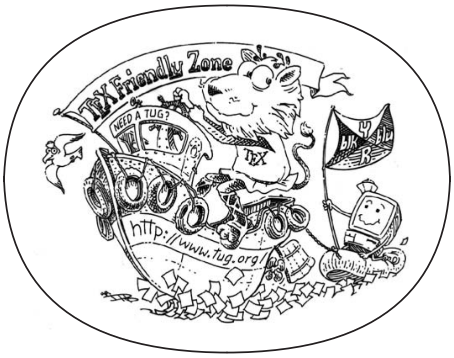
A CLEAN TITLE

LUCAS KERBS



A Fun Subtitle

February 2022 – LucasThesis v1

Ohana means family.
Family means nobody gets left behind, or forgotten.

— Lilo & Stitch

Dedicated to the loving memory of Rudolf Miede.

1939 – 2005

ABSTRACT

Short summary of the contents in English. . . a great guide by Kent Beck how to write good abstracts can be found here:

https://plg.uwaterloo.ca/~migod/research/beck00PSLA.html

We have seen that computer programming is an art,
because it applies accumulated knowledge to the world,
because it requires skill and ingenuity, and especially
because it produces objects of beauty.

— Donald E. Knuth [11]

ACKNOWLEDGMENTS

Put your acknowledgments here.

Many thanks to everybody who already sent me a postcard!

Regarding the typography and other help, many thanks go to Marco Kuhlmann, Philipp Lehman, Lothar Schlesier, Jim Young, Lorenzo Pantieri and Enrico Gregorio[1], Jörg Sommer, Joachim Köstler, Daniel Gottschlag, Denis Aydin, Paride Legovini, Steffen Prochnow, Nicolas Repp, Hinrich Harms, Roland Winkler, Jörg Weber, Henri Menke, Claus Lahiri, Clemens Niederberger, Stefano Bragaglia, Jörn Hees, Scott Lowe, Dave Howcroft, José M. Alcaide, David Carlisle, Ulrike Fischer, Hugues de Lassus, Csaba Hajdu, Dave Howcroft, and the whole LaTeX-community for support, ideas and some great software.

Regarding LyX: The LyX port was intially done by *Nicholas Mariette* in March 2009 and continued by *Ivo Pletikosić* in 2011. Thank you very much for your work and for the contributions to the original style.

1 Members of GuIT (Gruppo Italiano Utilizzatori di TeX e LaTeX)

CONTENTS

LIST OF FIGURES

Part I

OBJECTS AND THE MAPS BETWEEN THEM

*"Young man, in mathematics you don't understand things.
You just get used to them"*

— John von Neumann

1

A FIRST ATTEMPT

1.1 INTRODUCTION

As a note to the reader (and myself): things written in blue denote things I want to add/expand upon, things writteng in red denote things that I need to add/find out/fix, and things in green denote wording I don't like but want to come back to

Gotta do this at some point. Maybe here we define things like $\mathcal{U}_n$.

1.2 FUNCTIONAL CALCULUS

Functional Calculus refers to the process of extending the domain of a function on $\mathbb{R}$ to include matrices (or in some cases operators). The most basic formulation uses the fact that the space $n \times n$ matrices forms a ring and so there is a natural way to evaluate polynomials $f \in \mathbb{C}[x]$. If we require that $A \in M_n(\mathbb{C})$ is self-adjoint—and hence diagonalizable as $A = U\Lambda U^*$—then it is a standard result that:

$$\begin{aligned} f(A) &= a_n A^n + \cdots + a_1 A + a_0 I_n \\ &= a_n (U\Lambda U^*)^n + \cdots + a_1 U\Lambda U^* + a_0 I_n \\ &= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 I_n \\ &= U (a_n \Lambda^n + \cdots + a_1 \Lambda + a_0 I_n) U^* \\ &= U (f(\Lambda)) U^* \end{aligned}$$

Further, since Λ is diagonal and f is a polynomial,

$$f\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right) = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}$$

Therefore, given a self-adjoint matrix A and a polynomial $f \in \mathbb{C}[x]$

$$f(A) = Uf(\Lambda)U^* = U \operatorname{diag}\{f(\lambda_1), \ldots, f(\lambda_n)\} U^*$$

Notice that can simply substitute A in for x without any trouble as long as we transform the constant term $a_0 \mapsto a_0 I_n$ when evaluation on $n \times n$ matrices.[1] Since self-adjoint matrices play such a vital role in free analysis, we will let $\mathbb{H}_n \subset M_n(\mathbb{C})$ denote the set of $n \times n$-matrices over $\mathbb{C}$. With the polynomial case in mind, we can extend a function $g : [a,b] \to \mathbb{C}$ to a function on self adjoint matrices with their spectrum in $[a,b]$. Let A be such a matrix (diagonalized by the unitary matrix U), and define

$$g(A) = U \begin{bmatrix} g(\lambda_1) & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^*$$

Thus, for each $n \in \mathbb{N}$, g induces a function on the self-adjoint $n \times n$ matrices with spectrum in $[a,b]$. The natural ordering on self-adjoint matrices is called the **Loewner Order**:

Definition i.1 (Loewner Ordering). *For like size self-adjoint matrices, we say that $A \preceq B$ if $B - A$ is positive semidefinite and $A \prec B$ is $B - A$ is positive definite.*

With this ordering in place, we can extend many of the familiar function theoretic properties (monotonicity, convexity) to these matrix-values functions. In fact, these properties are defined identically to their classical counterpart: We say that a function is *matrix-monotone* if $A \preceq B$ implies that $f(A) \preceq f(B)$ and *matrix-convex* (or *nc-convex*) if

$$f\left(\frac{X+Y}{2}\right) \preceq \frac{f(X) + f(Y)}{2}$$

for every pair of like-size matrice for which f is defined. These condition are rather restrictive (since the must hold for matrices of *all* sizes) so many functions which are convex/monotone (in the traditional sense) fail to be matrix-convex/monotone. For a full treatment of nc-convexity, see [8]. To illustrate the restrictiveness of nc-convexity, we will steal an example from Helton. (I don't like this phrasing. I don't mind the word steal, its just awkward)

Example i.2. *In contrast to the real (or even complex) case, $f(x) = x^4$ fails to be nc-convex. Indeed, if*

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \quad \textit{and} \quad Y = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

1 Technically we have $a_0 \mapsto a_0 \otimes I_n$ but they are identical in this case. It is common in free analysis to tensor by I_n to make matrices the same size.

Then

$$\frac{X^4+Y^4}{2} - \left(\frac{1}{2}X + \frac{1}{2}Y\right)^4 = \begin{bmatrix} 164 & 120 \\ 120 & 84 \end{bmatrix}$$

Which is not positive definite! Thus x^4 fails to be convex on even 2×2 matrices.

Further, a number of the standard constructions lift identically in this functional calculus.

Definition i.3 (Directional Derivative)**.** *The derivative of f in the direction H is*

$$Df(X)[H] = \lim_{t \to 0} \frac{f(X+tH) - f(X)}{t}$$

where H and X are like-size self-adjoint matices.

Often, the best way to compute these directional derivatives is via an equivalent formulation:

$$Df(X)[H] = \left.\frac{df(X+tH)}{dt}\right|_{t=0}$$

This version allows us to more easily define higher order derivatives

$$D^{(k)}f(X)[H] = \left.\frac{d^{(k)}f(X+tH)}{d^{(k)}t}\right|_{t=0}$$

Example i.4. *Just as in the classical case, the directional derivative is linear, so we will only show a calculation of a monomial. Let $f(x) = x^3$. Since X and H do not commute,*

$$\begin{aligned} f(X+tH) = X^3 + tX^2H + tXHX + t^2XH^2 \\ + tHX^2 + t^2HXH + t^2H^2X + t^3H^3. \end{aligned}$$

From here, we can calculate:

$$\begin{aligned} \frac{d}{dt}f(X+tH) = X^2H + XHX + 2tXH^2 + HX^2 \\ + 2tHXH + 2tH^2X + 3t^2H^3 \end{aligned}$$

$$\frac{d^2}{dt^2}f(X+tH) = 2XH^2 + 2HXH + 2H^2X + 6tH^3$$

$$\frac{d^3}{dt^3}f(X+tH) = 6H^3.$$

And so the first 3 directional derivatives are:

$$Df(X)[H] = X^2H + XHX + HX^2$$

$$D^{(2)}f(X)[H] = 2XH^2 + 2HXH + 2H^2X$$

$$D^{(3)}f(X)[H] = 6H^3$$

In general, the k-th derivative of a polynomial is degree k as a polynomial in H.

Just as in the classical case, the second derivative tells gives us information about the convexity of a function. A function $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is said to be **positive** if $0 \preceq A \implies 0 \preceq f(A)$. In the functional calculus, we say that f is **nc positive** if it is positive as a map on $M_n(\mathbb{C})$ for all n. Despite nc-convexity being so restrictive, Lemma 12 in [8] shows that the standard characterization of convexity via the second derivative: a function f is convex if and only if $D^2f(X)[H]$ is nc-positive. Unlike the classical case, however, the only convex polynomials are of degree 2.[2]

1.3 EXTENDING MULTI-VARIABLE FUNCTIONS

We can extend this same functional calculus to functions of several variables, although the details are a bit more subtle. We could simply "plug in" at tuple of matrices to a standard multivariable polynomial ring over $\mathbb{R}$ or $\mathbb{C}$, but this ignores the noncommutativity of $M_n(\mathbb{C})$. In light of this, let $x = (x_1, \dots x_g)$ be a g-tuples of noncommuting formal variables. The formal variables $x_1, \dots, x_n$ are *free* in the sense that there are no nontrivial relations between them.[3] A **word** in x is a product of these variables (e. g.$x_1x_3x_1x_4^2$ or $x_1^2x_5^3$). An **nc-polynomial** in x is a formal finite linear combination of words in x with coefficients in your favorite field. We use $\mathbb{R}\langle x\rangle$ and $\mathbb{C}\langle x\rangle$ to denote the set of nc-polynomials in x over $\mathbb{R}$ or $\mathbb{C}$ respectively.

With $\mathbb{C}\langle x\rangle$ constructed, we can define the functional calculus. Given a word $w(x) = x_{i_1}^{p_1} \cdots x_{i_d}^{p_d}$ and a g-tuple of self-adjoint matrices, X, we can evaluate w on X via $w(X) = X_{i_1}^{p_1} \cdots X_{i_d}^{p_d}$.

2 See [8] for details.

3 This becomes important in the eventual functional calculus—matrices *do* have nontrivial relations. See section [ALGEBRAIC CONSTRUCTION] for the details.

Since our nc-polynomials are linear combinations of these words, we can extend this evaluation to evaluation of entire polynomials. Algebraically, we have a natural evaluation map: Given some $f \in \mathbb{C}\langle x \rangle$ and $X = (X_1, \dots, X_g)$ a g-tuple of self-adjoint matrices, define

$$\begin{aligned} \varepsilon_f : \mathbb{H}_\bullet{}^g &\longrightarrow M_\bullet(\mathbb{C}) \\ X &\longmapsto f(X). \end{aligned}$$

Notice that our functions are **graded** in the sense that if X is a tuple of $n \times n$ matrices, then $f(X)$ is also a tuple of $n \times n$ matrices.

Example i.5. *Let $f(x,y) = x^2 - xyx + 1 \in \mathbb{R}\langle x, y \rangle$. If we define*

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

as before, then

$$\begin{aligned} f(X,Y) &= X^2 - XYX + I_2 \\ &= \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}^2 - \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -11 & -4 \\ -4 & 1 \end{bmatrix}. \end{aligned}$$

Additionally,

$$\begin{aligned} f(X \oplus X, Y \oplus Y) &= \begin{bmatrix} -11 & -4 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & -11 & -4 \\ 0 & 0 & -4 & 1 \end{bmatrix} \\ &= f(X,Y) \oplus f(X,Y). \end{aligned}$$

It is no accident that polynomials handle direct sums of matrices well. As in the classical case, they are the "well behaved" example which we would like general objects to emulate. In the next chapter, we will define free functions—which behave like nc polynomials.

In the context of these multivariate functions, our definition of the Directional Derivative still makes sense (although our direction H now becomes a tuple of directions). We also inherit (from multi-variable calculus) a notion of the **gradient** of a function—but this will require a bit more work.

1.3.1 *The Natural Involution on nc-Polynomials*

Given our ring of nc polynomials, we may define an involution * which we may view as an extension of the conjugate transpose. Let * reverse the order of words (i. e.$(x_1 x_3 x_2^2)^* = x_2^2 x_3 x_1$) and extend linearly to all of $\mathbb{R}\langle x\rangle$. We consider the formal variables $x_1, \ldots, x_n$ *symmetric* in the sense that $x_i^* = x_i$. We say that a polynomial $p \in \mathbb{R}\langle x\rangle$ is symmetric if $p^* = p$. For example, if

$$p(x) = 5x_1^2 x_3 x_2 + x_3 x_2 x_3 \qquad q(x) = 3x_2 x_1 x_2 + x_3^2 - x_1,$$

then a cursory inspection tells that q is symmetric while p is not.

Notice that the majority of the previous two sections breaks down if we try to extend functions to non-self-adjoint matrices. The act of "plugging in" a tuple of arbitry matrices to some element of $\mathbb{R}\langle x\rangle$ via the same functional calculus described above still works, but $\mathbb{R}\langle x\rangle$ is no longer the natural algebra for these evaluations.

Let $x = (x_1, \ldots, x_g)$ be formal variables and let $x^* = (x_1^*, \ldots, x_g^*)$ denote their formal adjoints. Once again, we let the ring $\mathbb{R}\langle x, x^*\rangle$ be the finite formal sums of words in $x_1, x_1^*, \ldots, x_g, x_g^*$ with coefficients in $\mathbb{R}$. Endow $\mathbb{R}\langle x, x^*\rangle$ with an involution * which sends $x_i \mapsto x_i^*$ and $x_i^* \mapsto x_i$ and reverses the order of words extended linearly. Notice that this involution behaves identically to the adjoint with respect to products and sums of matrices. This new ring inherits a natural functional calculus just like that in section 1.3 except it can accept *any* matrix as an input instead of simply self-adjoint matrices.

Example i.6. *Let* $f(x,y) = x^*y - xy^*x + 2$. *Then*

$$f^*(x,y) = y^*x - x^*yx^* + 2.$$

Evaluating f on a pair of non self-adjoint matrices is left to the reader.

1.3.2 *Matrices of nc-Polynomials*

It is occasionally useful in the larger theory of free analysis (e. g. when construction the free topology in section 2.3.1) to consider matrices where the matrices are nc polynomials. Formally, let $\mathbb{R}\langle x\rangle^{k\times k}$ denote the set of $k \times k$ matrices with entries in $\mathbb{R}\langle x\rangle$.[4] We can naturally extend the involution * on $\mathbb{R}\langle x\rangle$ to our matrices

4 Some sources additionally consider non-square matrices but this is rare.

by applying * component wise and taking the transpose of the matrix.[5]

Given some $\delta \in \mathbb{R}\langle x \rangle^{k\times k}$ a matrix of nc polynomials, and $X \in \mathbb{H}_n^g$ there is a natural evaluation map.

$$\begin{aligned} \varepsilon_\delta : \mathbb{H}_n{}^g &\longrightarrow M_{nk}(\mathbb{C}) \\ X &\longmapsto \delta(X) \end{aligned}$$

given by evaluating each polynomial in δ at X and then viewing the result at a block $k \times k$ where each block is an $n \times n$ matrix.

Example i.7. *Define $\delta \in \mathbb{R}\langle x, y \rangle^{2\times 2}$ as*

$$\delta(x,y) = \begin{bmatrix} x^2 - xyx + 1 & xy - yx \\ x^4 & y^3 - 5xy + 3 \end{bmatrix}$$

Then

$$\delta^*(x,y) = \begin{bmatrix} x^2 - xyx + 1 & yx - xy \\ x^4 & y^3 - 5yx + 3 \end{bmatrix}$$

For an evaluation, we will once again let

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

We already know what the evaluations of the first column from examples i.2 and i.5, so we need only complute the second column.

$$XY - YX = \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$$

$$Y^3 - 5XY + 3 = \begin{bmatrix} -29 & 0 \\ -20 & 3 \end{bmatrix}$$

And thus

$$\delta(X,Y) = \begin{bmatrix} -11 & -4 & 0 & -2 \\ -4 & 1 & 4 & 0 \\ 164 & 120 & -29 & 0 \\ 120 & 84 & -20 & 3 \end{bmatrix}.$$

5 We could likewise define $\mathbb{R}\langle x, x^* \rangle$ and extend the corresponding involution.

2 A SECOND ATTEMPT

In seeking a more general theory, the functional calculus defined last chapter is insufficient—it would be useful to be able to define *new* functions instead of simply lifting polynomials to matrix domains. In a move that will feel familiar to any good student of mathematics, we will trade treat the set of self-adjoint matrices and polynomial and rational functions on them as prototypical examples of a more general mathematical object, the so-called *Matrix Universe*. After defining this new space and the natural maps in sections 2.1 and 2.2, we turn our attention to various topologies places on matrix universes in section 2.3. While the genesis of free analysis followed chapter 1 (albeit with the usual bumps in the road that accompany research) modern free analysis looks much more like this chapter.

2.1 MATRIX UNIVERSES

Beyond the functional calculus, it becomes useful to construct general functions on spaces of matrices—to do so, we must make this idea of "spaces of matrices" concrete. The largest such space is the so-called **Matrix Universe**—consisting of g-tuples of matrices of all sizes:

$$\mathcal{M}^g = \bigcup_{n=1}^{\infty} (M_n(\mathbb{C}))^g$$

By convention, when we consider some $X = \left(X_1, \ldots, X_g\right) \in \mathcal{M}^g$, we require that the X_i are all the same size. Since $\mathcal{M}^g$ is such a large set, we often want to deal with subsets that still carry some of the implicit structure of $\mathcal{M}^g$.

Definition i.8 (Free Set)**.** *We say $D \subset \mathcal{M}^g$ is a* ***free set*** *(also called an nc set) if it is closed with respect to direct sums and unitary conjugation. That it*

1. *$X, Y \in D$ means $X \oplus Y \in D$.*

2. *For X, U like-size matrices with U unitary and $X \in D$, then $UXU^* = \left(UX_1U^*, \ldots, UX_gU^*\right) \in D$.*

For the remainder of this text, D will denote some free set. Using the terminology of [14], let $D_n = D \cap M_n(\mathbb{C})^g$ be the level-wise slice of all $n \times n$ matrices in D. We say that D is **nc-open**[1] (resp. **connected**, **simply connected**, **bounded**) if each D_n is open (resp. connected, simply connected, bounded). Finally, we say that D is **differentiable** if each D_n is an open C^1 manifold where the complex tangent space of every $X \in D_n$ is all of $M_n(\mathbb{C})^g$. Given some $X \in \mathcal{M}^g$, there are three associated sets which capture the structure of free sets.

Definition i.9 (Similarity Envelope). *Given $X \in \mathcal{M}^g$, a tuple of $n \times n$ matrices, the **similarity envelope** of X is the set*

$$\{U^* X U \mid U \in \mathcal{U}_n\}.$$

Definition i.10 (Fiber). *Given $X \in \mathcal{M}^g$, a tuple of $n \times n$ matrices, the **fiber** of X is the set*

$$\{X^{\oplus k} \mid k \in \mathbb{N}\}.$$

This is my definition. Is this okay? Can I just define things?

Definition i.11 (Envelope). *Given $X \in \mathcal{M}^g$, a tuple of $n \times n$ matrices, the **envelope** of X is the set*

$$\{U^* X^{\oplus k} U \mid k \in \mathbb{N}, U \in \mathcal{U}_{kn}\}.$$

Notice that if $X \in D$, then the entire envelope of X is automatically in D as well! Notice that (as shown in example i.5) polynomials respect the envelope of a matrix in a particularly well-behaved way. Colloquially, we think of all points in the envelope of X as "the same"—this notion is explored in section 2.3 and throughout chapter 4.

In the context of sections 1.2 and 1.3 , the domains in the functional calculus were $\mathbb{H}^g = \bigcup_{n=1}^{\infty} \mathbb{H}_n{}^g$. $\mathbb{H}^g$ is a differentiable a free.

On $\mathcal{M}^g$, we define a product that resembles the inner product on $\mathbb{C}^n$. Given $A, B \in \mathcal{M}^g$ which are g-tuples of $n \times n$ matrices:

$$\cdot : \mathcal{M}^g \times \mathcal{M}^g \longrightarrow M_n(\mathbb{C})$$
$$\cdot(A, B) = A \cdot B \longmapsto \sum_{i=1}^{g} A_i B_i$$

James uses this product, but like what in the world is going on with it???

$tr(A \otimes Id)$ But its more complicated than that bc A is a "row vector" of sorts

1 The topology of $\mathcal{M}^g$ is still in flux and there is not a canonical topology. See section 2.3 for the details

2.2 TRACIAL FUNCTIONS AND UNIQUENESS OF THE GRADIENT

Now that we have $\mathcal{M}^d$, we can work with general functions on our matrix universe. As a whole, free analysis is concerned with so-called *free functions*, which respect the direct sums and unitary conjugation. Do they need to be graded?

Definition i.12 (Free Function). *A function $f : D \to \mathcal{M}^{\hat{d}}$ is called* ***free*** *if*

1. $f(X \oplus Y) = f(X) \oplus f(Y)$
2. $f(UXU^*) = f(U)f(X)f(U^*)$ *where X and U are like-size and U is unitary.*

The two other classes of functions we are concerned with are those that act like the trace and the determinant:

Definition i.13 (Determinantal Free Function). *A function $f : D \to \mathbb{C}$ is a* ***determinantal free function*** *if*

1. $f(X \oplus Y) = f(X)f(Y)$
2. $f(UXU^*) = f(X)$ *where X and U are like-size and U is unitary.*

Definition i.14 (Tracial Free Function). *A function $f : D \to \mathbb{C}$ is a* ***tracial free function*** *if*

1. $f(X \oplus Y) = f(X) + f(Y)$
2. $f(UXU^*) = f(X)$ *where X and U are like-size and U is unitary.*

Given a free function of any type, we can define the directional derivative (Definition i.3) identically. It is worth noting that, while they share the moniker of *free*, determinantal and tracial functions are *not* free functions. It is only these tracial functions which inherit the gradient mentioned above. Similarly to traditional multivariable calculus we define the gradient via its relationship to the directional derivative:

Definition i.15 (Free Gradient). *Given a tracial free function f, the free gradient, ∇f, is the unique free function satisfying*

$$\operatorname{tr}(H \cdot \nabla f(X)) = \operatorname{tr} Df(X)[H]$$

It is not-at-all obvious that such a ∇f should be unique—after all any linear combination of commutator is has trace zero. should I explain this? In the case that f is a single-variable function we can replace ∇f with the traditional derivative, f', as seen in [16, Thm 3.3].

Theorem i.16. *Let $f : (a, b) \to \mathbb{R}$ be a C^1 function. Then*

$$\operatorname{tr} Df(X)[H] = \operatorname{tr}\left(Hf'(X)\right)$$

The proof in [16] simply asserts the uniqueness of a function $g(X)$ and then shows that $g(x) = f'(x)$ for $x \in (a, b)$. Instead, we can construct such a g and recover the theorem along the way:

Proof. We start with a construction from Bhatia's Matrix Analysis: Let $f \in C^1(I)$ and define $f^{[1]}$ on $I \times I$ by

$$f^{[1]}(\lambda, \mu) = \begin{cases} \frac{f(\lambda)-f(\mu)}{\lambda-\mu} & \lambda \neq \mu \\ f'(\lambda) & \lambda = \mu. \end{cases}$$

We call $f^{[1]}(\lambda, \mu)$ the *first divided difference* of f at (λ, μ). If Λ is a diagonal matrix with entries $\{\lambda_i\}$, We may extend f to accept Λ by defining the (i, j)-entry of $f^{[1]}(\Lambda)$ to be $f^{[1]}(\lambda_i, \lambda_j)$. If A is a self adjoint matrix with $A = U\Lambda U^*$, then we define $f^{[1]}(A) = Uf^{[1]}(\Lambda)U^*$. Now we borrow a theorem from Bhatia [4]:

Theorem i.17 (Bhatia V.3.3)**.** *Theorem numbering? Let $f \in C^1(I)$ and let A be a self adjoint matrix with all eigenvalues in I. Then*

$$Df(A)[H] = f^{[1]}(A) \circ H,$$

where $\circ$ denotes the Schur-product[2] *in a basis where A is diagonal.*

That is, if $A = U\Lambda U^*$, then

$$Df(A)[H] = U\left(f^{[1]}(\Lambda) \circ (U^*HU)\right)U^*.$$

To prove our claim, we need to take the trace of both sides. Since trace is invariant under a change of basis, it is clear that

$$\operatorname{tr} Df(A)[H] = \operatorname{tr}\left(f^{[1]}(\Lambda) \circ (U^*HU)\right).$$

2 Entrywise

If $U = u_{ij}$, $U^* = \overline{u}_{ij}$ and $H = h_{ij}$, then the (i, j)-entry of U^*HU is

$$(U^*HU)_{ij} = \overline{u}_{ik} h_{k\ell} u_{\ell j}$$

Where we sum over the duplicate indices k and ℓ. While the structure of $f^{[1]}(\Lambda)$ is a bit unruly, our diagonal entries are $f'(\lambda)$. This means that when we take the trace of the Schur product, we have

$$\sum_k \sum_\ell \sum_i f'(\lambda_i) \overline{u}_{ik} h_{k\ell} u_{\ell i}$$

Now consider the matrix product $U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\} U^*H$. Since one of our terms is diagonal, the trace of this multiplication is simple:

$$\operatorname{tr} U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\} U^*H = \sum_k \sum_\ell \sum_i u_{ik} f'(\lambda_k) \overline{u}_{k\ell} h_{\ell i}$$

Since $u_{ik}, \overline{u}_{k\ell}, h_{\ell i} \in \mathbb{C}$ they commute. We can then relabel our indices $i \mapsto \ell\ \ell \mapsto k\ k \mapsto i$ to get

$$\operatorname{tr}\ U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\} U^*H = \sum_k \sum_\ell \sum_i f'(\lambda_i) \overline{u}_{ik} h_{k\ell} u_{\ell i},$$

So, for every direction H, we have that $\operatorname{tr}\left(U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\} U^*H\right) = \operatorname{tr}\left(f^{[1]}(\Lambda) \circ (U^*HU)\right)$. overfull hbox :eyeroll: By picking the "correct" H, [3] we conclude that our unique quantity $g(X)$ is $U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\} U^*$. But, recall that $X = U\Lambda U$ so, in the functional calculus, $g(X) = f'(X)$. This recovers theorem 3.3 of [16] as we have constructed a g such that

$$\operatorname{tr}\ Df(X)[H] = \operatorname{tr} Hg(X)$$

■

With our theorem proven, we turn our attention back to the ∇f. The single variable case motivates that ∇f should correspond to the standard gradient from vector calculus. With some work, the above proof lifts the multi-variable case. It will be instructive, however, to consider a different proof.

Theorem i.18 (Trace Duality). *Let f, g be free functions $\mathcal{M}^g \to \mathcal{M}^{\tilde{g}}$. If $\operatorname{tr} H \cdot f = \operatorname{tr} H \cdot g$ for all tuples H, then $f = g$.*

3 See the proof of i.18 for the details of how to pick the H's

Proof. Since the trace relation holds for all H, we may choose our H carefully to show the equality of f and g. Say that $H, f(X), g(X)$ are g-tuples of matrices—we will first show that $f_1 = g_1$ and we will do so entry by entry. Let E_{ij} be the matrix will all zeroes and a 1 in the (i,j)-entry. Now let $H = (E_{ji}, 0, \ldots, 0)$. So $\operatorname{tr} E_{ji} f_1(X) = \operatorname{tr} E_{ji} g_1(X)$. In our products, the only elements on the diagonal are $(f_1(X))_{ij}$ and $(g_1(X))_{ij}$, so when we take the trace we have $(f_1(X))_{ij} = (g_1(X))_{ij}$. If we do this for every (i,j), we see that $f_1(X) = g_1(X)$. Similarly, we can choose $H = (0, E_{ji}, 0, \ldots, 0)$ for each i, j to show that $f_2(X) = g_2(X)$ and so on. Since $f(X) = g(X)$ for each $X \in \mathcal{M}^g$, it follows that $f = g$. ∎

Admittedly, there is a slight complication that is overlooked in the above proof when it comes to the domains of f and g. Where these domains overlap, we can consider them as the same function (and therefore ∇f is unique) but if f is defined on D and g is defined on $\tilde{D}$, then the above proof only holds on $D \cap \tilde{D}$. Examples of such f and g abound when considering rational functions, which are explored in section 2.5.

2.3 THE TOPOLOGY OF MATRIX UNIVERSES

At the time of writing, there is no "canonical topology" for $\mathcal{M}^g$. For a long time it seemed like the *free* topology (to be defined below) was the obvious choice, but recent work (c.f. [13]) has shown that the free topology does not put enough structure on $\mathcal{M}^g$. See [2] for a full treatment of the common topologies on $\mathcal{M}^g$.

A naive approach to a topology on $\mathcal{M} = \bigcup_n M_n(\mathbb{C})$ would be the disjoint union topology—which is then extended do a topology on $\mathcal{M}^g$ via the product topology. Notice, however that this ignores a significant amount of the implicit structure of nc-sets as we get a disconnected space with countable many connected components. Topologically, this is means that means that

$$H_\bullet(D) = \bigoplus_{n \in \mathbb{N}} H_\bullet(D_n).$$

At first glance, this seems fine enough, but it ignores the fact that for $X \in D$ we require $X^{\oplus k} \in D$ for all k and $U^* X U \in D$ for all unitary U. In a sense, we think of the all the direct sums of X and its similarity envelope as "the same." In light of this, if $\sim$ is the equivalence relation that $X \sim Y$ if $Y = X^{\oplus k}$ or

$Y = U^*XU$ is this actually an equivalence relation? The second statement is immediate but the first isn't an eq. rel., then any useful topological theory on $D \subset \mathcal{M}^g$ should descend to classic theory on $D/\sim$. One needs only look at $H_0(D)$ to see that the naive approach fails to give useful information. It should be the case that $H_0(\mathcal{M}^g)$ is trivial but in the disjoint union topology it is easy to see

$$H_0(\mathcal{M}^g) = \bigoplus_{n \in \mathbb{N}} \mathbb{Z},$$

which does not behave as we would expect.

a note on convergence somewhere.

2.3.1 *Admissible Topologies*

a cool example is showing that $\mathbb{H}$ is dense in $\mathcal{M}$

In light of the above discussion, we will present some of the candidate topologies which show some promise in understanding the topology on $\mathcal{M}^g$ and its subsets. We say that a topology τ is **admissible** if it has a basis of nc bounded open sets, D (recall that this means that D is closed under direct sums and unitary conjugation, and that each D_n is a bounded open set in $M_n(\mathbb{C})^g$). The finest admissible topology is the so-called **fine topology**, the basis of which consists of *all* nc open sets.

A slightly more restrictive topology (that seems to show some promise in the eyes of the author) is the **fat** topology. For $n \in \mathbb{N}$, $r \in \mathbb{R}^+$, and $X \in \mathcal{M}^g_n$, we first define a matricial polydisc

$$D_n(X, r) := \{A \in \mathcal{M}^g \mid \max_{1 \le i \le g} \|X_i - A_i\| < r\}.$$

Now we sweep D_n through all direct sum copies of X:

$$D(X, r) := \bigcup_{k=1}^{\infty} D_{kn}(X^{\oplus k}, r)$$

Finally, we take the similarity envelope of $D(X, r)$

$$F(X, r) := \bigcup_{n=1}^{\infty} \bigcup_{U \in \mathcal{U}_n} U^* \left(D(X, r) \cap \mathcal{M}^g_n\right) U$$

Both the fine and the fat topologies admit implicit function theorems.

The final candidate topology is the aforementioned **free** topology. Recall that $\mathbb{R}\langle x \rangle$ is the algebra of nc polynomials over the real number and that $\mathbb{R}\langle x\rangle^{k\times k}$ is the set of $k \times k$ matrices with entries in $\mathbb{R}\langle x\rangle$. Let $\delta \in \mathbb{R}\langle x\rangle^{k\times k}$ and define

$$G_\delta = \{x \in \mathcal{M}^g \mid \|\delta(x)\| < 1\}$$

The set of all G_δ as k ranges over $\mathbb{Z}^+$ form the basis for the free topology. Indeed, any $X \in \mathcal{M}^g$ is trivially in one of the G_δ (take $\delta = X$) and with some work one can show that $G_{\delta_1} \cap G_{\delta_2} = G_{\delta_1 \oplus \delta_2}$ (prove this) so we do, indeed, have a basis.

2.4 FREE ANALOGUES OF CLASSICAL RESULTS

In [1], Agler and McCarthy proved an free analogue of the Oka-Weil theorem: any holomorphic function on a compact set in the free topology can be uniformly approximated by polynomials. Unfortunately, it was later proven in [13] and [3] that the only compact sets in the $\mathcal{M}^g$ are the envelope of finitely many points, trivializing the result of Agler and McCarthy.

For the rest of this thesis, we will be using the conventions mentioned in section 2.1: $D \subset \mathcal{M}^g$ open if each D_n is open—these are precisely the basic open sets in the fine topology. Given $X, Y \in D$, it is not generally true that we can separate X and Y with open sets. However if Y is not in the similarity envelope of X and X and Y have disjoint fibers, then we *can* separate them! Motivated by definitions in section 4.4 we call a topology satisfying this condition (Hausdorff outside of the similarity envelope and fiber) **essentially Hausdorff**.

2.5 NC RATIONAL FUNCTIONS

Short review about defining rational functions via equivalence classes. We need this bc rational functions give the nc picard group and divisors and all that

Free polynomial rings are covered in detail (as algebraic objects) in [5]. Among their results is a characterization of rational functions:

Theorem i.19. *For any nondegenerate rational expression, r, there is a linear square matrix of polynomials, L, and rectangular constants b, c such that $r = b^*L^{-1}c$—where L^{-1} is defined wherever r is defined.*

Part II

THE ALGEBRAIC GEOMETRY AND TOPOLOGY OF MATRIX DOMAINS

"Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine."

— Michael Francis Atiyah

3 ZERO SETS AND PRINCIPLE DIVISORS

3.1 VARIETIES, CLASSICAL AND FREE

In the classical case, varieties are fairly easily to classify. Given some (commutative) polynomial, $f \in \mathbb{C}[x_1, \ldots, x_g]$ we define the zero set

$$V(f) = \{a \in \mathbb{A}^n \mid f(a) = 0\},$$

where $\mathbb{A}^n$ is complex affine n-space. Varieties (both affine and projective) are well studied in algebraic geometry (Hartshorne's *Alegbraic Geometry* [6] is a standard introduction). Of particular interest is a geometric invariant of a variety called a *divisor*. While divisors require robust machinery to construct formally[1] one can think of them (loosely) as formal sums of codimension one subvarieties. The concept of a divisor lift naturally to the noncommutative setting, although varieties are a touch more complex.

Let f be a matrix of polynomials on $\mathcal{M}^g$. Unlike the classical case, it is not immediate what should be meant by $f(X) = 0$—is it enough for $f(X)$ to be singular, or should $f(X)$ be the zero matrix? In light of this ambiguity, we make three definitions.

Definition ii.1 (Singular Set). *Let f be a matrix of polynomials function. The* ***n-Singular Set*** *of f is*

$$\mathscr{Z}_n(f) = \{X \in M_n(\mathbb{C}) \mid \det f(X) = 0\}.$$

The ***Singular Set*** *of f is*

$$\mathscr{Z}(f) = \bigcup_{n \in \mathbb{N}} \mathscr{Z}_n(f).$$

Associated with the singular set is the ***Directional Singular Set****:*

$$\mathscr{Z}_{dir}(f) = \{(X, v) \mid f(X)v = 0\}.$$

Definition ii.2 (Zero Set). *Let f be a matrix of polynomials function. The* ***n-Zero Set*** *of f is*

$$\mathscr{V}_n(f) = \{X \in M_n(\mathbb{C}) \mid f(X) = 0\}.$$

1 Schemes, in particular.

*The **Zero Set** of f is*

$$\mathscr{V}(f) = \bigcup_{n\in\mathbb{N}} \mathscr{V}_n(f).$$

While the singular set encodes the matrices for which $f(X)$ has a nontrivial kernel, the directional singular set bundles this information together with the kernel itself. Section 6 of Helton's *Free Convex Algebraic Geometry* [8] shows how this can is analogous to the tangent plane of a classical variety. While it may seem counter intuitive to use a script "Z" for the singular set instead of the zero set, the singular set of a free function is (in many cases) a more natural generalization of varieties. One needs to be careful when interfacing with the literature as these definitions (including which of these three sets it the "zero set") are not universal and each author seems to make their own choices.

Lots of work has been done in the past decade generating Null- and Positivstellensatz for these three sets. In particular, [9] treats singular and zero sets while [10] treats the directional zero set.

3.2 PRINCIPAL DIVISORS

Recall that given a differentiable traical free function f, the free gradient, ∇f is the unique free functions satisfying

$$\operatorname{tr}(H \cdot \nabla f) = Df(X)[H]$$

for all directions H. On the other hand, for every square [2] *free* function, g, we can associate a determinantal function $\det g$—which is defined in the obvious way. If f is a nontrivial *determinantal* function, then there is an induced tracial function, $\log f$ wherever f is nonzero.

Definition ii.3 (Principal Divisors)**.** *Let f be a nonzero* determinantal *free function. Then the **principal divisor** of f is*

$$\operatorname{div} f = \nabla \log f.$$

Alternatively, if g is square free *function, then the principal divisor of g is*

$$\operatorname{div} g = \nabla \log \det g$$

2 Meaning the output of g is a square matrix.

Before exploring the properties of $\operatorname{div} f$, it is worth acknowledging that the notation is overloaded. Unfortunately, the principal divisors of both free and determinantal functions have significant utility. One has to be careful whether theorems concern the divisors of free functions or determinantal ones. In light of this, the author has elected to italicize "free" and "determinantal" for the remainder of this section whenever there could be ambiguity should one not read too carefully.

While it is trivial to verify, (simply use the properties of log and the linearity of ∇), observe that

$$\operatorname{div} fg = \operatorname{div} f + \operatorname{div} g.$$

We will use this fact to give some characterzations of divisors

Lemma ii.4. *Let f, g be C^1 nonzero* determinantal *free functions. Then,*

1. *There exists an inverible locally constant determinantal functions c such that $f = cg$ if and only if $\operatorname{div} f = \operatorname{div} g$.*
2. *$\frac{f}{g}$ has a C^1 extension to the whole domain if and only if there is a C^1 determinantal function h on the whole domain such that $\operatorname{div} f - \operatorname{div} g = \operatorname{div} h$.*
3. *$\frac{f}{g}$ and $\frac{g}{f}$ have a C^1 extension to the whole domain if and only if $\operatorname{div} f - \operatorname{div} g$ has a continuous extension to the whole domain.*

Proof.

1. Suppose such a c existed. Then

$$\operatorname{div} f = \operatorname{div} cg = \operatorname{div} c + \operatorname{div} g.$$

But because c is locally constant, the presence of ∇ makes $\operatorname{div} c = 0$, so $\operatorname{div} f = \operatorname{div} g$.

Conversely, suppose $\operatorname{div} f = \operatorname{div} g$. But then

$$\begin{aligned} 0 &= \operatorname{div} f - \operatorname{div} g \\ &= \nabla (\log f - \log g) \\ &= \nabla \log \frac{f}{g}. \end{aligned}$$

And so $\log \frac{f}{g}$ is locally constant! It follows that $\frac{f}{g}$ is locally constant and hence we can write $g = cf$ for some locally constant functions c.

2. Suppose there is a function h on the whole domain such that $\operatorname{div} h = \operatorname{div} f - \operatorname{div} g$—then by part 1, h differs from $\frac{f}{g}$ by a constant but is definied on the entire domain. It is immediate, then, that $\frac{f}{g}$ extends to the whole domain.

   Conversely, suppose h is the continuous extension to the entire domain. But then

$$\begin{gathered} \frac{f}{g} = h \\ \Downarrow \\ \log f - \log g = \log h \\ \Downarrow \\ \operatorname{div} f - \operatorname{div} g = \operatorname{div} h. \end{gathered}$$

3. Part 3 follows immediately from part 2.

■

Example ii.5. *Consider the* free *functions $f(X,Y) = e^X e^Y, g(X,Y) = e^{X+Y}$. In significant contrast to the classical case, X and Y do not commute, so $f \neq g$. Before we look at the divisors of f and g it is pertinant to consider how f, g are actually defined—f and g are free functions defined on all of $\mathcal{M}^2$, so we are we are outside the functional calculus of section 1.3, which required self-adjoint matrices. For the values for which X, Y are diagonlizable, we can evaluate $f(X,Y)$ with the usual functional calculus. For an X or Y which is and nondiagonalizable, recall that $\mathbb{H}_n^2$ is dense in $M_n(\mathbb{C})^2$ so we have level-wise continuous extension of f (and of course g) to all of $\mathcal{M}^2$.*

Now we consider the divisors of f and g. Since they are free functions, recall that div *is actually* div det. *But then,*

$$\begin{aligned} \operatorname{div} e^X e^Y &= \nabla \log \det \left(e^X e^Y\right) \\ &= \nabla \log \left(e^{\operatorname{tr} X} e^{\operatorname{tr} Y}\right) \\ &= \nabla \left(\log e^{\operatorname{tr} X} + \log e^{\operatorname{tr} Y}\right) \\ &= \nabla \operatorname{tr} X + \nabla \operatorname{tr} Y \end{aligned} \qquad \begin{aligned} \operatorname{div} e^{X+Y} &= \nabla \log \det \left(e^{X+Y}\right) \\ &= \nabla \log \left(e^{\operatorname{tr}(X+Y)}\right) \\ &= \nabla \operatorname{tr}(X+Y) \\ &= \nabla \operatorname{tr} X + \nabla \operatorname{tr} Y \end{aligned}$$

And so we see that $\operatorname{div} e^X e^Y = \operatorname{div} e^{X+Y}$.

This example relies on the fact that log plays nicely with $e^{\operatorname{tr} X}$ and one might wonder if there is an easier way to compute principal divisors.

Theorem ii.6. *Let* $f : D \to \mathcal{M}^{\hat{d}\times\hat{d}}$ *be a* C^1 *free function*[3] *such that* $\det f \not\equiv 0$. *Then*

$$\operatorname{tr}\left(H \cdot \operatorname{div} f\right) = \operatorname{tr}\left(Df(X)[H]f(X)^{-1}\right)$$

Proof. We begin by recalling Jacobi's formula, which gives us a way to understand the directional derivative of the determant in terms of the adjugate[4] of a matrix. For a matrix X,

$$D\det X[H] = \operatorname{tr}\left(H \operatorname{adj} A\right).$$

It will be imperative later in the proof to recall the following property of the adjugate. For an inevitable matrix X,

$$\operatorname{adj}(X) = \det(X)X^{-1}.$$

With these preliminaries sorted, we continue with the proof. Unraveling the definitions given above, the principal divisor of f (a *free function*) is the unique free function on its nonsigular set satisfying

$$D\log\det f(X)[H] = \operatorname{tr}(H \cdot \operatorname{div} f).$$

We compute

$$\begin{aligned}
D\log\det f(X)[H] &= \frac{d}{dt}\left[\log\det f(X+tH)\right]\Big|_{t=0} \\
&= \frac{1}{\det f(X)}\left(\frac{d}{dt}\left[\det f(X+tH)\right]\Big|_{t=0}\right) \\
&= \frac{1}{\det f(X)}\operatorname{tr}\left(\frac{d}{dt}\left[f(X+tH)\right]\operatorname{adj} f(X+tH)\Big|_{t=0}\right) \\
&= \frac{1}{\det f(X)}\operatorname{tr}\left(Df(X)[H]\operatorname{adj} f(X)\right) \\
&= \operatorname{tr}\left(Df(X)[H]\frac{\operatorname{adj} f(X)}{\det f(X)}\right) \\
&= \operatorname{tr}\left(Df(X)[H]f^{-1}(X)\right)
\end{aligned}$$

■

The next section will treat divisors of polynomial and rational functions in detail. Before continuing, we give one more example.

3 Since the codomain is $\mathcal{M}^{\hat{d}\times\hat{d}}$, one can view f as a $\hat{d}\times\hat{d}$ matrix of free functions.

4 The transpose of the cofactor matrix.

Example ii.7. *Let $f(X,Y) = 1 + XY$ and $g(X,Y) = 1 + YX$. Using the previous theorem, we have that*

$$\begin{aligned}\operatorname{tr}\left((H_1, H_2) \cdot \operatorname{div} f\right) &= \operatorname{tr}\left(Df(X,Y)[H_1, H_2]f(X,Y)^{-1}\right) \\ &= \operatorname{tr}\left((H_1 Y + XH_2)(1+XY)^{-1}\right) \\ &= \operatorname{tr}\left(H_1 Y(1+XY)^{-1} + H_2(1+XY)^{-1}X\right)\end{aligned}$$

Appealing to trace duality (theorem i.18), we see that

$$\operatorname{div} f = \left(Y(1+XY)^{-1}, (1+XY)^{-1}X\right).$$

With a nearly identical computation, we recover the principal divisor of g as well:

$$\operatorname{div} g = \left((1+YX)^{-1}Y, X(1+YX)^{-1}\right).$$

Since $Y(1+XY) = (1+XY)Y$, it follows that $Y(1+XY)^{-1} = (1+XY)^{-1}Y$, and so $\operatorname{div} f = \operatorname{div} g$!

3.3 THE GROUP OF DIVISORS

For the remainder of this chapter, we will concern ourselves with the divisors of square matrices of nc polynomials and nc rational functions. These are all f free functions, so div will denote div det. We begin with a theorem.

Theorem ii.8. *Let f, g be square matrices of nc polynomials such that $\det f, \det g \not\equiv 0$. If $\frac{\det f}{\det g}$ and $\frac{\det g}{\det f}$ are entire, then $\operatorname{div} f = \operatorname{div} g$.*

Proof. Consider $\frac{\det f}{\det g}$ and $\frac{\det g}{\det f}$ as functions $M_n(\mathbb{C}) \to \mathbb{C}$. Since both of these are entire, $\det f, \det g$ are both never 0—hence any zeroes or poles that they possess must be at infinity. Suppose that $\frac{\det f}{\det g}$ is unbounded. Depending on how the degrees of $\det f$ and $\det g$ compare, there is either a zero or a pole at infinity. But this means that $\frac{\det g}{\det f}$ has either a zero or a pole at 0. Either way we have a contradiction, and so $\frac{\det f}{\det g}$ is bounded (and entire)—hence constant.

We now appeal to lemma ii.4, part 3. Since $\frac{\det f}{\det g}$ and its reciprocal both have C^1 extension (namely themselves), we have a levelwise constant function h such that $\operatorname{div} f - \operatorname{div} g = \operatorname{div} h$. But clearly $\operatorname{div} h$ is 0, so $\operatorname{div} f = \operatorname{div} g$! ∎

One of the major themes of the development of principal divisor of free functions (like in [14]) is that much of the structure of divisors is an immediate corollary of the structure of $\det f$. For example, the following theorem is proven almost entirely by its lemma.

Theorem ii.9. *Let r be a nondenerate square matrix of nc rational expressions, such that $\det X \not\equiv 0$. Then there exists square matrices of nc polynomials such that*

$$\operatorname{div} r = \operatorname{div} p - \operatorname{div} q$$

Proof. We begin with a lemma.

Lemma ii.10. *Let r be a nondenerate square matrix of nc rational expressions, such that $\det X \not\equiv 0$. Then there exists square matrices of nc polynomials such that*

$$\det r = \frac{\det p}{\det q} = \det(pq^{-1})$$

Proof. Recalling theorem i.19, let $r = b^* L^{-1} c$. We claim that

$$p = \begin{bmatrix} L & c \\ -b & 0 \end{bmatrix} \qquad \text{and} \qquad q = L.$$

We see that

$$\det \begin{bmatrix} L & c \\ -b & 0 \end{bmatrix} \Big/ \det L = \det \begin{bmatrix} L & c \\ -b & 0 \end{bmatrix} \det \begin{bmatrix} L^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \det \begin{bmatrix} 1 & c \\ -bL^{-1} & 0 \end{bmatrix}.$$

Now we recall the formula for the determinant of a block matrix:

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \det D - CA^{-1}B.$$

With this in hand, we see that $\det \left[\begin{smallmatrix} 1 & c \\ -bL^{-1} & 0 \end{smallmatrix}\right] = \det b^* L^{-1} c$, and we are done. ∎

Now take the div of both sides of the lemma to get the required result. ∎

Just as in the classical case, these is a deep link between factorization of polynomials, subvarieties, and principal divisors. Before we can explore this link in the noncommutative setting, we need a definition.

Definition ii.11 (Atomic)**.** *A square matrix of polynomials p is **atomic** if* $\det p \not\equiv 0$ *and if $p_1 p_2 = p$, then either* $\det p_1$ *or* $\det p_2$ *is locally constant.*

Atomic square matrices of polynomials function like irrecudible factors of tradition (commutative) polynomials. While we cannot have truly "unique" factorization, we do have factorization into atoms. In [9], Helton et al. prove the following theorem:

Theorem ii.12. *Let f is a square matrix of nc polynomials and p an atom. If we $f = p_1 \cdots p_k$ is the factorzation of f into atoms, then*

$$\mathscr{Z}(p) \subset \mathscr{Z}(f)$$

if and only if $\det p = \det(cp_i)$ *for one of the atoms of f and c a nonzero constant.*

With the help of lemma ii.4, part 1, this says that factorization is unique up to equivalence of principal divisors.

4 MONODROMY, GLOBAL GERMS, ALGEBRAIC TOPOLOGY

The results of the last three chapters seem hopeful—free analysis seems to be able to generalize many classical results, as listed in section 2.4. As previously mentioned, there are the free topology admits an Oka-Weil-type theorem. While this is promising, the only compact sets in the free topology are the envelopes of a finite collection of points.

It is the opinion of the author that all of these topologies (fine, fat, free, etc.) are definitively broken. As shown above, the free topology lacks a wealth of compact sets. The fine topology (and therefore any admissible topology) fails to be T_1, let alone Hausdorff—notice that any open set containing X must also contain $X \oplus X$. Further, given any free function f on an nc-domain D, if f is locally bounded on each D_n then f is analytic (admits a power series representation.) There are two ways so view this result: First, one can accept that analytic functions are a dime a dozen on $\mathcal{M}^g$. Alternatively, one can be skeptical that the topological structures put on $\mathcal{M}^g$ are indeed the natural choice. The work of J.E. Pascoe in [14] seeks to solve some of these issues by extending some of the concepts of traditional algebraic topology.

4.1 CLASSICAL MONODROMY

In the study of functions of a single complex variable, many of the central theorems surround the idea of analytic continuation. Given some analytic function f on a domain $\Omega \subset \mathbb{C}$ and a larger domain $\overline{\Omega} \supset \Omega$, we can (with sufficient "niceness" conditions) extend f to an analytic function g on $\overline{\Omega}$. In particular, given some path γ which start in Ω we wish to continue f *along* γ by recomputing the power series on overlapping disks with their centers on γ.

Our path γ must avoid any potential poles of f so that we may compute the power series, but the uniqueness of such an extension is not obvious. This is where the aforementioned niceness conditions come into play! For example, consider the follow setup:

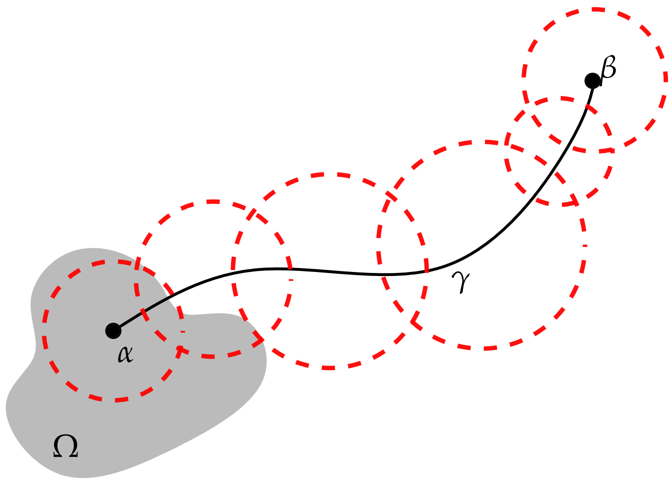


Figure 4.1: Analytic continuation along a curve

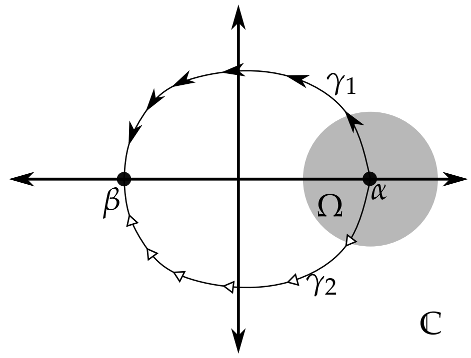


Figure 4.2: Two paths in $\mathbb{C}$

Example ii.13. *If we let $f(x) = \operatorname{Log} x$ be the principle branch of the complex logarithm the defined on the right half plane, and continue f along γ_1 and γ_2 we get two functions f_1 and f_2 which are analytic at β, but they don't agree! In this case, $f_1(\beta)$ and $f_2(\beta)$ disagree by exactly $2\pi i$.*

The monodromy theorem gives sufficient conditions for the continuation along two curves to agree:

Theorem ii.14 (Monodromy I)**.** *Let γ_1, γ_2 be two paths from α to β and Γ_s be a fixed-endpoint homotopy between them. If f can be continued along Γ_s for all $s \in [0, 1]$, then the continuations along γ_1 and γ_2 agree at β.*

In the example above, any homotopy between the two paths must pass through the origin—where $\operatorname{Log} x$ fails to be analytic—and hence the two continuations disagree at β. An equivalent formulation of the monodromy theorem concerns extending a functions to a larger domain:

Theorem ii.15 (Monodromy II)**.** *Let $U \subset \mathbb{C}$ be a disk in $\mathbb{C}$ centered at z_0 and $f : U \to \mathbb{C}$ an analytic function. If W is an open, simply connected set containing U and f continues along any path $\gamma \subset W$ starting at z_0, then f has a unique extension to all of W.*

This second formulation gives another perspective on $\operatorname{Log} x$. In the example, U is a disk around α that stays in the right half plane and W is $\mathbb{C} \setminus \{0\}$. While $\operatorname{Log} x$ continues along any path in $\mathbb{C} \setminus \{0\}$, the larger domain is *not* simply connected, so monodromy fails.

In practice, after the initial exposure in a first course in complex variables, no one computes continuations by hand. This could be a paragraph, but is it necessary?

4.2 FREE MONODROMY

There is an analogous theorem to theorems ii.14 and ii.15 in the free settings initial proven by J.E. Pasocoe in [15]. In the classic case, the larger set W must be simply connected. In the free setting, however, the theorem is much more powerful. Before we state and prove the theorem, recall that free functions respect direct sums—so if $f : D \to \mathcal{M}^{\tilde{g}}$ is a free function,

$$f(X \oplus Y) = f(X) \oplus f(Y).$$

Given two paths $\gamma_1, \gamma_2 \in D_n$, we can take their direct sum in the obvious way

$$(\gamma_1 \oplus \gamma_2)(t) = \gamma(t) := \begin{bmatrix} \gamma_1(t) & \\ & \gamma_2(t) \end{bmatrix}$$

to obtain a path in D_{2n}. If f is a free function defined on $B \subset D$, and then we can analytically continue f along γ (presuming that γ originates in B). If F is the resulting function defined at $\gamma(1)$, and F_1, F_2 are the continuations at $\gamma_1(1), \gamma_2(1)$ respectively, then a routine computation shows that

$$F(\gamma(1)) = \begin{bmatrix} F_1(\gamma_1(1)) & \\ & F_2(\gamma_2(1)) \end{bmatrix}.$$

With this preliminary result, we can introduce Universal Monodromy.

Theorem ii.16 (Free Univeresal Monodromy)**.** *If f is an analytic free function defined on some ball $B \subset D$, for D an open, connected free set. Then f analytically continues along every path in D if and only if f has a unique analytic continuation to all of D.*

Proof (Should I cite that talk?) The fact that a unique extension to all of D implies that f has a continuation along any γ is immediate.

Now suppose that f, a free function, analytically continues along every path in D. Fix $X \in B_n$ and pick some and let γ_1, γ_2 be two paths taking X to some $Y \in D_n$. Let F_1, F_2 be the analytic continuation of f along γ_1, γ_2 respectively. If $F_1 = F_2$, then we are done! Define $\hat{\gamma}, \gamma$ be path in D_{2n} defined by

$$\hat{\gamma} = \begin{bmatrix} \gamma_1 & \\ & \gamma_2 \end{bmatrix} \qquad \gamma = \begin{bmatrix} \gamma_2 & \\ & \gamma_1 \end{bmatrix}.$$

But we have a homotopy between $\hat{\gamma}$ and γ given by

$$\Gamma(t,s) = \begin{bmatrix} \cos(s\frac{\pi}{2}) & \sin(s\frac{\pi}{2}) \\ -\sin(s\frac{\pi}{2}) & \cos(s\frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} \gamma_1 & \\ & \gamma_2 \end{bmatrix} \begin{bmatrix} \cos(s\frac{\pi}{2}) & -\sin(s\frac{\pi}{2}) \\ \sin(s\frac{\pi}{2}) & \cos(s\frac{\pi}{2}) \end{bmatrix}.$$

Indeed, one easily checks that

$$\Gamma(t,0) = \hat{\gamma} \qquad \Gamma(t,1) = \gamma \qquad \Gamma(0,s) = X \oplus X \qquad \Gamma(1,s) = Y \oplus Y$$

But since $\hat{\gamma}$ and γ are homotopic we can apply the classical (albiet multivariable) monodromy theorem—so we know that the analytic continuations of f along $\hat{\gamma}, \gamma$ must agree near $Y \oplus Y$. Since free functions respect direct sums, if we let $\hat{F}$ and F denote the continuations of f along $\hat{\gamma}, \gamma$ respectively, we obtain the following chain of equalities:

$$\begin{bmatrix} F_1(\gamma_1) & \\ & F_2(\gamma_2) \end{bmatrix} = \hat{F}(\gamma_1 \oplus \gamma_2) = F(\gamma_2 \oplus \gamma_1) = \begin{bmatrix} F_2(\gamma_2) & \\ & F_1(\gamma_1) \end{bmatrix}$$

In particular, we see that $F_1(\gamma_1) = F_2(\gamma_2)$—so F_1 and F_2 agree! ∎

In the free case, the "larger" set need not be simply connected. Analytic continuations of free functions, then, cannot be used to detect holes in matrix domains. It will turn out, however, that the tracial and determinental functions introduced in section 2.2 can detect holes and produce an analogue of the fundamental group!

4.3 THE GERM OF FUNCTION

As studied in complex analytic and measure theoretic settings, if our space is structured enough functions are defined by their local behavior. This idea can be generalized to arbitrary topological spaces by stealing from sheaf theory fix that wording.

Let X be a topological space. For any open set U we can have $C(U)$, the ring of continuous functions $f : U \to \mathbb{R}$ (where addition and multiplication are defined point-wise) Tracial functions fail to be a ring but they *are* a group—should I just change this to a group?. Given any $V \subset U$, notice that a continuous function f on U, we can restrict f to V and maintain continuity. This gives two maps:

$$\begin{aligned} V &\hookrightarrow U \\ v &\longmapsto v \end{aligned} \qquad\qquad \begin{aligned} C(U) &\hookrightarrow C(V) \\ f &\longmapsto f\,|_V \end{aligned}$$

Notice that the induced function goes the "other way." This construction is an example of a sheaf of rings[1]—since $C(U)$ has a ring structure. We can similarly define sheaves of abelian groups or sets: to each open set in X we assign a group (or set) such that there are analogous restriction maps. For our purposes, these will always be groups/sets of functions and the restriction maps are the natural ones.

We are interested in the general behavior of continuous functions at some $x \in X$. Define $\mathfrak{C}_x$ to be the set of all functions defined on a neighborhood of x:

$$\mathfrak{C}_x = \{f \in C(U) \mid x \in U \subset X \text{ is open}\}.$$

By convention, we refer to elements of $\mathfrak{C}_x$ as a pair, (f, U) of a continuous function and the open set on which it is defined. In light of the inclusion maps given above, it obvious that $\mathfrak{C}_x$ will have "duplicate" elements. Therefore, we define an equivalence relation on $\mathfrak{C}_x$ by $(f, U) \sim (g, V) \Leftrightarrow$ there exists $W \subset U \cap V$ where $f\,|_W = g\,|_W$. In a sheaf-theoretic context, $\mathfrak{C}_x/\sim$ is called the **stalk** at x and elements of the stalk are **germs** at x. If we are dealing with sheaves of groups or sets, this construction remains unchanged! We can still define the stalk at given point. While it will not come into play, it is worth noting that the stalk inherits the algebraic structure of the original sheaf—e. g.for a sheaf of rings (or group), the stalk has a natural ring (group) structure.

Sheafs of rings/groups/sets of functions arise naturally in many areas of mathematics. For example, if X happens to be a

1 To be completely rigorous, a sheaf needs additional axioms, but the sheaf of continuous functions is one of the prototypical examples so the full defition is not needed in this context.

smooth manifold, we may replace $C(U)$ with $C^\infty(U)$, the ring of smooth functions into $\mathbb{R}$ and then obtain germs of smooth functions. Similarly, if X is a complex manifold we can construct germs of holomorphic functions.

Example ii.17. *Should I use the same number?*

Consider, again, example ii.13. Our function $f(x) = \operatorname{Log} x$ has a germ in Ω. In particular, both f_1 and f_2 belong to the equivalence class $[(f,\Omega)]$ as all three functions agree on Ω. From this, we see the genesis of the name germ: germs capture the local *behavior of function. Colloquially, this is the "heart" of a function similar to the germ of seed.*[2]

Link to monodromy again?

As usual, lifting this construction to the free context requires some nuance. For $U \subset D$ open, the set of tracial functions on U (denoted $C_{\text{tr}}(U)$) does not for a ring—it is closed under addition but not multiplication. Given two tracial functions, $f, g \in C_{\text{tr}}(U)$, we see that

$$\begin{aligned}(f+g)(X \oplus Y) &= f(X \oplus Y) + g(X \oplus Y) \\ &= f(X) + f(Y) + g(X) + g(Y) \\ &= (f+g)(X) + (f+g)(Y)\end{aligned}$$

but,

$$\begin{aligned}(fg)(X \oplus Y) &= f(X \oplus Y)g(X \oplus Y) \\ &= (f(X) + f(Y))(g(X) + g(Y)) \\ &= (fg)(X) + (fg)(Y) + f(X)g(Y) + f(Y)g(X).\end{aligned}$$

Thankfully, however, the construction remains unchanged if we substitute a ring of functions for an abelian group of functions (with the identity being $f \equiv 0$ and inverses given by simply negating the output). In the case of determinental and free functions (which play a lesser role in the theory to be developed) there is not a natural algebraic structure for the corresponding sheaves, so they are simply sheaves of sets.

4.4 THE TRACIAL FUNDAMENTAL GROUP

While Free Monodromy means that free functions cannot detect the topology of free sets, the same is not true for a general tracial function! Following [14], we will need some definitions.

2 Sheaf theory abounds with agrarian nomenclature.

Definition ii.18 (Anchored)**.** *Let $D \subset \mathcal{M}^g$ be a connected, open, free set. If there exists a nonempty, simply-connected, open, free $B \subset D$, then we say that D is **anchored**.*

Definition ii.19 (Global Germ)**.** *For D an open set, and $B \subset D$ its anchor, we call a tracial function $f : B \to \mathbb{C}$ a **global germ** if it analytically continues along every path in D which starts in B.*

In order to define the fundamental group, we need a notion of a path in D. Traditionally, a path taking X to Y is a continuous function $\gamma : [0,1] \to D$ such that $\gamma(0) = X$ and $\gamma(1) = Y$. Unfortunately, this disregards the fiber of X and Y. An mentioned in section 2.3, a proper topological theory should account for identification of the fibers.

Definition ii.20 (Essential Path)**.** *A continuous function $\gamma : [0,1] \to D$ **essentially takes** X to Y if*

$$\gamma(0) = X^{\oplus \ell}, \text{ for some } \ell \in \mathbb{N}$$
$$\gamma(1) = Y^{\oplus k}, \text{ for some } k \in \mathbb{N}.$$

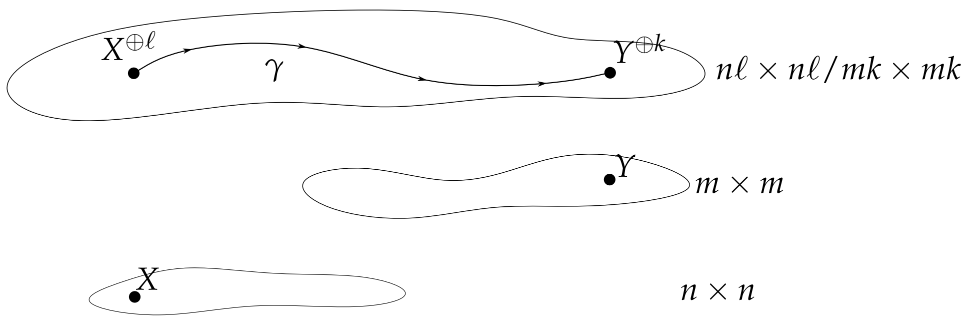


Figure 4.3: A path essentially taking X to Y

A path essentially taking X to Y is a path from some element of the fiber of X to some element of the fiber of Y. Just as in the classical case, essential paths have product. First, we need a way to take the direct sum of paths.

Definition ii.21 (Direct Sum of Paths)**.** *Given γ essentially taking X to Y and β taking Z to W, define*

$$\gamma \oplus \beta(t) = \begin{bmatrix} \gamma(t) & 0 \\ 0 & \beta(t) \end{bmatrix}.$$

It is not, in general, true that $\gamma \oplus \beta$ essentially takes $X \oplus Z$ to $Y \oplus W$. However, if γ essentially takes X to Y, then so does $\gamma \oplus \gamma$. As with matrices we define

$$\gamma^{\oplus k} := \underbrace{\gamma \oplus \cdots \oplus \gamma}_{k \text{ times}}$$

With these preliminaries, we can now define a concatenation product for essential paths:

Definition ii.22 (Concatenation Product)**.** *Let γ and β be paths taking X to Y and Y to Z respectively. We define their product to be the path essentially taking X to Z given by*

$$\beta\gamma(t) := \begin{cases} \gamma^{\oplus k}(2t) & t \in [0, 0.5) \\ \beta^{\oplus \ell}(2t-1) & t \in (0.5, 1] \end{cases}$$

where k and ℓ are positive integers chosen to maintain continuity.

With essential paths and their product we can build the first analogue of the fundamental group. Let D be an anchored space with B its anchor. For $X \in B$, we define $\pi_1(D, X)$ to be the set of path essentially taking X to X up to traditional homotopy equivalence and the relation $\gamma = \gamma^{\oplus k}$. Section 6 of [14] explores this construction in detail, including proving its commutativity.

Given a path essentially taking X to Y we can view the path as coupled with its endpoint. For B and anchor and f a global germ, we can reasonably define $f(\gamma)$: analytically continue f along γ and define

$$f(\gamma) := \frac{1}{k} f(Y^{\oplus k}).$$

Since we can evaluate paths with global germs, we can use global germs to something paths.

Definition ii.23 (Trace Equivalent)**.** *Let $B \subset D$ be an anchor and fix $X \in D$. If γ and β both essentially take X to Y, we say they are* ***trace equivalent*** *if, for every global germ f and every path δ taking Y to Z, $f(\delta\gamma) = f(\delta\beta)$.*

That it, trace equivalent paths are those which cannot be told apart via analytic continuation of global germ.

Under trace equivalence, the normalization given above means $\gamma = \gamma^{\oplus k}$ since both essentially take X to Y. Further, since homotopic paths have the same analytic continuation, homotopic paths are trace equivalent. This allows us to define a second fundamental group which will be our central object of study.

Definition ii.24 (Tracial Fundamental Group)**.** *Let D be an anchored space with B its anchor. For $X \in B$ define $\pi_1^{tr}(D, X)$ to be the group of trace equivalent paths essentially taking X to X.*

If D is connected, then π_1^{tr} is independent of our choice of base point—in fact, the isomorphism from the classical case works here as well. The identity is given by γ^X, the constant path at X and inverses given by

$$\gamma^{-1}(t) = \gamma(1-t).$$

Note that, since fixed endpoint homotopic paths are trace equivalent, π_1^{tr} is a quotient of π_1. We can construct a covering space for D with respect to π_1^{tr} similar to the construction of the universal cover in [7].

Definition ii.25 (Tracial Covering Space). *For $X \in B \subset D$, the **tracial covering space** of D is the set of paths (up to tracial equivalence [3]) in D starting at X:*

$$C^{tr}(D) = \{[\gamma] \mid \gamma \textit{ a path essentially taking } X \textit{ to } Y\}$$

Since we identify paths with their terminal endpoint, we have the natural covering space map $\rho : C^{\text{tr}}(D) \to D, [\gamma] \mapsto Y$. In order for this map to be continuous (and obey the rest of the axioms of a covering space), we need to endow $C^{\text{tr}}(D)$ with a topology. A metric for $C^{\text{tr}}(D)$ is given in [14], but the details are not particularly enlightening. With the topology induced by the metric, one can easily verify that we do, indeed, have a covering space. do we? I don't think its obvious but to prove it would distract from the point.

Because B is simply connected, for any $Y \in B$ there is exactly one path essentially taking X to Y. In light of this, there is a natural inclusion $B \hookrightarrow C^{\text{tr}}(D)$. Given a global germ, f, we induce a function on the covering space (given by $f(\gamma)$), which the norm on $C^{\text{tr}}(D)$ forces to be analytic.

4.5 A BIT OF COHOMOLOGY

For a treatment of Cohomology, see Allan Hatcher's famous *Algebraic Topology* [7]. As a quick review, given some chain complex

$$\cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} \cdots$$

we can form the for the **cochain complex** as follows: First, fix some abelian group G. The object in out cochain complex are

3 From here on, unless otherwise specified, we will only refer to paths up to trace equivalence. is it appropriate to put this in a footnote? Do people other than me read footnotes?

$C^\bullet = \mathrm{Hom}(C_\bullet, G)$ the group of morphisms $C_\bullet \to G$. The maps are induced ones

$$d = \partial^* : \mathrm{Hom}(C_n, G) \to \mathrm{Hom}(C_{n+1}, G).$$

Put together, this gives us a complex with the arrows reversed

$$\cdots \xrightarrow{d_{n-2}} C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \xrightarrow{d_{n+1}} \cdots$$

Computing the homology of this dual complex gives the *cohomology* groups, $H^\bullet(C; G) = \mathrm{Ker}\, d / \mathrm{Im}\, d$—which only depend on the homology of the original complex and the choice of G. Of particular interest is **De Rham Cohomology**, where $C^k = \Omega^k$, the set of k-forms on a manifold, and $G = \mathbb{R}$. In this case, the boundary map is given by the exterior derivative. For a full construction of De Rham Cohomology, see [12]. In the classical case we say that a k-form is **closed** if $\mathrm{d}f = 0$ and **exact** if there exists a k–1-form, g, such that $\mathrm{d}g = f$. The kth De Rham cohomology group, then, is the vector space of closed forms moduluo the exact forms.

A full cohomology theory has yet to be developed in the free setting. Lifting De Rham cohomology appears promising given that considering $\mathcal{M}^g$ carries a natural (if complex) manifold structure. While there is not a full generalization of the exterior derivative, recall that for any global germ f, we have that ∇f is a free function. If $\mathcal{G}$ is the set of global germs and $\mathcal{F}$ is the set of free functions, we have the beginning of a cochain complex

$$0 \to \mathcal{G} \xrightarrow{\nabla} \mathcal{F} \to \cdots .$$

While we cannot define "closed" and "exact" for general functions on $\mathcal{M}^g$ (in part because we don't know what the general cochain groups are) we can define them on $\mathcal{F}$.

Definition ii.26 (Exact)**.** *A free function f is **exact** if there exists a global germ g such that $\nabla g = f$.*

Definition ii.27 (Closed)**.** *A free function f is **closed** if*

$$\mathrm{tr}\,(K \cdot Df(X)[H]) = \mathrm{tr}\,(H \cdot Df(X)[K])$$

for all directions H, K.

While exactness is a direct lift of the classical condition, our definition of closed is decidedly unenlightening. I am still unenlightened. I will add more when I know what is going on. With our definitions, we can define the first tracial cohomology group.

Definition ii.28 (First Tracial Cohomology Group)**.** *The **first tracial cohomology group** is the vector space of closed free functions moduluo the exact free function. We write $H^1_{tr}(D)$.*

At first glance, H^1_{tr} seems rather convoluted, arbitrary, and not particularly useful. [4] Thankfully, we can put the tracial cohomology group to immediate use in understanding the structure of π_1^{tr}. Recall that, by definition, a global germ $f : B \to \mathbb{C}$ analytically continues along every path. It follows, then, that ∇f must analytically continue along every path as well—simply analytically continue f along the path and then take its gradient. Since ∇f is a free function, univeral monodromy (theorem ii.16) tells us that ∇f has a unique continuation to all of D. Because of this, we know that for any $\gamma \in \pi^1_{\mathrm{tr}}$ and γ' a path starting our anchor point X

$$f(\gamma'\gamma) - f(\gamma')$$

is locally constant. Since $\gamma'\gamma$ and γ have the same terminal end point and ∇ is linear, we know that $\nabla f(\gamma'\gamma) = \nabla(f)$, which tells us that our expression must be locally constant.

Before the homomorphism, talk about the action of traical pi 1 on global germs.

Given f a global germ and $X \in B_n$ the anchor point of

$$c^f(\gamma) := \frac{f(\gamma) - f(\gamma^X)}{n}.$$

c^f maps into $\mathbb{C}$ and some routine work with ∇ shows that only that if $c^f = c^{f'}$ then, $\nabla f, \nabla f'$ are in the same tracial cohomology class—i. e., c^f only depends on the class of ∇f in H^1_{tr}. If we define $\phi_g : \pi_1^{\mathrm{tr}} \to \mathbb{C}, \gamma \mapsto c^f(\gamma)$—where ∇f is in the tracial cohomology class of g—we get a *homomorphism* into $\mathbb{C}$, as

$$\begin{aligned} c^f(\gamma_1\gamma_2) &= \frac{f(\gamma_1\gamma_2) - f(\gamma_X)}{n} \\ &= \frac{f(\gamma_1\gamma_2) - f(\gamma_1) + f(\gamma_1) - f(\gamma_X)}{n} \\ &= \text{then some black magic} \\ &= c^f(\gamma_2) + c^f(\gamma_1). \end{aligned}$$

The fact that ϕ_g is a homomorphism is the first step to characterizing π_1^{tr}.

4 In all fairness, this is most people's reaction when the encounter cohomology for the first time. Is this allowed? I think it's a fair comment, but I have a bad habit of putting jokes in footnotes

Lemma ii.29. *The map*

$$\begin{aligned} \Phi : \prod_{g \in H^1_{\mathrm{tr}}} \pi_1^{\mathrm{tr}}(D) &\longrightarrow \prod_{g \in H^1_{\mathrm{tr}}} \mathbb{C} \\ \prod_{g \in H^1_{\mathrm{tr}}} \gamma &\longmapsto \prod_{g \in H^1_{\mathrm{tr}}} \phi_g(\gamma) \end{aligned}$$

is an injective homomophism.

Proof. The fact that Φ is a homomorphism is immediate, as each of the ϕ_g are. For injectivity, let $\alpha, \beta \in \pi_1^{\mathrm{tr}}(D)$ such that $\prod \phi_g(\alpha) = \prod \phi_g(\beta)$. Seeking to show that α and β are trace equivalent, let f be a global germ and γ essentially take X to Z. Then,

$$\begin{aligned} f(\gamma\alpha) - f(\gamma\beta) &= f(\gamma\alpha) - f(\gamma) - (f(\gamma\beta) - f(\gamma)) \\ &= c^f(\alpha) - c^f(\beta) \end{aligned}$$

This requires the trick from showings its a homom

But since $\prod \phi_g(\alpha) = \prod \phi_g(\beta)$, $c^f(\alpha) = c^f(\beta)$. put a sentence as to why. Thus, α and β are trace equivalent and we have shown injectivity. ■

Note that lemma also tells us that $\pi_1^{\mathrm{tr}}(D)$ is both commutative and torsion free as is injects into a commutative, torsion free group (namely a product of $\mathbb{C}$'s). With this, we can also show that $\pi_1^{\mathrm{tr}}(D)$ needs to be divisible as well. First, note that for any path γ,

$$\gamma \oplus \gamma_X = \gamma_X \oplus \gamma,$$

since

$$H(t, \theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} (\gamma \oplus \gamma_X) \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^*$$

is a homotopy between the paths. Then we see that

$$
\begin{aligned}
\gamma &= \underbrace{\begin{bmatrix} \gamma & & & \\ & \gamma & & \\ & & \ddots & \\ & & & \gamma \end{bmatrix}}_{k+1\text{-times}} \\
&= \begin{bmatrix} \gamma & & & \\ & \gamma x & & \\ & & \ddots & \\ & & & \gamma x \end{bmatrix} \begin{bmatrix} \gamma x & & & \\ & \gamma & & \\ & & \ddots & \\ & & & \gamma \end{bmatrix} \cdots \begin{bmatrix} \gamma & & & \\ & \gamma & & \\ & & \ddots & \\ & & & \gamma x \end{bmatrix} \\
&= \begin{bmatrix} \gamma & & & \\ & \gamma x & & \\ & & \ddots & \\ & & & \gamma x \end{bmatrix} \begin{bmatrix} \gamma & & & \\ & \gamma x & & \\ & & \ddots & \\ & & & \gamma x \end{bmatrix} \cdots \begin{bmatrix} \gamma & & & \\ & \gamma x & & \\ & & \ddots & \\ & & & \gamma x \end{bmatrix} \\
&= \begin{bmatrix} \gamma & & & \\ & \gamma x & & \\ & & \ddots & \\ & & & \gamma x \end{bmatrix}^k,
\end{aligned}
$$

and so $\pi_1^{\mathrm{tr}}(D)$ is divisible. As there is only one way (up to isomorphism, of course) to be a divisible, torsion free subgroup of $\mathbb{C}$, we have completely characterized $\pi_1^{\mathrm{tr}}(D)$! Why is this true? Certianly $\mathbb{R}$ satisfies this as well? After some googling, this is true if you have *enough* copies of Q (particularly $2^{\aleph}$). Should I mention this?

Theorem ii.30. *For D an anchored free set,*

$$\pi_1^{\mathrm{tr}}(D) \simeq \bigoplus_{i \in I} \mathbb{Q} = \mathbb{Q}^I$$

for some set I.

4.6 COMPUTING THE TRACIAL FUNDAMENTAL GROUP

While this structure theorem is useful, it gets us no closer to actually *computing* $\pi_1^{\mathrm{tr}}(D)$ or $H^1_{\mathrm{tr}}(D)$. Unfortunately, there is nothing analogous to Van Kampen's theorem or the Mayer Vietoris sequence. For simple domains, we have some basic tools. Recall the following definion relating to abelian groups

Definition ii.31 (Rank)**.** *For an abelian group, G, the **rank** of G is the maximal size of a linearly independent subset. That is, it the maximal size of a set $\{g_1, g_2, \ldots, g_k\} \subset G$ such that*

$$\sum_{i=0}^{k} n_i g_i = 0 \implies n_i = 0 \text{ for all } i.$$

Theorem ii.32. *Let D be a free anchored set. Then,*

1. $\dim H^1_{\mathrm{tr}}(D) \neq 0$ *if and only if* $\operatorname{rk} \pi_1^{\mathrm{tr}}(D) \neq 0$
2. $\dim H^1_{\mathrm{tr}}(D) \leq \operatorname{rk} \pi_1^{\mathrm{tr}}(D)$ *whenever both quantities are at most countably infinite*

Proof.

1. Suppose that $\operatorname{rk} \pi_1^{\mathrm{tr}}(D) \neq 0$, so for distinct paths γ_1, γ_2, there are corresponding global germs f_1, f_2 and a path β essentially taking X to Z such that $f_1(\beta\gamma_1) \neq f_2(\beta\gamma_2)$. something about monodromy.
2. We can restrict ourselves to the case where $\operatorname{rk} \pi_1^{\mathrm{tr}}(D)$ is finite. Let $\gamma_1, \ldots, \gamma_k$ be a maximally linearly independent set of paths, and suppose that $g_1, \ldots, g_{k+1} \in H^1_{\mathrm{tr}}(D)$ is linearly independent. Now consider the matrix $[\phi_{g_j}(\gamma_i)]_{ij}$, which is clearly singular. If $(\alpha_1, \alpha_2, \ldots, \alpha_{k+1})$ is a nontrivial vector in its kernel, then we can define $g = \sum_{j=0}^{k+1} \alpha_j g_j$. By construction, $g(\gamma_i) = 0$ for all i. Since $\{\gamma_j\}$ is maximal, it follows that g is the zero function, contradicting our assumption that $\{g_i\}$ is linearly independent.

■

This bound on the dimension of $H^1_{\mathrm{tr}}(D)$ is useful, but only if we have some way to reliably compute $\pi_1^{\mathrm{tr}}(D)$. Under certain circumstances—which are not particularly difficult to satisfy—we can compute $\pi_1^{\mathrm{tr}}(D)$ as a direct limit by looking at the level-wise homology groups. Classically, the Hurewicz theorem first homology group of a path connected manifold is isomorphic to the abelization of the fundamental group. Since we require the domain D to be path connected, we can leverage this fact to compute $\pi_1^{\mathrm{tr}}(D)$.

Let D be an anchored, free, path connected set such that each D_n is nonempty. Choose an anchor $B \subset D$ such that each B_n is also nonempty. If $X \in B_1$ is our base point, then we have a natural gradation on $\pi_1^{\mathrm{tr}}(D)$. Let $\pi_1^{\mathrm{tr}}(D)_n$ denote the subgroup of paths contained in D_n. For any $m \in \mathbb{N}$, there is a natural inclusion map $\pi_1^{\mathrm{tr}}(D)_n \hookrightarrow \pi_1^{\mathrm{tr}}(D)_{mn}$ given by $\gamma \mapsto \gamma^{\oplus m}$. Since our base point is in on the "scalar" level, we get a sequence of maps

$$\pi_1^{\mathrm{tr}}(D)_1 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_2 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_6 \hookrightarrow \cdots \hookrightarrow \pi_1^{\mathrm{tr}}(D)_{n!} \hookrightarrow \cdots$$

As long as one isn't too fearful of universal properties, it is not difficult to show that the direct limit of this sequence is isomorphic to $\pi_1^{\mathrm{tr}}(D)$. Using this result for computation requires understanding the structure of $\pi_1^{\mathrm{tr}}(D)_n$. Since $\pi_1^{\mathrm{tr}}(D)_n$ only contains paths in D_n, it is isomorphic to a quotient of some subgroup of $\pi_1(D_n)$! Moreover, since we are restricting ourselves to a fixed level, we can leverage the Hurewicz theorem. Since $\pi_1^{\mathrm{tr}}(D)_n$ is abelian is actually a quotient of $H_1(D_n)$! Thus, we can realize $\pi_1^{\mathrm{tr}}(D)$ as a direct limit of quotients of $H_1(D_{n!})$.

4.7 SOME EXAMPLES

The tracial fundamental group is a very new idea, so examples of computation don't abound. Pascoe's paper provides two examples as exercises. We present a "topological" proofs and invite the reader to fill in the details.

Example ii.33. *Let $D = GL(\mathbb{C}) = \bigcup_{\mathbb{N}} GL_n(\mathbb{C})$. Consider the case where $n = 1$. If we view complex numbers as 1×1 matrices, then $\{\det z = z \neq 0\} = \mathbb{C} \setminus \{0\}$. Then $\pi_1^{\mathrm{tr}}(GL)_1$ is a quotient of $H_1(GL_1(\mathbb{C})) \simeq \mathbb{Z}$. Since there are no torsion free subgroups of $\mathbb{Z}$, it must be the case that $\pi_1^{\mathrm{tr}}(GL)_1 \simeq \mathbb{Z}$ as well.*

Additionally, we know that there is a natural inclusion $\pi_1^{\mathrm{tr}}(GL)_1 \hookrightarrow \pi_1^{\mathrm{tr}}(GL)_2$, and so $\pi_1^{\mathrm{tr}}(GL)_2$ contains a copy of $\mathbb{Z}$. Moreover, given some $\gamma \in \pi_1^{\mathrm{tr}}(GL)_1$, we have that

$$\begin{bmatrix} \gamma & \\ & \gamma x \end{bmatrix} \in \pi_1^{\mathrm{tr}}(GL)_2.$$

Recall that if we square this element, then we get γ—so $\pi_1^{\mathrm{tr}}(GL)_2$ is isomorphic to the group

$$\mathbb{Z}\left[\frac{1}{2}\right]$$

Our next inclusion $\pi_1^{\text{tr}}(GL)_2 \hookrightarrow \pi_1^{\text{tr}}(GL)_6$ picks up cube roots for the same reason—since

$$\begin{bmatrix} \gamma & & \\ & \gamma_X & \\ & & \gamma_X \end{bmatrix} = \begin{bmatrix} \gamma & & & & & \\ & \gamma & & & & \\ & & \gamma_X & & & \\ & & & \gamma_X & & \\ & & & & \gamma_X & \\ & & & & & \gamma_X \end{bmatrix}.$$

Taking the square and cube roots simultaneously, we also obtain 6th roots.

$$\pi_1^{\text{tr}}(GL)_6 \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$$

In the n-th inclusion, then, we pick a $\frac{1}{n}\mathbb{Z}$ and any other factors needed for closure. The direct limit is, therefore,

$$\pi_1^{\text{tr}}(GL) \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right] \simeq \mathbb{Q}$$

Example ii.34. *Let $\Lambda \subset \mathbb{C}$ be finite and define*

$$G_\Lambda = \{X \in \mathcal{M} \mid detX - \lambda \neq 0 \, for \ all \ \lambda \in \Lambda\}.$$

We compute $\pi_1^{\text{tr}}(G_\Lambda)$ similarly to above. First, see that $(G_\Lambda)_1 = \mathbb{C} \setminus \Lambda$, and so $\pi_1^{\text{tr}}(G_\Lambda)_1 \simeq H_1((G_\Lambda)_1) \simeq \mathbb{Z}^{|\Lambda|}$. Inclusion into $\pi_1^{\text{tr}}(G_\Lambda)_2$ picks up square roots, so

$$\pi_1^{\text{tr}}(G_\Lambda)_2 \simeq \mathbb{Z}^{|\Lambda|}\left[\frac{1}{2}\right]$$

Inclusion into $\pi_1^{\text{tr}}(G_\Lambda)_6$ picks up cube and 6th roots, and on. Therefore, in the direct limit, we see

$$\pi_1^{\text{tr}}(G_\Lambda) \simeq \mathbb{Q}^{|\Lambda|}.$$

Part III

APPENDIX

BIBLIOGRAPHY

[1] Jim Agler and John E. McCarthy. "Global Holomorphic Functions in Several Non-Commuting Variables." In: *arXiv:1305.1636 [math]* (July 2013). arXiv: 1305.1636 [math].

[2] Jim Agler and John E. McCarthy. "Aspects of Non-Commutative Function Theory." In: *Concrete Operators* 3.1 (Jan. 2016). ISSN: 2299-3282. DOI: 10.1515/conop-2016-0003.

[3] Meric Augat, Sriram Balasubramanian, and Scott McCullough. "Compact Sets in the Free Topology." In: *arXiv:1604.04580 [math]* (Oct. 2017). arXiv: 1604.04580 [math].

[4] Rajendra Bhatia. *Matrix Analysis*. Vol. 169. Graduate Texts in Mathematics. New York, NY: Springer New York, 1997. ISBN: 978-1-4612-6857-4 978-1-4612-0653-8. DOI: 10.1007/978-1-4612-0653-8.

[5] P. M. Cohn. *Free Ideal Rings and Localization in General Rings*. New Mathematical Monographs 3. Cambridge, UK ; New York: Cambridge University Press, 2006. ISBN: 978-0-521-85337-8.

[6] Robin Hartshorne. *Algebraic Geometry*. Fourteenth. Graduate Texts in Mathematics 52. New York, NY: Springer, 2008. ISBN: 978-0-387-90244-9.

[7] Allen Hatcher. *Algebraic Topology*. Cambridge ; New York: Cambridge University Press, 2002. ISBN: 978-0-521-79160-1 978-0-521-79540-1.

[8] J. William Helton, Igor Klep, and Scott McCullough. "Free Convex Algebraic Geometry." In: (Apr. 2013). DOI: 10.48550/arXiv.1304.4272.

[9] J. William Helton, Igor Klep, and Jurij Volčič. "Factorization of Noncommutative Polynomials and Nullstellens\"atze for the Free Algebra." In: (July 2019).

[10] J. William. Helton, Scott McCullough, and Mihai Putinar. "Strong Majorization in a Free *-Algebra." In: *Mathematische Zeitschrift* 255.3 (Jan. 2007), pp. 579–596. ISSN: 0025-5874, 1432-1823. DOI: 10.1007/s00209-006-0032-0.

[11] Donald E. Knuth. "Computer Programming as an Art." In: *Communications of the ACM* 17.12 (1974), pp. 667–673.

[12] John M. Lee. *Introduction to Smooth Manifolds*. 2nd ed. Graduate Texts in Mathematics 218. New York ; London: Springer, 2013. ISBN: 978-1-4419-9981-8 978-1-4419-9982-5.

[13] J. E. Pascoe. "An Entire Free Holomorphic Function Which Is Unbounded on the Row Ball." In: *arXiv:1908.06753 [math]* (Aug. 2019). arXiv: 1908.06753 [math].

[14] J. E. Pascoe. "Free Noncommutative Principal Divisors and Commutativity of the Tracial Fundamental Group." In: (Oct. 2020).

[15] J. E. Pascoe. "Noncommutative Free Universal Monodromy, Pluriharmonic Conjugates, and Plurisubharmonicity." In: (Feb. 2020).

[16] J. E. Pascoe. "Trace Minmax Functions and the Radical Laguerre-P\'olya Class." In: (Aug. 2020). DOI: 10.48550/arXiv.2008.05469.