Searching for Holes in the Matrix Universe

Lucas Kerbs

Spring 2022

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1. Eventual goal: lift the tools of algebraic topology to spaces of matrices
2. If we have a fixed size, use classical results but this doesn't work for multiple sizes.
3. For multiple (which hopefully I can convince you is interesting) we need to develop some hefty tools
4. To do so, we need to go back to our mathematical roots

Objects and Maps

A Naive Attempt

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1. That's right—objects and maps.
2. Our Naive attempt involves that looking at lifting functions on $\mathbb{R}$ or $\mathbb{C}$ to accept matrices as their input.

Functional Calculus

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- Since we are SA, we can diagonalize
- Watch what happens when we plug this into our polynomial

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$$f(A) = a_n A^n + \cdots + a_1 A + a_0 I_k$$

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Let $g : [a, b] \to \mathbb{C}$ and $D \subset \mathbb{H}$ denote the set of self adjoint matrices with their spectrum in $[a, b]$.

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Directional Derivative

Definition: *Directional Derivative*
Fix some $X \in \mathbb{H}_n$. The derivative of f at X in the direction $H \in M_n(\mathbb{C})$ is

$$Df(X)[H] = \lim_{t \to 0} \frac{f(X + tH) - f(X)}{t}$$

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- Now we consider an example. Since $Df(X)[H]$ is linear, we can just work with a single monomial
- First we expand $(x + th)^3$—but we can't use the binomial theorem since x and h don't commute
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From here, we can calculate:

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And so the first 3 directional derivatives are:

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How should we treat multivariable functions?

What about $f(x, y) = xy \in \mathbb{C}[x, y]$? For $X, Y \in M_n(\mathbb{C})$, what is $f(X, Y)$?

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└Multivarible Functions
└How should we treat multivariable functions?

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Example: $f(x, y) = x^2 - xyx - 1 \in \mathbb{R}\langle x, y\rangle$

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

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Objects and Maps

A Second Attempt

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Objects and Maps
A Second Attempt

1. Second attempt takes polynomials as an example object instead of as a thing to lift

Some Definitions

Definition: *Matrix Universe*
The g-dimensional **Matrix Universe** is

$$\mathcal{M}^g = \bigcup_{n \in \mathbb{N}} (M_n(\mathbb{C}))^g$$

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We say $D \subset \mathcal{M}^g$ is a **free set** if it is closed with respect to direct sums and unitary conjugation. That is,

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