sense that there are no nontrivial relations between them.<sup>2</sup> A **word** in  $x$  is a product of these variables (e.g.  $x_1 x_3 x_1 x_4^2$  or  $x_1^2 x_5^3$ ). An **nc-polynomial** in  $x$  is a formal finite linear combination of words in  $x$  with coefficients in your favorite field. We use  $\mathbb{R}\langle x \rangle$  and  $\mathbb{C}\langle x \rangle$  to denote the set of nc-polynomials in  $x$  over  $\mathbb{R}$  or  $\mathbb{C}$  respectively.

With  $\mathbb{C}\langle x \rangle$  constructed, we can define the functional calculus. Given a word  $w(x) = x_{i_1}^{p_1} \cdots x_{i_d}^{p_d}$  and a  $g$ -tuple of self-adjoint matrices,  $X$ , we can evaluate  $w$  on  $X$  via  $w(X) = X_{i_1}^{p_1} \cdots X_{i_d}^{p_d}$ . Since our nc-polynomials are linear combinations of these words, we can extend this evaluation to evaluation of entire polynomials. Algebraically, we have a natural evaluation map: Given some  $f \in \mathbb{C}\langle x \rangle$  and  $X = (X_1, \dots, X_n)$  a  $g$ -tuple of self-adjoint matrices, define

$$\begin{aligned} \varepsilon_f : \mathbb{H}_n^g &\longrightarrow M_n(\mathbb{C}) \\ X &\longmapsto f(X). \end{aligned}$$

In the context of these multivariate functions, our definition of the Directional Derivative still makes sense (although our direction  $H$  now becomes a tuple of directions). We also inherit (from multi-variable calculus) a notion of the **gradient** of a function—but this will require a bit more work.

## 1.3 THE NATURAL INVOLUTION ON NC-POLYNOMIALS

[Short section on the involution. Could also put this in the algebra errata](#)

<sup>2</sup> This becomes important in the eventual functional calculus—matrices *do* have non-trivial relations. See section [ALGEBRAIC CONSTRUCTION] for the details.





## A SECOND ATTEMPT

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### 2.1 MATRIX UNIVERSES

Beyond the functional calculus, it becomes useful to construct general functions on spaces of matrices—to do so, we must make this idea of “spaces of matrices” concrete. The largest such space is the so-called **Matrix Universe**—consisting of  $g$ -tuples of matrices of all sizes:

$$\mathcal{M}^g = \bigcup_{n=1}^{\infty} (M_n(\mathbb{C}))^g$$

By convention, when we consider some  $X = (X_1, \dots, X_g) \in \mathcal{M}^g$ , we require that the  $X_i$  are all the same size. Since  $\mathcal{M}^g$  is such a large set, we often want to deal with subsets that still carry some of the implicit structure of  $\mathcal{M}^g$ .

**Definition i.3** (Free Set). *We say  $D \subset \mathcal{M}^g$  is a **free set** if it is closed with respect to direct sums and unitary conjugation. That it*

1.  $X, Y \in D$  means  $X \oplus Y \in D$ .
2. For  $X, U$  like-size matrices with  $U$  unitary and  $X \in D$ , then  $UXU^* = (UX_1U^*, \dots, UX_gU^*) \in D$ .

For the remainder of this text,  $D$  will denote some free set. Using the terminology of [4], let  $D_n = D \cap M_n(\mathbb{C})^g$  be the level-wise slice of all  $n \times n$  matrices in  $D$ . We say that  $D$  is **open**<sup>1</sup> (resp. **connected**, **simply connected**) if each  $D_n$  is open (resp. connected, simply connected). Finally, we say that  $D$  is **differentiable** if each  $D_n$  is an open  $C^1$  manifold where the complex tangent space of every  $X \in D_n$  is all of  $M_n(\mathbb{C})^g$ .

In the context of sections 1.1 1.2 how to ref multiple sections, the domains in the functional calculus were  $\mathbb{H}^g = \bigcup_{n=1}^{\infty} \mathbb{H}_n^g$ .  $\mathbb{H}^g$  is a differentiable a free set with two connected components.

On  $\mathcal{M}^g$ , we define a product that resembles the inner product on  $\mathbb{C}^n$ . Given  $A, B \in \mathcal{M}^g$  which are  $g$ -tuples of  $n \times n$  matrices:

$$\begin{aligned} \cdot : \mathcal{M}^g \times \mathcal{M}^g &\longrightarrow M_n(\mathbb{C}) \\ \cdot(A, B) = A \cdot B &\longmapsto \sum_{i=1}^g A_i B_i \end{aligned}$$

James uses this product, but like what in the world is going on with it???

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<sup>1</sup> The topology of  $\mathcal{M}^g$  is still in flux and there is not a canonical topology. See section NUMBER for the details

## 2.2 THE TOPOLOGY OF MATRIX UNIVERSES

How much detail here? Just what we are doing or the basics of the other topologies? This could also talk about the nc varieties/singular sets

## 2.3 TRACIAL FUNCTIONS AND UNIQUENESS OF THE GRADIENT

Now that we have  $\mathcal{M}^d$ , we can work with general functions on our matrix universe. As a whole, free analysis is concerned with so-called *free functions*, which respect the direct sums and unitary conjugation.

Do they need to be graded?

**Definition i.4** (Free Function). *A function  $f : D \rightarrow \mathcal{M}^{\text{something}}$  is called free if*

1.  $f(X \oplus Y) = f(X) \oplus f(Y)$
2.  $f(UXU^*) = f(U)f(X)f(U^*)$  where  $X$  and  $U$  are like-size and  $U$  is unitary.

The two other classes of functions we are concerned with are those that act like the trace and the determinant:

**Definition i.5** (Determinantal Free Function). *A function  $f : D \rightarrow \mathbb{C}$  is a determinantal free function if*

1.  $f(X \oplus Y) = f(X)f(Y)$
2.  $f(UXU^*) = f(X)$  where  $X$  and  $U$  are like-size and  $U$  is unitary.

**Definition i.6** (Tracial Free Function). *A function  $f : D \rightarrow \mathbb{C}$  is a tracial free function if*

1.  $f(X \oplus Y) = f(X) + f(Y)$
2.  $f(UXU^*) = f(X)$  where  $X$  and  $U$  are like-size and  $U$  is unitary.

Given a free function of any type, we can define the directional derivative (Definition i.2) identically. It is worth noting that, while they share the moniker of *free*, determinantal and tracial functions are *not* free functions. It is only these tracial functions which inherit the gradient mentioned above. Similarly to traditional multivariable calculus we define the gradient via its relationship to the directional derivative:

**Definition i.7** (Free Gradient). *Given a tracial free function  $f$ , the free gradient,  $\nabla f$ , is the unique free function satisfying*

$$\text{tr}(H \cdot \nabla f(X)) = \text{tr} Df(X)[H]$$

It is not-at-all obvious that such a  $\nabla f$  should be unique—after all any linear combination of commutator is has trace zero. **should I explain this?** In the case that  $f$  is a single-variable function we can replace  $\nabla f$  with the traditional derivative,  $f'$ , as seen in [5, Thm 3.3].

**Theorem i.8.** *Let  $f : (a, b) \rightarrow \mathbb{R}$  be a  $C^1$  function. Then*

$$\text{tr } Df(X)[H] = \text{tr } (Hf'(X))$$

The proof in [5] simply asserts the uniqueness of a function  $g(X)$  and then shows that  $g(x) = f'(x)$  for  $x \in (a, b)$ . Instead, we can construct such a  $g$  and recover the theorem along the way:

*Proof.* We start with a construction from Bhatia's Matrix Analysis: Let  $f \in C^1(I)$  and define  $f^{[1]}$  on  $I \times I$  by

$$f^{[1]}(\lambda, \mu) = \begin{cases} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} & \lambda \neq \mu \\ f'(\lambda) & \lambda = \mu. \end{cases}$$

We call  $f^{[1]}(\lambda, \mu)$  the *first divided difference* of  $f$  at  $(\lambda, \mu)$ . If  $\Lambda$  is a diagonal matrix with entries  $\{\lambda_i\}$ , We may extend  $f$  to accept  $\Lambda$  by defining the  $(i, j)$ -entry of  $f^{[1]}(\Lambda)$  to be  $f^{[1]}(\lambda_i, \lambda_j)$ . If  $A$  is a self adjoint matrix with  $A = U\Lambda U^*$ , then we define  $f^{[1]}(A) = Uf^{[1]}(\Lambda)U^*$ . Now we borrow a theorem from Bhatia [1]:

**Theorem i.9** (Bhatia V.3.3). ***Theorem numbering?** Let  $f \in C^1(I)$  and let  $A$  be a self adjoint matrix with all eigenvalues in  $I$ . Then*

$$Df(A)[H] = f^{[1]}(A) \circ H,$$

where  $\circ$  denotes the Schur-product<sup>2</sup> in a basis where  $A$  is diagonal.

That is, if  $A = U\Lambda U^*$ , then

$$Df(A)[H] = U \left( f^{[1]}(\Lambda) \circ (U^* H U) \right) U^*.$$

To prove our claim, we need to take the trace of both sides. Since trace is invariant under a change of basis, it is clear that

$$\text{tr } Df(A)[H] = \text{tr } \left( f^{[1]}(\Lambda) \circ (U^* H U) \right).$$

If  $U = u_{ij}$ ,  $U^* = \bar{u}_{ij}$  and  $H = h_{ij}$ , then the  $(i, j)$ -entry of  $U^* H U$  is

$$(U^* H U)_{ij} = \bar{u}_{ik} h_{k\ell} u_{\ell j}$$

Where we sum over the duplicate indices  $k$  and  $\ell$ . While the structure of  $f^{[1]}(\Lambda)$  is a bit unruly, our diagonal entries are  $f'(\lambda)$ . This means that when we take the trace of the Schur product, we have

$$\sum_k \sum_\ell \sum_i f'(\lambda_i) \bar{u}_{ik} h_{k\ell} u_{\ell i}$$

<sup>2</sup> Entrywise

Now consider the matrix product  $U \operatorname{diag}\{f'(\lambda_1), \dots, f'(\lambda_n)\} U^* H$ . Since one of our terms is diagonal, the trace of this multiplication is simple:

$$\operatorname{tr} U \operatorname{diag}\{f'(\lambda_1), \dots, f'(\lambda_n)\} U^* H = \sum_k \sum_\ell \sum_i u_{ik} f'(\lambda_k) \bar{u}_{k\ell} h_{\ell i}$$

Since  $u_{ik}, \bar{u}_{k\ell}, h_{\ell i} \in \mathbb{C}$  they commute. We can then relabel our indices  $i \mapsto \ell$   $\ell \mapsto k$   $k \mapsto i$  to get

$$\operatorname{tr} U \operatorname{diag}\{f'(\lambda_1), \dots, f'(\lambda_n)\} U^* H = \sum_k \sum_\ell \sum_i f'(\lambda_i) \bar{u}_{ik} h_{k\ell} u_{\ell i},$$

So, for every direction  $H$ , we have that  $\operatorname{tr} (U \operatorname{diag}\{f'(\lambda_1), \dots, f'(\lambda_n)\} U^* H) = \operatorname{tr} \left( f^{[1]}(\Lambda) \circ (U^* H U) \right)$ . **overfull hbox :eyeroll:** By picking the “correct”  $H^3$ , we conclude that our unique quantity  $g(X)$  is  $U \operatorname{diag}\{f'(\lambda_1), \dots, f'(\lambda_n)\} U^*$ . But, recall that  $X = U \Lambda U$  so, in the functional calculus,  $g(X) = f'(X)$ . This recovers theorem 3.3 of [5] as we have constructed a  $g$  such that

$$\operatorname{tr} Df(X)[H] = \operatorname{tr} H g(X)$$

■

[Connect this to the gradient and this section is done](#)

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<sup>3</sup> See example EXAMPLE NUMBER for details

## Part II

### THE SHOWCASE

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## Part III

### APPENDIX





## BIBLIOGRAPHY

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