Arc Vector Algebra - Warm-Burn Primitive (Constant Turn-Rate) with Apollo-11 Style Example

1. Setup

Work in an inertial frame at epoch t_0 with state $(\mathbf{r}_0, \mathbf{v}_0)$ and local RTN unit basis $(\hat{\mathbf{R}}, \hat{\mathbf{T}}, \hat{\mathbf{N}})$. Let $\mathbf{v}_{\text{req}}$ be the required post-burn velocity at t_0. Define the goal vector

$$\mathbf{g} \;=\; \mathbf{v}_{\text{req}} - \mathbf{v}_0. \tag{1}$$

2. Impulsive ("cold") AVA primitive

Given a Δv budget $\rho > 0$, the one-shot canted impulse is

$$\Delta\mathbf{v}_{\text{imp}} \;=\; \rho\,\frac{\mathbf{g}}{\|\mathbf{g}\|}. \tag{2}$$

If $\mathbf{v}_0$ and $\mathbf{v}_{\text{req}}$ have magnitudes $v_i = \|\mathbf{v}_0\|$, $v_f = \|\mathbf{v}_{\text{req}}\|$ and enclose angle Δi, then by the law of cosines

$$\rho_* \;=\; \|\mathbf{v}_{\text{req}} - \mathbf{v}_0\| \;=\; \sqrt{v_i^2 + v_f^2 - 2\,v_i v_f \cos\Delta i}\,. \tag{3}$$

3. Finite ("warm") AVA primitive: constant turn-rate

Assume thrust acceleration magnitude a_t is approximately constant during the burn, and the thrust *direction* rotates at constant angular rate s about a fixed unit axis $\hat{\mathbf{n}}$:

$$\hat{\mathbf{u}}(t) \;=\; \mathcal{R}(\hat{\mathbf{n}},\, st)\,\hat{\mathbf{u}}_0, \qquad 0 \le t \le T, \quad \Theta \equiv sT, \quad \hat{\mathbf{u}}_0 \equiv \frac{\mathbf{g}}{\|\mathbf{g}\|}.$$

Choose $\hat{\mathbf{n}}$ normal to the plane change, for example $\hat{\mathbf{n}} = \mathbf{v}_0 \times \mathbf{v}_{\text{req}} \|\mathbf{v}_0 \times \mathbf{v}_{\text{req}}\|$ (when $\Delta i \neq 0$). With Rodrigues' formula,

$$\mathcal{R}(\hat{\mathbf{n}}, \theta)\,\hat{\mathbf{u}}_0 = \hat{\mathbf{u}}_0 \cos\theta + (\hat{\mathbf{n}} \times \hat{\mathbf{u}}_0)\sin\theta + \hat{\mathbf{n}}\,(\hat{\mathbf{n}}\cdot\hat{\mathbf{u}}_0)\,(1 - \cos\theta),$$

the delivered velocity change is

$$\Delta\mathbf{v}_{\text{fin}} = \int_0^T a_t\,\hat{\mathbf{u}}(t)\,dt = \frac{a_t}{s}\left[\sin\Theta\,\hat{\mathbf{u}}_0 + (1 - \cos\Theta)\,(\hat{\mathbf{n}} \times \hat{\mathbf{u}}_0) + (\Theta - \sin\Theta)\,(\hat{\mathbf{n}}\cdot\hat{\mathbf{u}}_0)\,\hat{\mathbf{n}}\right]. \tag{4}$$

Special case $\hat{\mathbf{n}}\cdot\hat{\mathbf{u}}_0 = 0$ (typical for pure plane-change coupling):

$$\|\Delta\mathbf{v}_{\text{fin}}\| = \frac{a_t}{s}\,2\sin\frac{\Theta}{2} = \frac{\rho}{\Theta}\,2\sin\frac{\Theta}{2}, \qquad \rho \equiv a_t T. \tag{5}$$

To match the impulsive target magnitude ρ_* from (3), choose

$$\rho \;=\; \rho_* \,\frac{\Theta}{2\sin(\Theta/2)}. \tag{6}$$

4. Arc vector (real Hilbert-space form) and "best next arc"

Represent a warm burn by the 6D arc vector

$$\mathbf{a} \;=\; [\,\rho\,\hat{\mathbf{u}}_0 \;;\; \Theta\,\hat{\mathbf{n}}\,] \in R^6, \quad splitas \mathbf{a} = [\,\mathbf{t}\,;\,\mathbf{c}\,],\ \mathbf{t},\mathbf{c} \in R^3. \tag{7}$$

Endow R^6 with inner product

$$\langle \mathbf{a}, \mathbf{b} \rangle_W \;=\; \mathbf{t}_a^\top \mathbf{t}_b \;+\; \alpha\,\mathbf{c}_a^\top \mathbf{c}_b, \qquad \alpha > 0, \quad \|\mathbf{a}\|_W = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle_W}. \tag{8}$$

With budget $\|\mathbf{a}\|_W \le \rho$, the best next arc is the W-metric projection of a goal arc $\mathbf{g}_W$:

$$\mathbf{a}^* \;=\; \rho\,\frac{\mathbf{g}_W}{\|\mathbf{g}_W\|_W}. \tag{9}$$

The impulsive ("cold") case is the limit $\Theta \to 0$, i.e. $\mathbf{a} = [\,\rho\,\hat{\mathbf{u}}_0;\, \mathbf{0}\,]$.

5. Worked numerical example (Apollo-11 style, perigee TLI)

Earth $\mu = 398600.4418$ km^3/s^2, radius $R = 6378.137$ km. Parking orbit altitude $h = 185$ km: $r_p = R + h = 6563.137$ km. Circular speed $v_i = \sqrt{\mu/r_p} = 7.793152$ km/s. Target: translunar ellipse with apogee $r_a \approx 384400$ km: $a = 12(r_p + r_a)$, perigee speed $v_f = \sqrt{\mu(2/r_p - 1/a)} = 10.928283$ km/s. Consider a small plane change $\Delta i = 5^\circ$ at perigee.

Impulsive comparison. Split baseline (pure tangential change $|v_f - v_i|$ plus pure plane change $2v_{\min}\sin(\Delta i/2)$ at the lower speed $v_{\min} = v_i$):

$$\Delta v_{\text{split}} \approx 3.814996 \text{ km/s}.$$

Single canted burn from (3):

$$\Delta v_{\text{single}} \approx 3.236852 \text{ km/s}.$$

Saving $\approx 15.15\%$ by coupling energy and plane change in one shot.

Warm-burn sizing (constant turn-rate). Let $\Theta = \Delta i = 5^\circ = 0.0872665$ rad. Using (6),

$$\frac{\Theta}{2\sin(\Theta/2)} \approx 1.000317, \qquad \rho \;=\; \rho_* \times 1.000317 \;\approx\; 3.237879 \text{ km/s}.$$

Thus a constant-turn warm burn needs only about 0.032% more integrated thrust than the impulsive ideal at 5°.

Remarks. Apollo timed TLI so that $\Delta i \approx 0$; this example is a stress test showing AVA's default one-shot coupling benefit when a small plane change is unavoidable. In high-fidelity design (J2/three-body and constraints), use this AVA arc as a warm start; the full optimizer enforces the detailed physics.