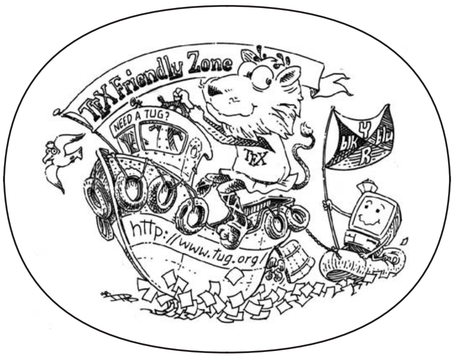
A CLEAN TITLE

LUCAS KERBS



A Fun Subtitle

February 2022 – LucasThesis v1

Ohana means family.
Family means nobody gets left behind, or forgotten.

— Lilo & Stitch

Dedicated to the loving memory of Rudolf Miede.

1939 – 2005

ABSTRACT

Short summary of the contents in English. . . a great guide by Kent Beck how to write good abstracts can be found here:

https://plg.uwaterloo.ca/~migod/research/beckOOPSLA.html

We have seen that computer programming is an art,
because it applies accumulated knowledge to the world,
because it requires skill and ingenuity, and especially
because it produces objects of beauty.

— Donald E. Knuth [2]

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Put your acknowledgments here.

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1 Members of GuIT (Gruppo Italiano Utilizzatori di TeX e LaTeX)

CONTENTS

LIST OF FIGURES

LIST OF TABLES

Part I

PART I: *NAME*

1 INTRODUCTION

1.1 FUNCTIONAL CALCULUS

Functional Calculus refers to the process of extending the domain of a function on $\mathbb{R}$ to include matrices (or in some cases operators). The most basic formulation uses the fact that the space $n \times n$ matrices forms a ring and so there is a natural way to evaluate polynomials $f \in \mathbb{C}[x]$. If we require that $A \in M_n(\mathbb{C})$ is self-adjoint—and hence diagonalizable as $A = U\Lambda U^*$—then it is a standard result that:

$$\begin{aligned} f(A) &= a_n A^n + \cdots + a_1 A + a_0 \\ &= a_n \left(U\Lambda U^*\right)^n + \cdots + a_1 U\Lambda U^* + a_0 \\ &= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 \\ &= U\left(a_n \Lambda^n + \cdots + a_1 \Lambda + a_0\right) U^* \\ &= U\left(f(\Lambda)\right) U^* \end{aligned}$$

Further, since Λ is diagonal and f is a polynomial,

$$f\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right) = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}$$

Therefore, given a self-adjoint matrix A and a polynomial $f \in \mathbb{C}[x]$

$$f(A) = Uf(\Lambda)U^* = U\,\mathrm{diag}\{f(\lambda_1), \ldots, f(\lambda_n)\}\,U^*$$

With this in mind, we can extend a function $g : [a, b] \to \mathbb{C}$ to a function on self adjoint (normal?) matrices with their spectrum in $[a, b]$. Let A be such a matrix (diagonalized by the unitary matrix U), and define

$$g(A) = U \begin{bmatrix} g(\lambda_1) & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^*$$

Thus, for each $n \in \mathbb{N}$, g induces a function on the self-adjoint $n \times n$ matrices with spectrum in $[a, b]$. The natural ordering explain why natural? on self-adjoint matrices is called the **Loewner Order**:

Definition i.1 (Loewner Ordering)**.** *For like size self-adjoint matrices, we say that $A \preceq B$ if $B - A$ is positive semidefinite and $A \prec B$ is $B - A$ is positive definite.*

With this ordering in place, we can extend many of the familiar function theoretic properties (monotonicity, convexity) to these matrix-values functions. In fact, these properties are defined identically to their classical counterpart: We say that a function is *matrix-monotone* if $A \preceq B$ implies that $f(A) \preceq f(B)$ and *matrix-convex* (or *nc-convex*) if

$$f\left(\frac{X+Y}{2}\right) \preceq \frac{f(X)+f(Y)}{2}$$

for every pair of like-size matrice for which f is defined. These condition are rather restrictive (since the must hold for matrices of *all* sizes) so many functions which are convex/monotone (in the traditional sense) fail to be matrix-convex/monotone. For example, $f(x) = x^4$ fails to be nc-convex. Below is an example from FCAC (Helton). What is the best way to refer to this?

Indeed, if

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$\frac{X^4+Y^4}{2} - \left(\frac{1}{2}X + \frac{1}{2}Y\right)^4 = \begin{bmatrix} 164 & 120 \\ 120 & 84 \end{bmatrix}$$

Which is not positive definite! Thus x^4 fails to be convex on even 2×2 matrices.

nc-positive if positive for all matrices

Further, a number of the standard constructions lift identically in this functional calculus.

Definition i.2 (Directional Derivative)**.** *The derivative of f in the direction H is*

$$Df(X)[H] = \lim_{t\to 0} \frac{f(X+tH)-f(X)}{t}$$

where H and X are like-size self-adjoint matices.

Often, the best way to compute these directional derivatives is via an equivalent formulation:

$$Df(X)[H] = \left.\frac{df(X+tH)}{dt}\right|_{t=0}$$

This version allows us to more easily define higher order derivatives

$$D^{(k)}f(X)[H] = \left.\frac{d^{(k)}f(X+tH)}{d^{(k)}t}\right|_{t=0}$$

Despite nc-convexity being so restrictive, Lemma 12 in [1] shows that the standard characterization of convexity via the second derivativie: a function f is convex if and only if $D^2 f(X)[H]$ is nc-positive. Unlike the classical case, however, the only convex polynomials are of degree 2[1].

1 See [1] for details.

1.2 EXTENDING MULTI-VARIABLE FUNCTIONS

We can extend this same functional calculus to functions of several variables, although the details are a bit more subtle. Let $x = (x_1, \ldots x_g)$ be a $g - tuples$ of noncommuting formal variables and let $\mathbb{R}\langle x \rangle$ why does everyone use $\mathbb{R}$ and now $\mathbb{C}$? be the ring of nc-polynomials with coefficients in $\mathbb{R}$.

Part II

THE SHOWCASE

You can put some informational part preamble text here. Illo principalmente su nos. Non message *occidental* anglo-romanic da. Debitas effortio simplificate sia se, auxiliar summarios da que, se avantiate publicationes via. Pan in terra summarios, capital interlingua se que. Al via multo esser specimen, campo responder que da. Le usate medical addresses pro, europa origine sanctificate nos se.

Part III

APPENDIX