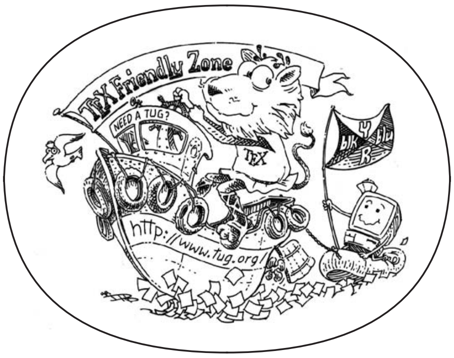
A CLEAN TITLE

LUCAS KERBS



A Fun Subtitle

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All our dignity consists therefore of thought.
It is from there that we must be lifted up and not
from time and space, which we could never fill.
So let us work on thinking well.

— Blaise Pascal

For all those who taught me to love learning.

ABSTRACT

A newcomer the greater story of mathematics, free analysis seeks to understand functions that accept matrices of arbitrary sizes as inputs. Despite the fact that these functions are naturally noncommutative, many classical results from the study of complex variables and real algebraic geometry have direct analogues in this new setting. Unfortunately, the topological underpinnings are shaky at best. In this thesis we cover recent efforts to address these issues—including the first developments of an theory of algebraic topology of matrix domains.

I would contend at all costs both in word and deed as far as I could that we will be better men, braver and less idle, if we believe that one must search for the things one does not know, rather than if we believe that it is not possible to find out what we do not know and that we must not look for it.

— Plato

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I would also like to extend my deepest gratitude to my family. I am who I am because of your influence—those of you who have not yet made it to my arm, know that your flower lives in my heart. I cannot begin to express my thanks to my fellow graduate students. To Karl and Caroline, thank you for welcoming me into your little family with such open arms. To Kelsey, for the many late night McDonalds runs. And to Klig, both for the sandwiches and the conversations. Finally, I would like to thank Christian Leonard—his contributions to my life are immeasurable, but by far the greatest is introducing me to Häagen-Dazs Vanilla Swiss Almond.

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OBJECTS AND THE MAPS BETWEEN THEM

"Young man, in mathematics you don't understand things.
You just get used to them"

— John von Neumann

1 A FIRST ATTEMPT

The fields of free analysis and noncommutative function theory are in their (comparative) infancy. Seeking to understand functions of noncommuting indeterminants, so-called "free functions" have a natural evaluation on tulpes of matrices. In this thesis, we introduce the basic objects and maps that underly free analysis—chapter 1 first does this through the generalization of polynomials and chapter 2 generalizes the work of chapter 1 to include more exotic functions. Then in part ii we explore some of the major results in generalizing algebraic geometry and topology.

Before we get into the meat, a few preliminary pieces of notation. $\mathcal{U}_n$ will denote the set of $n \times n$ unitary matrices. Unless otherwise stated, all maps are morphisms in their respective categories. [1]

1.1 FUNCTIONAL CALCULUS

Functional Calculus is the process of extending the domain of a function on $\mathbb{R}$ to include matrices (or in some cases operators). The most basic formulation uses the fact that the space $n \times n$ matrices forms a ring and so there is a natural way to evaluate polynomials $f \in \mathbb{C}[x]$. If we require that $A \in M_n(\mathbb{C})$ is self-adjoint—and hence diagonalizable as $A = U\Lambda U^*$—then it is a standard result that:

$$\begin{aligned} f(A) &= a_n A^n + \cdots + a_1 A + a_0 I_n \\ &= a_n \left(U\Lambda U^*\right)^n + \cdots + a_1 U\Lambda U^* + a_0 I_n \\ &= a_n U\Lambda^n U^* + \cdots + a_1 U\Lambda U^* + a_0 I_n \\ &= U\left(a_n \Lambda^n + \cdots + a_1 \Lambda + a_0 I_n\right) U^* \\ &= U\left(f(\Lambda)\right) U^* \end{aligned}$$

1 Those fearful of category theory need not despair, this thesis only invokes a universal property once.

Further, since Λ is diagonal and f is a polynomial,

$$f\left(\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}\right) = \begin{bmatrix}f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n)\end{bmatrix}$$

Therefore, given a self-adjoint matrix A and a polynomial $f \in \mathbb{C}[x]$

$$f(A) = Uf(\Lambda)U^* = U\ \mathrm{diag}\{f(\lambda_1), \ldots, f(\lambda_n)\}\ U^*$$

Notice that can simply substitute A in for x without any trouble as long as we transform the constant term $a_0 \mapsto a_0 I_n$ when evaluating on $n \times n$ matrices.[2] Since self-adjoint matrices play such a vital role in free analysis, we will let $\mathbb{H}_n \subset M_n(\mathbb{C})$ denote the set of $n \times n$ self adjoint matrices over $\mathbb{C}$. With the polynomial case in mind, we can extend a function $g : [a, b] \to \mathbb{C}$ to a function on self adjoint matrices with their spectrum in $[a, b]$. Let A be such a matrix (diagonalized by the unitary matrix U), and define

$$g(A) := U \begin{bmatrix}g(\lambda_1) & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} U^*$$

Thus, for each $n \in \mathbb{N}$, g induces a function on the self-adjoint $n \times n$ matrices with spectrum in $[a, b]$. Many properties of functions of a single real variable utilize the fact that $\mathbb{R}$ it totally ordered. While there is not a total ordering on $\mathbb{H}$, there is a partial ordering. The natural ordering on self-adjoint matrices is called the **Loewner Order**:

Definition i.1 (Loewner Ordering)**.** *For like size self-adjoint matrices, we say that $A \preceq B$ if $B - A$ is positive semidefinite and $A \prec B$ is $B - A$ is positive definite.*

With this ordering in place, we can extend many of the familiar function theoretic properties (monotonicity, convexity) to these matrix-values functions. In fact, these properties are defined identically to their classical counterpart: We say that a function is *matrix-monotone* if $A \preceq B$ implies that $f(A) \preceq f(B)$ and *matrix-convex* (or *nc-convex*) if

$$f\left(\frac{A+B}{2}\right) \preceq \frac{f(A) + f(B)}{2}$$

2 Technically we have $a_0 \mapsto a_0 \otimes I_n$ but they are identical in this case. It is common in free analysis to tensor by I_n to make the matrices compatible.

for every pair of like-size matrices, A and B, for which f is defined. These condition are rather restrictive (since the must hold for matrices of *all* sizes) so many functions which are convex/monotone (in the traditional sense) fail to be matrix-convex/monotone. For a full treatment of nc-convexity, see [10]. To illustrate the restrictiveness of nc-convexity,

Example i.2. *In contrast to the real (or even complex) case, $f(x) = x^4$ fails to be nc-convex. Indeed, if*

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \qquad \textit{and} \qquad Y = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$\frac{X^4 + Y^4}{2} - \left(\frac{1}{2}X + \frac{1}{2}Y\right)^4 = \begin{bmatrix} 164 & 120 \\ 120 & 84 \end{bmatrix}$$

Which is not positive definite! Thus x^4 fails to be convex on even 2×2 matrices.

Further, a number of the standard constructions lift identically in this functional calculus.

Definition i.3 (Directional Derivative)**.** *The derivative of f in the direction H is*

$$Df(X)[H] := \lim_{t \to 0} \frac{f(X + tH) - f(X)}{t}$$

where H and X are like-size self-adjoint matices.

Often, the best way to compute these directional derivatives is via an equivalent formulation:

$$Df(X)[H] = \left. \frac{df(X + tH)}{dt} \right|_{t=0}$$

This version allows us to more easily define higher order derivatives

$$D^{(k)} f(X)[H] = \left. \frac{d^{(k)} f(X + tH)}{d^{(k)} t} \right|_{t=0}$$

Notice that this formulations requires each derivative to be in the same direction. Higher order derivatives in different directions clearly exist (simply nest the limits), but their complexity grows quickly. For example, the second derivative in two directions, first H and then K is given by

$$D^2 f(X)[H][K] = \lim_{t \to 0} \frac{Df(X + tK)[H] - Df(X)[H]}{t}.$$

Example i.4. *Just as in the classical case, the directional derivative is linear, so we will only show a calculation of a monomial. Let* $f(x) = x^3$*. Since X and H do not commute,*

$$f(X+tH) = X^3 + tX^2H + tXHX + t^2XH^2 + tHX^2 + t^2HXH + t^2H^2X + t^3H^3.$$

From here, we can calculate:

$$\frac{d}{dt}f(X+tH) = X^2H + XHX + 2tXH^2 + HX^2 + 2tHXH + 2tH^2X + 3t^2H^3$$

$$\frac{d^2}{dt^2}f(X+tH) = 2XH^2 + 2HXH + 2H^2X + 6tH^3$$

$$\frac{d^3}{dt^3}f(X+tH) = 6H^3.$$

And so the first 3 directional derivatives are:

$$Df(X)[H] = X^2H + XHX + HX^2$$

$$D^{(2)}f(X)[H] = 2XH^2 + 2HXH + 2H^2X$$

$$D^{(3)}f(X)[H] = 6H^3$$

In general, the k-th derivative of a polynomial is degree k as a polynomial in H.

Just as in the classical case, the second derivative gives us information about the convexity of a function. A function $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is said to be **positive** if $0 \preceq A \implies 0 \preceq f(A)$. In the functional calculus, we say that f is **nc positive** if it is positive as a map on $M_n(\mathbb{C})$ for all n. Despite nc-convexity being so restrictive, Lemma 12 in [10] shows that the standard characterization of convexity via the second derivative: a function f is convex if and only if $D^2 f(X)[H]$ is nc-positive. Unlike the classical case, however, the only convex polynomials are of degree 2.[3]

3 See [10] for details.

1.2 EXTENDING MULTI-VARIABLE FUNCTIONS

We can extend this same functional calculus to functions of several variables, although the details are a bit more subtle. We could simply "plug in" at tuple of matrices to a standard multivariable polynomial ring over $\mathbb{R}$ or $\mathbb{C}$, but this ignores the noncommutativity of $M_n(\mathbb{C})$.

For example, consider $p \in \mathbb{C}[x, y]$ defined by

$$p(x, y) = xy = yx.$$

If we were to evaluate p on $X, Y \in \mathbb{H}^2$, should it be

$$p(X, Y) = XY, \quad P(X, Y) = YX, \text{ or } p(X, Y) = \frac{XY + YX}{2}?$$

It is evident, then, that $C[x_1, \ldots, x_n]$ is not the algebra of polynomials that we should use. In light of this, let $x = (x_1, \ldots x_g)$ be a g-tuples of noncommuting formal variables. The formal variables $x_1, \ldots, x_n$ are *free* in the sense that there are no nontrivial relations between them.[4] A **word** in x is a product of these variables (e. g.$x_1 x_3 x_1 x_4^2$ or $x_1^2 x_5^3$). An **nc polynomial** in x is a formal finite linear combination of words in x with coefficients in your favorite field. We use $\mathbb{R}\langle x \rangle$ and $\mathbb{C}\langle x \rangle$ to denote the set of nc-polynomials in x over $\mathbb{R}$ or $\mathbb{C}$ respectively.

With $\mathbb{C}\langle x \rangle$ constructed, we can define the functional calculus. Given a word $w(x) = x_{i_1}^{p_1} \cdots x_{i_d}^{p_d}$ and a g-tuple of self-adjoint matrices, X, we can evaluate w on X via $w(X) = X_{i_1}^{p_1} \cdots X_{i_d}^{p_d}$. Since our nc-polynomials are linear combinations of these words, we can extend this evaluation to evaluation of entire polynomials. Algebraically, we have a natural evaluation map: Given some $f \in \mathbb{C}\langle x \rangle$ and $X = \left(X_1, \ldots, X_g\right)$ a g-tuple of self-adjoint matrices, define

$$\begin{aligned} \varepsilon_f : \mathbb{H}_\bullet{}^g &\longrightarrow M_\bullet(\mathbb{C}) \\ X &\longmapsto f(X). \end{aligned}$$

Notice that our functions are **graded** in the sense that if X is a tuple of $n \times n$ matrices, then $f(X)$ is also a tuple of $n \times n$ matrices.

4 This becomes important in the eventual functional calculus—matrices *do* have nontrivial relations. We solve this by creating so-called "generic matrix" rings. [14] and [13] contain the construction as well as a list of further sources.

Example i.5. *Let* $f(x,y) = x^2 - xyx + 1 \in \mathbb{R}\langle x,y\rangle$*. If we define*

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

as before, then

$$\begin{aligned} f(X,Y) &= X^2 - XYX + I_2 \\ &= \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}^2 - \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -11 & -4 \\ -4 & 1 \end{bmatrix}. \end{aligned}$$

Additionally,

$$\begin{aligned} f(X \oplus X, Y \oplus Y) &= \begin{bmatrix} -11 & -4 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & -11 & -4 \\ 0 & 0 & -4 & 1 \end{bmatrix} \\ &= f(X,Y) \oplus f(X,Y). \end{aligned}$$

It is no accident that polynomials handle direct sums of matrices well. As in the classical case, they are the "well behaved" example which we would like general objects to emulate. In the next chapter, we will define free functions—which behave like nc polynomials.

In the context of these multivariate functions, our definition of the Directional Derivative still makes sense (although our direction H now becomes a tuple of directions). We also inherit (from multi-variable calculus) a notion of the **gradient** of a function—but this will require a bit more work.

1.2.1 *The Natural Involution on nc Polynomials*

Given our ring of nc polynomials, we may define an involution * which we may view as an extension of the conjugate transpose. Let * reverse the order of words (i. e.$(x_1x_3x_2^2)^* = x_2^2x_3x_1$) and extend linearly to all of $\mathbb{R}\langle x\rangle$. We consider the formal variables $x_1, \dots, x_n$ *symmetric* in the sense that $x_i^* = x_i$. We say that a polynomial $p \in \mathbb{R}\langle x\rangle$ is symmetric if $p^* = p$. For example, if

$$p(x) = 5x_1^2x_3x_2 + x_3x_2x_3 \qquad q(x) = 3x_2x_1x_2 + x_3^2 - x_1,$$

then a cursory inspection tells that q is symmetric while p is not.

Notice that the majority of the previous two sections breaks down if we try to extend functions to non-self-adjoint matrices. The act of "plugging in" a tuple of arbitry matrices to some element of $\mathbb{R}\langle x\rangle$ via the same functional calculus described above still works, but $\mathbb{R}\langle x\rangle$ is no longer the natural algebra for these evaluations.

Let $x = (x_1, \ldots, x_g)$ be formal variables and let $x^* = (x_1^*, \ldots, x_g^*)$ denote their formal adjoints. Once again, we let the ring $\mathbb{R}\langle x, x^*\rangle$ be the finite formal sums of words in $x_1, x_1^*, \ldots, x_g, x_g^*$ with coefficients in $\mathbb{R}$. Endow $\mathbb{R}\langle x, x^*\rangle$ with an involution * which sends $x_i \mapsto x_i^*$ and $x_i^* \mapsto x_i$ and reverses the order of words extended linearly. Notice that this involution behaves identically to the adjoint with respect to products and sums of matrices. This new ring inherits a natural functional calculus just like that in section 1.2 except it can accept *any* matrix as an input instead of simply self-adjoint matrices.

Example i.6. *Let* $f(x,y) = x^*y - xy^*x + 2$. *Then*

$$f^*(x,y) = y^*x - x^*yx^* + 2.$$

Evaluating f on a pair of non self-adjoint matrices is left to the reader.

1.2.2 *Matrices of nc Polynomials*

It is occasionally useful in the larger theory of free analysis (e. g. when construction the free topology in section 2.3.1 and when characterizing the zero sets of nc polynomials in 3) to consider matrices where the matrices are nc polynomials. Formally, let $\mathbb{R}\langle x\rangle^{k\times k}$ denote the set of $k \times k$ matrices with entries in $\mathbb{R}\langle x\rangle$.[5] We can naturally extend the involution * on $\mathbb{R}\langle x\rangle$ to our matrices by applying * component wise and taking the transpose of the matrix.[6]

Given some $\delta \in \mathbb{R}\langle x\rangle^{k\times k}$ a matrix of nc polynomials, and $X \in \mathbb{H}_n^g$ there is a natural evaluation map.

$$\begin{aligned} \varepsilon_\delta : \mathbb{H}_n{}^g &\longrightarrow M_{nk}(\mathbb{C}) \\ X &\longmapsto \delta(X) \end{aligned}$$

given by evaluating each polynomial in δ at X and then viewing the result at a block $k \times k$ where each block is an $n \times n$ matrix.

5 Some sources additionally consider non-square matrices but this is rare.
6 We could likewise define $\mathbb{R}\langle x, x^*\rangle$ and extend the corresponding involution.

Example i.7. *Define $\delta \in \mathbb{R}\langle x, y\rangle^{2\times 2}$ as*

$$\delta(x,y) = \begin{bmatrix} x^2 - xyx + 1 & xy - yx \\ x^4 & y^3 - 5xy + 3 \end{bmatrix}$$

Then

$$\delta^*(x,y) = \begin{bmatrix} x^2 - xyx + 1 & yx - xy \\ x^4 & y^3 - 5yx + 3 \end{bmatrix}$$

For an evaluation, we will once again let

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \qquad \textit{and} \qquad Y = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

We already know what the evaluations of the first column from examples i.2 and i.5, so we need only complute the second column.

$$XY - YX = \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$$

$$Y^3 - 5XY + 3 = \begin{bmatrix} -29 & 0 \\ -20 & 3 \end{bmatrix}$$

And thus

$$\delta(X,Y) = \begin{bmatrix} -11 & -4 & 0 & -4 \\ -4 & 1 & 4 & 0 \\ 164 & 120 & -29 & 0 \\ 120 & 84 & -20 & 3 \end{bmatrix}.$$

2 A SECOND ATTEMPT

In seeking a more general theory, the functional calculus defined last chapter is insufficient—it would be useful to be able to define *new* functions instead of simply lifting polynomials to matrix domains. In a move that will feel familiar to any good student of mathematics, we will trade treat the set of self-adjoint matrices and nc polynomials as prototypical examples of a more general mathematical object, the so-called *Matrix Universe*. After defining this new space and the natural maps in sections 2.1 and 2.2, we turn our attention to various topologies places on matrix universes in section 2.3. While the genesis of free analysis followed chapter 1 (albeit with the usual bumps in the road that accompany research) modern free analysis looks much more like this chapter.

2.1 MATRIX UNIVERSES

Beyond the functional calculus, it becomes useful to construct general functions on spaces of matrices—to do so, we must make this idea of "spaces of matrices" concrete. The largest such space is the so-called **Matrix Universe**—consisting of g-tuples of matrices of all sizes:

$$\mathcal{M}^g := \bigcup_{n=1}^{\infty} (M_n(\mathbb{C}))^g$$

By convention, when we consider some $X = \left(X_1, \ldots, X_g\right) \in \mathcal{M}^g$, we require that the X_i are all the same size. Since $\mathcal{M}^g$ is such a large set, we often want to deal with subsets that still carry some of the implicit structure of $\mathcal{M}^g$.

Definition i.8 (Free Set)**.** *We say $D \subset \mathcal{M}^g$ is a **free set** (also called an nc set) if it is closed with respect to direct sums and unitary conjugation. That is*

1. *$X, Y \in D$ means $X \oplus Y \in D$.*
2. *For X, U like-size matrices with U unitary and $X \in D$, then $UXU^* = \left(UX_1U^*, \ldots, UX_gU^*\right) \in D$.*

For the remainder of this text, D will denote some free set. Using the terminology of [17], let $D_n = D \cap M_n(\mathbb{C})^g$ be the level-wise slice of all $n \times n$ matrices in D. We say that D is **nc-open**[1] (resp. **connected**, **simply connected**, **bounded**) if each D_n is open (resp. connected, simply connected, bounded). Finally, we say that D is **differentiable** if each D_n is an open C^1 manifold where the complex tangent space of every $X \in D_n$ is all of $M_n(\mathbb{C})^g$. Given some $X \in \mathcal{M}^g$, there are three associated sets which capture the structure of free sets.

Definition i.9 (Similarity Envelope)**.** *Given $X \in \mathcal{M}^g$, a tuple of $n \times n$ matrices, the **similarity envelope** of X is the set*

$$\{U^* X U \mid U \in \mathcal{U}_n\}.$$

Definition i.10 (Fiber)**.** *Given $X \in \mathcal{M}^g$, a tuple of $n \times n$ matrices, the **fiber** of X is the set*

$$\{X^{\oplus k} \mid k \in \mathbb{N}\}.$$

Definition i.11 (Envelope)**.** *Given $X \in \mathcal{M}^g$, a tuple of $n \times n$ matrices, the **envelope** of X is the set*

$$\{U^* X^{\oplus k} U \mid k \in \mathbb{N}, U \in \mathcal{U}_{kn}\}.$$

Notice that if $X \in D$, then the entire envelope of X is automatically in D as well! Further, as shown in example i.5, polynomials respect the envelope of a matrix in a particularly well-behaved way. Colloquially, we think of all points in the envelope of X as "the same"—this notion is explored in section 2.3 and throughout chapter 4.

In the context of sections 1.1 and 1.2 , the domains in the functional calculus were $\mathbb{H}^g = \bigcup_{n=1}^{\infty} \mathbb{H}_n{}^g$. $\mathbb{H}^g$ is a differentiable, connected free set.

On $\mathcal{M}^g$, we define a product that resembles the inner product on $\mathbb{C}^n$ which will be extensively throughout chapters 3 and 4. Given $A, B \in \mathcal{M}^g$ which are g-tuples of $n \times n$ matrices:

$$\begin{aligned} \cdot : \mathcal{M}^g \times \mathcal{M}^g &\longrightarrow M_n(\mathbb{C}) \\ \cdot(A,B) = A \cdot B &\longmapsto \sum_{i=1}^{g} A_i B_i \end{aligned}$$

If we combine this product with the trace, we get a bilinear form on $\mathcal{M}^g$ which functions like an inner product given by $\operatorname{tr}(A \cdot B)$. It will be particularly useful in chapter 4 as well for defining the gradient of a function.

1 The topology of $\mathcal{M}^g$ is still in flux and there is not a canonical topology. See section 2.3 for the details

2.2 TRACIAL FUNCTIONS AND UNIQUENESS OF THE GRADIENT

Now that we have $\mathcal{M}^d$, we can work with general functions on our matrix universe. As a whole, free analysis is concerned with so-called *free functions*, which are graded and respect direct sums and unitary conjugation.

Definition i.12 (Free Function)**.** *A function* $f : D \to \mathcal{M}^{\hat{d}}$ *is called* ***free*** *if*

1. $f(X \oplus Y) = f(X) \oplus f(Y)$
2. $f(UXU^*) = f(U)f(X)f(U^*)$ *where* X *and* U *are like-size and* U *is unitary.*

The two other classes of functions we are concerned with are those that act like the trace and the determinant:

Definition i.13 (Determinantal Free Function)**.** *A function* $f : D \to \mathbb{C}$ *is a* ***determinantal free function*** *if*

1. $f(X \oplus Y) = f(X)f(Y)$
2. $f(UXU^*) = f(X)$ *where* X *and* U *are like-size and* U *is unitary.*

Definition i.14 (Tracial Free Function)**.** *A function* $f : D \to \mathbb{C}$ *is a* ***tracial free function*** *if*

1. $f(X \oplus Y) = f(X) + f(Y)$
2. $f(UXU^*) = f(X)$ *where* X *and* U *are like-size and* U *is unitary.*

It is worth noting that, while they share the moniker of *free*, determinantal and tracial functions are *not* free functions. Since these three classes of functions all contain the word "free," we will often drop this qualifier and only refer to determinantal, tracial, and free function. Given a free function of any type, we can define the directional derivative (Definition i.3) identically. It is only these tracial functions which inherit the gradient mentioned section 1.1. Similarly to traditional multivariable calculus we define the gradient via its relationship to the directional derivative:

Definition i.15 (Free Gradient)**.** *Given a tracial free function* f*, the free gradient,* ∇f*, is the unique free function satisfying*

$$\operatorname{tr}(H \cdot \nabla f(X)) = Df(X)[H]$$

It is not-at-all obvious that such a ∇f should be unique—after all any linear combination of commutator has trace zero. To show the uniqueness of ∇f, we will first restrict ourselves to single variable functions. In the case that f is a single-variable function we can replace ∇f with the traditional derivative, f', as seen in [19, Thm 3.3].

Theorem i.16. *Let $f : (a,b) \to \mathbb{R}$ be a C^1 function. Then*

$$\operatorname{tr} Df(X)[H] = \operatorname{tr}\left(Hf'(X)\right)$$

The proof in [19] simply asserts the uniqueness of a function $g(X)$ satisfying $\operatorname{tr} Df(X)[H] = \operatorname{tr}(Hg(X))$ and then shows that $g(x) = f'(x)$ for $x \in (a,b)$. Instead, we can construct such a g and recover the theorem along the way:

Proof. We start with a construction from Bhatia's Matrix Analysis [5]: Let $f \in C^1(I)$ and define $f^{[1]}$ on $I \times I$ by

$$f^{[1]}(\lambda,\mu) = \begin{cases} \frac{f(\lambda)-f(\mu)}{\lambda-\mu} & \lambda \neq \mu \\ f'(\lambda) & \lambda = \mu. \end{cases}$$

We call $f^{[1]}(\lambda,\mu)$ the *first divided difference* of f at (λ,μ). If Λ is a diagonal matrix with entries $\{\lambda_i\}$, We may extend f to accept Λ by defining the (i,j)-entry of $f^{[1]}(\Lambda)$ to be $f^{[1]}(\lambda_i,\lambda_j)$. If A is a self adjoint matrix with $A = U\Lambda U^*$, then we define $f^{[1]}(A) = Uf^{[1]}(\Lambda)U^*$. Now we borrow a theorem from Bhatia:

Theorem i.17 (Bhatia V.3.3)**.** *Let $f \in C^1(I)$ and let A be a self adjoint matrix with all eigenvalues in I. Then*

$$Df(A)[H] = f^{[1]}(A) \circ H,$$

where $\circ$ denotes the Schur-product[2] in a basis where A is diagonal.

That is, if $A = U\Lambda U^*$, then

$$Df(A)[H] = U\left(f^{[1]}(\Lambda) \circ (U^*HU)\right)U^*.$$

To prove our claim, we need to take the trace of both sides. Since trace is invariant under a change of basis, it is clear that

$$\operatorname{tr} Df(A)[H] = \operatorname{tr}\left(f^{[1]}(\Lambda) \circ (U^*HU)\right).$$

2 Entrywise

If $U = u_{ij}$, $U^* = \overline{u}_{ij}$ and $H = h_{ij}$, then the (i, j)-entry of U^*HU is

$$(U^*HU)_{ij} = \sum_k \sum_\ell \overline{u}_{ik} h_{k\ell} u_{\ell j}$$

Where we sum over the duplicate indices k and ℓ. While the structure of $f^{[1]}(\Lambda)$ is a bit unruly, our diagonal entries are $f'(\lambda)$. This means that when we take the trace of the Schur product, we have

$$\sum_k \sum_\ell \sum_i f'(\lambda_i) \overline{u}_{ik} h_{k\ell} u_{\ell i}$$

Now consider the matrix product $U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\}\, U^*H$. Since one of our terms is diagonal, the trace of this multiplication is simple:

$$\operatorname{tr} U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\}\, U^*H = \sum_k \sum_\ell \sum_i u_{ik} f'(\lambda_k) \overline{u}_{k\ell} h_{\ell i}$$

Since $u_{ik}, \overline{u}_{k\ell}, h_{\ell i} \in \mathbb{C}$ they commute. We can then relabel our indices $i \mapsto \ell$ $\ell \mapsto k$ $k \mapsto i$ to get

$$\operatorname{tr}\ U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\}\, U^*H = \sum_k \sum_\ell \sum_i f'(\lambda_i) \overline{u}_{ik} h_{k\ell} u_{\ell i},$$

So, for every direction H, we have that

$$\operatorname{tr}\left(U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\}\, U^*H\right) = \operatorname{tr}\left(f^{[1]}(\Lambda) \circ (U^*HU)\right).$$

By picking the "correct" H, [3] we conclude that there is a unique quantity $g(X)$ satisfying

$$\operatorname{tr}\ Df(X)[H] = \operatorname{tr}(Hg(X)).$$

In particular, $g(X) = U \operatorname{diag}\{f'(\lambda_1), \ldots, f'(\lambda_n)\}\, U^*$. But, recall that $X = U\Lambda U$ so, in the functional calculus, $g(X) = f'(X)$. Making this substitution, we have the required result:

$$\operatorname{tr}\ Df(X)[H] = \operatorname{tr}(Hf'(X))$$

■

With our theorem proven, we turn our attention back to the ∇f. The single variable case motivates that ∇f should correspond to the standard gradient from vector calculus. With some work, the above proof lifts the multi-variable case. It will be instructive, however, to consider a different proof.

3 See the proof of i.18 for the details of how to pick the H's

Theorem i.18 (Trace Duality)**.** *Let f, g be free functions $\mathcal{M}^g \to \mathcal{M}^{\tilde{g}}$. If $\operatorname{tr} H \cdot f = \operatorname{tr} H \cdot g$ for all tuples H, then $f = g$.*

Proof. Since the trace relation holds for all H, we may choose our H carefully to show the equality of f and g. Say that $H, f(X), g(X)$ are g-tuples of matrices—we will first show that $f_1 = g_1$ and we will do so entry by entry. Let E_{ij} be the matrix will all zeroes and a 1 in the (i, j)-entry. Now let $H = (E_{ji}, 0, \ldots, 0)$. So $\operatorname{tr} E_{ji} f_1(X) = \operatorname{tr} E_{ji} g_1(X)$. In our products, the only elements on the diagonal are $(f_1(X))_{ij}$ and $(g_1(X))_{ij}$, so when we take the trace we have $(f_1(X))_{ij} = (g_1(X))_{ij}$. If we do this for every (i, j), we see that $f_1(X) = g_1(X)$. Similarly, we can choose $H = (0, E_{ji}, 0, \ldots, 0)$ for each i, j to show that $f_2(X) = g_2(X)$ and so on. Since $f(X) = g(X)$ for each $X \in \mathcal{M}^g$, it follows that $f = g$. ■

Admittedly, there is a slight complication that is overlooked in the above proof when it comes to the domains of f and g. Where these domains overlap, we can consider them as the same function (and therefore ∇f is unique) but if f is defined on D and g is defined on $\tilde{D}$, then the above proof only holds on $D \cap \tilde{D}$. This complication occurs semi-frequently in free analysis, but in generally swept under the rug—if two free functions agree the intersection of their domains, it is convention to consider them equivalent. Examples of such f and g abound when considering rational functions, which are explored in section 2.5.

2.3 THE TOPOLOGY OF MATRIX UNIVERSES

At the time of writing, there is no "canonical topology" for $\mathcal{M}^g$. For a long time it seemed like the *free* topology (to be defined below) was the obvious choice, but recent work (c.f. [16]) has implied that the free topology does not put enough structure on $\mathcal{M}^g$. See [2] for a full treatment of the common topologies on $\mathcal{M}^g$.

A naive approach to a topology on $\mathcal{M} = \bigcup_n M_n(\mathbb{C})$ would be the disjoint union topology—which is then extended do a topology on $\mathcal{M}^g$ via the product topology. Notice, however that this ignores a significant amount of the implicit structure of nc sets as we get a disconnected space with countable many connected components. Topologically, this is means that means that

$$H_\bullet(D) = \bigoplus_{n \in \mathbb{N}} H_\bullet(D_n).$$

At first glance, this seems fine enough, but it ignores the fact that for $X \in D$ we require $X^{\oplus k} \in D$ for all k and $U^*XU \in D$ for all unitary U. In a sense, we think of the all the direct sums of X and its similarity envelope as "the same." In light of this, if $\sim$ is the equivalence relation that $X \sim Y$ if $Y = X^{\oplus k}$ or $Y = U^*XU$, then any useful topological theory on $D \subset \mathcal{M}^g$ should descend to classic theory on $D/\sim$. One needs only look at $H_0(D)$ to see that the naive approach fails to give useful information. It should be the case that $H_0(\mathcal{M}^g)$ is trivial but in the disjoint union topology it is easy to see

$$H_0(\mathcal{M}^g) = \bigoplus_{n\in\mathbb{N}} \mathbb{Z},$$

which does not behave as we would expect.

2.3.1 *Admissible Topologies*

In light of the above discussion, we will present some of the candidate topologies which show some promise in understanding the topology on $\mathcal{M}^g$ and its subsets. Let $D \subset \mathcal{M}^g$ be a nc bounded open set (recall that this means that D is closed under direct sums and unitary conjugation, and that each D_n is a bounded open set in $M_n(\mathbb{C})^g$) and let

$$\mathcal{B} = \{D \mid D \text{ is an nc bounded open set}\}.$$

It is not difficult to se that $\mathcal{B}$ is the basis for a topology on $\mathcal{M}^g$, called the **fine** topology. Currently, there is not a "standard" topology for $\mathcal{M}^g$. Any topology on, τ, $\mathcal{M}^g$ is considered **admissable** if it basis is a subset of $\mathcal{B}$—i. e.it has a basis of nc bounded open sets.

Example i.19. *The fine topology is often very convenient as we can leverage the level-wise open-ness of the basis. Let $\mathbb{D} \subset \mathcal{M}$ denote the set of diagonalizable matrices. We seek to show that $\mathbb{D}$ is dense in $\mathcal{M}$. Let $D \in \mathcal{B}$ be nonempty. Since D is open in $\mathcal{M}$, we know that D_n is open in $M_n(\mathbb{C})$. But $\mathbb{D}_n$ is dense in $M_n(\mathbb{C})$, so $D_n \cap \mathbb{D} \neq \emptyset$. It follows that $D \cap \mathbb{D} \neq \emptyset$, and so $\mathbb{D}$ is dense in $\mathcal{M}$.*

Of course, a similar argument works in $\mathcal{M}^g$. This gives an powerful boost to the functional calculi discussed in chapter 1. While we worked over the self adjoint matrices, one can develop a nearly identical functional calculus for diagonalizable matrices. Since the diagonalizable matrices are dense in $\mathcal{M}^g$, however, we obtain a function on the en-

tire domain by via a continuous extension off of the diagonalizable matrices! This gives us a way to make sense of the free functions

$$f(X,Y) = e^X e^Y, \qquad g(X,Y) = e^{X+Y}$$

both of which map $\mathcal{M}^2 \to \mathcal{M}$.

A slightly more restrictive topology (that seems to show some promise in the eyes of the author) is the **fat** topology. For $n \in \mathbb{N}$, $r \in \mathbb{R}^+$, and $X \in \mathcal{M}_n^g$, we first define a matricial polydisc

$$D_n(X,r) := \{A \in \mathcal{M}^g \mid \max_{1 \le i \le g} \|X_i - A_i\| < r\}.$$

Now we sweep D_n through all direct sum copies of X:

$$D(X,r) := \bigcup_{k=1}^{\infty} D_{kn}(X^{\oplus k}, r)$$

Finally, we take the similarity envelope of $D(X,r)$

$$F(X,r) := \bigcup_{n=1}^{\infty} \bigcup_{U \in \mathcal{U}_n} U^* \left(D(X,r) \cap \mathcal{M}_n^g\right) U$$

Both the fine and the fat topologies admit implicit function theorems—which are discussed (in brief) in section 2.4.

The final candidate topology is the aforementioned **free** topology. Recall that $\mathbb{R}\langle x \rangle$ is the algebra of nc polynomials over the real numbers and that $\mathbb{R}\langle x \rangle^{k \times k}$ is the set of $k \times k$ matrices with entries in $\mathbb{R}\langle x \rangle$. Let $\delta \in \mathbb{R}\langle x \rangle^{k \times k}$ and define

$$G_\delta = \{x \in \mathcal{M}^g \mid \|\delta(x)\| < 1\}.$$

For $x \in M_n(\mathbb{C})$, $\|\cdot\|$ is the operator norm in $\mathcal{B}(\mathbb{C}^k \otimes \mathbb{C}^n)$. This may seem strange [4] but the level-wise definition allows the norm to "play nice" with direct sums. The set of all G_δ as k ranges over $\mathbb{Z}^+$ form the basis for the free topology. Indeed, any $X \in \mathcal{M}^g$ is trivially in one of the G_δ (take $\delta = X$) and with some work one can show that $G_{\delta_1} \cap G_{\delta_2} = G_{\delta_1 \oplus \delta_2}$ so we do, indeed, have a basis. All that is needed to satisfy the axioms for a base is that $G_{\delta_1} \cap G_{\delta_2} \supset G_{\delta_1 \oplus \delta_2}$. In fact if one chose a more "standard" norm for $\delta(X)$ above (e. g.the Frobenius norm) one gets the needed inclusion. The benefit of the strange norm, however, is that we get equality here instead of inclusion.

4 The author would like to note that it is, in fact, strange.

2.4 FREE ANALOGUES OF CLASSICAL RESULTS

In general, efforts to reprove classical results from single and several varible complex analysis have been successful. A full treatise of the results proven before 2020 can be found in [3]—but we will include a handful here.

Among the many astounding results is the following characterization of holomorphic functions in admissible topologies:

Theorem i.20 (Locally Bounded Implies Analytic)**.** *Let D be an nc domain and f a free function on D. If f is locally bounded on each D_n, then f is an analytic function of the entries of the matrices at each level n.*

A proof for this (rather suprising) result is given in [9]. With different topologies come different analytic functions. In light of this, if the topology is not understood functions are usually referred to as {fine/fat/free} holomorphic. As mentioned above, both the fine and fat topologies have implicit function theorems. The fat implicit function theorem requires a significant amount more work to state, but it can be found in [3].

Theorem i.21 (Fine Implicit Function Theorem)**.** *Let $D \subset \mathcal{M}^g$ be an nc domain. Let Φ be a fine holomorphic map $D \to \mathcal{M}^g$. The following equivalent:*

1. *Φ is injective on D*
2. *$D\Phi(X)[H]$ is nonsingular for every $X \in D$ and like-size $H \in \mathcal{M}^g$.*
3. *the function Φ^{-1} exists and is a fine holomorphic map.*

Various Null- and Positivstellensatz [5] exist throughout the literature extending Hilbert's famous Nullstellensatz in algebraic geometry—many of which utilize the idea of so-called "atomic" matrices of nc polynomials (defined in theorem ii.12). See [11] for the specifics.

In [1], Agler and McCarthy prove a free analogue of the Oka-Weil theorem: any free holomorphic function on a compact set can be uniformly approximated by polynomials. Unfortunately, it was later proven in [20] and [4] that the only compact sets in the $\mathcal{M}^g$ are the envelope of finitely many points, trivializing the result of Agler and McCarthy.

5 And even a QuadratischePositivstellensatz!

2.5 NC RATIONAL FUNCTIONS

While polynomials were fairly simply to lift to the noncommutative setting, dealing with rational functions is a bit more complex. In depth discussion of nc rational functions descends quickly into abstract nonsense, so we will only cover the basics and will not go beyond what is needed for the work done in the rest of this thesis.

Definition i.22 (nc Rational Expression)**.** *An **nc rational expression** (alternativey a free rational expression) in noncommuting indeterminants, $x_1, \dots, x_g$ is a syntactically valid expression of those indeterminants involving addition, multiplicatoin, inverses, and scalar multiplication.*

For example, the following are examples of free rational expressions in two variables:

$$x_1x_2 + x_2x_1(x_1 - x_2)^{-1},\ (2x_2^2x_1^{-1} + (x_1x_2 - x_2x_1)^{-1}),$$
$$(x_2(1 - x_1x_2) - (1 - x_2x_1)x_2)^{-1}.$$

The careful reader will notice, however, that one of these expressions is not like the other. If we were to evaluate these expression on tuples of matrices, there is no pair for which $(x_2(1 - x_1x_2) - (1 - x_2x_1)x_2)^{-1}$ is defined. This is an example of a **degenerate** rational expression. Formally, an nc rational expression is **nondegenerate** if there is at least one $X \in \mathcal{M}^g$ such that $r(X)$ is defined.

When seeking to make rational *functions* out of a nondegenerate rational expression, one encounters significant difficulties. For example,

$$x_1 - x_1 + x_2^{-1}x_2 - 1 \quad \text{and} \quad 0$$

are syntactically distinct nc rational expressions, have wildly different domains, yet whenever their domains of evaluation align, they will give the same evaluation. With the likes of the Identity Theorem from complex analysis one would like to say that these are the same function. In light of this, nc rational functions are defined via equivalence classes. We say two rational expressions, r_1, r_2, are equivalent if $r_1(X) = r_2(X)$ whenever both expressions are defined.

Unfortunately, this introduces a new wrinkle. There is, of course, the issue of what the domain of the equivalence class is—usually one simply works with the domain of the representative chosen. Moreover, given two representatives of the same equivalence class of rational functions, there is no guarantee of an algebraic manipulation to transform one into the other.

Just like the polynomial case, we often consider *matrices* of nc rational functions. Chapter 3 explores the algebraic geometry of nc polynomials and rational functions. Thankfully, however, we do not need any of the in depth theory of rational functions. [10] contains a slightly broader introduction from an analytical lens while [6] provides a (more complete) algebraic treatment. The only theorem we will need comes from the latter:

Theorem i.23. *For any nondegenerate rational expression, r, there is a linear square matrix of polynomials, L, and rectangular constants b, c such that $r = b^* L^{-1} c$—where L^{-1} is defined wherever r is defined.*

Part II

THE ALGEBRAIC GEOMETRY AND TOPOLOGY OF MATRIX DOMAINS

"Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine."

— Michael Francis Atiyah

3 ZERO SETS AND PRINCIPLE DIVISORS

3.1 VARIETIES, CLASSICAL AND FREE

In the classical case, varieties are fairly easily to classify. Given some (commutative) polynomial, $f \in \mathbb{C}[x_1, \ldots, x_g]$ we define the zero set

$$V(f) = \{a \in \mathbb{A}^n \mid f(a) = 0\},$$

where $\mathbb{A}^n$ is complex affine n-space. Varieties (both affine and projective) are well studied in algebraic geometry (Hartshorne's *Alegbraic Geometry* [7] is a standard introduction). Of particular interest is a geometric invariant of a variety called a *divisor*. While classical divisors require robust machinery to construct formally[1] one can think of them (loosely) as formal sums of codimension one subvarieties. The concept of a divisor lift naturally to the noncommutative setting, although varieties are a touch more complex.

Before we return to the noncommutative setting, however, it is work making a quick remark on what topology we will adopt for the rest of this thesis. We will be using the conventions mentioned in section 2.1: $D \subset \mathcal{M}^g$ open if each D_n is open—these are precisely the basic open sets in the fine topology. Given $X, Y \in D$, it is not generally true that we can separate X and Y with open sets. However if Y is not in the similarity envelope of X and X and Y have disjoint fibers, then we *can* separate them! Motivated by definitions in section 4.4 we call a topology satisfying this condition (Hausdorff outside of the similarity envelope and fiber) **essentially Hausdorff**.

Let f be a matrix of polynomials on $\mathcal{M}^g$. Unlike the classical case, it is not immediate what should be meant by $f(X) = 0$—is it enough for $f(X)$ to be singular, or should $f(X)$ be the zero matrix? In light of this ambiguity, we make three definitions.

Definition ii.1 (Singular Set). *Let f be a matrix of nc polynomials. The **n-Singular Set** of f is*

$$\mathscr{Z}_n(f) = \{X \in M_n(\mathbb{C}) \mid \det f(X) = 0\}.$$

1 Schemes, in particular.

*The **Singular Set** of f is*

$$\mathscr{Z}(f) = \bigcup_{n \in \mathbb{N}} \mathscr{Z}_n(f).$$

Definition ii.2 (Directional Singular Set)**.** *Let f be a matrix of nc polynomials. Associated with the singular set is the **Directional Singular Set**:*

$$\mathscr{Z}_{dir}(f) = \{(X, v) \mid f(X)v = 0\}.$$

Definition ii.3 (Zero Set)**.** *Let f be a matrix of nc polynomials. The **n-Zero Set** of f is*

$$\mathscr{V}_n(f) = \{X \in M_n(\mathbb{C}) \mid f(X) = 0\}.$$

*The **Zero Set** of f is*

$$\mathscr{V}(f) = \bigcup_{n \in \mathbb{N}} \mathscr{V}_n(f).$$

While the singular set encodes the matrices for which $f(X)$ has a nontrivial kernel, the directional singular set bundles this information together with the kernel itself. Section 6 of Helton's *Free Convex Algebraic Geometry* [10] shows how this can is analogous to the tangent plane of a classical variety. While it may seem counter intuitive to use a script "Z" for the singular set instead of the zero set, the singular set of a free function is (in many cases) a more natural generalization of varieties. One needs to be careful when interfacing with the literature as these definitions (including which of these three sets it the "zero set") are not universal and each author seems to make their own choices.

Over the past decade, many author have generated Null- and Positivstellensatz for these three sets. In particular, [11] treats singular and zero sets while [12] treats the directional singular set.

3.2 PRINCIPAL DIVISORS

Recall that given a differentiable traical free function f, the free gradient, ∇f is the unique free functions satisfying

$$\operatorname{tr}(H \cdot \nabla f) = Df(X)[H]$$

for all directions H. On the other hand, for every square [2] *free* function, g, we can associate a determinantal function $\det g$—which is defined in the obvious way. If f is a nontrivial *determinantal* function, then there is an induced tracial function, $\log f$ wherever f is nonzero.

Definition ii.4 (Principal Divisors)**.** *Let f be a nonzero* determinantal *free function. Then the **principal divisor** of f is*

$$\operatorname{div} f = \nabla \log f.$$

Alternatively, if g is square free *function, then the principal divisor of g is*

$$\operatorname{div} g = \nabla \log \det g$$

Before exploring the properties of $\operatorname{div} f$, it is worth acknowledging that the notation is overloaded. Unfortunately, the principal divisors of both free and determinantal functions have significant utility. One has to be careful whether theorems concern the divisors of free functions or determinantal ones. In light of this, the author has elected to italicize "free" and "determinantal" for the remainder of this section whenever there could be ambiguity should one not read too carefully.

While it is trivial to verify, (simply use the properties of log and the linearity of ∇), observe that

$$\operatorname{div} fg = \operatorname{div} f + \operatorname{div} g.$$

We will use this fact to partially characterize divisors.

Lemma ii.5. *Let f, g be C^1 nonzero* determinantal *free functions. Then,*

1. *There exists an inverible locally constant determinantal function c such that $f = cg$ if and only if $\operatorname{div} f = \operatorname{div} g$.*
2. *$\frac{f}{g}$ has a C^1 extension to the whole domain if and only if there is a C^1 determinantal function h on the whole domain such that $\operatorname{div} f - \operatorname{div} g = \operatorname{div} h$.*
3. *$\frac{f}{g}$ and $\frac{g}{f}$ have a C^1 extension to the whole domain if and only if $\operatorname{div} f - \operatorname{div} g$ has a continuous extension to the whole domain.*

Proof.

2 Meaning the output of g is a square matrix.

1. Suppose such a c existed. Then

$$\operatorname{div} f = \operatorname{div} cg = \operatorname{div} c + \operatorname{div} g.$$

But because c is locally constant, the presence of ∇ makes $\operatorname{div} c = 0$,[3] so $\operatorname{div} f = \operatorname{div} g$.

Conversely, suppose $\operatorname{div} f = \operatorname{div} g$. But then

$$\begin{aligned} 0 &= \operatorname{div} f - \operatorname{div} g \\ &= \nabla \left(\log f - \log g\right) \\ &= \nabla \log \frac{f}{g}. \end{aligned}$$

And so $\log \frac{f}{g}$ is locally constant! It follows that $\frac{f}{g}$ is locally constant and hence we can write $g = cf$ for some locally constant function c.

2. Suppose there is a function h on the whole domain such that $\operatorname{div} h = \operatorname{div} f - \operatorname{div} g$—then by part 1, h differs from $\frac{f}{g}$ by a constant but is definied on the entire domain. It is immediate, then, that $\frac{f}{g}$ extends to the whole domain.

Conversely, suppose h is the continuous extension to the entire domain. But then

$$\begin{gathered} \frac{f}{g} = h \\ \Downarrow \\ \log f - \log g = \log h \\ \Downarrow \\ \operatorname{div} f - \operatorname{div} g = \operatorname{div} h. \end{gathered}$$

3. Part 3 follows immediately from part 2.

■

Example ii.6. *Consider the* free *functions $f(X,Y) = e^X e^Y, g(X,Y) = e^{X+Y}$. In significant contrast to the classical case, X and Y do not commute, so $f \neq g$. Before we look at the divisors of f and g it is pertinant to consider how f, g are actually defined. Recall the discussion of example i.19: f and g are free functions defined on all of $\mathcal{M}^2$, so we are we are outside the functional calculus of section 1.2,*

3 The fact that c locally constant implies $\nabla c = 0$ is not immediately obvious from the definition. Thankfully, it is very quick to verify.

which required self-adjoint matrices. For the values for which X, Y are diagonalizable, we can evaluate $f(X,Y)$ with the usual functional calculus. For an X or Y which is and nondiagonalizable, recall that $\mathbb{D}_n^2$ is dense in $M_n(\mathbb{C})^2$ so we have level-wise continuous extension of f (and of course g) to all of $\mathcal{M}^2$.

Now we consider the divisors of f and g. Since they are free functions, recall that div *is actually* div det. *But then,*

$$\begin{aligned} \operatorname{div} e^X e^Y &= \nabla \log\det\left(e^X e^Y\right) \\ &= \nabla \log\left(e^{\operatorname{tr} X} e^{\operatorname{tr} Y}\right) \\ &= \nabla\left(\log e^{\operatorname{tr} X} + \log e^{\operatorname{tr} Y}\right) \\ &= \nabla \operatorname{tr} X + \nabla \operatorname{tr} Y \end{aligned} \qquad \begin{aligned} \operatorname{div} e^{X+Y} &= \nabla \log\det\left(e^{X+Y}\right) \\ &= \nabla \log\left(e^{\operatorname{tr}(X+Y)}\right) \\ &= \nabla \operatorname{tr}(X+Y) \\ &= \nabla \operatorname{tr} X + \nabla \operatorname{tr} Y \end{aligned}$$

And so we see that $\operatorname{div} e^X e^Y = \operatorname{div} e^{X+Y}$.

This example relies on the fact that log plays nicely with $e^{\operatorname{tr} X}$ and one might wonder if there is an easier way to compute principal divisors.

Theorem ii.7. *Let $f : D \to \mathcal{M}^{\hat{d}\times\hat{d}}$ be a C^1 free function[4] such that* $\det f \not\equiv 0$. *Then*

$$\operatorname{tr}(H \cdot \operatorname{div} f) = \operatorname{tr}\left(Df(X)[H]f(X)^{-1}\right)$$

Proof. We begin by recalling Jacobi's formula, which gives us a way to understand the directional derivative of the determant in terms of the adjugate[5] of a matrix. For a matrix X,

$$D\det X[H] = \operatorname{tr}(H \operatorname{adj} X).$$

It will be imperative later in the proof to recall the following property of the adjugate: for an invertable matrix X,

$$\operatorname{adj}(X) = \det(X)X^{-1}.$$

With these preliminaries sorted, we continue with the proof. Unraveling the definitions given above, the principal divisor of f (a *free function*) is the unique free function on its nonsigular set satisfying

$$D\log\det f(X)[H] = \operatorname{tr}(H \cdot \operatorname{div} f).$$

4 Since the codomain is $\mathcal{M}^{\hat{d}\times\hat{d}}$, one can view f as a $\hat{d}\times\hat{d}$ matrix of free functions.

5 The transpose of the cofactor matrix.

We compute

$$\begin{aligned}
D\log\det f(X)[H] &= \frac{d}{dt}\left[\log\det f(X+tH)\right]\Big|_{t=0} \\
&= \frac{1}{\det f(X)}\left(\frac{d}{dt}\left[\det f(X+tH)\right]\Big|_{t=0}\right) \\
&= \frac{1}{\det f(X)}\operatorname{tr}\left(\frac{d}{dt}\left[f(X+tH)\right]\operatorname{adj} f(X+tH)\Big|_{t=0}\right) \\
&= \frac{1}{\det f(X)}\operatorname{tr}\left(Df(X)[H]\operatorname{adj} f(X)\right) \\
&= \operatorname{tr}\left(Df(X)[H]\frac{\operatorname{adj} f(X)}{\det f(X)}\right) \\
&= \operatorname{tr}\left(Df(X)[H]f^{-1}(X)\right).
\end{aligned}$$

■

The next section will treat divisors of polynomial and rational functions in detail. Before continuing, we give one more example.

Example ii.8. *Let $f(X,Y) = 1 + XY$ and $g(X,Y) = 1 + YX$. Using the previous theorem, we have that*

$$\begin{aligned}
\operatorname{tr}\left((H_1,H_2)\cdot\operatorname{div} f\right) &= \operatorname{tr}\left(Df(X,Y)[H_1,H_2]f(X,Y)^{-1}\right) \\
&= \operatorname{tr}\left((H_1Y + XH_2)(1+XY)^{-1}\right) \\
&= \operatorname{tr}\left(H_1Y(1+XY)^{-1} + H_2(1+XY)^{-1}X\right) \\
&= \operatorname{tr}\left((H_1,H_2)\cdot\left(Y(1+XY)^{-1},(1+XY)^{-1}X\right)\right)
\end{aligned}$$

Appealing to trace duality (theorem i.18), we see that

$$\operatorname{div} f = \left(Y(1+XY)^{-1},(1+XY)^{-1}X\right).$$

With a nearly identical computation, we recover the principal divisor of g as well:

$$\operatorname{div} g = \left((1+YX)^{-1}Y, X(1+YX)^{-1}\right).$$

Since $Y(1+XY) = (1+XY)Y$, it follows that $Y(1+XY)^{-1} = (1+XY)^{-1}Y$, and so $\operatorname{div} f = \operatorname{div} g$*!*

3.3 THE GROUP OF DIVISORS

For the remainder of this chapter, we will concern ourselves with the divisors of square matrices of nc polynomials and nc rational functions. These are all f free functions, so div will denote div det. We begin with a theorem.

Theorem ii.9. *Let f, g be square matrices of nc polynomials such that $\det f, \det g \not\equiv 0$. If $\frac{\det f}{\det g}$ and $\frac{\det g}{\det f}$ are entire, then $\operatorname{div} f = \operatorname{div} g$.*

Proof. Consider $\frac{\det f}{\det g}$ and $\frac{\det g}{\det f}$ as functions $M_n(\mathbb{C}) \to \mathbb{C}$. Since both of these are entire, $\det f, \det g$ are both never 0—hence any zeroes or poles that they possess must be at infinity. Suppose that $\frac{\det f}{\det g}$ is unbounded. Depending on how the degrees of $\det f$ and $\det g$ compare, there is either a zero or a pole at infinity. But this means that $\frac{\det g}{\det f}$ has either a zero or a pole at 0. Either way we have a contradiction, and so $\frac{\det f}{\det g}$ is bounded (and entire)—hence constant.

We now appeal to lemma ii.5, part 3. Since $\frac{\det f}{\det g}$ and its reciprocal both have C^1 extension (namely themselves), we have a levelwise constant function h such that $\operatorname{div} f - \operatorname{div} g = \operatorname{div} h$. But clearly $\operatorname{div} h$ is 0, so $\operatorname{div} f = \operatorname{div} g$! ■

One of the major themes of the development of principal divisor of free functions (like in [17]) is that much of the structure of divisors is an immediate corollary of the structure of $\det f$. For example, the following theorem is proven almost entirely by its lemma.

Theorem ii.10. *Let r be a nondenerate square matrix of nc rational expressions, such that $\det r(X) \not\equiv 0$. Then there exists square matrices of nc polynomials p, q such that*

$$\operatorname{div} r = \operatorname{div} p - \operatorname{div} q$$

Proof. We begin with a lemma.

Lemma ii.11. *Let r be a nondenerate square matrix of nc rational expressions, such that $\det r(X) \not\equiv 0$. Then there exists square matrices of nc polynomials p, q such that*

$$\det r = \frac{\det p}{\det q} = \det(pq^{-1})$$

Proof. Recalling theorem i.23, let $r = b^* L^{-1} c$. We claim that

$$p = \begin{bmatrix} L & c \\ -b & 0 \end{bmatrix} \qquad \text{and} \qquad q = L.$$

We see that

$$\det \begin{bmatrix} L & c \\ -b & 0 \end{bmatrix} \Big/ \det L = \det \begin{bmatrix} L & c \\ -b & 0 \end{bmatrix} \det \begin{bmatrix} L^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & c \\ -bL^{-1} & 0 \end{bmatrix}.$$

Now we recall the formula for the determinant of a block matrix:

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \det D - CA^{-1}B.$$

With this in hand, we see that $\det \left[\begin{smallmatrix} 1 & c \\ -bL^{-1} & 0 \end{smallmatrix}\right] = \det b^* L^{-1} c$, and we are done. ■

Now take the div of both sides of the lemma to get the required result. ■

Just as in the classical case, these is a deep link between factorization of polynomials, subvarieties, and principal divisors. Before we can explore this link in the noncommutative setting, we need a definition.

Definition ii.12 (Atomic)**.** *A square matrix of polynomials p is **atomic** if* $\det p \not\equiv 0$ *and if $p_1 p_2 = p$, then either* $\det p_1$ *or* $\det p_2$ *is locally constant.*

Atomic square matrices of polynomials function like irrecudible factors of tradition (commutative) polynomials. While we cannot have truly "unique" factorization, we do have factorization into atoms. In [11], Helton et al. prove the following theorem:

Theorem ii.13. *Let f is a square matrix of nc polynomials and p an atom. If we let $f = p_1 \cdots p_k$ be the factorzation of f into atoms, then*

$$\mathscr{Z}(p) \subset \mathscr{Z}(f)$$

if and only if $\det p = \det(cp_i)$ *for one of the atoms of f and c a nonzero constant.*

With the help of lemma ii.5 (part 1) this says that factorization is unique up to equivalence of principal divisors. Given any square matrix of nc rational expressions, theorem ii.10 allows us to express $\operatorname{div} r$ as a linear combination of divisors of matrices of nc polynomials. Better yet, if we consider the set of all non-degenerate square matrices of rational expressions, the set of divisors is a free abelian group generated by the atomic square matrices of nc polynomials!

4 MONODROMY, GLOBAL GERMS, ALGEBRAIC TOPOLOGY

The results of the last three chapters seem hopeful—free analysis seems to be able to generalize many classical results, as listed in section 2.4. As previously mentioned, the free topology admits an Oka-Weil-type theorem. While this is promising, the only compact sets in the free topology are the envelopes of a finite collection of points.

It is the opinion of the author that all of these topologies (fine, fat, free, etc.) are definitively broken. As shown above, the free topology lacks a wealth of compact sets. The fine topology (and therefore any admissible topology) fails to be T_1, let alone Hausdorff—notice that any open set containing X must also contain $X \oplus X$. Further, given any free function f on an nc-domain D, if f is locally bounded on each D_n then f is analytic (admits a power series representation). There are two ways so view this result: First, one can accept that analytic functions are a dime a dozen on $\mathcal{M}^g$. Alternatively, one can be skeptical that the topological structures put on $\mathcal{M}^g$ are indeed the natural choice. The work of J.E. Pascoe in [17] seeks to solve some of these issues by extending some of the concepts of traditional algebraic topology.

4.1 CLASSICAL MONODROMY

In the study of functions of a complex variable(s),[1] many of the central theorems surround the idea of analytic continuation. Given some analytic function f on a domain $\Omega \subset \mathbb{C}$ and a larger domain $\overline{\Omega} \supset \Omega$, we can (with sufficient "niceness" conditions) extend f to an analytic function g on $\overline{\Omega}$. In particular, given some path γ which start in Ω we can analytically continue f *along* γ by recomputing the power series on overlapping disks with their centers on γ. The standard picture of this process is fig. 4.1.

1 There are nearly identical "monodromy" theorems for functions of one and several complex variables, but we will only treat functions of a single variable in this section.

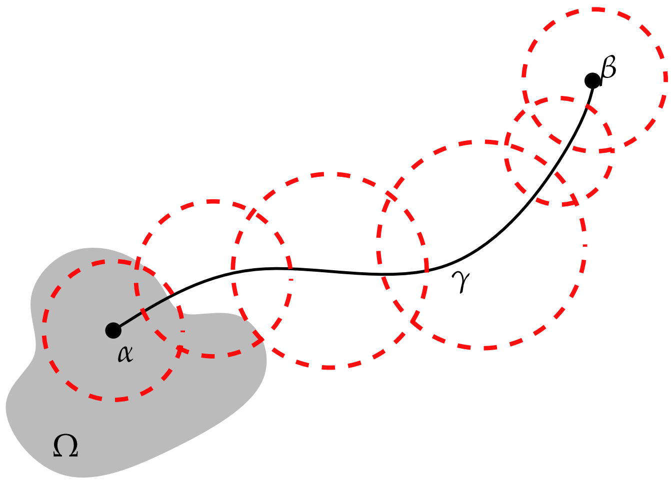

Figure 4.1: Analytic continuation along a curve

Our path γ must avoid any potential poles of f so that we may compute the power series, but the uniqueness of such an extension is not obvious. This is where the aforementioned niceness conditions come into play! For example, consider the setup of fig. 4.2:

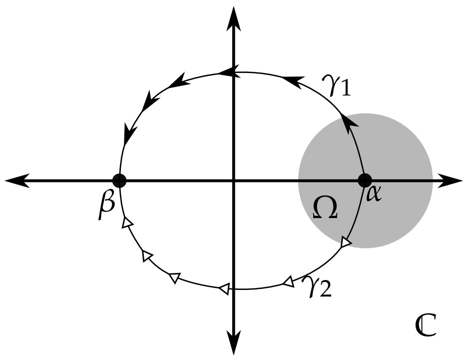

Figure 4.2: Two paths in $\mathbb{C}$

Example ii.14. *If we let $f(x) = \operatorname{Log} x$ be the principle branch of the complex logarithm the defined on the right half plane, and continue f along γ_1 and γ_2 we get two functions f_1 and f_2 which are analytic at β, but they don't agree! In this case, $f_1(\beta)$ and $f_2(\beta)$ disagree by exactly $2\pi i$.*

The monodromy theorem gives sufficient conditions for the continuation along two curves to agree:

Theorem ii.15 (Monodromy I)**.** *Let γ_1, γ_2 be two paths from α to β and Γ_s be a fixed-endpoint homotopy between them. If f can be continued along Γ_s for all $s \in [0,1]$, then the continuations along γ_1 and γ_2 agree at β.*

In the example above, any homotopy between the two paths must pass through the origin—where $\operatorname{Log} x$ fails to be analytic—and hence the two continuations disagree at β. An equivalent formulation of the monodromy theorem concerns extending a functions to a larger domain:

Theorem ii.16 (Monodromy II)**.** *Let $U \subset \mathbb{C}$ be a disk in $\mathbb{C}$ centered at z_0 and $f : U \to \mathbb{C}$ an analytic function. If W is an open, simply connected set containing U and f continues along any path $\gamma \subset W$ starting at z_0, then f has a unique extension to all of W.*

This second formulation gives another perspective on $\operatorname{Log} x$. In the example, U is a disk around α that stays in the right half plane and W is $\mathbb{C} \setminus \{0\}$. While $\operatorname{Log} x$ continues along any path in $\mathbb{C} \setminus \{0\}$, the larger domain is *not* simply connected, so monodromy fails.

4.2 FREE MONODROMY

There is an analogous theorem to theorems ii.15 and ii.16 in the free settings initial proven by J.E. Pasocoe in [18]. In the classic case, the larger set W must be simply connected. In the free setting, however, the theorem is much more powerful. Before we state and prove the theorem, recall that free functions respect direct sums—so if $f : D \to \mathcal{M}^{\tilde{g}}$ is a free function,

$$f(X \oplus Y) = f(X) \oplus f(Y).$$

Given two paths $\gamma_1, \gamma_2 \in D_n$, we can take their direct sum in the obvious way

$$(\gamma_1 \oplus \gamma_2)(t) = \gamma(t) := \begin{bmatrix} \gamma_1(t) & \\ & \gamma_2(t) \end{bmatrix}$$

to obtain a path in D_{2n}. If f is a free function defined on $B \subset D$, and then we can analytically continue f along γ (presuming that γ originates in B). If F is the resulting function defined at $\gamma(1)$, and F_1, F_2 are the continuations at $\gamma_1(1), \gamma_2(1)$ respectively, then a routine computation shows that

$$F(\gamma(1)) = \begin{bmatrix} F_1(\gamma_1(1)) & \\ & F_2(\gamma_2(1)) \end{bmatrix}.$$

With this preliminary result, we can introduce Universal Monodromy.

Theorem ii.17 (Free Univeresal Monodromy)**.** *If f is an analytic free function defined on some ball $B \subset D$, for D an open, connected free set. Then f analytically continues along every path in D if and only if f has a unique analytic continuation to all of D.*

Proof (Should I cite that talk?) The fact that a unique extension to all of D implies that f has a continuation along any γ is immediate.

Now suppose that f, a free function, analytically continues along every path in D. Fix $X \in B_n$ and pick some and let γ_1, γ_2 be two paths taking X to some $Y \in D_n$. Let F_1, F_2 be the analytic continuation of f along γ_1, γ_2 respectively. We seek to show that F_1 and F_2 agree in some neighborhood of $\gamma_1(1)$! Let $\hat{\gamma}, \gamma$ be paths in D_{2n} defined by

$$\hat{\gamma} = \begin{bmatrix} \gamma_1 & \\ & \gamma_2 \end{bmatrix} \qquad \gamma = \begin{bmatrix} \gamma_2 & \\ & \gamma_1 \end{bmatrix}.$$

We have a homotopy between $\hat{\gamma}$ and γ given by

$$\Gamma(t,s) = \begin{bmatrix} \cos(s\frac{\pi}{2}) & \sin(s\frac{\pi}{2}) \\ -\sin(s\frac{\pi}{2}) & \cos(s\frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} \gamma_1(t) & \\ & \gamma_2(t) \end{bmatrix} \begin{bmatrix} \cos(s\frac{\pi}{2}) & -\sin(s\frac{\pi}{2}) \\ \sin(s\frac{\pi}{2}) & \cos(s\frac{\pi}{2}) \end{bmatrix}.$$

Indeed, one easily checks that

$$\Gamma(t,0) = \hat{\gamma} \qquad \Gamma(t,1) = \gamma \qquad \Gamma(0,s) = X \oplus X \qquad \Gamma(1,s) = Y \oplus Y$$

But since $\hat{\gamma}$ and γ are homotopic we can apply the classical (albiet multivariable) monodromy theorem—so we know that the analytic continuations of f along $\hat{\gamma}, \gamma$ must agree near $Y \oplus Y$. Since free functions respect direct sums, if we let $\hat{F}$ and F denote the continuations of f along $\hat{\gamma}, \gamma$ respectively, we obtain the following chain of equalities:

$$\begin{bmatrix} F_1(\gamma_1) & \\ & F_2(\gamma_2) \end{bmatrix} = \hat{F}(\gamma_1 \oplus \gamma_2) = F(\gamma_2 \oplus \gamma_1) = \begin{bmatrix} F_2(\gamma_2) & \\ & F_1(\gamma_1) \end{bmatrix}$$

In particular, we see that $F_1(\gamma_1) = F_2(\gamma_2)$—so F_1 and F_2 agree! ∎

In the free case, the "larger" set need not be simply connected. Analytic continuations of free functions, then, cannot be used to detect holes in matrix domains. It will turn out, however, that the tracial and determinental functions introduced in section 2.2 can detect holes and produce an analogue of the fundamental group!

4.3 THE GERM OF FUNCTION

As studied in complex analytic and measure theoretic settings, if our space is structured enough functions are defined by their local behavior. This idea can be generalized to arbitrary topological spaces with a construction from algebraic geometry.

Let X be a topological space. To any open set U we associate $C(U)$, the ring of continuous functions $f : U \to \mathbb{R}$ (where addition and multiplication are defined point-wise). Given any $V \subset U$, notice that a continuous function f on U, we can restrict f to V and maintain continuity. This gives two maps:

$$\begin{aligned} V &\hookrightarrow U \\ v &\longmapsto v \end{aligned} \qquad\qquad \begin{aligned} C(U) &\hookrightarrow C(V) \\ f &\longmapsto f\,|_V \end{aligned}$$

Notice that the induced function goes the "other way." This construction is an example of a sheaf of rings[2]—since $C(U)$ has a ring structure. We can similarly define sheaves of abelian groups or sets: to each open set in X we assign a group (or set) such that there are analogous restriction maps. For our purposes, these will always be groups/sets of functions and the restriction maps are the natural ones.

We are interested in the general behavior of continuous functions at some $x \in X$. Define $\mathfrak{C}_x$ to be the set of all functions defined on a neighborhood of x:

$$\mathfrak{C}_x = \{f \in C(U) \mid x \in U \subset X \text{ is open}\}.$$

By convention, we refer to elements of $\mathfrak{C}_x$ as a pair, (f, U) of a continuous function and the open set on which it is defined. In light of the inclusion maps given above, it obvious that $\mathfrak{C}_x$ will have "duplicate" elements. Therefore, we define an equivalence relation on $\mathfrak{C}_x$ by $(f, U) \sim (g, V) \Leftrightarrow$ there exists $W \subset U \cap V$ where $f\,|_W = g\,|_W$. In a sheaf-theoretic context, $\mathfrak{C}_x/\sim$ is called the **stalk** at x and elements of the stalk are **germs** at x. If we are dealing with sheaves of groups or sets, this construction remains unchanged—i. e. can still define the stalk at given point. While it will not come into play, it is worth noting that the stalk inherits the algebraic structure of the original sheaf—e. g. for a sheaf of rings, the stalk has a natural ring structure.

2 To be completely rigorous, a sheaf needs additional axioms, but the sheaf of continuous functions is one of the prototypical examples so the full defition is not needed in this context.

Sheafs of rings/groups/sets of functions arise naturally in many areas of mathematics. For example, if X happens to be a smooth manifold, we may replace $C(U)$ with $C^\infty(U)$, the ring of smooth functions into $\mathbb{R}$ and then obtain germs of smooth functions. Similarly, if X is a complex manifold we can construct germs of holomorphic functions.

Example ii.18. *Consider, again, example ii.14. Our function $f(x) = \operatorname{Log} x$ has a germ in Ω. In particular, both f_1 and f_2 belong to the equivalence class $[(f, \Omega)]$ as all three functions agree on Ω. From this, we see the aptness of the name germ: germs capture the local behavior of function. Colloquially, this is the "heart" of a function similar to the germ of seed.*[3]

As usual, lifting this construction to the free context requires some nuance. For $U \subset D$ open, the set of tracial functions on U (denoted $C_{\mathrm{tr}}(U)$) does not form a ring—it is closed under addition but not multiplication. Given two tracial functions, $f, g \in C_{\mathrm{tr}}(U)$, we see that

$$\begin{aligned}(f+g)(X \oplus Y) &= f(X \oplus Y) + g(X \oplus Y) \\ &= f(X) + f(Y) + g(X) + g(Y) \\ &= (f+g)(X) + (f+g)(Y)\end{aligned}$$

but,

$$\begin{aligned}(fg)(X \oplus Y) &= f(X \oplus Y)g(X \oplus Y) \\ &= (f(X) + f(Y))(g(X) + g(Y)) \\ &= (fg)(X) + (fg)(Y) + f(X)g(Y) + f(Y)g(X).\end{aligned}$$

Thankfully, however, the construction remains unchanged if we substitute a ring of functions for an abelian group of functions (with the identity being $f \equiv 0$ and inverses given by simply negating the output). In the case of determinental and free functions (which play a lesser role in the theory to be developed) there is not a natural algebraic structure for the corresponding sheaves, so they are simply sheaves of sets.

4.4 THE TRACIAL FUNDAMENTAL GROUP

While free monodromy means that free functions cannot detect the topology of free sets, the same is not true for a general tracial function! Following [17], we will need some definitions.

3 Sheaf theory abounds with agrarian nomenclature.

Definition ii.19 (Anchored)**.** *Let $D \subset \mathcal{M}^g$ be a connected, open, free set. If there exists a nonempty, simply-connected, open, free $B \subset D$, then we say that D is **anchored**.*

Definition ii.20 (Global Germ)**.** *For D an anchored set, and $B \subset D$ its anchor, we call a tracial function $f : B \to \mathbb{C}$ a **global germ** if it analytically continues along every path in D which starts in B.*

In order to define the fundamental group, we need a notion of a path in D. Traditionally, a path taking X to Y is a continuous function $\gamma : [0,1] \to D$ such that $\gamma(0) = X$ and $\gamma(1) = Y$. Unfortunately, this disregards the fiber of X and Y. An mentioned in section 2.3, a proper topological theory should account for identification of the fibers.

Definition ii.21 (Essential Path)**.** *A continuous function $\gamma : [0,1] \to D$ **essentially takes** X to Y if*

$$\gamma(0) = X^{\oplus \ell}, \text{ for some } \ell \in \mathbb{N}$$
$$\gamma(1) = Y^{\oplus k}, \text{ for some } k \in \mathbb{N}.$$

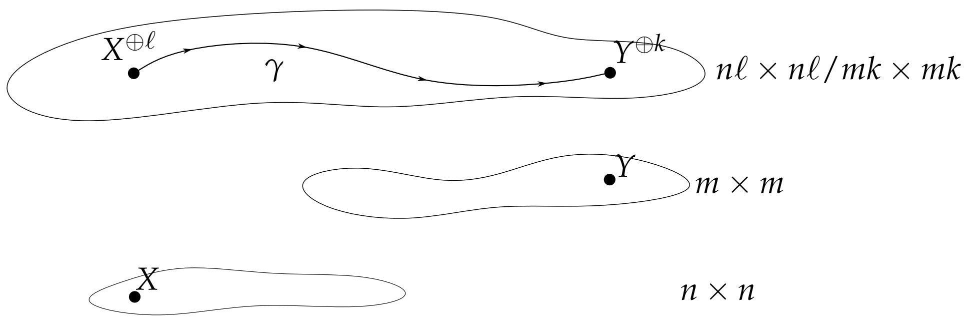

Figure 4.3: A path essentially taking X to Y

A path essentially taking X to Y is a path from some element of the fiber of X to some element of the fiber of Y. Just as in the classical case, essential paths have product. First, we need a way to take the direct sum of paths.

Definition ii.22 (Direct Sum of Paths)**.** *Given γ essentially taking X to Y and β taking Z to W, define*

$$\gamma \oplus \beta(t) := \begin{bmatrix} \gamma(t) & 0 \\ 0 & \beta(t) \end{bmatrix}.$$

It is not, in general, true that $\gamma \oplus \beta$ essentially takes $X \oplus Z$ to $Y \oplus W$. However, if γ essentially takes X to Y, then so does $\gamma \oplus \gamma$. As with matrices we define

$$\gamma^{\oplus k} := \underbrace{\gamma \oplus \cdots \oplus \gamma}_{k \text{ times}}$$

With these preliminaries, we can now define a concatenation product for essential paths:

Definition ii.23 (Concatenation Product)**.** *Let γ and β be paths taking X to Y and Y to Z respectively. We define their product to be the path essentially taking X to Z given by*

$$\beta\gamma(t) := \begin{cases} \gamma^{\oplus k}(2t) & t \in [0, 0.5) \\ \beta^{\oplus \ell}(2t-1) & t \in [0.5, 1] \end{cases}$$

where k and ℓ are positive integers chosen to make $\gamma^{\oplus k}$ and $\beta^{\oplus \ell}$ like size matrices for each t.

With essential paths and their product we can build the first analogue of the fundamental group. Let D be an anchored space with B its anchor. For $X \in B$, we define $\pi_1(D, X)$ to be the set of path essentially taking X to X up to traditional homotopy equivalence and the relation $\gamma = \gamma^{\oplus k}$. Section 6 of [17] explores this construction in detail, including proving its commutativity. At the moment, not much can be said about the full fundamental group. Instead, we restrict ourselves to a different (abliet related) group of paths which are determined by the analytic continuation of tracial functions.

Given a path essentially taking X to Y we can view the path as coupled with its endpoint. For B and anchor and f a global germ, we can reasonably define $f(\gamma)$: analytically continue f along γ and define

$$f(\gamma) := \frac{1}{k} f(Y^{\oplus k}).$$

Since we can evaluate paths with global germs, we can use global germs to distinguish between certain paths.

Definition ii.24 (Trace Equivalent)**.** *Let $B \subset D$ be an anchor and fix $X \in D$. If γ and β both essentially take X to Y, we say they are* ***trace equivalent*** *if, for every global germ f and every path δ taking Y to Z, $f(\delta\gamma) = f(\delta\beta)$.*

That is, trace equivalent paths are those which cannot be told apart via analytic continuation of global germ.

Under trace equivalence, the normalization given above implies $\gamma = \gamma^{\oplus k}$ since both essentially take X to Y. Further, since homotopic paths have the same analytic continuation, homotopic paths are trace equivalent. This allows us to define a second fundamental group which will be our central object of study.

Definition ii.25 (Tracial Fundamental Group)**.** *Let D be an anchored space with B its anchor. For $X \in B$ define $\pi_1^{tr}(D, X)$ to be the group of trace equivalent paths essentially taking X to X.*

If D is connected, then $\pi_1^{\text{tr}}(D)$ is independent of our choice of base point—in fact, the isomorphism from the classical case works here as well. The identity is given by γ^X, the constant path at X and inverses given by

$$\gamma^{-1}(t) = \gamma(1-t).$$

Note that, since fixed endpoint homotopic paths are trace equivalent, $\pi_1^{\text{tr}}(D)$ is a quotient of $\pi_1(D)$. We can construct a covering space for D with respect to $\pi_1^{\text{tr}}(D)$ similar to the construction of the universal cover in [8].

Definition ii.26 (Tracial Covering Space)**.** *For $X \in B \subset D$, the **tracial covering space** of D is the set of paths (up to tracial equivalence[4]) in D starting at X:*

$$C^{tr}(D) = \{[\gamma] \mid \gamma \textit{ a path essentially taking } X \textit{ to } Y\}$$

Since we identify paths with their terminal endpoint, we have the natural covering space map $\rho : C^{\text{tr}}(D) \to D, [\gamma] \mapsto Y$. In order for this map to be continuous (and obey the rest of the axioms of a covering space), we need to endow $C^{\text{tr}}(D)$ with a topology. We do so be defining a metric $d : C^{\text{tr}}(D) \times C^{\text{tr}}(D) \to \mathbb{R} \cup \{\infty\}$. Let γ_1 be a path essentially taking X to Y and γ_2 a path essentially taking X to Z. If Y and Z are different sizes, set $d(\gamma_1, \gamma_2) = \infty$. On the other hand, say that both Y and Z are $n \times n$ matrices and let $\Gamma_{Y,Z}$ be the set of path in D_n taking Y to Z. If $\|\gamma\|$ denotes the length of γ in $\mathbb{C}^{n^2}$, then we set

$$d(\gamma_1, \gamma_2) := \inf\{\|\gamma\| \mid \gamma \in \Gamma_{Y,Z} \text{ such that } \gamma\gamma_1 = \gamma_2\}.$$

With the topology induced by the metric, one can easily verify that we do, indeed, have a covering space.

Because B is simply connected, for any $Y \in B$ there is exactly one path essentially taking X to Y. In light of this, there is a natural inclusion $B \hookrightarrow C^{\text{tr}}(D)$. Given a global germ, f, we induce a function on the covering space (given by $f(\gamma)$), which the norm on $C^{\text{tr}}(D)$ forces to be analytic.

4 From here on, unless otherwise specified, we will only refer to paths up to trace equivalence.

4.5 A BIT OF COHOMOLOGY

For a complete treatment of Cohomology, see Allan Hatcher's famous *Algebraic Topology* [8]. As a quick review, given some chain complex

$$\cdots \xleftarrow{\partial_{n-1}} C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} \cdots$$

we can form the for the **cochain complex** as follows: First, fix some abelian group G. The objects in out cochain complex are $C^\bullet = \mathrm{Hom}(C_\bullet, G)$ the group of morphisms $C_\bullet \to G$. The maps are induced ones

$$d = \partial^* : \mathrm{Hom}(C_n, G) \to \mathrm{Hom}(C_{n+1}, G).$$

Put together, this gives us a complex with the arrows reversed

$$\cdots \xrightarrow{d_{n-2}} C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \xrightarrow{d_{n+1}} \cdots$$

Computing the homology of this dual complex gives the *cohomology* groups, $H^\bullet(C; G) = \mathrm{Ker}\ d / \mathrm{Im}\ d$—which only depend on the homology of the original complex and the choice of G. Of particular interest is **De Rham Cohomology**, where $C^k = \Omega^k$, the set of k-forms on a manifold, and $G = \mathbb{R}$. In this case, the boundary map is given by the exterior derivative. For a full construction of De Rham Cohomology, see [15]. In the classical case we say that a k-form is **closed** if $\mathrm{d}f = 0$ and **exact** if there exists a k–1-form, g, such that $\mathrm{d}g = f$. The kth De Rham cohomology group, then, is the vector space of closed forms moduluo the exact forms.

A full cohomology theory has yet to be developed in the free setting. Lifting De Rham cohomology appears promising given that considering $\mathcal{M}^g$ carries a natural (if complex) manifold structure. While there is not a full generalization of the exterior derivative, recall that for any global germ f, we have that ∇f is a free function. If $\mathcal{T}$ is the set of tracial functions and $\mathcal{F}$ is the set of free functions, we have the beginning of a cochain complex

$$0 \to \mathcal{T} \xrightarrow{\nabla} \mathcal{F} \to \cdots.$$

While we cannot define "closed" and "exact" for general functions on $\mathcal{M}^g$ (in part because we don't know what the general cochain groups are) we can define them on $\mathcal{F}$.

Definition ii.27 (Exact)**.** *A free function $g : D \to \mathcal{M}^g$ is **exact** if there exists a tracial function $f : D \to \mathbb{C}$ such that $\nabla f = g$.*

Definition ii.28 (Closed)**.** *A free function $g : D \to \mathcal{M}^g$ is **closed** if*

$$\operatorname{tr}(K \cdot Dg(X)[H]) = \operatorname{tr}(H \cdot Dg(X)[K])$$

for all directions H, K.

While exactness is a direct lift of the classical condition, our definition of closed is decidedly unenlightening. Recall from the discussion at the end of section 2.1 that the combination of the trace of a dot product is a bilinear form on $\mathcal{M}^g$ which we can think of as an inner product.[5] more after we talk today. With our definitions, we can define the first tracial cohomology group.

Definition ii.29 (First Tracial Cohomology Group)**.** *The **first tracial cohomology group** is the vector space of closed free functions moduluo the exact free function. We write $H^1_{tr}(D)$.*

At first glance, H^1_{tr} seems rather convoluted, arbitrary, and not particularly useful. [6] Thankfully, we can put the tracial cohomology group to immediate use in understanding the structure of $\pi_1^{\text{tr}}(D)$. Recall that, by definition, a global germ $f : B \to \mathbb{C}$ analytically continues along every path. It follows, then, that ∇f must analytically continue along every path as well—simply analytically continue f along the path and then take its gradient. Since ∇f is a free function, univeral monodromy (theorem ii.17) tells us that ∇f has a unique continuation to all of D.

At first glance it would seem that ∇f (for f a global germ) is trivial in $H^1_{\text{tr}}(D)$. This, however, is not the case. A free function g, is exact if there is a *tracial function*, $\hat{f}$ defined on *all of* D such that $g = \nabla \hat{f}$. Given a *global germ* $f : B \to \mathbb{C}$, it is not always the case that f has a unique extension to all of D—hence ∇f is not necessarily exact!

Because monodromy holds for the gradient of a global germ we know that for any $\gamma \in \pi^1_{\text{tr}}(D)$, $f : B \to \mathbb{C}$ a global germ, and γ' a path starting our anchor point X, the function

$$\Phi^f_\gamma(\gamma') = f(\gamma'\gamma) - f(\gamma')$$

is locally constant as a function on $C^{\text{tr}}(D)$. Because of the metric we put on $C^{\text{tr}}(D)$, showing that this function is locally constant

5 Albeit, without conjugates.

6 In all fairness, this is most people's reaction when the encounter cohomology for the first time.

is rather finicky. This result has a direct analogue in classical complex analysis, and the proof similar so it will be omitted. We remark that Φ_γ^f measure the action of γ on a global germ f when continuing along a path γ'. We can use to prove the following technical lemma.

Lemma ii.30. *Let D be an nc domain. For any $\alpha, \beta \in \pi_1^{\mathrm{tr}}(D)$ and global germ f,*

$$f(\alpha\beta) - f(\alpha) = f(\beta) - f(\gamma_X).$$

Proof. First, see that $f(\beta) = f(\gamma_X\beta)$, so we need to show that $\Phi_\beta^f(\alpha) = \Phi_\beta^f(\gamma_X)$. Let Γ be a path in $C^{\mathrm{tr}}(D)$ defined by

$$\Gamma(t) = \alpha\Big|_{[0,t]}.$$

That is, Γ is the path (in the space of path) where, at each time step Γ is the path given by going t through α. Continuity of Γ is immediate from our metric—since the distance between $\Gamma(t)$ and $\Gamma(t+\varepsilon)$ can be made arbitrarily small.

Since we have path between γ_X and α, they are in the same path component of $C^{\mathrm{tr}}(D)$. Therefore, since Φ_β^f is locally constant, it must be the case that $\Phi_\beta^f(\alpha) = \Phi_\beta^f(\gamma_X)$. ∎

Given f a global germ and $X \in B_n$ the anchor point and $\gamma \in \pi_1^{\mathrm{tr}}$, define

$$c^f(\gamma) := \frac{f(\gamma) - f(\gamma^X)}{n}.$$

We remark that, while it is a notation nightmare, $c^f(\gamma) = \Phi_{\gamma_X}^f(\gamma)$. c^f maps into $\mathbb{C}$ and some routine work with ∇ shows that only that if $c^f = c^{f'}$ then, ∇f and $\nabla f'$ are in the same tracial cohomology class—i. e., c^f only depends on the class of ∇f in $H^1_{\mathrm{tr}}(D)$. If we define $\phi_g : \pi_1^{\mathrm{tr}} \to \mathbb{C}, \gamma \mapsto c^f(\gamma)$—where ∇f is in the tracial cohomology class of g—we get a *homomorphism* into $\mathbb{C}$, as

$$\begin{aligned} c^f(\gamma_1\gamma_2) &= \frac{f(\gamma_1\gamma_2) - f(\gamma_X)}{n} \\ &= \frac{f(\gamma_1\gamma_2) - f(\gamma_1) + f(\gamma_1) - f(\gamma_X)}{n} \\ &= \frac{f(\gamma_X\gamma_2) - f(\gamma_X) + f(\gamma_1) - f(\gamma_X)}{n} \\ &= c^f(\gamma_2) + c^f(\gamma_1). \end{aligned}$$

Where the penultimate equality uses lemma ii.30. The fact that ϕ_g is a homomorphism is the first step to characterizing $\pi_1^{\mathrm{tr}}(D)$.

Lemma ii.31. *The map*

$$\begin{aligned}\Phi : \prod_{g \in H^1_{\mathrm{tr}}} \pi_1^{\mathrm{tr}}(D) &\longrightarrow \prod_{g \in H^1_{\mathrm{tr}}} \mathbb{C} \\ \prod_{g \in H^1_{\mathrm{tr}}} \gamma &\longmapsto \prod_{g \in H^1_{\mathrm{tr}}} \phi_g(\gamma)\end{aligned}$$

is an injective homomophism.

Proof. The fact that Φ is a homomorphism is immediate, as each of the ϕ_g are. For injectivity, let $\alpha, \beta \in \pi_1^{\mathrm{tr}}(D)$ such that $\prod \phi_g(\alpha) = \prod \phi_g(\beta)$. Seeking to show that α and β are trace equivalent, let f be a global germ and γ essentially take X to Z. Then, once again using lemma ii.30,

$$\begin{aligned} f(\gamma\alpha) - f(\gamma\beta) &= f(\gamma\alpha) - f(\gamma) - (f(\gamma\beta) - f(\gamma)) \\ &= c^f(\alpha) - c^f(\beta) \end{aligned}$$

But since $\prod \phi_g(\alpha) = \prod \phi_g(\beta)$, and c^f only depends on the class of ∇f, $c^f(\alpha) = c^f(\beta)$. Thus, α and β are trace equivalent and we have shown injectivity. ∎

Note that lemma also tells us that $\pi_1^{\mathrm{tr}}(D)$ is both commutative and torsion free as is injects into a commutative, torsion free group (namely a product of $\mathbb{C}$'s). With this, we can also show that $\pi_1^{\mathrm{tr}}(D)$ needs to be divisible as well. First, note that for any path γ,

$$\gamma \oplus \gamma_X = \gamma_X \oplus \gamma,$$

since

$$H(t,\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} (\gamma \oplus \gamma_X) \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^*$$

is a homotopy between the paths. Then we see that

$$\begin{aligned}
\gamma &= \underbrace{\begin{bmatrix} \gamma & & & \\ & \gamma & & \\ & & \ddots & \\ & & & \gamma \end{bmatrix}}_{k+1\text{-times}} \\
&= \begin{bmatrix} \gamma & & & \\ & \gamma\chi & & \\ & & \ddots & \\ & & & \gamma\chi \end{bmatrix} \begin{bmatrix} \gamma\chi & & & \\ & \gamma & & \\ & & \ddots & \\ & & & \gamma \end{bmatrix} \cdots \begin{bmatrix} \gamma & & & \\ & \gamma & & \\ & & \ddots & \\ & & & \gamma\chi \end{bmatrix} \\
&= \begin{bmatrix} \gamma & & & \\ & \gamma\chi & & \\ & & \ddots & \\ & & & \gamma\chi \end{bmatrix} \begin{bmatrix} \gamma & & & \\ & \gamma\chi & & \\ & & \ddots & \\ & & & \gamma\chi \end{bmatrix} \cdots \begin{bmatrix} \gamma & & & \\ & \gamma\chi & & \\ & & \ddots & \\ & & & \gamma\chi \end{bmatrix} \\
&= \begin{bmatrix} \gamma & & & \\ & \gamma\chi & & \\ & & \ddots & \\ & & & \gamma\chi \end{bmatrix}^k,
\end{aligned}$$

and so $\pi_1^{\mathrm{tr}}(D)$ is divisible. As there is only one way (up to isomorphism, of course) to be a divisible, torsion free subgroup of $\mathbb{C}$, we have completely characterized $\pi_1^{\mathrm{tr}}(D)$!

Theorem ii.32. *For D an anchored free set,*

$$\pi_1^{\mathrm{tr}}(D) \simeq \bigoplus_{i \in I} \mathbb{Q} = \mathbb{Q}^I$$

for some set I.

4.6 COMPUTING THE TRACIAL FUNDAMENTAL GROUP

While this structure theorem is useful, it gets us no closer to actually *computing* $\pi_1^{\mathrm{tr}}(D)$ or $H^1_{\mathrm{tr}}(D)$. Unfortunately, there is nothing analogous to Van Kampen's theorem or the Mayer Vietoris sequence. For simple domains, we have some basic tools. Recall the following definion relating to abelian groups:

Definition ii.33 (Rank). *For an abelian group, G, the **rank** of G is the maximal size of a linearly independent subset. That is, it the maximal size of a set $\{g_1, g_2, \ldots, g_k\} \subset G$ such that*

$$\sum_{i=0}^{k} n_i g_i = 0 \implies n_i = 0 \text{ for all } i.$$

Theorem ii.34. *Let D be a free anchored set. Then,*

1. $\dim H^1_{\mathrm{tr}}(D) \leq \operatorname{rk} \pi_1^{\mathrm{tr}}(D)$ *whenever both quantities are at most countably infinite*
2. $\dim H^1_{\mathrm{tr}}(D) \neq 0$ *if and only if* $\operatorname{rk} \pi_1^{\mathrm{tr}}(D) \neq 0$

Proof.

1. We can restrict ourselves to the case where $\operatorname{rk} \pi_1^{\mathrm{tr}}(D)$ is finite. Let $\gamma_1, \ldots, \gamma_k$ be a maximally linearly independent set of paths, and suppose that $g_1, \ldots, g_{k+1} \in H^1_{\mathrm{tr}}(D)$ is linearly independent. Now consider the matrix $[\phi_{g_j}(\gamma_i)]_{ij}$, which is clearly singular. If $(\alpha_1, \alpha_2, \ldots, \alpha_{k+1})$ is a nontrivial vector in its kernel, then we can define $g = \sum_{j=0}^{k+1} \alpha_j g_j$. By construction, $g(\gamma_i) = 0$ for all i. Since $\{\gamma_j\}$ is maximal, it follows that g is the zero function, contradicting our assumption that $\{g_i\}$ is linearly independent.

2. Suppose that $\operatorname{rk} \pi_1^{\mathrm{tr}}(D) \neq 0$—hence there is at least one γ, a global germ, f, and a path β essentially taking X to Z such that $f(\beta) \neq f(\beta\gamma)$. It follows that f does not have a global extension to all of D—hence ∇f is nontrivial in $H^1_{\mathrm{tr}}(D)$.

   Conversely, suppose that $\operatorname{rk} \pi_1^{\mathrm{tr}}(D) = 0$. By part 1 of this same theorem, $\dim H^1_{\mathrm{tr}}(D) = 0$ as well.

■

This bound on the dimension of $H^1_{\mathrm{tr}}(D)$ is useful, but only if we have some way to reliably compute $\pi_1^{\mathrm{tr}}(D)$. Under certain circumstances—which are not particularly difficult to satisfy—we can compute $\pi_1^{\mathrm{tr}}(D)$ as a direct limit of groups by looking at the levelwise homology groups. Classically, the Hurewicz theorem says that first homology group of a path connected manifold is isomorphic to the abelization of the fundamental group. Since we require the domain D to be path connected, we can leverage this fact to compute $\pi_1^{\mathrm{tr}}(D)$.

Let D be an anchored, free, path connected set such that each D_n is nonempty. Choose an anchor $B \subset D$ such that each B_n is also nonempty. If $X \in B_1$ is our base point, then we have a natural gradation on $\pi_1^{\text{tr}}(D)$. Let $\pi_1^{\text{tr}}(D)_n$ denote the subgroup of paths contained in D_n. For any $m \in \mathbb{N}$, there is a natural inclusion map $\pi_1^{\text{tr}}(D)_n \hookrightarrow \pi_1^{\text{tr}}(D)_{mn}$ given by $\gamma \mapsto \gamma^{\oplus m}$. Since our base point is in on the "scalar" level, we get a sequence of maps

$$\pi_1^{\text{tr}}(D)_1 \hookrightarrow \pi_1^{\text{tr}}(D)_2 \hookrightarrow \pi_1^{\text{tr}}(D)_6 \hookrightarrow \cdots \hookrightarrow \pi_1^{\text{tr}}(D)_{n!} \hookrightarrow \cdots$$

As long as one isn't too fearful of universal properties, it is not difficult to show that the direct limit of this sequence is isomorphic to $\pi_1^{\text{tr}}(D)$. Using this result for computation requires understanding the structure of $\pi_1^{\text{tr}}(D)_n$. Since $\pi_1^{\text{tr}}(D)_n$ only contains paths in D_n, it is isomorphic to a quotient of some subgroup of $\pi_1(D_n)$! Moreover, since we are restricting ourselves to a fixed level, we can leverage the Hurewicz theorem. Since $\pi_1^{\text{tr}}(D)_n$ is abelian is actually a quotient of $H_1(D_n)$. Thus, we can realize $\pi_1^{\text{tr}}(D)$ as a direct limit of quotients of $H_1(D_{n!})$!

4.7 SOME EXAMPLES

The tracial fundamental group is a very new idea, so examples of computation don't abound. Pascoe's paper provides two examples as exercises. We present a "topological" proofs and invite the reader to fill in the details.

Example ii.35. *Let $D = GL(\mathbb{C}) = \bigcup_{n\in\mathbb{N}} GL_n(\mathbb{C})$. Consider the case where $n = 1$. If we view complex numbers as 1×1 matrices, then $\{\det z = z \neq 0\} = \mathbb{C} \setminus \{0\}$. Then $\pi_1^{\text{tr}}(GL)_1$ is a quotient of $H_1(GL_1(\mathbb{C})) \simeq \mathbb{Z}$. Since there are no nontrivial torsion free quotients of $\mathbb{Z}$, it must be the case that $\pi_1^{\text{tr}}(GL)_1 \simeq \mathbb{Z}$ as well.*

Additionally, we know that there is a natural inclusion $\pi_1^{\text{tr}}(GL)_1 \hookrightarrow \pi_1^{\text{tr}}(GL)_2$, and so $\pi_1^{\text{tr}}(GL)_2$ contains a copy of $\mathbb{Z}$. Moreover, given some $\gamma \in \pi_1^{\text{tr}}(GL)_1$, we have that

$$\begin{bmatrix} \gamma & \\ & \gamma_X \end{bmatrix} \in \pi_1^{\text{tr}}(GL)_2.$$

Recall that if we square this element, then we get γ—so $\pi_1^{\text{tr}}(GL)_2$ is isomorphic to the group

$$\mathbb{Z}\left[\frac{1}{2}\right].$$

Our next inclusion $\pi_1^{\text{tr}}(GL)_2 \hookrightarrow \pi_1^{\text{tr}}(GL)_6$ *picks up cube roots for the same reason—since*

$$\begin{bmatrix} \gamma & & \\ & \gamma X & \\ & & \gamma X \end{bmatrix} = \begin{bmatrix} \gamma & & & & & \\ & \gamma & & & & \\ & & \gamma X & & & \\ & & & \gamma X & & \\ & & & & \gamma X & \\ & & & & & \gamma X \end{bmatrix}.$$

Taking the square and cube roots simultaneously, we also obtain 6th roots.

$$\pi_1^{\text{tr}}(GL)_6 \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$$

In the n-th inclusion, then, we pick up n-th roots and any other factors needed for closure—and so we adjoin $\frac{1}{n}$ *to the preceeding group. The direct limit is, therefore,*

$$\pi_1^{\text{tr}}(GL) \simeq \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right] \simeq \mathbb{Q}$$

Example ii.36. *Let* $\Lambda \subset \mathbb{C}$ *be finite and define*

$$G_\Lambda = \{X \in \mathcal{M} \mid detX - \lambda \neq 0 \textit{ for all } \lambda \in \Lambda\}.$$

We compute $\pi_1^{\text{tr}}(G_\Lambda)$ *just as above. First, see that* $(G_\Lambda)_1 = \mathbb{C} \setminus \Lambda$, *and so* $\pi_1^{\text{tr}}(G_\Lambda)_1 \simeq H_1((G_\Lambda)_1) \simeq \mathbb{Z}^{|\Lambda|}$. *Inclusion into* $\pi_1^{\text{tr}}(G_\Lambda)_2$ *picks up square roots, so*

$$\pi_1^{\text{tr}}(G_\Lambda)_2 \simeq \mathbb{Z}^{|\Lambda|}\left[\frac{1}{2}\right]$$

Inclusion into $\pi_1^{\text{tr}}(G_\Lambda)_6$ *picks up cube and 6th roots, and so on. Therefore, in the direct limit, we see*

$$\pi_1^{\text{tr}}(G_\Lambda) \simeq \mathbb{Q}^{|\Lambda|}.$$

Part III

APPENDIX

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