Arc Vector Algebra (AVA): Foundations, Theory, and Applications in Differential Geometry

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Foreword

The idea of treating vectors as functions, i.e *arcs*, rather than as traditional finite coordinate lists, first originated during the author's studies at Bergen University College, Faculty of Engineering (2007–2010). The initial question was rather simple:

> *If we interpret a vector as a smooth, bounded function segment $\vec{f} = (f(x), a, b)$ rather than a finite list of scalars, can we extend classical linear operations — such as addition, projection, and inner products into a richer functional space? Does such a redefinition allow us to harness both the intuitive power of linear algebra and the expressive reach of real Hilbert spaces, particularly in applications like trajectory design, curvature analysis, and signal encoding?*

That thought experiment evolved into a deeper goal: could the same formalism ease the leap from linear algebra to differential geometry—especially for those learning the math behind Einstein's field equations?

Through extensive dialogue[1] and gradual formalization, the present paper took shape. Its purpose is two-fold:

1. to frame *Arc Vector Algebra* (AVA) as a rigorous functional Hilbert-space setting, and
2. to provide an intuitive "bridge" that engineering, mathematics, and physics students can traverse in the limited time allotted to their foundational courses.

Although the underlying Hilbert-space machinery is well established (see, for example, [2, 3]), the present work is, to the author's knowledge, the first to *package curve–segment operations in a vector-style API* ("arc vectors") and to demonstrate their didactic power across mechanics, electromagnetism, and power-system phasors within one consistent notation. The author hopes that AVA will serve both as an accessible entry point to differential geometry and as a basis for rigorous testing, critique, and refinement by the scientific community.

[1] Including iterative drafting with the assistance of state-of-the-art large language models.

Appendix A provides formal mathematical proofs of AVA, developed with assistance from the GPT model o3. To keep the main text accessible, these detailed theoretical justifications are included there for mathematically inclined readers.

Abstract

Arc Vector Algebra (AVA) re-casts a *curve segment*— an *arc vector*—as the basic element of a linear-looking, but geometrically nonlinear, vector space. After proving that the space of smooth arc vectors is a real *inner-product space whose norm-completion is the Hilbert space* $L^2[a, b]$,and after introducing a Wronskian determinant analogue, we develop the core toolkit:

- construction of *resultant arcs* by simple addition,
- Rayleigh-quotient eigen-arcs that quantify global curvature,
- arc length and inter-arc angle as one-number descriptors, and
- a linear-independence test via the AVA determinant.

A self-contained TikZ case study—ballistic flight from a 10-foot platform with and without a head-wind—shows how AVA manipulates entire non-linear trajectories as if they were ordinary vectors, instantly generating modified paths and curvature metrics without re-solving the underlying dynamics.

We close by outlining current application domains (trajectory design, animation, aero-elastic analysis, ML embeddings, and pedagogy) and sketching future research directions (higher-dimension "surf-vectors," non-linear operator spectra, GPU-accelerated libraries, categorical reformulations, and automatic- differentiation-driven optimisation). Thus AVA serves both as an intuitive gateway to differential geometry and as a practical algebraic platform for curved-motion problems across science and engineering.

1 Arc Vectors and Basic Operations

Definition. An *arc vector* is a triple

$$\vec{f} = (f(x), a, b), \qquad a < b, \qquad f \in C^\infty[a, b],$$

interpreted as the directed curve $y = f(x)$ for $x \in [a, b]$.

For two arc vectors on the *same* interval $[a, b]$:

$$\vec{f} + \vec{g} = (f + g, a, b), \qquad \lambda \vec{f} = (\lambda f, a, b), \quad \lambda \in \mathbb{R}.$$

Proposition 1. *With these operations the set* $\mathcal{V} = \{(f, a, b) \mid f \in C^\infty[a, b]\}$ *is a real vector space.*

Inner product (AVA dot product).

$$\langle \vec{f}, \vec{g} \rangle = \int_a^b f(x)\, g(x)\, dx.$$

It is bilinear, symmetric, and positive-definite; hence $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ is a real *inner-product space.* Its norm-completion is the real Hilbert space $L^2[a, b]$.

2 Resultant Arc Example

Take two arcs on $[0,1]$

$$\vec{u} = (x^2, 0, 1), \qquad \vec{v} = (\sin x, 0, 1).$$

Their **resultant arc** (vector sum) is

$$\vec{w} = \vec{u} + \vec{v} = (x^2 + \sin x, 0, 1).$$

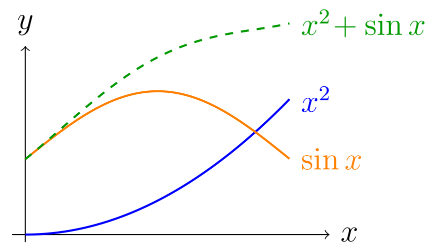

Figure 1: Two AVA vectors (solid) and their resultant arc (dashed).

3 Eigen-Analysis of the Resultant Arc

Domain convention. Throughout this section we restrict to arc vectors f satisfying $f(a) = f(b) = 0$ and $f \in H^2 \cap H_0^1([a,b])$. With this domain the differential operator

$$T = \frac{d^2}{dx^2}$$

is self-adjoint, so the Rayleigh quotient below is well-defined.

In AVA an **eigen-arc** is defined relative to a linear operator $T : C^\infty[a,b] \to C^\infty[a,b]$:

$$T\,\vec{f} = \lambda\,\vec{f} \quad \Longleftrightarrow \quad T\,f(x) = \lambda\,f(x).$$

A natural choice for trajectory problems is the second derivative $T = \dfrac{d^2}{dx^2}$, which appears in beam theory and classical vibration.

Rayleigh–quotient eigenvalue

For any non-zero arc vector $\vec{f}$ we can assign an *effective* eigenvalue by the Rayleigh quotient

$$\lambda_{\text{eff}}(\vec{f}) \;=\; \frac{\langle T\vec{f}, \vec{f}\rangle}{\langle \vec{f}, \vec{f}\rangle} \;=\; \frac{\displaystyle\int_a^b f''(x)\,f(x)\,dx}{\displaystyle\int_a^b f^2(x)\,dx}.$$

Component arcs

Over $[0,1]$ we defined $\vec{u} = (x^2, 0, 1)$, $\vec{v} = (\sin x, 0, 1)$, and the resultant $\vec{w} = \vec{u} + \vec{v}$.

$$\lambda_{\text{eff}}(\vec{u}) = \frac{\displaystyle\int_0^1 2\,x^2\,dx}{\displaystyle\int_0^1 x^4\,dx} = \frac{2/3}{1/5} = \frac{10}{3} \approx 3.33,$$

$$\lambda_{\text{eff}}(\vec{v}) = \frac{\displaystyle\int_0^1 (-\pi^2)\sin^2 x\,dx}{\displaystyle\int_0^1 \sin^2 x\,dx} = -\,\pi^2 \approx -9.87.$$

Resultant arc

Write $w(x) = x^2 + \sin x$. Routine calculus gives

$$\langle\, T\vec{w}, \vec{w}\rangle = \frac{2}{3} - \frac{\pi^2}{2}, \qquad \langle\, \vec{w}, \vec{w}\rangle = \frac{1}{5} + \frac{8}{\pi^3} + \frac{1}{2},$$

so

$$\boxed{\lambda_{\text{eff}}(\vec{w}) \;\approx\; -4.46}$$

Interpretation

- $\vec{u}$ (pure parabola) is *compressive* ($\lambda_{\text{eff}} > 0$) under $T = f''$, matching the fact that x^2 is *concave up.*
- $\vec{v}$ (sine) is an *exact eigen-arc* with $\lambda = -\pi^2$ (concave–down oscillation).
- The resultant $\vec{w}$ is *not* an eigen-arc, but its Rayleigh value lies between those of its constituents, reflecting the superposed curvatures.

Thus eigen-analysis in AVA offers a compact, quantitative measure of *overall curvature tendency* for any nonlinear trajectory: one number per arc, obtained without solving a new differential equation.

4 Ball Thrown from 10 ft with Head-On Wind (AVA Demo)

Launch data

$$h_0 = 10 \text{ ft} \;=\; 3.048 \text{ m}, \qquad v_0 = 5 \text{ m/s}, \qquad \theta = 45^\circ, \quad g = 9.81 \text{ m/s}^2,$$
$$v_x = v_0 \cos\theta \approx 3.535 \text{ m/s}, \quad v_y = v_0 \sin\theta \approx 3.535 \text{ m/s}.$$

Arc vectors

$$\begin{aligned}
\vec{x}(t) &= (v_x t,\ 0,\ T),\\
\vec{y}(t) &= \left(h_0 + v_y t - \tfrac{1}{2} g t^2,\ 0,\ T\right),\\
\vec{w}(t) &= (-1\cdot t,\ 0,\ T) \qquad \text{(head-wind arc)},\\
\vec{x}_{\text{tot}} &= \vec{x} + \vec{w}.
\end{aligned}$$

The flight ends when $y(t) = 0$: $T \approx 1.226$ s.
Functional inner product (no wind)

$$\langle \vec{x}, \vec{y} \rangle = \int_0^T x(t)\, y(t)\, dt.$$

With wind, replace $x(t)$ by $x_{\text{tot}}(t)$.

Trajectories

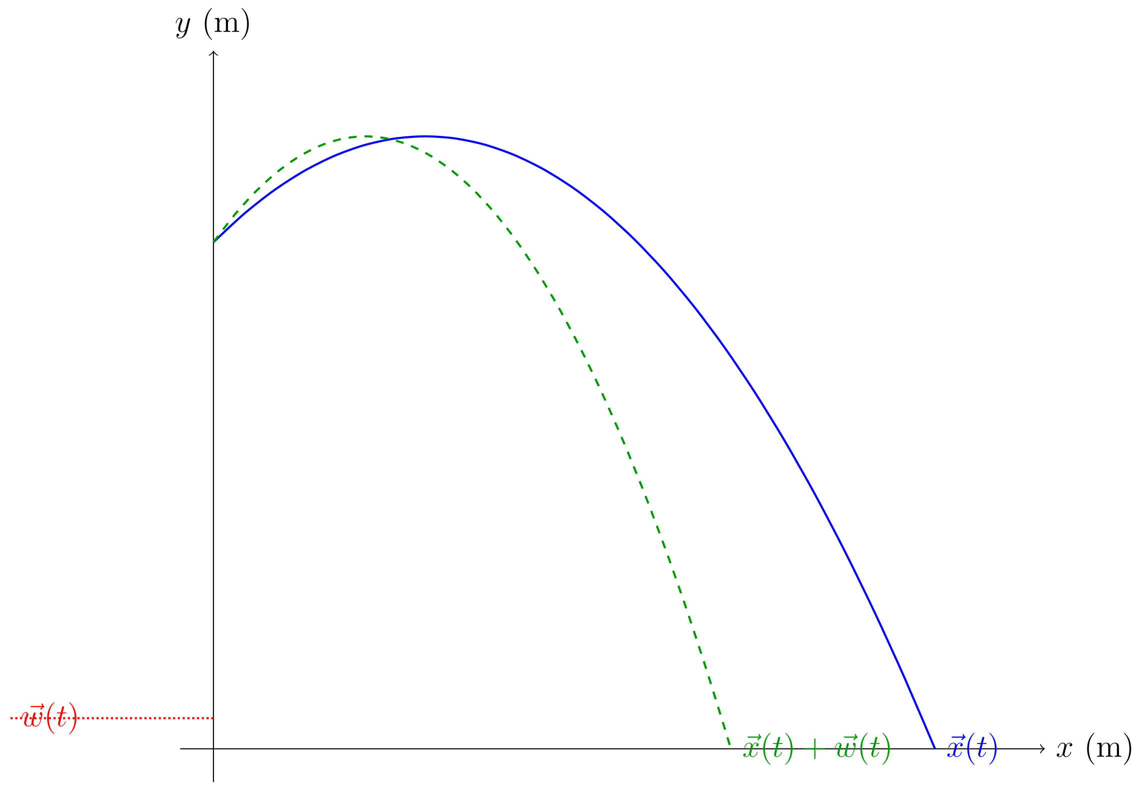

Figure 2: Blue: original AVA arc vector. Red dotted: head-wind arc vector (-1 m/s). Green dashed: resultant arc under wind.

5 Metrics and Independence in AVA

Throughout $[0,1]$ we use

$$u(x) = x^2, \qquad v(x) = \sin x, \qquad w(x) = u(x) + v(x) = x^2 + \sin x.$$

Arc length (geometric size)

For a graph $y = f(x)$ the AVA arc length is $L(f) = \int_a^b \sqrt{1 + f'(x)^2}\, dx$.

$$L(u) = \int_0^1 \sqrt{1 + (2x)^2}\, dx \; = \; \frac{x}{2}\sqrt{1 + 4x^2} + \tfrac{1}{4}\sinh^{-1}(2x)\Big|_0^1 \approx 1.479,$$

$$L(v) = \int_0^1 \sqrt{1 + \cos^2 x}\, dx \approx 1.312,$$

$$L(w) = \int_0^1 \sqrt{1 + \big(2x + \cos x\big)^2}\, dx \approx 2.11.$$

Hence the resultant arc is physically the longest of the three.

Angle between two nonlinear arcs

With the AVA inner product, $\cos\Theta = \dfrac{\langle \vec{u}, \vec{v}\rangle}{\|\vec{u}\|\,\|\vec{v}\|}$.

$$\langle \vec{u}, \vec{v}\rangle = \int_0^1 x^2 \sin x\, dx = \Big[-x^2\cos x + 2x\sin x + 2\cos x\Big]_0^1 \approx 0.2245,$$

$$\|\vec{u}\| = \sqrt{\tfrac{1}{5}}, \qquad \|\vec{v}\| = \sqrt{\tfrac{1}{2} - \tfrac{\sin 2}{4}}, \qquad \Longrightarrow \qquad \Theta \approx 16^\circ.$$

Thus the parabola and sine arcs are almost (but not exactly) aligned in AVA space.

Wronskian determinant

The AVA determinant for two arcs on $[0,1]$ is the classical Wronskian:

$$W(u,v)(x) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = 2x\cos x - \sin x.$$

Because $W(u,v)$ is *not* identically zero on $(0,1)$, the set $\{\vec{u}, \vec{v}\}$ is linearly independent; hence every arc of the form $A\,x^2 + B\,\sin x$ is uniquely representable in the span $\operatorname{span}\{\vec{u}, \vec{v}\}$.

Together, these three quick metrics—length, angle, and Wronskian— illustrate how AVA supplies one-number descriptors for curvature, alignment, and algebraic independence, all without re-solving the underlying differential equations.

6 Building Arc Vectors in Practice

- **Polynomials**: fit data points $\to$ cubic arc.
- **Splines/Bézier**: control points $\to$ piecewise arc vectors.
- **Fourier arcs**: periodic signal $\to$ Fourier-series arc on $[0, 2\pi]$.
- **Taylor arcs**: local cubic/truncated series around x_0.

7 Magnetic Flux: Classical vs. Arc-Vector Calculation

We illustrate how AVA can repackage the standard magnetic-flux surface integral into inner products of arc vectors.

Classical Magnetic Flux

For a static magnetic field $\mathbf{B}(x, y)$ through a planar loop $\mathcal{C} = \partial S$, the flux is

$$\Phi = \iint_S \mathbf{B} \cdot d\mathbf{A} = \oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{r},$$

where $\mathbf{A}$ is a vector potential for $\mathbf{B}$ and $d\mathbf{r} = (dx, dy)$. Parameterise the boundary by $t \in [0, 1]$, $(x(t), y(t))$, and write $\mathbf{A} = (A_x, A_y)$. Then

$$\Phi = \int_0^1 \Big[A_x\big(x(t), y(t)\big)\, \dot{x}(t) + A_y\big(x(t), y(t)\big)\, \dot{y}(t) \Big]\, dt.$$

AVA formulation

Define four scalar arc vectors on $[0, 1]$:

$$\vec{A}_x = \big(A_x(x(t), y(t)), 0, 1\big), \quad \vec{A}_y = \big(A_y(x(t), y(t)), 0, 1\big), \quad \vec{x}' = \big(\dot{x}(t), 0, 1\big), \quad \vec{y}' = \big(\dot{y}(t), 0, 1\big).$$

Using the AVA inner product $\langle \vec{f}, \vec{g} \rangle = \int_0^1 f(t) g(t)\, dt$, we obtain

$$\boxed{\Phi = \langle \vec{A}_x, \vec{x}' \rangle + \langle \vec{A}_y, \vec{y}' \rangle}.$$

Single-object notation (optional). Group the two field components and two velocity components:

$$\vec{\boldsymbol{A}} := (\vec{A}_x, \vec{A}_y), \qquad \vec{\boldsymbol{r}}' := (\vec{x}', \vec{y}').$$

Define a component-wise AVA inner product

$$\langle\!\langle (\vec{f}_1, \vec{f}_2), (\vec{g}_1, \vec{g}_2) \rangle\!\rangle := \langle \vec{f}_1, \vec{g}_1 \rangle + \langle \vec{f}_2, \vec{g}_2 \rangle.$$

Then

$$\Phi = \langle\!\langle \vec{\boldsymbol{A}}, \vec{\boldsymbol{r}}' \rangle\!\rangle.$$

Interpretation

- The classical flux integral becomes two scalar AVA inner products, each pairing a field-component arc with its matching velocity arc.
- All manipulations (addition, projection, orthogonalisation) now act on entire curves rather than on pointwise data, so complex boundary deformations or time-dependent fields can be handled by algebraic operations on arc vectors.
- The paired-vector formulation collapses the computation to a single inner product while staying free of cross-terms.

Thus AVA offers a compact algebraic viewpoint on magnetic flux (and, by analogy, any line or surface integral that factors into component-wise products).

A App A: Mathematical Foundations of Arc Vector Algebra - Deep Research by OpenAI o3

This appendix furnishes a rigorous justification that the *Arc Vector Algebra* (AVA)—defined as smooth real-valued curve segments on a fixed interval $[a, b]$—admits the structure of a real Hilbert space equipped with a standard L^2 – inner product and a Wronskian–determinant analogue for testing linear independence.

A.1 The Vector Space of Arc Vectors

[Arc-vector space] Let

$$AVA := C^\infty([a,b],\mathbb{R}) \quad = \quad \big\{ f : [a,b] \to \mathbb{R} \bigm| f \text{ infinitely differentiable} \big\}.$$

For $f, g \in AVA$ and $\alpha \in \mathbb{R}$ define

$$(f+g)(x) := f(x) + g(x), \qquad (\alpha f)(x) := \alpha f(x) \quad (x \in [a,b]).$$

$(AVA, +, \cdot)$ is a real vector space.

Proof. All vector–space axioms (commutativity, associativity, existence of a zero element and additive inverses, distributivity, scalar compatibility) hold pointwise because $\mathbb{R}$ is a field and the operations are defined component-wise on $[a, b]$. □

A.2 Inner Product and Hilbert–Space Completeness

[Functional inner product] For $f, g \in AVA$ set

$$\langle f, g \rangle \ := \int_a^b f(x)\, g(x)\, dx, \qquad \|f\|_2 := \sqrt{\langle f, f \rangle}.$$

The form $\langle \cdot, \cdot \rangle$ of Definition A is *symmetric*, *bilinear*, and *positive–definite*. Hence $(AVA, \langle \cdot, \cdot \rangle)$ is a real inner–product space.

Proof. Symmetry follows from commutativity of multiplication. Bilinearity follows from linearity of the integral. Since $f^2(x) \geq 0$ and $f \in C^\infty[a, b]$ is continuous, the integral of f^2 vanishes only when $f \equiv 0$ on $[a, b]$. □

[Hilbert–space completion] AVA is dense in $L^2[a, b]$; its norm completion is therefore isometric to the real Hilbert space $L^2[a, b]$.

Sketch. Continuous functions (hence C^∞ functions) are dense in $L^2[a, b]$ by the Weierstrass approximation theorem and the Riesz–Fischer property of L^2. Because $L^2[a, b]$ is complete, the closure $\overline{AVA}$ (with respect to $\|\cdot\|_2$) equals $L^2[a, b]$. Thus the completion of AVA inherits the Hilbert-space structure of $L^2[a, b]$. □

A.3 Wronskian-Determinant Criterion

[Wronskian determinant] For arc vectors $f_1, \dots, f_n \in AVA$ define

$$W[f_1, \dots, f_n](x) = \det \begin{pmatrix} f_1(x) & \dots & f_n(x) \\ f_1'(x) & \dots & f_n'(x) \\ \vdots & & \vdots \\ f_1^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{pmatrix}.$$

[Wronskian criterion] Let $f_1, \dots, f_n \in AVA$ be *real-analytic* on (a, b). If $W[f_1, \dots, f_n] \equiv 0$ on (a, b), then the family is linearly dependent on (a, b). For functions that are merely C^∞ the converse can fail (Peano-type counter-examples).

Idea of proof. If the functions are dependent, a non-trivial $\sum c_i f_i = 0$ yields a non-trivial vector in the kernel of the Wronskian matrix at every x, hence $\det = 0$. Conversely, Abel's identity plus analyticity shows that an identically vanishing Wronskian forces a constant linear relation among the f_i. □

A.4 Summary

- AVA with pointwise operations satisfies all vector–space axioms.
- Equipped with the L^2 inner product it forms a real inner–product space whose completion is Hilbert.
- The Wronskian determinant provides a practical criterion for linear independence under the stated analytic hypothesis.

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References

[1] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw–Hill, 1955.

[2] W. Rudin, *Functional Analysis*, 2nd ed., McGraw–Hill, 1991.

[3] R. Kress, *Linear Integral Equations*, 3rd ed., Springer, 1999.