Searching for Holes in the Matrix Universe

Lucas Kerbs

Spring 2022

Searching for Holes in the Matrix Universe

Lucas Kerbs

Spring 2022

- Eventual goal: lift the tools of algebraic topology to spaces of matrices
- If we only consider 2×2 matrices we can use classical theory
- The moment we want more than one size, things the classical theory breaks down
- Today we will develop some *fairly heftly* tools to do just that
- Along the way, hopefully I can convince you that this is an interesting question.
- To do so, we need to go back to our mathematical roots

Part I: Objects and Maps

A Naive Attempt

- That's right—objects and maps.
- Our Naive attempt involves that looking at lifting functions on $\mathbb{R}$ or $\mathbb{C}$ to accept matrices as their input.
- An operator theorist would call this a "functional calculus"

Functional Calculus

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.

Functional Calculus
Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.

- Polynomials are the most well behaved functions we have, so lets start with a polynomial and a self adjoint ($A = A^*$) matrix.
- You might say that SA is unnecessary bc we can already evaluate a polynomial on a matrix

Functional Calculus

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U \Lambda U^*$

Functional Calculus
Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U \Lambda U^*$

- Polynomials are the most well behaved functions we have, so lets start with a polynomial and a self adjoint ($A = A^*$) matrix.
- You might say that SA is unnecessary bc we can already evaluate a polynomial on a matrix
- Since we are SA, we can diagonalize with unitary matrices.

Functional Calculus

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U\Lambda U^*$

$$f(A) = a_n A^n + \cdots + a_1 A + a_0 I_k$$

Functional Calculus
Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U\Lambda U^*$
$f(A) = a_n A^n + \cdots + a_1 A + a_0 I_k$

- Polynomials are the most well behaved functions we have, so lets start with a polynomial and a self adjoint ($A = A^*$) matrix.
- You might say that SA is unnecessary bc we can already evaluate a polynomial on a matrix
- Since we are SA, we can diagonalize with unitary matrices.
- Watch what happens when we plug this into our polynomial — need to be careful with the constant term.

Functional Calculus

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U\Lambda U^*$

$$\begin{aligned} f(A) &= a_n A^n + \cdots + a_1 A + a_0 I_k \\ &= a_n \left(U\Lambda U^*\right)^n + \cdots + a_1 U\Lambda U^* + a_0 I_k \end{aligned}$$

Functional Calculus
Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U\Lambda U^*$
$f(A) = a_n A^n + \cdots + a_1 A + a_0 I_k$
$= a_n (U\Lambda U^*)^n + \cdots + a_1 U\Lambda U^* + a_0 I_k$

- Polynomials are the most well behaved functions we have, so lets start with a polynomial and a self adjoint ($A = A^*$) matrix.
- You might say that SA is unnecessary bc we can already evaluate a polynomial on a matrix
- Since we are SA, we can diagonalize with unitary matrices.
- Watch what happens when we plug this into our polynomial — need to be careful with the constant term.

Functional Calculus

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U\Lambda U^*$

$$\begin{aligned} f(A) &= a_n A^n + \cdots + a_1 A + a_0 I_k \\ &= a_n \left(U\Lambda U^*\right)^n + \cdots + a_1 U\Lambda U^* + a_0 I_k \\ &= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 I_k \end{aligned}$$

Functional Calculus
Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U\Lambda U^*$
$f(A) = a_n A^n + \cdots + a_1 A + a_0 I_k$
$= a_n \left(U\Lambda U^*\right)^n + \cdots + a_1 U\Lambda U^* + a_0 I_k$
$= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 I_k$

- Polynomials are the most well behaved functions we have, so lets start with a polynomial and a self adjoint ($A = A^*$) matrix.
- You might say that SA is unnecessary bc we can already evaluate a polynomial on a matrix
- Since we are SA, we can diagonalize with unitary matrices.
- Watch what happens when we plug this into our polynomial — need to be careful with the constant term.

Functional Calculus

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U\Lambda U^*$

$$\begin{aligned} f(A) &= a_n A^n + \cdots + a_1 A + a_0 I_k \\ &= a_n \left(U\Lambda U^*\right)^n + \cdots + a_1 U\Lambda U^* + a_0 I_k \\ &= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 I_k \\ &= U\left(a_n\Lambda^n + \cdots + a_1\Lambda + a_0 I_k\right) U^* \end{aligned}$$

Functional Calculus
Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U\Lambda U^*$
$f(A) = a_n A^n + \cdots + a_1 A + a_0 I_k$
$= a_n \left(U\Lambda U^*\right)^n + \cdots + a_1 U\Lambda U^* + a_0 I_k$
$= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 I_k$
$= U\left(a_n\Lambda^n + \cdots + a_1\Lambda + a_0 I_k\right) U^*$

- Polynomials are the most well behaved functions we have, so lets start with a polynomial and a self adjoint ($A = A^*$) matrix.
- You might say that SA is unnecessary bc we can already evaluate a polynomial on a matrix
- Since we are SA, we can diagonalize with unitary matrices.
- Watch what happens when we plug this into our polynomial — need to be careful with the constant term.

Functional Calculus

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U\Lambda U^*$

$$\begin{aligned} f(A) &= a_n A^n + \cdots + a_1 A + a_0 I_k \\ &= a_n \left(U\Lambda U^*\right)^n + \cdots + a_1 U\Lambda U^* + a_0 I_k \\ &= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 I_k \\ &= U\left(a_n\Lambda^n + \cdots + a_1\Lambda + a_0 I_k\right) U^* \\ &= U\left(f(\Lambda)\right) U^* \end{aligned}$$

Functional Calculus
Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U\Lambda U^*$
$f(A) = a_n A^n + \cdots + a_1 A + a_0 I_k$
$= a_n \left(U\Lambda U^*\right)^n + \cdots + a_1 U\Lambda U^* + a_0 I_k$
$= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 I_k$
$= U\left(a_n\Lambda^n + \cdots + a_1\Lambda + a_0 I_k\right) U^*$
$= U\left(f(\Lambda)\right) U^*$

- Polynomials are the most well behaved functions we have, so lets start with a polynomial and a self adjoint ($A = A^*$) matrix.
- You might say that SA is unnecessary bc we can already evaluate a polynomial on a matrix
- Since we are SA, we can diagonalize with unitary matrices.
- Watch what happens when we plug this into our polynomial — need to be careful with the constant term.

Functional Calculus

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U\Lambda U^*$

$$\begin{aligned} f(A) &= a_n A^n + \cdots + a_1 A + a_0 I_k \\ &= a_n \left(U\Lambda U^*\right)^n + \cdots + a_1 U\Lambda U^* + a_0 I_k \\ &= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 I_k \\ &= U\left(a_n\Lambda^n + \cdots + a_1\Lambda + a_0 I_k\right) U^* \\ &= U\left(f(\Lambda)\right) U^* \end{aligned}$$

$$f\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right)$$

Functional Calculus
Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U\Lambda U^*$
$f(A) = a_n A^n + \cdots + a_1 A + a_0 I_k$
$= a_n \left(U\Lambda U^*\right)^n + \cdots + a_1 U\Lambda U^* + a_0 I_k$
$= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 I_k$
$= U\left(a_n\Lambda^n + \cdots + a_1\Lambda + a_0 I_k\right) U^*$
$= U\left(f(\Lambda)\right) U^*$
$f\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right)$

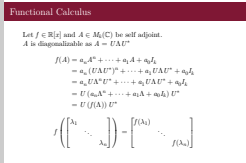
- Polynomials are the most well behaved functions we have, so lets start with a polynomial and a self adjoint ($A = A^*$) matrix.
- You might say that SA is unnecessary bc we can already evaluate a polynomial on a matrix
- Since we are SA, we can diagonalize with unitary matrices.
- Watch what happens when we plug this into our polynomial — need to be careful with the constant term.

Functional Calculus

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U\Lambda U^*$

$$\begin{aligned} f(A) &= a_n A^n + \cdots + a_1 A + a_0 I_k \\ &= a_n \left(U\Lambda U^*\right)^n + \cdots + a_1 U\Lambda U^* + a_0 I_k \\ &= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 I_k \\ &= U\left(a_n\Lambda^n + \cdots + a_1\Lambda + a_0 I_k\right) U^* \\ &= U\left(f(\Lambda)\right) U^* \end{aligned}$$

$$f\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right) = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}$$



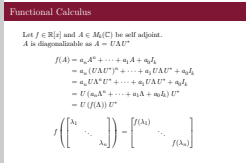
- Polynomials are the most well behaved functions we have, so lets start with a polynomial and a self adjoint ($A = A^*$) matrix.
- You might say that SA is unnecessary bc we can already evaluate a polynomial on a matrix
- Since we are SA, we can diagonalize with unitary matrices.
- Watch what happens when we plug this into our polynomial — need to be careful with the constant term.
- This application along the diagonal is precisely the behavior we want to emulate in the functional calculus.

Functional Calculus

Let $f \in \mathbb{R}[x]$ and $A \in M_k(\mathbb{C})$ be self adjoint.
A is diagonalizable as $A = U\Lambda U^*$

$$\begin{aligned} f(A) &= a_n A^n + \cdots + a_1 A + a_0 I_k \\ &= a_n \left(U\Lambda U^*\right)^n + \cdots + a_1 U\Lambda U^* + a_0 I_k \\ &= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 I_k \\ &= U\left(a_n\Lambda^n + \cdots + a_1\Lambda + a_0 I_k\right) U^* \\ &= U\left(f(\Lambda)\right) U^* \end{aligned}$$

$$f\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right) = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}$$



- Polynomials are the most well behaved functions we have, so lets start with a polynomial and a self adjoint ($A = A^*$) matrix.
- You might say that SA is unnecessary bc we can already evaluate a polynomial on a matrix
- Since we are SA, we can diagonalize with unitary matrices.
- Watch what happens when we plug this into our polynomial — need to be careful with the constant term.
- This application along the diagonal is precisely the behavior we want to emulate in the functional calculus.

Let $\mathbb{H}_n$ be the set of $n \times n$ self adjoint matrices, and define

$$\mathbb{H} = \bigcup_{n \in \mathbb{N}} \mathbb{H}_n, \qquad \mathcal{M} = \bigcup_{n \in \mathbb{N}} M_n(\mathbb{C})$$

Let $\mathbb{H}_n$ be the set of $n \times n$ self adjoint matrices, and define
$\mathbb{H} = \bigcup_{n \in \mathbb{N}} \mathbb{H}_n, \qquad \mathcal{M} = \bigcup_{n \in \mathbb{N}} M_n(\mathbb{C})$

- With the polynomial case in mind, we can extend a general function. First, a piece of notation
- Lets grab a function on the real line and the self adjoint matrices with their spectrum in that domain
- Then we can lift g by emulating the behavior of polynomials.
- unwrap a self adjoint matrix, apply g to the diagonal, then wrap it back up
- Something to notice about this functional calculus—it treats direct sums *very* well
- This is all well and good, but can we do anything with these new functions?

Let $\mathbb{H}_n$ be the set of $n \times n$ self adjoint matrices, and define

$$\mathbb{H} = \bigcup_{n \in \mathbb{N}} \mathbb{H}_n, \qquad \mathcal{M} = \bigcup_{n \in \mathbb{N}} M_n(\mathbb{C})$$

Definition:
Let $g : [a, b] \to \mathbb{C}$ and $D \subset \mathbb{H}$ denote the set of self adjoint matrices with their spectrum in $[a, b]$.

Let $\mathbb{H}_n$ be the set of $n \times n$ self adjoint matrices, and define
$\mathbb{H} = \bigcup_{n \in \mathbb{N}} \mathbb{H}_n, \quad \mathcal{M} = \bigcup_{n \in \mathbb{N}} M_n(\mathbb{C})$
Definition:
Let $g : [a, b] \to \mathbb{C}$ and $D \subset \mathbb{H}$ denote the set of self adjoint matrices with their spectrum in $[a, b]$.

- With the polynomial case in mind, we can extend a general function. First, a piece of notation
- Lets grab a function on the real line and the self adjoint matrices with their spectrum in that domain
- Then we can lift g by emulating the behavior of polynomials.
- unwrap a self adjoint matrix, apply g to the diagonal, then wrap it back up
- Something to notice about this functional calculus—it treats direct sums *very* well
- This is all well and good, but can we do anything with these new functions?

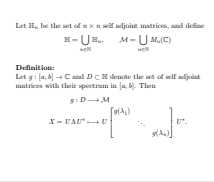
Let $\mathbb{H}_n$ be the set of $n \times n$ self adjoint matrices, and define

$$\mathbb{H} = \bigcup_{n \in \mathbb{N}} \mathbb{H}_n, \qquad \mathcal{M} = \bigcup_{n \in \mathbb{N}} M_n(\mathbb{C})$$

Definition:
Let $g : [a, b] \to \mathbb{C}$ and $D \subset \mathbb{H}$ denote the set of self adjoint matrices with their spectrum in $[a, b]$. Then

$$g : D \longrightarrow \mathcal{M}$$

$$X = U \Lambda U^* \longmapsto U \begin{bmatrix} g(\lambda_1) & & \\ & \ddots & \\ & & g(\lambda_n) \end{bmatrix} U^*.$$



- With the polynomial case in mind, we can extend a general function. First, a piece of notation
- Lets grab a function on the real line and the self adjoint matrices with their spectrum in that domain
- Then we can lift g by emulating the behavior of polynomials.
- unwrap a self adjoint matrix, apply g to the diagonal, then wrap it back up
- Something to notice about this functional calculus—it treats direct sums *very* well
- This is all well and good, but can we do anything with these new functions?

Let $\mathbb{H}_n$ be the set of $n \times n$ self adjoint matrices, and define

$$\mathbb{H} = \bigcup_{n\in\mathbb{N}} \mathbb{H}_n, \qquad \mathcal{M} = \bigcup_{n\in\mathbb{N}} M_n(\mathbb{C})$$

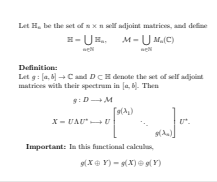
Definition:
Let $g : [a, b] \to \mathbb{C}$ and $D \subset \mathbb{H}$ denote the set of self adjoint matrices with their spectrum in $[a, b]$. Then

$$g : D \longrightarrow \mathcal{M}$$

$$X = U\Lambda U^* \longmapsto U \begin{bmatrix} g(\lambda_1) & & \\ & \ddots & \\ & & g(\lambda_n) \end{bmatrix} U^*.$$

Important: In this functional calculus,

$$g(X \oplus Y) = g(X) \oplus g(Y)$$



- With the polynomial case in mind, we can extend a general function. First, a piece of notation
- Lets grab a function on the real line and the self adjoint matrices with their spectrum in that domain
- Then we can lift g by emulating the behavior of polynomials.
- unwrap a self adjoint matrix, apply g to the diagonal, then wrap it back up
- Something to notice about this functional calculus—it treats direct sums *very* well
- This is all well and good, but can we do anything with these new functions?

Directional Derivative

Definition: *Directional Derivative*
Fix some $X \in \mathbb{H}_n$. The derivative of f at X in the direction $H \in M_n(\mathbb{C})$ is

$$Df(X)[H] = \lim_{t \to 0} \frac{f(X + tH) - f(X)}{t}$$

Directional Derivative
Definition: *Directional Derivative*
Fix some $X \in \mathbb{H}_n$. The derivative of f at X in the direction $H \in M_n(\mathbb{C})$ is
$Df(X)[H] = \lim_{t \to 0} \frac{f(X + tH) - f(X)}{t}$

- We can define a directional derivative—as long as we are careful to have the direction in the same "level-wise" slice.
- Notice that, with some special attention to what operation we are carrying our, this is the exact same definition as classic multivariable calculus.
- There is another formulation that is (generally) more useful for computation

Directional Derivative

Definition: *Directional Derivative*
Fix some $X \in \mathbb{H}_n$. The derivative of f at X in the direction $H \in M_n(\mathbb{C})$ is

$$Df(X)[H] = \lim_{t \to 0} \frac{f(X + tH) - f(X)}{t}$$

Alternatively,

$$Df(X)[H] = \left. \frac{df(X + tH)}{dt} \right|_{t=0}$$

Directional Derivative
Definition: *Directional Derivative*
Fix some $X \in \mathbb{H}_n$. The derivative of f at X in the direction $H \in M_n(\mathbb{C})$ is
$$Df(X)[H] = \lim_{t \to 0} \frac{f(X + tH) - f(X)}{t}$$
Alternatively,
$$Df(X)[H] = \left. \frac{df(X + tH)}{dt} \right|_{t=0}$$

- We can define a directional derivative—as long as we are careful to have the direction in the same "level-wise" slice.
- Notice that, with some special attention to what operation we are carrying our, this is the exact same definition as classic multivariable calculus.
- There is another formulation that is (generally) more useful for computation

Example: $g(x) = x^3$

$$g(X + tH) =$$

Example: $g(x) = x^3$
$g(X + tH) =$

- Now we consider an example. Since $Df(X)[H]$ is linear, we can just work with a single monomial
- First we expand $(x + th)^3$—but we can't use the binomial theorem since x and h don't commute
- Once we expand, we take standard derivatives w.r.t t—treating X and H as formal symbols.

Example: $g(x) = x^3$

$$g(X + tH) = X^3 + tX^2H + tXHX + t^2XH^2 + tHX^2 + t^2HXH + t^2H^2X + t^3H^3.$$

Example: $g(x) = x^3$
$g(X + tH) = X^3 + tX^2H + tXHX + t^2XH^2 + tHX^2 + t^2HXH + t^2H^2X + t^3H^3.$

- Now we consider an example. Since $Df(X)[H]$ is linear, we can just work with a single monomial
- First we expand $(x + th)^3$—but we can't use the binomial theorem since x and h don't commute
- Once we expand, we take standard derivatives w.r.t t—treating X and H as formal symbols.

Example: $g(x) = x^3$

$$g(X + tH) = X^3 + tX^2H + tXHX + t^2XH^2 + tHX^2 + t^2HXH + t^2H^2X + t^3H^3.$$

From here, we can calculate:

Example: $g(x) = x^3$
$g(X + tH) = X^3 + tX^2H + tXHX + t^2XH^2 + tHX^2 + t^2HXH + t^2H^2X + t^3H^3.$
From here, we can calculate:

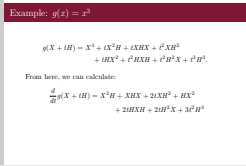
- Now we consider an example. Since $Df(X)[H]$ is linear, we can just work with a single monomial
- First we expand $(x + th)^3$—but we can't use the binomial theorem since x and h don't commute
- Once we expand, we take standard derivatives w.r.t t—treating X and H as formal symbols.

Example: $g(x) = x^3$

$$g(X + tH) = X^3 + tX^2H + tXHX + t^2XH^2 + tHX^2 + t^2HXH + t^2H^2X + t^3H^3.$$

From here, we can calculate:

$$\frac{d}{dt}g(X + tH) = X^2H + XHX + 2tXH^2 + HX^2 + 2tHXH + 2tH^2X + 3t^2H^3$$



- Now we consider an example. Since $Df(X)[H]$ is linear, we can just work with a single monomial
- First we expand $(x + th)^3$—but we can't use the binomial theorem since x and h don't commute
- Once we expand, we take standard derivatives w.r.t t—treating X and H as formal symbols.

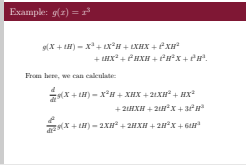
Example: $g(x) = x^3$

$$g(X + tH) = X^3 + tX^2H + tXHX + t^2XH^2 + tHX^2 + t^2HXH + t^2H^2X + t^3H^3.$$

From here, we can calculate:

$$\frac{d}{dt}g(X + tH) = X^2H + XHX + 2tXH^2 + HX^2 + 2tHXH + 2tH^2X + 3t^2H^3$$

$$\frac{d^2}{dt^2}g(X + tH) = 2XH^2 + 2HXH + 2H^2X + 6tH^3$$



- Now we consider an example. Since $Df(X)[H]$ is linear, we can just work with a single monomial
- First we expand $(x + th)^3$—but we can't use the binomial theorem since x and h don't commute
- Once we expand, we take standard derivatives w.r.t t—treating X and H as formal symbols.

Example: $g(x) = x^3$

$$g(X+tH) = X^3 + tX^2H + tXHX + t^2XH^2 + tHX^2 + t^2HXH + t^2H^2X + t^3H^3.$$

From here, we can calculate:

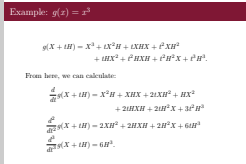
$$\frac{d}{dt}g(X+tH) = X^2H + XHX + 2tXH^2 + HX^2 + 2tHXH + 2tH^2X + 3t^2H^3$$

$$\frac{d^2}{dt^2}g(X+tH) = 2XH^2 + 2HXH + 2H^2X + 6tH^3$$

$$\frac{d^3}{dt^3}g(X+tH) = 6H^3.$$

Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Functional Calculus
Example: $g(x) = x^3$

2022-06-02



- Now we consider an example. Since $Df(X)[H]$ is linear, we can just work with a single monomial
- First we expand $(x+th)^3$—but we can't use the binomial theorem since x and h don't commute
- Once we expand, we take standard derivatives w.r.t t—treating X and H as formal symbols.

Example: $g(x) = x^3$

And so the first 3 directional derivatives are:

Example: $g(x) = x^3$
And so the first 3 directional derivatives are:

- Now all we have to do is set $t = 0$ in the previous expressions and we get the first three directional derivatives
- Note that they are all in the same direction—mixes derivatives are possible but we won't need them.

Example: $g(x) = x^3$

And so the first 3 directional derivatives are:

$$Df(X)[H] = X^2H + XHX + HX^2$$

Example: $g(x) = x^3$
And so the first 3 directional derivatives are:
$Df(X)[H] = X^2H + XHX + HX^2$

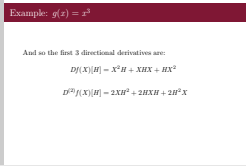
- Now all we have to do is set $t = 0$ in the previous expressions and we get the first three directional derivatives
- Note that they are all in the same direction—mixes derivatives are possible but we won't need them.

Example: $g(x) = x^3$

And so the first 3 directional derivatives are:

$$Df(X)[H] = X^2H + XHX + HX^2$$

$$D^{(2)}f(X)[H] = 2XH^2 + 2HXH + 2H^2X$$



- Now all we have to do is set $t = 0$ in the previous expressions and we get the first three directional derivatives
- Note that they are all in the same direction—mixes derivatives are possible but we won't need them.

Example: $g(x) = x^3$

And so the first 3 directional derivatives are:

$$Df(X)[H] = X^2H + XHX + HX^2$$

$$D^{(2)}f(X)[H] = 2XH^2 + 2HXH + 2H^2X$$

$$D^{(3)}f(X)[H] = 6H^3$$

Example: $g(x) = x^3$
And so the first 3 directional derivatives are:
$Df(X)[H] = X^2H + XHX + HX^2$
$D^{(2)}f(X)[H] = 2XH^2 + 2HXH + 2H^2X$
$D^{(3)}f(X)[H] = 6H^3$

- Now all we have to do is set $t = 0$ in the previous expressions and we get the first three directional derivatives
- Note that they are all in the same direction—mixes derivatives are possible but we won't need them.

Part I.5: Objects and Maps

A Second Attempt

- In seeking a more general theory we need to leave the world of this "SA functional calculus" behind.
- Rather than lifting functions to be matrix valued, we will define *new* objects that behave like those we just looked at.

Some Definitions

Definition: *Matrix Universe*
The g-dimensional **Matrix Universe** is

$$\mathcal{M}^g = \bigcup_{n\in\mathbb{N}} (M_n(\mathbb{C}))^g$$

Some Definitions
Definition: *Matrix Universe*
The g-dimensional **Matrix Universe** is
$\mathcal{M}^g = \bigcup_{n\in\mathbb{N}} (M_n(\mathbb{C}))^g$

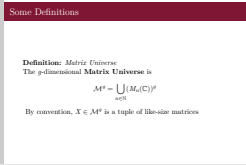
- Definition of the matrix universe — g tuples of matrices of all sizes
- By convetion, tuples are likes size

Some Definitions

Definition: *Matrix Universe*
The g-dimensional **Matrix Universe** is

$$\mathcal{M}^g = \bigcup_{n\in\mathbb{N}} (M_n(\mathbb{C}))^g$$

By convention, $X \in \mathcal{M}^g$ is a tuple of like-size matrices



- Definition of the matrix universe — g tuples of matrices of all sizes
- By convetion, tuples are likes size

Definition: *Free Set*
We say $D \subset \mathcal{M}^g$ is a **free set** if it is closed with respect to direct sums and unitary conjugation. That is,

.

Definition: *Free Set*
We say $D \subset \mathcal{M}^g$ is a **free set** if it is closed with respect to direct sums and unitary conjugation. That is,

.

- In math we often think about substructures that capture the implicit structure our space (subgroup, subspace, etc)
- In the nc setting, this is a *free set*, also called nc set
- direct sums and unitary conjugation are component wise
- If you see a D, you can assume that it is a free set.
- A subscript denotes a level-wise slice
- Note that this requires a lot of structure on free sets—we want to put a name to these structure.

Definition: *Free Set*
We say $D \subset \mathcal{M}^g$ is a **free set** if it is closed with respect to direct sums and unitary conjugation. That is,

1. $X, Y \in D$ means $X \oplus Y = (X_1 \oplus Y_1, \ldots, X_g \oplus Y_g) \in D$.

.

Definition: *Free Set*
We say $D \subset \mathcal{M}^g$ is a **free set** if it is closed with respect to direct sums and unitary conjugation. That is,
1. $X, Y \in D$ means $X \oplus Y = (X_1 \oplus Y_1, \ldots, X_g \oplus Y_g) \in D$.

- In math we often think about substructures that capture the implicit structure our space (subgroup, subspace, etc)
- In the nc setting, this is a *free set*, also called nc set
- direct sums and unitary conjugation are component wise
- If you see a D, you can assume that it is a free set.
- A subscript denotes a level-wise slice
- Note that this requires a lot of structure on free sets—we want to put a name to these structure.

Definition: *Free Set*
We say $D \subset \mathcal{M}^g$ is a **free set** if it is closed with respect to direct sums and unitary conjugation. That is,

1. $X, Y \in D$ means $X \oplus Y = (X_1 \oplus Y_1, \ldots, X_g \oplus Y_g) \in D$.
2. For X, U like-size matrices with U unitary and $X \in D$, then $UXU^* = (UX_1U^*, \ldots, UX_gU^*) \in D$.

.

2022-06-02
Searching for Holes in the Matrix Universe
Part I: Objects and Maps
Matrix Universe

Definition: *Free Set*
We say $D \subset \mathcal{M}^g$ is a **free set** if it is closed with respect to direct sums and unitary conjugation. That is,
- $X, Y \in D$ means $X \oplus Y = (X_1 \oplus Y_1, \ldots, X_g \oplus Y_g) \in D$.
- For X, U like-size matrices with U unitary and $X \in D$, then $UXU^* = (UX_1U^*, \ldots, UX_gU^*) \in D$.

.

- In math we often think about substructures that capture the implicit structure our space (subgroup, subspace, etc)
- In the nc setting, this is a *free set*, also called nc set
- direct sums and unitary conjugation are component wise
- If you see a D, you can assume that it is a free set.
- A subscript denotes a level-wise slice
- Note that this requires a lot of structure on free sets—we want to put a name to these structure.

Definition: *Free Set*
We say $D \subset \mathcal{M}^g$ is a **free set** if it is closed with respect to direct sums and unitary conjugation. That is,

1. $X, Y \in D$ means $X \oplus Y = (X_1 \oplus Y_1, \ldots, X_g \oplus Y_g) \in D$.
2. For X, U like-size matrices with U unitary and $X \in D$, then $UXU^* = (UX_1U^*, \ldots, UX_gU^*) \in D$.

For D a free set, define $D_n = D \cap (M_n(\mathbb{C}))^g$.

Definition: *Free Set*
We say $D \subset \mathcal{M}^g$ is a **free set** if it is closed with respect to direct sums and unitary conjugation. That is,
- $X, Y \in D$ means $X \oplus Y = (X_1 \oplus Y_1, \ldots, X_g \oplus Y_g) \in D$.
- For X, U like-size matrices with U unitary and $X \in D$, then $UXU^* = (UX_1U^*, \ldots, UX_gU^*) \in D$.

For D a free set, define $D_n = D \cap (M_n(\mathbb{C}))^g$.

- In math we often think about substructures that capture the implicit structure our space (subgroup, subspace, etc)
- In the nc setting, this is a *free set*, also called nc set
- direct sums and unitary conjugation are component wise
- If you see a D, you can assume that it is a free set.
- A subscript denotes a level-wise slice
- Note that this requires a lot of structure on free sets—we want to put a name to these structure.

Definition: *Fiber*
Given $X \in \mathcal{M}^g$, a tuple of $n \times n$ matrices, the **fiber** of X is the set

$$\{X^{\oplus k} \mid k \in \mathbb{N}\}.$$

Definition: *Fiber*
Given $X \in \mathcal{M}^g$, a tuple of $n \times n$ matrices, the **fiber** of X is the set
$\{X^{\oplus k} \mid k \in \mathbb{N}\}$.

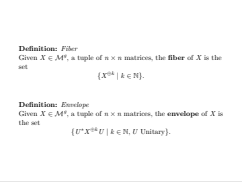
- The fiber is all the points "above" a given point. Conceptually, we imagine identification along the fiber—this will become important when we start doing topology
- The envelope (which will be less important to us) is the unitary smearing of the fiber at each level.

Definition: *Fiber*
Given $X \in \mathcal{M}^g$, a tuple of $n \times n$ matrices, the **fiber** of X is the set

$$\{X^{\oplus k} \mid k \in \mathbb{N}\}.$$

Definition: *Envelope*
Given $X \in \mathcal{M}^g$, a tuple of $n \times n$ matrices, the **envelope** of X is the set

$$\{U^* X^{\oplus k} U \mid k \in \mathbb{N},\ U \text{ Unitary}\}.$$



- The fiber is all the points "above" a given point. Conceptually, we imagine identification along the fiber—this will become important when we start doing topology
- The envelope (which will be less important to us) is the unitary smearing of the fiber at each level.

A bit of topology

A bit of topology

- If we are going to look for holes and build up the algebraic topology, we need a point set topology first—so there is a natural question.
- Bad news: there isn't a natural choice
- There are a handful of candidates (fine, fat, free, nc Zariski). I wish we had time to go into detail.
- For us, free sets are open if their level-wise restriction is open
- Other point-set characterizations are similar.

A bit of topology

What does it mean for $D \subset \mathcal{M}^g$ to be open?

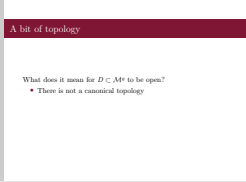
A bit of topology
What does it mean for $D \subset \mathcal{M}^g$ to be open?

- If we are going to look for holes and build up the algebraic topology, we need a point set topology first—so there is a natural question.
- Bad news: there isn't a natural choice
- There are a handful of candidates (fine, fat, free, nc Zariski). I wish we had time to go into detail.
- For us, free sets are open if their level-wise restriction is open
- Other point-set characterizations are similar.

A bit of topology

What does it mean for $D \subset \mathcal{M}^g$ to be open?

- There is not a canonical topology



- If we are going to look for holes and build up the algebraic topology, we need a point set topology first—so there is a natural question.
- Bad news: there isn't a natural choice
- There are a handful of candidates (fine, fat, free, nc Zariski). I wish we had time to go into detail.
- For us, free sets are open if their level-wise restriction is open
- Other point-set characterizations are similar.

A bit of topology

What does it mean for $D \subset \mathcal{M}^g$ to be open?

- There is not a canonical topology
- For us, we will say that D is open if each D_n is open.

A bit of topology
What does it mean for $D \subset \mathcal{M}^g$ to be open?
- There is not a canonical topology
- For us, we will say that D is open if each D_n is open.

- If we are going to look for holes and build up the algebraic topology, we need a point set topology first—so there is a natural question.
- Bad news: there isn't a natural choice
- There are a handful of candidates (fine, fat, free, nc Zariski). I wish we had time to go into detail.
- For us, free sets are open if their level-wise restriction is open
- Other point-set characterizations are similar.

A bit of topology

What does it mean for $D \subset \mathcal{M}^g$ to be open?

- There is not a canonical topology
- For us, we will say that D is open if each D_n is open.
- Simply connected, connected, bounded, etc. are defined similarly.

A bit of topology
What does it mean for $D \subset \mathcal{M}^g$ to be open?
- There is not a canonical topology
- For us, we will say that D is open if each D_n is open.
- Simply connected, connected, bounded, etc. are defined similarly.

- If we are going to look for holes and build up the algebraic topology, we need a point set topology first—so there is a natural question.
- Bad news: there isn't a natural choice
- There are a handful of candidates (fine, fat, free, nc Zariski). I wish we had time to go into detail.
- For us, free sets are open if their level-wise restriction is open
- Other point-set characterizations are similar.

What the natural functions on $\mathcal{M}^g$?

What the natural functions on $\mathcal{M}^g$?

- We have our objects, but what are the maps?
- Free functions are defined to be anything that behaves like a polynomial
- tracial functions look like traces
- For both of these maps, the directional derivative is defined identically as before—but tracial functions get something extra.

What the natural functions on $\mathcal{M}^g$?

Definition:
A function $f : D \to \mathcal{M}^{\hat{g}}$ is called **free** if

1. $f(X \oplus Y) = f(X) \oplus f(Y)$
2. $f(UXU^*) = f(U)f(X)f(U^*)$ where X and U are like-size and U is unitary.

What the natural functions on $\mathcal{M}^g$?
Definition:
A function $f : D \to \mathcal{M}^{\hat{g}}$ is called **free** if
1. $f(X \oplus Y) = f(X) \oplus f(Y)$
2. $f(UXU^*) = f(U)f(X)f(U^*)$ where X and U are like-size and U is unitary.

- We have our objects, but what are the maps?
- Free functions are defined to be anything that behaves like a polynomial
- tracial functions look like traces
- For both of these maps, the directional derivative is defined identically as before—but tracial functions get something extra.

What the natural functions on $\mathcal{M}^g$?

Definition:
A function $f : D \to \mathcal{M}^{\hat{g}}$ is called **free** if

1. $f(X \oplus Y) = f(X) \oplus f(Y)$
2. $f(UXU^*) = f(U)f(X)f(U^*)$ where X and U are like-size and U is unitary.

Definition:
A function $f : D \to \mathbb{C}$ is a **tracial function** if

1. $f(X \oplus Y) = f(X) + f(Y)$
2. $f(UXU^*) = f(X)$ where X and U are like-size and U is unitary.

What the natural functions on $\mathcal{M}^g$?

Definition:
A function $f : D \to \mathcal{M}^{\hat{g}}$ is called **free** if
- $f(X \oplus Y) = f(X) \oplus f(Y)$
- $f(UXU^*) = f(U)f(X)f(U^*)$ where X and U are like-size and U is unitary.

Definition:
A function $f : D \to \mathbb{C}$ is a **tracial function** if
- $f(X \oplus Y) = f(X) + f(Y)$
- $f(UXU^*) = f(X)$ where X and U are like-size and U is unitary.

- We have our objects, but what are the maps?
- Free functions are defined to be anything that behaves like a polynomial
- tracial functions look like traces
- For both of these maps, the directional derivative is defined identically as before—but tracial functions get something extra.

Definition: *Free Gradient*
Given a tracial function f, the free gradient, ∇f, is the unique free function satisfying

$$\operatorname{tr}\left(H \cdot \nabla f(X)\right) = Df(X)[H],$$

where, if $A = (A_1, \ldots, A_g)$ and $B = (B_1, \ldots, B_g)$ are tuples of like-size matrices then $A \cdot B = \sum_{i=1}^{g} A_i B_i$.

Definition: *Free Gradient*
Given a tracial function f, the free gradient, ∇f, is the unique free function satisfying

$$\operatorname{tr}\left(H \cdot \nabla f(X)\right) = Df(X)[H],$$

where, if $A = (A_1, \ldots, A_g)$ and $B = (B_1, \ldots, B_g)$ are tuples of like-size matrices then $A \cdot B = \sum_{i=1}^{g} A_i B_i$.

- The ∇ of a free function is the unique free function satisfy this equation—where the * is just like the dot product
- Whenever you see tr(*) I want you to think of the inner product—it is slightly distinct but it will make a lot of things make more sense
- Some of you may be hesitant at the fact that I claim ∇ is unique. Why should this be true?

Why should ∇f be unique?

Theorem (Trace Duality)

Let f, g be free functions $\mathcal{M}^g \to \mathcal{M}^{\tilde{g}}$. If $\operatorname{tr}(H \cdot f) = \operatorname{tr}(H \cdot g)$ for all tuples H, then $f = g$.

Why should ∇f be unique?
Theorem (Trace Duality)
Let f, g be free functions $\mathcal{M}^g \to \mathcal{M}^{\tilde{g}}$. If $\operatorname{tr}(H \cdot f) = \operatorname{tr}(H \cdot g)$ for all tuples H, then $f = g$.

- $f = g$ whenever the domains overlap
- In the vector space setting—with an inner product—this is a fairly immediate result. You would show it by picking vectors of all 0's and a single 1.
- You prove this identically—but with coordinate matrices instead of coordinate vectors.

Part II: Analytic Continuation and Monodromy

Searching for Holes in the Matrix Universe
└Part II: Monodromy

Analytic Continuation

Figures of generic analytic continuation

2022-06-02
Searching for Holes in the Matrix Universe
Part II: Monodromy
Analytic Continuation
Analytic Continuation

Analytic Continuation
Figures of generic analytic continuation

figures of the example of continuing log(x) through UHP and LHP

2022-06-02
Searching for Holes in the Matrix Universe
Part II: Monodromy
Analytic Continuation
figures of the example of continuing log(x) through UHP and LHP

When are two analytic continuations equal?

Theorem (Monodromy I)

Let γ_1, γ_2 be two paths from α to β and Γ_s be a fixed-endpoint homotopy between them. If f can be continued along Γ_s for all $s \in [0,1]$, then the continuations along γ_1 and γ_2 agree at β.

When are two analytic continuations equal?
Theorem (Monodromy I)
Let γ_1, γ_2 be two paths from α to β and Γ_s be a fixed-endpoint homotopy between them. If f can be continued along Γ_s for all $s \in [0,1]$, then the continuations along γ_1 and γ_2 agree at β.

When are two analytic continuations equal?

Theorem (Monodromy I)

Let γ_1, γ_2 be two paths from α to β and Γ_s be a fixed-endpoint homotopy between them. If f can be continued along Γ_s for all $s \in [0,1]$, then the continuations along γ_1 and γ_2 agree at β.

figure of the homotopy

When are two analytic continuations equal?

Theorem (Monodromy I)

Let γ_1, γ_2 be two paths from α to β and Γ_s be a fixed-endpoint homotopy between them. If f can be continued along Γ_s for all $s \in [0,1]$, then the continuations along γ_1 and γ_2 agree at β.

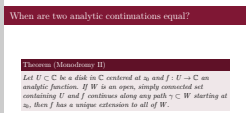
figure of the homotopy

When are two analytic continuations equal?

Theorem (Monodromy II)

Let $U \subset \mathbb{C}$ be a disk in $\mathbb{C}$ centered at z_0 and $f : U \to \mathbb{C}$ an analytic function. If W is an open, simply connected set containing U and f continues along any path $\gamma \subset W$ starting at z_0, then f has a unique extension to all of W.

2022-06-02
Searching for Holes in the Matrix Universe
Part II: Monodromy
Monodromy
When are two analytic continuations equal?



When are two analytic continuations equal?

Theorem (Monodromy II)

Let $U \subset \mathbb{C}$ be a disk in $\mathbb{C}$ centered at z_0 and $f : U \to \mathbb{C}$ an analytic function. If W is an open, simply connected set containing U and f continues along any path $\gamma \subset W$ starting at z_0, then f has a unique extension to all of W.

figure of the simply connected larger set

2022-06-02
Searching for Holes in the Matrix Universe
Part II: Monodromy
Monodromy
When are two analytic continuations equal?

When are two analytic continuations equal?

Theorem (Monodromy II)

Let $U \subset \mathbb{C}$ be a disk in $\mathbb{C}$ centered at z_0 and $f : U \to \mathbb{C}$ an analytic function. If W is an open, simply connected set containing U and f continues along any path $\gamma \subset W$ starting at z_0, then f has a unique extension to all of W.

figure of the simply connected larger set

What about the nc case?

Before we look at a free analogue of the monodromy theorem, we need to ask an important question: What does it mean for a free function to be analytic?

What about the nc case?
Before we look at a free analogue of the monodromy theorem, we need to ask an important question: What does it mean for a free function to be analytic?

- be sure to comment on the theorem—it is powerful!

What about the nc case?

Before we look at a free analogue of the monodromy theorem, we need to ask an important question: What does it mean for a free function to be analytic?

A: $f : D \to \mathcal{M}^{\hat{g}}$ is analytic if it is analytic as a function of each D_n.

What about the nc case?
Before we look at a free analogue of the monodromy theorem, we need to ask an important question: What does it mean for a free function to be analytic?
A: $f : D \to \mathcal{M}^{\hat{g}}$ is analytic if it is analytic as a function of each D_n.

- be sure to comment on the theorem—it is powerful!

What about the nc case?

Before we look at a free analogue of the monodromy theorem, we need to ask an important question: What does it mean for a free function to be analytic?

A: $f : D \to \mathcal{M}^{\hat{g}}$ is analytic if it is analytic as a function of each D_n.

Theorem (Agler, McCarthy (2016))

Let $f : D \to \mathcal{M}^{\hat{g}}$ be a free function. If f is locally bounded on each D_n, then f is an analytic free function.

What about the nc case?
Before we look at a free analogue of the monodromy theorem, we need to ask an important question: What does it mean for a free function to be analytic?
A: $f : D \to \mathcal{M}^{\hat{g}}$ is analytic if it is analytic as a function of each D_n.
Theorem (Agler, McCarthy (2016))
Let $f : D \to \mathcal{M}^{\hat{g}}$ be a free function. If f is locally bounded on each D_n, then f is an analytic free function.

- be sure to comment on the theorem—it is powerful!

Q: What about the nc case? Can we say anything similar?

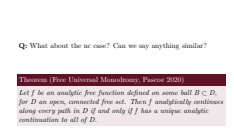
2022-06-02
Searching for Holes in the Matrix Universe
Part II: Monodromy
Free Monodromy
Q: What about the nc case? Can we say anything similar?

Q: What about the nc case? Can we say anything similar?

Theorem (Free Universal Monodromy, Pascoe 2020)

Let f be an analytic free function defined on some ball $B \subset D$, for D an open, connected free set. Then f analytically continues along every path in D if and only if f has a unique analytic continuation to all of D.

2022-06-02
Searching for Holes in the Matrix Universe
Part II: Monodromy
Free Monodromy



Consequences of Free Monodromy

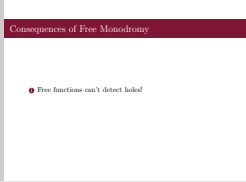
2022-06-02
Searching for Holes in the Matrix Universe
└Part II: Monodromy
└Free Monodromy
└Consequences of Free Monodromy

Consequences of Free Monodromy

Consequences of Free Monodromy

1. Free functions can't detect holes!

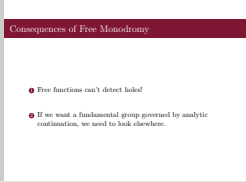
2022-06-02
Searching for Holes in the Matrix Universe
Part II: Monodromy
Free Monodromy
Consequences of Free Monodromy



Consequences of Free Monodromy

1. Free functions can't detect holes!
2. If we want a fundamental group governed by analytic continuation, we need to look elsewhere.

2022-06-02
Searching for Holes in the Matrix Universe
Part II: Monodromy
Free Monodromy
Consequences of Free Monodromy



Part III: Homotopy

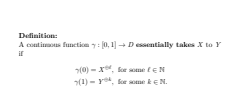
Searching for Holes in the Matrix Universe
└Part III: Homotopy

Part III: Homotopy

Definition:
A continuous function $\gamma : [0,1] \to D$ **essentially takes** X to Y if

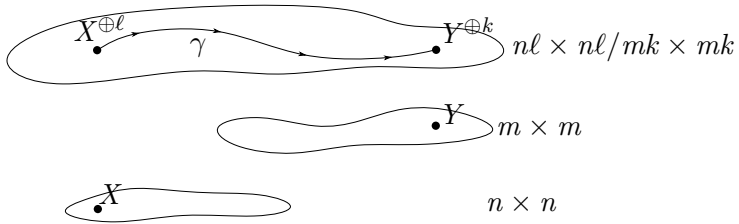
$$\gamma(0) = X^{\oplus \ell}, \text{ for some } \ell \in \mathbb{N}$$
$$\gamma(1) = Y^{\oplus k}, \text{ for some } k \in \mathbb{N}.$$

2022-06-02
Searching for Holes in the Matrix Universe
Part III: Homotopy
A First Fundamental Group

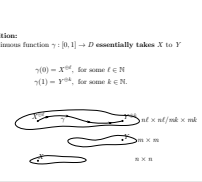


Definition:
A continuous function $\gamma : [0, 1] \to D$ **essentially takes** X to Y if

$$\gamma(0) = X^{\oplus \ell}, \text{ for some } \ell \in \mathbb{N}$$
$$\gamma(1) = Y^{\oplus k}, \text{ for some } k \in \mathbb{N}.$$



2022-06-02
Searching for Holes in the Matrix Universe
Part III: Homotopy
A First Fundamental Group



Given γ essentially taking X to Y and β taking Z to W, define

$$\gamma \oplus \beta(t) := \begin{bmatrix} \gamma(t) & 0 \\ 0 & \beta(t) \end{bmatrix}.$$

2022-06-02
Searching for Holes in the Matrix Universe
Part III: Homotopy
A First Fundamental Group

Given γ essentially taking X to Y and β taking Z to W, define
$\gamma \oplus \beta(t) := \begin{bmatrix} \gamma(t) & 0 \\ 0 & \beta(t) \end{bmatrix}.$

Given γ essentially taking X to Y and β taking Z to W, define

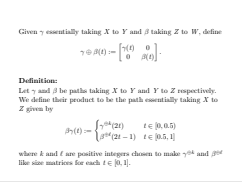
$$\gamma \oplus \beta(t) := \begin{bmatrix} \gamma(t) & 0 \\ 0 & \beta(t) \end{bmatrix}.$$

Definition:
Let γ and β be paths taking X to Y and Y to Z respectively. We define their product to be the path essentially taking X to Z given by

$$\beta\gamma(t) := \begin{cases} \gamma^{\oplus k}(2t) & t \in [0, 0.5) \\ \beta^{\oplus \ell}(2t-1) & t \in [0.5, 1] \end{cases}$$

where k and ℓ are positive integers chosen to make $\gamma^{\oplus k}$ and $\beta^{\oplus \ell}$ like size matrices for each $t \in [0, 1]$.

2022-06-02
Searching for Holes in the Matrix Universe
Part III: Homotopy
A First Fundamental Group



The Full Fundamental Group

For $D \subset \mathcal{M}^g$ a connected free set, the **full fundamenal group**, $\pi_1(D)$, is the group of paths essentially taking X to X up to homotopy equivalence and the relation $\gamma = \gamma^{\oplus k}$.

The Full Fundamental Group

For $D \subset \mathcal{M}^g$ a connected free set, the **full fundamenal group**, $\pi_1(D)$, is the group of paths essentially taking X to X up to homotopy equivalence and the relation $\gamma = \gamma^{\oplus k}$.

- remark about how this misses the link to analytic continuation of functions, but free functions won't do.

Let $D \subset \mathcal{M}^g$ be a connected, open, free set. If there exists a nonempty, simply-connected, open, free $B \subset D$, then we say that D is **anchored**.

2022-06-02
Searching for Holes in the Matrix Universe
Part III: Homotopy
A Second Fundamental Group

Let $D \subset \mathcal{M}^g$ be a connected, open, free set. If there exists a nonempty, simply-connected, open, free $B \subset D$, then we say that D is **anchored**.

Let $D \subset \mathcal{M}^g$ be a connected, open, free set. If there exists a nonempty, simply-connected, open, free $B \subset D$, then we say that D is **anchored**.

For D an anchored set, and $B \subset D$ its anchor, we call a tracial function $f : B \to \mathbb{C}$ a **global germ** if it analytically continues along every path in D which starts in B.

2022-06-02
Searching for Holes in the Matrix Universe
Part III: Homotopy
A Second Fundamental Group

Let $D \subset \mathcal{M}^g$ be a connected, open, free set. If there exists a nonempty, simply-connected, open, free $B \subset D$, then we say that D is **anchored**.

For D an anchored set, and $B \subset D$ its anchor, we call a tracial function $f : B \to \mathbb{C}$ a **global germ** if it analytically continues along every path in D which starts in B.

Let $D \subset \mathcal{M}^g$ be a connected, open, free set. If there exists a nonempty, simply-connected, open, free $B \subset D$, then we say that D is **anchored**.

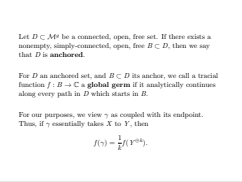
For D an anchored set, and $B \subset D$ its anchor, we call a tracial function $f : B \to \mathbb{C}$ a **global germ** if it analytically continues along every path in D which starts in B.

For our purposes, we view γ as coupled with its endpoint. Thus, if γ essentially takes X to Y, then

$$f(\gamma) = \frac{1}{k} f(Y^{\oplus k}).$$

Searching for Holes in the Matrix Universe
Part III: Homotopy
A Second Fundamental Group

2022-06-02

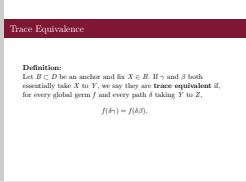


Trace Equivalence

Definition:
Let $B \subset D$ be an anchor and fix $X \in B$. If γ and β both essentially take X to Y, we say they are **trace equivalent** if, for every global germ f and every path δ taking Y to Z,

$$f(\delta\gamma) = f(\delta\beta).$$

2022-06-02
Searching for Holes in the Matrix Universe
Part III: Homotopy
A Second Fundamental Group
Trace Equivalence



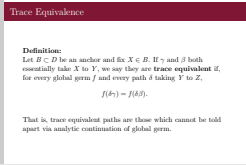
Trace Equivalence

Definition:
Let $B \subset D$ be an anchor and fix $X \in B$. If γ and β both essentially take X to Y, we say they are **trace equivalent** if, for every global germ f and every path δ taking Y to Z,

$$f(\delta\gamma) = f(\delta\beta).$$

That is, trace equivalent paths are those which cannot be told apart via analytic continuation of global germ.

2022-06-02
Searching for Holes in the Matrix Universe
Part III: Homotopy
A Second Fundamental Group
Trace Equivalence



The Tracial Fundamental Group

The Tracial Fundamental Group

- remark that the choice of base point is irrelevant. and that it is a quotient of the full fundamental group

The Tracial Fundamental Group

Let $D \subset \mathcal{M}^g$ be an anchored space with B is anchor. For $X \in B$ define $\pi_1^{\text{tr}}(D)$ to be the group of trace equivalent paths essentially taking X to X.

The Tracial Fundamental Group
Let $D \subset \mathcal{M}^g$ be an anchored space with B is anchor. For $X \in B$ define $\pi_1^{\text{tr}}(D)$ to be the group of trace equivalent paths essentially taking X to X.

- remark that the choice of base point is irrelevant. and that it is a quotient of the full fundamental group

The Tracial Fundamental Group

Let $D \subset \mathcal{M}^g$ be an anchored space with B is anchor. For $X \in B$ define $\pi_1^{\text{tr}}(D)$ to be the group of trace equivalent paths essentially taking X to X. Computationally, we are still stuck.

The Tracial Fundamental Group
Let $D \subset \mathcal{M}^g$ be an anchored space with B is anchor. For $X \in B$ define $\pi_1^{\text{tr}}(D)$ to be the group of trace equivalent paths essentially taking X to X. Computationally, we are still stuck.

- remark that the choice of base point is irrelevant. and that it is a quotient of the full fundamental group

Part IV: Cohomology

Searching for Holes in the Matrix Universe
└Part IV: Cohomology

Part IV: Cohomology

2022-06-02
Searching for Holes in the Matrix Universe
└Part IV: Cohomology
└A Short Review of Cohomology

Traditional homology considers a complex of the form

$$\cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} \cdots$$

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
A Short Review of Cohomology

Traditional homology considers a complex of the form
$\cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} \cdots$

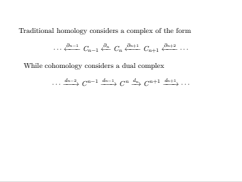
Traditional homology considers a complex of the form

$$\cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} \cdots$$

While cohomology considers a dual complex

$$\cdots \xrightarrow{d_{n-2}} C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \xrightarrow{d_{n+1}} \cdots$$

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
A Short Review of Cohomology



Traditional homology considers a complex of the form

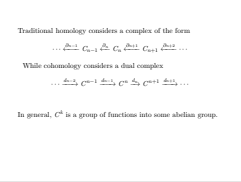
$$\cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} \cdots$$

While cohomology considers a dual complex

$$\cdots \xrightarrow{d_{n-2}} C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \xrightarrow{d_{n+1}} \cdots$$

In general, C^k is a group of functions into some abelian group.

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
A Short Review of Cohomology



Traditional homology considers a complex of the form

$$\cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} \cdots$$

While cohomology considers a dual complex

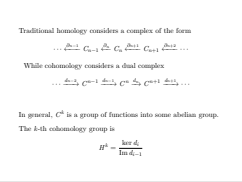
$$\cdots \xrightarrow{d_{n-2}} C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \xrightarrow{d_{n+1}} \cdots$$

In general, C^k is a group of functions into some abelian group.

The k-th cohomology group is

$$H^k = \frac{\ker d_i}{\operatorname{Im} d_{i-1}}$$

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
A Short Review of Cohomology



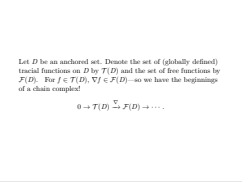
Let D be an anchored set. Denote the set of (globally defined) tracial functions on D by $\mathcal{T}(D)$ and the set of free functions by $\mathcal{F}(D)$.

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Tracial Cohomology
Let D be an anchored set. Denote the set of (globally defined) tracial functions on D by $\mathcal{T}(D)$ and the set of free functions by $\mathcal{F}(D)$.

Let D be an anchored set. Denote the set of (globally defined) tracial functions on D by $\mathcal{T}(D)$ and the set of free functions by $\mathcal{F}(D)$. For $f \in \mathcal{T}(D)$, $\nabla f \in \mathcal{F}(D)$—so we have the beginnings of a chain complex!

$$0 \to \mathcal{T}(D) \xrightarrow{\nabla} \mathcal{F}(D) \to \cdots .$$

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Tracial Cohomology



$$0 \to \mathcal{T}(D) \xrightarrow{\nabla} \mathcal{F}(D) \to \cdots .$$

Searching for Holes in the Matrix Universe
Part IV: Cohomology
Tracial Cohomology
2022-06-02

$0 \to \mathcal{T}(D) \xrightarrow{\nabla} \mathcal{F}(D) \to \cdots .$

- test text

$$0 \to \mathcal{T}(D) \xrightarrow{\nabla} \mathcal{F}(D) \to \cdots .$$

A free function $g : D \to \mathcal{M}^g$ is **exact** if there exists a tracial function $f : D \to \mathbb{C}$ such that $\nabla f = g$.

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Tracial Cohomology

$0 \to \mathcal{T}(D) \xrightarrow{\nabla} \mathcal{F}(D) \to \cdots .$

A free function $g : D \to \mathcal{M}^g$ is **exact** if there exists a tracial function $f : D \to \mathbb{C}$ such that $\nabla f = g$.

- test text

$$0 \to \mathcal{T}(D) \xrightarrow{\nabla} \mathcal{F}(D) \to \cdots .$$

A free function $g : D \to \mathcal{M}^g$ is **exact** if there exists a tracial function $f : D \to \mathbb{C}$ such that $\nabla f = g$.

A free function $g : D \to \mathcal{M}^g$ is **closed** if

$$\operatorname{tr}\left(K \cdot Dg(X)[H]\right) = \operatorname{tr}\left(H \cdot Dg(X)[K]\right)$$

for all directions H, K.

$0 \to \mathcal{T}(D) \xrightarrow{\nabla} \mathcal{F}(D) \to \cdots .$

A free function $g : D \to \mathcal{M}^g$ is **exact** if there exists a tracial function $f : D \to \mathbb{C}$ such that $\nabla f = g$.

A free function $g : D \to \mathcal{M}^g$ is **closed** if

$\operatorname{tr}\left(K \cdot Dg(X)[H]\right) = \operatorname{tr}\left(H \cdot Dg(X)[K]\right)$

for all directions H, K.

- test text

$$0 \to \mathcal{T}(D) \xrightarrow{\nabla} \mathcal{F}(D) \to \cdots .$$

A free function $g : D \to \mathcal{M}^g$ is **exact** if there exists a tracial function $f : D \to \mathbb{C}$ such that $\nabla f = g$.

A free function $g : D \to \mathcal{M}^g$ is **closed** if

$$\operatorname{tr}\left(K \cdot Dg(X)[H]\right) = \operatorname{tr}\left(H \cdot Dg(X)[K]\right)$$

for all directions H, K.

Definition:
The **first tracial cohomology group** is the vector space of closed free functions moduluo the exact free function. We write $H^1_{\mathrm{tr}}(D)$.

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Tracial Cohomology

- test text

What out global germs?

- For $f : B \to \mathbb{C}$ a global germ, since f analytically continues along every path, so does ∇f.

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Tracial Cohomology
What out global germs?

What out global germs?
- For $f : B \to \mathbb{C}$ a global germ, since f analytically continues along every path, so does ∇f.

What out global germs?

- For $f : B \to \mathbb{C}$ a global germ, since f analytically continues along every path, so does ∇f.
- Free Monodromy means that ∇f has a global extension.

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Tracial Cohomology
What out global germs?

What out global germs?
- For $f : B \to \mathbb{C}$ a global germ, since f analytically continues along every path, so does ∇f.
- Free Monodromy means that ∇f has a global extension.

What out global germs?

- For $f : B \to \mathbb{C}$ a global germ, since f analytically continues along every path, so does ∇f.
- Free Monodromy means that ∇f has a global extension.
- **Important:** If f is a global germ, then ∇f is not necessarily exact since $\mathcal{T}(D)$ is the set of tracial functions defined on *all of* D.

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Tracial Cohomology
What out global germs?

What out global germs?
- For $f : B \to \mathbb{C}$ a global germ, since f analytically continues along every path, so does ∇f.
- Free Monodromy means that ∇f has a global extension.
- **Important:** If f is a global germ, then ∇f is not necessarily exact since $\mathcal{T}(D)$ is the set of tracial functions defined on *all of* D.

Goal: Show that $\pi_1^{\mathrm{tr}}(D)$ injects into $\mathbb{C}$.

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Injecting into $\mathbb{C}$
Goal: Show that $\pi_1^{\mathrm{tr}}(D)$ injects into $\mathbb{C}$.

Goal: Show that $\pi_1^{\mathrm{tr}}(D)$ injects into $\mathbb{C}$.

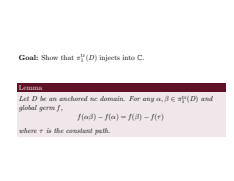
Lemma

Let D be an anchored nc domain. For any $\alpha, \beta \in \pi_1^{\mathrm{tr}}(D)$ and global germ f,

$$f(\alpha\beta) - f(\alpha) = f(\beta) - f(\tau)$$

where τ is the constant path.

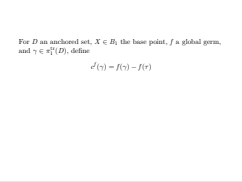
2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Injecting into $\mathbb{C}$



For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{\mathrm{tr}}(D)$, define

$$c^f(\gamma) = f(\gamma) - f(\tau)$$

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Injecting into $\mathbb{C}$



For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{\mathrm{tr}}(D)$, define

$$c^f(\gamma) = f(\gamma) - f(\tau)$$

c^f gives us a homomorphism into $\mathbb{C}$!

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Injecting into $\mathbb{C}$

For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{\mathrm{tr}}(D)$, define
$c^f(\gamma) = f(\gamma) - f(\tau)$
c^f gives us a homomorphism into $\mathbb{C}$!

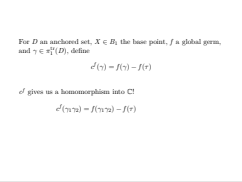
For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{\mathrm{tr}}(D)$, define

$$c^f(\gamma) = f(\gamma) - f(\tau)$$

c^f gives us a homomorphism into $\mathbb{C}$!

$$c^f(\gamma_1\gamma_2) = f(\gamma_1\gamma_2) - f(\tau)$$

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Injecting into $\mathbb{C}$



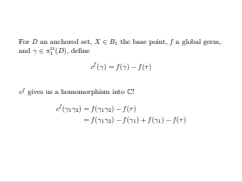
For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{\mathrm{tr}}(D)$, define

$$c^f(\gamma) = f(\gamma) - f(\tau)$$

c^f gives us a homomorphism into $\mathbb{C}$!

$$\begin{aligned} c^f(\gamma_1\gamma_2) &= f(\gamma_1\gamma_2) - f(\tau) \\ &= f(\gamma_1\gamma_2) - f(\gamma_1) + f(\gamma_1) - f(\tau) \end{aligned}$$

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Injecting into $\mathbb{C}$



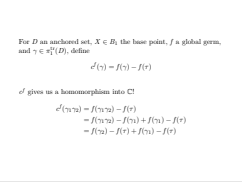
For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{\mathrm{tr}}(D)$, define

$$c^f(\gamma) = f(\gamma) - f(\tau)$$

c^f gives us a homomorphism into $\mathbb{C}$!

$$\begin{aligned} c^f(\gamma_1\gamma_2) &= f(\gamma_1\gamma_2) - f(\tau) \\ &= f(\gamma_1\gamma_2) - f(\gamma_1) + f(\gamma_1) - f(\tau) \\ &= f(\gamma_2) - f(\tau) + f(\gamma_1) - f(\tau) \end{aligned}$$

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Injecting into $\mathbb{C}$



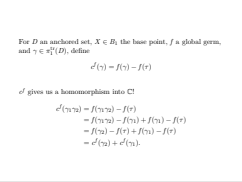
For D an anchored set, $X \in B_1$ the base point, f a global germ, and $\gamma \in \pi_1^{\mathrm{tr}}(D)$, define

$$c^f(\gamma) = f(\gamma) - f(\tau)$$

c^f gives us a homomorphism into $\mathbb{C}$!

$$\begin{aligned} c^f(\gamma_1\gamma_2) &= f(\gamma_1\gamma_2) - f(\tau) \\ &= f(\gamma_1\gamma_2) - f(\gamma_1) + f(\gamma_1) - f(\tau) \\ &= f(\gamma_2) - f(\tau) + f(\gamma_1) - f(\tau) \\ &= c^f(\gamma_2) + c^f(\gamma_1). \end{aligned}$$

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Injecting into $\mathbb{C}$



Lemma

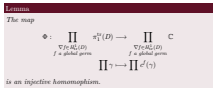
The map

$$\Phi : \prod_{\substack{\nabla f \in H^1_{\mathrm{tr}}(D) \\ f \text{ a global germ}}} \pi_1^{\mathrm{tr}}(D) \longrightarrow \prod_{\substack{\nabla f \in H^1_{\mathrm{tr}}(D) \\ f \text{ a global germ}}} \mathbb{C}$$

$$\prod \gamma \longmapsto \prod c^f(\gamma)$$

is an injective homomophism.

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Injecting into $\mathbb{C}$



Lemma

The map

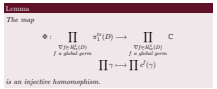
$$\Phi : \prod_{\substack{\nabla f \in H^1_{\mathrm{tr}}(D) \\ f \text{ a global germ}}} \pi_1^{\mathrm{tr}}(D) \longrightarrow \prod_{\substack{\nabla f \in H^1_{\mathrm{tr}}(D) \\ f \text{ a global germ}}} \mathbb{C}$$

$$\prod \gamma \longmapsto \prod c^f(\gamma)$$

is an injective homomophism.

So $\pi_1^{\mathrm{tr}}(D)$ is commutative and torsion free!

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Injecting into $\mathbb{C}$



$\pi_1^{\mathrm{tr}}(D)$ is divisible

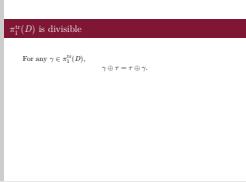
2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Characterizing $\pi_1^{\mathrm{tr}}(D)$
$\pi_1^{\mathrm{tr}}(D)$ is divisible

$\pi_1^{\mathrm{tr}}(D)$ is divisible

For any $\gamma \in \pi_1^{\mathrm{tr}}(D)$,

$$\gamma \oplus \tau = \tau \oplus \gamma.$$

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Characterizing $\pi_1^{\mathrm{tr}}(D)$
$\pi_1^{\mathrm{tr}}(D)$ is divisible



$\pi_1^{\mathrm{tr}}(D)$ is divisible

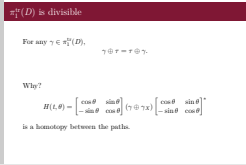
For any $\gamma \in \pi_1^{\mathrm{tr}}(D)$,

$$\gamma \oplus \tau = \tau \oplus \gamma.$$

Why?

$$H(t,\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} (\gamma \oplus \gamma_X) \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^*$$

is a homotopy between the paths.



$$\gamma = \underbrace{\begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \gamma \end{bmatrix}}_{k+1\text{-times}}$$

Searching for Holes in the Matrix Universe
└Part IV: Cohomology
└Characterizing $\pi_1^{\mathrm{tr}}(D)$

2022-06-02

$$\gamma = \underbrace{\begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \gamma \end{bmatrix}}_{k+1\text{-times}}$$
$$= \begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \tau \end{bmatrix} \begin{bmatrix} \tau & & \\ & \ddots & \\ & & \tau \end{bmatrix} \cdots \begin{bmatrix} \tau & & \\ & \ddots & \\ & & \gamma \end{bmatrix}$$

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Characterizing $\pi_1^{\mathrm{tr}}(D)$

$$\gamma = \underbrace{\begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \gamma \end{bmatrix}}_{k+1\text{-times}}$$
$$= \begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \tau \end{bmatrix} \begin{bmatrix} \tau & & \\ & \ddots & \\ & & \tau \end{bmatrix} \cdots \begin{bmatrix} \tau & & \\ & \ddots & \\ & & \gamma \end{bmatrix}$$
$$= \begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \gamma_X \end{bmatrix} \begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \tau \end{bmatrix} \cdots \begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \tau \end{bmatrix}$$

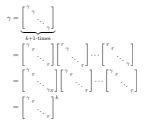
2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Characterizing $\pi_1^{\mathrm{tr}}(D)$

$$\gamma = \underbrace{\begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \gamma \end{bmatrix}}_{k+1\text{-times}}$$

$$= \begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \tau \end{bmatrix} \begin{bmatrix} \tau & & \\ & \ddots & \\ & & \tau \end{bmatrix} \cdots \begin{bmatrix} \tau & & \\ & \ddots & \\ & & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \gamma_X \end{bmatrix} \begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \tau \end{bmatrix} \cdots \begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \tau \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \tau \end{bmatrix}^k$$



Theorem

For D an anchored free set, $\pi_1^{\mathrm{tr}}(D)$ is a torsion free, abelian, divisible group. That is,

$$\pi_1^{\mathrm{tr}}(D) \simeq \bigoplus_{i \in I} \mathbb{Q} = \mathbb{Q}^I$$

for some set I.

2022-06-02
Searching for Holes in the Matrix Universe
Part IV: Cohomology
Characterizing $\pi_1^{\mathrm{tr}}(D)$

Theorem
For D an anchored free set, $\pi_1^{\mathrm{tr}}(D)$ is a torsion free, abelian, divisible group. That is,
$\pi_1^{\mathrm{tr}}(D) \simeq \bigoplus_{i \in I} \mathbb{Q} = \mathbb{Q}^I$
for some set I.

Part V: Computing $\pi_1^{\mathrm{tr}}(D)$

- hard bc no VC or MV

Let D be an anchored, free, path connected set such that each D_n is nonempty and choose an anchor $B \subset D$ such that each B_n is also nonempty.

2022-06-02
Searching for Holes in the Matrix Universe
└Computing $\pi_1^{\mathrm{tr}}(D)$
└The Direct Limit Approach

Let D be an anchored, free, path connected set such that each D_n is nonempty and choose an anchor $B \subset D$ such that each B_n is also nonempty.

Let D be an anchored, free, path connected set such that each D_n is nonempty and choose an anchor $B \subset D$ such that each B_n is also nonempty.

Let $\pi_1^{\mathrm{tr}}(D)_n$ is the subgroup of paths in D_n. Note that $\pi_1^{\mathrm{tr}}(D)_1$ is a quotient of $\pi_1(D_n)$.

2022-06-02
Searching for Holes in the Matrix Universe
Computing $\pi_1^{\mathrm{tr}}(D)$
The Direct Limit Approach

Let D be an anchored, free, path connected set such that each D_n is nonempty and choose an anchor $B \subset D$ such that each B_n is also nonempty.

Let $\pi_1^{\mathrm{tr}}(D)_n$ is the subgroup of paths in D_n. Note that $\pi_1^{\mathrm{tr}}(D)_1$ is a quotient of $\pi_1(D_n)$.

Let D be an anchored, free, path connected set such that each D_n is nonempty and choose an anchor $B \subset D$ such that each B_n is also nonempty.

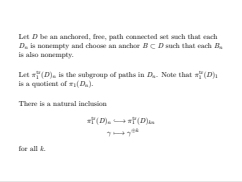
Let $\pi_1^{\mathrm{tr}}(D)_n$ is the subgroup of paths in D_n. Note that $\pi_1^{\mathrm{tr}}(D)_1$ is a quotient of $\pi_1(D_n)$.

There is a natural inclusion

$$\begin{aligned} \pi_1^{\mathrm{tr}}(D)_n &\hookrightarrow \pi_1^{\mathrm{tr}}(D)_{kn} \\ \gamma &\longmapsto \gamma^{\oplus k} \end{aligned}$$

for all k.

2022-06-02
Searching for Holes in the Matrix Universe
Computing $\pi_1^{\mathrm{tr}}(D)$
The Direct Limit Approach



Now consider the chain of inclusions:

$$\pi_1^{\mathrm{tr}}(D)_1 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_2 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_6 \hookrightarrow \cdots \hookrightarrow \pi_1^{\mathrm{tr}}(D)_{n!} \hookrightarrow \cdots$$

2022-06-02
Searching for Holes in the Matrix Universe
Computing $\pi_1^{\mathrm{tr}}(D)$
The Direct Limit Approach

Now consider the chain of inclusions:
$\pi_1^{\mathrm{tr}}(D)_1 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_2 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_6 \hookrightarrow \cdots \hookrightarrow \pi_1^{\mathrm{tr}}(D)_{n!} \hookrightarrow \cdots$

Now consider the chain of inclusions:

$$\pi_1^{\mathrm{tr}}(D)_1 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_2 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_6 \hookrightarrow \cdots \hookrightarrow \pi_1^{\mathrm{tr}}(D)_{n!} \hookrightarrow \cdots$$

The limit of this sequence isomorphic to $\pi_1^{\mathrm{tr}}(D)$!

Now consider the chain of inclusions:
$\pi_1^{\mathrm{tr}}(D)_1 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_2 \hookrightarrow \pi_1^{\mathrm{tr}}(D)_6 \hookrightarrow \cdots \hookrightarrow \pi_1^{\mathrm{tr}}(D)_{n!} \hookrightarrow \cdots$
The limit of this sequence isomorphic to $\pi_1^{\mathrm{tr}}(D)$!

Example: $\pi_1^{\mathrm{tr}}(GL)$

Let $GL = \bigcup_{n\in\mathbb{N}} GL_n(\mathbb{C})$.

as well.

Example: $\pi_1^{\mathrm{tr}}(GL)$
Let $GL = \bigcup_{n\in\mathbb{N}} GL_n(\mathbb{C})$.
as well.

Example: $\pi_1^{\mathrm{tr}}(GL)$

Let $GL = \bigcup_{n\in\mathbb{N}} GL_n(\mathbb{C})$.

$GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. Since $\pi_1(GL_1) \simeq \mathbb{Z}$, $\pi_1^{\mathrm{tr}}(GL)_1 \simeq \mathbb{Z}$ as well.

Example: $\pi_1^{\mathrm{tr}}(GL)$
Let $GL = \bigcup_{n\in\mathbb{N}} GL_n(\mathbb{C})$.
$GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. Since $\pi_1(GL_1) \simeq \mathbb{Z}$, $\pi_1^{\mathrm{tr}}(GL)_1 \simeq \mathbb{Z}$ as well.

Example: $\pi_1^{\mathrm{tr}}(GL)$

Let $GL = \bigcup_{n\in\mathbb{N}} GL_n(\mathbb{C})$.

$GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. Since $\pi_1(GL_1) \simeq \mathbb{Z}$, $\pi_1^{\mathrm{tr}}(GL)_1 \simeq \mathbb{Z}$ as well.

Inclusion into $\pi_1^{\mathrm{tr}}(GL)_2$ picks up square roots. If $\gamma \in \pi_1^{\mathrm{tr}}(GL)_1$, then

$$\begin{bmatrix} \gamma & \\ & \tau \end{bmatrix} \in \pi_1^{\mathrm{tr}}(GL)_2.$$

Example: $\pi_1^{\mathrm{tr}}(GL)$

Let $GL = \bigcup_{n\in\mathbb{N}} GL_n(\mathbb{C})$.

$GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. Since $\pi_1(GL_1) \simeq \mathbb{Z}$, $\pi_1^{\mathrm{tr}}(GL)_1 \simeq \mathbb{Z}$ as well.

Inclusion into $\pi_1^{\mathrm{tr}}(GL)_2$ picks up square roots. If $\gamma \in \pi_1^{\mathrm{tr}}(GL)_1$, then

$\begin{bmatrix} \gamma & \\ & \tau \end{bmatrix} \in \pi_1^{\mathrm{tr}}(GL)_2$.

Example: $\pi_1^{\mathrm{tr}}(GL)$

Let $GL = \bigcup_{n\in\mathbb{N}} GL_n(\mathbb{C})$.

$GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. Since $\pi_1(GL_1) \simeq \mathbb{Z}$, $\pi_1^{\mathrm{tr}}(GL)_1 \simeq \mathbb{Z}$ as well.

Inclusion into $\pi_1^{\mathrm{tr}}(GL)_2$ picks up square roots. If $\gamma \in \pi_1^{\mathrm{tr}}(GL)_1$, then

$$\begin{bmatrix} \gamma & \\ & \tau \end{bmatrix} \in \pi_1^{\mathrm{tr}}(GL)_2.$$

Thus, $\pi_1^{\mathrm{tr}}(GL)_2 \simeq \mathbb{Z}\left[\frac{1}{2}\right]$.

Example: $\pi_1^{\mathrm{tr}}(GL)$

Let $GL = \bigcup_{n\in\mathbb{N}} GL_n(\mathbb{C})$.

$GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. Since $\pi_1(GL_1) \simeq \mathbb{Z}$, $\pi_1^{\mathrm{tr}}(GL)_1 \simeq \mathbb{Z}$ as well.

Inclusion into $\pi_1^{\mathrm{tr}}(GL)_2$ picks up square roots. If $\gamma \in \pi_1^{\mathrm{tr}}(GL)_1$, then

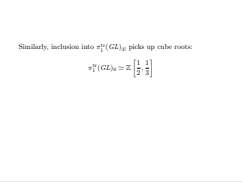
$$\begin{bmatrix} \gamma & \\ & \tau \end{bmatrix} \in \pi_1^{\mathrm{tr}}(GL)_2.$$

Thus, $\pi_1^{\mathrm{tr}}(GL)_2 \simeq \mathbb{Z}\left[\frac{1}{2}\right]$.

Similarly, inclusion into $\pi_1^{\mathrm{tr}}(GL)_{3!}$ picks up cube roots:

$$\pi_1^{\mathrm{tr}}(GL)_6 \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$$

2022-06-02
Searching for Holes in the Matrix Universe
Computing $\pi_1^{\mathrm{tr}}(D)$
An Example



Similarly, inclusion into $\pi_1^{\mathrm{tr}}(GL)_{3!}$ picks up cube roots:

$$\pi_1^{\mathrm{tr}}(GL)_6 \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$$

In the n-th inclusion, we pick up n-th roots and so we adjoin $\frac{1}{n}$ to the preceding group. Therefore,

$$\pi_1^{\mathrm{tr}}(GL) \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right]$$

Similarly, inclusion into $\pi_1^{\mathrm{tr}}(GL)_{3!}$ picks up cube roots:

$$\pi_1^{\mathrm{tr}}(GL)_6 \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$$

In the n-th inclusion, we pick up n-th roots and so we adjoin $\frac{1}{n}$ to the preceding group. Therefore,

$$\pi_1^{\mathrm{tr}}(GL) \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right]$$

Similarly, inclusion into $\pi_1^{\mathrm{tr}}(GL)_{3!}$ picks up cube roots:

$$\pi_1^{\mathrm{tr}}(GL)_6 \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$$

In the n-th inclusion, we pick up n-th roots and so we adjoin $\frac{1}{n}$ to the preceding group. Therefore,

$$\pi_1^{\mathrm{tr}}(GL) \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right] \simeq \mathbb{Q}.$$

Similarly, inclusion into $\pi_1^{\mathrm{tr}}(GL)_{3!}$ picks up cube roots:

$$\pi_1^{\mathrm{tr}}(GL)_6 \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$$

In the n-th inclusion, we pick up n-th roots and so we adjoin $\frac{1}{n}$ to the preceding group. Therefore,

$$\pi_1^{\mathrm{tr}}(GL) \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right] \simeq \mathbb{Q}.$$

Thank You!

Searching for Holes in the Matrix Universe
Computing $\pi_1^{\mathrm{tr}}(D)$
An Example

Thank You!